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Efficient domain decomposition based reliability analysis for polymorphic uncertain material parameters

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Realistic uncertainty description incorporating aleatoric and epistemic uncertainties can be described within the framework of polymorphic uncertainty, which is computationally demanding. Utilizing a domain decomposition approach for random field based uncertainty models the proposed level-based sampling method can reduce these computational costs significantly and shows good agreement with a standard sampling technique. While 2-level configurations tend to get unstable with decreasing sampling density 3-level setups show encouraging results for the investigated reliability analysis of a structural unit square.

1 Introduction

Realistic modeling of uncertainties for engineering models require appropriate uncertainty models. Uncertainties can be categorized into aleatoric and epistemic ones [1]. Aleatoric uncertainties are associated with the natural variability in processes or parameters and is usually governed by probability theory [2, 3]. Epistemic uncertainties are due to limited data and/or the lack of knowledge and can be modeled using intervals or fuzzy variables [4,5]. Realistic uncertainty models can be achieved by combining basic uncertainty approaches to polymorphic uncertainty models, like interval probability-based random variables or fields [6, 7]. Various approaches have been developed to remedy high computational costs of reliability analysis: Ranging from efficient failure probability estimation [2, 8] to domain decomposition (DD) approaches involving random field material description [9, 10]. Analyses involving polymorphic uncertainty are typically even more challenging due to the analysis procedures in the non-stochastic space, e.g. α-level optimization [11]. Surrogate-based fuzzy analysis have been proposed to boost numerically efficiency [12]. This contribution proposes a computational efficient level-based sampling technique utilizing a DD approach of interval probability-based random fields for a structural reliability analysis.

2 Methods

2.1 Polymorphic uncertainty models

Random fields A random field or process Z^{rf} comprises a set of random variables $\{Z^{\mathrm{rf}}(\theta),\ \theta \in \Theta\}$, where $Z^{\mathrm{rf}}(\theta)$ represents a random variable at every point θ in Θ. Random fields are usually defined over the Euklidian space ($\Theta \subseteq \mathbb{R}^3$), whereas random processes are associated with time ($\Theta \subseteq \mathcal{T}$). Random fields are defined on a common probability space $(\Omega,\ \Sigma,\ \mathrm{P})$, where Ω is the set of events, Σ_Ω is a sigma-algebra over Ω and $\mathrm{P} : \Sigma_\Omega \to [0, 1]$ a probability measure for the occurrence of an event $\omega \in \Sigma_\Omega$. In case of an univariate random field, at every point θ, the field is a random variable: $Z^{\mathrm{rf}}(\theta,\ \omega) = X^{\mathrm{r}}(\omega)$, whereas it is a random vector $Z^{\mathrm{rf}}(\theta,\ \omega) = \boldsymbol{X}^{\mathrm{r}}(\omega)$ for a multivariate random field.

A random field is characterized by its auto-covariance $\mathrm{Cov}\,(\theta_1,\ \theta_2) = \mathrm{E}\left(Z^{\mathrm{rf}}(\theta_1, \omega),\ Z^{\mathrm{rf}}(\theta_2, \omega)\right)$ of two points $\theta_1,\ \theta_2 \in \Theta$. An auto-correlation function $R_{Z^{\mathrm{rf}}} : \Theta \times \Theta \to [-1,\ 1]$ with $R_{Z^{\mathrm{rf}}}(\theta_1,\ \theta_2) = R_{Z^{\mathrm{rf}}}(\theta_2,\ \theta_1)$ depends on a distance measure d_Θ and an auto-correlation length l_Θ. Typically, the isotropic Euclidean distance is used for random fields in an engineering context.

Interval probability-based random fields Combinations of basic uncertainty models form polymorphic uncertainty models that offer the possibility to describe uncertain input of engineering models more realistically. Interval probability-based random fields are introduced exemplary for polymorphic uncertainty models. Further models (like fuzzy probability-based random variables) can be found in [6].

