

Lorentz-force approach to the Casimir force

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Abstract. An approach to the problem of the Casimir force on magnetodielectric bodies is outlined, which is based on the calculation of the ground-state Lorentz force acting on the polarization and magnetization charges and currents that constitute the bodies within the framework of linear, macroscopic electrodynamics. As an application, planar structures are considered and a correct generalization of Casimir's original formula to the case where the two highly reflecting plates are embedded in a medium is given.

1. Introduction

Stimulated by the progress in the fabrication and operation of nanomechanical devices, a renewed interest in QED vacuum forces has emerged over the last years. It is common to distinguish between van der Waals (vdW), Casimir-Polder (CP) and Casimir forces, although they are intimately related to each other (see, e.g., reference [1] for an overview). As is well known, the vdW force between unpolarized ground-state atoms can be regarded as the force between the atomic dipole moments that are induced due to the interaction with the fluctuating electromagnetic vacuum. The term CP force is used to characterize the force on an unpolarized ground-state atom near a neutral, unpolarized macroscopic body. It can be regarded as the force with which the fluctuating body-assisted electromagnetic vacuum field acts on the atomic dipole moment induced by that field. From a microscopic point of view, the CP force is nothing but a manifestation of the vdW force on a macroscopic level. Another macroscopic manifestation of the vdW force is the Casimir force between neutral, unpolarized macroscopic bodies.

Within the framework of macroscopic QED, different approaches to the calculation of the Casimir force have been developed. A frequently used approach (close to Casimir's ideas) is the so-called mode summation approach, where one starts from some geometry-dependent electromagnetic vacuum energy

$$E(d) = \frac{1}{2} \sum_m \hbar \omega_m(d) - \frac{1}{2} \sum_m \hbar \omega_m(d \rightarrow \infty) \quad (1)$$

and regards it (after suitable regularization) as the potential of the Casimir force. As the eigenvalue problem used for determining the geometry-dependent mode frequencies $\omega_m(d)$ shows some nonstandard features when the field in dispersing and absorbing media needs to be considered, the resulting (non-normal) mode formalism tends to become somewhat heuristic. Another, more fundamental problem is that the physical content of the energy expression (1) is not very clear in such cases. A rather different and physically much more transparent approach

is based on the so-called Rytov-Lifshitz fluctuation electrodynamics, which was first used by Lifshitz to calculate the Casimir stress between two dispersing and absorbing dielectric half-spaces separated by empty space [2]. To find the force on one of the half-spaces, only the stress tensor in the free-space region between the half-spaces is required.

The question how the Casimir force between bodies should be calculated if the interspace between them is not empty arises quite naturally. No matter what approach to this problem is used, it is clear that inclusion in the calculation of the field energy or stress inside the medium between the bodies is necessarily required. Typical previous approaches (see, e.g., the references in reference [3]), irrespective of their philosophical and calculational details, effectively amount to a simple replacement of the vacuum permittivity with the medium permittivity, $\varepsilon_0 \mapsto \varepsilon_0 \varepsilon(\mathbf{r}, \omega)$. It is obvious that this is very questionable, even for non-absorbing media. In our approach outlined in the following, the difficulty is circumvented by consistently employing exact QED in linear, causal media and calculating the Casimir force as the ground-state expectation value of the Lorentz force that acts on the charges and currents attributed to magnetodielectric bodies, as they are given – within the framework of the macroscopic description of the bodies – in terms of polarization and magnetization fields. Needless to say that in mechanical equilibrium the Casimir force calculated in this way must be balanced by additional external or internal forces that are not included here (for a controversial point of view, see reference [4]).

2. Casimir force as ground-state Lorentz force

As is well known, the momentum balance for Maxwell's equations reads

$$\mathbf{f}_L = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} = \nabla \cdot \mathbb{T} - \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}), \quad (2)$$

where \mathbf{f}_L is the Lorentz force density, and \mathbb{T} is the stress tensor

$$\mathbb{T} = \varepsilon_0 \mathbf{E} \mathbf{E} + \mu_0^{-1} \mathbf{B} \mathbf{B} - \frac{1}{2} (\varepsilon_0 E^2 + \mu_0^{-1} B^2) \mathbf{1}. \quad (3)$$

Under stationary conditions the Lorentz force acting on the charges and currents inside some space region of volume V is thus

$$\mathbf{F}_L = \int_V d^3r \mathbf{f}_L = \int_{\partial V} d\mathbf{a} \cdot \mathbb{T}. \quad (4)$$

