Corrigendum: Correlations of indistinguishable particles in non-Hermitian lattices

2013 *New J. Phys.* **15** 033008

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An inadvertent indexing error has occurred in the derivation of the quantum Langevin equation presented in appendix A of our paper (2013 *New J. Phys.* **15** 033008). In the previous version, the summation over the system modes $\hat{a}_q^\dagger$ and the reservoir modes $\hat{b}_r^\dagger$ was executed with independent indices. However, as each system mode is coupled to a different set of reservoir modes, the summation of the reservoir modes should rather depend on the system mode. Hence, we introduce a set of reservoir summation indices $\{r_q\}_{q=1,...,M}$, which are assigned to the individual system modes $\hat{a}_q^\dagger$.

While this modification slightly alters equations (A.1)–(A.5) and (B.4)–(B.6), it has no implications to the results of the derivations, nor does it affect any result or conclusion in the main part of the paper.

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Appendix A. Quantum Langevin equation for coupled systems

In the following, we deduce the quantum Langevin equation for our systems from a general approach by principally following similar considerations as in [1]. Regard the total Hamiltonian of the overall configuration under the assumption of weak coupling between reservoir and loss-free system with strengths g_{m,n_m},

$$
\begin{aligned}
\hat{H}_{\text{total}} &= \hat{H}_0 + \hat{H}_{\text{reservoir}} + \hat{H}_{\text{interaction}} \\
&= \hbar \sum_{m=1}^{M} \left\{ \beta_m \, \hat{a}_m^\dagger \hat{a}_m + \sum_{j=1}^{M} C_{j,m} \, \hat{a}_j^\dagger \hat{a}_m \right\} + \hbar \sum_{m,n_m} \bar{\beta}_{n_m} \, \hat{b}_{n_m}^\dagger \, \hat{b}_{n_m} \\
&\quad + \hbar \sum_{m,n_m} \left\{ g_{m,n_m} \hat{a}_m^\dagger \, \hat{b}_{n_m} + g_{m,n_m}^* \hat{a}_m \, \hat{b}_{n_m}^\dagger \right\},
\end{aligned}
\tag{A.1}
$$

where $\bar{\beta}_{n_m}$ denotes the propagation constants of the reservoir modes and β_m and $C_{j,m}$ are defined in the main text. Thus, we find the Heisenberg equations for the system field operators $\hat{a}_q^\dagger$ and the reservoir field operators $\hat{b}_{r_q}^\dagger$ as coupled differential equations:

$$
\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}[\hat{a}_q^\dagger(z),\ \hat{H}_{\text{total}}] = -\frac{\mathrm{i}}{\hbar}[\hat{a}_q^\dagger(z),\ \hat{H}_0] + \mathrm{i} \sum_{r_q} g_{q,r_q}^* \, \hat{b}_{r_q}^\dagger,
\tag{A.2}
$$

$$
\frac{\mathrm{d}\,\hat{b}_{r_q}^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}[\hat{b}_{r_q}^\dagger(z),\ \hat{H}_{\text{total}}] = \mathrm{i}\bar{\beta}_{r_q}\, \hat{b}_{r_q}^\dagger(z) + \mathrm{i} g_{q,r_q}\, \hat{a}_q^\dagger(z)\,.
\tag{A.3}
$$

Integrating equation (A.3) and plugging into equation (A.2) yields

$$
\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}[\hat{a}_q^\dagger(z),\ \hat{H}_0] + \mathrm{i} \sum_{r_q} g_{q,r_q}^* \, \hat{b}_{r_q}^\dagger(0)\, \mathrm{e}^{\mathrm{i}\bar{\beta}_{r_q} z} - \int_0^z \mathrm{d}\zeta \sum_{r_q} |g_{q,r_q}|^2 \mathrm{e}^{\mathrm{i}\bar{\beta}_{r_q}(z-\zeta)} \hat{a}_q^\dagger(\zeta)\,,
\tag{A.4}
$$

where the second term on the rhs acts as a noise operator $\hat{f}_q^\dagger = \mathrm{i} \sum_{r_q} g_{q,r_q}^* \, \hat{b}_{r_q}^\dagger(0)\mathrm{e}^{\mathrm{i}\bar{\beta}_{r_q} z}$. For the last term on the rhs we replace the summation by an integral $\sum_{r_q} \to \frac{V}{(2\pi)^3} \int \mathrm{d}^3 \bar{k}_{r_q} \to \int \mathrm{d}\bar{\beta}_{r_q} D(\bar{\beta}_{r_q})$ with the spectral density of states $D(\beta)$:

$$
\begin{aligned}
\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} &= -\frac{\mathrm{i}}{\hbar}[\hat{a}_q^\dagger(z),\ \hat{H}_0] - \int_0^z \mathrm{d}\zeta \int \mathrm{d}\bar{\beta}_{r_q} D(\bar{\beta}_{r_q}) |g_q(\bar{\beta}_{r_q})|^2 \mathrm{e}^{\mathrm{i}(\bar{\beta}_{r_q}-\beta_q)(z-\zeta)} \hat{a}_q^\dagger(\zeta)\mathrm{e}^{\mathrm{i}\beta_q(z-\zeta)} + \hat{f}_q^\dagger(z) \\
&= -\frac{\mathrm{i}}{\hbar}[\hat{a}_q^\dagger(z),\ \hat{H}_0] - \int_0^z \mathrm{d}\zeta\, D(\beta_q)|g_q(\beta_q)|^2 \pi \delta(z-\zeta)\, \hat{a}_q^\dagger(\zeta)\mathrm{e}^{\mathrm{i}\beta_q(z-\zeta)} + \hat{f}_q^\dagger(z) \\
&= \frac{\mathrm{i}}{\hbar}[\hat{a}_q^\dagger(z),\ \hat{H}_0] - \pi D(\beta_q)|g_q(\beta_q)|^2 \, \hat{a}_q^\dagger(z) + \hat{f}_q^\dagger(z) \\
&= -\frac{\mathrm{i}}{\hbar}[\hat{a}_q^\dagger(z),\ \hat{H}_0] - \frac{\gamma_q}{2}\, \hat{a}_q^\dagger(z) + \hat{f}_q^\dagger(z)\,.
\end{aligned}
\tag{A.5}
$$

With Fermi's golden rule decay rates $\gamma_q = 2\pi D(\beta_q)|g_q(\beta_q)|^2$ and the introduction of $\hat{H}_{\text{system}} = \hat{H}_0 + \mathrm{i}\hat{A}_{\text{loss}} = \hat{H}_0 + \mathrm{i}\hbar \sum_{m=1}^{M} \frac{\gamma_m}{2} \hat{a}_m^\dagger \hat{a}_m$ one obtains the quantum Langevin equation:

$$\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}[\hat{a}_q^\dagger(z),\ \hat{H}_{\text{system}}] + \hat{f}_q^\dagger(z) = \mathrm{i}\sum_{j=1}^{M} \tilde{C}_{q,j}\,\hat{a}_j^\dagger(z) + \hat{f}_q^\dagger(z), \tag{A.6}$$

where $\tilde{C}_{q,j} = (\beta_j + \mathrm{i}\frac{\gamma_j}{2})\delta_{j,q} + C_{q,j}$. With the linear propagation operator $U_{q,j}(z) = (\mathrm{e}^{\mathrm{i}\hat{\tilde{C}}z})_{q,j}$ the formal solution of (A.6) is

$$\hat{a}_q^\dagger(z) = \sum_{j=1}^{M} U_{q,j}(z)\,\hat{a}_j^\dagger(0) + \int_0^z \mathrm{d}\zeta \sum_{j=1}^{M} U_{q,j}(z-\zeta)\hat{f}_q^\dagger(\zeta). \tag{A.7}$$

