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Full Length Article

Characterizations of pointwise multipliers of Besov spaces in endpoint cases with an application to the duality principle ☆

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A B S T R A C T

Let $p,q \in (0,\infty]$, $s \in \mathbb{R}$, and $M(B^s_{p,q}(\mathbb{R}^n))$ denote the pointwise multiplier space of the Besov space $B^s_{p,q}(\mathbb{R}^n)$. In this article, the authors first establish the characterizations of both $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in \mathbb{R} \setminus \{0\}$ and $M(B^s_{\infty,1}(\mathbb{R}^n))$ with $s \in (-\infty, 0]$. Then, as an application, the authors give a corrected proof of the well-known duality principle for pointwise multiplier spaces of Besov spaces, namely the formula

$$M\left(B^s_{p,q}(\mathbb{R}^n)\right) = M\left(B^{-s}_{p',q'}(\mathbb{R}^n)\right),$$

where $p,q \in [1,\infty]$, $s \in \mathbb{R}$, and $1/a + 1/a' = 1$ for any $a \in [1,\infty]$, and, moreover, the authors also show that this duality principle is sharp in some sense. The proofs of all these results essentially depend on the duality theorem of Besov spaces themselves, some elaborate estimates of paraproducts as well as the relation between $M(B^s_{p,q}(\mathbb{R}^n))$ and the auxiliary

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multiplier space $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$, where $\widetilde{B}^s_{p,q}(\mathbb{R}^n)$ denotes the completion of the Schwartz function space $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$.

1. Introduction

Determining the set of all pointwise multipliers for a given function (or distribution) space belongs to the key problems in the theory of function spaces. Remarkable contributions have been made by Maz'ya and Shaposhnikova [18,19]. They described the set of all pointwise multipliers for Sobolev, Besov, and Bessel potential spaces for a wide range of parameters. For more information about pointwise multipliers on various function spaces, we refer to Strichartz [34], Kalyabin [12,13], Triebel [35,37,38], Netrusov [25], Gala and Sawano [10], Kawasumi and Nakai [14], Nakai and Yabuta [24], Nakai and Sadasue [23], and Nakai [20–22] as well as the monographs [28] and [31]. In particular, smoothness function spaces, such as Besov and Triebel–Lizorkin spaces, and their related multiplier spaces have been intensively investigated; see, for instance, [11] for weighted Besov and Triebel–Lizorkin spaces, [29] for Besov–Morrey and Triebel–Lizorkin–Morrey spaces, [43] for Besov-type and Triebel–Lizorkin-type spaces, [42] for Besov and Triebel–Lizorkin spaces in metric spaces as well as [5] for Zygmund spaces.

It is well known that pointwise multipliers have been widely used in the study of partial differential equations; see, for instance, the monographs [18,19] and [1]. Here, we present another recent very interesting application of pointwise multipliers in the study of the so-called bilinear decomposition for multiplications of elements in the Hardy space $H^1(\mathbb{R}^n)$ and its dual space $\mathrm{BMO}\,(\mathbb{R}^n)$, which is just [4, Conjecture 1.7]. This means to find the "smallest" linear space $\mathcal{Y}$ such that $H^1(\mathbb{R}^n) \times \mathrm{BMO}\,(\mathbb{R}^n)$ has the following bilinear decomposition of the form:

$$H^1(\mathbb{R}^n) \times \mathrm{BMO}\,(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + \mathcal{Y}.$$

Indeed, via making full use of the characterization of pointwise multipliers of $\mathrm{BMO}\,(\mathbb{R}^n)$, which was obtained by Nakai and Yabuta [24] (see also the recent survey [22] of Nakai), Bonami et al. [3] showed $\mathcal{Y} = H^{\log}(\mathbb{R}^n)$ (a special case of Musielak–Orlicz–Hardy spaces originally introduced by Ky [16]) and hence completely solved this conjecture. The corresponding conjectures on the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1)$ and their dual spaces as well as their local versions were completely solved in [2,6,41,44] via first establishing the characterizations of pointwise multipliers of Campanato spaces. For more studies on the real-variable theory of Musielak–Orlicz–Hardy spaces and their relations with multiplier spaces, we also refer to [16,40].

Here in our work we will concentrate on the multiplier space $M(B^s_{p,q}(\mathbb{R}^n))$ of the Besov space $B^s_{p,q}(\mathbb{R}^n)$. Studied already in various articles (see, for instance, [15,17,19, 25–27,33,37]), the knowledge is still rather incomplete. In particular, in the endpoint cases $p = \infty$ or $q = \infty$, there still exist many cases in which a concrete characterization of $M(B^s_{p,q}(\mathbb{R}^n))$ is unknown.

In this article we establish characterizations of the following three multiplier spaces:

- $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in (0, \infty)$ (see Theorem 3.1),
- $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in (-\infty, 0)$ (see Theorem 3.5),
- $M(B^s_{\infty,1}(\mathbb{R}^n))$ with $s \in (-\infty, 0]$ (see Theorems 3.9 and 3.13).

Recall that the characterizations of both $M(B^0_{1,\infty}(\mathbb{R}^n))$ and $M(B^s_{\infty,1}(\mathbb{R}^n))$ with $s \in (0, \infty)$ were given, respectively, in [15, Theorem 5] (see also [17, Theorem 3.3 and Remark 3.4]) and [26, Theorem 1.7]. As an application of the preceding characterizations we will give a corrected proof of the well-known duality principle for pointwise multiplier spaces of Besov spaces, namely the formula

$$M\left(B^s_{p,q}(\mathbb{R}^n)\right) = M\left(B^{-s}_{p',q'}(\mathbb{R}^n)\right), \tag{1.1}$$

where $p, q \in [1, \infty]$, $s \in \mathbb{R}$, and, as usual, $1/a + 1/a' = 1$ for any $a \in [1, \infty]$. In [9, p. 134], Frazier and Jawerth stated an even more general result. To be precise, let X be a Banach space such that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ and denote by X' its dual space, then Frazier and Jawerth claimed that $M(X) = M(X')$. But this is not true in this generality as was first observed by Triebel [37]. Formula (1.1) was also mentioned in [28, Lemma 4.9], but the given proof there contains an essential error (see Remark 4.2 below for more details).

Our proofs of all these results obtained in this article essentially rely on the duality theorem of Besov spaces themselves, some elaborate estimates of paraproducts as well as the relation between $M(B^s_{p,q}(\mathbb{R}^n))$ and the auxiliary multiplier space $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$, where $\widetilde{B}^s_{p,q}(\mathbb{R}^n)$ denotes the completion of the Schwartz function space $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$.

The remainder of this article is organized as follows.

In Section 2, we first recall some basic concepts and properties about both Besov spaces and associated pointwise multiplier spaces. We then establish an identity between $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$ and $M(B^{-s}_{p',q'}(\mathbb{R}^n))$ (see Lemma 2.12) and an embedding result between $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$ (see Lemma 2.15), which play essential roles in this article.

The target of Section 3 is to characterize both the pointwise multiplier spaces $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in \mathbb{R} \setminus \{0\}$ and $M(B^s_{\infty,1}(\mathbb{R}^n))$ with $s \in (-\infty, 0]$. On the one hand this complements the knowledge of pointwise multiplier spaces of Besov spaces, on the other hand it also plays a key role in our proof of the duality principle. First, via carefully estimating paraproducts and constructing the auxiliary functions $\{g_k\}_{k \in \mathbb{N} \cap [2,\infty)}$ [see (3.14) below for the definition], we obtain a characterization of the multiplier space $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in (0, \infty)$. Then, by some elaborate estimates of paraproducts again and several embedding relations between the two multiplier spaces $M(B^s_{p,q}(\mathbb{R}^n))$ and

$M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$ (see, for instance, Lemma 3.8), we will obtain a Fourier analytic characterization of both $M(B^s_{1,\infty}(\mathbb{R}^n))$ for any $s \in (-\infty, 0)$ and $M(B^s_{\infty,1}(\mathbb{R}^n))$ for any $s \in (-\infty, 0]$.

In Section 4, based on the characterizations obtained in the last section as well as the identity between $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$ and $M(B^{-s}_{p',q'}(\mathbb{R}^n))$ established in Lemma 2.12, we give a proof of the duality principle (1.1). Moreover, we give a counterexample to show that the identity (1.1) is no longer true for values $p, q \in (0, 1)$.

Finally, we make some conventions on notation. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}_+$ the natural numbers including 0, $\mathbb{Z}$ the integers, and $\mathbb{R}$ the real numbers. Let $n \in \mathbb{N}$. Then $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{Z}^n$ and $\mathbb{Z}^n_+$ denote, respectively, the spaces of vectors in $\mathbb{R}^n$ with integer and with nonnegative integer components. For any $s \in \mathbb{R}$, the symbol $\lceil s \rceil$ denotes the smallest integer not less than s, and the symbol $\lfloor s \rfloor$ denotes the largest integer not greater than s. For any $j \in \mathbb{Z}$ and $\nu \in \mathbb{Z}^n$, define the dyadic cube $Q_{j,\nu} := 2^{-j}(\nu + [0, 1)^n)$. The symbol $\mathscr{Q}$ denotes the set of all cubes with edges parallel to the coordinate axes. In addition, for any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$B(x, r) := \{y \in \mathbb{R}^n : \ |y - x| < r\}.$$

For any locally integrable function u and any measurable set $E \subset \mathbb{R}^n$, let

$$\fint_E u(x)\, dx := \frac{1}{|E|} \int_E u(x)\, dx$$

and the characteristic function of E is denoted by $\mathbf{1}_E$. Also, throughout this article, for any $q \in (0, \infty]$, let

$$q' := \begin{cases} \dfrac{q}{q-1} & \text{when } q \in (1, \infty), \\ \infty & \text{when } q \in (0, 1]. \end{cases} \tag{1.2}$$

Then, for any $q \in [1, \infty]$, q' is just the usual *conjugate index* of q, that is, $1/q + 1/q' = 1$. Furthermore, the symbols $c, c_1, \dots$ denote positive constants which depend only on the fixed parameters n, s, b, and p, and probably also on auxiliary functions unless otherwise stated, their values may vary from line to line. Sometimes we use the symbol "$\lesssim$" instead of "$\le$". The meaning of $A \lesssim B$ is given by: there exists a positive constant C such that $A \le C\, B$. Similarly we shall use $\gtrsim$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g = h$ or $f \lesssim g \le h$, rather than $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$. The symbol $A \sim B$ will be used as an abbreviation of $A \lesssim B \lesssim A$.

2. Besov spaces and pointwise multipliers

The main targets of this section are twofold. The first one is to recall some basic concepts and results about Besov spaces as well as pointwise multiplier spaces, which are frequently used in this article. The second one is to establish several useful properties of the multiplier spaces $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$, which also play key roles later.

2.1. Besov spaces

In this subsection, we recall some basic concepts and related auxiliary lemmas about the Besov space $B^s_{p,q}(\mathbb{R}^n)$, which are important in this article. In particular, we present the concept of the subspace $\widetilde{B}^s_{p,q}(\mathbb{R}^n)$ of $B^s_{p,q}(\mathbb{R}^n)$ and also recall the dual theorem of Besov spaces, both of which play essential roles in the investigation of Besov multiplier spaces in this article. To do this, let $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support, and let $\mathcal{S}(\mathbb{R}^n)$ denote the set of all Schwartz functions on $\mathbb{R}^n$, whose topology is determined by a family of norms, $\{\|\cdot\|_{\mathcal{S}_M(\mathbb{R}^n)}\}_{M\in\mathbb{Z}_+}$, where, for any $M\in\mathbb{Z}_+$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$,

$$\|\varphi\|_{\mathcal{S}_M(\mathbb{R}^n)} := \sup_{\{\alpha\in\mathbb{Z}^n_+,\, |\alpha|\le M\}} \sup_{x\in\mathbb{R}^n} \left\{(1+|x|)^{n+M+|\alpha|}\,|\partial^\alpha\varphi(x)|\right\}$$

with the multi-index $\alpha:=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}^n_+$, $|\alpha|:=\alpha_1+\cdots+\alpha_n$, and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$. Also, let $\mathcal{S}'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology. Here and thereafter, for any $\gamma:=(\gamma_1,\dots,\gamma_n)\in\mathbb{Z}^n_+$ and $x:=(x_1,\dots,x_n)\in\mathbb{R}^n$, let $x^\gamma := x_1^{\gamma_1}\cdots x_n^{\gamma_n}$. For any $f\in\mathcal{S}'(\mathbb{R}^n)$, we use $\mathcal{F}f$ to denote its *Fourier transform* and $\mathcal{F}^{-1}f$ to denote its *inverse Fourier transform* in $\mathcal{S}'(\mathbb{R}^n)$; in particular, for any $f\in L^1(\mathbb{R}^n)$ and $\xi\in\mathbb{R}^n$,

$$\mathcal{F}f(\xi) := (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(x)\,e^{-ix\cdot\xi}\,dx$$

and

$$\mathcal{F}^{-1}f(\xi) := (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(x)\,e^{ix\cdot\xi}\,dx.$$

Let $\phi_0\in\mathcal{S}(\mathbb{R}^n)$ be a radial and real-valued function satisfying that

$$0\le\phi_0\le 1,\ \phi_0\equiv 1 \text{ on } \{x\in\mathbb{R}^n:\ |x|\le 1\},\text{ and } \phi_0\equiv 0 \text{ on } \{x\in\mathbb{R}^n:\ |x|\ge 3/2\}. \tag{2.1}$$

