Aspects of Holography and Higher Spins in Three-Dimensional Asymptotically Flat Spacetimes

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Why, sometimes I've believed as many as six impossible things before breakfast.

("Alice's Adventures in Wonder Land",
Lewis Carroll)

Abstract

This thesis deals with the application of the holographic principle to asymptotically flat spacetimes in three dimensions, with a particular focus on higher-spin generalisations of Einstein gravity. The inquiry about different examples of holographic theories apart from well-studied AdS/CFT implementations, in particular such models that include (quantum) gravitational theories relevant to observable physics, is an imperative step towards a broadened understanding of the overall applicability of the holographic principle. Simultaneously, the existence of non-trivially interacting higher-spin theories in flat spacetimes poses an interesting question on itself.

Within the three-dimensional setup, a novel higher-spin symmetry algebra is constructed from a universal-enveloping-algebra approach, upon which a theory of higher-spin gravity is defined in terms of a Chern-Simons theory, which describes an infinite tower of massless fields of ever higher spin. This adds an important piece to the insights obtained in previous literature, where only the case of a single spin-three field was explicitly considered so far.

Since gravitational theories in three dimensions are of topological nature, it is desirable to be in possession of simple mechanisms to introduce additional degrees of freedom, in particular matter fields propagating in the respective spacetime. In the holographic context, these give rise to observables of the dual field theory. Within this work, said algebra structure is shown to provide a suitable matter-coupling prescription, appearing in the form of unfolded massive equations. It is demonstrated how an infinite set of Fierz-Pauli fields in a classical (Einstein) background is described by these equations.

The holographic calculation of field-theory observables using Wilson lines will be revisited in the case of entanglement entropy. The respective result known to the literature is reproduced but using a refined formal framework, which clarifies the role of a massive, spinning probe field in order to obtain contributions to the entanglement entropy proportional to both the central charges of the asymptotic symmetry algebra. The probe action used here will be generalised to the case of higher-spin gravity constructed earlier.

In order to derive field-theory observables in the semi-classical limit, techniques are needed to efficiently handle the action of the respective symmetry. This work presents an oscillator form of the highest-weight representation of the flat-space asymptotic symmetry algebra $\mathfrak{bms}_3$ that allows the calculation of (BMS-)conformal blocks and the proof of their exponentiation. As initial steps towards the case of induced, thus unitary, representations, coherent states of the Poincaré group are studied. This gives rise to an oscillator form of an induced representation of the three-dimensional Poincaré algebra.

Part of the results presented here were originally published in [1–3].

Zusammenfassung

Diese Dissertation behandelt die Anwendung des Holographischen Prinzips auf asymptotisch flache Raumzeiten in drei Dimensionen, unter besonderer Berücksichtigung von Verallgemeinerungen der Einstein'schen Gravitationstheorie durch höhere Spins. Die Untersuchung verschiedener Beispiele holographischer Theorien, abgesehen von wohlbekannten Realisierungen der AdS/CFT-Dualität, im Besonderen solcher Modelle, die für die beobachtbare Physik relevante (Quanten-)Gravitationstheorien enthalten, ist ein unerlässlicher Schritt in Richtung eines erweiterten Verständnisses der generellen Anwendbarkeit des Holographischen Prinzips. Gleichzeitig stellt allein die Existenz nichttrivialer, wechselwirkender Theorien höheren Spins in flachen Raumzeiten eine interessante Frage dar.

Hier wird die Konstruktion einer neuartigen Symmetriealgebra höherer Spins in drei Dimensionen auf Grundlage einer universellen einhüllenden Algebra vorgenommen, vermittels derer eine Theorie der Gravitation höherer Spins in Form einer Chern-Simons-Theorie formuliert wird, welche eine unendliche Auftürmung masseloser Felder zunehmend höheren Spins beschreibt. Das fügt der bisherigen Literatur, in welcher lediglich der Fall eines einzelnen Spin-drei-Feldes explizit behandelt wurde, einen wesentlichen Baustein hinzu.

Da dreidimensionale Gravitationstheorie topologischer Natur ist, ist es wünschenswert, über Mechanismen zur Einführung zusätzlicher Freiheitsgrade, insbesondere auf der Raumzeit propagierender Materiefelder, zu verfügen. Solche stehen holographisch in Beziehung zu dualen Feldtheorie-Observablen. In dieser Arbeit wird gezeigt, dass besagte Algebrastruktur einen geeigneten Formalismus der Materie-Kopplung in der Form entfalteter, massiver Gleichungen mit sich bringt und wie diese Gleichungen eine unendliche Anzahl an Fierz-Pauli-Feldern auf einem klassischen (d.i. Einstein'schen) Hintergrund beschreiben.

Die holographische Berechnung von Feldtheorie-Observablen mittels Wilson-Linien wird für den Falle der Verschränkungsentropie aufgegriffen. Das der Literatur bekannte Ergebnis wird unter Verwendung eines überarbeiteten Formalismus, der die Rolle massiver Testfelder nichtverschwindenden Spins zur Berechnung der den beiden zentralen Ladungen der asymptotischen Symmetriealgebra proportionalen Anteile der Verschränkungsentropie hervorhebt, reproduziert. Die hierzu verwendete Wirkung wird auf den Falle der zuvor konstruierten Gravitation höherer Spins verallgemeinert.

Die Berechnung von Feldtheorie-Observablen im halbklassischen Grenzfall bedarf besonderer Techniken im Umgang mit der asymptotischen Symmetrie. Diese Arbeit stellt eine Oszillator-Form einer Darstellung höchsten Gewichts der asymptotischen Symmetriealgebra flacher Raumzeiten ($\mathfrak{bms}_3$) vor, welche die Berechnung BMS-konformer Blöcke sowie den Beweis der Exponenzierung selbiger erlaubt. Als Schritt in Richtung induzierter, unitärer Darstellungen werden kohärente Zustände der Poincaré-Gruppe untersucht. Diese führen auf eine Oszillator-Form einer induzierten Darstellung der Poincaré-Algebra.

Teile der hier vorgestellten Resultate wurden in den Publikationen [1–3] veröffentlicht.

Contents

Chapter 1

Introduction

The fundamental building blocks of our universe are believed to be described by quantum fields on a microscopic level. The material constituents of the observable world and their interactions are thoroughly represented within the realm of Quantum Field Theory (QFT), which governs the classification of elementary particles as well as the strong, weak and electromagnetic interactions in terms of the standard model of particle physics [4–7]. Predictions of the standard model are confirmed in various experiments to great accuracy, one of the most popular verifications of the twenty-first century being the detection of the Higgs boson [8–11].

At the same time, the behaviour of our universe on magnitudes ranging from cosmological scales, including its long-term time evolution, down to planetary scales is explained by the theory of General Relativity (GR) in a geometric manner [12–14]. It describes both the propagation of matter in curved spacetime as well as its back-reaction on the curvature and furthermore predicts the existence of black holes and gravitational waves [15–18], both of which have become subject of direct astronomical observation in recent years [19–23].

Although both QFT and GR apparently provide a suitable description of experimental data, it is unavoidable that at least one of these two theories will have to be modified on accordingly high energy scales (small length scales, respectively), when the radius of curvature becomes relevant on a microscopic level. In this regime, a quantisation of the gravitational interaction appears to become necessary – a procedure that is addressed by Quantum Gravity (QG). How such a theory is challenging our basic understanding of the principles of quantum mechanics, relativity and locality, all three of which are cornerstones of current theoretical physics, is probably best exemplified in the context of black-hole formation and evaporation [24].

It is nowadays widely accepted that a black hole should be assigned an entropy proportional to the area of its event horizon [25]. But, according to quantum statistics, this entropy should give rise to the existence of black-hole micro-states, which are however prohibited by the so-called no-hair theorem in classical GR. A second contradiction emerges from the existence of Hawking radiation, causing a black hole to evaporate under the emission of thermal radiation; this implies the loss of information about pure quantum states (that collapsed to a black hole at some time), at the very latest after the black hole completely disappeared, which is in conflict with unitary time evolution. An attempt to resolve the information-loss paradox at times later than the Page time [26, 27] (no later than complete evaporation of the black hole), immediately entails a third obstacle: Assuming unitary time evolution within quantum gravity and a mechanism such that the information stored in the black hole may escape at late times (in form of correlations between early and late Hawking radiation), the late radiation needs to be maximally entangled with the early radiation as well as with the black hole interior. This can however not be the case due to properties of entanglement entropy and quantum information, implying the presence of a mechanism breaking up the entanglement between late Hawking radiation and the black hole interior, which, in turn, implies a drastic violation of the equivalence principle and would cause an in-falling observer to burn up when passing the event horizon [28].

There have been various approaches to address such problems and formulate a consistent theory of quantum gravity [29]. An interesting and promising approach comes in the form of the holographic principle, which states that the entire information about a d-dimensional (quantum) gravitational system can be encoded on the $(d-1)$-dimensional boundary of that system (in analogy to optical holography). The idea is conceptionally most interesting and can be traced back to the works of Bekenstein and Hawking [25, 30], who found that the entropy of a black hole scales with the surface of its event horizon instead of the volume, as might have been expected, and was generalised by 't Hooft, Thorne and Susskind [31–34] to theories of quantum gravity.

The holographic principle can be made more rigorous in the context of string theory and the renowned AdS/CFT correspondence. The correspondence describes a duality of the anti-de Sitter spacetime (a maximally symmetric solution of Einstein's equations with negative cosmological constant, AdS for short) to a conformal field theory (a field theory that is invariant under conformal transformations, CFT for short). To be precise, Maldacena conjectured in 1997 [35] that type-IIB string theory on $AdS_5 \times \mathbb{S}^5$, a five-dimensional theory of quantum gravity, is dual to $\mathcal{N} = 4$ super-Yang Mills theory, a four-dimensional QFT.

An appealing feature of the AdS/CFT duality is its explanation of geometry on the gravity side as an emergent concept from the field-theory side, particularly the emergence of Einstein gravity as a semi-classical limit of CFT. Different states of the CFT can be identified with different geometries within the dual gravity theory, for instance, the CFT-vacuum state corresponds to pure AdS spacetime, while states of thermal equilibrium at a given temperature correspond to black-hole configurations in AdS, where the temperature is given by the temperature of the respective Hawking radiation [36]. Note that, in this sense, the full string-theoretic version of AdS/CFT provides a manifestly background-independent definition of string theory, which is to be regarded as a key ingredient to any reasonable theory of quantum gravity.

Yet another aspect of holography, more precisely of gauge/gravity duality, is its property of posing a strong-weak duality [37]. That is to be understood in the sense that strongly coupled field theories can be translated into weakly coupled gravitational theories in the bulk, where they can be solved using standard perturbative methods, and the result can be translated back to the field-theory side, a procedure that already led to important applications, for example in the realm of heavy-ion physics [38]. From this point of view, AdS/CFT does not necessarily restrict to its role as an important stage in the search for a theory of quantum gravity but possibly provides a practical mathematical tool for completely different investigations on strongly coupled systems.

Though there are many more examples, dualities involving three-dimensional theories of (quantum) gravity and two-dimensional field theories, such as $\mathrm{AdS}_3/\mathrm{CFT}_2$, will be of main interest for the present work. The advantage of studying the three-dimensional case comes through the purely topological character of the free gravitational theory, i.e. the absence of any propagating local degrees of freedom, such as gravitational waves. This actually allows the quantisation of Einstein gravity [39, 40]. At the same time the dual field theories in two dimensions are highly constrained by the huge symmetry involved, which actually allows to solve these theories completely. In order to gain deeper insights into the fundamental working mechanisms of holography, it is of the utmost interest to have examples of dualities at our disposal, in which both sides are completely under our control. In this regard, lower-dimensional settings, though certainly of limited applicability as models of quantum gravity in real-world scenarios, provide valuable toy models to improve our understanding of holography and quantum gravity. It is for example possible to address fundamental problems of quantum-gravitational nature, such as the information-loss paradox, in the three-dimensional case, see e.g. [41].

Chapter 2

Motivation

Having given a general argumentation in favour of the holographic principle and AdS/CFT, I will in the following try and motivate the particular steps taken within the work presented here. I will explain how this thesis ties in with previously known results and give an outline of its structure.

2.1 The Holographic Principle and Flat Space

The main motivation for the present work stems from the basic question to which extend the holographic principle can be applied in more general circumstances than AdS/CFT. Though many of its prime examples evolved from string theory [35], the holographic principal itself promises a much wider range of applicability [34, 42, 43]. One may ask which of the features we are witnessing in AdS/CFT are generic to holography and which are merely accidental in this particular instance.

Of course there are many possibilities of generalising AdS/CFT. Taking the viewpoint that holographic dualities should be capable of defining a theory of quantum gravity, an apparent step to take is in direction of more realistic models on the gravity side. As far as cosmological scales are concerned, it would be beneficial to understand dualities that involve (asymptotically) de Sitter spacetimes, since this appears to be the kind of universe we happen to live in. Attempts in this direction have been made, see [44–52] for a selection. A second example of a theory that might be considered more realistic would be a duality that involves asymptotically flat spacetimes on the gravity side [53–61] since such a theory would be believed to be of significance for earth-scale experiments, where the curvature of the universe can be neglected.

Studying holography in a flat-space setting is significantly more complicated than AdS/CFT, thus making it a formidable research subject. While some tools and principle ideas of AdS/CFT may be translated to the case of flat spacetimes, in many instances the specific techniques have to be thoroughly revised. Let me collect some aspects in which this case is different from the prime example. First, there is no string-theoretic interpretation of a putative duality, thus a huge guiding principle is missing. Second, in the case of negative and positive curvature, the cosmological constant provides a natural length scale that is absent in flat space, which does pose a problem in some circumstances, such as the regularisation of one-loop partition functions [62, 63]. As a third point it is to be mentioned that the causal structure of asymptotically flat spacetimes exhibits several asymptotic regions on which to put a dual field theory, including past and future null infinity. The latter are light-like surfaces rather than time-like ones, making the dual field theories in question, so-called Carrollian field theories, considerably harder to study. Also, conformal field theories were well known and extensively studied long before the AdS/CFT correspondence [64, 65], something that cannot be said about Carrollian field theories [66–68]. Lastly, in connection to the absence of the cosmological constant (which renders the underlying symmetries non-semisimple) and the peculiarities of the dual field theory, much greater care is to be taken with respect to representation theory.

The passage to other maximally symmetric solutions is of course only one way of pursuing alternative realisations of the holographic principle. Another ansatz is to try and go away from the purely topological gravitational setup of AdS_3/CFT_2, which might be considered too simple for some kinds of questions. There are, of course, higher-dimensional models in AdS/CFT, in which local degrees of freedom are present. But if one wishes to stay in three dimensions in order to not complicate the gravitational part of the theory too much, there is also the option to introduce propagating degrees of freedom by coupling in additional, possibly back-reacting fields. Taking a scalar field, say, this newly introduced degree of freedom gives rise to additional entries in the holographic dictionary, namely it allows the calculation of certain correlation functions of the field theory, an example that has been successfully implemented in AdS/CFT [69–73]. A translation of the analogous reasoning to the flat case is an intriguing thought, in particular it introduces a length scale to the theory through the mass of the scalar field.

2.2 Higher-Spin Gravity

Sticking to the role that AdS/CFT plays in the context of string theory for a moment, an interesting observation can be made: In a certain large-energy limit of string theory, in which the string length may be taken to infinity (known as tensionless limit), the massless spin-two modes of closed strings get accompanied by an infinite tower of massless higher-spin modes, thus providing an enhancement of symmetry. This makes it possible to identify a theory of higher-spin gravity [74–76], which could alternatively be viewed as a generalisation of Einstein gravity. In that sense, a theory of higher-spin gravity lies somewhat in-between general relativity and full-fledged string theory, in which the higher-spin symmetries might be dynamically broken [77]. Therefore, it is to hope that we will broaden our understanding of the latter by studying theories that allow higher-spin fields to exist and couple to the graviton. This amounts to viewing string theory as higgsed version of higher-spin gravity and, as it is done in the standard model, one should first study the un-higgsed (or symmetric) phase before investigating the higgsed phase, namely higher-spin gravity.

These theories pose a most interesting field of study on themselves and can be viewed as a generalisation of Einstein gravity through an enhancement of symmetry. In three-dimensional AdS this enhancement introduces a finite or infinite number of additional (bosonic) massless fields of ever larger spin to the theory, accompanying the usual spin-two metric field [78–83]. It is to note that the larger symmetry that is imposed on the theory, of which usual diffeomorphism invariance is only a sub-sector, renders many of the familiar invariant quantities known from classical GR non-gauge invariant[1]; a notion of proper distances, for instance, ceases to make sense and the whole concept of geometry, obscured by the presence of higher-spin symmetry transformations, needs to be replaced by a different, gauge invariant prescription [85, 86].

The great advantage of the AdS case, not only in three but in higher dimensions as well, is that there exists a complete theory of interacting massless higher-spin fields, known as Prokushkin-Vasiliev theory [82, 83]. This theory is capable of describing an infinite collection of massless higher-spin fields as well as massive scalar fields, all non-linearly and non-locally interacting. While the full theory presents itself as an unbearably complicated subject, its linearisation makes it tractable and allows to study a particularly nice realisation of holography [69, 73, 87–89]: Given a family of theories of higher-spin gravity, linearly coupled to two complex scalar fields of a certain mass, these theories are dual to so-called vector-like, coset CFTs that possess conserved currents of higher spin, which give rise to so-called $\mathscr{W}$-symmetry algebras [90, 91]. In a semi-classical (large-central charge) limit the spectrum of both kinds of theory agrees and it is possible to match results for correlation functions and entropies. The scalar fields on the gravity side then encode information of scalar conformal operators on the field-theory side, where the mass of the scalars is linked to the conformal dimension of these operators.

The situation described above makes a strong case for taking the tensionless limit of string theory in an AdS vacuum [92–94]. But it is tempting to ask to which extend a similar line of thought can be applied within the case of flat spacetimes. First of all, in flat space neither a guiding string-theory interpretation nor a counterpart to the fully interacting theory of Vasiliev exist. Accordingly, one may want to investigate this question rather from a bottom-up perspective, first looking at the simplest possible higher-spin generalisations of asymptotically flat spacetimes and their field-theory duals. However, the problem in flat space lies much deeper than that since one is dealing with the fundamental question of whether or not non-trivially interacting (massless) higher-spin theories can exist in flat

[1] Note that the words “asymptotically” and “AdS” still make sense in higher-spin gravity, though. While the notion of the asymptotic behaviour of a spacetime can be generalised to the higher-spin case by consideration of a suitable set of boundary/fall-off conditions, the term AdS merely refers to the presence of a cosmological constant [84].

space, in the first place.

Already in the early stages of quantum theory the question of how to describe particles of higher spin has arisen [95, 96]. First attempts to formulate equations describing massive, freely propagating fields of arbitrary integer spin by Fierz and Pauli [97, 98] and later free massless fields by Fronsdal and Fang [99, 100] were counteracted by various no-go theorems concerning the consistent interaction of higher-spin fields [101–106]. Yet, a variety of working examples could be established [75, 107–111] and, as of today, we are in possession of at least one example of a fully interacting theory of massless higher-spin fields, as well as their interaction with massive matter fields in the case of a negative cosmological constant in form of Prokushkin-Vasiliev theory [78, 79, 82] described above. The question of non-trivially interacting (massless) higher-spin fields in the case of a vanishing cosmological constant, however, remains yet to be answered.

2.3 Objective and Outline of Thesis

The objective of this thesis lies in the above described idea to generalise AdS/CFT. In particular, this work will take three steps at once: the transition to asymptotically flat spacetimes, the generalisation to higher-spin gravity, and the coupling to massive scalar (and higher-spin) degrees of freedom. As it turns out, all three aspects are mutually intertwined.

Despite the existence of earlier works on spin-three gravity in asymptotically flat spacetimes, no rigorous treatment of infinite towers of massless higher-spin fields has been performed in the literature before. Such a treatment will be initiated here, with the mathematical foundation being laid by construction of a suitable higher-spin Lie algebra in chapter 4. This Lie algebra stems from an underlying associative algebra that presents a much richer structure than the one usually discussed in the AdS context, which is a characteristic feature of the non-semisimple nature of flat-space symmetries. I will present properties of this novel algebra and its relation to other constructions in the literature.

Note that a guiding principle of (almost) all explorations undertaken here is the derivation of flat-space quantities purely from first principles, without relying on any sort of vanishing-cosmological constant limit. Though a number of results within flat-space holography are due to such limiting procedures, which can be performed both as a limit on the gravitational side (taking the cosmological constant to zero, or, equivalently the radius of the (A)dS curvature to infinity) and as a limit on the field-theory side (taking the speed of light to zero or infinity), it is often not a priori clear which starting point to take and which scalings of relevant quantities to perform. Some comments are given in section 4.3 on how to obtain the algebra structure here developed from a contraction of an AdS higher-spin algebra, thereby demonstrating how the precise nature of such a procedure can be subtle.

Obviously, having formulated a proposal for a flat-space higher-spin algebra, the first application lies in the definition of a theory of higher-spin gravity by thinking up appropriate boundary conditions. This is done in section 5.1, where an infinite set of higher-spin charges is introduced that naturally generalises Einstein gravity.

A formalism for the introduction of matter fields may be motivated by the linearised form of Vasiliev's equations in AdS, where the coupling of the matter content to the background gauge fields is realised in terms of an underlying associative algebra structure, on which the respective coupling equation takes on the form of a covariant-constancy condition [82, 112]. In section 5.2 an analogue of this framework is build upon the associative algebra that was constructed beforehand. The benefits of such a matter-coupling prescription are multifold: From the perspective of flat-space holography it constitutes a first step towards an enhanced holographic dictionary since it will be necessary to holographically calculate three-point functions involving scalar (and higher-spin) currents. From a pure higher-spin viewpoint, an unfolded version of massive (higher-spin) wave equations is a necessary step in a bottom-up approach to higher-spin theory in flat space.

A particularly interesting technique for holographic calculations of field-theory observables, such as entanglement entropy, is the utilisation of Wilson lines [60, 113–115]. It is the dynamics of a system on the Wilson line that determine the respective observables in a semi-classical limit (if appropriate boundary conditions are set). In section 5.3 existing proposals on Wilson lines in flat space and their generalisation to higher spins will be revisited. I will construct a suitable probe system from a worldline action on the Poincaré group manifold and clarify a couple of technicalities in connection with the calculation of entanglement entropy. A generalisation to the higher-spin theory defined earlier is initiated.

Within chapter 6 the focus will be shifted towards the field-theory side, in particular to representations of the flat-space asymptotic symmetry algebra $\mathfrak{bms}_3$. Since it is known from the case of conformal symmetry that a particular realisation of symmetry generators is needed in order to calculate quantities like conformal blocks (in a semi-classical limit), an analogous construction in the case of $\mathfrak{bms}_3$-symmetry is presented. The underlying representation is however inherently non-unitary and the construction of an induced, unitary version remains an open problem. As an initial exploration in this direction, I will study induced representations of the Poincaré algebra and its generalised coherent states.

Chapter 3

Foundations

It is the purpose of this chapter to introduce some fundamentals that lay the ground for the results obtained in the present work and to partly fix the notation. I will first review gravity in three spacetime dimensions in the form of Chern-Simons gauge theories, followed by a brief survey of higher-spin theory, in particular higher-spin gravity in three dimensions. Finally, I will give a glance on some relevant aspects of holography in the case of vanishing and non-vanishing cosmological constant.

3.1 Three-Dimensional Gravity in the Chern-Simons Formalism

Gravity can be viewed as a gauge theory and this property is most apparent in the case of three spacetime dimensions. It is long known that in this case the vacuum theory can be cast into the form of a Chern-Simons theory [39, 40, 116, 117], which is due to its being completely topological, in the sense that there are no propagating degrees of freedom, i.e. gravitational waves, in three dimensions.

3.1.1 Basics of Chern-Simons Theory

It is a peculiarity of (2+1)-dimensional physics in general that an additional type of gauge theory, other than Maxwell or Yang-Mills theory, makes an appearance [118]. These so-called Chern-Simons theories first appeared in the study of four-dimensional, closed, oriented manifolds [119] and since then found wide application in the theory of anyons [120], the fractional quantum-Hall effect [121], the modelling of atmospheric dynamics [122] or the horizontal flow of shallow water [123], the most interesting realisation for the purposes of the present thesis, however, being its application to various aspects of quantum gravity in 2+1 dimensions [40].

I will closely follow the introduction presented in [118]. Considering an abelian gauge field A_μ as well as a conserved matter current J_μ, $\partial_\mu J^\mu = 0$, that reside in 2+1 dimensions, an action can be written down that is of the form

$$S_{\rm CS} = \int \mathrm{d}^3x \left(\frac{k_{\rm CS}}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu \right), \tag{3.1.1}$$

which is invariant under gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu \lambda$, as long as boundary terms are omitted,[1] since the Lagrangian changes by a total derivative. The real constant $k_{\rm CS}$ is called Chern-Simons level. The most obvious difference to the Maxwell action is the appearance of derivatives of the gauge field only to first power, which would not be possible in higher dimensions. This implies rather simple equations of motion; introducing the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the respective Euler-Lagrange equations read

$$\frac{k_{\rm CS}}{2\pi} F_{\mu\nu} - \varepsilon_{\mu\nu\rho} J^\rho = 0, \tag{3.1.2}$$

such that, in the source-free case $J_\mu = 0$, the vanishing of the field-strength tensor, $F_{\mu\nu} = 0$, states that all solutions are pure gauge.