Appendix B. Calculation of the particle number correlation

Here we present an extensive derivation of the particle number correlation and prove that this quantity does not depend on the noise operators. Calculating the particle number correlations by using (A.7)

$$\Gamma_{q,r}(z) = \langle \hat{a}_q^\dagger(z)\,\hat{a}_r^\dagger(z)\hat{a}_r(z)\hat{a}_q(z)\rangle = \sum_{\alpha=1}^{16} \Gamma_{q,r}^{(\alpha)}(z) \tag{B.1}$$

yields 16 terms $\Gamma_{q,r}^{(\alpha)}$. They can be categorized in how often a noise operator appears. One finds contributions without noise operator ($\Gamma_{q,r}^{(1)}$), with one ($\Gamma_{q,r}^{(2...5)}$), two ($\Gamma_{q,r}^{(6...11)}$), three ($\Gamma_{q,r}^{(12...15)}$) and four ($\Gamma_{q,r}^{(16)}$) noise operators. In the following we will show exemplarily the contribution of each group by an extended calculation for one member. For the sake of clarity we slip the z-argument of the propagation operator elements $U_{j,k}$. Any other dependence will be written explicitly. Taking into account that the noise operators are Markovian and the average value of the noise equals zero as well as the zero temperature approximation (ZTA) for the reservoir, we get:

$\alpha = 1$:

$$\Gamma_{q,r}^{(1)} = \sum_{j,k,l,p=1}^{M} U_{q,j} U_{r,k} U_{r,l}^* U_{q,p}^* \langle \hat{a}_j^\dagger(0)\,\hat{a}_k^\dagger(0)\hat{a}_l(0)\hat{a}_p(0)\rangle. \tag{B.2}$$

$\alpha = 2, \ldots, 5$:

$$\Gamma_{q,r}^{(2)} = \sum_{k,l,p=1}^{M} U_{r,k} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \sum_{j=1}^{M} U_{q,j}(z-\zeta)\langle \hat{f}_q^\dagger(\zeta)\,\hat{a}_k^\dagger(0)\hat{a}_l(0)\hat{a}_p(0)\rangle$$

$$= \sum_{k,l,p=1}^{M} U_{r,k} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \sum_{j=1}^{M} U_{q,j}(z-\zeta) \underbrace{\langle \hat{f}_q^\dagger(\zeta)\rangle}_{=0}\langle \hat{a}_k^\dagger(0)\hat{a}_l(0)\hat{a}_p(0)\rangle$$

$$= 0. \tag{B.3}$$

$\alpha = 6, \ldots, 11$:

$$
\begin{aligned}
\Gamma_{q,r}^{(6)} &= \sum_{l,p=1}^{M} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \sum_{j,k=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta') \hat{a}_l(0)\hat{a}_p(0) \rangle \\
&= \sum_{l,p=1}^{M} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \zeta\,\mathrm{d}\zeta' \sum_{j,k=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta') \rangle \langle \hat{a}_l(0)\hat{a}_p(0) \rangle \\
&= \sum_{l,p=1}^{M} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \sum_{j,k=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') \\
&\quad \times \sum_{\iota_q,\kappa_r} g_{q,\iota_q}^* g_{r,\kappa_r}^* \underbrace{\langle \hat{b}_{\iota_q}^\dagger(0)\, \hat{b}_{\kappa_r}^\dagger(0) \rangle}_{=0} \langle \mathrm{e}^{\mathrm{i}\bar{\beta}_{\iota_q}\zeta} \mathrm{e}^{\mathrm{i}\bar{\beta}_{\kappa_r}\zeta'} \langle \hat{a}_l(0)\hat{a}_p(0) \rangle \\
&= 0.
\end{aligned}
\tag{B.4}
$$

$\alpha = 12, \ldots, 15$:

$$
\begin{aligned}
\Gamma_{q,r}^{(12)} &= \sum_{p=1}^{M} U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \sum_{j,k,l=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'') \\
&\quad \times \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta')\, \hat{f}_r(\zeta'') \hat{a}_p(0) \rangle \\
&= \sum_{p=1}^{M} U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \sum_{j,k,l=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'') \\
&\quad \times \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta')\, \hat{f}_r(\zeta'') \rangle \langle \hat{a}_p(0) \rangle \\
&= \sum_{p=1}^{M} U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \sum_{j,k,l=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'') \\
&\quad \times \sum_{\iota_q,\kappa_r,\lambda_r} g_{q,\iota_q}^* g_{r,\kappa_r}^* g_{r,\lambda_r} \underbrace{\langle \hat{b}_{\iota_q}^\dagger(0)\, \hat{b}_{\kappa_r}^\dagger(0)\hat{b}_{\lambda_r}(0) \rangle}_{=0} \mathrm{e}^{\mathrm{i}\bar{\beta}_{\iota_q}\zeta} \mathrm{e}^{\mathrm{i}\bar{\beta}_{\kappa_r}\zeta'} \mathrm{e}^{-\mathrm{i}\bar{\beta}_{\lambda_r}\zeta''} \langle \hat{a}_p(0) \rangle \\
&= 0.
\end{aligned}
\tag{B.5}
$$

$\alpha = 16$:

$$
\begin{aligned}
\Gamma_{q,r}^{(16)} &= \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \int_0^z \mathrm{d}\zeta''' \sum_{j,k,l,p=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'') U_{q,p}^*(z-\zeta''') \\
&\quad \times \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta')\, \hat{f}_r(\zeta'')\, \hat{f}_q(\zeta''') \rangle \\
&= \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \int_0^z \mathrm{d}\zeta''' \sum_{j,k,l,p=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'') U_{q,p}^*(z-\zeta''') \\
&\quad \times \sum_{\iota_q,\kappa_r,\lambda_r,\pi_q} g_{q,\iota_q}^* g_{r,\kappa_r}^* g_{r,\lambda_r} g_{q,\pi_q} \underbrace{\langle \hat{b}_{\iota_q}^\dagger(0)\, \hat{b}_{\kappa_r}^\dagger(0)\hat{b}_{\lambda_r}(0)\hat{b}_{\pi_q}(0) \rangle}_{=0 \,(\mathrm{ZTA})} \mathrm{e}^{\mathrm{i}\bar{\beta}_{\iota_q}\zeta} \mathrm{e}^{\mathrm{i}\bar{\beta}_{\kappa_r}\zeta'} \mathrm{e}^{-\mathrm{i}\bar{\beta}_{\lambda_r}\zeta''} \mathrm{e}^{-\mathrm{i}\bar{\beta}_{\pi_q}\zeta'''} \\
&= 0.
\end{aligned}
\tag{B.6}
$$

Note, $\langle \hat{b}^\dagger_{\iota_q}(0)\, \hat{b}^\dagger_{\kappa_r}(0) \hat{b}_{\lambda_r}(0) \hat{b}_{\pi_q}(0) \rangle = 0$ because in the ZTA the average particle number for each reservoir mode vanishes.