In particular, the classical (steady-state) radiation force acting on some macroscopic body for which

$$\rho = -\nabla \cdot \mathbf{P}, \quad \mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \quad (5)$$

(\mathbf{P} , polarization; \mathbf{M} , magnetization) may be calculated from equation (4) together with equation (2) or (3) if the constitutive relations $\mathbf{P} = \mathbf{P}[\mathbf{E}, \mathbf{B}]$, $\mathbf{M} = \mathbf{M}[\mathbf{E}, \mathbf{B}]$ are given and the corresponding solution of Maxwell's equations has been constructed and plugged into equation (2) or (3).

Here we focus on linearly and locally responding causal media (without any additional radiation sources). To be more specific, representing the fields in the form of

$$\mathbf{F}(\mathbf{r}, t) = \int_0^\infty d\omega \underline{\mathbf{F}}(\mathbf{r}, \omega) e^{-i\omega t} + \text{c.c.}, \quad (6)$$

where \mathbf{F} may be the electric field, the induction field or another field that enters equations (2)–(5), we consider the constitutive relations

$$\underline{\mathbf{P}}(\mathbf{r}, \omega) = \varepsilon_0 [\varepsilon(\mathbf{r}, \omega) - 1] \underline{\mathbf{E}}(\mathbf{r}, \omega) + \underline{\mathbf{P}}_N(\mathbf{r}, \omega), \quad (7)$$

$$\underline{\mathbf{M}}(\mathbf{r}, \omega) = \kappa_0[1 - \kappa(\mathbf{r}, \omega)]\underline{\mathbf{B}}(\mathbf{r}, \omega) + \underline{\mathbf{M}}_N(\mathbf{r}, \omega), \quad (8)$$

with $\underline{\mathbf{P}}_N(\mathbf{r}, \omega)$ and $\underline{\mathbf{M}}_N(\mathbf{r}, \omega)$ being the noise polarization and noise magnetization, respectively, which form the noise current as

$$\underline{\mathbf{j}}_N(\mathbf{r}, \omega) = -i\omega\underline{\mathbf{P}}_N(\mathbf{r}, \omega) + \nabla \times \underline{\mathbf{M}}_N(\mathbf{r}, \omega). \quad (9)$$

The unique solution of Maxwell's equations for the magnetodielectric bodies specified by equations (7) and (8) follows from

$$\underline{\mathbf{E}}(\mathbf{r}, \omega) = i\mu_0\omega \int d^3r' \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \cdot \underline{\mathbf{j}}_N(\mathbf{r}', \omega) \quad (10)$$

where $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$ is the (retarded) Green tensor. In particular, it can be shown that $\underline{\rho}(\mathbf{r}, \omega)$ and $\underline{\mathbf{j}}(\mathbf{r}, \omega)$ can be expressed in terms of $\underline{\mathbf{j}}_N(\mathbf{r}', \omega)$ as

$$\underline{\rho}(\mathbf{r}, \omega) = \frac{i\omega}{c^2} \nabla \cdot \int d^3r' \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \cdot \underline{\mathbf{j}}_N(\mathbf{r}', \omega) \quad (11)$$

and

$$\underline{\mathbf{j}}(\mathbf{r}, \omega) = \left(\nabla \times \nabla \times - \frac{\omega^2}{c^2} \right) \int d^3r' \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \cdot \underline{\mathbf{j}}_N(\mathbf{r}', \omega). \quad (12)$$

According to reference [5], the transition to QED may then be performed by regarding the classical fields as operator-valued ones, setting

$$\hat{\underline{\mathbf{P}}}_N(\mathbf{r}, \omega) = i[\hbar\varepsilon_0 \text{Im} \varepsilon(\mathbf{r}, \omega)/\pi]^{1/2} \hat{\mathbf{f}}_e(\mathbf{r}, \omega), \quad (13)$$

$$\hat{\underline{\mathbf{M}}}_N(\mathbf{r}, \omega) = [-\hbar\kappa_0 \text{Im} \kappa(\mathbf{r}, \omega)/\pi]^{1/2} \hat{\mathbf{f}}_m(\mathbf{r}, \omega), \quad (14)$$