In the theory written down above, it was assumed that the gauge-field components A_μ commute, i.e. one is dealing with an abelian Chern-Simons theory. This is in general not necessarily true; the gauge field may be considered to take values in some (representation of) a Lie algebra $\mathfrak{g}$. In that case, here for vanishing current, the appropriate action reads

$$S_{\rm CS} = \frac{k_{\rm CS}}{4\pi} \int \mathrm{d}^3x \, \varepsilon^{\mu\nu\rho} \, \mathrm{tr}\left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right), \tag{3.1.3}$$

where tr(.) refers to a trace in the respective representation of $\mathfrak{g}$. Note that the cubic term automatically vanishes in the abelian case.

Finally, one may store the gauge-field components in a Lie-algebra valued one-form, $A = A_\mu dx^\mu$, and exchange the trace by a bilinear form on the Lie algebra, $\langle .,. \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, such that no reference to an algebra representation is needed. The Chern-Simons action then reads

$$S_{\rm CS}[A] = \frac{k_{\rm CS}}{4\pi} \int \left\langle A \overset{\wedge}{,} \mathrm{d}A + \frac{2}{3} A \wedge A \right\rangle, \tag{3.1.4}$$

where I introduced the notation $\langle . \overset{\wedge}{,} . \rangle$ to highlight the simultaneous application of the bilinear form to the Lie-algebra elements and of the wedge product to the spacetime one-forms[2]. The equations of motion turn into the flatness condition

$$\mathrm{d}A + A \wedge A = 0, \tag{3.1.5}$$

[1] This can be achieved, for example, by imposing appropriate fall-off conditions. Here, large gauge transformations are excluded.

[2] One may easily be convinced that there is no difference between $\langle A \overset{\wedge}{,} A \wedge A \rangle$ and $\langle A \wedge A \overset{\wedge}{,} A \rangle$.

and the invariance of the theory under finite transformations, i.e. under the action of the respective gauge group G, is ensured by the transformation behaviour

$$A \mapsto g^{-1} A g + g^{-1} \mathrm{d} g \tag{3.1.6}$$

of the gauge field under the action of some $g \in G$.

3.1.2 Asymptotically AdS Spacetimes

The starting point for the definition of asymptotically AdS here is the presumption of a manifold[3] with the topology of a two-torus, reflected by a choice of coordinates (ρ, x^+, x^-). The most general solution of Einstein's equations with negative cosmological constant $\Lambda = -1/l^2$ under the boundary conditions given in [125] is given in terms of a so-called Fefferman-Graham metric [126, 127]

$$\begin{aligned} ds^2 = d\rho^2 + 8\pi G l \left(\mathscr{L}(x^+) \left(dx^+\right)^2 + \overline{\mathscr{L}}(x^-) \, (dx^-)^2 \right) \\ + \left(l^2 \, \mathrm{e}^{2\rho/l} + (8\pi G)^2 \mathscr{L}(x^+) \overline{\mathscr{L}}(x^-) \, \mathrm{e}^{-2\rho/l} \right) dx^+ dx^- , \end{aligned} \tag{3.1.7}$$

where the functions $\mathscr{L}(x^+)$ and $\overline{\mathscr{L}}(x^-)$ are arbitrary. For constants functions $\mathscr{L}$, $\overline{\mathscr{L}}$ this metric describes a black hole of mass $\mathcal{M}$ and angular momentum $\mathscr{J}$ given by

$$\mathcal{M} = 2\pi \left(\mathscr{L} + \overline{\mathscr{L}} \right) , \qquad \mathscr{J} = -2\pi \left(\mathscr{L} - \overline{\mathscr{L}} \right) , \tag{3.1.8}$$

called the BTZ black hole [128–131]. Its black-hole nature may be more apparent when changing to Schwarzschild-like coordinates through the transformations $x^\pm = \phi \pm t/l$ as well as $1 + r^2/l^2 = \left(\mathrm{e}^{\rho/l} + 1/4 \, \mathrm{e}^{-\rho/l}\right)^2$, such that

$$ds^2 = -N^2(r) \, \mathrm{d}t^2 + \frac{dr^2}{N^2(r)} + r^2 \left(N_\phi(r) dt + d\phi \right)^2 , \tag{3.1.9}$$

where lapse function and angular shift are given by

$$N^2(r) = -\mathcal{M} + \frac{r^2}{l^2} + \frac{\mathscr{J}^2}{4r^2} , \qquad N_\phi(r) = -\frac{\mathscr{J}}{2r^2} , \tag{3.1.10}$$

and the inner and outer event horizons are at radial coordinates

$$r_\pm^2 = \frac{\mathcal{M} l^2}{2} \left(1 \pm \sqrt{1 - \left(\frac{\mathscr{J}}{\mathcal{M} l} \right)^2} \right) . \tag{3.1.11}$$

[3]The reader interested in a rigorous introduction to the notion of asymptotic spacetimes may consult [124].

The relevant asymptotic region lies at spatial infinity, $r \to \infty$. Empty AdS spacetime is apparently included for $\mathscr{L} = \overline{\mathscr{L}} = -1/(4\pi)$.

The isometries of the spacetime (3.1.7) are given by the AdS-Lorentz group $SO(2,1) \times SO(2,1) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. The respective isometry Lie algebra is $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, spanned by generators L_m, $\bar{L}_m$ with $m \in \{0, \pm 1\}$ that fulfil the Lie brackets

$$[L_m, L_n] = (m-n) L_{m+n}, \qquad [L_m, \bar{L}_n] = 0, \qquad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}. \tag{3.1.12}$$

It is thus possible to express the three-dimensional, asymptotically AdS gravity theory as a Chern-Simons gauge theory, introducing two gauge fields[4] A and $\bar{A}$, each corresponding to one of the copies $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, and a suitable bilinear form on the algebra

$$\langle L_m, L_n \rangle = \eta_{mn}, \qquad \langle \bar{L}_m, \bar{L}_n \rangle = \eta_{mn}, \tag{3.1.13}$$

where $\eta = \text{antidiag}(-2, 1, -2)$. The Einstein-Hilbert action is then replaced by two copies of the Chern-Simons action (3.1.4), namely by $S = S_{\text{CS}}[A] - S_{\text{CS}}[\bar{A}]$. The dimensionless Chern-Simons level k_{CS} has to be identified with Newton's constant G_{N} as $k_{\text{CS}} = l/(4G_{\text{N}})$. Since the theory decomposes into two mutually commuting sectors, it is usually sufficient to consider only one of the copies.

The explicit form of the most general asymptotically AdS spacetime is given in terms of the gauge field as [84]

$$A = \left(\mathrm{e}^{\rho/l} L_1 - \frac{2\pi \mathscr{L}(x^+)}{k_{\text{CS}}} \mathrm{e}^{-\rho/l} L_{-1} \right) dx^+ + L_0 \frac{d\rho}{l}, \tag{3.1.14a}$$

$$\bar{A} = -\left(\mathrm{e}^{\rho/l} L_{-1} - \frac{2\pi \overline{\mathscr{L}}(x^-)}{k_{\text{CS}}} \mathrm{e}^{-\rho/l} L_1 \right) dx^- - L_0 \frac{d\rho}{l}. \tag{3.1.14b}$$

One may define spin connection and vielbein as

$$\omega = \frac{A + \bar{A}}{2}, \qquad e = l \frac{A - \bar{A}}{2} \tag{3.1.15}$$

and the metric (3.1.7) may be re-constructed from the gauge fields (3.1.14) with the help of the bilinear form, $g_{\mu\nu} = \langle e_\mu, e_\nu \rangle$.

3.1.3 Asymptotically Flat Spacetimes

To define a notion of a spacetime being asymptotically flat, one may first choose a set of coordinates. With the hindsight that the asymptotic regions of flat spacetimes will be light-like surfaces, one may choose outgoing or ingoing Eddington-Finkelstein coordinates,

[4] It is however necessary to impose an opposite gauge-transformation behaviour than (3.1.6) in the barred sector, namely $\bar{A} \mapsto g \bar{A} g^{-1} + g\, \mathrm{d}g^{-1}$.

denoted by (u,r,ϕ) and (v,r,φ), respectively. For the case of outgoing coordinates, a possible choice of gauge for the metric is the Newman-Unti gauge [132], which is $g_{ur}=-1$, $g_{rr}=g_{r\phi}=0$. Then, demanding the fall-off conditions $g_{uu}=g_{u\phi}=\mathcal{O}(1)$ and $g_{\phi\phi}=\mathcal{O}(r)$, the most general[5] solution of Einstein's equations with vanishing cosmological constant allowed by these conditions is given by [57, 134, 135]

$$\boxed{ds^2 = M(\phi)du^2 - 2du dr + 2N(u,\phi)du d\phi + r^2 d\phi^2\,,} \tag{3.1.16}$$

where the mass aspect $M(\phi)$ and the angular momentum aspect $N(u,\phi)$ are functions constraint by the integrability condition $\partial_\phi M(\phi) = 2\partial_u N(u,\phi)$, i.e. one may write $N(u,\phi) = \Xi(\phi) + u/2\,\partial_\phi M(\phi)$, where now $M(\phi)$ and $\Xi(\phi)$ are arbitrary functions on the circle. Special cases of these functions include [136]

- **Minkowski Spacetime**, $M(\phi)=-1\,,\ N(u,\phi)=0$: the vacuum spacetime that is in close analogy to its higher-dimensional counterparts, being in possession of two light-like asymptotic regions $\mathscr{I}^\pm$, space-like infinity i^0 as well as timelike infinities $i^\pm$, see the Penrose diagram in figure 3.1a;

- **Flat-Space Cosmologies**, $M(\phi)=M>0\,,\ N(u,\phi)=N\neq 0$: a space-time that possesses a cosmological horizon proportional to the angular momentum aspect and is causally different from its higher-dimensional counterpart, the Schwarzschild black hole, in that it is rotated by 90 degrees, as can be seen from the Penrose diagram in figure 3.1b;

- **Angular-Deficit Spacetimes**, $M(\phi)=M\,,\ -1<M<0\,,\ N(u,\phi)=N$: spacetimes describing localised sources (point particles) [137] .

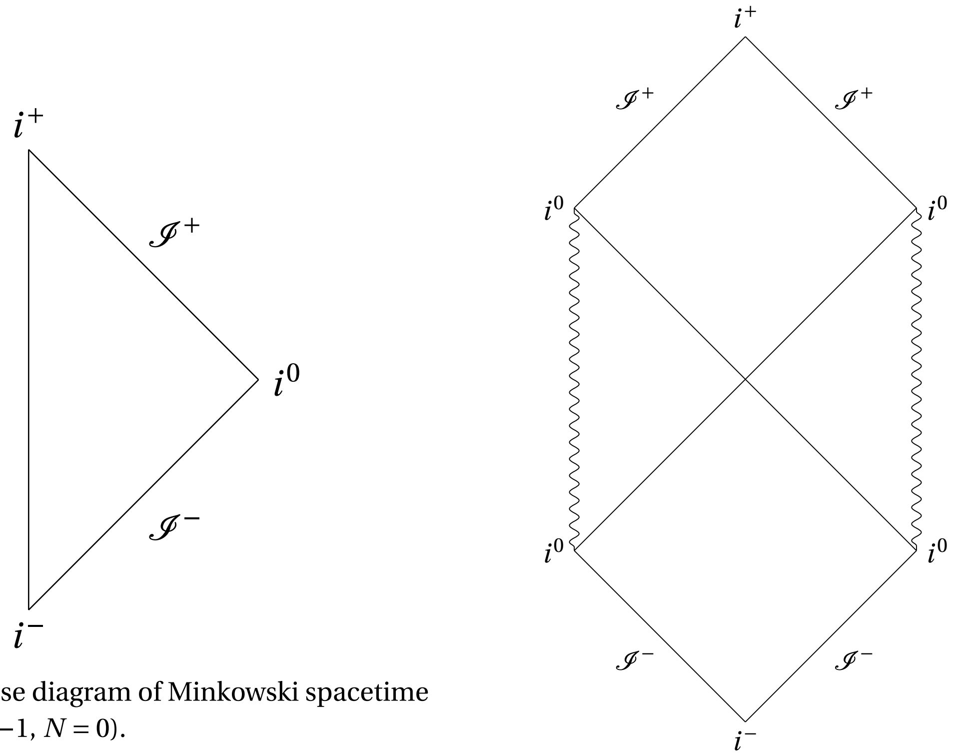
There are more cases to distinguish for $M=0$ or $M<0$ but these will not be discussed here. Note that there is no proper black-hole solution in three-dimensional gravity with vanishing cosmological constant [138].

Naturally, the isometries of the spacetime (3.1.16) are given by the Poincaré group in three dimensions $ISO(2,1) = SO(2,1)\ltimes\mathbb{R}^3$, which is the semi-direct product of Lorentz transformations and translations. The corresponding Lie algebra is $\mathfrak{iso}(2,1)=\mathfrak{so}(2,\mathbb{R})\inplus\mathbb{R}^3$, spanned by generators of infinitesimal Lorentz transformations J_a and generators of space and time translations P_a, where $a\in\{0,1,2\}$, equipped with the Lie brackets

$$[J_a\,,J_b]=\varepsilon_{ab}{}^c J_c\,,\qquad [J_a\,,P_b]=\varepsilon_{ab}{}^c P_c\,,\qquad [P_a\,,P_b]=0\,, \tag{3.1.17}$$

where $\varepsilon_{ab}{}^c=\eta^{cd}\varepsilon_{abd}$ with normalisation $\varepsilon_{012}=1$ and $\eta=\mathrm{diag}(-1,1,1)$ being the flat Minkowski metric in three dimensions. The Chern-Simons gauge field A is a one-form, valued in this Lie algebra, $A\in\mathfrak{iso}(2,1)$, and it can be decomposed into a spin connection $\omega=\omega^a J_a$

[5] A more general notion of asymptotically flat spacetimes in the three-dimensional case may be found in [133].



(a) Penrose diagram of Minkowski spacetime ($M = -1$, $N = 0$).

(b) Penrose diagram of flat-space cosmologies ($M > 0$, $N \neq 0$, both constant).

Figure 3.1: Penrose diagrams of Minkowski spacetime and flat-space cosmologies [136]. The latter are of a significantly different causal structure than their higher-dimensional (Schwarzschild) counterparts, in particular past and future light-like infinity are separated by the cosmological horizon.

and a vielbein $e = e^a P_a$, i.e. $A = \omega + e$. The Chern-Simons action (3.1.4) is equivalent to the Einstein-Hilbert action if the Poincaré algebra is equipped with the invariant bilinear form [39]

$$\langle J_a, J_b \rangle = 0\,, \qquad \langle J_a, P_b \rangle = \eta_{ab}\,, \qquad \langle P_a, P_b \rangle = 0\,. \tag{3.1.18}$$

The explicit form of the gauge fields corresponding to the metric (3.1.16) is best given in a different basis, namely since $\mathfrak{iso}(2,1) \simeq \mathfrak{isl}(2,\mathbb{R})$ one can write the algebra equivalently as

$$[J_m, J_n] = (m-n) J_{m+n}\,, \qquad [J_m, P_n] = (m-n) P_{m+n}\,, \qquad [P_m, P_n] = 0\,, \tag{3.1.19}$$

where $m, n \in \{0, \pm 1\}$. The basis change is performed by $J_{\pm 1} = J_0 \mp J_1$ and $J_0 = J_2$ and analogously for P_m and P_a. The bilinear form in terms of these generators is similar to (3.1.18), now with non-vanishing entries $\langle J_m, P_n \rangle = \eta_{mn}$ with $\eta = \text{antidiag}(-2, 1, -2)$. In the $\mathfrak{isl}(2,\mathbb{R})$-basis, the gauge fields corresponding to (3.1.16) read [139, 140]

$$\omega = \left(J_1 - \frac{M(\phi)}{4} J_{-1}\right) d\phi\,, \tag{3.1.20a}$$

$$e = \left(P_1 - \frac{M(\phi)}{4} P_{-1}\right) du + \frac{1}{2} P_{-1} dr + \left(r P_0 - \frac{N(u,\phi)}{2} P_{-1}\right) d\phi\,. \tag{3.1.20b}$$

In terms of these fields, the flatness condition (3.1.5) turns into vanishing-torsion and zero-curvature conditions

$$\mathrm{d}\omega + \omega \wedge \omega = 0\,, \qquad \mathrm{d}e + \omega \wedge e + e \wedge \omega = 0\,. \tag{3.1.21}$$

Furthermore, from the gauge transformation behaviour (3.1.6) of the Chern-Simons field A follows the transformation laws for spin connection and vielbein. Upon decomposition of a group element (of the one-parameter subgroup of Poincaré that is obtained through exponentiation of the Lie algebra) $g \in ISO(2,1) \simeq ISL(2,\mathbb{R})$ into a Lorentz and a translational part, $g = g_{\mathrm{T}} g_{\mathrm{L}}$, where

$$g_{\mathrm{L}} = \mathrm{e}^{\xi_{\mathrm{L}}} = \exp\left(\xi_{\mathrm{L}}^m J_m\right)\,, \qquad g_{\mathrm{T}} = \mathrm{e}^{\xi_{\mathrm{T}}} = \exp\left(\xi_{\mathrm{T}}^m P_m\right)\,, \tag{3.1.22}$$

the fields transform like

$$\omega \mapsto g_{\mathrm{L}}^{-1}\,(\omega + \mathrm{d})\,g_{\mathrm{L}}\,, \qquad e \mapsto g_{\mathrm{L}}^{-1}\,(e + [\omega, \xi_{\mathrm{T}}] + \mathrm{d}\xi_{\mathrm{T}})\,g_{\mathrm{L}}\,. \tag{3.1.23}$$

The way back to the metric formulation is simply achieved by the identification $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b = \eta_{mn} e_\mu^m e_\nu^n$.

3.2 Higher-Spin Gravity

An interesting generalisation of Einstein gravity consists in the introduction of massless fields of higher spin in addition to the metric field, i.e. a theory of higher-spin gravity. A convenient feature of the Chern-Simons formulation of gravity reviewed above is its straightforward generalisation to higher-spin gravity – one simply replaces the underlying isometry algebra with a suitable higher-spin algebra.

I will in the following review relevant aspects of this framework in the case of negative cosmological constant, i.e. asymptotically AdS spacetimes, first for the example of a single spin-three field coupled to gravity, then turning to the general case of an infinite tower of massless higher-spin fields. Furthermore, I will present some of the main aspects of the fully non-linear theory known in AdS and finally review recent developments in the case of asymptotically flat spacetimes.

3.2.1 Higher-Spin Chern-Simons Formulation

As a simple starting point one may replace the AdS isometry algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ with a finite-spin algebra $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$. Then the respective Chern-Simons theory describes massless, non-interacting fields of spin[6] $2, 3, \ldots, N$, i.e. a sector of higher-spin gauge fields coupled to Einstein gravity [80, 81, 84]. Let me consider the case $N = 3$ as a simple illustration.

Spin-3 Gravity

Consider a Chern-Simons theory with underlying gauge algebra $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$ and focus on one copy of $\mathfrak{sl}(3,\mathbb{R})$. The Lie brackets of this algebra read

$$[L_m, L_n] = (m-n)L_{m+n}, \tag{3.2.1a}$$

$$[L_m, W_n] = (2m-n)W_{m+n}, \tag{3.2.1b}$$

$$[W_m, W_n] = \frac{\sigma}{3}(m-n)\left(2m^2+2n^2-mn-8\right)L_{m+n}, \tag{3.2.1c}$$

with some normalisation factor σ. The indices of generators take values $m \in \{0, \pm 1\}$ for L_m and $m \in \{0, \pm 1, \pm 2\}$ for W_m. For simplicity, assume that we are given a matrix representation of these generators (see equations (B.3.1) of the Appendix). Then a bilinear form is given in terms of the trace and the field content consists of a spin-two field $g_{\mu\nu}$, namely the metric, and a spin-three field $\phi_{\mu\nu\rho}$, given by

$$g_{\mu\nu} \sim \operatorname{tr}\left(e_\mu e_\nu\right), \qquad \phi_{\mu\nu\rho} \sim \operatorname{tr}\left(e_\mu e_\nu e_\rho\right). \tag{3.2.2}$$

It is possible to write down explicit solutions A and $\bar{A}$ that provide natural generalisations of the asymptotically AdS boundary conditions [84] and, in particular, the BTZ solution (3.1.14), i.e. higher-spin BTZ black holes [141, 142]. One may then compare different embeddings of the $\mathfrak{sl}(2,\mathbb{R})$-algebra into its higher-spin generalisation, study the thermodynamics of these black holes, find their asymptotic symmetries and perform many more investigations that will not be reproduced here [85, 86, 143–148].

An important insight lies in the fact that the finite gauge transformations of the spin-three theory are now given in terms of the group $SL(3,\mathbb{R})$, which apparently is larger than $SL(2,\mathbb{R})$, the latter being a subgroup of the former. Thus the symmetry consists not only of diffeomorphisms but also of pure higher-spin transformations acting on the metric, implying that the usual geometric invariants, proper distances and the causal structure of spacetime are no longer gauge invariant concepts.

[6] Here I only consider the principle embedding of the spin-two algebra into $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$.

Infinite Field Content – the Algebra $\mathfrak{hs}(\lambda)$

An important intermediate step towards an understanding of interacting higher-spin fields and the role of higher-spin modes in the tensionless limit of string theory is the introduction of an infinite field content, $N \to \infty$. Since the description in terms of matrices may become involved in that limit (for infinite-dimensional matrix representations of that kind see [149–151]), a different approach should be taken. The theory of an infinite tower of massless fields of increasing spin is formulated as a Chern-Simons theory with gauge algebra $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\lambda)$. I will briefly describe the universal-enveloping-algebra construction of $\mathfrak{hs}(\lambda)$ in the following [152–154].

Starting point is the universal enveloping algebra (UEA) of the spin-two isometry algebra, $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}))$, which is the tensor algebra of $\mathfrak{sl}(2,\mathbb{R})$ modulo the commutation relations (3.1.12), i.e. it may be seen as an algebra build from all possible formal products of generators L_m where combinations that are connected by commutation relations are considered equivalent; see [155, 156] for introductions. There is one second-order Casimir element

$$\mathscr{C} = L_0 L_0 - L_1 L_{-1} + L_0 \tag{3.2.3}$$

that commutes with any element of the UEA. Setting it to a multiple of the identity defines an ideal, here compactly denoted $\langle \mathscr{C} \rangle$, that can be quotiented out[7] to form an associative algebra,[8]

$$\mathfrak{hs}(\lambda) = \frac{\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}))}{\langle \mathscr{C} \rangle}, \qquad \mathscr{C} = \frac{\lambda^2 - 1}{4}. \tag{3.2.4}$$

The parameter $\lambda \in \mathbb{R}$ is a particular choice for the parametrisation of the Casimir element.

It is common to present the algebra in a certain, so called highest-weight basis, in which elements are classified according to their behaviour under the adjoint action of L_m. One defines highest-weight generators as formal powers

$$\mathcal{V}^s_{s-1} := (L_1)^{s-1}, \qquad s \geq 1, \tag{3.2.5}$$

which span the complete set of all elements that commute with L_1. All remaining generators are defined by repeated adjoint action of L_{-1} and suitable pre-factors,

$$\mathcal{V}^s_m := (-1)^{s-1-m} \frac{(s-1+m)!}{(2s-2)!} \mathrm{ad}^{s-1-m}_{L_{-1}} \left(\mathcal{V}^s_{s-1} \right), \qquad -(s-1) \leq m \leq s-1, \tag{3.2.6}$$

[7] I will in the following not distinguish in the notation between the Casimir element itself (as a UEA element) and its parametrisation by a real number – the meaning of the symbol $\mathscr{C}$ should be self explanatory in all circumstances.

[8] The notation will not distinguish between the *associative* algebra $\mathfrak{hs}(\lambda)$, equipped with an associative product, and the *Lie* algebra $\mathfrak{hs}(\lambda)$, equipped with a Lie bracket and without the unit element since this distinction is of no practical relevance here.

where a mode index m was introduced. The associative product in this particular basis will be denoted by "$\star$" (originally dubbed *lone-star product* in [157]). The definition (3.2.6) implies the standard commutation relation

$$\left[\mathcal{V}^s_m, J_n\right]_\star = (m-(s-1)n)\mathcal{V}^s_{m+n}\,, \tag{3.2.7}$$

where $[.,.]_\star$ is the commutator with respect to the star product. It is possible to explicitly work out an expression for the products of arbitrary generators, which reads [157, 158]

$$\mathcal{V}^s_m \star \mathcal{V}^t_n = \sum_{u=0}^{s+t-|s-t|-2} g^{st}_u(m,n;\lambda)\mathcal{V}^{s+t-u-1}_{m+n}\,, \tag{3.2.8}$$

where the structure constants are defined as

$$g^{st}_u(m,n;\lambda) = \frac{\mathcal{N}^{st}_u(m,n)}{4^u u!}\,{}_4F_3\left[\begin{array}{cccc|c} 1/2+\lambda\,, & 1/2-\lambda\,, & 1/2-u/2\,, & -u/2 & \\ 3/2-s\,, & 3/2-t\,, & s+t-1/2-u & & 1\end{array}\right], \tag{3.2.9}$$

with mode functions

$$\mathcal{N}^{st}_u(m,n) = \sum_{k=0}^{u}(-1)^k\binom{u}{k}(s-1+m)^{\underline{u-k}}(s-1-m)^{\underline{k}}(t-1+n)^{\underline{k}}(t-1-n)^{\underline{u-k}}\,. \tag{3.2.10}$$

Here and throughout this entire work I compactly denote with $a^{\underline{k}} = a(a-1)\dots(a-k+1)$ the falling factorial. The commutator with respect to the star product can be used to define a Lie bracket in the usual way. Due to the identity $\mathcal{N}^{ts}_u(n,m) = (-1)^u\mathcal{N}^{st}_u(m,n)$ it reads

$$\left[\mathcal{V}^s_m, \mathcal{V}^t_n\right] = 2\sum_{\substack{u=1\\ u\text{ odd}}}^{s+t-|s-t|-2} g^{st}_u(m,n;\lambda)\mathcal{V}^{s+t-u-1}_{m+n}\,. \tag{3.2.11}$$

Thus, one is provided with a higher-spin Lie algebra $\mathfrak{hs}(\lambda)$, on two copies of which one may define a theory of higher-spin gravity that contains an infinite tower of massless fields of ever increasing spin.