As all terms with $\alpha > 1$ do not contribute it holds for the particle number correlation:

$$\Gamma_{q,r}(z) = \sum_{j,k,l,p=1}^{M} U_{q,j}(z) U_{r,k}(z) U^*_{r,l}(z) U^*_{q,p}(z) \langle \hat{a}^\dagger_j(0)\, \hat{a}^\dagger_k(0) \hat{a}_l(0) \hat{a}_p(0) \rangle \qquad \text{(B.7)}$$

which is independent of any noise operators.

Reference

[1] Yamamoto Y and İmamoğlu A 1999 *Mesoscopic Quantum Optics* (New York: Wiley)

Correlations of indistinguishable particles in non-Hermitian lattices

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Abstract. A novel approach to investigate the dynamics of indistinguishable particles in non-Hermitian lattice systems is presented, allowing an efficient calculation of quantum correlations between these particles in the presence of losses. Particular attention is paid to quasi-parity-time-symmetric systems, for which we numerically analyze two-particle quantum random walks for a variety of input states. Our results show how in some scenarios coherence is lost, inducing classical random walks, while in others the characteristic signatures of bosonic and fermionic exchange symmetry prevail.

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Contents

1. Introduction

Understanding quantum random walks of indistinguishable particles and their mutual correlation allows deep insight into the very nature of quantum mechanical processes in discretized space. In particular waveguiding lattices provide an exceptional tool for the study of such phenomena [1, 2]. To date, the correlations of identical particles have been explored in various systems, such as lattices with a transverse potential gradient [3] or disorder [4], Glauber–Fock lattices [5], two-dimensional lattices [6] and lattices with periodic boundaries [7]. However, an intrinsic property of all these quantum systems is their Hermiticity. Recently, parity-time($\mathcal{PT}$)-symmetric systems attracted much attention throughout the physics community. First proposed in 1998 [8], such Hamiltonians are not necessarily Hermitian, but still can exhibit purely real eigenvalue spectra. Since this concept was introduced in optics by anti-symmetric gain–loss distributions [9], numerous works were devoted to exploring optical $\mathcal{PT}$-symmetric phenomena [10–15], culminating in the first experimental demonstration of $\mathcal{PT}$-symmetry in physics [16, 17]. However, all of the reported effects have a purely classical origin, leaving the analysis of a true quantum evolution in $\mathcal{PT}$-symmetric lattices an open question.

Guo *et al* [12] showed that systems with anti-symmetric modulated loss distribution behave equivalently to those with gain–loss structure due to a transformation into an exponentially damped co-moving frame. For example a periodic gain–loss structure with equal gain and loss is equivalent to a structure with periodic no-loss–loss distribution where the loss is twice as before. We call such structures 'quasi-$\mathcal{PT}$-symmetric'. However, due to noise from vacuum fluctuations we are limited to transformations that result in vanishing net gain for the analysis of quantum evolution in non-Hermitian lattices. Nevertheless, despite this limitation, quasi-$\mathcal{PT}$-symmetric quantum systems give rise to fascinating dynamics. In this paper, we provide the first theoretical analysis of quantum random walks of identical particles in quasi-$\mathcal{PT}$-symmetric lattices. We further demonstrate how such dissipative quantum systems can be described by a set of coupled equations that allows the efficient simulation of extended configurations and investigate the correlation behavior of indistinguishable separable and entangled particles. We show how the symmetry of the lattice leads to independence or even decoherence of the quantum walkers for certain input configurations, while preserving strong correlations for others.

2. Theory

The general approach to describe a dissipative quantum systems is to weakly couple the loss-free system with a reservoir. Thus, the total Hamiltonian of the overall configuration is represented by the sum

$$\hat{H}_{\text{total}} = \hat{H}_0 + \hat{H}_{\text{reservoir}} + \hat{H}_{\text{interaction}} \tag{1}$$

with the Hamiltonian of the subsystem without losses $\hat{H}_0$. The evolution of such an open system can be analyzed by, e.g. the Lindblad formalism [18] which evaluates the evolution of the system's full density matrix subjected to the interaction with the reservoir. Hence, the numerical effort in solving the evolution equations grows strongly nonlinearly with system size. In a lattice with M sites the overhead is $\mathcal{O}(M^4)$, rendering even moderate system sizes computationally intractable. In our work, we follow a different approach, which scales more favorably.

We consider a lattice consisting of M coupled sites with an arbitrary transverse loss profile. The non-Hermitian Hamiltonian of the dissipative system (lattice with no back coupling from the reservoir) with the creation (annihilation) operator $\hat{a}_q^\dagger$ ($\hat{a}_q$) for mode q, is defined as $\hat{H}_{\text{system}} = \hat{H}_0 + \mathrm{i}\hat{A}_{\text{loss}}$ with the Hermitian Hamiltonian of the loss-free system

$$\hat{H}_0 = \hbar \sum_{m=1}^{M} \left\{ \beta_m \, \hat{a}_m^\dagger \, \hat{a}_m + \sum_{j=1}^{M} C_{j,m} \, \hat{a}_j^\dagger \, \hat{a}_m \right\} \tag{2}$$

and the Hermitian Operator describing the loss

$$\hat{A}_{\text{loss}} = \hbar \sum_{m=1}^{M} \frac{\gamma_m}{2} \, \hat{a}_m^\dagger \, \hat{a}_m. \tag{3}$$

Here β_m are the propagation constants in each lattice site and energy eigenvalues of $\hat{H}_0$, respectively, γ_m are the loss coefficients and $C_{j,m}$ are the coupling coefficients between sites j and m that form a matrix $\hat{C}$. Analogously to [19, 20] for weak coupling between system and reservoir, one can derive the quantum Langevin equation as the corresponding equation of motion (see appendix A)

$$\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar} \left[\hat{a}_q^\dagger(z), \, \hat{H}_{\text{system}} \right] + \hat{f}_q^\dagger(z) = \mathrm{i} \sum_{j=1}^{M} \tilde{C}_{q,j} \, \hat{a}_j^\dagger(z) + \hat{f}_q^\dagger(z) \tag{4}$$

with $\tilde{C}_{q,j} = (\beta_j + \mathrm{i}\frac{\gamma_j}{2})\delta_{j,q} + C_{q,j}$, z denoting the evolution coordinate of the system and the noise operators $\hat{f}_q^\dagger$ which ensure the commutator (anti-commutator) relations $[\hat{a}_j, \, \hat{a}_k^\dagger] = \delta_{j,k}$ ($\{\hat{a}_j, \, \hat{a}_k^\dagger\} = \delta_{j,k}$) (with the Kronecker delta $\delta_{j,k}$) for bosons (fermions). Formal integration yields

$$\hat{a}_q^\dagger(z) = \sum_{j=1}^{M} U_{q,j}(z) \, \hat{a}_j^\dagger(0) + \int_0^z \mathrm{d}\zeta \sum_{j=1}^{M} U_{q,j}(z - \zeta) \, \hat{f}_q^\dagger(\zeta), \tag{5}$$

where in analogy to [1, 21] $U_{q,j}(z) = (\mathrm{e}^{\mathrm{i}\hat{\tilde{C}}z})_{q,j}$ are the matrix elements of the linear z-propagation operator $\hat{U}$. The average occupation number $n_q(z)$ in mode q is

$$
\begin{aligned}
n_q(z) &= \langle \hat{a}_q^\dagger(z)\, \hat{a}_q(z) \rangle_{\psi_0} \\
&= \sum_{j,k=1}^{M} U_{q,j}(z) U_{q,k}^*(z) \langle \hat{a}_j^\dagger(0)\, \hat{a}_k(0) \rangle \\
&+ \underbrace{\sum_{j=1}^{M} U_{q,j}(z) \int_0^z \mathrm{d}\zeta \sum_{k=1}^{M} U_{q,k}^*(z-\zeta) \langle \hat{a}_j^\dagger(0)\, \hat{f}_q(\zeta) \rangle}_{=:n_q^{(1)}} \\
&+ \underbrace{\sum_{k=1}^{M} U_{q,k}^*(z) \int_0^z \mathrm{d}\zeta \sum_{j=1}^{M} U_{q,j}(z-\zeta) \langle \hat{f}_q^\dagger(\zeta)\, \hat{a}_k(0) \rangle}_{=:n_q^{(2)}} \\
&+ \underbrace{\int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \sum_{j,k=1}^{M} U_{q,j}(z-\zeta) U_{q,k}^*(z-\zeta') \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_q(\zeta') \rangle}_{=:n_q^{(3)}} .
\end{aligned}
\tag{6}
$$

