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On the use of curvature invariants in numerical relativity

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Abstract. The recent breakthroughs in numerical relativity have made it possible to generate gravitational wave signals coming from binary systems. The waveforms are extracted using the Newman-Penrose formalism and in particular the Weyl scalar Ψ_4. In order for Ψ_4 to contain the physical information on the gravitational wave signal, the tetrad chosen for its calculation should converge to the Kinnersley tetrad in the limit of single black hole space-time.

In this work we present a compact expression for Ψ_4 that automatically enforces this condition; the result is a simple function of curvature invariants that can be easily calculated in a numerical code.

1. Introduction

Given the recent successes of numerical relativity and the ability of current codes [1, 2, 3] to simulate a binary black hole merger, the accuracy of generated waveforms has become of primary importance. This is the so called problem of wave extraction which consists of calculating specific quantities from the numerically evolved variables that are directly related to the gravitational wave degrees of freedom. To achieve this, most groups use the Newman-Penrose (NP) formalism and in particular the Weyl scalar Ψ_4.

The relevant quantities in the NP formalism are the mentioned Weyl scalars, given by the contraction of the Weyl tensor over a specific combination of the four null tetrad vectors, and the connection coefficients (spin coefficients) related to the covariant derivatives of the same tetrad vectors. All of these quantities are scalars, making the choice of the coordinate system irrelevant for their calculation; they are however dependent on the tetrad choice which constitutes the gauge freedom in this formalism.

It is well known that these variable, under certain assumptions, acquire a precise physical meaning. For example Ψ_0 and Ψ_4 are related to the ingoing and outgoing gravitational wave contribution, while Ψ_2 is related to the background contribution to the curvature. This explains the use of Ψ_4 for numerical simulations of Einstein's equations. However, a suitable tetrad choice is fundamental for the calculation of these quantities. The importance of a robust wave extraction technique has been underlined by some recent articles [4, 5, 6, 7]. Recent works [8, 9, 10] have identified in *transverse tetrads*, i.e. those tetrads satisfying the condition $\Psi_1 = \Psi_3 = 0$, a convenient candidate for wave extraction and in general for a better understanding of the equations governing the Newman-Penrose formalism.

The advantages of the choice $\Psi_1 = \Psi_3 = 0$ have already been shown in [8, 9, 10], namely one of these three frames naturally converges to the frame where $\Psi_0 = \Psi_4 = 0$ when the spacetime approaches Petrov type D. This property ensures that the values for the two scalars Ψ_0 and Ψ_4, in the linear regime, are at first order tetrad invariant, and directly associated with the gravitational wave signal [11].

To determine the tetrad completely, however, one also needs to fix the spin-boost degree of freedom. The Kinnersley tetrad [12] identifies the spin-boost parameter by imposing the condition on one spin coefficient $\epsilon = 0$.

In this paper we show that, choosing the tetrads where $\Psi_1 = \Psi_3 = 0$, it is possible to find a connection between the spin-boost parameter and the expressions for the spin coefficients. This will allow us to impose the condition $\epsilon = 0$ and find the correspondent value for the spin-boost parameter. Applying this spin boost parameter to the expression of Ψ_4 will give us a compact result which is function of curvature invariants.

2. General definitions

Weyl scalars are given by contraction of the Weyl tensor over a certain combination of four null vectors, two real (ℓ^μ and n^μ) and two complex conjugates (m^μ and $\bar{m}^\mu$), according to

$$\Psi_0 = -C_{abcd}\ell^a m^b \ell^c m^d, \tag{1a}$$
$$\Psi_1 = -C_{abcd}\ell^a n^b \ell^c m^d, \tag{1b}$$
$$\Psi_2 = -C_{abcd}\ell^a m^b \bar{m}^c n^d, \tag{1c}$$
$$\Psi_3 = -C_{abcd}\ell^a n^b \bar{m}^c n^d, \tag{1d}$$
$$\Psi_4 = -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d. \tag{1e}$$

These expressions are obviously tetrad dependent, however it is possible to construct two quantities which are no longer dependent on the tetrad choice. Such quantities are the curvature invariants I and J, and their expressions as functions of the Weyl scalars are given by $I = \Psi_4\Psi_0 - 4\Psi_1\Psi_3 + 3\Psi_2^2$ and $J = \Psi_4\left(\Psi_0\Psi_2 - \Psi_1^2\right) - \Psi_3\left(\Psi_3\Psi_0 - \Psi_1\Psi_2\right) + \Psi_2\left(\Psi_1\Psi_3 - \Psi_2^2\right)$.