Indeed, it is possible to write down gauge fields that contain an infinite number of higher-spin charges and naturally generalise the BTZ black-hole solution to the case of $\mathfrak{hs}(\lambda)$ [159].

Note that the definition of a theory of higher-spin gravity as Chern-Simons theory without any interactions only requires the existence of a Lie algebra. We will however see in the following subsection how Vasiliev's theory of higher spins and, in particular, the coupling to a massive scalar degree of freedom, requires the underlying associative structure.

3.2.2 Aspects of Vasiliev Theory

This subsection shall provide a quick outline of the linearisation of the fully interacting higher-spin theory, as it was originally constructed by Fradkin, Prokushkin and Vasiliev [78, 79, 82, 83]. I will not discuss the theory in its entirety but only scratch its surface.

The complete nonlinear system of equations consists of the following generating functions: a spacetime one-form W that captures the gauge sector of the theory, a zero-form B describing its matter content and two further zero-forms S_α, $\alpha \in \{1,2\}$, that collect auxiliary fields in order to maintain internal symmetries. These fields are subject to the equations [82]

$$\mathrm{d}W = W \wedge_* W\,, \tag{3.2.12a}$$

$$\mathrm{d}B = [W\,,B]_*\,, \tag{3.2.12b}$$

$$\mathrm{d}S_\alpha = [W\,,S_\alpha]_*\,, \tag{3.2.12c}$$

$$S_\alpha * S^\alpha = -2\mathrm{i}(1+B*K)\,, \tag{3.2.12d}$$

$$[B\,,S_\alpha]_* = 0\,. \tag{3.2.12e}$$

All fields take values in the enveloping algebra that is generated by a set of (non-deformed) oscillator variables [160, 161], denoted y_α and z_α, which fulfil

$$\left[y_\alpha\,,y_\beta\right]_* = 2\mathrm{i}\varepsilon_{\alpha\beta}\,, \qquad \left[z_\alpha\,,z_\beta\right]_* = -2\mathrm{i}\varepsilon_{\alpha\beta}\,, \qquad \left[y_\alpha\,,z_\alpha\right]_* = 0\,, \tag{3.2.13}$$

as well as Clifford elements $\psi_{1/2}$, k and ρ with $\{\psi_i\,,\psi_j\} = \delta_{ij}$, $\{k\,,\rho\} = 0$ and $k^2 = 1 = \rho^2$. The product denoted "$*$" is a Moyal product [162, 163]. Finally, K is a Klein operator given by $K = k\,\mathrm{e}^{\mathrm{i}z_\alpha y^\alpha}$.

Without giving any further details about the properties of the above system of equations and its mathematical constitution, let me proceed by stating that this system can be linearised around the vacuum, which is the solution $S^{(0)}_\alpha = \rho z_\alpha$ and $B^{(0)} = \nu$, with some constant[9] ν, while the gauge field $W^{(0)}$ can be projected[10] to the Chern-Simons gauge fields A and $\bar{A}$ and its equation of motion results in the flatness conditions (3.1.5) for these fields. A linear fluctuation is induced by writing

$$B = \nu + \mathcal{C}\,. \tag{3.2.14}$$

Then the resulting equations for $\mathcal{C}$ can be evaluated by splitting it into a dynamical and an auxiliary part with the help of the Clifford element ψ_2, and finally projecting the dynamical

[9]There are different ways to present a solution; in this case one is actually led to an algebra of *deformed* oscillators that is of the form $[y_\alpha\,,y_\beta]_* = 2\mathrm{i}\varepsilon_{\alpha\beta}(1+\nu k)$, which is needed to construct the background solutions. For a proper treatment the reader is well advised to consult [82].

[10]Projection operators can be build from the Clifford element ψ_1 as $\mathscr{P}_\pm = (1\pm\psi_1)/2$.

part to two conjugate fields C and $\bar{C}$. These fields then fulfil the linear equations [71]

$$\mathrm{d}C + A * C - C * \bar{A} = 0\,, \qquad \mathrm{d}\bar{C} + \bar{A} * \bar{C} - \bar{C} * A = 0\,. \tag{3.2.15}$$

Note that, up to now, both the matter fields C, $\bar{C}$ and the gauge fields A, $\bar{A}$ are still elements of the enveloping algebra of oscillators y_α, z_α.

As it turns out, the Moyal product used in linearised Vasiliev theory can be identified with the associative star product of $\mathfrak{hs}(\lambda)$ introduced in the previous subsection, if the identification $\lambda = (1-\nu)/2$ is made [71, 154, 164, 165]. That is, both the matter fields and the gauge fields may be seen as elements of $\mathfrak{hs}(\lambda)$ and the equations (3.2.15) can be evaluated using the product rules (3.2.8). It was shown in [71] that the unfolded equations (3.2.15) can be reduced to Klein-Gordon equations for the scalar fields appearing as coefficients of the unit element in the expansion of C and $\bar{C}$, where the background geometry is given by A and $\bar{A}$ and the masses of the scalars need to be identified as $m^2 = \lambda^2 - 1$.

3.2.3 Higher Spins in Flat Space: State of the Art

The formulation of interacting (massless) higher-spin theories in flat spacetimes, if existing, is involved, since a variety of no-go theorems apply. Yet, there have been various attempts in the past to approach this issue. I will briefly review some of the recent findings in the context of the three-dimensional theory, excluding the insights that evolved around the project presented here.

İnönü-Wigner Contractions

A common path to obtain physics in asymptotically flat spacetimes, including their higher-spin generalisations, consists in taking the flat limit of known AdS results, i.e. sending the cosmological constant to zero, $\Lambda \to 0$, respectively the radius of curvature to infinity, $l \to \infty$ [134, 135, 166–170].

On the level of classical isometry algebras, such a limit is known as İnönü-Wigner contraction [171] and in the case of three-dimensional gravity provides a transition $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{isl}(2,\mathbb{R})$. Given the two copies of $\mathfrak{sl}(2,\mathbb{R})$ with Lie brackets (3.1.12), there are the following two possibilities:

- **Galilean Contraction:** Define new generators J_m and P_m by the combinations

$$J_m = L_m + \bar{L}_m\,, \qquad P_m = \varepsilon\left(L_m - \bar{L}_m\right). \tag{3.2.16}$$

  Then in the limit $\varepsilon \to 0$, these generators fulfil the $\mathfrak{isl}(2,\mathbb{R})$ algebra (3.1.19). This contraction is also called the *non-relativistic* limit.

- **Carrollian Contraction:** Define new generators J_m and P_m by the combinations

$$J_m = L_m - \bar{L}_{-m}\,, \qquad P_m = \varepsilon\left(L_m + \bar{L}_{-m}\right). \tag{3.2.17}$$

  Then in the limit $\varepsilon \to 0$, these generators fulfil the $\mathfrak{isl}(2,\mathbb{R})$ algebra (3.1.19), as well. This contraction is also called the *ultra-relativistic* limit.

Both contractions yield the same result, which is a coincidence of the three-dimensional case (the map $L_m \mapsto -L_{-m}$ leaves the $\mathfrak{sl}(2,\mathbb{R})$-commutation relations unaffected). There is, however, a manifest difference on the level of representation theory, since Galilean contractions typically result in non-unitary, Carrollian ones in unitary representations.

Note that both kinds of contractions can (on the algebra level) equally well be performed by introduction of a Grassmann-valued parameter that squares to zero, instead of taking a limit [172]. Then the translation generators P_m are of odd parity.

Though contractions from AdS to flat may be useful in order to check the consistency of flat-space calculations with known AdS results and in some circumstances provide a quick, ad-hoc possibility to derive quantities, both in a flat-space gravitational (higher-spin) theory and in a corresponding boundary field theory, such limiting procedures may be subtle or not even well defined. For example, it is long known that higher-spin interactions in 3+1 dimensions lead to a singular flat-space limit, since these interactions contain powers of the inverse cosmological constant at cubic order [78, 79].

Spin-Three Chern-Simons Theory

As may be expected, it is possible to find a generalisation of the Chern-Simons formulation of asymptotically flat gravity based on the gauge algebra $\mathfrak{isl}(2,\mathbb{R})$, as presented in subsection 3.1.3, to a finite-spin theory, particularly to the case of spin three [140, 173]. The algebra $\mathfrak{isl}(3,\mathbb{R})$ may be defined by the Lie brackets

$$[J_m, J_n] = (m-n)J_{m+n}\,, \tag{3.2.18a}$$

$$[J_m, P_n] = (m-n)P_{m+n}\,, \tag{3.2.18b}$$

$$[J_m, U_n] = (2m-n)U_{m+n}\,, \tag{3.2.18c}$$

$$[J_m, V_n] = (2m-n)V_{m+n}\,, \tag{3.2.18d}$$

$$[P_m, U_n] = (2m-n)V_{m+n}\,, \tag{3.2.18e}$$

$$[U_m, U_n] = (m-n)\left(2m^2+2n^2-mn-8\right)J_{m+n}\,, \tag{3.2.18f}$$

$$[U_m, V_n] = (m-n)\left(2m^2+2n^2-mn-8\right)P_{m+n}\,, \tag{3.2.18g}$$

where all remaining brackets are vanishing. The indices take values $m \in \{0,\pm1\}$ in J_m, P_m and $m \in \{0,\pm1,\pm2\}$ in U_m, V_m. Apparently, $\mathfrak{isl}(2,\mathbb{R})$ is contained as a Lie subalgebra. The Lie algebra (3.2.18) can be equipped with a bilinear form of the type $\langle J_m, P_n\rangle \sim \delta_{m+n,0}$ (as in the spin-two case) and $\langle U_m, V_n\rangle \sim \delta_{m+n,0}$.

One may then propose boundary conditions generalising the gauge fields (3.1.20) by introducing spin-three charges $Z(\phi)$ and $W(u,\phi)$ as [140]

$$\omega = \left(J_1 - \frac{M(\phi)}{4}J_{-1} - \frac{Z(\phi)}{4}U_{-2}\right)d\phi\,, \tag{3.2.19a}$$

$$\begin{aligned} e = &\left(P_1 - \frac{M(\phi)}{4}P_{-1} - \frac{Z(\phi)}{4}V_{-2}\right)du + \frac{1}{2}P_{-1}dr \\ &+ \left(rP_0 - \frac{N(u,\phi)}{2}P_{-1} - \frac{W(u,\phi)}{2}V_{-2}\right)d\phi\,, \end{aligned} \tag{3.2.19b}$$

where gauge flatness requires $\partial_\phi Z(\phi) = 2\partial_u W(u,\phi)$. Apart from the metric field $g_{\mu\nu}$ the theory now contains an additional spin-three field $\phi_{\mu\nu\rho}$, which reads

$$\phi_{\mu\nu\rho}dx^\mu dx^\nu dx^\rho = -Z(\phi)du^3 - 2W(u,\phi)du^2 d\phi\,. \tag{3.2.20}$$

The Lie algebra (3.2.18) can be derived as an İnönü-Wigner contraction of two copies of $\mathfrak{sl}(3,\mathbb{R})$ and, furthermore, the gauge fields (3.2.19) can be found as a limit of the spin-three boundary conditions in the AdS case [174].

Infinite Field Content

Though it should in principle be possible to define algebras $\mathfrak{isl}(N,\mathbb{R})$ for any finite spin N, the situation is unsatisfying as long as there is no prescription for infinite towers of massless higher-spin fields. Especially, in the course of circumventing no-go theorems the infinite set-up is expected to play a crucial role.

The lack of a suitable flat-space counterpart to the AdS higher-spin algebra $\mathfrak{hs}(\lambda)$ was attempted to be rectified in the literature by performing an İnönü-Wigner contraction from $\mathfrak{hs}(\lambda)\oplus\mathfrak{hs}(\lambda)$ to a Lie algebra that may be called $\mathfrak{ihs}(\lambda)$. In analogy to (3.2.16), define [170, 175, 176]

$$V^s_m = \mathcal{V}^s_m + \bar{\mathcal{V}}^s{}_m\,, \qquad W^s_m = \varepsilon\left(\mathcal{V}^s_m - \bar{\mathcal{V}}^s{}_m\right). \tag{3.2.21}$$

Using (3.2.11), in the limit $\varepsilon \to 0$ these definitions imply the commutation relations

$$\left[V^s_m, V^t_n\right] = 2\sum_{\substack{u=1\\u\text{ odd}}}^{s+t-|s-t|-2} g^{st}_u(m,n;\lambda)\,V^{s+t-u-1}_{m+n}\,, \tag{3.2.22a}$$

$$\left[V^s_m, W^t_n\right] = 2\sum_{\substack{u=1\\u\text{ odd}}}^{s+t-|s-t|-2} g^{st}_u(m,n;\lambda)\,W^{s+t-u-1}_{m+n}\,, \tag{3.2.22b}$$

$$\left[W^s_m, W^t_n\right] = 0\,. \tag{3.2.22c}$$

A Chern-Simons theory living on this Lie algebra can be seen as a candidate for a theory of higher-spin gravity in asymptotically flat spacetimes, but there are several issues with this ansatz, some of which I will discuss in the following.

First note that the definition (3.2.21) is merely an educated guess, originating in comparison with the spin-two case. In general, one could define different, more complicated linear combinations or switch to a different basis. A priori, it is not clear that a limit taken from the direct sum $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\lambda)$ does result in a meaningful flat-space algebra (see the discussion in section 4.3, in which the limit is discussed in the universal-enveloping-algebra picture).

A second remark concerns the role of the parameter λ: in the generic case it is possible to choose different values of the two Casimir elements appearing in the different copies of the AdS higher-spin algebra, i.e. to consider $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\bar{\lambda})$ with $\lambda \neq \bar{\lambda}$. In that case, however, the simple contraction (3.2.21) does not work, because it is not possible to get rid of divergent contributions without further, more complicated re-scalings. Therefore one free parameter is missing in the resulting theory, thus limiting its applicability to begin with.

Finally, in the light of the role played by the associative product of $\mathfrak{hs}(\lambda)$ in the linearised version of Vasiliev theory, one would like to be in possession of an associative product in the flat-space case, as well. Accordingly, one may try and perform the same contraction (3.2.21) on the level of the associative algebra, which indeed gives a closed structure. It will however turn out that this construction does not allow non-trivial equations of motion for the matter sector as far as a coupling equation related to (3.2.15) is considered, which is due to the vanishing of the products $W^s_m \star W^t_n = 0$.

3.3 Aspects of Holography

The exploration of the holographic principle in the case of three-dimensional gravitational theories and their two-dimensional field-theory duals has been an active research area in recent years and continues to be of vital importance to our understanding of fundamental holographic working mechanisms as well as to the attempt to postulate possible models of quantum gravity.

The holographic duality, in its strong version connecting boundary quantum field theories to theories of quantum gravity in the bulk, reduces in a semi-classical limit to a correspondence involving classical gravity. It is thus a first step in the development of holographic dictionaries to identify which quantities in gravitational theories compute which quantities in the (semi-classical limit of) corresponding field theories. It is the goal of this section to give a brief summary of some previously obtained results from the gravity point of view.

3.3.1 Asymptotic Symmetries

At the heart of the known holographic correspondences lies a matching of symmetries. More precisely, the asymptotic symmetries of a class of spacetimes simultaneously serve as the symmetries underlying the dual field theory. Accordingly, from the three-dimensional gravity perspective, a first step towards holographic dualities is to study the symmetries that are implied by our definitions of asymptotically AdS and asymptotically flat spacetimes, as given in section 3.1.

There are various mathematical frameworks to explicitly calculate the asymptotic symmetry algebra implied by a choice of boundary conditions. Since we are here studying three-dimensional gravity, the analysis can be carried out in the Chern-Simons formalism. To be more precise, the Chern-Simons theory can be treated as a constraint Hamiltonian system (see [177, 178] for basic introductions), the symmetry algebra of which is to be determined.

Asymptotic Symmetries of AdS

The boundary conditions (3.1.14) contain the charges $\mathscr{L}$ and $\bar{\mathscr{L}}$ as free functions. These may be expanded into Fourier modes L_m, $\bar{L}_m$ with respect to the angular coordinate ϕ (since the boundary topology is that of a cylinder), the zero-modes L_0 and $\bar{L}_0$ being related to mass and angular momentum of the respective spacetime. An analysis of the constraints then results in an asymptotic symmetry algebra given through the Dirac brackets [84, 127]. From canonical quantisation, i.e. replacing the Dirac brackets by commutators, follows the respective quantum version.

The Lie brackets of the asymptotic symmetry algebra are[11]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m\left(m^2-1\right)\delta_{m+n,0}\,, \tag{3.3.1a}$$

$$\left[L_m, \bar{L}_n\right] = 0\,, \tag{3.3.1b}$$

$$\left[\bar{L}_m, \bar{L}_n\right] = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12} m\left(m^2-1\right)\delta_{m+n,0}\,, \tag{3.3.1c}$$

where $m, n \in \mathbb{Z}$. This algebra is known as the two-dimensional conformal algebra and it consists of two copies of the so-called Virasoro algebra, $\mathfrak{vir}_2 \oplus \mathfrak{vir}_2$. The global symmetry algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ is apparently contained as the subalgebra spanned by L_m and $\bar{L}_m$ for $m \in \{0, \pm 1\}$. A quantum field theory that possesses the conformal algebra, respectively the conformal group generated from this algebra, as underlying symmetry is called a conformal field theory (CFT).

In the context of Einstein gravity discussed here, $c = \bar{c}$ and the parameter c, called central

[11] Here, a shift in the zero-mode generators L_0 and $\bar{L}_0$ has been performed to achieve the standard form of the central extension.

charge of the Virasoro algebra, is related to Newton's constant through [125]

$$c = \frac{3l}{2G_\mathrm{N}}\,. \tag{3.3.2}$$

This remarkable identification shows how a semi-classical limit on the gravity side, i.e. $G_\mathrm{N} \to 0$, is implemented as a large-central charge limit in the dual CFT. As an example, in the specific case of the BTZ black hole, the central charge appearing in (3.3.1) is an important ingredient in a microscopic derivation of black-hole entropy [179], since it can be related to a counting of micro states by application of Cardy's formula [180]. This provides a starting point for the calculation of quantum corrections to black-hole entropy [181].

Note that the addition of a Lorentz-Chern-Simons term to the Einstein-Hilbert action, i.e. the study of topologically massive gravity [182–185], leads to a difference in the central charges, $c \neq \bar{c}$, in the asymptotic analysis [186, 187]. A similar statement holds true for gravity with torsion [188].

An analogous derivation of asymptotic symmetries for the case of higher-spin gravity, in particular the case of spin three, has been performed in [84], where it was shown that an $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$ Chern-Simons theory will result in (two copies of) a $\mathscr{W}_3$ symmetry algebra[12] with the same central charge (3.3.2) as in the spin-two case. Similarly, the spin-N theory possesses $\mathscr{W}_N \oplus \mathscr{W}_N$ with the same central charge as asymptotic symmetry algebra [84, 143] and the gauge algebra $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\lambda)$ leads to a central extension of two copies of the $\mathscr{W}_\infty$ algebra [157, 189] as shown in [190].

Asymptotic Symmetries of Flat Spacetimes

The same procedure described for the derivation of asymptotic symmetries of AdS spacetimes can be applied to flat spacetimes. The boundary conditions (3.1.20) on the gauge fields contain the charges M and N, which are to be expanded into Fourier modes L_m and M_m with respect to the angular coordinate ϕ (note that the boundary topology at $\mathscr{I}^+$ or $\mathscr{I}^-$ is again that of a cylinder), with the zero-modes L_0 and M_0 being related to mass and angular momentum of the respective spacetime. Then a canonical quantisation of the Dirac brackets may be performed.

The asymptotic symmetry algebra is given by the Lie brackets[13]

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_\mathrm{L}}{12}m(m^2-1)\delta_{m+n,0} \tag{3.3.3a}$$

$$[L_m, M_n] = (m-n)M_{m+n} + \frac{c_\mathrm{M}}{12}m(m^2-1)\delta_{m+n,0}\,, \tag{3.3.3b}$$

$$[M_m, M_n] = 0\,, \tag{3.3.3c}$$

[12] Again, here we are only concerned with the principle embedding.

[13] A shift in the zero-mode generators L_0, M_0 has to be performed to arrive at the standard form of the central extension.

where $m, n \in \mathbb{Z}$. This algebra is known as the $\mathfrak{bms}_3$ algebra, dubbed after Bondi, van der Burg, Metzner and Sachs, who first discussed the asymptotic symmetries of flat spacetimes [191–194]. The L_m and M_m are generators of so-called super-rotations and super-translations, respectively. The global symmetry algebra $\mathfrak{isl}(2,\mathbb{R})$ is contained as the subalgebra spanned by L_m and M_m for $m \in \{0, \pm 1\}$. A quantum theory that has underlying $\mathfrak{bms}_3$ symmetry will be called BMS field theory or Carrollian field theory [66–68].

In the case of Einstein gravity discussed here, one of the central charges is actually vanishing, $c_{\mathrm{L}} = 0$, while the other one is to be identified with Newton's constant as [132]

$$c_{\mathrm{M}} = \frac{1}{4G_{\mathrm{N}}}. \tag{3.3.4}$$

Note that a non-vanishing central charge $c_{\mathrm{L}} \neq 0$ can be obtained by considering a chiral deformation of Einstein gravity [195, 196], similar to the case of topologically massive gravity in AdS.

Moreover, asymptotic symmetries of higher-spin generalisations have been discussed in the literature. In [140, 173] the asymptotic symmetry algebra $\mathscr{FW}_3$ of the spin-three generalisation presented in subsection 3.2.3 was derived, both through asymptotic analysis and by contraction of the AdS algebra $\mathscr{W}_3 \oplus \mathscr{W}_3$. A definition of an asymptotic symmetry algebra $\mathscr{FW}_\infty$ belonging to the $\mathfrak{ihs}(\lambda)$-higher spin theory discussed above was proposed in terms of an İnönü-Wigner contraction in [170, 175].

3.3.2 Holographic Probes

Given a field theory defined on the boundary of a spacetime, invariant under the respective asymptotic symmetry of the gravitational theory, i.e. a CFT in the case of AdS or a Carrollian field theory in the case of flat space, the question comes up how observables of the field theory (in a semi-classical limit) arise on the gravity side. Within this subsection I will present a few particular examples.

Holographic Entanglement Entropy

A well known observable in a generic field theory is entanglement entropy [197–199]. Consider a spatial interval A (in higher dimensions a spatial region) in the field theory, to which a reduced density matrix ρ_{A} may be assigned. Then the von-Neumann entropy of this reduced density matrix is the entanglement entropy associated to the interval A.

The computation of this quantity, though typically being a highly non-trivial task in a generic quantum field theory, can be performed holographically using the Ryu-Takayanagi prescription [200–203]. In general dimensions, the idea is to compute the area of a minimal surface of codimension two hanging into the radial direction of some higher-dimensional space and anchored to the boundary of the spatial region A.

In the case of an interval in a two-dimensional CFT, the entanglement entropy is computed by a so-called Wilson line attached to the endpoints of the interval A and hanging into the bulk spacetime. More precisely, following [113], the entanglement entropy of an interval A is computed as the logarithm of the Wilson-line operator $W_{\mathcal{R}}$ in a particular representation $\mathcal{R}$,

$$S_{\text{EE}} = -\ln\left(W_{\mathcal{R}}\right), \qquad W_{\mathcal{R}} = \text{tr}_{\mathcal{R}}\left(\mathscr{P}\exp\int_{\mathcal{C}} A\right). \tag{3.3.5}$$

Here $\mathcal{C}$ denotes the path of the Wilson line through the spacetime and the representation $\mathcal{R}$ is chosen as the Hilbert space of an auxiliary quantum mechanical system living on the Wilson line – more insights into the relation of Wilson lines in AdS to Hilbert-state representations can be found in [115]. In the initial calculation of [113] the representation was chosen to be an infinite-dimensional highest-weight representation, given by the dynamics of a single particle living on the $SL(2,\mathbb{R})$ group manifold and the trace was evaluated as a euclidean path integral. Setting appropriate boundary conditions on the probe field and using a saddle-point approximation (in accordance with the semi-classical limit), it could be shown that the entanglement entropy is calculated as the on-shell value of the probe action.

The result for the on-shell action in a limit where the holographic coordinate is taken to infinity is precisely the entanglement entropy of a CFT in a thermal state [204]. Furthermore, it was shown that the Wilson line can be wound around a BTZ black hole to compute its Bekenstein-Hawking entropy.