It is typically assumed that the noise operators are Markovian (no memory effect) and the average value of the noise is equal to zero. This yields $n_q^{(1)} = n_q^{(2)} = 0$. Following [19], one obtains $n_q^{(3)} \sim n_{\mathrm{R}}$ with the particle density of the reservoir $n_{\mathrm{R}} = \left(\exp\left(\frac{\hbar\omega}{kT} \right) - 1 \right)^{-1}$ in equilibrium at frequency ω and temperature T. In the zero temperature approximation (ZTA) this contribution vanishes[5]. Therefore, we find the remarkable result that $n_q(z) = \sum_{j,k=1}^{M} U_{q,j}(z) U_{q,k}^*(z) \langle \hat{a}_j^\dagger(0)\, \hat{a}_k(0) \rangle$ is independent of noise because $\hat{a}_q^\dagger(0)$ is not related to the noise operators. Thus, a numerical treatment of the average occupation number of any excitation—bosonic, fermionic and irrespective of the number of launched particles—in a lattice with loss can be performed by neglecting the noise operators in (4), i.e. the standard Heisenberg equation for the non-Hermitian Hamiltonian $\hat{H}_{\mathrm{system}}$:

$$
\frac{\mathrm{d}\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar} \left[\hat{a}_q^\dagger(z),\ \hat{H}_{\mathrm{system}} \right].
\tag{7}
$$

When only two indistinguishable particles evolve in the lattice, one can analyze their mutual particle number correlation $\Gamma_{q,r}(z)$, which characterizes the probability to find one particle at lattice site q and the other one at site r. We find (see appendix B)

$$
\Gamma_{q,r}(z) = \langle \hat{a}_q^\dagger(z)\, \hat{a}_r^\dagger(z)\, \hat{a}_r(z)\, \hat{a}_q(z) \rangle = \sum_{j,k,l,p=1}^{M} U_{q,j} U_{r,k} U_{r,l}^* U_{q,p}^* \langle \hat{a}_j^\dagger(0)\, \hat{a}_k^\dagger(0)\, \hat{a}_l(0)\, \hat{a}_p(0) \rangle,
\tag{8}
$$

which is also independent of the noise term and only determined by the z-propagation operator $\hat{U}$, i.e. by the coupling dynamics. Hence, both the evolution of the average particle number and

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[5] Note, that exactly the same approximations are made in order to derive the Lindblad formalism.

the particle number correlation can be computed using the simple equation (7). Note that our approach reduces the overhead to $\mathcal{O}(M^2)$, as compared to $\mathcal{O}(M^4)$ for the Lindblad method, enabling the numerical analysis of much larger systems without any additional assumptions. Furthermore, we emphasize that this concept is applicable to any non-Hermitian lattice with arbitrary loss profile, irrespective of its physical manifestation, e.g., spin chains, electrons in crystalline solids, Bose- or Fermi–Hubbard systems, molecular chains or photonic lattices.

3. Numerical investigations

Let us now consider two identical bosons (b)/fermions (f) being launched into the lattice at sites m and n ($m \neq n$). These separable input states $|\psi^{(b/f)}\rangle = \hat{a}_m^\dagger(0)\,\hat{a}_n^\dagger(0)\,|0\rangle$ (with the vacuum state $|0\rangle$) yield the particle number correlations [1, 4]

$$\Gamma_{q,r}^{(b/f)}(z) = \left| U_{q,m}U_{r,n} \pm U_{q,n}U_{r,m} \right|^2,$$ (9)

where '+' corresponds to bosons and '−' to fermions. In contrast, injecting two bosons simultaneously into either site m or n results in a symmetric (s) or anti-symmetric (a) path-entangled input state $|\psi^{(s/a)}\rangle = \frac{1}{2}\left((\hat{a}_m^\dagger)^2 \pm (\hat{a}_n^\dagger)^2 \right)|0\rangle$ [3, 21]

$$\Gamma_{q,r}^{(s/a)}(z) = \left| U_{q,m}U_{r,m} \pm U_{q,n}U_{r,n} \right|^2.$$ (10)

For all considered input states one finds $n_q(z) = \left| U_{q,m} \right|^2 + \left| U_{q,n} \right|^2$.

To answer the open question of a true quantum evolution in loss-modulated quasi-$\mathcal{PT}$-symmetric systems, we consider a lattice where only even numbered sites exhibit loss (see figure 1(a)). Furthermore, the coupling shall be restricted to adjacent sites such that $C_{j,m} = C\,\delta_{j,m\pm1}$. In order to investigate the impact of such a loss structure on the mutual correlation of bosons, we consider an array of evanescently coupled optical waveguides [22] with identical propagation constants $\beta_m = \beta$. Thus, the lattice can be described in a frame co-propagating with β so that rapid phase oscillations can be separated from the transverse dynamics ($\hat{\tilde{a}}_q^\dagger = \hat{a}_q^\dagger\,\mathrm{e}^{-\mathrm{i}\beta z}$). Fermions, on the other hand, can be investigated directly in cold atoms trapped in optical lattices [23], where the spatial coordinate z is replaced by time, or simulated in waveguides lattices [24, 25]. We performed extensive numerical simulations for an array of $M = 50$ sites, a length of 15 cm, a coupling constant $C = 1\,\mathrm{cm}^{-1}$ and alternating loss with $\gamma_q = 0\,\mathrm{cm}^{-1}$ for odd and $\gamma_q = 4\,\mathrm{cm}^{-1}$ for even q. In an envisioned experiment, this can be realized by a controlled variation of the fabrication parameters of the waveguides [26]. We consider three different scenarios: (i) two particles are injected into two adjacent channels $m = 25$ (lossless) and $n = 26$ (lossy) (figure 1(b)); (ii) two particles are injected into next-nearest sites $m = 24$ and $n = 26$ (both lossy) (figure 1(c)) and (iii) lossless next-nearest neighbors $m = 25$ and $n = 27$ (figure 1(d)). All other possible input states result in similar dynamics as long as the particles do not reach the boundary, depending on whether an even or an odd number of lattice sites is in between the input channels. A striking feature of these quasi-$\mathcal{PT}$-symmetric lattices is that they give rise to a diffusive wave evolution despite being invariant along z. This means the variance of the wave packet spreads linearly with the propagation which can only be achieved by longitudinal disorder, i. e., propagation constants randomly varying along z (corresponding to a temporal fluctuation in the quantum mechanical context) [27] in the Hermitian system.