In transverse frames (i.e. when $\Psi_1 = \Psi_3 = 0$) the non-vanishing Weyl scalars can be written directly as functions of curvature invariants, yielding

$$\Psi_0 = -\frac{i\mathcal{B}^{-2}}{2}\cdot\Psi_-, \tag{2a}$$
$$\Psi_2 = -\frac{1}{2\sqrt{3}}\cdot\Psi_+, \tag{2b}$$
$$\Psi_4 = -\frac{i\mathcal{B}^{2}}{2}\cdot\Psi_-, \tag{2c}$$

where

$$\Psi_\pm = I^{\frac{1}{2}}\left(e^{\frac{2\pi i k}{3}}\Theta \pm e^{-\frac{2\pi i k}{3}}\Theta^{-1}\right). \tag{3}$$

In these expressions $\Theta = \sqrt{3}PI^{-\frac{1}{2}}$, $P = \left[J + \sqrt{J^2 - (I/3)^3}\right]^{\frac{1}{3}}$ and k is an integer number assuming the values $\{0, 1, 2\}$ corresponding to the three different transverse frames. The spin-boost parameter is given by $\mathcal{B} = \left(\frac{\Psi_4}{\Psi_0}\right)^{\frac{1}{4}}$. The limit of type D corresponds to $\Theta \to 1$.

3. The connection between spin coefficients and spin-boost parameter

In this section, we will use the Ricci identities to understand how the spin coefficients ϵ, γ, α and β relate to the spin-boost parameter $\mathcal{B}$.

We assume to be in the Petrov type D limit, where $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and also, as a consequence of the Goldberg-Sachs theorem, the four spin coefficients λ, σ, ν and κ vanish. We begin with the following Ricci identity

$$D\tilde{\beta} - \delta\tilde{\epsilon} = \tilde{\epsilon}\left(\pi^* - \alpha^* - \beta\right) + \tilde{\beta}\left(\rho^* + \epsilon - \epsilon^*\right). \tag{4}$$

which is expressed in terms of the reduced spin coefficients $\tilde{\epsilon} = \epsilon + \frac{1}{2}D\ln\mathcal{B}$ and $\tilde{\beta} = \beta + \frac{1}{2}\delta\ln\mathcal{B}$.

Comparing Eq. (4) with the expression of the commutator $[D,\delta]$ (again assuming $\sigma = \kappa = 0$)

$$[D,\delta] = \left(\pi^* - \alpha^* - \beta\right)D + \left(\rho^* + \epsilon - \epsilon^*\right)\delta, \tag{5}$$

it is possible to see that the Ricci identity is consistent with having $\tilde{\epsilon} = D\mathcal{H}_1$ and $\tilde{\beta} = \delta\mathcal{H}_1$, where $\mathcal{H}_1$ is a function to be determined. Using the equivalent Ricci identity obtained after exchanging the tetrad vectors $\ell \leftrightarrow n$ and $m \leftrightarrow \bar{m}$ we obtain an equivalent result for the reduced spin coefficients $\tilde{\gamma} = \gamma + \frac{1}{2}\Delta\ln\mathcal{B}$ and $\tilde{\alpha} = \alpha + \frac{1}{2}\delta^*\ln\mathcal{B}$ and conclude that they also can be expressed as $\tilde{\gamma} = \Delta\mathcal{H}_2$ and $\tilde{\alpha} = \delta^*\mathcal{H}_2$, and from the properties of transformation of spin coefficients under exchange operation one must have $\mathcal{H}_1 = -\mathcal{H}_2 = \mathcal{H}$. The expression for $\mathcal{H}$ can be found using the following two other Ricci identities