The Wilson-line calculation here described has been applied in a variety of different holographic set-ups: While the generalisation to the spin-three case has already been performed in [113], the entanglement entropy of BMS field theories at zero temperature has been obtained from three-dimensional Minkowski spacetime [60], as well as for finite temperature from flat-space cosmologies [114]. In the latter work also a generalisation to flat-space spin-three gravity was introduced. Note however, that these calculations were performed in a non-relativistic, non-unitary set-up. A revised prescription for flat-space Wilson-line calculations will be given in section 5.3 of the main part of this work.

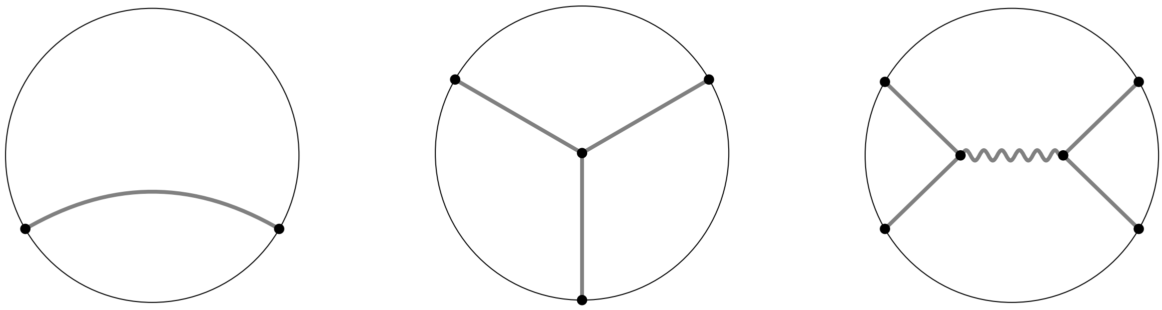
Holographic Correlation Functions

Entanglement entropy is only one example of a bi-local observable that can be computed by the help of Wilson lines in the two-dimensional case. A more general set of CFT observables comes in the form of n-point correlation functions of local operators $\mathcal{O}(z)$, i.e. objects of the form $\langle\mathcal{O}(z_1)\mathcal{O}(z_2)\dots\mathcal{O}(z_n)\rangle$, here scalar operators for simplicity.[14]

The holographic counterpart of such an n-point correlator in the bulk of the spacetime is an arrangement of n interacting (scalar) fields, whose interaction in the semi-classical limit is of perturbative nature and can be represented by Witten diagrams [208–211]. Let me

[14]For basic introductions to two- or higher-dimensional CFTs, see e.g. [205–207].

elaborate on the cases $n = 2$, $n = 3$ and $n = 4$. In the first case, the Witten diagram consists of a (scalar) boundary-boundary propagator that connects the operator insertions, as can be seen from figure 3.2a. In the case of a heavy scalar, i.e. $ml \gg 1$, this propagator relates to the length of the geodesic connecting both points on the boundary. In figure 3.2b the case $n = 3$ is sketched, in which three bulk-boundary propagators are joint in a bulk vertex that needs to be integrated over. In the case $n = 4$ the situation becomes more involved since the Witten diagram may contain a bulk-bulk propagator connecting two vertices, see figure 3.2c. In order to compute the four-point correlation function, the sum over all such intermediate fields needs to be performed – in general also higher-spin fields are included (for example in case of a free CFT).



(a) Witten diagram in the case $n = 2$.

(b) Witten diagram in the case $n = 3$.

(c) Witten diagram in the case $n = 4$.

Figure 3.2: Witten diagrams depicted for the cases of two-, three- and four-point correlation functions. The periphery of circles corresponds to the boundary of AdS.

Results on three- and four-point correlation functions of CFTs with a $\mathcal{W}$-symmetry have been obtained in various instances in the literature [70–72, 150, 212, 213], building on a proposal by Klebanov and Polyakov [69, 214, 215] that connects such correlators to a subsector of Vasiliev higher-spin theory. The presence of scalar fields in the bulk theory is a necessary ingredient, since it gives rise to primary operators in the CFT.

While correlation functions depend on the specific microscopics of the CFT at hand, conformal blocks only depend on the representation and therefore can be regarded as fundamental building blocks of any CFT. A modification of the Witten-diagram prescription also allows the calculation of conformal blocks in the semi-classical limit [216–218].

Analogously to AdS/CFT the n-point correlation functions of stress-tensor components of Carrollian field theories have been calculated [219]. Furthermore, computations of global (Poincaré) blocks [220] and of $\mathfrak{bms}_3$-blocks [221] have been performed. However, in these cases only one part of the asymptotic boundary (either past or future light-like infinity) have been included.

Chapter 4

An Algebraic Approach to Flat-Space Higher-Spin Symmetry

This first chapter of the thesis' main part is meant to cover the mathematical footing which the path to higher-spin physics here taken is build upon. It describes the algebraic construction developed for the case of three-dimensional asymptotically flat spacetimes and summarises properties of the obtained algebraic structures.

Main emphasis shall be laid on the construction of a higher-spin algebra based on a quotient of the universal enveloping algebra (UEA) of the classical isometries, which are embodied by the Poincaré algebra in three spacetime dimensions $\mathfrak{iso}(2,1) \simeq \mathfrak{isl}(2,\mathbb{R})$. This is indeed a well-known recipe for the step from classical to higher-spin physics, even in higher dimensions, see e.g. [222, 223].

The intention of this study is twofold: First, it provides a well-defined notion of flat-space higher-spin gravity in terms of a Chern-Simons theory, namely it produces as a sub-structure a Lie algebra which the corresponding gauge fields can take values in. Second, it will later on be shown that the associative structure obtained here serves as an indispensable ingredient to the introduction of massive matter fields (of any spin) to the gravitational theory.

This chapter contains results of [1].

4.1 An Associative Higher-Spin Algebra

In the case at hand we shall introduce a quotient of the UEA of the isometry algebra $\mathfrak{isl}(2,\mathbb{R})$. This quotient is taken with respect to the ideal generated by the second-order Casimir elements of the algebra, which are called mass squared $\mathscr{M}^2$ and spin $\mathbb{S}$. I propose to call this algebra $\mathfrak{ihs}(\mathscr{M}^2,\mathbb{S})$,

$$\mathfrak{ihs}(\mathscr{M}^2,\mathbb{S}) := \frac{\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))}{\langle \mathscr{M}^2,\mathbb{S}\rangle}. \tag{4.1.1}$$

In the following we will first study the UEA of $\mathfrak{isl}(2,\mathbb{R})$ and in a next step define the generators of its quotient in a certain basis, called highest-weight basis. These generators will then be equipped with a star product.

4.1.1 The Universal Enveloping Algebra of $\mathfrak{isl}(2,\mathbb{R})$

As introduced in subsection 3.1.3 of the Foundations, the Lie algebra $\mathfrak{isl}(2,\mathbb{R})$ is spanned by elements J_m and P_m with mode indices $m \in \{0,\pm 1\}$ and the defining Lie brackets (3.1.19). It has the structure of a semi-direct sum of Lorentz transformations and translations, $\mathfrak{isl}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \inplus \mathbb{R}^3$, and since the translations P_m form a non-trivial ideal of $\mathfrak{isl}(2,\mathbb{R})$, this Lie algebra is not semi-simple.

The (canonical) universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is generally defined as the quotient of the tensor algebra $\bigotimes(\mathfrak{g})$ with the ideal $\mathscr{I}$ generated by the commutation relations of $\mathfrak{g}$, i.e. $\mathscr{U}(\mathfrak{g}) = \bigotimes(\mathfrak{g})/\mathscr{I}$, see [155, 156] for neat introductions. Accordingly, one may think of $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$ as consisting of all *formal* products of generators J_m, P_m (elements of the tensor algebra), modulo identification of combinations that are related by commutation relations, in the sense of applying the commutator with respect to the formal product.

The Poincaré-Birkhoff-Witt theorem then allows us to choose an ordering relation for basis elements of the UEA, which basically amounts to choosing a representative of the respective equivalence class. Here I will choose to sort translational generators to the right and order both rotational and translational generators in descending order of indices. That is, a basis of $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$ is given by monomials of the form[1]

$$(J_1)^a (J_0)^b (J_{-1})^c (P_1)^d (P_0)^e (P_{-1})^f\,, \qquad a,b,c,d,e,f \in \mathbb{N}_0\,. \tag{4.1.2}$$

By simple combinatorial considerations it is possible to write down the necessary relations to express an arbitrary monomial in terms of a sum of ordered products, i.e. to write down product rules for basis elements (4.1.2). These relations are given in section A.1 of the Appendix, for a combinatorial derivation in the case of $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}))$ see [224].

The structure of $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$ can be depicted as an infinite wedge: counting the overall number of generators involved in a formal product by an index $s := a+b+c+d+e+f+1$ and the number of rotational generators by an index $l := a+b+c$, such that $s \geq 1$ and $0 \leq l \leq s-1$, while suppressing mode indices, we see how the algebra grows with rising s, in the sense that the allowed range of the index l is becoming larger. Moreover, it is apparent from the construction that the algebra $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}))$, from which the algebra $\mathfrak{hs}(\lambda)$ is build in the AdS case (see section 3.2 of the Foundations) is a subalgebra of $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$, consisting of elements with $l = s-1$. This situation is schematically depicted in figure 4.1.

A further comment regards the assignment of units to generators. Picturing a realisation of Lorentz transformations and translations in terms of differential operators, one would tend to view the generators J_m to be dimensionless, while the generators P_m were of inverse

[1] From here on, powers of generators denote powers with respect to the formal product.

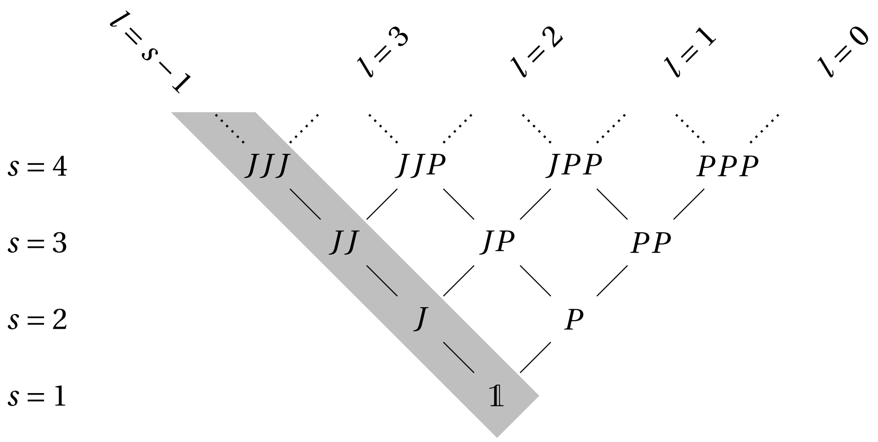


Figure 4.1: Visualisation of the infinite-wedge structure of $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$. Mode indices are suppressed. The subalgebra $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}))$ is depicted in grey.

length dimension one. This property is reflected in the semi-direct sum structure of $\mathfrak{isl}(2,\mathbb{R})$ and is as such carried over to $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$, such that a generic element of the UEA carries an inverse length dimension of $s-1-l$. This assignment proves helpful in many circumstances throughout the present work.

The second-order objects

$$\mathscr{M}^2 = (P_0)^2 - P_1 P_{-1}\,, \qquad\qquad \mathcal{S} = J_0 P_0 - \frac{1}{2}(J_1 P_{-1} + J_{-1} P_1) \tag{4.1.3}$$

are the Casimir elements of the algebra. They commute with all elements of $\mathfrak{isl}(2,\mathbb{R})$ and, consequently, with all elements of $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$. As such, they behave like a multiple of the identity and will be divided out to finally define $\mathfrak{ihs}(\mathscr{M}^2,\mathcal{S})$. The notation used here will not distinguish between these objects seen as elements of the UEA or just as real numbers (plus length scale), for the meaning should always be clear from the context.

The transition to the quotient algebra can be achieved on the level of formal products by choosing one respective representative of the equivalence classes defined through (4.1.3) and eliminating it whenever it appears; in view of the chosen ordering relation one may formally employ the identifications $(P_0)^2 \sim P_1 P_{-1} + \mathscr{M}^2$ and $J_{-1}P_1 \sim 2J_0P_0 - J_1P_{-1} - 2\mathcal{S}$. Note that such identifications need to be consistent, e.g. eliminating $(P_0)^2$ and J_0P_0 simultaneously cannot be consistent with associativity of the formal product.

4.1.2 Highest-Weight Basis and Star Product

A particular basis of $\mathfrak{ihs}(\mathscr{M}^2,\mathcal{S})$ that will turn out to be useful in this work can be obtained by classification of the algebra elements according to their behaviour under the adjoint action of Lorentz generators J_m. First identify all linearly independent elements that commute with

J_1 – these shall be called highest-weight generators[2] – and then define so-called descendant generators by repeated application of the adjoint action of J_{-1}.

For a classification of highest-weight generators one may start with the most obvious set of elements, namely the combinations of maximal mode-index sum

$$\left\{ (J_1)^l (P_1)^{s-1-l} \,\middle|\, s \geq 1,\ 0 \leq l \leq s-1 \right\}. \tag{4.1.4}$$

Next, since we have $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}))$ as a subalgebra of $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$ it is clear that the Casimir element of the former, which is not affected by the quotienting, is still an element of the algebra $\mathfrak{ihs}(\mathscr{M}^2,\mathbb{S})$ and commutes with all Lorentz generators, which implies that a further contribution to the set of highest-weight generators comes from powers of the element

$$\mathscr{C} \equiv (J_0)^2 - J_1 J_{-1} + J_0 . \tag{4.1.5}$$

Finally, the search for additional highest-weight elements of second order reveals the combination $J_0 P_1 - J_1 P_0$ as such an object, which is indeed linearly independent of all previous ones but will only contribute to linear order since

$$(J_0 P_1 - J_1 P_0)^2 = \mathscr{M}^2 (J_1)^2 - 2\mathbb{S} J_1 P_1 + \mathscr{C}(P_1)^2 . \tag{4.1.6}$$

Taking everything together, I propose the following definition of highest-weight generators:

$$\boxed{{}^{l}_{\xi}\mathbf{Q}^{s}_{s-1-\xi} := \begin{cases} (J_1)^{l-\xi}\mathscr{C}^{\left\lfloor \frac{\xi}{2} \right\rfloor}(P_1)^{s-1-l}, & \xi \text{ even}, \\ (J_1)^{l-\xi}\mathscr{C}^{\left\lfloor \frac{\xi}{2} \right\rfloor}(J_0 P_1 - J_1 P_0)(P_1)^{s-2-l}, & \xi \text{ odd}. \end{cases}} \tag{4.1.7}$$

The indices take integer values within ranges that are apparently given by

$$s \geq 1, \qquad 0 \leq \xi \leq 2\left\lfloor \frac{s-1}{2} \right\rfloor, \qquad \xi \leq l \leq s-1-\left(\xi - 2\left\lfloor \frac{\xi}{2} \right\rfloor\right) \tag{4.1.8}$$

and I will refer to ξ as the *level* of a generator.

We can now turn to the definition of descendant generators, which will be defined by the adjoint action of J_{-1}, modulo normalisation factors. Taking

$$\boxed{{}^{l}_{\xi}\mathbf{Q}^{s}_{m} := (-1)^{s-1-\xi-m}\frac{(s-\xi+m-1)!}{(2s-2\xi-2)!}\,\mathrm{ad}^{s-1-\xi-m}_{J_{-1}}\left({}^{l}_{\xi}\mathbf{Q}^{s}_{s-1-\xi}\right),} \tag{4.1.9}$$

where the mode index m is restricted to $|m| \leq s-1-\xi$, these definitions indeed cover all possible linearly independent elements of $\mathfrak{ihs}(\mathscr{M}^2,\mathbb{S})$, as one may verify by simple counting

[2]Occasionally, I will use the phrasing *lowest-weight generators* for elements that commute with J_{-1}.

arguments. From this construction one can, in principle, define a star product as the associative multiplication of generators[3],

$$ {}^{k}_{\xi}\mathbf{Q}^{s}_{m} \star {}^{l}_{\eta}\mathbf{Q}^{t}_{n} = \sum_{\lambda,\sigma,\zeta} \Gamma^{\zeta}_{\lambda,\sigma}(k,l;s,t;\xi,\eta;m,n)\, {}^{\lambda}_{\zeta}\mathbf{Q}^{\sigma}_{m+n}\,. \qquad (4.1.10) $$

Unfortunately, the derivation of closed-form expressions for the structure constants $\Gamma^{\zeta}_{\lambda,\sigma}$ is unreasonably complicated and no complete solution could be found within the scope of the present work. However, for the applications presented here it is actually not necessary to have full control over the associative algebra. It was possible to find expressions for star products in which at least one of the factors is a single $\mathfrak{isl}(2,\mathbb{R})$-generator (i.e. $s=2$ or $t=2$ in equation (4.1.10)), the full expressions, although rather unwieldy, can be found in section A.2.2 of the Appendix.

Moreover, the above construction automatically implies a standard form of the star-commutator with J_n,

$$ \left[{}^{l}_{\xi}\mathbf{Q}^{s}_{m}, J_n \right]_{\star} = (m-(s-1-\xi)n)\; {}^{l}_{\xi}\mathbf{Q}^{s}_{m+n}\,. \qquad (4.1.11) $$

As we will see later, this can be interpreted as the transformation behaviour of fields associated to generators ${}^{l}_{\xi}\mathbf{Q}^{s}_{m}$ under Lorentz transformations, namely as transformation in the $(2s-2\xi-1)$-dimensional adjoint representation, the associated fields therefore being of spin $s-\xi-1$.

For generators living in the outer right slice of the associative algebra, i.e. generators that carry an index $l=0$ or $l=1$, it is actually possible to derive closed-form expressions for their product. The easiest example is given by products of purely translational generators, which I reproduce here for later use:

$$ {}^{0}_{0}\mathbf{Q}^{s}_{m} \star {}^{0}_{0}\mathbf{Q}^{t}_{n} = \sum_{u=0}^{\lfloor\frac{s+t-2}{2}\rfloor} \frac{(-1)^u \mathcal{M}^{2u}}{4^{2u}u!} \frac{\mathcal{N}^{st}_{2u}(m,n)}{(s-3/2)^{\underline{u}}(t-3/2)^{\underline{u}}(s+t-u-3/2)^{\underline{u}}}\; {}^{0}_{0}\mathbf{Q}^{s+t-1-2u}_{m+n}\,. \qquad (4.1.12) $$

The mode functions $\mathcal{N}^{st}_{u}(m,n)$ are defined in equation (3.2.10). More product rules may be found in section A.2 of the Appendix.

Finally, let me comment that in the case of lowest-level generators the index l can be introduced through the adjoint action of P_0 like

$$ {}^{l}_{0}\mathbf{Q}^{s}_{s-1} = \frac{l!}{(s-1)!}\,\mathrm{ad}^{s-1-l}_{P_0}\left({}^{s-1}_{0}\mathbf{Q}^{s}_{s-1}\right). \qquad (4.1.13) $$

Further, possibly useful identities are collected in Appendix A.

[3] In the present phrasing, the star product is actually nothing special, in particular it is not connected to the theory of deformation quantisation but rather refers to the particular choice of basis we made. Apart from that, it can readily be identified with the formal product again.

4.1.3 A Quotient of the Associative Algebra

The associative algebra obtained in the previous subsections may at first sight appear to be too large for any reasonable application, in particular in view of the much simpler structure of the higher-spin algebra in the case of AdS. One may therefore ask the question if the algebra at hand can be further reduced in some sense. When searching for additional quotients, there are indeed not many formal relations between generators that one can write down without producing contradictions to the $\mathfrak{isl}(2,\mathbb{R})$-commutation relations, but there exists another quotient that can be taken, as I will briefly discuss it in the following. These considerations were first presented in [1] and further clarified in [225].

The idea is to reduce the number of independent objects in the UEA by imposing the identification $J_m P_n \sim P_m J_n$. Demanding consistency with the $\mathfrak{isl}(2,\mathbb{R})$-commutation relations, this however immediately forces the formal identification $P_m P_n \sim 0$, i.e. translational generators can only appear up to first power. Accordingly, one is now only dealing with the two outer left slices in the UEA picture given in figure 4.1.

The vanishing of higher powers of P_m implies $\mathcal{S}^2 = 0$ on the level of the UEA and thus restricts the parametrisation of the spin-Casimir element to the value zero (or a Grassmann-valued number, squaring to zero). Furthermore, the element $\mathscr{C}$ now becomes a Casimir element of the left over structure, which can as well be divided out. On a formal level, the additional quotienting can be performed by the subsequent replacements

$$ {}^{\,l}_{\xi}\mathbf{Q}^s_m \mapsto (-1)^{\xi}\mathscr{C}^{\left\lfloor\frac{\xi}{2}\right\rfloor}\begin{cases} V^{s-\xi}_m\,, & s-1-l=0\,,\\ W^{s-\xi}_m\,, & s-1-l=1\,,\\ 0\,, & s-1-l\geq 2\,,\end{cases} \tag{4.1.14}$$

as well as $\mathcal{S}V^s_m \mapsto \mathscr{C}W^s_m$ and $\mathcal{S}W^s_m \mapsto 0$, where now $\mathscr{C}$ is just a number. Implementing this prescription on the level of spin-s-spin-two product rules, the structure constants occurring in the respective expressions turn out to be precisely those of the Lie algebra $\mathfrak{ihs}(\lambda)$ that was introduced through an İnönü-Wigner contraction of $\mathfrak{hs}(\lambda)\oplus\mathfrak{hs}(\lambda)$ in subsection 3.2.3 of the Foundations. Thus follow the product rules of the structure so obtained,

$$ V^s_m \star V^t_n = \sum_{u=0}^{s+t-|s-t|-2} g^{st}_u(m,n;\lambda)\,V^{s+t-u-1}_{m+n}\,, \tag{4.1.15a}$$

$$ V^s_m \star W^t_n = \sum_{u=0}^{s+t-|s-t|-2} g^{st}_u(m,n;\lambda)\,W^{s+t-u-1}_{m+n}\,, \tag{4.1.15b}$$

$$ W^s_m \star W^t_n = 0\,, \tag{4.1.15c}$$

with the parametrisation $\mathscr{C} = (\lambda^2-1)/4$ and the constants $g^{st}_u(m,n;\lambda)$ given in (3.2.9). The vanishing of the Casimir elements $\mathscr{M}^2$ and $\mathcal{S}$ is in accordance with the viewpoint of the contraction: the former is of higher order in the contraction parameter, while the latter is proportional to the difference $\lambda - \bar{\lambda}$, which needs to be zero for the contraction to work without further re-scalings.

It is apparent that both of the original flat-space Casimir elements got lost in the passage to the quotient algebra, while at the same time the $\mathfrak{sl}(2,\mathbb{R})$-Casimir element entered as a new parameter. Since we expect the elements $\mathscr{M}^2$ and $\mathbb{S}$ to carry physically significant information (for instance, they should label single-particle representations of $\mathfrak{isl}(2,\mathbb{R})$), this quotient algebra will not be considered as underlying higher-spin symmetry algebra in the following investigations.[4]

4.2 A Lie-Subalgebra of the Associative Algebra

Rather than exploring quotients of $\mathfrak{ihs}(\mathscr{M}^2,\mathbb{S})$ one may seek for interesting subalgebras. From the viewpoint of higher-spin *gravity* in the sense of a Chern-Simons gauge theory it would be sufficient to have at hand a *Lie*-subalgebra, preferably one that still contains the parameters $\mathscr{M}^2$ and $\mathbb{S}$, which we expect to play an important role, at least in the context of $\mathfrak{isl}(2,\mathbb{R})$-representations.

With these preliminaries in mind, it seems natural to focus on the outer right slice of the infinite wedge in figure 4.1, i.e. on generators with indices $l = 0$ and $l = 1$, as well as $\xi = 0$. Indeed, these generators span a Lie-subalgebra and it turns out to be possible to fully calculate its structure constants. This Lie-subalgerbra, which, to avoid a growing stack of different notation, I will simply refer to as *Lie*-$\mathfrak{ihs}(\mathscr{M}^2,\mathbb{S})$, will be content of this section.