Define $\{\phi_k\}_{k\in\mathbb{N}}$ by setting, for any $x\in\mathbb{R}^n$,

$$\phi_1(x) := \phi_0\left(\frac{x}{2}\right) - \phi_0(x) \tag{2.2}$$

and, for any $k\in\mathbb{N}\cap[2,\infty)$,

$$\phi_k(x) := \phi_1\left(2^{-k+1}x\right). \tag{2.3}$$

Clearly, $\sum_{k\in\mathbb{Z}_+}\phi_k=1$ on $\mathbb{R}^n$. Furthermore, for any $k\in\mathbb{Z}_+$ and $f\in\mathcal{S}'(\mathbb{R}^n)$, we let

$$S_k f:=\mathcal{F}^{-1}\left(\phi_k\mathcal{F}f\right)=\varphi_k*f \tag{2.4}$$

and

$$S^k f:=\sum_{j=0}^{k}S_j f=2^{nk}\left[\varphi_0\left(2^k\cdot\right)\right]*f, \tag{2.5}$$

where, for any $k\in\mathbb{Z}_+$, $\varphi_k:=(2\pi)^{-n/2}\mathcal{F}^{-1}\phi_k$. It is easy to show that, for any $f\in\mathcal{S}'(\mathbb{R}^n)$,

$$\operatorname{supp}\mathcal{F}\left(S_0 f\right)\subset\left\{x\in\mathbb{R}^n:\ |x|\le 3/2\right\}, \tag{2.6}$$

for any $k\in\mathbb{N}$

$$\operatorname{supp}\mathcal{F}\left(S_k f\right)\subset\left\{x\in\mathbb{R}^n:\ 2^{k-1}\le|x|\le 3\cdot 2^{k-1}\right\} \tag{2.7}$$

and

$$\operatorname{supp}\mathcal{F}\left(S^k f\right)\subset\left\{x\in\mathbb{R}^n:\ |x|\le 3\cdot 2^{k-1}\right\}, \tag{2.8}$$

and, for any $k\in\mathbb{N}\cap[2,\infty)$

$$\begin{aligned}&\operatorname{supp}\mathcal{F}\left(\left[S^{k-2}f\right]S_k g\right)\subset\left\{x\in\mathbb{R}^n:\ 2^{k-3}\le|x|\le 2^{k+1}\right\},\\&\operatorname{supp}\mathcal{F}\left(\left[S_k f\right]S^{k-2}g\right)\subset\left\{x\in\mathbb{R}^n:\ 2^{k-3}\le|x|\le 2^{k+1}\right\},\end{aligned} \tag{2.9}$$

and

$$\operatorname{supp}\mathcal{F}\left(\sum_{j=k-1}^{k+1}\left[S_j f\right]S_k g\right)\subset\left\{x\in\mathbb{R}^n:\ |x|\le 5\cdot 2^k\right\}. \tag{2.10}$$

Now, we present the concept of Besov spaces as follows, see, for instance, [35, Definition 2.3.1/2].

Definition 2.1. Let $p,q\in(0,\infty]$ and $s\in\mathbb{R}$. Then the *Besov space* $B^s_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)}:=\left\{\sum_{k\in\mathbb{Z}_+}\left[2^{ks}\|S_k f\|_{L^p(\mathbb{R}^n)}\right]^q\right\}^{1/q}<\infty.$$

Remark 2.2. By [35, Proposition 2.3.2/1 and Theorem 2.3.3], we know that the Besov space $B^s_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space independent of the choice of the generator ϕ_0. For more information about Besov spaces we refer to the monographs by Peetre [27], Triebel [35,36,38], and Sawano [30].

Let $a \in (0, \infty)$, $j \in \mathbb{Z}_+$, and $f \in \mathcal{S}'(\mathbb{R}^n)$. Recall that the *maximal function of Peetre–Fefferman–Stein type* is defined by setting, for any $x \in \mathbb{R}^n$,

$$S_j^{*,a} f(x) := \sup_{y\in\mathbb{R}^n} \frac{|S_j f(x-y)|}{(1+2^j|y|)^a}, \tag{2.11}$$

where the operator S_j is defined in (2.4); see, for instance, [35, Definition 2.3.6/2]. Clearly, $|S_j f(x)| \le S_j^{*,a} f(x)$ for any $x \in \mathbb{R}^n$. The following lemma (see [35, 2.3.6/(22)]) means that, for any $p \in (0, \infty]$, $S_j^{*,a} f$ is bounded by $S_j f$ in the norm of $L^p(\mathbb{R}^n)$.

Lemma 2.3. *Let $p \in (0, \infty]$ and $a \in (\frac{n}{p}, \infty)$. Then there exists a positive constant C such that, for any $j \in \mathbb{Z}_+$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\left\|S_j^{*,a} f\right\|_{L^p(\mathbb{R}^n)} \le C \left\|S_j f\right\|_{L^p(\mathbb{R}^n)}.$$

The following useful conclusion is just [28, 4.2.1/(4) and (5)], which is important in this article.

Lemma 2.4. *Let $p, q \in (0, \infty]$ and $s \in \mathbb{R}$.*

(i) *There exist two positive constants C_1 and C_2 such that, for any $f \in B^s_{p,q}(\mathbb{R}^n)$,*

$$C_1 \|f\|_{B^s_{p,q}(\mathbb{R}^n)} \le \sup_{j\in\mathbb{Z}_+} \left\|S^j f\right\|_{B^s_{p,q}(\mathbb{R}^n)} \le C_2 \|f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

(ii) *If $q \in (0, \infty)$, then, for any $f \in B^s_{p,q}(\mathbb{R}^n)$, $\lim_{j\to\infty} \|S^j f - f\|_{B^s_{p,q}(\mathbb{R}^n)} = 0$.*

Furthermore, we need the following Fatou property of Besov spaces, see, e.g., [8, p. 40, Theorem 1].

Lemma 2.5. *Let $p, q \in (0, \infty]$ and $s \in \mathbb{R}$. Assume that $\{f_j\}_{j\in\mathbb{N}} \subset B^s_{p,q}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfy that both $f_j \to f$ in $\mathcal{S}'(\mathbb{R}^n)$ as $j \to \infty$ and $\liminf_{j\to\infty} \|f_j\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty$. Then $f \in B^s_{p,q}(\mathbb{R}^n)$ and there exists a positive constant C, independent of both $\{f_j\}_{j\in\mathbb{N}}$ and f, such that*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \le C \liminf_{j\to\infty} \|f_j\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Finally, we recall the duality theorem of Besov spaces. Let $X \subset \mathcal{S}'(\mathbb{R}^n)$ be a quasi-Banach space such that $\mathcal{S}(\mathbb{R}^n)$ is dense in X. Then the *dual space* of X, denoted by X', is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exists a positive constant C such that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $|\langle f, \varphi\rangle| \le C\|\varphi\|_X$. Moreover, for any $f \in X'$, let

$$\|f\|_{X'} := \sup_{\varphi\in\mathcal{S}(\mathbb{R}^n),\, \|\varphi\|_X\le 1} |\langle f, \varphi\rangle| .$$

Also, recall that, for any $q \in (0,\infty]$, q' is the same as in (1.2). Then the following is known, see, for instance, [35, 2.11.2/2.11.3].

Lemma 2.6. *Let $p, q \in (0,\infty)$ and $s \in \mathbb{R}$.*

(i) *If $p \in [1,\infty)$, then*

$$\left(B^s_{p,q}(\mathbb{R}^n)\right)' = B^{-s}_{p',q'}(\mathbb{R}^n).$$

(ii) *If $p \in (0,1)$, then*

$$\left(B^s_{p,q}(\mathbb{R}^n)\right)' = B^{\frac{n}{p}-n-s}_{p',q'}(\mathbb{R}^n).$$

Moreover, let $p, q \in (0,\infty]$ and $s \in \mathbb{R}$. Recall that $\widetilde{B}^s_{p,q}(\mathbb{R}^n)$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^n)$ in the Besov space $B^s_{p,q}(\mathbb{R}^n)$ and is equipped with the quasi-norm $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^n)}$. Then, if $p, q \in (0,\infty)$, from the density of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$ (see, for instance, [35, Theorem 2.3.3]), we infer that, in this case, $\widetilde{B}^s_{p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$. Furthermore, we have the following duality relation, we refer to [35, Remarks 2.11.2/2 and 2.11.3/3].

Lemma 2.7.

(i) *If $p \in [1,\infty)$, $q = \infty$, and $s \in \mathbb{R}$, then*

$$\left(\widetilde{B}^s_{p,\infty}(\mathbb{R}^n)\right)' = B^{-s}_{p',1}(\mathbb{R}^n).$$

(ii) *If $p = \infty$, $q \in (0,\infty]$, and $s \in \mathbb{R}$, then*

$$\left(\widetilde{B}^s_{\infty,q}(\mathbb{R}^n)\right)' = B^{-s}_{1,q'}(\mathbb{R}^n).$$

(iii) *If $p \in (0,1)$, $q = \infty$, and $s \in \mathbb{R}$, then*

$$\left(\widetilde{B}^s_{p,\infty}(\mathbb{R}^n)\right)' = B^{\frac{n}{p}-n-s}_{p',1}(\mathbb{R}^n).$$

2.2. Pointwise multiplication

In this subsection, we recall the definition of the product of two distributions as well as the concept of pointwise multiplier spaces. Then we give some basic properties of both the multiplier spaces $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$. In particular, via making full use of the duality relations of Besov spaces themselves, we establish an identity between $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$ and $M(B^{-s}_{p',q'}(\mathbb{R}^n))$.

First, we give the definition of the product as follows. For any $f, g \in \mathcal{S}'(\mathbb{R}^n)$, we put

$$fg := \lim_{j\to\infty} \left(S^j f\right) S^j g$$

if the limit on the right-hand side exists in $\mathcal{S}'(\mathbb{R}^n)$. Thus, at least formally, fg has a decomposition

$$\begin{aligned} fg &= \sum_{k=2}^{\infty} \left(S^{k-2} f\right) S_k g + \sum_{k=0}^{\infty} \sum_{l=\max\{0,k-1\}}^{k+1} \left(S_l f\right) S_k g + \sum_{k=2}^{\infty} \left(S_k f\right) S^{k-2} g \\ &=: \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g). \end{aligned} \tag{2.12}$$

The bilinear mappings $\Pi_1(f,g)$, $\Pi_2(f,g)$, and $\Pi_3(f,g)$ are called the *paraproducts* which are indispensable tools in the study of pointwise multipliers. If they exist in $\mathcal{S}'(\mathbb{R}^n)$, then the product exists as well.

Definition 2.8. Let $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ be a quasi-Banach space. Then the *pointwise multiplier space* $M(X)$ of X is defined to be the collection of all $m \in \mathcal{S}'(\mathbb{R}^n)$ such that $mg \in X$ for all $g \in X$ and

$$\|m\|_{M(X)} := \sup_{g\in X,\, g\neq \mathbf{0}} \frac{\|mg\|_X}{\|g\|_X} < \infty\,. \tag{2.13}$$

Here $\mathbf{0}$ denotes the zero element of X.

Remark 2.9. Recall that Maz'ya and Shaposhnikova [19, p. 33] used a slightly weaker definition. They did not require (2.13). But, this is due to the fact that they dealt with X being a subspace of $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$. For those spaces (2.13) follows from $mg \in X$ for all $g \in X$.

For the pointwise multiplier space $M(X)$, the following duality relation holds, we refer to [9, p. 134] (see also [37, Subsection 4.2]).

Lemma 2.10. *Let X be a quasi-Banach space satisfying that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Assume $\mathcal{S}(\mathbb{R}^n)$ is dense in X. Let X' denote the dual space of X. Then $M(X) \hookrightarrow M(X')$.*

The following lemma contains some basic properties of the multiplier spaces $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$. For proofs of parts (i) and (ii) we refer to [15, Lemma 9] and [28, Theorem 4.3.2 and Lemma 4.6.3/1]. Moreover, part (iii) was proved in [17, Lemma 2.7(iv)].

Lemma 2.11. *Let $p, q \in (0, \infty]$, $s \in \mathbb{R}$, and $X := B^s_{p,q}(\mathbb{R}^n)$ or $\widetilde{B}^s_{p,q}(\mathbb{R}^n)$.*

(i) *There exists a positive constant C such that, for any $f \in M(X)$, $f \in L^\infty(\mathbb{R}^n)$ and*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C \|f\|_{M(X)} .$$

(ii) *If further assume $p, q \in [1, \infty]$ and $h \in L^1(\mathbb{R}^n)$, then $h * f \in M(X)$ and*

$$\|h * f\|_{M(X)} \le \|h\|_{L^1(\mathbb{R}^n)} \|f\|_{M(X)} .$$

(iii) *If further assume $p, q \in [1, \infty]$, then $M(X)$ is a Banach space.*

In the remainder of this subsection, we turn to investigate the relation between $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$. First, we have the following relation based on the duality.

Lemma 2.12. *Let $p \in [1, \infty]$, $q \in [1, \infty)$, and $s \in \mathbb{R}$. Then*

$$M\left(\widetilde{B}^s_{p,q}(\mathbb{R}^n)\right) = M\left(B^{-s}_{p',q'}(\mathbb{R}^n)\right)$$

in the sense of equivalent norms.

Remark 2.13. As already mentioned in the introduction, Frazier and Jawerth [9, p. 134] stated a more general result than (1.1) or Lemma 2.12. Let X and X' be the same as in Lemma 2.10, and further assume X is a Banach space. Then Frazier and Jawerth claimed that $M(X) = M(X')$. But, this is not true in this generality as can be seen from the following counterexample, due to Triebel [37]. Let $s \in (0, \infty)$. Then it holds that

$$M\left(\widetilde{B}^s_{\infty,\infty}(\mathbb{R}^n)\right) \subsetneqq M\left(B^{-s}_{1,1}(\mathbb{R}^n)\right). \tag{2.14}$$

However, recall that $(\widetilde{B}^s_{\infty,\infty}(\mathbb{R}^n))' = B^{-s}_{1,1}(\mathbb{R}^n)$ by Lemma 2.7(ii). For an explicit example of a function $f \in M(B^{-s}_{1,1}(\mathbb{R}^n)) \setminus M(\widetilde{B}^s_{\infty,\infty}(\mathbb{R}^n))$, we refer to [37, (4.6)] (see also [17, Remark 2.14]).

To prove Lemma 2.12, we first show the following auxiliary conclusion about Schwartz functions.