Interval probability-based random fields (ip-rf) H^{iprf} are introduced based on the definition of random fields. Interval probability-based random fields are defined if at least one of these characteristics is fulfilled:

- An interval probability space $(\Omega,\ \Sigma,\ \mathrm{P}^{\mathrm{i}})$, e.g. by interval parametrization of the underlying distribution function.
- An interval auto-correlation function $R^{\mathrm{i}}_{H^{\mathrm{iprf}}}\left(\boldsymbol{\theta}_A,\ \boldsymbol{\theta}_B;\ d^{\mathrm{i}}_\Theta; l^{\mathrm{i}}_\Theta\right)$, defined by an interval variable in auto-correlation length l^{i}_Θ and/or an interval parametrization in the distance measure d^{i}_Θ.

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2.2 Random field generation using a Domain Decomposition Karhunen-Loève method

This section briefly summarizes the Domain Decomposition Karhunen-Loève (DD-KL) method from [9], which is used in this study. We consider non-overlapping subdomains of domain Θ with $\Theta = \bigcup_{d=1}^{D} \Theta_d, \Theta_i \cap \Theta_{i\neq j} = \emptyset$. The main idea of the DD-KL is to express a global random field (RF) via local (subdomain based) eigenmodes and random variables, enabling an independent parallelized computation of the global RF in the subdomains while reducing the stochastic dimension dramatically.

Random field by global KL Let $U \in L_2(\Theta, \Omega)$ be a centered random process with known covariance. The truncated KL approximation, U_N, consists of an N-term expansion where each term is composed of a product of a deterministic function of $L_2(\Theta)$ and a random variable of $L_2(\Omega)$. The KL approximation is defined as the minimization of the approximation error in a L_2 sense. A zero-mean RF can be expressed as such a KL expansion, which is given by $U_N(\vec{x}, \omega) \equiv \sum_{\alpha=1}^{N} \sqrt{\lambda_\alpha}\eta_\alpha(\omega)\Phi_\alpha(\vec{x})$, where λ_α and Φ_α are the dominant eigenvalues and their corresponding (normalized) eigenfunctions of the covariance and $\eta_\alpha(\omega)$ are the orthonormal stochastic coordinates of U. Due to the structure of the covariance function, the non-negative eigenvalues can be ordered in decreasing magnitude leading to a truncation of the expansion.

Random field by local KL Local eigenmodes $\widetilde{\phi}_\beta^{(d)}$ and their corresponding eigenvalues $\lambda_\beta^{(d)}$ are introduced on the subdomain Θ_d. The local eigenfunctions are extended to the global domain. Therfore, the eigenfunctions $\phi_\beta^{(d)}(\vec{x})$ are also orthonormal in Θ.

Choosing m_d dominant eigenpairs for each subdomain Θ_d, we get an orthonormal reduced basis. An approximation of the global modes is then given by $\Phi(\vec{x}) \approx \hat{\Phi}(\vec{x}) = \sum_{d=1}^{D}\sum_{\beta}^{m_d} a_\beta^{(d)}\phi_\beta^{(d)}(\vec{x})$, where the vector of local coordinates of $\hat{\Phi}$ for $\vec{x} \in \Theta_d$ is given by $\vec{a}^{(d)} = \left(a_1^{(d)}, \ldots, a_{m_d}^{(d)}\right)$. Applying the Galerkin method yields a global discrete eigenvalue probelm. Using its eigenvalues Λ_α in decreasing magnitude, we can set up a truncated approximation of the global RF U

$$U \approx \hat{U}_{\hat{N}}(\vec{x}, \omega) = \sum_{d=1}^{D}\left[\sum_{\beta=1}^{m_d}\sqrt{\lambda_\beta^{(d)}}\xi_\beta^{(d)}(\omega)\phi_\beta^{(d)}(\vec{x})\right], \text{ with } \xi_\beta^{(d)}(\omega) = \sum_{\alpha=1}^{\hat{N}}\sqrt{\frac{\Lambda_\alpha}{\lambda_\beta^{(d)}}}a_{\alpha,\beta}^{(d)}\hat{\eta}_\alpha(\omega), \tag{1}$$

where $\{\xi_\beta^{(d)}, \beta = 1, \ldots, m_d\}$ are called the local random variables. Hence, the global RF U is represented on the subdomain level in terms of local random variables, eigenmodes and eigenvalues. The proposed method of level-based sampling exploits this property.