$$\begin{aligned} D\tilde{\gamma} - \Delta\tilde{\epsilon} &= \tilde{\alpha}\left(\tau + \pi^*\right) + \tilde{\beta}\left(\tau^* + \pi\right) + \tau\pi && \text{(6a)}\\ &- \tilde{\gamma}\left(\epsilon + \epsilon^*\right) - \tilde{\epsilon}\left(\gamma + \gamma^*\right) + \Psi_2, \\ \delta\tilde{\alpha} - \delta^*\tilde{\beta} &= \mu\rho + \tilde{\alpha}\alpha^* + \tilde{\beta}\beta^* - 2\tilde{\alpha}\tilde{\beta} && \text{(6b)}\\ &+ \tilde{\gamma}\left(\rho - \rho^*\right) + \tilde{\epsilon}\left(\mu - \mu^*\right) - \Psi_2, \end{aligned}$$

which, together with the just found properties of these spin coefficients as directional derivatives of the function $\mathcal{H}$, lead to the following equation for $\mathcal{H}$:

$$\nabla^\mu\nabla_\mu\mathcal{H} + \nabla^\mu\ln\left(I^{\frac{1}{6}}\right)\nabla_\mu\left(2\mathcal{H} + \ln I^{\frac{1}{12}}\right) = -2\Psi_2, \tag{7}$$

where we have also used the fact that in the Petrov type D limit $\rho = D\ln I^{\frac{1}{6}}$, $\mu = -\Delta\ln I^{\frac{1}{6}}$, $\tau = \delta\ln I^{\frac{1}{6}}$ and $\pi = -\delta^*\ln I^{\frac{1}{6}}$. The solution for Eq. (7) in the case of Kerr space-time using Boyer-Lindquist coordinates, is given by

$$\mathcal{H} = \frac{1}{2}\ln\left(\Gamma^{\frac{1}{2}}I^{\frac{1}{6}}\sin\theta\right), \tag{8}$$

where $\Gamma = r^2 - 2Mr + a^2$, M and a being mass and spin parameter of the black hole. The spin coefficient ϵ is therefore given by

$$\epsilon = \frac{1}{2}D\ln\left(\Gamma^{\frac{1}{2}}I^{\frac{1}{6}}\mathcal{B}^{-1}\sin\theta\right). \tag{9}$$

Setting $\epsilon = 0$ leads to the following expression for the spin-boost parameter

$$\mathcal{B} = \mathcal{B}_0 f\left(\theta\right)I^{\frac{1}{6}}\Gamma^{\frac{1}{2}}\sin\theta, \tag{10}$$

where $\mathcal{B}_0$ is an integration constant and the function $f(\theta)$ can be found from the expression of the spin coefficient β and is given by $f\left(\theta\right) = \sin^{-1}\theta$. The final result for $\mathcal{B}$ is therefore $\mathcal{B} = \mathcal{B}_0 I^{\frac{1}{6}}\Gamma^{\frac{1}{2}}$.

When applied to the expression for the Weyl scalars given in Eq. (2), Eq. (10) yields

$$\begin{aligned}\Psi_0 &= \mathcal{B}_0^{-2} \cdot \Gamma^{-1} I^{\frac{1}{6}} \left(\Theta - \Theta^{-1}\right), &(11a)\\ \Psi_4 &= \mathcal{B}_0^{2} \cdot \Gamma I^{\frac{5}{6}} \left(\Theta - \Theta^{-1}\right), &(11b)\end{aligned}$$

for ingoing waves, and

$$\begin{aligned}\Psi_0 &= \mathcal{B}_0^{2} \cdot \Gamma I^{\frac{5}{6}} \left(\Theta - \Theta^{-1}\right), &(12a)\\ \Psi_4 &= \mathcal{B}_0^{-2} \cdot \Gamma^{-1} I^{\frac{1}{6}} \left(\Theta - \Theta^{-1}\right), &(12b)\end{aligned}$$

for outgoing waves.

These expressions for the scalars immediately give the correct radial fall-offs at future null infinity once the peeling behavior of the Weyl tensor is assumed, corresponding to $I \propto r^{-6}$ and $J \propto r^{-9}$. The function Γ is defined in the limit of Petrov type D and gives no radial contribution at future null infinity; the variable Θ gives no contribution either as it is the ratio of quantities that have the same radial behavior at future null infinity. In conclusion, Eq. (12) gives $\Psi_0 \propto r^{-5}$ and $\Psi_4 \propto r^{-1}$ for outgoing waves, as is expected from perturbation theory.

Future work will address the determination of the integration constant $\mathcal{B}_0$ as well as relating the function Γ to other invariant properties of the space-time under consideration.

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