Lemma 2.14. *Let $f \in L^\infty(\mathbb{R}^n)$ and $g, h \in \mathcal{S}(\mathbb{R}^n)$. Then $(fg) * h \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. Applying [7, Proposition 8.10], we find that $(fg) * h \in C^\infty(\mathbb{R}^n)$ and, for any $\beta \in \mathbb{Z}_+^n$,

$$\partial^\beta([fg] * h) = (fg) * \left(\partial^\beta h\right).$$

Thus, for any $\alpha, \beta \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\left|x^\alpha \partial^\beta([fg] * h)(x)\right| \\
&\quad = \left| x^\alpha \int_{\mathbb{R}^n} f(y)g(y)\partial^\beta h(x-y)\,dy \right| \\
&\quad \le \|f\|_{L^\infty(\mathbb{R}^n)} |x|^{|\alpha|} \int_{\mathbb{R}^n} |g(y)| \left|\partial^\beta h(x-y)\right| \,dy \\
&\quad \lesssim \int_{\mathbb{R}^n} \frac{|x|^{|\alpha|}|g(y)|}{(1+|x-y|)^{n+1+|\alpha|}}\,dy \lesssim \int_{\mathbb{R}^n} \frac{(|x-y|^{|\alpha|}+|y|^{|\alpha|})}{(1+|x-y|)^{n+1+|\alpha|}} |g(y)|\,dy \\
&\quad \lesssim \int_{\mathbb{R}^n} \left[\frac{1}{(1+|x-y|)^{n+1}} + |y|^{|\alpha|}|g(y)|\right] dy \sim 1,
\end{aligned}$$

where the implicit positive constants are independent of x. This further implies that, for any $\alpha, \beta \in \mathbb{Z}_+^n$, $\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta([fg] * h)(x)| < \infty$. Therefore, $(fg) * h \in \mathcal{S}(\mathbb{R}^n)$, which then completes the proof of Lemma 2.14. □

Now, we show Lemma 2.12.

Proof of Lemma 2.12. From Lemmas 2.6(i), 2.7(ii), and 2.10, we infer that

$$\left(\widetilde{B}^s_{p,q}(\mathbb{R}^n)\right)' = B^{-s}_{p',q'}(\mathbb{R}^n),$$

and hence

$$M\left(\widetilde{B}^s_{p,q}(\mathbb{R}^n)\right) \hookrightarrow M\left(B^{-s}_{p',q'}(\mathbb{R}^n)\right). \tag{2.15}$$

Conversely, we next prove $M(B^{-s}_{p',q'}(\mathbb{R}^n)) \hookrightarrow M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$. To do this, let $f \in M(B^{-s}_{p',q'}(\mathbb{R}^n))$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Then, by the definition of the product of both Schwartz functions and distributions, we conclude that fg exists in $\mathcal{S}'(\mathbb{R}^n)$ and $fg = \lim_{j\to\infty}(S^j f)g$ in $\mathcal{S}'(\mathbb{R}^n)$. Notice that, for any $j \in \mathbb{Z}_+$, $(S^j f)g \in \mathcal{S}(\mathbb{R}^n) \subset \widetilde{B}^s_{p,q}(\mathbb{R}^n)$. Thus, applying the fact that $(\widetilde{B}^s_{p,q}(\mathbb{R}^n))' = B^{-s}_{p',q'}(\mathbb{R}^n)$, [7, Theorem 5.8(ii)], Lemma 2.11(ii), and (2.5), we find that, for any $j \in \mathbb{Z}_+$,

$$
\begin{aligned}
\left\|\left(S^j f\right) g\right\|_{\widetilde{B}^s_{p,q}(\mathbb{R}^n)} &= \sup_{h\in(\widetilde{B}^s_{p,q}(\mathbb{R}^n))',\, \|h\|_{(\widetilde{B}^s_{p,q}(\mathbb{R}^n))'}=1} \left|\left\langle h, \left(S^j f\right) g\right\rangle\right| \\
&\sim \sup_{h\in B^{-s}_{p',q'}(\mathbb{R}^n),\, \|h\|_{B^{-s}_{p',q'}(\mathbb{R}^n)}=1} \left|\left\langle h, \left(S^j f\right) g\right\rangle\right| \\
&= \sup_{h\in B^{-s}_{p',q'}(\mathbb{R}^n),\, \|h\|_{B^{-s}_{p',q'}(\mathbb{R}^n)}=1} \left|\left\langle hS^j f, g\right\rangle\right| \\
&\lesssim \sup_{h\in B^{-s}_{p',q'}(\mathbb{R}^n),\, \|h\|_{B^{-s}_{p',q'}(\mathbb{R}^n)}=1} \left\|hS^j f\right\|_{B^{-s}_{p',q'}(\mathbb{R}^n)} \|g\|_{B^s_{p,q}(\mathbb{R}^n)} \\
&\le \left\|S^j f\right\|_{M(B^{-s}_{p',q'}(\mathbb{R}^n))} \|g\|_{B^s_{p,q}(\mathbb{R}^n)} \\
&\le 2^{nj} \left\|\varphi_0\left(2^j\cdot\right)\right\|_{L^1(\mathbb{R}^n)} \|f\|_{M(B^{-s}_{p',q'}(\mathbb{R}^n))} \|g\|_{B^s_{p,q}(\mathbb{R}^n)} \\
&\sim \|f\|_{M(B^{-s}_{p',q'}(\mathbb{R}^n))} \|g\|_{B^s_{p,q}(\mathbb{R}^n)} .
\end{aligned}
$$

This, combined with the fact that $fg = \lim_{j\to\infty}(S^j f)g$ in $\mathcal{S}'(\mathbb{R}^n)$ and also with Lemma 2.5, further implies that $fg \in B^s_{p,q}(\mathbb{R}^n)$ and

$$
\|fg\|_{B^s_{p,q}(\mathbb{R}^n)} = \|fg\|_{\widetilde{B}^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{M(B^{-s}_{p',q'}(\mathbb{R}^n))} \|g\|_{B^s_{p,q}(\mathbb{R}^n)} . \tag{2.16}
$$

Now, we show that $fg \in \widetilde{B}^s_{p,q}(\mathbb{R}^n)$. Indeed, from the assumption $q < \infty$ and Lemma 2.4(ii), it follows that

$$
\lim_{j\to\infty} \left\|S^j(fg) - fg\right\|_{B^s_{p,q}(\mathbb{R}^n)} = 0. \tag{2.17}
$$

In addition, for any $j \in \mathbb{Z}_+$, by Lemmas 2.11(i) and 2.14, we conclude that $S^j(fg) \in \mathcal{S}(\mathbb{R}^n)$. Therefore, using (2.17), we obtain $fg \in \widetilde{B}^s_{p,q}(\mathbb{R}^n)$. Combining this, the assumption that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\widetilde{B}^s_{p,q}(\mathbb{R}^n)$, and (2.16), we further have $f \in M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$ and $\|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))} \lesssim \|f\|_{M(B^{-s}_{p',q'}(\mathbb{R}^n))}$. This completes the proof of Lemma 2.12. $\square$

Finally, we establish the following relation between $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$.

Lemma 2.15. *Let $p \in (0,\infty]$, $q \in (0,\infty)$, and $s \in \mathbb{R}$. Then $M(B^s_{p,q}(\mathbb{R}^n)) \hookrightarrow M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$. Moreover, for any $f \in M(B^s_{p,q}(\mathbb{R}^n))$,*

$$
\|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))} \le \|f\|_{M(B^s_{p,q}(\mathbb{R}^n))} .
$$

Proof. Let $f \in M(B^s_{p,q}(\mathbb{R}^n))$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Then, by Lemmas 2.11(i) and 2.14, we find that, for any $j \in \mathbb{Z}_+$, $S^j(fg) \in \mathcal{S}(\mathbb{R}^n)$. Moreover, applying the assumption $q < \infty$ and Lemma 2.4(ii), we conclude that $\|S^j(fg) - fg\|_{B^s_{p,q}(\mathbb{R}^n)} \to 0$ as $j \to \infty$. Therefore, $fg \in \widetilde{B}_{p,q}(\mathbb{R}^n)$. This, combined with the density of $\mathcal{S}(\mathbb{R}^n)$ in $\widetilde{B}^s_{p,q}(\mathbb{R}^n)$, further implies that $f \in M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$ and hence, as sets, $M(B^s_{p,q}(\mathbb{R}^n)) \subset M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$. Furthermore, for any $f \in M(B^s_{p,q}(\mathbb{R}^n))$ and $g \in \widetilde{B}^s_{p,q}(\mathbb{R}^n)$, we have

$$\|fg\|_{B^s_{p,q}(\mathbb{R}^n)} \le \|f\|_{M(B^s_{p,q}(\mathbb{R}^n))} \|g\|_{B^s_{p,q}(\mathbb{R}^n)},$$

which then implies that, for any $f \in M(B^s_{p,q}(\mathbb{R}^n))$, $\|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))} \le \|f\|_{M(B^s_{p,q}(\mathbb{R}^n))}$. This finishes the proof of Lemma 2.15. □

Remark 2.16. We should point out that, in Lemma 2.15, if $q = \infty$, then the conclusion may not hold. Indeed, let $p \in (1, \infty]$ and $s \in \mathbb{R}$. Then, in this case, by [17, Proposition 2.13], we have the embedding

$$M\left(\widetilde{B}^s_{p,\infty}(\mathbb{R}^n)\right) \hookrightarrow M\left(B^s_{p,\infty}(\mathbb{R}^n)\right), \tag{2.18}$$

which is completely contrary to the conclusion of Lemma 2.15. Moreover, as mentioned above in Remark 2.13, we have

$$M\left(\widetilde{B}^s_{\infty,\infty}(\mathbb{R}^n)\right) \subsetneqq M\left(B^s_{\infty,\infty}(\mathbb{R}^n)\right).$$

3. Characterizations of pointwise multipliers of Besov spaces in endpoint cases

In this section, we study characterizations of pointwise multiplier spaces of Besov spaces in the following endpoint cases:

- $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in (0, \infty)$ (see Theorem 3.1);
- $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in (-\infty, 0)$ (see Theorem 3.5);
- $M(B^s_{\infty,1}(\mathbb{R}^n))$ with $s \in (-\infty, 0]$ (see Theorems 3.9 and 3.13).

Moreover, for the simplicity of the representation, in what follows, for any $s \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$\|f\|_s := \sup_{k \in \mathbb{N} \cap [2,\infty)} 2^{ks} \sum_{l=0}^{k-2} 2^{-ls} \sup_{m \in \mathbb{Z}^n} \left[\fint_{Q_{l,m}} |S_k f(x)|\, dx \right]. \tag{3.1}$$

3.1. A characterization of $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in (0, \infty)$

In this subsection, we obtain the characterization of the multiplier space $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in (0, \infty)$. The proof essentially depends on both some elaborate estimates of paraproducts and the construction of the auxiliary functions $\{g_k\}_{k \in \mathbb{N} \cap [2,\infty)}$ [see (3.14) below for the definition] which are motivated by the intrinsic structure of the term $\|\cdot\|_s$ defined in (3.1).

Indeed, we characterize $M(B^s_{1,\infty}(\mathbb{R}^n))$ for any $s \in (0, \infty)$ as follows.

Theorem 3.1. *Let $s \in (0, \infty)$. Then $f \in M(B^s_{1,\infty}(\mathbb{R}^n))$ if and only if $f \in L^\infty(\mathbb{R}^n)$ and*

$$\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_s < \infty.$$

Moreover, there exist two positive constants C_1, C_2 such that, for any $f \in M(B^s_{1,\infty}(\mathbb{R}^n))$,

$$C_1\|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))} \le \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_s \le C_2\|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))}.$$

Remark 3.2. Netrusov in [25, Theorem 3(1)] has found the following characterization of the multiplier space $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s \in (0,\infty)$. Let $s \in (0,\infty)$. Then $f \in M(B^s_{1,\infty}(\mathbb{R}^n))$ if and only if $f \in L^\infty(\mathbb{R}^n)$ and

$$\sup_{j\in\mathbb{Z}_+} 2^{js} \int_{2^{-j-1}}^{1} \left[\sup_{x\in\mathbb{R}^n} \int_{B(x,t)} |f_j(x)|\,dx\right] t^{s-n}\,\frac{dt}{t} < \infty, \tag{3.2}$$

where $\{f_j\}_{j\in\mathbb{Z}_+} \subset \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $\operatorname{supp}\mathcal{F} f_0 \subset B(\mathbf{0}, 2)$,

$$\operatorname{supp}\mathcal{F} f_j \subset B(\mathbf{0}, 2^{j+1}) \setminus B(\mathbf{0}, 2^{j-1}),\ \forall\, j \in \mathbb{N},$$

and $f = \sum_{j\in\mathbb{Z}_+} f_j$ in $\mathcal{S}'(\mathbb{R}^n)$. We should point out that, compared with (3.2), our characterization in Theorem 3.1 has a minimal change in the formulation. However, Netrusov never published a proof.

To show Theorem 3.1, we first give some notation. In what follows, for any $Q \in \mathcal{Q}$, we always use $l(Q)$ to denote its edge length. Furthermore, for any $x \in \mathbb{R}^n$ and $l \in (0,\infty)$, the *symbol* $Q(x, l)$ denotes the cube with center x and edge length l, and the *symbol* $\mathbf{e} := (1, \dots, 1)$ denotes the *unit* of $\mathbb{R}^n$. Let $h \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ be such that

$$\mathbf{1}_{Q_{0,\mathbf{0}}} \le h \le \mathbf{1}_{Q(\frac{1}{2}\mathbf{e},2)}.$$

Then we have the following estimates about h, which play key roles in the proof of Theorem 3.1.

Lemma 3.3. *Let $M \in \mathbb{Z}_+$. For any $l \in \mathbb{Z}_+$ and $x_l, x \in \mathbb{R}^n$, let*

$$h_{l,x_l}(x) := h\left(2^l[x - x_l]\right).$$

Then there exists a positive constant C, depending only on n and M, such that

(i) *for any $j \in \mathbb{Z}_+ \cap [0, l]$,*

$$\|S_j\left(h_{l,x_l}\right)\|_{L^1(\mathbb{R}^n)} \le C2^{-ln};$$

(ii) *for any* $j \in \mathbb{N} \cap [l+1, \infty)$,

$$\|S_j(h_{l,x_l})\|_{L^1(\mathbb{R}^n)} \leq C 2^{(M+1-n)l-(M+1)j}.$$

Proof. We first prove (i). Indeed, using the Young inequality, we find that

$$\|S_j(h_{l,x_l})\|_{L^1(\mathbb{R}^n)} \lesssim \|h_{l,x_l}\|_{L^1(\mathbb{R}^n)} = 2^{-ln} \|h\|_{L^1(\mathbb{R}^n)} \sim 2^{-ln},$$

which completes the proof of (i).