Figure 2 shows the correlation functions $\Gamma_{q,r}^{(b/f)}$ for separable bosonic/fermionic particles as well as for the symmetric/anti-symmetric path-entangled bosons $\Gamma_{q,r}^{(s/a)}$ after a quantum
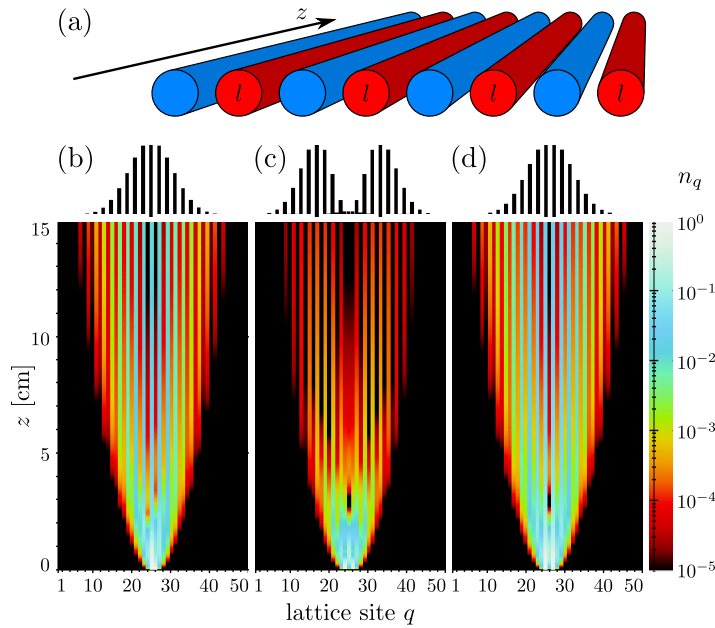
Figure 1. (a) Arrangement of alternating loss (red) and loss-free (blue) lattice sites. (b)–(d) Output distribution and evolution of the average particle number n_q for two adjacent inputs (b) and next neighbor injection in lossy (c) or loss-free (d) channels for coupling constant of $C = 1\,\text{cm}^{-1}$ and loss coefficient $\gamma_q = 4\,\text{cm}^{-1}$ for even q. The output distributions are normalized to their maximum value respectively.

random walk with an evolution distance of $z = 15\,\text{cm}$, for the three input conditions described above. As a first intuitive result we mention that in all observed cases the average particle number distribution n_q almost vanishes in the lossy channels (see figure 1) and the particle number correlation $\Gamma_{q,r}$ approaches significant values only in the loss-free sites (see figure 2). Comparing these results with the loss-free case ($\gamma_q = 0,\ \forall\,q$) one can clearly observe circumstances where the shapes of the correlations are changed drastically due to loss whereas being basically preserved in others.

The correlation function of bosons launched into adjacent waveguides (figure 2(a)) maintains the characteristic bimodal shape of the Hermitian loss-free case [1], with the peaks being slightly broadened and closer to the excited sites due to the diffusive propagation. In contrast, for next-nearest neighbor excitation of lossy channels the bosons are no longer correlated and thus evolve as independent but coherent wave packets through the structure as evident from the equal height of the four peaks (figure 2(b)). The situation changes completely when both bosons are launched into loss-free next-nearest channels. In this case, the non-classical correlation of a coherent random walker is destroyed and the classical broad Gaussian-shaped correlation of two incoherent diffusive particles is formed around the excitation center (figure 2(c)).

For fermions, we find that launching two separable particles into adjacent lattice sites again preserves the main features of the anti-bunching correlation pattern of the Hermitian system, (see figure 2(d)). However, fermions injected in next-nearest channels show a behavior that is very different from that in the Hermitian system, where the correlation pattern shows a
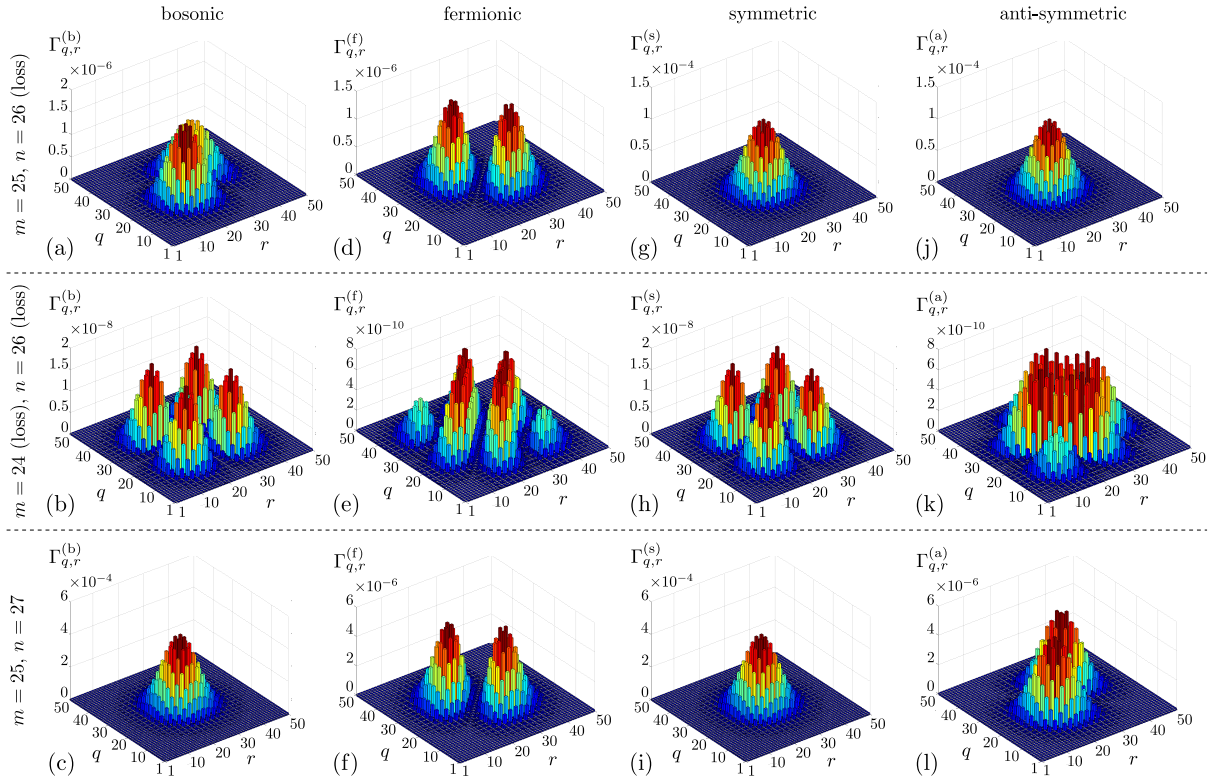
Figure 2. Correlation patterns for a separable bosonic input state launched in (a) two neighboring lattice sites, (b) two next-nearest lossy channels and (c) two next-nearest loss-free sites. (d)–(l) Correlation maps with the same input configurations for separable fermions (d)–(f), an entangled symmetric (g)–(i) and anti-symmetric (j)–(l) input state.

square shape indicating that both fermions will travel with high probability far from the excited channels and will always be found in different guides (due to the Pauli exclusion principle). In contrast, in our system, the correlation function $\Gamma_{q,r}^{(f)}$ changes significantly. When both input sites exhibit loss, we find a complex correlation pattern that shows signatures of anti-bunching as well as distinct correlation ridges parallel to the bunching diagonal (figure 2(e)). Note that the row sum of the correlation function $\sum_r \Gamma_{q,r}$ does not equal the average particle number n_q (see figure 3) as in a Hermitian system [1, 2]. While the former quantity accounts for probabilities of two particle events, the latter also includes scenarios where one particle is lost.