and imposing the requirement that the dynamical variables $\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega)$ and $\hat{\mathbf{f}}_\lambda^\dagger(\mathbf{r}, \omega)$ ($\lambda = e, m$) obey bosonic commutation relations [and evolve as $\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega, t) = \hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega, t')e^{-i\omega(t-t')}$ in the Heisenberg picture]. The eigenvectors of the operators $\hat{f}_{\lambda k}^\dagger(\mathbf{r}, \omega)\hat{f}_{\lambda k}(\mathbf{r}, \omega)$ can be used to define the (Fock) state space of the combined system, with the ground state being defined by $\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega)|0\rangle = 0$. From the above it is near at hand that the ground-state expectation value of the Lorentz force acting on the (polarization and magnetization) charge and current densities $\hat{\rho}$ and $\hat{\mathbf{j}}$ that are attributed [according to equation (5)] to the macroscopic bodies can be regarded as being the Casimir force that the bodies are subject to. Hence the Casimir force on a body of volume V is

$$\mathbf{F} = \int_V d^3r \langle 0|\hat{\mathbf{f}}_L|0\rangle = \int_V d^3r \langle 0|\hat{\rho}\hat{\mathbf{E}} + \hat{\mathbf{j}} \times \hat{\mathbf{B}}|0\rangle = \int_{\partial V} \mathbf{da} \cdot \mathbf{T}(\mathbf{r}, \mathbf{r}), \quad (15)$$

where [3]

$$\mathbf{T}(\mathbf{r}, \mathbf{r}) = \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \left[\theta(\mathbf{r}, \mathbf{r}') - \frac{1}{2} \text{Tr} \theta(\mathbf{r}, \mathbf{r}') \right], \quad (16)$$

$$\theta(\mathbf{r}, \mathbf{r}') = -\frac{\hbar}{\pi} \int_0^\infty d\xi \left[\frac{\xi^2}{c^2} \mathbf{G}^{(S)}(\mathbf{r}, \mathbf{r}', i\xi) + \nabla \times \mathbf{G}^{(S)}(\mathbf{r}, \mathbf{r}', i\xi) \times \overleftarrow{\nabla}' \right], \quad (17)$$

with $\mathbf{G}^{(S)}(\mathbf{r}, \mathbf{r}', i\xi)$ being the scattering part of the Green tensor. Note that it is not required that the body is embedded in vacuum (the surrounding medium should, however, be homogeneous at least in the neighborhood of the body).

An independent confirmation of equations (15)–(17) can be given by studying the quantized (microscopic) electromagnetic field interacting with dielectric matter that is modeled by damped

harmonic oscillators. The coupled field-matter equations of motion can be solved exactly in this model. Considering a coarse-grained time scale on which the damping terms in the equations of motion of the medium oscillators can be characterized by a simple damping constant, a Drude–Lorentz-type permittivity is obtained for the medium. The calculation of the steady-state Lorentz force

$$\mathbf{F} = \lim_{t \rightarrow \infty} \int_V d^3r \langle \hat{\rho}(\mathbf{r}, t) \hat{\mathbf{E}}(\mathbf{r}, t) + \hat{\mathbf{j}}(\mathbf{r}, t) \times \hat{\mathbf{B}}(\mathbf{r}, t) \rangle \quad (18)$$

that acts on the (model) dielectric matter in a space region of volume V is then found to be in full agreement with the general formulas (15)–(17), see reference [3] for details.

3. Casimir force in planar structures

Let us apply the theory to planar structures of the type

$$\varepsilon(\mathbf{r}, \omega) = \begin{cases} \varepsilon_-(z, \omega), & z < 0, \\ \varepsilon_j(\omega), & 0 < z < d_j, \\ \varepsilon_+(z, \omega), & z > d_j, \end{cases} \quad \mu(\mathbf{r}, \omega) = \begin{cases} \mu_-(z, \omega), & z < 0, \\ \mu_j(\omega), & 0 < z < d_j, \\ \mu_+(z, \omega), & z > d_j. \end{cases} \quad (19)$$