Next, we show (ii). To this end, let $j \in \mathbb{N} \cap [l+1, \infty)$. Then, from (2.7) with $k := j$, we infer that, for any $\gamma \in \mathbb{Z}_+^n$,

$$\int_{\mathbb{R}^n} \varphi_j(x) x^\gamma \, dx = 0.$$

This, together with both the Taylor remainder theorem and the Tonelli theorem, further implies that, for any $y \in \mathbb{R}^n$, there exists $t_y \in (0,1)$ such that

$$\begin{aligned}
&\|S_j(h_{l,x_l})\|_{L^1(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \varphi_j(y) h_{l,x_l}(x-y) \, dy \right| dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \varphi_j(y) \left[h_{l,x_l}(x-y) - \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| \leq M}} \frac{\partial^\gamma (h_{l,x_l}(x-\cdot))(\mathbf{0})}{\gamma!} y^\gamma \right] dy \right| dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \varphi_j(y) \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = M+1}} \frac{\partial^\gamma (h(2^l(x-x_l-\cdot)))(t_y y)}{\gamma!} y^\gamma \, dy \right| dx \\
&\lesssim 2^{(M+1)l+jn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{M+1} \left| \varphi_1(2^j y) \right| \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = M+1}} \left| \partial^\gamma h\left(2^l(x-x_l-t_y y)\right) \right| dx \, dy \\
&= 2^{(M+1-n)l-(M+1)j} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{M+1} |\varphi_1(y)| \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = M+1}} \left| \partial^\gamma h(x - 2^{l-j} t_y y) \right| dx \, dy \\
&= 2^{(M+1-n)l-(M+1)j} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^{M+1} |\varphi_1(y)| \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = M+1}} |\partial^\gamma h(x)| \, dx \, dy.
\end{aligned}$$

Applying this and the assumption $h \in C_c^\infty(\mathbb{R}^n)$, we further conclude that

$$\max_{\substack{\gamma\in\mathbb{Z}_+^n\\|\gamma|=M+1}}\left\{\|\partial^\gamma h\|_{L^\infty(\mathbb{R}^n)}\right\}<\infty$$

and hence

$$\begin{aligned}&\left\|S_j\left(h_{l,x_l}\right)\right\|_{L^1(\mathbb{R}^n)}\\&\quad\lesssim 2^{(M+1-n)l-(M+1)j}\int_{\mathbb{R}^n}\int_{\operatorname{supp}h}|y|^{M+1}|\varphi_1(y)|\sum_{\substack{\gamma\in\mathbb{Z}_+^n\\|\gamma|=M+1}}|\partial^\gamma h(x)|\,dx\,dy\\&\quad\lesssim 2^{(M+1-n)l-(M+1)j}\int_{\mathbb{R}^n}\int_{Q(\frac{1}{2}\mathbf{e},2)}|y|^{M+1}|\varphi_1(y)|\,dx\,dy\\&\quad\sim 2^{(M+1-n)l-(M+1)j}\int_{\mathbb{R}^n}|y|^{M+1}|\varphi_1(y)|\,dy\sim 2^{(M+1-n)l-(M+1)j}.\end{aligned}$$

This finishes the proof of (ii) and hence Lemma 3.3. □

We also need the following simple estimate of the paraproduct $\Pi_1(\cdot,\cdot)$, see, e.g., [28] or [17, Lemma 4.1].

Lemma 3.4. *Let $p,q\in[1,\infty]$ and $s\in\mathbb{R}$. Then there exists a positive constant C such that, for any $f\in L^\infty(\mathbb{R}^n)$ and $g\in B^s_{p,q}(\mathbb{R}^n)$,*

$$\|\Pi_1(f,g)\|_{B^s_{p,q}(\mathbb{R}^n)}\le C\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Via above preparations, we now prove Theorem 3.1.

Proof of Theorem 3.1. We first show the sufficiency. To this end, let $f\in L^\infty(\mathbb{R}^n)$ be such that $\|f\|_{L^\infty(\mathbb{R}^n)}+\|f\|_s<\infty$ and let $g\in B^s_{1,\infty}(\mathbb{R}^n)$. Then, applying (2.12), we find that

$$\|fg\|_{B^s_{1,\infty}(\mathbb{R}^n)}\le\sum_{j=1}^3\|\Pi_j(f,g)\|_{B^s_{1,\infty}(\mathbb{R}^n)}=:\sum_{j=1}^3\mathrm{I}_j.\tag{3.3}$$

By Lemma 3.4 with $p:=1$ and $q:=\infty$, we conclude that

$$\mathrm{I}_1\lesssim\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}\le\left[\|f\|_{L^\infty(\mathbb{R}^n)}+\|f\|_s\right]\|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}.\tag{3.4}$$

Now, we deal with I_2. Indeed, using (2.6), (2.7), and (2.10), we obtain, for any $l\in\mathbb{Z}_+$,

$$S_l(\Pi_2(f,g))=S_l\left(\sum_{k=\max\{0,l-3\}}^\infty\sum_{j=\max\{0,k-1\}}^{k+1}[S_jf]\,S_kg\right).\tag{3.5}$$

From this, both the Young and the Hölder inequalities, the definition of $\|\cdot\|_{B^s_{1,\infty}(\mathbb{R}^n)}$, and the assumption $s\in(0,\infty)$, we deduce that, for any $l\in\mathbb{Z}_+$,

$$\begin{aligned}
&\|S_l\,(\,\Pi_2\,(f,g))\|_{L^1(\mathbb{R}^n)}\\
&\quad\le\sum_{k=\max\{0,l-3\}}^{\infty}\ \sum_{j=\max\{0,k-1\}}^{k+1}\|S_l\,([S_jf]\,S_kg)\|_{L^1(\mathbb{R}^n)}\\
&\quad\lesssim\sum_{k=\max\{0,l-3\}}^{\infty}\ \sum_{j=\max\{0,k-1\}}^{k+1}\|S_jf\|_{L^\infty(\mathbb{R}^n)}\,\|S_kg\|_{L^1(\mathbb{R}^n)}\\
&\quad\lesssim\sum_{k=\max\{0,l-3\}}^{\infty}\ \sum_{j=\max\{0,k-1\}}^{k+1}\|f\|_{L^\infty(\mathbb{R}^n)}\,\|S_kg\|_{L^1(\mathbb{R}^n)}\\
&\quad\lesssim\|f\|_{L^\infty(\mathbb{R}^n)}\,\|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}\sum_{k=l-3}^{\infty}2^{-ks}\sim 2^{-ls}\,\|f\|_{L^\infty(\mathbb{R}^n)}\,\|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}\,.
\end{aligned}$$

This, combined with the definitions of both I_2 and $\|\cdot\|_{B^s_{1,\infty}(\mathbb{R}^n)}$, further implies that

$$\begin{aligned}
\mathrm{I}_2&=\sup_{l\in\mathbb{Z}_+}\left\{2^{ls}\,\|S_l\,(\,\Pi_2\,(f,g))\|_{L^1(\mathbb{R}^n)}\right\}\\
&\lesssim\|f\|_{L^\infty(\mathbb{R}^n)}\,\|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}\le\left[\|f\|_{L^\infty(\mathbb{R}^n)}+\|f\|_s\right]\|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}\,,
\end{aligned}\tag{3.6}$$

which completes the estimation of I_2. We next estimate I_3. Indeed, by (2.6), (2.7), and (2.9), we conclude that, for any $j\in\mathbb{Z}_+$,

$$S_j\,(\,\Pi_3\,(f,g))=S_j\left(\sum_{k=\max\{2,j-1\}}^{j+3}[S_kf]\,S^{k-2}g\right).\tag{3.7}$$

Applying this and the Young inequality, we find that, for any $j\in\mathbb{Z}_+$,

$$\begin{aligned}
\|S_j\,(\,\Pi_3\,(f,g))\|_{L^1(\mathbb{R}^n)}&\le\sum_{k=\max\{2,j-1\}}^{j+3}\left\|S_j\left([S_kf]\,S^{k-2}g\right)\right\|_{L^1(\mathbb{R}^n)}\\
&\lesssim\sum_{k=\max\{2,j-1\}}^{j+3}\left\|(S_kf)\,S^{k-2}g\right\|_{L^1(\mathbb{R}^n)}.
\end{aligned}\tag{3.8}$$

To finish the estimation of I_3, we now estimate $\|(S_kf)S^{k-2}g\|_{L^1(\mathbb{R}^n)}$ with $k\in\mathbb{N}\cap[2,\infty)$. Indeed, for any $k\in\mathbb{N}\cap[2,\infty)$, by (2.5) with k and f therein replaced, respectively, by $k-2$ and g, we conclude that

$$\begin{aligned}
\left\|(S_k f)\, S^{k-2} g\right\|_{L^1(\mathbb{R}^n)} &\le \sum_{l=0}^{k-2} \int_{\mathbb{R}^n} |S_k f(x)|\, |S_l g(x)|\, dx \\
&= \sum_{l=0}^{k-2} \sum_{m\in\mathbb{Z}^n} \int_{Q_{l,m}} |S_k f(x)|\, |S_l g(x)|\, dx \\
&\le \sum_{l=0}^{k-2} \sum_{m\in\mathbb{Z}^n} \max_{x\in Q_{l,m}} \{|S_l g(x)|\} \int_{Q_{l,m}} |S_k f(x)|\, dx. \qquad (3.9)
\end{aligned}$$

Noticing that, for any given $l \in \mathbb{Z}_+$ and $m \in \mathbb{Z}^n$ and for any $x, y \in Q_{l,m}$, we have

$$|y - x| \le 2^{-l}\sqrt{n},$$

which further implies that

$$|S_l g(x)| \lesssim \frac{|S_l g(x)|}{(1 + 2^l |y - x|)^{n+1}} \le S_l^{*,n+1} g(y)$$

with the implicit positive constant independent of both l and m, where $S_j^{*,n+1} g(y)$ is the same as in (2.11) with a, j, f, and x therein replaced, respectively, by $n+1$, l, g, and y. From this, we then infer that, for any $l \in \mathbb{Z}_+$ and $m \in \mathbb{Z}^n$,

$$\max_{x\in Q_{l,m}} \{|S_l g(x)|\} \lesssim \min_{y\in Q_{l,m}} \left\{ S_l^{*,n+1} g(y) \right\}.$$

This, together with (3.9), Lemma 2.3(i) with p, a, j, and f therein replaced, respectively, by 1, $n+1$, l, and g, and the definition of $\|\cdot\|_{B^s_{1,\infty}(\mathbb{R}^n)}$, further implies that, for any $k \in \mathbb{N} \cap [2, \infty)$,

$$\begin{aligned}
\left\|(S_k f)\, S^{k-2} g\right\|_{L^1(\mathbb{R}^n)} &\lesssim \sum_{l=0}^{k-2} \sum_{m\in\mathbb{Z}^n} \min_{y\in Q_{l,m}} \left\{ \left| S_l^{*,n+1} g(y) \right| \right\} \int_{Q_{l,m}} |S_k f(x)|\, dx \\
&\le \sum_{l=0}^{k-2} \sum_{m\in\mathbb{Z}^n} \int_{Q_{l,m}} S_l^{*,n+1} g(y)\, dy \fint_{Q_{l,m}} |S_k f(x)|\, dx \\
&\le \sum_{l=0}^{k-2} \left\| S_l^{*,n+1} g \right\|_{L^1(\mathbb{R}^n)} \sup_{m\in\mathbb{Z}^n} \left\{ \fint_{Q_{l,m}} |S_k f(x)|\, dx \right\} \\
&\lesssim \sum_{l=0}^{k-2} \|S_l g\|_{L^1(\mathbb{R}^n)} \sup_{m\in\mathbb{Z}^n} \left\{ \fint_{Q_{l,m}} |S_k f(x)|\, dx \right\}
\end{aligned}$$

$$\leq \|g\|_{B^s_{1,\infty}(\mathbb{R}^n)} \sum_{l=0}^{k-2} 2^{-ls} \sup_{m\in\mathbb{Z}^n} \left\{ \fint_{Q_{l,m}} |S_k f(x)|\, dx \right\}, \quad (3.10)$$

which is the desired estimate of $\|S_k f S^{k-2} g\|_{L^1(\mathbb{R}^n)}$.

On the other hand, observe that, for any $j \in \mathbb{Z}_+$ and $k \in \mathbb{N} \cap [\max\{2, j-1\}, j+3]$, $2^{js} \sim 2^{ks}$. Combining this, (3.8), and (3.10), we find that, for any $j \in \mathbb{Z}_+$,

$$\begin{aligned}
&2^{js} \|S_j(\Pi_3(f,g))\|_{L^1(\mathbb{R}^n)} \\
&\lesssim 2^{js} \|g\|_{B^s_{1,\infty}(\mathbb{R}^n)} \sum_{k=\max\{2,j-1\}}^{j+2} \sum_{l=0}^{k-2} 2^{-ls} \sup_{m\in\mathbb{Z}^n} \left\{ \fint_{Q_{l,m}} |S_k f(x)|\, dx \right\} \\
&\sim \|g\|_{B^s_{1,\infty}(\mathbb{R}^n)} \sum_{k=\max\{2,j-1\}}^{j+2} 2^{ks} \sum_{l=0}^{k-2} 2^{-ls} \sup_{m\in\mathbb{Z}^n} \left\{ \fint_{Q_{l,m}} |S_k f(x)|\, dx \right\} \\
&\lesssim \|f\|_s \|g\|_{B^s_{1,\infty}(\mathbb{R}^n)} \leq \left[\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_s \right] \|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}.
\end{aligned}$$

Using this and the definition of I_3, we further obtain

$$\mathrm{I}_3 = \sup_{j\in\mathbb{Z}_+} \left\{ 2^{js} \|S_j(\Pi_3(f,g))\|_{L^1(\mathbb{R}^n)} \right\} \lesssim \left[\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_s \right] \|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}.$$

From this, (3.3), (3.4), and (3.6), it follows that

$$\|fg\|_{B^s_{1,\infty}(\mathbb{R}^n)} \lesssim \left[\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_s \right] \|g\|_{B^s_{1,\infty}(\mathbb{R}^n)}.$$

Hence, $f \in M(B^s_{1,\infty}(\mathbb{R}^n))$ and

$$\|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_s. \quad (3.11)$$

Thus, we have completed the proof of the sufficiency part.