Nevertheless, also in the quasi-$\mathcal{PT}$-symmetric system the fermions always occupy different sites. When the input channels are loss-free, a distinct anti-bunching behavior of the correlation function is observed (figure 2(f)), very similar to the case where two neighboring modes are excited. These transitions strikingly demonstrate how the loss profile affects the correlation patterns of the evolving particles.

In the case of the symmetric and anti-symmetric input states one finds similarly a strong dependence of the correlations on the input configuration. For adjacent site excitation $\Gamma_{q,r}^{(s/a)}$ shows how the presence of losses can lead to full decoherence as heralded from the single Gaussian peak in figures 2(g)–(j). On the other hand for next-nearest neighbor excitation the particle number correlations explicitly show the random walkers' behavior of indistinguishable
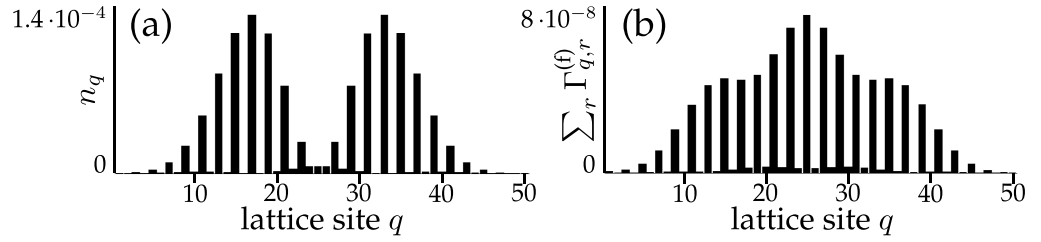
Figure 3. (a) The average particle number and (b) the row sum of the particle number correlation at $z = 15$ cm for injection of fermions in next-nearest lossy sites.

bosons (figure 2(l)) as well as the behavior of distinguishable but coherent walkers (figure 2(h)) and fully classical incoherent walkers (figure 2(i)) and additional an intricate shape exhibiting bunching as well as anti-bunching properties (figure 2(k)).

4. Conclusion

Summarized, we have introduced a new method to efficiently investigate the quantum random walks of identical particles in lattice structures with arbitrary loss profile. By virtue of this method, we analyzed the correlations of separable bosonic and fermionic input states as well as entangled symmetric and anti-symmetric input states in a quasi-$\mathcal{PT}$-symmetric lattice structure. We found that, depending on the position of the excited sites, the quantum correlations of a corresponding homogeneous loss-free system may either prevail, alter significantly or get destroyed forming the correlation of distinguishable or even classical random walkers.

Acknowledgments

Appendix A. Quantum Langevin equation for coupled systems

In the following, we deduce the quantum Langevin equation for our systems from a general approach by principally following similar considerations as in [19]. Regard the total Hamiltonian of the overall configuration under the assumption of weak coupling between reservoir and loss-free system with strengths $g_{m,n}$

$$
\begin{aligned}
\hat{H}_{\text{total}} =&\ \hat{H}_0 + \hat{H}_{\text{reservoir}} + \hat{H}_{\text{interaction}} \\
=&\ \hbar \sum_{m=1}^{M} \left\{ \beta_m\, \hat{a}_m^\dagger\, \hat{a}_m + \sum_{j=1}^{M} C_{j,m}\, \hat{a}_j^\dagger\, \hat{a}_m \right\} + \hbar \sum_n \bar{\beta}_n\, \hat{b}_n^\dagger\, \hat{b}_n \\
&+ \hbar \sum_{m,n} \left\{ g_{m,n}\, \hat{a}_m^\dagger\, \hat{b}_n + g_{m,n}^*\, \hat{a}_m\, \hat{b}_n^\dagger \right\},
\end{aligned}
\tag{A.1}
$$

where $\bar{\beta}_n$ denotes the propagation constants of the reservoir modes and β_m and $C_{j,m}$ are defined in the main text. Thus, we find the Heisenberg equations for the system field operators $\hat{a}_q^\dagger$ and the reservoir field operators $\hat{b}_r^\dagger$ as coupled differential equations

$$\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}\left[\hat{a}_q^\dagger(z),\ \hat{H}_{\text{total}}\right] = -\frac{\mathrm{i}}{\hbar}\left[\hat{a}_q^\dagger(z),\ \hat{H}_0\right] + \mathrm{i}\sum_r g_{q,r}^*\,\hat{b}_r^\dagger, \tag{A.2}$$

$$\frac{\mathrm{d}\,\hat{b}_r^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}\left[\hat{b}_r^\dagger(z),\ \hat{H}_{\text{total}}\right] = \mathrm{i}\bar{\beta}_r\,\hat{b}_r^\dagger(z) + \mathrm{i}\sum_q g_{q,r}\,\hat{a}_q^\dagger(z). \tag{A.3}$$

Integrating equation (A.3) and plugging into equation (A.2) yields

$$\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}\left[\hat{a}_q^\dagger(z),\ \hat{H}_0\right] + \mathrm{i}\sum_r g_{q,r}^*\,\hat{b}_r^\dagger(0)\,\mathrm{e}^{\mathrm{i}\bar{\beta}_r z} - \int_0^z \mathrm{d}\zeta\sum_{q,r}|g_{q,r}|^2\,\mathrm{e}^{\mathrm{i}\bar{\beta}_r(z-\zeta)}\,\hat{a}_q^\dagger(\zeta), \tag{A.4}$$

where the second term on the rhs acts as a noise operator $\hat{f}_q^\dagger = \mathrm{i}\sum_r g_{q,r}^*\,\hat{b}_r^\dagger(0)\,\mathrm{e}^{\mathrm{i}\bar{\beta}_r z}$. For the last term on the rhs we replace one summation by an integral $\sum_r \to \frac{V}{(2\pi)^3}\int \mathrm{d}^3\bar{k}_r \to \int \mathrm{d}\bar{\beta}_r D(\bar{\beta}_r)$ with the spectral density of states $D(\beta)$

$$\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}\left[\hat{a}_q^\dagger(z),\ \hat{H}_0\right] - \sum_q\int_0^z \mathrm{d}\zeta\int \mathrm{d}\bar{\beta}_r D(\bar{\beta}_r)|g_{q,r}(\bar{\beta}_r)|^2\,\mathrm{e}^{\mathrm{i}(\bar{\beta}_r-\beta_q)(z-\zeta)}$$

$$\times\,\hat{a}_q^\dagger(\zeta)\,\mathrm{e}^{\mathrm{i}\beta_q(z-\zeta)} + \hat{f}_q^\dagger(z)$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{a}_q^\dagger(z),\ \hat{H}_0\right] - \sum_q\int_0^z \mathrm{d}\zeta\,D(\beta_q)|g_{q,r}(\beta_q)|^2\pi\,\delta(z-\zeta)\,\hat{a}_q^\dagger(\zeta)\,\mathrm{e}^{\mathrm{i}\beta_q(z-\zeta)}$$

$$+ \hat{f}_q^\dagger(z)$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{a}_q^\dagger(z),\ \hat{H}_0\right] - \sum_q\pi\,D(\beta_q)|g_{q,r}(\beta_q)|^2\,\hat{a}_q^\dagger(z) + \hat{f}_q^\dagger(z)$$

$$= -\frac{\mathrm{i}}{\hbar}\left[\hat{a}_q^\dagger(z),\ \hat{H}_0\right] - \sum_q\frac{\gamma_q}{2}\,\hat{a}_q^\dagger(z) + \hat{f}_q^\dagger(z). \tag{A.5}$$