From equation (16) together with equation (17), the Casimir stress within the interspace ($0 < z < d_j$) is then found in the form [3]

$$T_{zz}(\mathbf{r}, \mathbf{r}) = \frac{\hbar}{8\pi^2} \int_0^\infty d\xi \int_0^\infty dq q \frac{\mu_j(i\xi)}{i\beta_j(i\xi, q)} g_j(z, i\xi, q), \quad (20)$$

where

$$\begin{aligned} g_j(z, \omega, q) = & 2[\beta_j^2(1 + n_j^{-2}) - q^2(1 - n_j^{-2})] D_{js}^{-1} r_{j+}^s r_{j-}^s e^{2i\beta_j d_j} \\ & + 2[\beta_j^2(1 + n_j^{-2}) + q^2(1 - n_j^{-2})] D_{jp}^{-1} r_{j+}^p r_{j-}^p e^{2i\beta_j d_j} \\ & - (\beta_j^2 + q^2)(1 - n_j^{-2}) D_{js}^{-1} [r_{j-}^s e^{2i\beta_j z} + r_{j+}^s e^{2i\beta_j(d_j - z)}] \\ & + (\beta_j^2 + q^2)(1 - n_j^{-2}) D_{jp}^{-1} [r_{j-}^p e^{2i\beta_j z} + r_{j+}^p e^{2i\beta_j(d_j - z)}], \end{aligned} \quad (21)$$

with

$$n_j^2 = n_j^2(\omega) = \varepsilon_j(\omega) \mu_j(\omega), \quad (22)$$

$$\beta_j = \beta_j(\omega, q) = (\omega^2 n_j^2 / c^2 - q^2)^{1/2}, \quad (23)$$

and

$$D_{j\sigma} = D_{j\sigma}(\omega, q) = 1 - r_{j+}^\sigma r_{j-}^\sigma e^{2i\beta_j d_j}. \quad (24)$$

The $r_{j\pm}^\sigma = r_{j\pm}^\sigma(\omega, q)$ ($\sigma = s, p$) are generalized reflection coefficients (which reduce to the standard Fresnel coefficients for an interspace between two homogeneous half-spaces). As an important result, it is seen that the stress depends on the position z within the interspace unless the interspace is empty. Since the presence of a medium in the interspace means the existence of charges and currents [cf. equation (5)], this result is very natural. Note that in a number of papers and textbooks the Casimir stress in the interspace is considered to be position-independent even if the interspace is filled with a medium (see reference [3] for details).

To illustrate the results, let us calculate the Casimir force acting on a plate embedded in a planar cavity that is filled with a medium. In particular, restricting attention to the large-distance regime and considering the limiting case of an almost perfectly reflecting plate embedded in a cavity with almost perfectly reflecting mirrors, we find from equation (20) that the Casimir force per unit area on the embedded plate (whose thickness is irrelevant in that limit) is

$$F = \frac{\hbar c \pi^2}{240} \sqrt{\frac{\mu}{\varepsilon}} \left(\frac{2}{3} + \frac{1}{3\varepsilon\mu} \right) \left(\frac{1}{d_R^4} - \frac{1}{d_L^4} \right), \quad (25)$$

where ε and μ are the static (real) values of the permittivity and permeability of the cavity medium, respectively, and d_L and d_R are the distances from the embedded plate to the left and right cavity mirror, respectively. If one of the cavity mirrors is moved away to infinity (say $d_L \rightarrow \infty$), equation (25) reduces to ($d_R \mapsto d$)

$$F = \frac{\hbar c \pi^2}{240} \sqrt{\frac{\mu}{\varepsilon}} \left(\frac{2}{3} + \frac{1}{3\varepsilon\mu} \right) \frac{1}{d^4}. \quad (26)$$

Equation (26) extends Casimir's famous formula for the force between two perfectly reflecting plates in otherwise empty space to the case where the interspace between the plates is filled with a magnetodielectric medium. Note that equation (26) differs essentially from the formula [6] ($\mu = 1$)

$$F^{(M)} = \frac{\hbar c \pi^2}{240} \frac{1}{\sqrt{\varepsilon}} \frac{1}{d^4}, \quad (27)$$

which in our opinion takes the screening effect of the medium not properly into account.

4. Summary

Within the framework of exact QED in dispersing and absorbing linear media, we have calculated the Casimir force between (neutral, unpolarized) bodies as the ground-state expectation value of the Lorentz force that acts on the charges and currents attributed to the bodies via the fluctuating polarization and magnetization fields. The Lorentz-force approach has enabled us to correctly calculate the Casimir force acting on bodies embedded in media. The reason why previous work has not properly handled medium effects can be seen in the application of arguable energy and/or stress expressions for the (quantized) electromagnetic field in media.

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