Conversely, we next prove the necessity. To this end, let $f \in M(B^s_{1,\infty}(\mathbb{R}^n))$. Then, using Lemma 2.11(i) with $p := 1$ and $q := \infty$, we find that $f \in L^\infty(\mathbb{R}^n)$ and

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))}. \quad (3.12)$$

On the other hand, for any given $k \in \mathbb{N} \cap [2, \infty)$ and for any $l \in \mathbb{Z}_+ \cap [0, k-2]$, there exists $m_{k,l} \in \mathbb{Z}^n$ such that

$$\sup_{m\in\mathbb{Z}^n} \left\{ \fint_{Q_{l,m}} |S_k f(x)|\, dx \right\} \leq 2 \fint_{Q_{l,m_{k,l}}} |S_k f(x)|\, dx. \quad (3.13)$$

For any $k \in \mathbb{N} \cap [2, \infty)$ and $x \in \mathbb{R}^n$, let

$$g_k(x) := \sum_{l=0}^{k-2} 2^{(n-s)l} h\left(2^l x - m_{k,l}\right). \tag{3.14}$$

Now, we show that, for any $k \in \mathbb{N} \cap [2, \infty)$, $g_k \in B^s_{1,\infty}(\mathbb{R}^n)$ and

$$\sup_{k \in \mathbb{N} \cap [2,\infty)} \left\{ \|g_k\|_{B^s_{1,\infty}(\mathbb{R}^n)} \right\} \lesssim 1. \tag{3.15}$$

Indeed, for any $k \in \mathbb{N} \cap [2, \infty)$, by Lemma 3.3(i) with $j := 0$ and $x_l := 2^{-l} m_{k,l}$ for any $l \in \mathbb{Z}_+ \cap [0, k-2]$ and by the assumption $s \in (0, \infty)$, we conclude that

$$\begin{aligned} \|S_0 g_k\|_{L^1(\mathbb{R}^n)} &\le \sum_{l=0}^{k-2} 2^{(n-s)l} \left\| S_0 \left(h \left(2^l \left[\cdot - 2^{-l} m_{k,l} \right] \right) \right) \right\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \sum_{l=0}^{k-2} 2^{(n-s)l} 2^{-ln} \le \sum_{l \in \mathbb{Z}_+} 2^{-ls} \sim 1. \end{aligned} \tag{3.16}$$

On the other hand, for any $j \in \mathbb{N}$ and $k \in \mathbb{N} \cap [2, \infty)$, from Lemma 3.3 with $M := \lfloor s \rfloor$ and $x_l := 2^{-l} m_{k,l}$ for any $l \in \mathbb{Z}_+ \cap [0, k-2]$, we deduce that

$$\begin{aligned} \|S_j g_k\|_{L^1(\mathbb{R}^n)} &\le \sum_{l=0}^{k-2} 2^{(n-s)l} \left\| S_j \left(h \left(2^l \left[\cdot - 2^{-l} m_{k,l} \right] \right) \right) \right\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \sum_{l=0}^{j-1} 2^{(n-s)l} 2^{(\lfloor s \rfloor + 1 - n)l - (\lfloor s \rfloor + 1)j} + \sum_{l=j}^{\infty} 2^{(n-s)l} 2^{-ln} \\ &= \sum_{l=0}^{j-1} 2^{(\lfloor s \rfloor + 1 - s)l - (\lfloor s \rfloor + 1)j} + \sum_{l=j}^{\infty} 2^{-ls} \sim 2^{-js}. \end{aligned}$$

Combining this and (3.16), we further obtain, for any $k \in \mathbb{N} \cap [2, \infty)$,

$$\|g_k\|_{B^s_{1,\infty}(\mathbb{R}^n)} = \sup_{j \in \mathbb{Z}_+} \left\{ 2^{js} \|S_j g_k\|_{L^1(\mathbb{R}^n)} \right\} \lesssim 1,$$

which implies that $g_k \in B^s_{1,\infty}(\mathbb{R}^n)$ and hence completes the proof of (3.15).

In addition, observe that, for any $k \in \mathbb{N} \cap [2, \infty)$, $l \in \mathbb{Z}_+ \cap [0, k-2]$, and $x \in \mathbb{R}^n$, we have

$$\mathbf{1}_{Q_{l,m_{k,l}}}(x) \le h\left(2^l x - m_{k,l}\right).$$

This, together with (3.13), further implies that, for any $k \in \mathbb{N} \cap [2, \infty)$,

$$
\begin{aligned}
&2^{ks}\sum_{l=0}^{k-2}2^{-ls}\sup_{m\in\mathbb{Z}^n}\left\{\fint_{Q_{l,m}}|S_kf(x)|\,dx\right\}\\
&\lesssim 2^{ks}\sum_{l=0}^{k-2}2^{(n-s)l}\int_{Q_{l,m_{k,l}}}|S_kf(x)|\,dx\\
&=2^{ks}\int_{\mathbb{R}^n}|S_kf(x)|\sum_{l=0}^{k-2}2^{(n-s)l}\mathbf{1}_{Q_{l,m_{k,l}}}(x)\,dx\\
&\le 2^{ks}\int_{\mathbb{R}^n}|S_kf(x)|\sum_{l=0}^{k-2}2^{(n-s)l}h\left(2^lx-m_{k,l}\right)\,dx\\
&=2^{ks}\int_{\mathbb{R}^n}|S_kf(x)|\,|g_k(x)|\,dx\\
&=2^{ks}\left\|(S_kf)\,S^{k-2}g_k\right\|_{L^1(\mathbb{R}^n)}+2^{ks}\sum_{j=k-1}^{\infty}\|(S_kf)\,S_jg_k\|_{L^1(\mathbb{R}^n)}\\
&=:\mathrm{I}_{k,1}+\mathrm{I}_{k,2}.
\end{aligned}\tag{3.17}
$$

We first deal with $\mathrm{I}_{k,1}$. Indeed, for any $k\in\mathbb{N}\cap[2,\infty)$, applying (2.9), (2.6), (2.7), (2.13) with p, q, f, and g therein replaced, respectively, by 1, ∞, S_kf, and $S^{k-2}g_k$, Lemma 2.11(ii) with $p:=1$, $q:=\infty$, and $h:=\varphi_k$, Lemma 2.4(i) with $p:=1$, $q:=\infty$, and $f:=g$, and (3.15), we find that

$$
\begin{aligned}
\mathrm{I}_{k,1}&\le 2^{ks}\sum_{j\in\mathbb{Z}_+}\left\|S_j\left([S_kf]\,S^{k-2}g_k\right)\right\|_{L^1(\mathbb{R}^n)}\\
&\sim\sum_{j=\max\{0,k-3\}}^{k+1}2^{js}\left\|S_j\left([S_kf]\,S^{k-2}g_k\right)\right\|_{L^1(\mathbb{R}^n)}\\
&\lesssim\sup_{j\in\mathbb{Z}_+}\left\{2^{js}\left\|S_j\left([S_kf]\,S^{k-2}g_k\right)\right\|_{L^1(\mathbb{R}^n)}\right\}=\left\|(S_kf)\,S^{k-2}g_k\right\|_{B^s_{1,\infty}(\mathbb{R}^n)}\\
&\le\|\varphi_k*f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))}\left\|S^{k-2}g_k\right\|_{B^s_{1,\infty}(\mathbb{R}^n)}\\
&\lesssim\|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))}\|g_k\|_{B^s_{1,\infty}(\mathbb{R}^n)}\sim\|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))}.
\end{aligned}\tag{3.18}
$$

This finishes the estimation of $\mathrm{I}_{k,1}$ with $k\in\mathbb{N}\cap[2,\infty)$. Now, we turn to the estimation of $\mathrm{I}_{k,2}$, $k\in\mathbb{N}\cap[2,\infty)$. Indeed, for any $k\in\mathbb{N}\cap[2,\infty)$, by the Hölder inequality, Lemma 3.3(ii) with $M:=\lfloor s\rfloor$ and $x_l:=2^{-l}m_{k,l}$ for any $l\in\mathbb{Z}_+\cap[0,k-2]$, the Young inequality, and (3.12), we conclude that

$$
\begin{aligned}
\mathrm{I}_{k,2} &\le 2^{ks}\|S_k f\|_{L^\infty(\mathbb{R}^n)} \sum_{j=k-1}^{\infty} \|S_j g_k\|_{L^1(\mathbb{R}^n)} \\
&\le 2^{ks}\|S_k f\|_{L^\infty(\mathbb{R}^n)} \sum_{j=k-1}^{\infty}\sum_{l=0}^{k-2} 2^{(n-s)l}\left\|S_j\left(h\left(2^l\left[\cdot-2^{-l}m_{k,l}\right]\right)\right)\right\|_{L^1(\mathbb{R}^n)} \\
&\lesssim 2^{ks}\|S_k f\|_{L^\infty(\mathbb{R}^n)} \sum_{j=k-1}^{\infty}\sum_{l=0}^{k-2} 2^{(n-s)l}2^{(\lfloor s\rfloor+1-n)l-(\lfloor s\rfloor+1)j} \\
&\sim \|S_k f\|_{L^\infty(\mathbb{R}^n)} \le \|\varphi_k\|_{L^1(\mathbb{R}^n)}\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))},
\end{aligned}
$$

which then completes the estimation of $\mathrm{I}_{k,2}$. From this, (3.12), (3.17), (3.18), and the definition of $\|\cdot\|_s$, we infer that

$$
\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_s \lesssim \|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))}. \tag{3.19}
$$

This finishes the proof of the necessity. Moreover, applying (3.11) and (3.19), we further obtain

$$
\|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))} \sim \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_s,
$$

which completes the proof of Theorem 3.1. □

3.2. *A characterization of $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s\in(-\infty,0)$*

In this subsection, we characterize $M(B^s_{1,\infty}(\mathbb{R}^n))$ with $s\in(-\infty,0)$. Our main tool is an embedding between $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$ (see Lemma 3.8 below).

Let $s\in(-\infty,0)$. Then the Fourier analytic characterization of $M(B^s_{1,\infty}(\mathbb{R}^n))$ is as follows.

Theorem 3.5. *Let $s\in(-\infty,0)$. Then*

$$
M\left(B^s_{1,\infty}(\mathbb{R}^n)\right) = B^{-s}_{\infty,1}(\mathbb{R}^n)
$$

in the sense of equivalent norms.

Recall that Nguyen and Sickel [26, Theorem 1.7] obtained the following characterization of $M(B^s_{\infty,1}(\mathbb{R}^n))$ for any $s\in(0,\infty)$, which will be used in the proofs of both Theorem 3.5 and the duality principle Theorem 4.1 below.

Proposition 3.6. *Let $s\in(0,\infty)$. Then*

$$
M\left(B^s_{\infty,1}(\mathbb{R}^n)\right) = B^{s}_{\infty,1}(\mathbb{R}^n)
$$

in the sense of equivalent norms.

Also, to prove Theorem 3.5, we shall need some properties of differences. Let $N \in \mathbb{Z}_+$ and $h \in \mathbb{R}^n$. Recall that the *difference operator* Δ_h^N is defined by setting, for any complex-valued function f and any $x \in \mathbb{R}^n$,

$$\Delta_h^N f(x) := \sum_{j=0}^{N} (-1)^{N-j} \binom{N}{j} f(x+jh),$$

where, for any $j \in \{0,\dots,N\}$, $\binom{N}{j} := \frac{N!}{j!(N-j)!}$. In addition, recall that the *symbol* $C(\mathbb{R}^n)$ denotes the set of all continuous functions on $\mathbb{R}^n$. The following characterization of $B^s_{\infty,q}(\mathbb{R}^n)$ by differences is a simple application of both [36, 2.6.1/(3)] and [32, Theorem 3.3.1(ii)]; we omit the details.

Lemma 3.7. *Let $q \in (0,\infty]$, $s \in (0,\infty)$, and $N \in \mathbb{N} \cap (s,\infty)$. Then $f \in B^s_{\infty,q}(\mathbb{R}^n)$ if and only if $f \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and, when $q \in (0,\infty)$,*

$$\|f\|^*_{B^s_{\infty,q}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} |f(x)| + \left[\int_0^1 t^{-sq} \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} \left|\Delta_h^N f(x)\right|^q \frac{dt}{t}\right]^{\frac{1}{q}} < \infty$$

or, when $q = \infty$,

$$\|f\|^*_{B^s_{\infty,\infty}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{t\in(0,1)} \left\{t^{-s} \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} \left|\Delta_h^N f(x)\right|\right\} < \infty.$$

Moreover, there exist two positive constants C_1 and C_2 such that, for any $f \in B^s_{\infty,q}(\mathbb{R}^n)$,

$$C_1 \|f\|_{B^s_{\infty,q}(\mathbb{R}^n)} \le \|f\|^*_{B^s_{\infty,q}(\mathbb{R}^n)} \le C_2 \|f\|_{B^s_{\infty,q}(\mathbb{R}^n)}.$$

Based on Lemma 3.7, we now establish the following embedding between $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$, which will be essential for the proof of Theorem 3.5.

Lemma 3.8. *Let $p, q \in (0,\infty]$, $s \in \mathbb{R}$, and $f \in M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$. If fg exists in $\mathcal{S}'(\mathbb{R}^n)$ for any $g \in B^s_{p,q}(\mathbb{R}^n)$, then $f \in M(B^s_{p,q}(\mathbb{R}^n))$ and there exists a positive constant C, independent of f, such that*

$$\|f\|_{M(B^s_{p,q}(\mathbb{R}^n))} \le C\|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))}.$$

Proof. Let $g \in B^s_{p,q}(\mathbb{R}^n)$. Then, applying the assumption that fg exists in $\mathcal{S}'(\mathbb{R}^n)$ and also applying [17, Remark 2.9 and Corollary 2.11], we conclude that

$$fg = \lim_{j\to\infty} f S^j g \tag{3.20}$$

in $\mathcal{S}'(\mathbb{R}^n)$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $\psi(\mathbf{0}) = 1$. Then, from [7, Proposition 9.9], it follows that, for any $j \in \mathbb{N}$, $fS^jg = \lim_{l\to\infty} f(S^jg)\psi(2^{-l}\cdot)$ in $\mathcal{S}'(\mathbb{R}^n)$. Combining this and (3.20), we further obtain

$$fg = \lim_{j\to\infty} \lim_{l\to\infty} f\left(S^jg\right)\psi\left(2^{-l}\cdot\right) \tag{3.21}$$

in $\mathcal{S}'(\mathbb{R}^n)$.