One obtains with decay rates $\gamma_q = 2\pi D(\beta_q)|g_{q,r}(\beta_q)|^2$ from Fermi's golden rule and the introduction of $\hat{H}_{\text{system}} = \hat{H}_0 + \mathrm{i}\hat{A}_{\text{loss}} = \hat{H}_0 + \mathrm{i}\hbar\sum_{m=1}^M \frac{\gamma_m}{2}\hat{a}_m^\dagger\hat{a}_m$ the quantum Langevin equation

$$\frac{\mathrm{d}\,\hat{a}_q^\dagger(z)}{\mathrm{d}z} = -\frac{\mathrm{i}}{\hbar}\left[\hat{a}_q^\dagger(z),\ \hat{H}_{\text{system}}\right] + \hat{f}_q^\dagger(z) = \mathrm{i}\sum_{j=1}^M \tilde{C}_{q,j}\,\hat{a}_j^\dagger(z) + \hat{f}_q^\dagger(z), \tag{A.6}$$

where $\tilde{C}_{q,j} = (\beta_j + \mathrm{i}\frac{\gamma_j}{2})\delta_{j,q} + C_{q,j}$. With the linear propagation operator $U_{q,j}(z) = (\mathrm{e}^{\mathrm{i}\tilde{C}z})_{q,j}$ the formal solution of (4) is

$$\hat{a}_q^\dagger(z) = \sum_{j=1}^M U_{q,j}(z)\,\hat{a}_j^\dagger(0) + \int_0^z \mathrm{d}\zeta\sum_{j=1}^M U_{q,j}(z-\zeta)\,\hat{f}_q^\dagger(\zeta). \tag{A.7}$$

Appendix B. Calculation of the particle number correlation

Here we present an extensive derivation of the particle number correlation and prove that this quantity does not depend on the noise operators. Calculating the particle number correlations by using (5)

$$\Gamma_{q,r}(z) = \langle\, \hat{a}_q^\dagger(z)\, \hat{a}_r^\dagger(z)\, \hat{a}_r(z)\, \hat{a}_q(z)\,\rangle = \sum_{\alpha=1}^{16} \Gamma_{q,r}^{(\alpha)}(z) \tag{B.1}$$

yields 16 terms $\Gamma_{q,r}^{(\alpha)}$. They can be categorized in how often a noise operator appears. One finds contributions without noise operator ($\Gamma_{q,r}^{(1)}$), with one ($\Gamma_{q,r}^{(2\ldots5)}$), two ($\Gamma_{q,r}^{(6\ldots11)}$), three ($\Gamma_{q,r}^{(12\ldots15)}$) and four ($\Gamma_{q,r}^{(16)}$) noise operators. In the following we will show exemplarily the contribution of each group by an extended calculation for one member. Taking into account that the noise operators are Markovian and the average value of the noise equals zero as well as the ZTA for the reservoir, we get

$\alpha = 1$:

$$\Gamma_{q,r}^{(1)} = \sum_{j,k,l,p=1}^{M} U_{q,j} U_{r,k} U_{r,l}^* U_{q,p}^* \langle\, \hat{a}_j^\dagger(0)\, \hat{a}_k^\dagger(0)\, \hat{a}_l(0)\, \hat{a}_p(0)\,\rangle, \tag{B.2}$$

$\alpha = 2, \ldots, 5$:

$$\Gamma_{q,r}^{(2)} = \sum_{k,l,p=1}^{M} U_{r,k} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \sum_{j=1}^{M} U_{q,j}(z-\zeta)\, \langle\, \hat{f}_q^\dagger(\zeta)\, \hat{a}_k^\dagger(0)\, \hat{a}_l(0)\, \hat{a}_p(0)\,\rangle$$

$$= \sum_{k,l,p=1}^{M} U_{r,k} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \sum_{j=1}^{M} U_{q,j}(z-\zeta)\, \underbrace{\langle\, \hat{f}_q^\dagger(\zeta)\,\rangle}_{=\,0}\, \langle\, \hat{a}_k^\dagger(0)\, \hat{a}_l(0)\, \hat{a}_p(0)\,\rangle$$

$$= 0, \tag{B.3}$$

$\alpha = 6, \ldots, 11$:

$$\Gamma_{q,r}^{(6)} = \sum_{l,p=1}^{M} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \sum_{j,k=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta')\, \langle\, \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta')\, \hat{a}_l(0)\, \hat{a}_p(0)\,\rangle$$

$$= \sum_{l,p=1}^{M} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \zeta\,\mathrm{d}\zeta' \sum_{j,k=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta')\, \langle\, \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta')\,\rangle \langle\, \hat{a}_l(0)\, \hat{a}_p(0)\,\rangle$$

$$= \sum_{l,p=1}^{M} U_{r,l}^* U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \sum_{j,k=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta')$$

$$\times \sum_{\iota,\kappa} g_{q,\iota}^* g_{r,\kappa}^* \underbrace{\langle\, \hat{b}_\iota^\dagger(0)\, \hat{b}_\kappa^\dagger(0)\,\rangle}_{=\,0\ (\mathrm{ZTA})}\, \mathrm{e}^{\mathrm{i}\bar{\beta}_\iota\zeta}\, \mathrm{e}^{\mathrm{i}\bar{\beta}_\kappa\zeta'}\, \langle\, \hat{a}_l(0)\, \hat{a}_p(0)\,\rangle$$

$$= 0. \tag{B.4}$$

Obviously $\langle\, \hat{b}_\iota^\dagger(0)\, \hat{b}_\kappa^\dagger(0)\,\rangle = 0$. In this group one can get expressions like $\langle\, \hat{f}_q^\dagger(\zeta)\, \hat{f}_r(\zeta')\,\rangle$ which are consequently direct proportional to the average particle density in the reservoir n_R which equals zero in the ZTA as mentioned in the main text.

$\alpha = 12, \ldots, 15$:

$$\Gamma_{q,r}^{(12)} = \sum_{p=1}^{M} U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \sum_{j,k,l=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'')$$

$$\times \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta')\, \hat{f}_r(\zeta'')\, \hat{a}_p(0) \rangle$$

$$= \sum_{p=1}^{M} U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \sum_{j,k,l=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'')$$

$$\times \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta')\, \hat{f}_r(\zeta'') \rangle \langle \hat{a}_p(0) \rangle$$

$$= \sum_{p=1}^{M} U_{q,p}^* \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \sum_{j,k,l=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'')$$

$$\times \sum_{\iota,\kappa,\lambda} g_{q,\iota}^* g_{r,\kappa}^* g_{r,\lambda} \underbrace{\langle \hat{b}_\iota^\dagger(0)\, \hat{b}_\kappa^\dagger(0)\, \hat{b}_\lambda(0) \rangle}_{=\,0}\, \mathrm{e}^{\mathrm{i}\bar{\beta}_\iota \zeta}\, \mathrm{e}^{\mathrm{i}\bar{\beta}_\kappa \zeta'}\, \mathrm{e}^{-\mathrm{i}\bar{\beta}_\lambda \zeta''}\, \langle \hat{a}_p(0) \rangle$$