4.2.1 Lie Brackets from the Associative Product

It is easily seen from the structure of the UEA that the set $\{ {}^1_0\mathbf{Q}^s_m \,,\, {}^0_0\mathbf{Q}^s_m \}$ spans a Lie subalgebra. In order to clear up the notation, introduce $\mathbf{J}^s_m := (s-1)\, {}^1_0\mathbf{Q}^s_m$ and $\mathbf{P}^s_m := {}^0_0\mathbf{Q}^s_m$. Then the star-commutation relations read

$$\left[\mathbf{J}^s_m, \mathbf{J}^t_n\right]_\star = \frac{1}{2}\sum_{u=0}^{\lfloor\frac{s+t-4}{2}\rfloor} g^{st}_u(m,n)\,\mathbf{J}^{s+t-2u-2}_{m+n} + \frac{\mathbb{S}}{\mathscr{M}^2}\sum_{u=0}^{\lfloor\frac{s+t-3}{2}\rfloor} u\, g^{st}_u(m,n)\,\mathbf{P}^{s+t-2u-2}_{m+n}\,, \tag{4.2.1a}$$

$$\left[\mathbf{J}^s_m, \mathbf{P}^t_n\right]_\star = \frac{1}{2}\sum_{u=0}^{\lfloor\frac{s+t-3}{2}\rfloor} g^{st}_u(m,n)\,\mathbf{P}^{s+t-2u-2}_{m+n}\,, \tag{4.2.1b}$$

$$\left[\mathbf{P}^s_m, \mathbf{P}^t_n\right]_\star = 0\,, \tag{4.2.1c}$$

[4]Note, however, its recent holographic application in the case of a conformal Carrollian scalar field living on the boundary [226].

where I defined the constants

$$g_u^{st}(m,n) \equiv \frac{(-1)^u \mathscr{M}^{2u}}{4^{2u} u!} \frac{\mathscr{N}_{2u+1}^{st}(m,n)}{(s-3/2)^{\underline{u}}(t-3/2)^{\underline{u}}(s+t-u-5/2)^{\underline{u}}} \tag{4.2.2}$$

together with the mode function as defined in equation (3.2.10) of the Foundations. Note that $\mathscr{N}_u^{st}(m,n) = (-1)^u \mathscr{N}_u^{ts}(n,m)$, from which it is apparent that

$$\left[\mathbf{J}_m^s, \mathbf{P}_n^t\right]_\star = \left[\mathbf{P}_m^s, \mathbf{J}_n^t\right]_\star . \tag{4.2.3}$$

The case $\mathcal{S} = 0$ may be of special interest because the Lie algebra then assumes the structure of a semi-direct sum, in this regard resembling $\mathfrak{isl}(2,\mathbb{R})$. In particular, the generators $\mathbf{J}_m^s$ then span a Lie-subalgebra, while the generators $\mathbf{P}_m^s$ span an ideal of the Lie algebra. As stressed in [225] it turns out that in this case the value of $\mathscr{M}^2$ looses its meaning, since it can be scaled out completely. Then the Lie algebra so obtained can be obtained in a limit from the quotient algebra discussed in the previous section (namely by sending $\lambda \to \infty$). However, we will later see that the generic case $\mathcal{S} \neq 0$ appears be of importance in the application to Wilson lines in section 5.3.

Another interesting case is $\mathscr{M}^2 = 0$, where nearly all structure constants vanish, leaving only

$$\left[\mathbf{J}_m^s, \mathbf{J}_n^t\right]_\star = ((t-1)m - (s-1)n)\, \mathbf{J}_{m+n}^{s+t-2} + \frac{\mathcal{S}}{\mathscr{M}^2} g_1^{st}(m,n)\, \mathbf{P}_{m+n}^{s+t-4}\,, \tag{4.2.4a}$$

$$\left[\mathbf{J}_m^s, \mathbf{P}_n^t\right]_\star = ((t-1)m - (s-1)n)\, \mathbf{P}_{m+n}^{s+t-2}\,, \tag{4.2.4b}$$

$$\left[\mathbf{P}_m^s, \mathbf{P}_n^t\right]_\star = 0\,. \tag{4.2.4c}$$

Note that $g_1^{st}(m,n) \sim \mathscr{M}^2$ and $g_1^{s2}(m,n) = 0$. This implies the existence of an infinite set of ideals $\mathscr{I}_{(s,t)}$, consisting of generators with spin index greater or equal s and t, respectively, where $s > t$:

$$\mathscr{I}_{(s,t)} = \left\{ \mathbf{J}_{m'}^{s'}, \mathbf{P}_{n'}^{t'} \,\middle|\, s' \geq s,\ t' \geq t,\ |m'| \leq s'-1,\ |n'| \leq t'-1 \right\}, \qquad s > t,\ t \geq 1\,. \tag{4.2.5}$$

Thus, taking the largest possible ideal that does not contain $\mathfrak{isl}(2,\mathbb{R})$-generators and building the quotient, all that remains are the original $\mathfrak{isl}(2,\mathbb{R})$-generators, together with the spin-3 Lorentz-like generators $\mathbf{J}_m^3$.

We see that a key difference between the higher-spin Lie algebra here discussed and its AdS companion $\mathfrak{hs}(\lambda)$ is that in the latter case one can fix the parameter λ to some integer, $\lambda = N$, such that it produces an ideal that can be quotiented out and leaves one with a finite-spin algebra $\mathfrak{sl}(N,\mathbb{R})$. Apart from the seemingly trivial case $\mathscr{M}^2 = 0$ we do not have that option in the flat-space case – we always need to consider the complete infinite tower.[5]

[5] It is of course still possible to manually define finite-spin algebras, such as $\mathfrak{isl}(3,\mathbb{R})$, as in [140, 173].

4.2.2 Bilinear Form and Dual Lie Algebra

Having derived the Lie algebra from an underlying associative algebra, it is possible to define a bilinear form on it from identification of the coefficient of the unit element in the product of two generators [161]. That is, one may take as a starting point

$$\left\langle {}^{k}_{\xi}\mathbf{Q}^{s}_{m}\,,\,{}^{l}_{\eta}\mathbf{Q}^{t}_{n}\right\rangle \sim [\mathbb{1}]\left({}^{k}_{\xi}\mathbf{Q}^{s}_{m}\star{}^{l}_{\eta}\mathbf{Q}^{t}_{n}\right), \tag{4.2.6}$$

where $\mathbb{1}\equiv{}^{0}_{0}\mathbf{Q}^{1}_{0}$. However, there are issues in the present case of $\mathfrak{ihs}(\mathscr{M}^2,\mathbb{S})$, since the Lie algebra is defined in a different set than the associative product.

In the standard case, in which a Lie algebra is given through identification of the Lie bracket with the commutator of some associative product that closes within *the same set* as the Lie algebra, one can immediately show key properties of a bilinear form defined from the product. Let $\mathfrak{g}$ be equipped with an associative product and $X,Y,Z\in\mathfrak{g}$, then the definition $\langle X\,,Y\rangle := [\mathbb{1}](X\star Y)$ allows the following statements:

1. **Symmetry:** If the commutator of two elements can never produce a unit element, $[\mathbb{1}][X\,,Y]=0$, then the bilinear form is symmetric, $\langle X\,,Y\rangle=\langle Y\,,X\rangle$.

2. **Non-Degeneracy:** If the associative product is non-degenerate, i.e. there is no element X apart from zero, such that $X\star Y=0$ for all Y, then the bilinear form is non-degenerate, as well.

3. **Ad-Invariance:** If $[\mathbb{1}][X\,,Y]=0$ and the algebra does not possess an abelian subalgebra, then the bilinear form is ad-invariant, i.e.

$$\langle[X\,,Y]\,,Z\rangle=\langle X\,,[Y\,,Z]\rangle\,, \tag{4.2.7}$$

which follows simply from $[X\,,Y]\star Z=X\star[Y\,,Z]+[X\star Z\,,Y]$. Note that this argument fails if the coefficient in front of the unit element on the left-hand side is non-zero and Y and Z commute.

The last property, ad-invariance, will be at the centre of discussion in this subsection.

From the product rules and identities given in section A.3 of the Appendix one may extract the respective expressions for the unit-element coefficients, reading

$$[\mathbb{1}]\left(\mathbf{J}^{s}_{m}\star\mathbf{J}^{t}_{n}\right)=\frac{(-1)^{m}(s-1)^{\underline{2}}(s-1-m)!(s-1+m)!\mathscr{M}^{2(s-3)}\mathbb{S}^{2}}{2\cdot4^{s-1}(s-{}^{1}\!/\!_{2})^{\underline{s-1}}(s-{}^{3}\!/\!_{2})^{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,, \tag{4.2.8a}$$

$$[\mathbb{1}]\left(\mathbf{J}^{s}_{m}\star\mathbf{P}^{t}_{n}\right)=\frac{(-1)^{m}(s-1)(s-1-m)!(s-1+m)!\mathscr{M}^{2(s-2)}\mathbb{S}}{4^{s-1}(s-{}^{1}\!/\!_{2})^{\underline{s-1}}(s-{}^{3}\!/\!_{2})^{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,, \tag{4.2.8b}$$

$$[\mathbb{1}]\left(\mathbf{P}^{s}_{m}\star\mathbf{P}^{t}_{n}\right)=\frac{(-1)^{m}(s-1-m)!(s-1+m)!\mathscr{M}^{2(s-1)}}{4^{s-1}(s-{}^{1}\!/\!_{2})^{\underline{s-1}}(s-{}^{3}\!/\!_{2})^{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,. \tag{4.2.8c}$$

These expressions are symmetric and generically non-degenerate but not ad-invariant. First of all, the presence of an abelian subalgebra automatically spoils ad-invariance, as long as $\langle \mathbf{P}^s_m , \mathbf{P}^t_n \rangle \neq 0$. Secondly, one may suspect another violation of ad-invariance in view of the argument made in the third point of the above listing: the product of two generators does in general produce generators of level $\xi = 1$, which, in turn, do not have the property that $[\mathbb{1}][X, Y] = 0$. One may however check that this can be fixed by including appropriate s-dependent re-scalings (since it is merely the effect of an inconvenient choice of basis).

These remarks in mind, I define a symmetric, non-degenerate, ad-invariant form on the Lie algebra $\mathfrak{ihs}(\mathcal{M}^2, \mathcal{S})$ as

$$\langle \mathbf{J}^s_m , \mathbf{J}^t_n \rangle = \frac{(-1)^m (s-2)(s-1-m)!(s-1+m)!\mathcal{M}^{2(s-3)}\mathcal{S}^2}{2\cdot 4^{s-2}(s-1/2)_{\underline{s-1}}(s-3/2)_{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,, \tag{4.2.9a}$$

$$\langle \mathbf{J}^s_m , \mathbf{P}^t_n \rangle = \frac{(-1)^m (s-1-m)!(s-1+m)!\mathcal{M}^{2(s-2)}\mathcal{S}}{4^{s-1}(s-1/2)_{\underline{s-1}}(s-3/2)_{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,, \tag{4.2.9b}$$

$$\langle \mathbf{P}^s_m , \mathbf{P}^t_n \rangle = 0\,. \tag{4.2.9c}$$

Note that the case $\mathcal{S} = 0$ would lead to a degenerate form, in which case one would exclude $\mathcal{S}$ through appropriate normalisation factors in the first place. Note furthermore that the case $\mathcal{M}^2 = 0$ leads to a degeneration, in accordance to the discussion about ideals appearing in this case at the end of subsection 4.2.1.

Though the property of ad-invariance is generally considered to be of importance, for it is linked to gauge invariance of expressions formulated in terms of the bilinear form, there may be circumstances where the usage of a non-ad-invariant form is advantageous. In general, the absence of ad-invariance does not prevent one to write down gauge invariant expressions and it may generically be replaced by a more general property. I will discuss this in the following in the case of the Poincaré subalgebra.

Poincaré Case and Skew Symmetry

The specification of the bilinear form that can be obtained directly from the unit-element coefficients as in (4.2.8) to the case of the $\mathfrak{isl}(2, \mathbb{R})$-subalgebra (together with the inclusion of a constant normalisation factor for convenience) gives

$$\langle J_m , J_n \rangle = 0\,, \tag{4.2.10a}$$

$$\langle J_m , P_n \rangle = (-1)^m (1+m)!(1-m)!\mathcal{S}\delta_{m+n,0}\,, \tag{4.2.10b}$$

$$\langle P_m , P_n \rangle = (-1)^m (1+m)!(1-m)!\mathcal{M}^2\delta_{m+n,0}\,. \tag{4.2.10c}$$

This is in contrast to the classical form usually found in the literature[6] that goes back to Witten [39] and corresponds to setting $\mathcal{M}^2 = 0$ as well as $\mathcal{S} = 1$ (in disregard of the length scale contained in $\mathcal{S}$) in equations (4.2.10). It is easy to see that the bilinear form defined above is not ad-invariant due to the non-vanishing expression in the last line.

However, this issue is only present when translations are involved and, since the action of translations is particularly simple, turns out not to cause any problems in writing down gauge invariant expressions later on. In a sense, this particular behaviour of the bilinear form might even be expected in the present case as an imprint of the semi-direct-sum structure of the Poincaré algebra[7]. In fact, a non-vanishing contribution of the form (4.2.10c) has implicitly been used in literature on flat-space holography before, namely in [60] as well as in [114, 133, 227].

Let me elaborate more on this and stress that the ad-invariance property is actually only a special instance of a more general property, which is introduced under the name of *skew-symmetry* in [228, appendix 2] in the context of invariant metrics on group manifolds. From the definition (4.2.10) one may readily write down identities such as

$$\langle P_a , [P_b , J_c] \rangle = \langle [P_a , J_b] , P_c \rangle = \frac{\mathcal{M}^2}{\mathcal{S}} \langle [P_a , J_b] , J_c \rangle \tag{4.2.11a}$$

$$= \langle [J_a , P_b] , P_c \rangle = \frac{\mathcal{M}^2}{\mathcal{S}} \langle [J_a , P_b] , J_c \rangle , \tag{4.2.11b}$$

which can be put into a more general form. Denoting the generators J_m and P_m collectively by X_A, we find that the form defined above fulfils the identity

$$\langle B(X_A , X_B) , X_C \rangle = \langle B(X_C , X_A) , X_B \rangle \tag{4.2.12}$$

with a bilinear function B on the Lie algebra. This is precisely the property of skew-symmetry. The bilinear B can apparently be identified[8] as a commutator, in which translations do not commute,

$$B(J_m , J_n) = [J_m , J_n] , \qquad B(J_m , P_n) = [J_m , P_n] , \qquad B(P_m , P_n) = \frac{\mathcal{M}^2}{\mathcal{S}} [J_m , P_n] . \tag{4.2.13}$$

From this one can read off the expansion coefficients b_{AB}^C in $B(X_A , X_B) = b_{AB}^C X_C$. Finally, note that there is already one obvious example for an application of the bilinear form (4.2.10), namely the identification of the metric $g_{\mu\nu} \sim \langle e_\mu , e_\nu \rangle$.

[6]Keep in mind that the Killing form of $\mathfrak{isl}(2,\mathbb{R})$, which can be calculated to be $\mathcal{K}(J_m, J_n) = (-1)^m (1+m)!(1-m)!\delta_{m+n,0}$ and zero otherwise, is not a good choice for a bilinear form because it is obviously degenerate, as is expected for a Lie algebra that is not semi-simple.

[7]As a simple example one may keep in mind that the scalar product of vectors in euclidean $\mathbb{R}^3$ is invariant under rotations, but obviously not under translations.

[8]Note however that the definition of the function B given by Arnold [228], $\langle [X , Y] , Z \rangle =: \langle B(Z, X) , Y \rangle \ \forall X \in \mathfrak{g}$, fails in the case where $\mathfrak{g}$ possesses an abelian Lie-subalgebra.

Having seen in the Poincaré case that the property of ad-invariance can be replaced by skew-symmetry with an appropriate bilinear B, one can check that the same argumentation can be applied in the higher-spin case. Thus, I will in addition to (4.2.9) define a skew-symmetric bilinear form on $\mathfrak{ihs}(\mathscr{M}^2, \mathcal{S})$,

$$\left\langle \mathbf{J}^s_m , \mathbf{J}^t_n \right\rangle = \frac{(-1)^m (s-2)(s-1-m)!(s-1+m)!\mathscr{M}^{2(s-3)}\mathcal{S}^2}{2\cdot 4^{s-2}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,, \tag{4.2.14a}$$

$$\left\langle \mathbf{J}^s_m , \mathbf{P}^t_n \right\rangle = \frac{(-1)^m (s-1-m)!(s-1+m)!\mathscr{M}^{2(s-2)}\mathcal{S}}{4^{s-1}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,, \tag{4.2.14b}$$

$$\left\langle \mathbf{P}^s_m , \mathbf{P}^t_n \right\rangle = \frac{(-1)^m (s-1-m)!(s-1+m)!\mathscr{M}^{2(s-1)}}{4^{s-1}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,. \tag{4.2.14c}$$

Similar to the Poincaré case, the function B is given as

$$B\left(\mathbf{P}^s_m , \mathbf{P}^t_n\right) = \frac{\mathscr{M}^2}{\mathcal{S}}\left[\mathbf{J}^s_m , \mathbf{P}^t_n\right] \tag{4.2.15}$$

and as the commutators (4.2.1a) and (4.2.1b) otherwise.

Inverse Bilinear Form and Dual Lie Algebra

Being in possession of a bilinear form, the foundations are laid to define a dual Lie algebra, using its inverse. To do so, let me return to the ad-invariant bilinear form (4.2.9) and introduce the multi-index $A = (l, s, m)$ via $\mathbf{P}^s_m \equiv X_{(0,s,m)}$ and $\mathbf{J}^s_m \equiv X_{(1,s,m)}$. Then call $\langle X_A , X_B\rangle = \gamma_{AB}$, where

$$\gamma_{(0,s,m)(0,t,n)} = 0\,, \tag{4.2.16a}$$

$$\gamma_{(0,s,m)(1,t,n)} = \frac{(-1)^m (s-2)(s-1-m)!(s-1+m)!\mathscr{M}^{2(s-3)}\mathcal{S}^2}{2\cdot 4^{s-2}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,, \tag{4.2.16b}$$

$$\gamma_{(1,s,m)(1,t,n)} = \frac{(-1)^m (s-1-m)!(s-1+m)!\mathscr{M}^{2(s-2)}\mathcal{S}}{4^{s-1}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}\delta_{s,t}\delta_{m+n,0}\,, \tag{4.2.16c}$$

which can be put into matrix form,

$$\gamma \equiv \left(\gamma_{(l,s,m)(k,t,n)}\right) = \begin{pmatrix} 0 & \left(\gamma_{(0,s,m)(1,t,n)}\right) \\ \left(\gamma_{(0,s,m)(1,t,n)}\right) & \left(\gamma_{(1,s,m)(1,t,n)}\right) \end{pmatrix}. \tag{4.2.17}$$

Each matrix block is block diagonal with respect to the indices s and t,

$$\left(\gamma_{(0,s,m)(1,t,n)}\right) = \bigoplus_{s=2}^{\infty} \frac{\mathscr{M}^{2(s-2)}\mathcal{S}}{4^{s-1}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}\gamma^{(s)}\,, \tag{4.2.18a}$$

$$\left(\gamma_{(1,s,m)(1,t,n)}\right) = \bigoplus_{s=2}^{\infty} \frac{(s-2)\mathscr{M}^{2(s-3)}\mathcal{S}^2}{2\cdot 4^{s-2}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}\gamma^{(s)}\,, \tag{4.2.18b}$$

and each of these blocks consists of a symmetric, anti-diagonal matrix with respect to the mode indices m and n,

$$\gamma^{(s)} \equiv \text{antidiag}\left((-1)^m (s-1-m)!(s-1+m)!\right)_{m=-s+1}^{s-1} . \tag{4.2.19}$$

The inverse bilinear form $\gamma^{(l,s,m)(k,t,n)}$ is given in terms of the inverse matrix

$$\gamma^{-1} = \begin{pmatrix} -\left(\gamma_{(0,s,m)(1,t,n)}\right)^{-1}\left(\gamma_{(1,s,m)(1,t,n)}\right)\left(\gamma_{(0,s,m)(1,t,n)}\right)^{-1} & \left(\gamma_{(0,s,m)(1,t,n)}\right)^{-1} \\ \left(\gamma_{(0,s,m)(1,t,n)}\right)^{-1} & 0 \end{pmatrix} \tag{4.2.20}$$

and reads

$$\gamma^{(0,s,m)(0,t,n)} = \frac{(-1)^{m+1} 2\cdot 4^{s-1}(s-2)(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}{(s-1-m)!(s-1+m)!\mathcal{M}^{2(s-1)}}\delta_{s,t}\delta_{m+n,0} , \tag{4.2.21a}$$

$$\gamma^{(0,s,m)(1,t,n)} = \frac{(-1)^m 4^{s-1}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}{(s-1-m)!(s-1+m)!\mathcal{M}^{2(s-2)}\mathcal{S}}\delta_{s,t}\delta_{m+n,0} , \tag{4.2.21b}$$

$$\gamma^{(1,s,m)(1,t,n)} = 0 . \tag{4.2.21c}$$

Introducing the abbreviation

$$\gamma(s,m) \equiv \frac{(-1)^m 4^{s-1}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}{(s-1-m)!(s-1+m)!\mathcal{M}^{2(s-1)}} \tag{4.2.22}$$

the generators of the inverse Lie algebra can now be defined as $X^A = \gamma^{AB} X_B$, that is

$$\mathbf{P}^m_s = \gamma(s,m)\frac{\mathcal{M}^2}{\mathcal{S}}\left(\mathbf{J}^s_{-m} - (s-2)\frac{2\mathcal{S}}{\mathcal{M}^2}\mathbf{P}^s_{-m}\right), \qquad \mathbf{J}^m_s = \gamma(s,m)\frac{\mathcal{M}^2}{\mathcal{S}}\mathbf{P}^s_{-m} \tag{4.2.23}$$

and their Lie brackets can be found from the $\mathfrak{ihs}(\mathcal{M}^2,\mathcal{S})$-commutation relations, reading

$$\left[\mathbf{P}^m_s, \mathbf{P}^n_t\right] = \frac{1}{2}\sum_{u=0}^{\lfloor\frac{s+t-4}{2}\rfloor} \tilde{g}^{st}_u(m,n)\mathbf{P}^{m+n}_{s+t-2u-2} - \frac{\mathcal{S}}{\mathcal{M}^2}\sum_{u=0}^{\lfloor\frac{s+t-3}{2}\rfloor} u\tilde{g}^{st}_u(m,n)\mathbf{J}^{m+n}_{s+t-2u-2} , \tag{4.2.24a}$$

$$\left[\mathbf{P}^m_s, \mathbf{J}^n_t\right] = \frac{1}{2}\sum_{u=0}^{\lfloor\frac{s+t-3}{2}\rfloor} \tilde{g}^{st}_u(m,n)\mathbf{J}^{m+n}_{s+t-2u-2} , \tag{4.2.24b}$$

$$\left[\mathbf{J}^m_s, \mathbf{J}^n_t\right] = 0 , \tag{4.2.24c}$$

with inverse structure constants

$$\tilde{g}^{st}_u(m,n) \equiv -\frac{\gamma(s,m)\gamma(t,n)g^{st}_u(m,n)}{\gamma(s+t-2u-2,m+n)}\frac{\mathcal{M}^2}{\mathcal{S}} . \tag{4.2.25}$$

Dual Poincaré Algebra – Skew-Symmetric Case

One may wonder how the choice of bilinear form affects the dual Lie algebra defined through it. Let me elaborate on this in the case of $\mathfrak{isl}(2,\mathbb{R})$ and the bilinear form (4.2.10). Introducing a multi-index $A=(l,m)$, the bilinear form γ_{AB} is written

$$\gamma_{(0,m)(0,n)} = (-1)^m(1+m)!(1-m)!\mathcal{M}^2\delta_{m+n,0}\,, \tag{4.2.26a}$$
$$\gamma_{(0,m)(1,n)} = (-1)^m(1+m)!(1-m)!\mathcal{S}\delta_{m+n,0}\,, \tag{4.2.26b}$$
$$\gamma_{(1,m)(1,n)} = 0\,, \tag{4.2.26c}$$

and, abbreviating $1/\gamma(m) \equiv (-1)^m(1+m)!(1-m)!\mathcal{S}$, its inverse γ^{AB} reads

$$\gamma^{(0,m)(0,n)} = 0\,, \qquad \gamma^{(0,m)(1,n)} = \gamma(m)\delta^{m+n,0}\,, \qquad \gamma^{(1,m)(1,n)} = -\frac{\mathcal{M}^2}{\mathcal{S}}\gamma(m)\delta^{m+n,0}\,. \tag{4.2.27}$$

One may check that the defining relation $\gamma_{AC}\gamma^{CB} = \delta^B_A$ is fulfilled. Accordingly, the generators of the dual of $\mathfrak{isl}(2,\mathbb{R})$ with respect to this bilinear form are

$$P^m = \gamma(m)J_{-m}\,, \qquad\qquad J^m = \gamma(m)\left(P_{-m} - \frac{\mathcal{M}^2}{\mathcal{S}}J_{-m}\right) \tag{4.2.28}$$

and their commutation relations read

$$\left[P^m, P^n\right] = -(m-n)\frac{\gamma(m)\gamma(n)}{\gamma(m+n)}P^{m+n}\,, \tag{4.2.29a}$$
$$\left[J^m, P^n\right] = -(m-n)\frac{\gamma(m)\gamma(n)}{\gamma(m+n)}J^{m+n}\,, \tag{4.2.29b}$$
$$\left[J^m, J^n\right] = (m-n)\frac{\gamma(m)\gamma(n)\mathcal{M}^2}{\gamma(m+n)\mathcal{S}}\left(2J^{m+n} + \frac{\mathcal{M}^2}{\mathcal{S}}P^{m+n}\right)\,, \tag{4.2.29c}$$

from which we can read off the inverse structure constants.

In the present case one may simply recover the results that would have been obtained with the standard, ad-invariant bilinear form by replacing $\mathcal{M}^2 \mapsto 0$. Indeed, the dual Lie algebra (4.2.29) is isomorphic to the one obtained in the ad-invariant case; both versions are connected by the mapping $\tilde{J}^m := J^m + \mathcal{M}^2/\mathcal{S}\,P^m$.