2.3 Level-based sampling for interval reliability analysis

The analysis of any uncertain model requires sampling over the underlying uncertainty domain, for example Monte Carlo Sampling (MCS) on a stochastic domain.

Considering a model with a heterogeneous material described by an interval probability-based random field $Z^{\mathrm{iprf}}(\vec{x}, \omega, p^{\mathrm{i}})$, where $\vec{x} \in \mathbb{R}^3$, sampling on an interval-stochastic space is required. For every sample p_k from the interval space p^{i} a (usually computationally demanding) stochastic analysis is performed, for example a failure probability $p_{F,k}$ is estimated.

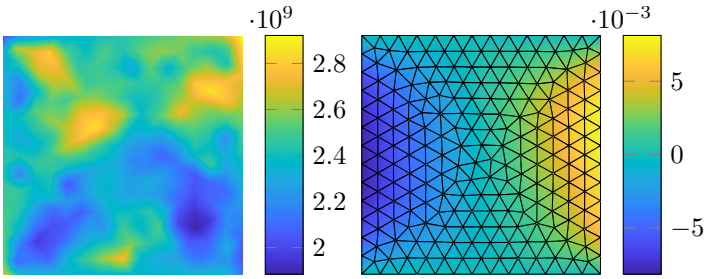
This contribution suggests a level-based sampling approach of the interval space to reduce the computational costs. For the considered case of a reliability analysis, it is assumed that the a critical region can be identified, where the QoI (Quantaty of Interest) is most likely to reach the limit state threshold. We define L levels $l = 1, \ldots, L$ with level samplings $p_1, \ldots p_L$, s.t. the number of samples n_l satisfies $n_1 \geq \cdots \geq n_L$, with $l = 1$ corresponding to the most critical level. D_l subdomains are associated to each level l based on the distance between the sub-domains' centroid and the critical region. The level sampling p_l are combined to $p = \{p_1, \ldots, p_L\}$, s.t. $\min \sum_{l=2}^{L}(p_l - p_1)^2$.

The subdomain based RF generation method enables a localization in the physical space of the RF parameters (like auto- and cross-correlation). Therefor the level-based sampling can be applied to interval valued RF parameters. Depending on the number of subdomains D_l and samples n_l per level the computational costs are given in terms of $\sum_{l=1}^{L} n_l D_l$ subdomain calculations.

3 Numerical example

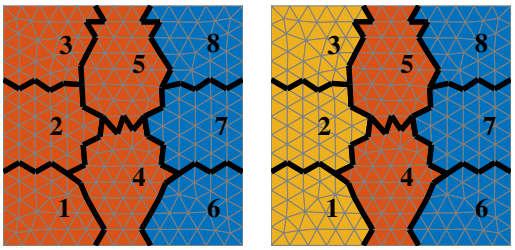
As an illustrative example we consider a structural unit square modeled using standard finite element method with a plane-stress linear elastic material law. A uniform vertical displacement is applied at the top while the system is fixed at bottom. The Young's modulus is described via lognormally distributed random field E^{iprf} with interval valued correlation length $l_c^{\mathrm{i}} = [0.2,\ 0.6]$ as parameter of a squared-exponential auto-correlation function, cf. Fig. 2; Poisson's ratio is aussumed as constant. The QoI is the maximum horizontal displacement $u_{x,\mathrm{mx}}$. The unit square is divided into $D = 8$ subdomains. The subdomains are associated to $L = 2$ and $L = 3$ levels, cf. Fig. 3a and Fig. 3b, respectively, where subdomains $d \in 6, 7, 8$ correspond to

$l = 1$ etc. For this case study, n_l samples are distributed evenly over the whole interval width $l_{\mathrm{c,w}} = 0.6 - 0.2 = 0.4$, where the sample points are calculated according to $p_i = l_{\mathrm{c,w}}/n_l(i - 0.5)$ for $i = 1, \ldots, n_l$.