$$= 0, \tag{B.5}$$

$\alpha = 16$:

$$\Gamma_{q,r}^{(16)} = \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \int_0^z \mathrm{d}\zeta''' \sum_{j,k,l,p=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'') U_{q,p}^*(z-\zeta''')$$

$$\times \langle \hat{f}_q^\dagger(\zeta)\, \hat{f}_r^\dagger(\zeta')\, \hat{f}_r(\zeta'')\, \hat{f}_q(\zeta''') \rangle$$

$$= \int_0^z \mathrm{d}\zeta \int_0^z \mathrm{d}\zeta' \int_0^z \mathrm{d}\zeta'' \int_0^z \mathrm{d}\zeta''' \sum_{j,k,l,p=1}^{M} U_{q,j}(z-\zeta) U_{r,k}(z-\zeta') U_{r,l}^*(z-\zeta'') U_{q,p}^*(z-\zeta''')$$

$$\times \sum_{\iota,\kappa,\lambda,\pi} g_{q,\iota}^* g_{r,\kappa}^* g_{r,\lambda} g_{q,\pi} \underbrace{\langle \hat{b}_\iota^\dagger(0)\, \hat{b}_\kappa^\dagger(0)\, \hat{b}_\lambda(0)\, \hat{b}_\pi(0) \rangle}_{=\,0\ (\mathrm{ZTA})}\, \mathrm{e}^{\mathrm{i}\bar{\beta}_\iota \zeta}\, \mathrm{e}^{\mathrm{i}\bar{\beta}_\kappa \zeta'}\, \mathrm{e}^{-\mathrm{i}\bar{\beta}_\lambda \zeta''}\, \mathrm{e}^{-\mathrm{i}\bar{\beta}_\pi \zeta'''}$$

$$= 0. \tag{B.6}$$

Note, $\langle \hat{b}_\iota^\dagger(0)\, \hat{b}_\kappa^\dagger(0)\, \hat{b}_\lambda(0)\, \hat{b}_\pi(0) \rangle = 0$ because in the ZTA the average particle number for each reservoir mode vanishes.

As all terms with $\alpha > 1$ do not contribute it holds for the particle number correlation

$$\Gamma_{q,r}(z) = \sum_{j,k,l,p=1}^{M} U_{q,j} U_{r,k} U_{r,l}^* U_{q,p}^* \langle \hat{a}_j^\dagger(0)\, \hat{a}_k^\dagger(0)\, \hat{a}_l(0)\, \hat{a}_p(0) \rangle, \tag{B.7}$$

which is independent of any noise operators.

References

[1] Bromberg Y, Lahini Y, Morandotti R and Silberberg Y 2009 Quantum and classical correlations in waveguide lattices *Phys. Rev. Lett.* **102** 253904

[2] Peruzzo A *et al* 2010 Quantum walks of correlated photons *Science* **329** 1500–3

[3] Bromberg Y, Lahini Y and Silberberg Y 2010 Bloch oscillations of path-entangled photons *Phys. Rev. Lett.* **105** 263604

[4] Lahini Y, Bromberg Y, Christodoulides D N and Silberberg Y 2010 Quantum correlations in two-particle anderson localization *Phys. Rev. Lett.* **105** 163905

[5] Keil R, Perez-Leija A, Dreisow F, Heinrich M, Moya-Cessa H, Nolte S, Christodoulides D N and Szameit A 2011 Classical analogue of displaced Fock states and quantum correlations in Glauber–Fock photonic lattices *Phys. Rev. Lett.* **107** 103601

[6] Keil R, Szameit A, Dreisow F, Heinrich M, Nolte S and Tuennermann A 2010 Photon correlations in two-dimensional waveguide arrays and their classical estimate *Phys. Rev.* A **81** 023834

[7] Owens J O *et al* 2011 Two-photon quantum walks in an elliptical direct-write waveguide array *New J. Phys.* **13** 075003

[8] Bender C M and Boettcher S 1998 Real spectra in non-Hermitian Hamiltonians having PT symmetry *Phys. Rev. Lett.* **80** 5243–6

[9] El-Ganainy R, Makris K G, Christodoulides D N and Musslimani Z H 2007 Theory of coupled optical PT-symmetric structures *Opt. Lett.* **32** 2632–4

[10] Makris K G, El-Ganainy R, Christodoulides D N and Musslimani Z H 2008 Beam dynamics in PT symmetric optical lattices *Phys. Rev. Lett.* **100** 103904

[11] Musslimani Z H, Makris K G, El-Ganainy R and Christodoulides D N 2008 Optical solitons in PT periodic potentials *Phys. Rev. Lett.* **100** 030402

[12] Guo A, Salamo G J, Duchesne D, Morandotti R, Volatier-Ravat M, Aimez V, Siviloglou G A and Christodoulides D N 2009 Observation of PT-symmetry breaking in complex optical potentials *Phys. Rev. Lett.* **103** 093902

[13] Longhi S 2009 Bloch oscillations in complex crystals with PT symmetry *Phys. Rev. Lett.* **103** 123601

[14] Zheng M C, Christodoulides D N, Fleischmann R and Kottos T 2010 PT optical lattices and universality in beam dynamics *Phys. Rev.* A **82** 010103

[15] Dmitriev S V, Sukhorukov A A and Kivshars Y S 2010 Binary parity-time-symmetric nonlinear lattices with balanced gain and loss *Opt. Lett.* **35** 2976–8

[16] Ruter C E, Makris K G, El-Ganainy R, Christodoulides D N, Segev M and Kip D 2010 Observation of parity-time symmetry in optics *Nature Phys.* **6** 192–5

[17] Regensburger A, Bersch C, Miri M-A, Onishchukov G, Christodoulides D N and Peschel U 2012 Parity–time synthetic photonic lattices *Nature* **488** 167–71

[18] Strunz W 2002 *Coherent Evolution in Noisy Environments* (Berlin: Springer)

[19] Yamamoto Y and İmamoğlu A 1999 *Mesoscopic Quantum Optics* (New York: Wiley)

[20] Vogel W and Welsch D-G 2006 *Quantum Optics* (Weinheim: Wiley-VCH) chapter 5

[21] Keil R, Dreisow F, Heinrich M, Tuennermann A, Nolte S and Szameit A 2011 Classical characterization of biphoton correlation in waveguide lattices *Phys. Rev.* A **83** 013808

[22] Lederer F, Stegeman G I, Christodoulides D N, Assanto G, Segev M and Silberberg Y 2008 Discrete solitons in optics *Phys. Rep.* **463** 1–126

[23] Bloch I, Dalibard J and Zwerger W 2008 Many-body physics with ultracold gases *Rev. Mod. Phys.* **80** 885

[24] Matthews J C F, Poulios K, Meinecke J D A, Politi A, Peruzzo A, Ismail N, Wörhoff K, Thompson M G and O'Brien J L 2011 Simulating quantum statistics with entangled photons: a continuous transition from bosons to fermions arXiv:1106.1166v1 [quant-ph]

[25] Longhi S and Giuseppe Della Valle 2012 Anyons in one-dimensional lattices: a photonic realization *Opt. Lett.* **37** 2160–2

[26] Eichelkraut T *et al* 2013 Mobility transition in non-Hermitian lattices: ballistic transport versus diffusion, unpublished

[27] Amir A, Lahini Y and Perets H B 2009 Classical diffusion of a quantum particle in a noisy environment *Phys. Rev.* E **79** 050105