Observe that, for any $j, l \in \mathbb{N}$, $(S^jg)\psi(2^{-l}\cdot) \in C_c^\infty(\mathbb{R}^n) \subset \widetilde{B}^s_{p,q}(\mathbb{R}^n)$. Using this, the assumption $f \in M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$, [35, Theorem 2.8.2], and Lemma 2.4(i), we find that there exists $\rho \in (\max\{0, s, \frac{n}{p} - s\}, \infty)$ such that, for any $j, l \in \mathbb{N}$,

$$\begin{aligned}
&\left\|f\left(S^jg\right)\psi\left(2^{-l}\cdot\right)\right\|_{B^s_{p,q}(\mathbb{R}^n)} \\
&\quad\le \|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))}\left\|\left(S^jg\right)\psi\left(2^{-l}\cdot\right)\right\|_{B^s_{p,q}(\mathbb{R}^n)} \\
&\quad\lesssim \|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))}\left\|S^jg\right\|_{B^s_{p,q}(\mathbb{R}^n)}\left\|\psi\left(2^{-l}\cdot\right)\right\|_{B^\rho_{\infty,\infty}(\mathbb{R}^n)} \\
&\quad\lesssim \|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))}\|g\|_{B^s_{p,q}(\mathbb{R}^n)}\left\|\psi\left(2^{-l}\cdot\right)\right\|_{B^\rho_{\infty,\infty}(\mathbb{R}^n)}.
\end{aligned} \tag{3.22}$$

In addition, for any $N \in \mathbb{Z}_+$, $l \in \mathbb{N}$, and $x, h \in \mathbb{R}^n$, we have

$$\Delta_h^N\left(\psi\left(2^{-l}\cdot\right)\right)(x) = \sum_{j=0}^N (-1)^{N-j}\binom{N}{j}\psi\left(2^{-l}x + 2^{-l}jh\right) = \Delta^N_{2^{-l}h}\psi\left(2^{-l}x\right).$$

By this and [43, (4.28)], we conclude that, for any $N \in \mathbb{Z}_+$, $l \in \mathbb{N}$, $t \in (0,1)$, and $x, h \in \mathbb{R}^n$ with $|h| \le t$,

$$\left|\Delta_h^N\left(\psi\left(2^{-l}\cdot\right)\right)(x)\right| \lesssim \left(2^{-l}|h|\right)^N \sup_{\alpha\in\mathbb{Z}^n_+,\,|\alpha|=N}\ \sup_{y\in\mathbb{R}^n,\,|2^{-l}x-y|\le N|h|} |\partial^\alpha\psi(y)| \lesssim |h|^N \le t^N.$$

From this with $N := \lfloor\rho\rfloor + 1$ and also from Lemma 3.7 with s replaced by ρ, we then deduce that, for any $l \in \mathbb{N}$,

$$\begin{aligned}
&\left\|\psi\left(2^{-l}\cdot\right)\right\|_{B^\rho_{\infty,\infty}(\mathbb{R}^n)} \\
&\quad\sim \sup_{x\in\mathbb{R}^n}\left|\psi\left(2^{-l}x\right)\right| + \sup_{t\in(0,1)}\left\{t^{-\rho}\sup_{h\in\mathbb{R}^n,\,|h|\le t}\ \sup_{x\in\mathbb{R}^n}\left|\Delta_h^{\lfloor\rho\rfloor+1}\left(\psi\left(2^{-l}\cdot\right)\right)(x)\right|\right\} \\
&\quad\lesssim \sup_{x\in\mathbb{R}^n}\left|\psi\left(2^{-l}x\right)\right| + \sup_{t\in(0,1)} t^{\lfloor\rho\rfloor+1-\rho} \sim 1.
\end{aligned}$$

This, together with (3.22), further implies that, for any $j, l \in \mathbb{N}$,

$$\left\|f\left(S^jg\right)\psi\left(2^{-l}\cdot\right)\right\|_{B^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))}\|g\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

By this, (3.21), and Lemma 2.5, we conclude that $fg \in B^s_{p,q}(\mathbb{R}^n)$ and

$$\|fg\|_{B^s_{p,q(\mathbb{R}^n)}} \lesssim \|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))} \|g\|_{B^s_{p,q}(\mathbb{R}^n)},$$

which implies that $f \in M(B^s_{p,q}(\mathbb{R}^n))$ and $\|f\|_{M(B^s_{p,q}(\mathbb{R}^n))} \lesssim \|f\|_{M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))}$. This then finishes the proof of Lemma 3.8. □

Now, we are in position to give the proof of Theorem 3.5.

Proof of Theorem 3.5. We first show

$$M\left(B^s_{1,\infty}(\mathbb{R}^n)\right) \hookrightarrow B^{-s}_{\infty,1}(\mathbb{R}^n). \tag{3.23}$$

To do this, let $f \in M(B^s_{1,\infty}(\mathbb{R}^n))$ and $g \in B^{-s}_{\infty,1}(\mathbb{R}^n)$. Then, applying Lemmas 2.11(i) and 3.7, we find that $f \in L^\infty(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. These, combined with [28, Proposition 4.2.1], further imply that fg exists in $\mathcal{S}'(\mathbb{R}^n)$. In addition, from Lemma 2.12, it follows that $f \in M(\widetilde{B}^{-s}_{\infty,1}(\mathbb{R}^n))$. Thus, by Lemma 3.8, we further conclude that $f \in M(B^{-s}_{\infty,1}(\mathbb{R}^n))$ and $\|f\|_{M(B^{-s}_{\infty,1}(\mathbb{R}^n))} \lesssim \|f\|_{M(\widetilde{B}^{-s}_{\infty,1}(\mathbb{R}^n))}$. Combining these, Proposition 3.6, and Lemma 2.12 again, we obtain

$$\|f\|_{B^{-s}_{\infty,1}(\mathbb{R}^n)} \sim \|f\|_{M(B^{-s}_{\infty,1}(\mathbb{R}^n))} \lesssim \|f\|_{M(\widetilde{B}^{-s}_{\infty,1}(\mathbb{R}^n))} \sim \|f\|_{M(B^s_{1,\infty}(\mathbb{R}^n))}.$$

This finishes the proof of (3.23).

By Proposition 3.6 and Lemmas 2.15 and 2.12, we find that

$$B^{-s}_{\infty,1}(\mathbb{R}^n) = M\left(B^{-s}_{\infty,1}(\mathbb{R}^n)\right) \hookrightarrow M\left(\widetilde{B}^{-s}_{\infty,1}(\mathbb{R}^n)\right) = M\left(B^s_{1,\infty}(\mathbb{R}^n)\right),$$

which proves $B^{-s}_{\infty,1}(\mathbb{R}^n) \hookrightarrow M(B^s_{1,\infty}(\mathbb{R}^n))$. The proof of Theorem 3.5 is then complete. □

3.3. A characterization of $M(B^s_{\infty,1}(\mathbb{R}^n))$ with $s \in (-\infty, 0]$

In this subsection, by some elaborate estimates of paraproducts and several relations between two multiplier spaces $M(B^s_{p,q}(\mathbb{R}^n))$ and $M(\widetilde{B}^s_{p,q}(\mathbb{R}^n))$, we obtain the Fourier analytic characterization of $M(B^s_{\infty,1}(\mathbb{R}^n))$ for any $s \in (-\infty, 0]$. First, we characterize $M(B^0_{\infty,1}(\mathbb{R}^n))$ as follows.

Theorem 3.9. *Let $\|\cdot\|_0$ be the same as in (3.1) with s replaced by 0. Then $f \in M(B^0_{\infty,1}(\mathbb{R}^n))$ if and only if $f \in L^\infty(\mathbb{R}^n)$ and*

$$\|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} + \|f\|_0 < \infty.$$

Moreover, there exist two positive constants C_1 and C_2 such that, for any $f \in M(B^0_{\infty,1}(\mathbb{R}^n))$,

$$C_1 \|f\|_{M(B^0_{\infty,1}(\mathbb{R}^n))} \le \|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} + \|f\|_0 \le C_2 \|f\|_{M(B^0_{\infty,1}(\mathbb{R}^n))} .$$

To prove this theorem, we need the following characterization of $M(B^0_{1,\infty}(\mathbb{R}^n))$, see, for instance, [15, Theorem 5] or [17, Theorem 3.3 and Remark 3.4].

Proposition 3.10. *Let $\|\cdot\|_0$ be the same as in (3.1) with s therein replaced by* 0*. Then $f \in M(B^0_{1,\infty}(\mathbb{R}^n))$ if and only if $f \in L^\infty(\mathbb{R}^n)$ and*

$$\|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} + \|f\|_0 < \infty.$$

Furthermore, there exist two positive constants C_1 and C_2 such that, for any $f \in M(B^0_{1,\infty}(\mathbb{R}^n))$,

$$C_1 \|f\|_{M(B^0_{1,\infty}(\mathbb{R}^n))} \le \|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} + \|f\|_0 \le C_2 \|f\|_{M(B^0_{1,\infty}(\mathbb{R}^n))} .$$

The following auxiliary estimate of the operators $\{S_j\}_{j\in\mathbb{Z}_+}$ on cubes also plays a key role in the proof of Theorem 3.9.

Lemma 3.11. *There exists a positive constant C such that, for any $f, g \in \mathcal{S}'(\mathbb{R}^n)$, $l, k, j \in \mathbb{Z}_+$, $\nu \in \mathbb{Z}^n$, and $x \in Q_{l,\nu}$,*

$$|S_l([S_k f]\, S_j g)(x)| \le C \|S_j g\|_{L^\infty(\mathbb{R}^n)} \sup_{m\in\mathbb{Z}^n} \fint_{Q_{l,m}} |S_k f(y)|\, dy.$$

Proof. Let $f, g \in \mathcal{S}'(\mathbb{R}^n)$, $l, k, j \in \mathbb{Z}_+$, and $\nu \in \mathbb{Z}^n$. Then, for any $x \in Q_{l,\nu}$, we have

$$\begin{aligned} |S_l([S_k f]\, S_j g)(x)| &\le \int_{\mathbb{R}^n} |\varphi_l(x-y)|\, |S_k f(y)|\, |S_j g(y)|\, dy \\ &\lesssim \|S_j g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{2^{ln}}{(1+2^l|x-y|)^{n+1}} |S_k f(y)|\, dy \\ &= \|S_j g\|_{L^\infty(\mathbb{R}^n)} \sum_{m\in\mathbb{Z}^n} \fint_{Q_{l,m}} \frac{1}{(1+2^l|x-y|)^{n+1}} |S_k f(y)|\, dy. \end{aligned} \tag{3.24}$$

Notice that, for any $m \in \mathbb{Z}^n$, $x \in Q_{l,\nu}$, and $y \in Q_{l,m}$, $1 + 2^l|x-y| \sim 1 + |\nu - m|$. From this, (3.24), and the fact that $\sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|\nu-m|)^{n+1}} \sim 1$, we then infer that, for any $x \in Q_{l,\nu}$,

$$|S_l([S_k f]\, S_j g)(x)| \lesssim \|S_j g\|_{L^\infty(\mathbb{R}^n)} \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|\nu-m|)^{n+1}} \fint_{Q_{l,m}} |S_k f(y)|\, dy$$

$$\lesssim \|S_j g\|_{L^\infty(\mathbb{R}^n)} \sup_{m\in\mathbb{Z}^n} \left[\fint_{Q_{l,m}} |S_k f(y)|\, dy \right].$$

This finishes the proof of Lemma 3.11. □

Now, we turn to the estimation of the paraproduct $\Pi_2(\cdot,\cdot)$, which will be used in the proof of Theorem 3.9.

Lemma 3.12. *Let $s \in (-\infty, 0]$. Assume that $f \in L^\infty(\mathbb{R}^n)$ satisfies $\|f\|_{-s} < \infty$ and that $g \in B^s_{\infty,1}(\mathbb{R}^n)$. Then there exists a positive constant C, independent of both f and g, such that*

$$\|\,\Pi_2(f,g)\|_{B^s_{\infty,1}(\mathbb{R}^n)} \le C\left[\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{-s}\right]\|g\|_{B^s_{\infty,1}(\mathbb{R}^n)}.$$

Proof. Using (3.5) and the Tonelli theorem, we have

$$\begin{aligned}
&\|\,\Pi_2(f,g)\|_{B^s_{\infty,1}(\mathbb{R}^n)}\\
&\le \sum_{l\in\mathbb{Z}_+}\sum_{j=\max\{0,l-3\}}^{\infty}\sum_{k=\max\{0,j-1\}}^{j+1} 2^{ls}\|S_l([S_k f]\, S_j g)\|_{L^\infty(\mathbb{R}^n)}\\
&= \sum_{l=0}^{5}\sum_{j=\max\{0,l-3\}}^{l+2}\sum_{k=\max\{0,j-1\}}^{j+1} 2^{ls}\|S_l([S_k f]\, S_j g)\|_{L^\infty(\mathbb{R}^n)}\\
&\quad + \sum_{l=0}^{5}\sum_{j=l+3}^{\infty}\sum_{k=j-1}^{j+1}\cdots + \sum_{l=6}^{\infty}\sum_{j=l-3}^{\infty}\sum_{k=j-1}^{j+1}\cdots\\
&\lesssim \sum_{l=0}^{5}\sum_{j=\max\{0,l-3\}}^{l+2}\sum_{k=\max\{0,j-1\}}^{j+1} 2^{ls}\|S_l([S_k f]\, S_j g)\|_{L^\infty(\mathbb{R}^n)}\\
&\quad + \sum_{l=0}^{5}\sum_{j=l+3}^{\infty}\sum_{k=j-1}^{j+1}\cdots + \sum_{j=3}^{\infty}\sum_{k=j-1}^{j+1}\sum_{l=0}^{k-2}\cdots + \sum_{j=3}^{\infty}\sum_{k=j-1}^{j+1}\sum_{l=k-1}^{j+3}\cdots\\
&=: \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3 + \mathrm{III}_4.
\end{aligned} \tag{3.25}$$

Notice that, for any $l \in \mathbb{Z}$ and $j \in \mathbb{Z} \cap [l-3, l+2]$, $2^{ls} \sim 2^{js}$. Applying this and the Young inequality, we obtain

$$\begin{aligned}
\mathrm{III}_1 &\lesssim \sum_{l=0}^{5}\sum_{j=\max\{0,l-3\}}^{l+2}\sum_{k=\max\{0,j-1\}}^{j+1} 2^{ls}\|S_k f\|_{L^\infty(\mathbb{R}^n)}\|S_j g\|_{L^\infty(\mathbb{R}^n)}\\
&\lesssim \|f\|_{L^\infty(\mathbb{R}^n)}\sum_{j=0}^{7} 2^{js}\|S_j g\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{B^s_{\infty,1}(\mathbb{R}^n)},
\end{aligned} \tag{3.26}$$

which is the desired estimation of III_1.