4.3 Limiting Procedure from AdS

On the level of symmetry algebras, it should be possible to express flat-space quantities in terms of vanishing-cosmological-constant limits of AdS quantities. Of course, one would not expect that such a transition is guaranteed to be possible, for instance when higher-spin interactions are turned on, which are not analytic in the cosmological constant and lead to singularities in a flat limit [229].

And yet, even on the level of symmetry algebras such a flat-space limit may be subtle, since it is not always a priori clear how the respective quantities should scale. In this section I will comment on the limiting procedure from which the associative algebra $\mathfrak{ihs}(\mathscr{M}^2, \mathbb{S})$ as well as the Lie algebra discussed in the previous section can be obtained, thereby illustrating such a subtlety, namely the choice of an appropriate starting point from which a contraction is initiated.

4.3.1 The Large AdS Higher-Spin Algebra

It is clear already from its size that the associative algebra $\mathfrak{ihs}(\mathscr{M}^2, \mathbb{S})$ cannot be defined as a contraction of $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\bar{\lambda})$, since mixed products of generators from different sectors of the direct sum vanish. It is therefore reasonable to first consider a quotient of the larger universal enveloping algebra $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))$, which also contains formal products of the form $L_m \bar{L}_n$.

The construction of a larger AdS higher-spin algebra may be performed along the same lines as in the flat-space case. Recall the commutation relations of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ given in (3.1.12). The UEA of the direct sum consists of all formal products of generators L_m and $\bar{L}_m$ and it contains two second-order Casimir elements,

$$\mathscr{C}_{\text{AdS}} = (L_0)^2 - L_1 L_{-1} + L_0\,, \qquad \overline{\mathscr{C}}_{\text{AdS}} = (\bar{L}_0)^2 - \bar{L}_1 \bar{L}_{-1} + \bar{L}_0\,. \tag{4.3.1}$$

A classification of elements of the quotient of $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))$ with the ideal generated by parametrisation of these two Casimir elements as multiples of the identity may be achieved by studying their behaviour under the adjoint action of combinations $L_m + \bar{L}_m$. We first collect all highest-weight elements, i.e. expressions that commute with $L_1 + \bar{L}_1$. Apart from powers of L_1 and $\bar{L}_1$ one finds that the second-order objects

$$\mathscr{D} \equiv 2L_0\bar{L}_0 - L_1\bar{L}_{-1} - L_{-1}\bar{L}_1 \qquad \text{and} \qquad L_0\bar{L}_1 - L_1\bar{L}_0 \tag{4.3.2}$$

are highest weight, as well. While any formal power of the object[9] $\mathscr{D}$ is an independent object in the UEA, for the second combination in (4.3.2) only the linear order needs to be included. Accordingly, a highest-weight basis can be defined from

$${}^{l}_{\xi}\mathbf{V}^{s}_{s-1-\xi} := (L_1)^{l-\left\lfloor\frac{\xi+1}{2}\right\rfloor} \mathscr{D}^{\left\lfloor\frac{\xi}{2}\right\rfloor} (L_0\bar{L}_1 - L_1\bar{L}_0)^{\left\lfloor\frac{\xi+1}{2}\right\rfloor - \left\lfloor\frac{\xi}{2}\right\rfloor} (\bar{L}_1)^{s-1-l-\left\lfloor\frac{\xi+1}{2}\right\rfloor} \tag{4.3.3}$$

by repeated application of the commutator with $L_{-1} + \bar{L}_{-1}$,

$${}^{l}_{\xi}\mathbf{V}^{s}_{m} := (-1)^{s-\xi-m-1} \frac{(s-\xi+m-1)!}{(2s-2\xi-2)!} \operatorname{ad}^{s-1-\xi-m}_{L_{-1}+\bar{L}_{-1}} \left({}^{l}_{\xi}\mathbf{V}^{s}_{s-1-\xi}\right). \tag{4.3.4}$$

[9] Note that $\mathscr{D}$ actually commutes with any $L_m + \bar{L}_m$ and can therefore be viewed as the analogue to the element $\mathscr{C}$ in the flat-space case.

The allowed range of indices is by construction given by

$$s \geq 1\,, \qquad 0 \leq \xi \leq 2\left\lfloor \frac{s-1}{2} \right\rfloor, \qquad |m| \leq s-1-\xi\,, \qquad \left\lfloor \frac{\xi+1}{2} \right\rfloor \leq l \leq s-1-\left\lfloor \frac{\xi+1}{2} \right\rfloor. \qquad (4.3.5)$$

These definitions then imply the standard commutation relation

$$\left[{}^{l}_{\xi}\mathbf{V}^{s}_{m}, L_n + \bar{L}_n\right] = (m-(s-\xi-1)n)\,{}^{l}_{\xi}\mathbf{V}^{s}_{m+n}\,. \qquad (4.3.6)$$

The associative algebra spanned by ${}^{l}_{\xi}\mathbf{V}^{s}_{m}$ naturally contains two subalgebras isomorphic to $\mathfrak{hs}(\lambda)$, namely one spanned by $\mathcal{V}^{s}_{m} = {}^{s-1}_{0}\mathbf{V}^{s}_{m}$ and one spanned by $\bar{\mathcal{V}}^{s}_{m} = {}^{0}_{0}\mathbf{V}^{s}_{m}$. The direct sum $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\bar{\lambda})$ is obtained as quotient under the formal identification $L_m\bar{L}_n \sim 0$, where λ and $\bar{\lambda}$ are parametrising the Casimir elements $\mathscr{C}_{\text{AdS}}$ and $\bar{\mathscr{C}}_{\text{AdS}}$, respectively.

In principle, it is again possible to derive multiplication rules from the above construction. For the present purposes, however, this will not be necessary. The main message to take away from these considerations is that it is indeed possible to define an associative algebra as a quotient of a UEA in the AdS, which behaves very much the same as $\mathfrak{ihs}(\mathscr{M}^2, \mathcal{S})$. In the following subsection I will argue that this algebra provides the appropriate starting point to define $\mathfrak{ihs}(\mathscr{M}^2, \mathcal{S})$ in terms of a contraction.

4.3.2 Comments on Contractions

The larger AdS higher-spin algebra introduced in the previous subsection will be the point of departure for a contraction to flat space. I will only consider the Galilean limit (3.2.16) for simplicity. First, it is obvious that a contraction from $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))$ to $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$ exists in the trivial sense that any element of the latter (of the form (4.1.2), say) can be expressed through elements of the former and powers of the contraction parameter ε by imposing (3.2.16). This relation remains valid when taking the quotients with respect to the Casimir elements on both sides.

Furthermore, from a UEA perspective it is quite apparent that the second quotienting of $\mathscr{U}(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))$, namely the one that leads to $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\bar{\lambda})$, and the second quotienting of $\mathscr{U}(\mathfrak{isl}(2,\mathbb{R}))$, discussed in subsection 4.1.3, are in one-to-one correspondence to each other, since the formal identification $J_m P_n \sim P_m J_n$ immediately translates into $L_m\bar{L}_n \sim L_n\bar{L}_m$, which forces $L_m\bar{L}_n \sim 0$ for consistency with the commutation relations.

The crucial quantities for the construction of $\mathfrak{ihs}(\mathscr{M}^2, \mathcal{S})$, namely $\mathscr{C}$, $\mathcal{S}$ and $\mathscr{M}^2$, can be expressed in terms of the AdS side as

$$\mathscr{C} = \mathscr{C}_{\text{AdS}} + \bar{\mathscr{C}}_{\text{AdS}} + \mathscr{D}\,, \quad \mathcal{S} = \varepsilon\left(\mathscr{C}_{\text{AdS}} - \bar{\mathscr{C}}_{\text{AdS}}\right), \quad \mathscr{M}^2 = \varepsilon^2\left(\mathscr{C}_{\text{AdS}} + \bar{\mathscr{C}}_{\text{AdS}} - \mathscr{D} - L_0 - \bar{L}_0\right). \quad (4.3.7)$$

This makes clear how the element $\mathscr{C}$ becomes a Casimir element in the second quotient by setting $\mathscr{D} \sim 0$ through the second quotient on the AdS side. Moreover, the expression for

$\mathcal{M}^2$ illustrates how its definition from a purely contraction-based point of view, i.e. without reference to the UEA of Poincaré, is a not at all straightforward; in particular, it only becomes a Casimir element *after* the limit $\varepsilon \to 0$ is taken.

One may wonder how a formulation of the contraction in terms of generators ${}^{l}_{\xi}\mathbf{V}^{s}_{m}$ and ${}^{l}_{\xi}\mathbf{Q}^{s}_{m}$ looks like. Such an identification necessarily exists, but may look rather complicated. However, for the most simple cases of zero-level generators, $\xi = 0$, it is possible to derive a respective expression; it is

$$ {}^{l}_{0}\mathbf{Q}^{s}_{m} = \varepsilon^{s-1-l} \sum_{k=0}^{s-1} \sum_{i+j=k} (-1)^{j} \binom{l}{i} \binom{s-1-l}{j} {}^{k}_{0}\mathbf{V}^{s}_{m} . \tag{4.3.8} $$

Though one may as well try and derive expressions for higher levels, this will not lead to any deeper insights at the present level of understanding, since the behaviour of higher levels, on both sides of the contraction, is inherently dependent on the particular basis choices made in the definitions of ${}^{l}_{\xi}\mathbf{V}^{s}_{m}$ and ${}^{l}_{\xi}\mathbf{Q}^{s}_{m}$ when the objects $\mathcal{D}$ and $\mathcal{C}$ are introduced.

The essential message of the contraction formula (4.3.8) is that, in order to derive the Lie algebra $\mathfrak{ihs}(\mathcal{M}^2, \mathcal{S})$, i.e. the span of $\mathbf{J}^{s}_{m}$ and $\mathbf{P}^{s}_{m}$, from a contraction, the starting point must necessarily be the larger higher-spin algebra on the AdS side, rather than just $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\bar{\lambda})$.

4.4 Towards Supersymmetric Extensions

The UEA-construction presented in section 4.1 may as well be performed with a supersymmetric extension of the three-dimensional Poincaré algebra. Here I will consider the case of $\mathcal{N} = 1$ supersymmetry. However, I will only provide the essential ingredients to the construction; the remaining steps in determining the (anti-)commutators of the resulting higher-spin algebra can be taken along the same lines as in the bosonic case. For more information on $\mathcal{N} = 2$ higher-spin algebras in AdS and their applications, see [165, 230–235].

Start with the supersymmetrically extended algebra $\mathfrak{sisl}(1|2)$ with (anti-)commutation relations [236, 237]

$$ [J_m, J_n] = (m-n) J_{m+n} , \qquad [J_m, M_n] = \left(\frac{m}{2} - n\right) M_{m+n} , \tag{4.4.1a} $$

$$ [J_m, P_n] = (m-n) P_{m+n} , \qquad [P_m, M_n] = 0 , \tag{4.4.1b} $$

$$ [P_m, P_n] = 0 , \qquad \{M_m, M_n\} = P_{m+n} , \tag{4.4.1c} $$

where the indices on fermionic generators M_m take the values $m \in \{\pm 1/2\}$. The second-order Casimir elements are

$$ \mathcal{M}^2 = P_0 P_0 - P_1 P_{-1} , \qquad \mathcal{S} = J_0 P_0 - \frac{1}{2}(J_1 P_{-1} + J_{-1} P_1) + \frac{1}{4}\left(M_{1/2} M_{-1/2} - M_{-1/2} M_{1/2}\right) , \tag{4.4.2} $$

and the anti-commutation relation implies the UEA-relations

$$M_{1/2}M_{1/2} = \frac{1}{2}P_1\,, \qquad\qquad M_{-1/2}M_{-1/2} = \frac{1}{2}P_{-1}\,, \tag{4.4.3}$$

while the element $\mathscr{M} \equiv 2M_{1/2}M_{-1/2} - P_0$ provides a square-root of the mass Casimir. It commutes with all bosonic generators, $[\mathscr{M}\,, J_m] = 0 = [\mathscr{M}\,, P_m]$, and anti-commutes with all fermionic generators, $\{\mathscr{M}, M_m\} = 0$, thus being a so-called *Scasimir element* of the algebra [238, 239].

We have now collected all ingredients to define a highest-weight basis of the quotient

$$\mathfrak{sihs}(\mathscr{M}^2, \mathcal{S}) := \frac{\mathscr{U}(\mathfrak{sisl}(2|1))}{\langle \mathscr{M}^2, \mathcal{S}\rangle}\,. \tag{4.4.4}$$

The new ingredients relevant for the classification of highest-weight generators are the bosonic element $\mathscr{M}$ and the fermionic element $M_{1/2}$. Accordingly, there are now four classes of highest-weight generators, namely

$$\left.\begin{aligned}&(J_1)^{l-\xi}\mathscr{C}^{\left\lfloor\frac{\xi}{2}\right\rfloor}(J_0P_1-J_1P_0)^{\xi-2\left\lfloor\frac{\xi}{2}\right\rfloor}(P_1)^{s-1-l-\left(\xi-2\left\lfloor\frac{\xi}{2}\right\rfloor\right)}\\&(J_1)^{l-\xi}\mathscr{C}^{\left\lfloor\frac{\xi}{2}\right\rfloor}(J_0P_1-J_1P_0)^{\xi-2\left\lfloor\frac{\xi}{2}\right\rfloor}(P_1)^{s-1-l-\left(\xi-2\left\lfloor\frac{\xi}{2}\right\rfloor\right)}\mathscr{M}\end{aligned}\right\}\ \text{(bosonic)}\,, \tag{4.4.5a}$$

$$\left.\begin{aligned}&(J_1)^{l-\xi}\mathscr{C}^{\left\lfloor\frac{\xi}{2}\right\rfloor}(J_0P_1-J_1P_0)^{\xi-2\left\lfloor\frac{\xi}{2}\right\rfloor}(P_1)^{s-1-l-\left(\xi-2\left\lfloor\frac{\xi}{2}\right\rfloor\right)}M_{1/2}\\&(J_1)^{l-\xi}\mathscr{C}^{\left\lfloor\frac{\xi}{2}\right\rfloor}(J_0P_1-J_1P_0)^{\xi-2\left\lfloor\frac{\xi}{2}\right\rfloor}(P_1)^{s-1-l-\left(\xi-2\left\lfloor\frac{\xi}{2}\right\rfloor\right)}\mathscr{M}M_{1/2}\end{aligned}\right\}\ \text{(fermionic)}\,. \tag{4.4.5b}$$

Let me define the normalised (dimensionless) Scasimir operator as

$$\mathbb{M} \equiv \frac{\mathscr{M}}{\sqrt{\mathscr{M}^2}}\,, \qquad \mathbb{M}^2 = \mathbb{1}\,, \qquad [\mathbb{M}, J_n] = [\mathbb{M}, P_n] = 0\,, \qquad \{\mathbb{M}, M_n\} = 0\,. \tag{4.4.6}$$

Then one may define highest-weight generators ${}^{l}_{\xi}\mathbf{Q}^{s}_{s-1-\xi}$ the same way as in (4.1.7) and the additional three classes as

$${}^{l}_{\xi}\bar{\mathbf{Q}}^{s}_{s-1-\xi} = {}^{l}_{\xi}\mathbf{Q}^{s}_{s-1-\xi}\,\mathbb{M}\,, \quad {}^{l}_{\xi}\mathbf{R}^{s+1/2}_{s-1/2-\xi} = {}^{l}_{\xi}\mathbf{Q}^{s}_{s-1-\xi}\,M_{1/2}\,, \quad {}^{l}_{\xi}\bar{\mathbf{R}}^{s+1/2}_{s-1/2-\xi} = {}^{l}_{\xi}\mathbf{Q}^{s}_{s-1-\xi}\,\mathbb{M}M_{1/2}\,. \tag{4.4.7}$$

The definition of descendant generators will then be the same as in (4.1.9) and one may, as far as possible, work out the structure constants of the associative product or the commutation and anti-commutation rules of $(l=0)$- and $(l=1)$-generators.

The present thesis is however not further concerned with the supersymmetric case and the precise form of products and (anti-)commutators will be worked out elsewhere, probably for the more interesting case of $\mathcal{N}=2$ supersymmetry [196].

Chapter 5

Applications

In chapter 4 the mathematical foundations were laid for a large spectrum of applications in the context of higher-spin physics and holography. In this chapter I will demonstrate the applicability of the novel higher-spin Lie algebra as the gauge algebra of a Chern-Simons theory, the role of the associative higher-spin algebra in the context of (higher-spin) matter coupling, and elaborate on the construction of holographic probes. Some of the concepts introduced in the previous chapter, though being motivated by the study of higher-spin symmetries, will provide useful insights into the classical, i.e. spin-two, theory, as well.

This chapter contains results of [1, 3].

5.1 Higher-Spin Gravity as Chern-Simons Gauge Theory

As a first step one may re-consider the Chern-Simons action (3.1.4) and write it using the new, not ad-invariant bilinear form of $\mathfrak{isl}(2,\mathbb{R})$ given in (4.2.10). We then have

$$S_{\mathrm{CS}} = \frac{k_{\mathrm{CS}}}{4\pi}\int \left(\left\langle \omega \overset{\wedge}{,} \mathrm{d}e + \frac{2}{3}(\omega\wedge e + e\wedge\omega)\right\rangle + \left\langle e \overset{\wedge}{,} \mathrm{d}\omega + \frac{2}{3}\omega\wedge\omega\right\rangle\right). \tag{5.1.1}$$

Compared to the Einstein-Hilbert action (modulo boundary contributions) one can identify the Chern-Simons level as[1]

$$k_{\mathrm{CS}} = \frac{3}{4\mathcal{S}G_{\mathrm{N}}}\,. \tag{5.1.2}$$

Written out in components, the above action is identical to (3.1.4). One may check that it is invariant under finite Poincaré transformations, under which spin connection and vielbein transform according to (3.1.23).

[1] Recall that Newton's constant in three dimensions is of length dimension one; thus, k_{CS} is dimension-less.

5.1.1 Boundary Conditions for Higher-Spin Gravity

The classical, spin-two gauge fields (3.1.20) provide suitable boundary conditions of asymptotically flat space-times and it is straightforward to introduce deformations by higher-spin charges $Z^{(s)}(\phi)$ and $W^{(s)}(u,\phi)$ in a Drinfeld-Sokolov-like gauge. Consider

$$\omega = \left(J_1 - \frac{1}{4}\sum_{s=2}^{\infty} Z^{(s)}(\phi)\,\mathbf{J}^s_{-s+1}\right) d\phi\,, \tag{5.1.3a}$$

$$e = \left(P_1 - \frac{1}{4}\sum_{s=2}^{\infty} Z^{(s)}(\phi)\,\mathbf{P}^s_{-s+1}\right) du + \frac{1}{2}P_{-1}\,dr + \left(r P_0 - \frac{1}{2}\sum_{s=2}^{\infty} W^{(s)}(u,\phi)\,\mathbf{P}^s_{-s+1}\right) d\phi\,, \tag{5.1.3b}$$

where the spin-2 charges $M(\phi) \equiv Z^{(2)}(\phi)$ and $N(u,\phi) \equiv W^{(2)}(u,\phi)$ are included. The gauge fields obey the flatness conditions (3.1.21), provided $\partial_\phi Z^{(s)}(\phi) = 2\partial_u W^{(s)}(u,\phi)$. Note that, though the underlying Lie algebra does not allow consistent truncations to finite spins $s > 2$, the charges $Z^{(s)}(\phi)$ and $W^{(s)}(u,\phi)$ can readily be set to zero for arbitrary parameter ranges.

As in the classical case, it is possible to find a gauge where the connection does not contain any dependency on r. In particular, writing $a = g^{-1}Ag + g^{-1}\,\mathrm{d}g$ with $g = \mathrm{e}^{-(r/2)P_{-1}}$ the respective gauge field reads

$$\begin{aligned} a = &\left(P_1 - \frac{1}{4}\sum_{s=2}^{\infty} Z^{(s)}(\phi)\,\mathbf{P}^s_{-s+1}\right) du \\ &+ \left(J_1 - \frac{1}{4}\sum_{s=2}^{\infty}\left(Z^{(s)}(\phi)\,\mathbf{J}^s_{-s+1} + 2W^{(s)}(u,\phi)\,\mathbf{P}^s_{-s+1}\right)\right) d\phi\,, \end{aligned} \tag{5.1.4}$$

which can conveniently be written as $a = [\mathscr{A}, P_0]\,du + [\mathscr{A}, J_0]\,d\phi$, where

$$\mathscr{A} = J_1 + \frac{1}{4}\sum_{s=2}^{\infty}\frac{1}{s-1}\left(Z^{(s)}(\phi)\,\mathbf{J}^s_{-s+1} + 2W^{(s)}(u,\phi)\,\mathbf{P}^s_{-s+1}\right). \tag{5.1.5}$$

This last property shows that there is in fact only one independent gauge field, which illustrates the role of the u-component as a Lagrange multiplier in the Chern-Simons action.

It is furthermore apparent that the connection is locally pure gauge, i.e. $a = g^{-1}\,\mathrm{d}g$, since it obeys gauge flatness. If the charges are taken to be constants, the explicit form of the gauge transformation is $g = g_u g_\phi$ with

$$g_u = \exp\left(P_1 - \frac{1}{4}\sum_{s=2}^{\infty} Z^{(s)}\,\mathbf{P}^s_{-s+1}\right) u\,, \tag{5.1.6a}$$

$$g_\phi = \exp\left(J_1 - \frac{1}{4}\sum_{s=2}^{\infty}\left(Z^{(s)}\,\mathbf{J}^s_{-s+1} + 2W^{(s)}\,\mathbf{P}^s_{-s+1}\right)\right)\phi\,. \tag{5.1.6b}$$

Note that $[a_u, a_\phi] = 0$.

5.1.2 Higher-Spin Soft Hair

The soft-hair proposal [240–242] is a step towards a possible resolution of the information-loss paradox and it states the existence of zero-energy excitations near the horizon of a black hole, which is implied by the presence of super-translations in the asymptotic symmetry algebra: since Hawking radiation carries super-translation charge to null infinity, there is a conservation law involving black-hole charge and super-translation charge, in contrast to the no-hair theorem [243].

Soft hair were studied in the three-dimensional setting in [244], where a particular choice of boundary conditions was made for the near-horizon region of the BTZ black hole. This ansatz was further generalised to higher-spin gravity in AdS [245], as well as to three-dimensional asymptotically flat spacetimes [246] and a possible higher-spin deformation thereof [176]. It thus appears to be a natural question whether or not the respective construction is possible in the theory of higher-spin gravity proposed within the present work.