(a) Young's modulus E [Pa] **(b)** Horizontal displacement u_x [m]

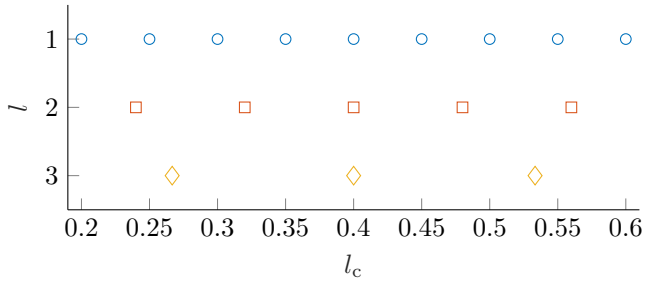
Fig. 2 Sample plots of Young's modulus random field E^{iprf} and the resulting horizontal displacement u_x



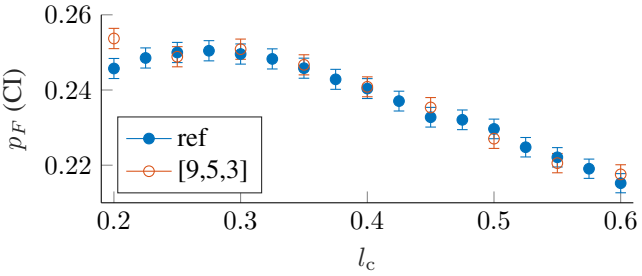
(a) $L = 2$ **(b)** $L = 3$

Fig. 4 Domain decomposition of a unit square, subdomains associated to $L = 2$ and $L = 3$ levels

For different configurations n_l reliability analyses are performed using a limit state function $g = u_{x,\mathrm{th}} - u_{x,\mathrm{mx}}$, where $g < 0$ defines the failure region. In this study $u_{x,\mathrm{th}} = 8.8\,\mathrm{mm}$ is used. A configuration with the maximum number of samples on all subdomains serves as a reference ($L = 1, D_1 = D$). In Fig. 5 a representative result for $L = 3$ levels with $n_l = \{9, 5, 3\}$ is shown, where Fig. 5a illustrates the level-based sampling and Fig. 5b shows the estimated failure probabilities of this configuration compared to the reference. It can be noticed that the reference and the $[9, 5, 3]$ configuration are in very good agreement for the most l_{c} sample points but show some discrepancies at the edges of $l_{\mathrm{c}}^{\mathrm{i}}$. The computational costs are calculated to $N_{\mathrm{ref}} = 72$ subdomain calculation for the reference compared to $N_{D_l,n_l} = \sum_{l=1}^{3} D_l \cdot n_l = 3 \cdot 9 + 2 \cdot 5 + 3 \cdot 3 = 46$ subdomain calculations, which constitutes only 64% of the reference's computational costs.



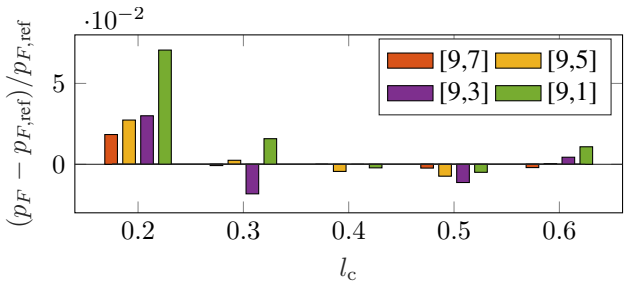
(a) Level-based sampling configuration using $L = 3$ levels with $n_l = [9, 5, 3]$