Now, we estimate III_2. Indeed, from the Tonelli theorem, Lemma 3.11, and the fact that, for any $j \in \mathbb{Z}$ and $k \in \mathbb{Z} \cap [j-1, j+1]$, $2^{js} \sim 2^{ks}$, we deduce that

$$\begin{aligned}\mathrm{III}_2 &\le \sum_{j=3}^{\infty} \sum_{k=j-1}^{j+1} \sum_{l=0}^{5} 2^{ls} \left\| S_l \left([S_k f]\, S_j g \right) \right\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \sum_{j=3}^{\infty} 2^{js} \left\| S_j g \right\|_{L^\infty(\mathbb{R}^n)} \sum_{k=j-1}^{j+1} 2^{-ks} \sum_{l=0}^{5} 2^{ls} \sup_{m\in\mathbb{Z}^n} \fint_{Q_{l,m}} |S_k f(y)|\, dy. \end{aligned} \tag{3.27}$$

Obviously we have

$$\begin{aligned}&\sum_{j=9}^{\infty} 2^{js} \left\| S_j g \right\|_{L^\infty(\mathbb{R}^n)} \sum_{k=j-1}^{j+1} 2^{-ks} \sum_{l=0}^{5} 2^{ls} \sup_{m\in\mathbb{Z}^n} \fint_{Q_{l,m}} |S_k f(y)|\, dy \\ &\quad\lesssim \|f\|_{-s} \|g\|_{B^s_{\infty,1}(\mathbb{R}^n)}, \end{aligned}$$

because in this case $5 \le k-2$. Furthermore,

$$\begin{aligned}&\sum_{j=3}^{9} 2^{js} \left\| S_j g \right\|_{L^\infty(\mathbb{R}^n)} \sum_{k=j-1}^{j+1} 2^{-ks} \sum_{l=0}^{5} 2^{ls} \sup_{m\in\mathbb{Z}^n} \fint_{Q_{l,m}} |S_k f(y)|\, dy \\ &\quad\lesssim \|g\|_{B^s_{\infty,1}(\mathbb{R}^n)} \sum_{l=0}^{5} 2^{ls} \sup_{m\in\mathbb{Z}^n} \fint_{Q_{l,m}} |S_k f(y)|\, dy \\ &\quad\lesssim \|g\|_{B^s_{\infty,1}(\mathbb{R}^n)} \|f\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

Inserting these inequalities into (3.27) we obtain

$$\mathrm{III}_2 \lesssim \left(\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{-s} \right) \|g\|_{B^s_{\infty,1}(\mathbb{R}^n)}. \tag{3.28}$$

This finishes the estimation of III_2.

Next, for III_3, we can proceed as in the case III_2 and find

$$\begin{aligned}\mathrm{III}_3 &\lesssim \sum_{j=3}^{\infty} \sum_{k=j-1}^{j+1} \sum_{l=0}^{k-2} 2^{ls} \left\| S_j g \right\|_{L^\infty(\mathbb{R}^n)} \sup_{m\in\mathbb{Z}^n} \fint_{Q_{l,m}} |S_k f(y)|\, dy \\ &\sim \sum_{j=3}^{\infty} 2^{js} \left\| S_j g \right\|_{L^\infty(\mathbb{R}^n)} \sum_{k=j-1}^{j+1} 2^{-ks} \sum_{l=0}^{k-2} 2^{ls} \sup_{m\in\mathbb{Z}^n} \fint_{Q_{l,m}} |S_k f(y)|\, dy \\ &\lesssim \|f\|_{-s} \|g\|_{B^s_{\infty,1}(\mathbb{R}^n)}. \end{aligned} \tag{3.29}$$

Finally, we deal with III_4. Indeed, using the Young inequality, we conclude that

$$\mathrm{III}_4 \lesssim \sum_{j=3}^{\infty} \sum_{k=j-1}^{j+1} \sum_{l=k-1}^{j+3} 2^{ls} \|S_k f\|_{L^\infty(\mathbb{R}^n)} \|S_j g\|_{L^\infty(\mathbb{R}^n)}$$

$$\sim \|f\|_{L^\infty(\mathbb{R}^n)} \sum_{j=3}^{\infty} 2^{js} \|S_j g\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{B^s_{\infty,1}(\mathbb{R}^n)}. \quad (3.30)$$

Combining the estimates (3.25) - (3.30) the claim in Lemma 3.12 follows. □

Based on the above lemmas, we next show Theorem 3.9.

Proof of Theorem 3.9. We first prove the necessity. Indeed, applying both Lemmas 2.15 and 2.12 with $p := \infty$, $q := 1$, and $s := 0$, we find that

$$M\left(B^0_{\infty,1}(\mathbb{R}^n)\right) \hookrightarrow M\left(\widetilde{B}^0_{\infty,1}(\mathbb{R}^n)\right) = M\left(B^0_{1,\infty}(\mathbb{R}^n)\right).$$

By this and Proposition 3.10, we then have, for any $f \in M(B^0_{\infty,1}(\mathbb{R}^n))$,

$$\|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} + \|f\|_0 \sim \|f\|_{M(B^0_{1,\infty}(\mathbb{R}^n))} \lesssim \|f\|_{M(B^0_{\infty,1}(\mathbb{R}^n))}, \quad (3.31)$$

which completes the proof of the necessity.

Conversely, we now show the sufficiency. For this purpose, let $f \in L^\infty(\mathbb{R}^n)$ be such that $\|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} + \|f\|_0 < \infty$ and let $g \in B^0_{\infty,1}(\mathbb{R}^n)$. Then

$$\|fg\|_{B^0_{\infty,1}(\mathbb{R}^n)} \le \sum_{j=1}^{3} \|\Pi_j(f,g)\|_{B^0_{\infty,1}(\mathbb{R}^n)} =: \sum_{j=1}^{3} \mathrm{I}_j. \quad (3.32)$$

To estimate I_1, we use Lemma 3.4 and the well-known embedding $B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. These yield

$$\mathrm{I}_1 \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{B^0_{\infty,1}(\mathbb{R}^n)} \le \|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} \|g\|_{B^0_{\infty,1}(\mathbb{R}^n)}. \quad (3.33)$$

Next, from Lemma 3.12 with $s := 0$ and the aforementioned embedding $B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ it follows that

$$\mathrm{I}_2 \lesssim \left[\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_0\right] \|g\|_{B^0_{\infty,1}(\mathbb{R}^n)} \le \left[\|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} + \|f\|_0\right] \|g\|_{B^0_{\infty,1}(\mathbb{R}^n)}. \quad (3.34)$$

Finally, again by the embedding $B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and by Lemma 3.4, we conclude that

$$\mathrm{I}_3 = \|\Pi_1(g,f)\|_{B^0_{\infty,1}(\mathbb{R}^n)} \lesssim \|g\|_{L^\infty(\mathbb{R}^n)} \|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} \le \|f\|_{B^0_{\infty,1}(\mathbb{R}^n)} \|g\|_{B^0_{\infty,1}(\mathbb{R}^n)}.$$

A combination of the estimates for I_1, I_2 and I_3 yields the claimed inequality for $\|fg\|_{B^0_{\infty,1}(\mathbb{R}^n)}$. This completes the proof of Theorem 3.9. □

Finally, we deal with $M(B^s_{\infty,1}(\mathbb{R}^n))$, $s\in(-\infty,0)$.

Theorem 3.13. *Let $s\in(-\infty,0)$ and $\|\cdot\|_{-s}$ be the same as in (3.1) with s therein replaced by $-s$. Then $f\in M(B^s_{\infty,1}(\mathbb{R}^n))$ if and only if $f\in L^\infty(\mathbb{R}^n)$ and*

$$\|f\|_{L^\infty(\mathbb{R}^n)}+\|f\|_{-s}<\infty.$$

Moreover, there exist two positive constants C_1 and C_2 such that, for any $f\in M(B^s_{\infty,1}(\mathbb{R}^n))$,

$$C_1\|f\|_{M(B^s_{\infty,1}(\mathbb{R}^n))}\le\|f\|_{L^\infty(\mathbb{R}^n)}+\|f\|_{-s}\le C_2\|f\|_{M(B^s_{\infty,1}(\mathbb{R}^n))}.$$

Compared to $s=0$, we need a modified estimate of $\Pi_3(\cdot,\cdot)$.

Lemma 3.14. *Let $s\in(-\infty,0)$. Then there exists a positive constant C such that, for any $f\in L^\infty(\mathbb{R}^n)$ and $g\in B^s_{\infty,1}(\mathbb{R}^n)$,*

$$\|\Pi_3(f,g)\|_{B^s_{\infty,1}(\mathbb{R}^n)}\le C\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{B^s_{\infty,1}(\mathbb{R}^n)}.$$

Proof. Applying both (3.7) and the Young inequality, we obtain, for any $j\in\mathbb{Z}_+$, $f\in L^\infty(\mathbb{R}^n)$, and $g\in B^s_{\infty,1}(\mathbb{R}^n)$,

$$\begin{aligned}\|S_j(\Pi_3(f,g))\|_{L^\infty(\mathbb{R}^n)}&\le\sum_{k=\max\{2,j-1\}}^{j+3}\left\|S_j\left([S_kf]\,S^{k-2}g\right)\right\|_{L^\infty(\mathbb{R}^n)}\\&\lesssim\sum_{k=\max\{2,j-1\}}^{j+3}\|S_kf\|_{L^\infty(\mathbb{R}^n)}\left\|S^{k-2}g\right\|_{L^\infty(\mathbb{R}^n)}\\&\lesssim\|f\|_{L^\infty(\mathbb{R}^n)}\sum_{l=0}^{j+1}\|S_lg\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Combining this, the Young inequality, the Tonelli theorem, and the assumption $s\in(-\infty,0)$, we conclude that, for any $f\in L^\infty(\mathbb{R}^n)$ and $g\in B^s_{\infty,1}(\mathbb{R}^n)$,

$$\begin{aligned}\|\Pi_3(f,g)\|_{B^s_{\infty,1}(\mathbb{R}^n)}&\lesssim\|f\|_{L^\infty(\mathbb{R}^n)}\sum_{j\in\mathbb{Z}_+}2^{js}\sum_{l=0}^{j+1}\|S_lg\|_{L^\infty(\mathbb{R}^n)}\\&\lesssim\|f\|_{L^\infty(\mathbb{R}^n)}\sum_{j\in\mathbb{Z}_+}2^{js}\sum_{l=0}^{j+1}\|S_lg\|_{L^\infty(\mathbb{R}^n)}\end{aligned}$$

$$= \|f\|_{L^\infty(\mathbb{R}^n)} \sum_{l=0}^{\infty} \|S_l g\|_{L^\infty(\mathbb{R}^n)} \sum_{j=\max\{0,l-1\}}^{\infty} 2^{js}$$

$$\sim \|f\|_{L^\infty(\mathbb{R}^n)} \sum_{l=0}^{\infty} 2^{ls} \|S_l g\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{B^s_{\infty,1}(\mathbb{R}^n)} .$$

This completes the proof of Lemma 3.14. □

We now show Theorem 3.13.

Proof of Theorem 3.13. We first prove the necessity. Indeed, from both Lemmas 2.15 and 2.12 with p and q therein replaced, respectively, by ∞ and 1, we infer that

$$M\left(B^s_{\infty,1}(\mathbb{R}^n)\right) \hookrightarrow M\left(\widetilde{B}^s_{\infty,1}(\mathbb{R}^n)\right) = M\left(B^{-s}_{1,\infty}(\mathbb{R}^n)\right),$$

which, together with Theorem 3.1 with s replaced by $-s$, further implies that, for any $f \in M(B^s_{\infty,1}(\mathbb{R}^n))$,

$$\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{-s} \sim \|f\|_{M(B^{-s}_{1,\infty}(\mathbb{R}^n))} \lesssim \|f\|_{M(B^s_{\infty,1}(\mathbb{R}^n))} . \tag{3.35}$$

This finishes the proof of the necessity. In addition, using Lemmas 3.4, 3.12, and 3.14, we find that, for any $f \in L^\infty(\mathbb{R}^n)$ with $\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{-s} < \infty$ and for any $g \in B^s_{\infty,1}(\mathbb{R}^n)$,

$$\|fg\|_{B^s_{\infty,1}(\mathbb{R}^n)} \leq \sum_{j=1}^{3} \|\Pi_j(f,g)\|_{B^s_{\infty,1}(\mathbb{R}^n)} \lesssim \left[\|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{-s}\right] \|g\|_{B^s_{\infty,1}(\mathbb{R}^n)} ,$$

which further implies that $f \in M(B^s_{\infty,1}(\mathbb{R}^n))$ and

$$\|f\|_{M(B^s_{\infty,1}(\mathbb{R}^n))} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{-s} .$$

This finishes the proof of the sufficiency. By this estimate and (3.35), we conclude that, for any $f \in M(B^s_{\infty,1}(\mathbb{R}^n))$,

$$\|f\|_{M(B^s_{\infty,1}(\mathbb{R}^n))} \sim \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{-s} ,$$

which completes the proof of Theorem 3.13. □

4. The duality principle for multiplier spaces of Besov spaces

In this section, as an application of several characterizations of pointwise multipliers obtained in Section 3, we shall give now a proof of the duality principle for multiplier spaces of Besov spaces. Moreover, we show that the duality principle does not extend to quasi-Banach spaces by considering $M(B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n))$ and $M(B^0_{\infty,\infty}(\mathbb{R}^n))$.