Proceeding closely to [176], one may choose the following near-horizon boundary conditions in the r-free gauge:

$$a=\sum_{s=2}^{\infty}\left(\mathcal{V}^{(s)}\,\mathbf{J}_0^s+\mathcal{W}^{(s)}\,\mathbf{P}_0^s\right)d\phi+\sum_{s=2}^{\infty}\left(\mu_{\mathrm{J}}^{(s)}\,\mathbf{J}_0^s+\mu_{\mathrm{P}}^{(s)}\,\mathbf{P}_0^s\right)du\,, \tag{5.1.7}$$

where the charges $\mathcal{V}^{(s)}=\mathcal{V}^{(s)}(u,\phi)$, $\mathcal{W}^{(s)}=\mathcal{W}^{(s)}(u,\phi)$ as well as the chemical potentials $\mu_{\mathrm{J}}^{(s)}=\mu_{\mathrm{J}}^{(s)}(u,\phi)$, $\mu_{\mathrm{P}}^{(s)}=\mu_{\mathrm{P}}^{(s)}(u,\phi)$ are allowed to depend on both coordinates u and ϕ. Flatness of the gauge field, $\mathrm{d}a+a\wedge a=0$, implies constraints on these functions,

$$\partial_u\mathcal{V}^{(s)}=\partial_\phi\mu_{\mathrm{J}}^{(s)}\,,\qquad\qquad \partial_u\mathcal{W}^{(s)}=\partial_\phi\mu_{\mathrm{P}}^{(s)}\,. \tag{5.1.8}$$

The radial dependence may be re-instated by the gauge transformation

$$g=\exp\left(\frac{1}{\mu_{\mathrm{P}}^{(2)}}P_1\right)\exp\left(\frac{r}{2}P_{-1}\right), \tag{5.1.9}$$

under which the gauge field transforms in the usual way, $A=g^{-1}ag+g^{-1}\,\mathrm{d}g$. A simple way to determine the asymptotic (near-horizon) symmetries is by imposing an infinitesimal gauge transformation that leaves the boundary conditions (5.1.7) invariant and to read off the Dirac brackets from the transformation behaviour of the canonical charge.[2]

Taking such a gauge transformation of the form

$$\varepsilon=g^{-1}\sum_{s=2}^{\infty}\left(\varepsilon_{\mathrm{J}}^{(s)}\,\mathbf{J}_0^s+\varepsilon_{\mathrm{P}}^{(s)}\,\mathbf{P}_0^s\right)g\,, \tag{5.1.10}$$

[2]Alternatively, one may treat the theory as a constraint Hamiltonian system; see Appendix C for related considerations.

where $\varepsilon_{\mathrm{J}}^{(s)} = \varepsilon_{\mathrm{J}}^{(s)}(u,\phi)$ and $\varepsilon_{\mathrm{P}}^{(s)} = \varepsilon_{\mathrm{P}}^{(s)}(u,\phi)$, the gauge field is required to transform as $\delta A = \mathrm{d}\varepsilon + [A,\varepsilon]$, from which the transformation behaviour of the charges and potentials follows to be

$$\delta\mathcal{V}^{(s)} = \partial_\phi \varepsilon_{\mathrm{J}}^{(s)}\,, \qquad \delta\mathcal{W}^{(s)} = \partial_\phi \varepsilon_{\mathrm{P}}^{(s)}\,, \qquad \delta\mu_{\mathrm{J}}^{(s)} = \partial_u \varepsilon_{\mathrm{J}}^{(s)}\,, \qquad \delta\mu_{\mathrm{P}}^{(s)} = \partial_u \varepsilon_{\mathrm{P}}^{(s)}\,. \tag{5.1.11}$$

At the same time, the gauge transformation behaviour of the canonical charge is generally of the form $\delta_Y Q[X] = \left\{Q[X], Q[Y]\right\}_{\mathrm{DB}}$, where

$$\delta Q[\varepsilon] = \frac{k_{\mathrm{CS}}}{2\pi}\int \mathrm{d}\phi \left\langle \varepsilon, \delta a_\phi \right\rangle. \tag{5.1.12}$$

Here, one has to use the ad-invariant bilinear form (4.2.9). One thus arrives, after functional integration, at the canonical charge

$$Q = \frac{k_{\mathrm{CS}}}{2\pi}\sum_{s=2}^{\infty}\alpha(s)\int \mathrm{d}\phi \left(\left(\varepsilon_{\mathrm{P}}^{(s)} + (s-2)\frac{2\mathcal{S}}{\mathcal{M}^2}\varepsilon_{\mathrm{J}}^{(s)}\right)\mathcal{V}^{(s)} + \varepsilon_{\mathrm{J}}^{(s)}\mathcal{W}^{(s)}\right), \tag{5.1.13}$$

where I abbreviated

$$\alpha(s) \equiv \frac{(s-1)!^2 \mathcal{M}^{2(s-2)}\mathcal{S}}{4^{s-1}(s-{}^1\!/_2)^{\underline{s-1}}(s-{}^3\!/_2)^{\underline{s-1}}}\,. \tag{5.1.14}$$

For the Dirac brackets of the charges $\mathcal{V}^{(s)}$, $\mathcal{W}^{(s)}$ one may take an ansatz proportional to the first derivative of a delta distribution in the angular coordinate and arrive at

$$\left\{\mathcal{V}^{(s)}(\phi), \mathcal{V}^{(t)}(\phi')\right\}_{\mathrm{DB}} = 0\,, \tag{5.1.15a}$$

$$\left\{\mathcal{V}^{(s)}(\phi), \mathcal{W}^{(t)}(\phi')\right\}_{\mathrm{DB}} = \frac{2\pi}{k_{\mathrm{CS}}\alpha(s)}\delta_{s,t}\delta'(\phi-\phi')\,, \tag{5.1.15b}$$

$$\left\{\mathcal{W}^{(s)}(\phi), \mathcal{W}^{(t)}(\phi')\right\}_{\mathrm{DB}} = -\frac{4\pi\mathcal{S}(s-2)}{k_{\mathrm{CS}}\mathcal{M}^2\alpha(s)}\delta_{s,t}\delta'(\phi-\phi')\,. \tag{5.1.15c}$$

Finally, expanding these functions into Fourier modes,

$$\mathcal{V}^{(s)}(\phi) = \frac{1}{k_{\mathrm{CS}}\sqrt{\alpha(s)}}\sum_{m\in\mathbb{Z}} V_m^s\,\mathrm{e}^{-\mathrm{i}m\phi}\,, \qquad \mathcal{W}^{(s)}(\phi) = \frac{1}{k_{\mathrm{CS}}\sqrt{\alpha(s)}}\sum_{m\in\mathbb{Z}} W_m^s\,\mathrm{e}^{-\mathrm{i}m\phi}\,, \tag{5.1.16}$$

and performing a canonical quantisation, $\{.,.\}_{\mathrm{DB}} \mapsto -\mathrm{i}[.,.]$, the asymptotic symmetry algebra reads

$$\left[V_m^s, V_n^t\right] = 0\,, \tag{5.1.17a}$$

$$\left[V_m^s, W_n^t\right] = k_{\mathrm{CS}}\, m\,\delta_{s,t}\delta_{m+n,0}\,, \tag{5.1.17b}$$

$$\left[W_m^s, W_n^t\right] = -(s-2)\frac{2\mathcal{S}}{\mathcal{M}^2}k_{\mathrm{CS}}\, m\,\delta_{s,t}\delta_{m+n,0}\,. \tag{5.1.17c}$$

Though the result appears to disagree with the result of [176] at first sight, the simple redefinition $\widetilde{W}^s_m = W^s_m + (s-2)\mathcal{S}/\mathcal{M}^2 V^s_m$ shows that the asymptotic symmetry algebra is actually isomorphic to the former and, thus, to an infinite set of decoupled $\mathfrak{u}(1)$-current algebras, as is expected for soft hair.

5.2 Coupling to Massive Higher-Spin Fields

This section deals with one main application of the algebra construction of chapter 4, the coupling of massive degrees of freedom to an Einstein or higher-spin theory of gravity in the form of a completely algebraic on-shell prescription. The introduction of such additional degrees of freedom is motivated by the analogous framework in $\mathrm{AdS}_3/\mathrm{CFT}_2$.

In the following it will become clear that the associative algebra introduced in the previous chapter is a necessary ingredient in the description of matter coupling. Previously constructed higher-spin algebras, be it through contraction [170, 176] or as the quotient discussed in subsection 4.1.3 (as well as in [1, 225]), do not supply an associative product that would give rise to non-trivial equations of motion.

5.2.1 Unfolded Klein-Gordon Equations

Let me start considerations with a massive scalar field that propagates linearly on a given gauge background. The new ingredient to the theory that captures the massive degrees of freedom is an $\mathfrak{ihs}(\mathcal{M}^2, \mathcal{S})$-valued zero form C,

$$C = \sum_{s,\xi,l,m} {}^{l}_{\xi}c^s_m(u, r, \phi)\, {}^{l}_{\xi}\mathbf{Q}^s_m \,. \tag{5.2.1}$$

I will refer to C as the master field.

Matter-Coupling Equations

Being in possession of a very limited number of building blocks, which are the gauge fields ω and e, the master field C and an associative algebra product to connect these elements, there are not too many possibilities to write down a linear, first-order equation of motion. Indeed, the equation to write down is a covariant-constancy condition for the master field [93, 112, 247] on the associative algebra,

$$\boxed{\mathrm{D}C \equiv \mathrm{d}C + [\omega, C]_\star + e \star C = 0\,.} \tag{5.2.2}$$

The operator so defined fulfils the integrability condition $\mathrm{D}^2 C = 0$, given the vanishing-torsion and vanishing-curvature conditions (3.1.21), which ensures consistency of the equations.

Under an $ISL(2,\mathbb{R})$-group transformation $g = g_{\mathrm{T}} g_{\mathrm{L}}$ the gauge fields transform as given in (3.1.23). One may fix the behaviour of the master field under such transformations as

$$C \mapsto g_{\mathrm{L}}^{-1} g_{\mathrm{T}}^{-1} C g_{\mathrm{L}} . \tag{5.2.3}$$

Then equation (5.2.2) remains invariant. In particular, one may use the pure-gauge property of the Chern-Simons fields to gauge $\omega \mapsto \tilde{\omega} = 0$ and $e \mapsto \tilde{e} = 0$, such that the master field in that gauge, $\tilde{C}$, fulfils $\mathrm{d}\tilde{C} = 0$. The argumentation, of course, works the other way around, as well: given the constant field $\tilde{C}$ and its gauge transformation behaviour, one can re-instate finite ω and e, thereby deriving the equation of motion (5.2.2).

The multiplicative action of the vielbein on the master field from the left is of course not unique. One could equally well define the product from the right, in which case the transformation behaviour has to be adjusted to be $C \mapsto g_{\mathrm{L}}^{-1} C g_{\mathrm{T}}^{-1} g_{\mathrm{L}}$; similarly, if e acts through an anti-commutator, one has to demand $C \mapsto g_{\mathrm{L}}^{-1} g_{\mathrm{T}}^{-1} C g_{\mathrm{T}}^{-1} g_{\mathrm{L}}$.

For the following considerations it will be of vital importance to find sub-structures of the algebra, in which the equation (5.2.2) closes, even if the associative product does not close within the sub-structure under question. In the case of a classical (spin two) background, i.e. $e, \omega \in \mathfrak{isl}(2,\mathbb{R})$, the commutator cannot increase the index l of any given algebra generator ${}^{l}_{\xi}\mathbf{Q}^{s}_{m}$, due to the standard spin-s-spin-two commutation relation (4.1.11). The same apparently holds true for multiplication with the purely translational vielbein and, consequently, it is possible to truncate the expansion of the master field C at any finite value of l and still maintain closed equations of motion.

If vielbein and spin connection are higher-spin, however, i.e. elements of the Lie-subalgebra spanned by generators with index $l \in \{0,1\}$ and $\xi = 0$ but of arbitrary s, the commutator does in general not close, except if the expansion of C is truncated to $l \leq 1$. Still, the product term produces generators of $\xi = 1$. The case is even simpler if the truncation is to $l = 0$, since the equations of motion then reside completely in the right-slice Lie algebra.

Klein-Gordon Field on Classical Background

Let me start with the simplest case, namely a truncation of the master field to the outer right slice of the algebra, $l = 0$, i.e.

$$C = \sum_{s=1}^{\infty} \sum_{|m| \leq s-1} c^{s}_{m}(u, r, \phi)\, \mathbf{P}^{s}_{m} \tag{5.2.4}$$

and a classical background, namely the most general asymptotically flat solutions of Einstein gravity (3.1.20). Then equation (5.2.2) together with the commutator (4.1.11) and the product rules (A.2.9) implies the following unfolded equations on the components of the master field (from now on suppressing coordinate dependence in the notation):

$$0 = \left(\partial_u + \frac{M}{2}\partial_r\right) c^s_m + c^{s-1}_{m-1} - \frac{(s-m+1)^{\underline{2}}\mathscr{M}^2}{4(s+{}^1\!/\!{}_2)^{\underline{2}}} c^{s+1}_{m-1}\,, \tag{5.2.5a}$$

$$0 = \partial_r c^s_m + \frac{1}{2} c^{s-1}_{m+1} - \frac{(s+m+1)^{\underline{2}}\mathscr{M}^2}{8(s+{}^1\!/\!{}_2)^{\underline{2}}} c^{s+1}_{m+1}\,, \tag{5.2.5b}$$

$$\begin{aligned} 0 = {} & \left(N\partial_r + \partial_\phi\right) c^s_m + (s-m) c^s_{m-1} + (s+m)\frac{M}{4} c^s_{m+1} \\ & + r\left(c^{s-1}_m + \frac{(s+m)(s-m)\mathscr{M}^2}{4(s+{}^1\!/\!{}_2)^{\underline{2}}} c^{s+1}_m\right). \end{aligned} \tag{5.2.5c}$$

Fields carrying index combinations that are note allowed have to be identified with zero. One may then combine the equations at lowest orders, $s = 1$ and $s = 2$, into a second-order equation for the coefficient $c \equiv c^1_0$. More systematic, one may take the general ansatz $\left(\alpha^{\mu\nu}\partial_\mu\partial_\nu + \alpha^\mu\partial_\mu + \alpha\right) c = 0$ and solve for the unknown coefficients. In any case, one arrives at the Klein-Gordon equation

$$\left(\Box^{(0)} - \mathscr{M}^2\right) c = 0\,, \tag{5.2.6}$$

where the operator $\Box^{(0)}$ denotes the d'Alembert operator in the background metric (3.1.16),

$$\Box^{(0)} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right) \tag{5.2.7}$$

$$= \left(-M + \frac{N^2}{r^2}\right)\partial_r^2 - 2\partial_u\partial_r + \frac{2N}{r^2}\partial_r\partial_\phi + \frac{\partial_\phi^2}{r^2} - \frac{\partial_u}{r} + \left(-M - \frac{N^2}{r^2} + \frac{\partial_\phi N}{r}\right)\frac{\partial_r}{r} - \frac{N}{r^3}\partial_\phi\,. \tag{5.2.8}$$

This shows that the matter-coupling equation (5.2.2) can indeed be seen as an unfolded system for the Klein-Gordon equation in the respective gauge background, where the scalar field is to be identified with the lowest-spin component in the $(l = 0)$-slice, in which the master field is expanded. Moreover, the mass of the Klein-Gordon field is to be identified with the parametrisation of the mass Casimir element.

All fields c^s_m with $s > 1$ are auxiliary fields that can be expressed through derivatives of the Klein-Gordon field. In the unfolded equations (5.2.5) it is easy to see that highest- or lowest-weight fields can simply be expressed as

$$c^s_{s-1} = (-1)^{s-1}\left(\partial_u + \frac{M}{2}\partial_r\right)^{s-1} c\,, \qquad c^s_{-s+1} = (-2)^{s-1}\partial_r^{s-1} c\,. \tag{5.2.9}$$

For non-highest- or lowest-weight fields any of the equations (5.2.5) can be used to iteratively reduce the spin index, eventually leading back to c.

Note that the above result is a non-trivial sanity check of the algebra structure that has been used to derive it. For example, the same calculation would not be possible with the associative algebra obtained as the quotient algebra discussed in subsection 4.1.3.

Klein-Gordon Field on Higher-Spin Background

An interesting generalisation of the above unfolding of the Klein-Gordon equation is to evaluate the master equation in the case of higher-spin gravity, i.e. to ask the question how the propagation of a scalar field is affected by the presence of higher-spin symmetry.

Using the gauge fields (5.1.3) that contain the higher-spin charges $Z^{(s)}(\phi)$ and $W^{(s)}(u,\phi)$ as background fields in (5.2.2), the master equation written out in spacetime components can be arranged into the system

$$0 = \left(\partial_u + \sum_{s=2}^{\infty}(-1)^s 2^{s-3} Z^{(s)} \partial_r^{s-1}\right) C + P_1 \star C\,, \tag{5.2.10a}$$

$$0 = \partial_r C + \frac{1}{2} P_{-1} \star C\,, \tag{5.2.10b}$$

$$0 = \left(\partial_\phi + \sum_{s=2}^{\infty}(-1)^s 2^{s-2} W^{(s)} \partial_r^{s-1}\right) C + [J_1\,, C] - \frac{1}{4}\sum_{s=2}^{\infty} Z^{(s)}\left[\mathbf{J}^s_{-s+1}\,, C\right] + r P_0 \star C\,, \tag{5.2.10c}$$

which shows that it is still sufficient to have information about the spin-s-spin-2 product rules as well as the commutation relations of the higher-spin Lie algebra.

Though the above equations can be written out in components, see (B.1.1) in the Appendix, the easiest way to proceed is to stay at the level of the master field, i.e. not plugging in the expansion (5.2.4) yet, and to try and assemble the charge-free part of the d'Alembert operator $-2\partial_u\partial_r - \partial_u/r + \partial_\phi^2/r^2$ acting on the master field out of the equations (5.2.10). It then turns out to be possible to reduce the equation to one that contains derivatives of C or commutators, only. An equation for c can then readily be read off, since commutators in the outer right slice of the algebra cannot produce the unit element $\mathbf{P}^1_0$ (as is apparent from the Lie brackets (4.2.1)). The equation thus obtained reads

$$\left(\Box^{(\text{hs})} - \mathcal{M}^2\right) c = 0\,, \tag{5.2.11}$$

with the higher-spin deformed d'Alembert operator

$$\begin{aligned}\Box^{(\text{hs})} \equiv &\sum_{s=2}^{\infty}(-1)^{s-1}2^{s-2}\left(\frac{Z^{(s)}}{r}\partial_r\left(r\partial_r^{s-1}\right) - \frac{1}{r}\partial_\phi\left(W^{(s)}\partial_r^{s-1}\right) - \frac{W^{(s)}}{r}\partial_r^{s-1}\left(\frac{\partial_\phi}{r}\right)\right)\\ &+\frac{1}{r}\sum_{s,s'=2}^{\infty}(-1)^{s+s'}2^{s+s'-4}W^{(s)}W^{(s')}\partial_r^{s-1}\left(\frac{\partial_r^{s'-1}}{r}\right) - 2\partial_u\partial_r - \frac{\partial_u}{r} + \frac{\partial_\phi^2}{r^2}\,.\end{aligned} \tag{5.2.12}$$

This is the supposed generalised Klein-Gordon equation fulfilled by a scalar field in a three-dimensional, asymptotically flat higher-spin gravity. Recall the coordinate dependence $Z^{(s)} = Z^{(s)}(\phi)$ and $W^{(s)} = W^{(s)}(u,\phi)$ suppressed in the notation. It is interesting to note that higher-spin contributions only occur in the form of higher-order derivatives with respect to the radial coordinate.

5.2.2 Unfolded Fierz-Pauli System

Up to this point only the outer right slice of the algebra $\mathfrak{ihs}(\mathcal{M}^2, \mathbb{S})$, i.e. the set of generators with index $l = 0$, has been used to expand the master field. In the following, the general set-up of an un-truncated master field, expanded into the complete algebra, will be considered in the case of a classical gauge background. Note that, as I argued above, one will still encounter a closed set of equations if the master field is restricted to some arbitrary finite value of l, as long as the gauge fields ω and e are restricted to spin two.

The expansion of the master field (5.2.1) within the complete algebra $\mathfrak{ihs}(\mathcal{M}^2, \mathbb{S})$ introduces fields ${}^{l}_{\xi}c^{s}_{m}$ that are coupled together by first-order equations. From now on, I will use a different version of the master equation, namely one in which the vielbein is multiplied from the right,

$$\mathrm{d}C + [\omega, C]_\star + C \star e = 0\,. \tag{5.2.13}$$

Given the form of the spin-s-spin-2 product rules (A.2.9), this choice will simplify the resulting equations considerably, without altering the physical content of the theory (since it is only a question of the choice of basis for the algebra). Note that this choice does not affect the previous considerations of the Klein-Gordon case, since C and e commuted there.

Extraction of Fundamental Equations

Equations (5.2.13) in the classical background (3.1.20) can be put into a particularly compact form by introducing the operator

$$\mathrm{D}_m \equiv e^\mu_m \partial_\mu + \frac{\delta_{m,0}}{r}\,\mathrm{ad}_{\omega_\phi}\,, \tag{5.2.14}$$

which utilises the inverse vielbein $e^\mu_m = \eta_{mn} g^{\mu\nu} e^n_\nu$. Then the master equation reads

$$\mathrm{D}_m C + C \star P_m = 0\,, \tag{5.2.15}$$

which, written out in algebra components, can be found as equation (B.1.4) in the Appendix. The conjecture now is that at each new slice of the algebra, enumerated by the index l, at least one new degree of freedom is introduced to the theory in form of the lowest-spin component ${}^{s-1}_{0}c^{s}_{m}$, and that these fields embody massive fields of spin $l = s-1$. Moreover, given that our notion of spin is tied to the standard commutation relation $\left[\,{}^{l}_{\xi}\mathbf{Q}^{s}_{m}\,, J_n\right] = (m-(s-1-\xi)n)\,{}^{l}_{\xi}\mathbf{Q}^{s}_{m+n}$, the appearance of the additional index ξ introduces even more degrees of freedom – there are in general infinitely many possibilities to realise a field of fixed spin $s-1-\xi$ by choosing combinations of s and ξ. This is a consequence of the presence of the element $\mathscr{C}$ in the UEA that commutes with all Lorentz generators. Note that in the case $l = s-1$ the parameter ξ is restricted to be an even number.

At lowest l, the following picture emerges: The first algebra slice, $l=0$, contains as fundamental field a single scalar ${}^{0}_{0}c^{1}_{0}$. The second slice, $l=1$, introduces a Proca field with components ${}^{1}_{0}c^{2}_{m}$. The third slice, $l=2$, gives rise to field components of topologically massive gravity (TMG) ${}^{2}_{0}c^{3}_{m}$ as well as to an additional scalar field ${}^{2}_{2}c^{3}_{0}$. This scheme goes on and on, up to arbitrary values of l (if a truncation is imposed) or up to infinity.

This idea in mind, one may go ahead and check whether or not the unfolded equations (5.2.15) allow for partial differential equations of first and second order for the fields ${}^{s-1}_{\ \xi}c^{s}_{m}$, where $s \geq 1$, $\xi = 0, 2, \ldots, 2\lfloor (s-1)/2 \rfloor$ and $|m| \leq s-1-\xi$; I will refer to these fields as *fundamental*. Indeed, taking a general ansatz one encounters the set of first-order equations

$$\begin{aligned} 0 = &\left(\partial_u + \frac{M}{2}\partial_r + \frac{M}{2r}(s-\xi-m)\right) {}^{s-1}_{\ \xi}c^{s}_{m} + \frac{2}{r}\frac{s-\xi-m}{s-\xi+m-1}\left(\partial_\phi + N\partial_r\right) {}^{s-1}_{\ \xi}c^{s}_{m-1} \\ &+ 2\frac{(s-\xi-m+1)^{\underline{2}}}{(s-\xi+m-1)^{\underline{2}}}\left(\partial_r + \frac{s-\xi+m-2}{r}\right) {}^{s-1}_{\ \xi}c^{s}_{m-2}\,. \end{aligned} \tag{5.2.16}$$

These are valid in the range $-(s-\xi-3) \leq m \leq s-\xi-1$, such that, at fixed $s-\xi$, there are $2s-2\xi-3$ equations. Similar to the strategy described in the earlier case of a higher-spin background, in order to extract a set of second-order equations it is advantageous to first stay at the level of the master field and assemble the scalar Klein-Gordon operator acting on it, which results in the expression

$$\left(\Box^{(0)} - \mathscr{M}^2\right) C = \frac{1}{r^2}\,\mathrm{ad}^2_{\omega_\phi}(C) + \frac{2}{r}\,\mathrm{ad}_{\omega_\phi}\left(C \star P_0 + \frac{N}{2r^2}C\right) + \frac{\partial_\phi M}{4r^2}\,[J_{-1}, C]\,, \tag{5.2.17}$$

where $\Box^{(0)}$ denotes the d'Alembert operator (5.2.8). One can then extract the coefficients of ${}^{s-1}_{\ \xi}\mathbf{Q}^{s}_{m}$ arising on the right-hand side of (5.2.17) and finally use a number of first-order equations to completely decouple the fields from each other. The result is the set of equations

$$\begin{aligned} 0 = &\left(\Box^{(0)} + \frac{M}{2r^2}\left((s-\xi)^{\underline{2}} - m^2\right) - \mathscr{M}^2\right) {}^{s-1}_{\ \xi}c^{s}_{m} \\ &+ \frac{2}{r^2}(s-\xi-m)\left(\partial_\phi + N\partial_r - \frac{N}{2r}\right) {}^{s-1}_{\ \xi}c^{s}_{m-1} \\ &+ \frac{M}{2r^2}(s-\xi+m)\left(\partial_\phi + N\partial_r - \frac{N}{2r} + \frac{\partial_\phi M}{2M}\right) {}^{s-1}_{\ \xi}c^{s}_{m+1} \\ &+ \frac{1}{r^2}(s-\xi-m+1)^{\underline{2}}\, {}^{s-1}_{\ \xi}c^{s}_{m-2} + \left(\frac{M}{4r}\right)^2 (s-\xi+m+1)^{\underline{2}}\, {}^{s-1}_{\ \xi}c^{s}_{m+2}\,. \end{aligned} \tag{5.2.18}$$

In the following we will see that these equations are indeed the Fierz-Pauli equations for massive higher-spin fields, freely propagating on the asymptotically flat spacetime, written down in a particular basis.

The free propagation of higher-spin fields of mass $\mathscr{M}$ is described by the equations of

Fierz and Pauli [97, 98],

$$\left(\Box-\mathscr{M}^2\right)\phi_{\mu_1\ldots\mu_\sigma}=0\,,\qquad \nabla^\mu\phi_{\mu\mu_2\ldots\mu_\sigma}=0\,,\qquad g^{\mu\nu}\phi_{\mu\nu\mu_3\ldots\mu_\sigma}=0\,. \tag{5.2.19}$$

The fields $\phi_{\mu_1\ldots\mu_\sigma}$ are totally symmetric, rank-σ tensors, each describing a collection of $2\sigma+1$ spin degrees of freedom. While the divergence equation in (5.2.19) ensures positivity of the energy or, equivalently, unitarity of the associated group representation, the zero-trace condition in (5.2.19) prevents the appearance of additional trace degrees of freedom or, equivalently, ensures irreducibility of the associated representation. If the latter was not imposed, the trace part of a tensor would be coupled in as an additional field of spin $\sigma-2$.