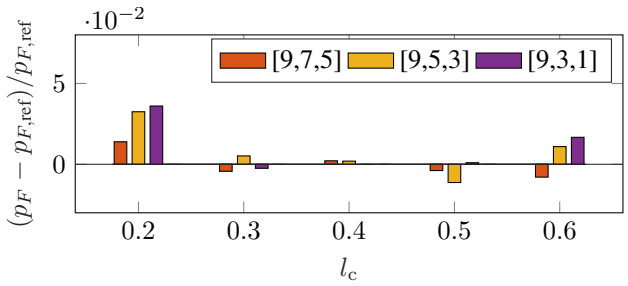
(b) Failure probabilities for reference (ref) and level-based sampling configuration ($[9, 5, 3]$

Fig. 5 Results of a reliability analysis: (a) level-based sampling configuration of $L = 3$, (b) comparison to reference

Figure 6 shows the relative error of the estimates p_F (w.r.t. to the reference $p_{F,\mathrm{ref}}$) for different level sampling configurations for $L = 2$ and $L = 3$ levels, respectively. It can be observed that the relative error increases with decreasing sampling density (e.g. $[9, 7] \rightarrow [9, 1]$) and at the lower edge of the auto-correlation length interval $l_{\mathrm{c}}^{\mathrm{i}}$. Finally, the Mean Squared Error (MSE) of p_F over all sample points in $l_{\mathrm{c}}^{\mathrm{i}}$ is evaluated as a function of the relative computational costs $N_{D_l,n_l}/N_{\mathrm{ref}}$ for each sampling configurations on $L = 2$ and $L = 3$ sampling levels, respectively, cf. Fig. 7. Obviously, the MSE increases with decreasing sampling density (i.e. relative computational costs). While the three-level configurations outperform the two-level



(a) $L = 2$



(b) $L = 3$

Fig. 6 Relative error of p_F for different configurations n_l

ones in the high accuracy regime (i.e. low MSE), the large MSE for $n_l = [9, 1]$ indicates a higher robustness of the sparse sampling configurations with relative computational costs of <50% for $L = 3$ levels, where the number of samples per level is decreased in a smoother fashion.

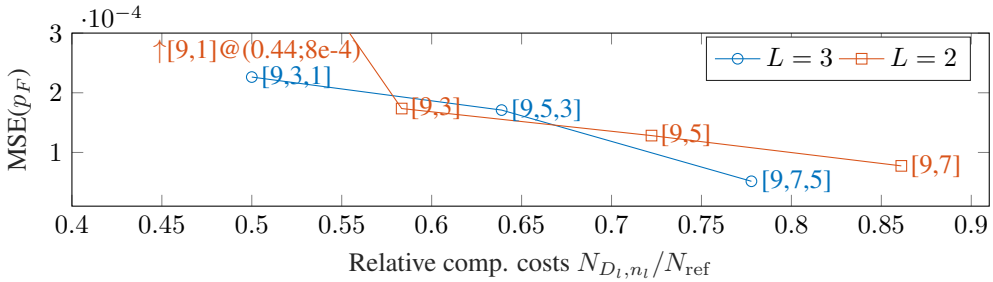


Fig. 7 Mean squared error (MSE) of estimated failure probability p_F as a function of relative computational costs $N_{D_l,n_l}/N_{\text{ref}}$

4 Conclusion

Uncertainty quantification within the scope of polymorphic uncertainty, which combines basic uncertainty models like interval, fuzzy and random variables, is computationally challenging. In the case of interval or fuzzy probability-based random fields a domain decomposition approach can be exploited, which enables localized, subdomain-based random field definition. The proposed method of level-based sampling was applied to a reliability analysis of a structural unit square where its Young's modulus was modeled via an interval probability-based random field. It could be shown that the computational costs were reduced up to 50% with a mean squared error of the estimated failure probabilities of 2×10^{-4} using a 3-level based sampling configuration. Sampling configurations with only 2 sampling levels were found to be unstable for lower sampling densities.

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