To be precise, by the duality principle we mean the following theorem.

Theorem 4.1. *Let $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Then*

$$M\left(B^s_{p,q}(\mathbb{R}^n)\right) = M\left(B^{-s}_{p',q'}(\mathbb{R}^n)\right)$$

in the sense of equivalent norms.

Proof. Almost all work is done. We consider the following three cases.

Case 1) $p, q \in [1, \infty)$. Because of $B^s_{p,q}(\mathbb{R}^n) = \widetilde{B}^s_{p,q}(\mathbb{R}^n)$ and Lemma 2.12, we find that

$$M\left(B^s_{p,q}(\mathbb{R}^n)\right) = M\left(\widetilde{B}^s_{p,q}(\mathbb{R}^n)\right) = M\left(B^{-s}_{p',q'}(\mathbb{R}^n)\right).$$

Case 2) $p = 1$ and $q = \infty$ or $p = \infty$ and $q = 1$. In this situation, from Theorems 3.1, 3.5, 3.9, 3.13 and Propositions 3.6 and 3.10 we further deduce that $M(B^s_{1,\infty}(\mathbb{R}^n)) = M(B^{-s}_{\infty,1}(\mathbb{R}^n))$.

Case 3) $p = q = \infty$. Case 1) yields $M(B^{-s}_{1,1}(\mathbb{R}^n)) = M(B^s_{\infty,\infty}(\mathbb{R}^n))$. The proof is then complete. □

Remark 4.2. As mentioned above, generalizations need some care in view of

$$M\left(\widetilde{B}^s_{\infty,\infty}(\mathbb{R}^n)\right) \subsetneqq M\left(B^{-s}_{1,1}(\mathbb{R}^n)\right), \qquad s \in (0, \infty),$$

and $(\widetilde{B}^s_{\infty,\infty}(\mathbb{R}^n))' = B^{-s}_{1,1}(\mathbb{R}^n)$, see Remark 2.13.

Moreover, we should point out that this principle can not extend to quasi-Banach spaces. Here is a counterexample. In case $p \in (0, 1)$ we know that

$$\left(B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)\right)' = B^0_{\infty,\infty}(\mathbb{R}^n),$$

see Lemma 2.6(ii).

Theorem 4.3. *Let $p \in (0, 1)$. Then $M(B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n))$ is a proper subspace of $M(B^0_{\infty,\infty}(\mathbb{R}^n))$.*

To show this theorem, we require the Fourier analytic characterization of the multiplier space $M(B^0_{\infty,\infty}(\mathbb{R}^n))$. Let $b \in \mathbb{R}$. Recall that the *Triebel–Lizorkin space* $F^0_{\infty,1}(\mathbb{R}^n)$ (resp. the *Besov space with logarithmic smoothness* $B^{0,b}_{\infty,\infty}(\mathbb{R}^n)$) is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^0_{\infty,1}(\mathbb{R}^n)} := \sup_{j\in\mathbb{Z},\, m\in\mathbb{Z}^n} \fint_{Q_{j,m}} \sum_{k=\max\{0,j\}}^{\infty} |S_k f(x)|\, dx < \infty$$

$$\left(\text{resp. } \|f\|_{B^{0,b}_{\infty,\infty}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}_+} (1+k)^b \|S_k f\|_{L^\infty(\mathbb{R}^n)} < \infty\right).$$

Then the following characterization of $M(B^0_{\infty,\infty}(\mathbb{R}^n))$ is just [15, Theorem 4].

Proposition 4.4. *$f \in M(B^0_{\infty,\infty}(\mathbb{R}^n))$ if and only if $f \in L^\infty(\mathbb{R}^n)$ and*

$$\|f\|^\star_{M(B^0_{\infty,\infty}(\mathbb{R}^n))} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{F^0_{\infty,1}(\mathbb{R}^n)} + \|f\|_{B^{0,1}_{\infty,\infty}(\mathbb{R}^n)} < \infty.$$

Moreover, there exist two positive constants C_1 and C_2 such that, for any $f \in M(B^0_{\infty,\infty}(\mathbb{R}^n))$,

$$C_1 \|f\|_{M(B^0_{\infty,\infty}(\mathbb{R}^n))} \le \|f\|^\star_{M(B^0_{\infty,\infty}(\mathbb{R}^n))} \le C_2 \|f\|_{M(B^0_{\infty,\infty}(\mathbb{R}^n))}.$$

Via this proposition, we now prove Theorem 4.3.

Proof of Theorem 4.3. Using Lemma 2.6(ii) and using Lemma 2.10 with $X := B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)$, we have

$$M\left(B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)\right) \hookrightarrow M\left(B^0_{\infty,\infty}(\mathbb{R}^n)\right).$$

Next, we show that

$$M\left(B^0_{\infty,\infty}(\mathbb{R}^n)\right) \setminus M\left(B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)\right) \neq \emptyset. \tag{4.1}$$

To do this, for any $x \in \mathbb{R}^n$, let

$$f(x) := \sum_{j\in\mathbb{N}} (1+5j)^{-\frac{1}{p}} e^{i2^{5j}x_1}.$$

By this and the assumption $p \in (0,1)$, we find that

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le \sum_{j\in\mathbb{N}} \frac{1}{(1+5j)^{\frac{1}{p}}} < \infty,$$

which further implies that $f \in L^\infty(\mathbb{R}^n)$. On the other hand, observe that, for any $k \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$S_k f(x) = \begin{cases} (1+k)^{-\frac{1}{p}} e^{i2^k x_1} & \text{when } k = 5j \text{ for some } j \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

From this and the assumption $p \in (0,1)$ again, we deduce that

$$\|f\|_{F^0_{\infty,1}(\mathbb{R}^n)} = \sum_{j\in\mathbb{N}} \frac{1}{(1+5j)^{\frac{1}{p}}} < \infty$$

and

$$\|f\|_{B^{0,1}_{\infty,\infty}(\mathbb{R}^n)} = \sup_{j\in\mathbb{N}} \left\{(1+5j)^{1-\frac{1}{p}}\right\} < \infty.$$

These, combined with both the fact that $f \in L^\infty(\mathbb{R}^n)$ and Proposition 4.4, further imply

$$\|f\|^{\star}_{M(B^0_{\infty,\infty}(\mathbb{R}^n))} < \infty$$

and hence $f \in M(B^0_{\infty,\infty}(\mathbb{R}^n))$.

We now prove that $f \notin M(B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n))$. Noticing that $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \subset B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)$, we only need to show $f\varphi_0 \notin B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)$. Indeed, applying [15, (5.4)], [28, Proposition 4.4.2/4], and (4.2), we have

$$\|\,\Pi_1\,(f,\varphi_0)\|_{B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}\,\|\varphi_0\|_{B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)} < \infty$$

and

$$\|\,\Pi_2\,(f,\varphi_0)\|_{B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)} \lesssim \|f\|_{B^0_{\infty,\infty}(\mathbb{R}^n)}\,\|\varphi_0\|_{B^{\varepsilon+\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)} = \|\varphi_0\|_{B^{\varepsilon+\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)} < \infty.$$

Here $\varepsilon \in (0,\infty)$ is arbitrary. Therefore, to prove $f\varphi_0 \notin B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)$, it suffices to show $\Pi_3\,(f,\varphi_0) \notin B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)$. For this purpose, let $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\mathbf{1}_{B(\mathbf{0},5)} \le \psi_0 \le \mathbf{1}_{B(\mathbf{0},6)}$$

and

$$\mathbf{1}_{B(\mathbf{0},\frac{159}{32})\setminus B(\mathbf{0},\frac{33}{32})} \le \psi_1 \le \mathbf{1}_{B(\mathbf{0},5)\setminus B(\mathbf{0},1)}.$$

For any $h \in \mathcal{S}'(\mathbb{R}^n)$ and $k \in \mathbb{N}$, let

$$\widetilde{S}_0 h := \mathcal{F}^{-1}\left(\psi_0 \mathcal{F} h\right) \text{ and } \widetilde{S}_k h := \mathcal{F}^{-1}\left(\psi_1\left(2^{-k+1}\cdot\right)\mathcal{F} h\right). \tag{4.3}$$

Observe that, for any $k \in \mathbb{N}$,

$$\operatorname{supp}\mathcal{F}\left(\widetilde{S}_k f\right) \subset \left\{x \in \mathbb{R}^n :\ 2^{k-1} \le |x| \le 5\cdot 2^{k-1}\right\}.$$

By this, the local mean characterization of Besov spaces (see, for instance, [39, Theorem 2.7]), (2.9), and (4.2), we conclude that

$$\begin{aligned}
&\|\,\Pi_3\,(f,\varphi_0)\|^p_{B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)} \\
&\quad\sim \sum_{i\in\mathbb{Z}_+} 2^{ni(1-p)} \left\|\widetilde{S}_i\left(\Pi_3\,(f,\varphi_0)\right)\right\|^p_{L^p(\mathbb{R}^n)}
\end{aligned}$$

$$\geq \sum_{j\in\mathbb{N}} 2^{5nj(1-p)} \left\| \widetilde{S}_{5j} \left(\sum_{k=5j-1}^{5j+4} [S_k f]\, S^{k-2}\varphi_0 \right) \right\|_{L^p(\mathbb{R}^n)}^p$$
$$= \sum_{j\in\mathbb{N}} 2^{5nj(1-p)} (1+5j)^{-1} \left\| \widetilde{S}_{5j} \left(e^{i(2^{5j},0,\ldots,0)\cdot(\cdot)} S^{5j-2}\varphi_0 \right) \right\|_{L^p(\mathbb{R}^n)}^p . \tag{4.4}$$

Using (2.1), (2.2), and (2.3), we find that, for any $k \in \mathbb{N} \cap [4,\infty)$,

$$\operatorname{supp} \phi_k \subset B(\mathbf{0}, 3\cdot 2^{k-1}) \setminus B(\mathbf{0}, 2^{k-1}) \subset [B(\mathbf{0},8)]^{\complement},$$

which, together with (2.1) again, further implies that, for any $k \in \mathbb{N} \cap [4,\infty)$, $\operatorname{supp} \phi_0 \cap \operatorname{supp} \phi_k = \emptyset$ and hence $\phi_k\phi_0 = 0$. From this and the fact that $\sum_{k\in\mathbb{Z}_+} \phi_k = 1$, it follows that, for any $j \in \mathbb{N}$,

$$\mathcal{F}\left(S^{5j-2}\varphi_0\right) = (2\pi)^{-\frac{n}{2}} \sum_{k=0}^{5j-2} \phi_k\phi_0 = (2\pi)^{-\frac{n}{2}} \sum_{k\in\mathbb{Z}_+} \phi_k\phi_0 = (2\pi)^{-\frac{n}{2}} \phi_0.$$

Combining this and (4.4), we obtain

$$\| \Pi_3\ (f,\varphi_0)\|^p_{B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)} \gtrsim \sum_{j\in\mathbb{N}} 2^{5nj(1-p)}(1+5j)^{-1} \left\| \widetilde{S}_{5j} \left(e^{i(2^{5j},0,\ldots,0)\cdot(\cdot)} \varphi_0 \right) \right\|^p_{L^p(\mathbb{R}^n)} . \tag{4.5}$$

In addition, for any $j \in \mathbb{N}$, we have

$$\mathcal{F}\left(\widetilde{S}_{5j} \left(e^{i(2^{5j},0,\ldots,0)\cdot(\cdot)} \varphi_0 \right) \right) = (2\pi)^{-\frac{n}{2}} \psi_1\left(2^{-5j+1}\cdot\right) \phi_0\left(\cdot - \left(2^{5j},0,\ldots,0\right)\right). \tag{4.6}$$

Observe that, for any given $j \in \mathbb{N}$ and for any $x \in \mathbb{R}^n$ with $|x - (2^{5j},0,\ldots,0)| < \frac{3}{2}$, we have

$$\left|2^{-5j+1}x\right| \leq 2^{-5j+1}\left|x - \left(2^{5j},0,\ldots,0\right)\right| + 2 < \frac{3}{32} + 2 < \frac{159}{32}$$

and

$$\left|2^{-5j+1}x\right| \geq 2 - 2^{-5j+1}\left|x - \left(2^{5j},0,\ldots,0\right)\right| > 2 - \frac{3}{32} > \frac{33}{32}.$$

From this and the assumption $\psi_1 \mathbf{1}_{B(\mathbf{0},\frac{159}{32})\setminus B(\mathbf{0},\frac{33}{32})} = 1$, we infer that, for any $j \in \mathbb{N}$,

$$\psi_1\left(2^{-5j+1}\cdot\right)\phi_0\left(\cdot - \left(2^{5j},0,\ldots,0\right)\right) = \phi_0\left(\cdot - \left(2^{5j},0,\ldots,0\right)\right),$$

which, combined with (4.6), (4.5), and the assumption $p \in (0,1)$, further implies that, for any $j \in \mathbb{N}$,

$$\widetilde{S}_{5j}\left(e^{i(2^{5j},0,\ldots,0)\cdot(\cdot)}\varphi_0\right) = e^{i(2^{5j},0,\ldots,0)\cdot(\cdot)}\varphi_0$$

and hence

$$\| \Pi_3 \ (f, \varphi_0)\|^p_{B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)} \gtrsim \|\varphi_0\|^p_{L^p(\mathbb{R}^n)} \sum_{j\in\mathbb{N}} 2^{5nj(1-p)}(1+5j)^{-1} = \infty.$$

Thus, $f\varphi_0 \notin B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)$ and therefore

$$f \in M\left(B^0_{\infty,\infty}(\mathbb{R}^n)\right) \setminus M\left(B^{\frac{n}{p}-n}_{p,p}(\mathbb{R}^n)\right).$$

The proof is then complete. □

Data availability

Data will be made available on request.

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