From Spacetime to Algebra Indices

Let me call $\sigma=s-\xi-1$ and perform the basis change from spacetime indices $\mu=u,r,\phi$ to flat $\mathfrak{isl}(2,\mathbb{R})$-indices $m\in\{0,\pm1\}$, which is given by the vielbein,

$$\phi_{\mu_1\ldots\mu_{s-\xi-1}}=\eta_{m_1n_1}\ldots\eta_{m_{s-\xi-1}n_{s-\xi-1}}e^{n_1}_{\mu_1}\ldots e^{n_{s-\xi-1}}_{\mu_{s-\xi-1}}\phi^{m_1\ldots m_{s-\xi-1}}\,. \tag{5.2.20}$$

The covariant derivative now acts via the spin connection $\omega_{mn}=-\varepsilon_{mnk}\omega^k$ (for more information, conventions and explicit expressions, see appendix B.4) and the wave equation in (5.2.19) and the divergence condition in (5.2.19) turn into

$$\left(\nabla^\mu\nabla_\mu-\mathscr{M}^2\right)\phi^{m_1\ldots m_{s-\xi-1}}=0\,,\qquad e^\mu_m\nabla_\mu\phi^{mm_2\ldots m_{s-\xi-1}}=0\,, \tag{5.2.21}$$

while the trace condition in (5.2.19) simply reads

$$\phi^{00m_3\ldots m_{s-\xi-1}}=4\phi^{1-1m_3\ldots m_{s-\xi-1}}\,. \tag{5.2.22}$$

Due to this identity and the symmetry of the indices it is sufficient to consider fields of the form $\phi^{1\ldots1-1\ldots-1}$ and $\phi^{1\ldots10-1\ldots-1}$; let me introduce the notation

$$(\pm1)_k\equiv\underbrace{\pm1\ldots\pm1}_{k}\,. \tag{5.2.23}$$

Then it is indeed possible to find an identification of the Fierz-Pauli fields in this particular basis with the fundamental fields ${}^{s-1}_{\ \xi}c^s_m$. Explicitly,

$$\phi^{(1)_{\frac{s-\xi-1+m}{2}}(-1)_{\frac{s-\xi-1-m}{2}}}=\frac{(s-\xi-1-m)!(s-\xi-1+m)!}{(2s-2)!}\,{}^{s-1}_{\ \xi}c^s_m\,, \tag{5.2.24a}$$

$$\phi^{(1)_{\frac{s-\xi-1+m}{2}}0(-1)_{\frac{s-\xi-1-m}{2}}}=\frac{2(s-\xi-1-m)!(s-\xi-1+m)!}{(2s-2)!}\,{}^{s-1}_{\ \xi}c^s_m\,. \tag{5.2.24b}$$

Key observation to arrive at this identification is that the field component $\phi_{uu\ldots u}$ always fulfils a Klein-Gordon equation and so does a particular linear combination of field components ${}^{s-1}_{\xi}c^{s}_{m}$; then, comparing both expressions in the $\mathfrak{isl}(2,\mathbb{R})$-basis, one can read off (5.2.24a) and from this simply guess equation (5.2.24b).

Taking everything together, there is an agreement between the fundamental equations derived above and the Fierz-Pauli system. In particular, equations (5.2.18) are precisely the wave equations in (5.2.19), while equations (5.2.16) are precisely the divergence conditions in (5.2.19) for massive fields of spin $s-\xi-1$, written down in the algebra basis. The trace condition in (5.2.19) is in a sense partly imposed: though there are additional degrees of freedom, namely in the form of fields with index $\xi > 0$ (precisely the same number of fields that would arise if the trace condition was abandoned), these additional fields do not couple amongst each other or to the $(\xi = 0)$-fields. In principle, any fundamental field of higher-ξ index can be consistently set to zero (not the corresponding auxiliary fields, however).

All in all, these considerations show that the master equation (5.2.2) on $\mathfrak{ihs}(\mathscr{M}^2,\mathbb{S})$ provides a mechanism of unfolding for a complete Fierz-Pauli system of an infinity of higher-spin fields, all of the same mass that equals the parametrisation of the mass Casimir element.

On the Decoupling of Equations

Up to now I have only shown that the Fierz-Pauli system emerges as the set of equations for the fundamental fields, thereby not discussing the actual independence of the fields ${}^{s-1}_{\xi}c^{s}_{m}$ from lower-spin fields. Since the first-order equations (B.1.4) couple together auxiliary and fundamental fields of all different indices l, s and ξ, it is not clear on first sight that (a) the fields ${}^{s-1}_{\xi}c^{s}_{m}$ cannot be expressed through derivatives of lower-spin fields, possibly even through the Klein-Gordon field ${}^{0}_{0}c^{1}_{m}$, which would render them auxiliary; (b) there are no additional first- or second-order partial differential equations for the fundamental fields, possibly coupling different fields together; and (c) there are no higher-order partial differential equations on the fundamental fields (that are not a consequence of the wave and the divergence equation), possibly coupling different fields together. These questions will be addressed in the following.

The uniqueness of the divergence and wave equations can readily be shown by performing a general ansatz. Any up-to-second-order partial differential equation, possibly coupling together different fundamental fields, must be of the form

$$\sum_{s=1}^{\infty}\sum_{\substack{\xi=0\\ \xi\text{ even}}}^{2\lfloor\frac{s-1}{2}\rfloor}\sum_{|m|\leq s-\xi-1}\left(\alpha^{\mu\nu}\partial_{\mu}\partial_{\nu}\,{}^{s-1}_{\xi}c^{s}_{m}+\alpha^{\mu}\partial_{\mu}\,{}^{s-1}_{\xi}c^{s}_{m}+\alpha\,{}^{s-1}_{\xi}c^{s}_{m}\right)=0\,, \tag{5.2.25}$$

where the coefficients $\alpha^{\mu\nu}$, α^{μ} and α all depend on the indices (s,ξ,m). Using the first-order equations (B.1.4), one can eliminate all derivatives and by comparison of coefficients in front of different fields – the assumption that there is no purely algebraic relation between different fields is inherent in the expansion of the master field to begin with – one may derive recurrence relations for the unknown coefficients. As it turns out, these recurrence relations only involve coefficients of the same indices s and ξ, thus showing that there is indeed no coupling between different fundamental fields at the level of first- or second-order differential equations.

In principle, one could carry on with this procedure to ever higher order in derivatives, but the necessary calculations become unreasonably involved due to more and more auxiliary fields entering. Alternatively, one may look at explicit examples. An implementation of a general ansatz of the form (5.2.25) up to third and fourth order in derivatives in *Mathematica* in the case $l \leq 2$, i.e. when the fundamental fields involved are the two scalars ${}^{0}_{0}c^{1}_{0}$ and ${}^{2}_{2}c^{3}_{0}$, the Proca field ${}^{1}_{0}c^{2}_{m}$ and the TMG field ${}^{2}_{0}c^{3}_{m}$, leads to solutions for the unknown coefficients $\alpha^{\mu\cdots}$. These solutions turn out to not mix coefficients of different indices s and ξ, thus showing that there is indeed no coupling of fundamental fields through differential equations up to order four.

5.3 Wilson Lines as Holographic Probes

As spelled out in subsection 3.3.2 of the Foundations, Wilson lines provide a valuable tool to perform holographic calculations of dual field-theory observables. One may utilise the dynamics of a probe field living on a Wilson line whose endpoints are attached to the boundary of a spacetime to determine the entanglement entropy of the interval that is cut out at the boundary that way.

Note that, additionally, Wilson lines may be used to re-introduce a concept of geometry to a Chern-Simons formulation. Particularly in the case of higher-spin gravity, these objects may serve as a gauge-invariant replacement for the metric field, which no longer holds an invariant description of spacetime when higher-spin symmetries are present.

In this section I will reproduce known results for the entanglement entropy derived from Minkowski or flat-space cosmology spacetimes [60, 114] but in a revised setting. I will use a single-particle action on the group manifold as starting point, thus introducing an auxiliary system that automatically fulfils the necessary invariance properties and furnishes a unitary (induced) representation of the Poincaré symmetry. The entanglement entropy of the dual BMS field theory is then calculated from the euclidean on-shell value of the probe action. Eventually, I will discuss the generalisation to the theory of higher-spin gravity defined in section 5.1.

5.3.1 Construction of a Probe Action

Starting point will be a one-parameter action for a massive, spinning particle on the group manifold $ISO(2,1)$ as it was given in [248, 249] (for the AdS case see [250, 251]). In terms of the non-deformed bilinear form, i.e. (3.1.18), or (4.2.10) with $\mathcal{S} = 1$ and $\mathcal{M}^2 = 0$, the Lagrange density reads

$$\mathcal{L} = \left\langle K, g^{-1}\dot{g}\right\rangle , \tag{5.3.1}$$

where $K = mJ_0 - \kappa P_0$ captures the mass m of the particle and its spin κ. The Cartan-Maurer element $g^{-1}\dot{g}$ is valued in the Lie algebra $\mathfrak{iso}(2,1)$. As in earlier instances we use the decomposition $g = g_\text{T} g_\text{L}$ of a general group element into a Lorentz part $g_\text{L} = \exp(\xi_\text{L})$ and a translation[3] part $g_\text{T} = \exp(\xi_\text{T})$. Abbreviating expansion coefficients like $g_\text{L} J_0 g_\text{L}^{-1} = \vartheta^a J_a$ and $g_\text{L} P_0 g_\text{L}^{-1} = \vartheta^a P_a$, the Lagrange density written out in components reads

$$\mathcal{L} = m\eta_{ab}\vartheta^a \left(\dot{g}_\text{T} g_\text{T}^{-1}\right)^b - \kappa\eta_{ab}\vartheta^a \left(\dot{g}_\text{L} g_\text{L}^{-1}\right)^b . \tag{5.3.2}$$

This will be the starting point for all following considerations.

Given the two contributions to the Lagrangian it seems reasonable to use the deformed bilinear form (4.2.10), at least as a short-hand, since it naturally contains parameters that can be identified with the mass and spin parameters of the probe[4]. From now on using this non-ad-invariant bilinear form, the Lagrange density simply reads

$$\mathcal{L} = \frac{1}{\mathcal{M}} \left\langle P_0, g^{-1}\dot{g}\right\rangle \tag{5.3.3}$$

and agrees with (5.3.2) upon identification $m = \mathcal{M}$ and $\kappa = -\mathcal{S}/\mathcal{M}$, as expected.[5] Thus, the action of the full theory is

$$S = S_\text{CS} + \frac{1}{\mathcal{M}} \int \mathrm{d}s \left\langle P_0, g^{-1}\dot{g}\right\rangle . \tag{5.3.4}$$

The next logical step is to introduce appropriate momenta. Under left-action of some infinitesimal group element $h(s)$, where we write $h^{-1}\dot{h} = \varepsilon_\text{T} + \varepsilon_\text{L}$, the Lagrange density transforms like

$$\mathcal{L} \mapsto \mathcal{L} + \mathcal{M}\eta_{ab}\vartheta^a\varepsilon_\text{T}^b + \frac{\mathcal{S}}{\mathcal{M}}\eta_{ab}\vartheta^a\varepsilon_\text{L}^b + \mathcal{M}\eta_{ab}\varepsilon^a{}_{cd}\xi_\text{T}^c\vartheta^d\varepsilon_\text{L}^b \tag{5.3.5}$$

and one can read off a set of charges. For convenience one may take a certain linear combination of these and define

$$p^a = \vartheta^a , \qquad j^a = -\frac{\mathcal{S}}{\mathcal{M}^2}\vartheta^a + \varepsilon^a{}_{bc}\vartheta^b\xi_\text{L}^c . \tag{5.3.6}$$

[3]Note that the translational part of the Cartan-Maurer element gives rise to the spacetime coordinates.
[4]Note that a similar procedure was used in [60, 114], in the latter called "twisted trace". However, there was also a deformation of the form $\langle J, J\rangle \neq 0$ being used, therefore turning to chiral gravity (see [195, 196]), while the actual (un-deformed) $\mathfrak{iso}(2,1)$-form has not been used.
[5]Apparently, κ is the (arbitrary) value of spin of the probe particle, while $\mathcal{S}$ is its helicity [252].

These can be viewed as components of spatial momentum and total angular momentum. Their algebraic versions are defined to be $\mathcal{P} = p^a P_a$ and $\mathcal{J} = j^a P_a$ and their normalisation can be expressed as $\langle \mathcal{P}, \mathcal{P} \rangle = -\mathscr{M}^2$ and $\langle \mathcal{P}, \mathcal{J} \rangle = \mathcal{S}$. When used as dynamical quantities in the action, these normalisation conditions have to be enforced by inclusion of Lagrange multipliers. Let me denote the group elements g_T and g_L by C_T and C_L from now on to highlight their role as matter fields, then the action expressed through the above defined momenta reads

$$S = \frac{1}{\mathscr{M}} \int \mathrm{d}s \Big(\big\langle \mathcal{P}, \dot{C}_\mathrm{T} C_\mathrm{T}^{-1} + [\xi_\mathrm{T}, \dot{C}_\mathrm{L} C_\mathrm{L}^{-1}] \big\rangle - \frac{\mathscr{M}^2}{\mathcal{S}} \langle \mathcal{J}, \dot{C}_\mathrm{L} C_\mathrm{L}^{-1} \rangle \\ + \lambda_\mathrm{P} \left(\langle \mathcal{P}, \mathcal{P} \rangle + \mathscr{M}^2 \right) + \lambda_\mathrm{J} \left(\langle \mathcal{P}, \mathcal{J} \rangle - \mathcal{S} \right) \Big). \tag{5.3.7}$$

Finally, I will introduce covariant derivatives. It may not be surprising at this point that covariant derivatives have to be introduced separately for the translational field component and the Lorentz field component. In analogy to the findings in section 5.2 for the coupling of a massive scalar field to the spacetime, define

$$\boxed{\mathrm{D}_s C_\mathrm{T} = \dot{C}_\mathrm{T} + [\omega_s, C_\mathrm{T}] + e_s C_\mathrm{T}, \qquad \nabla_s C_\mathrm{L} = \dot{C}_\mathrm{L} + [\omega_s, C_\mathrm{L}],} \tag{5.3.8}$$

with worldline spin connection and worldline vielbein

$$\omega_s = \omega_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}, \qquad e_s = e_\mu \frac{\mathrm{d}y^\mu}{\mathrm{d}s}, \tag{5.3.9}$$

where x^μ and y^μ are coordinates on the group manifold corresponding to Lorentz transformations and translations, respectively. Then the covariant worldline-action of a massive spinning particle reads

$$\boxed{S = \frac{1}{\mathscr{M}} \int \mathrm{d}s \Big(\langle \mathcal{P}, (\mathrm{D}_s C) C^{-1} \rangle - \Big\langle \mathcal{P} + \frac{\mathscr{M}^2}{\mathcal{S}} \mathcal{J}, (\nabla_s C_\mathrm{L}) C_\mathrm{L}^{-1} \Big\rangle \\ + \lambda_\mathrm{P} \left(\langle \mathcal{P}, \mathcal{P} \rangle + \mathscr{M}^2 \right) + \lambda_\mathrm{J} \left(\langle \mathcal{P}, \mathcal{J} \rangle - \mathcal{S} \right) \Big).} \tag{5.3.10}$$

Under an $ISO(2,1)$ group transformation the worldline spin connection and vielbein transform as

$$\omega_s \mapsto \left(g_\mathrm{L}^{-1} \omega_\mu g_\mathrm{L} + g_\mathrm{L}^{-1} \partial_\mu g_\mathrm{L} \right) \frac{\mathrm{d}x^\mu}{\mathrm{d}s}, \tag{5.3.11a}$$

$$e_s \mapsto g_\mathrm{L}^{-1} \left[\left(e_\mu + \partial_\mu \xi_\mathrm{T} \right) \frac{\mathrm{d}y^\mu}{\mathrm{d}s} + [\omega_\mu, \xi_\mathrm{T}] \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \right] g_\mathrm{L}, \tag{5.3.11b}$$

and the momenta behave like

$$\mathcal{P} \mapsto g_L^{-1}\mathcal{P} g_L\,, \qquad\qquad \mathcal{J} \mapsto g_L^{-1}\mathcal{J} g_L\,. \tag{5.3.12}$$

Then, assigning to the matter field components the transformation behaviour

$$C_T \mapsto g_L^{-1} g_T^{-1} C_T g_L\,, \tag{5.3.13a}$$
$$C_L \mapsto g_L^{-1} C_L g_L\,, \tag{5.3.13b}$$

the above action is indeed invariant.

The equations of motion can now be obtained by variation with respect to the different fields, momenta and Lagrange multipliers. Variation with respect to the momenta $\mathcal{J}$ and $\mathcal{P}$ yields

$$\mathcal{P} = \frac{\left(\nabla_s C_L \cdot C_L^{-1}\right)^a P_a}{\lambda_J}\,, \qquad \mathcal{J} = -\frac{1}{\lambda_J}\left(D_s C_T \cdot C_T^{-1} + \left[\xi_T, \nabla_s C_L \cdot C_L^{-1}\right] + 2\lambda_P \mathcal{P}\right) \tag{5.3.14}$$

and variation with respect to the fields C_T and C_L yields

$$\dot{\mathcal{P}} = [\mathcal{P}, \omega_s]\,, \qquad \dot{\mathcal{J}} = [\mathcal{J}, \omega_s] - \frac{1}{\lambda_J}\left[\nabla_s C_L \cdot C_L^{-1}, e_s\right]. \tag{5.3.15}$$

The consequence of the Lagrange multipliers is obvious. Using the equations of motion one may determine the on-shell action to be

$$S_{\text{on-shell}} = \frac{2}{\mathcal{M}} \int \mathrm{d}s \,\left(\mathcal{M}^2 \lambda_P - \mathcal{S}\lambda_J\right). \tag{5.3.16}$$

To actually calculate the on-shell value of the action, it is necessary to specify boundary conditions on the probe fields C_L and C_T at some fixed endpoints, an initial point $s = s_i$ and a final point $s = s_f$.

5.3.2 Entanglement Entropy from Flat-Space Cosmologies

It is now time to solve the equations of motion noted down in the previous subsection in order to find the on-shell value of the action, which should then correspond to the entanglement entropy of an interval in the dual field theory. I will first illustrate the strategy in the case of Minkowski spacetime, i.e. the background given by (3.1.20) with $M(\phi) = -1$ and $N(u,\phi) = 0$. Afterwards, I will make sure that the results of [114] are reproduced in the case of a flat-space cosmology background ((3.1.20) with $M(\phi) = M > 0$ and $N(u,\phi) = N \neq 0$), as well.

The key observation is that the gauge field $A = \omega + e$ is pure gauge, i.e. it can be written as $A = g^{-1}\,\mathrm{d}g$, or, in other words, it can be gauged to the value $a = 0$. I will refer to this as *nothingness gauge*. Since the action we are dealing with as well as the equations of motion are gauge invariant, we may solve them in nothingness gauge and subsequently transform the solutions back to the original gauge. In nothingness gauge, both the worldline vielbein and spin connection vanish and all covariant derivatives become simple derivatives, thus transforming the equations of motion to

$$\dot{\mathcal{P}} = 0\,, \qquad \dot{\mathcal{J}} = 0\,, \tag{5.3.17}$$

as well as

$$\left(\dot{C}_{\mathrm{L}} C_{\mathrm{L}}^{-1}\right)^a = \lambda_{\mathrm{J}} p^a\,, \qquad \dot{\xi}_{\mathrm{T}} + \left[\xi_{\mathrm{T}}\,, \dot{C}_{\mathrm{L}} C_{\mathrm{L}}^{-1}\right] = -\left(\lambda_{\mathrm{J}} \mathcal{J} + 2\lambda_{\mathrm{P}} \mathcal{P}\right)\,. \tag{5.3.18}$$

The equation for $\dot{C}_{\mathrm{L}}$ in (5.3.18) together with (5.3.17) can immediately be solved,

$$C_{\mathrm{L}} = c_0\, \mathrm{e}^{\alpha p^a J_a}\,, \qquad \dot{\alpha} = \lambda_{\mathrm{J}}\,, \tag{5.3.19}$$

with a constant group element c_0 that will be determined later. Applying $\langle \mathcal{P}\,, .\rangle$ to the equation for $\dot{\xi}_{\mathrm{T}}$ in (5.3.18) together with (5.3.17) and the normalisation conditions for $\mathcal{P}$ and $\mathcal{J}$ one can extract

$$\langle \mathcal{P}\,, \Delta\xi_{\mathrm{T}}\rangle = 2\mathscr{M}^2 \int_{s_{\mathrm{i}}}^{s_{\mathrm{f}}} \mathrm{d}s\; \lambda_{\mathrm{P}} - \mathcal{S} \int_{s_{\mathrm{i}}}^{s_{\mathrm{f}}} \mathrm{d}s\; \lambda_{\mathrm{J}}\,, \tag{5.3.20}$$

where the Wilson line is attached to some initial and final points, s_{i} and s_{f}, and the difference $\Delta\xi_{\mathrm{T}} \equiv \xi_{\mathrm{T}}(s_{\mathrm{f}}) - \xi_{\mathrm{T}}(s_{\mathrm{i}})$ will depend on the boundary conditions chosen. Thus, the on-shell action can be written

$$\mathscr{M} S_{\text{on-shell}} = \langle \mathcal{P}\,, \Delta\xi_{\mathrm{T}}\rangle - \mathcal{S}\Delta\alpha\,, \tag{5.3.21}$$

where $\Delta\alpha \equiv \alpha(s_{\mathrm{f}}) - \alpha(s_{\mathrm{i}})$ is determined by the boundary conditions. Note that, so far, we have not specified the gravitational background (apart from demanding that the corresponding gauge field is pure gauge).

In the following we specify boundary conditions. First, we put the endpoints of the Wilson line to two distinct points of the spacetime, but at fixed radial coordinate $r = r_0$, i.e. we parametrise the initial point as $s_{\mathrm{i}} = (u_{\mathrm{i}}\,, r_0\,, \phi_{\mathrm{i}})$ and the final point as $s_{\mathrm{f}} = (u_{\mathrm{f}}\,, r_0\,, \phi_{\mathrm{f}})$. For the probe fields C_{L} and C_{T} we choose the same boundary conditions as given in [114] (only correcting a typo in the final condition), namely

$$C_{\mathrm{L}}(s_{\mathrm{i}}) = C_{\mathrm{L}}(s_{\mathrm{f}}) = \mathrm{e}^{-\frac{a r_0}{2} J_{-1}}\,, \qquad C_{\mathrm{T}}(s_{\mathrm{i}}) = C_{\mathrm{T}}(s_{\mathrm{f}}) = \mathrm{e}^{-\frac{r_0}{2} P_{-1}}\,. \tag{5.3.22}$$

Here I included some inconsequential constant a of inverse length scale one in order to have a scale-free exponent. The boundary conditions (5.3.22) now have to be transformed into nothingness gauge using the gauge transformation behaviour (5.3.13) of the probe fields. Thus, for the translational part we find

$$C_{\mathrm{T}}(s_{\mathrm{i}}) = g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})g_{\mathrm{T}}^{-1}(s_{\mathrm{i}})C_{\mathrm{T},0}(s_{\mathrm{i}})g_{\mathrm{L}}(s_{\mathrm{i}}) \quad\rightarrow\quad \mathrm{e}^{\xi_{\mathrm{T}}(s_{\mathrm{i}})} = g_{\mathrm{T}}(s_{\mathrm{i}})g_{\mathrm{L}}(s_{\mathrm{i}})C_{\mathrm{T}}(s_{\mathrm{i}})g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})\,, \tag{5.3.23a}$$

$$C_{\mathrm{T}}(s_{\mathrm{f}}) = g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})g_{\mathrm{T}}^{-1}(s_{\mathrm{f}})C_{\mathrm{T},0}(s_{\mathrm{f}})g_{\mathrm{L}}(s_{\mathrm{f}}) \quad\rightarrow\quad \mathrm{e}^{\xi_{\mathrm{T}}(s_{\mathrm{f}})} = g_{\mathrm{T}}(s_{\mathrm{f}})g_{\mathrm{L}}(s_{\mathrm{f}})C_{\mathrm{T}}(s_{\mathrm{f}})g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})\,, \tag{5.3.23b}$$

such that

$$\mathrm{e}^{\Delta\xi_{\mathrm{T}}} = g_{\mathrm{T}}(s_{\mathrm{f}})g_{\mathrm{L}}(s_{\mathrm{f}})C_{\mathrm{T}}(s_{\mathrm{f}})g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})g_{\mathrm{L}}(s_{\mathrm{i}})C_{\mathrm{T}}^{-1}(s_{\mathrm{i}})g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})g_{\mathrm{T}}^{-1}(s_{\mathrm{i}}) \tag{5.3.24}$$

$$= g_{\mathrm{T}}(s_{\mathrm{f}})\exp\left(-\frac{r_0}{2}g_{\mathrm{L}}(s_{\mathrm{f}})P_{-1}g_{\mathrm{L}}^{-1}(s_{\mathrm{f}}) + \frac{r_0}{2}g_{\mathrm{L}}(s_{\mathrm{i}})P_{-1}g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})\right)g_{\mathrm{T}}^{-1}(s_{\mathrm{i}})\,. \tag{5.3.25}$$

The boundary conditions on the Lorentz probe field provide us with information about the constant group element c_0,

$$C_{\mathrm{L}}(s_{\mathrm{i}}) = g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})C_{\mathrm{L},0}(s_{\mathrm{i}})g_{\mathrm{L}}(s_{\mathrm{i}}) \quad\rightarrow\quad c_0 = g_{\mathrm{L}}(s_{\mathrm{i}})C_{\mathrm{L}}(s_{\mathrm{i}})g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})\,\mathrm{e}^{\alpha(s_{\mathrm{i}})p^a J_a}\,, \tag{5.3.26a}$$

$$C_{\mathrm{L}}(s_{\mathrm{f}}) = g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})C_{\mathrm{L},0}(s_{\mathrm{f}})g_{\mathrm{L}}(s_{\mathrm{f}}) \quad\rightarrow\quad c_0 = g_{\mathrm{L}}(s_{\mathrm{f}})C_{\mathrm{L}}(s_{\mathrm{f}})g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})\,\mathrm{e}^{\alpha(s_{\mathrm{f}})p^a J_a}\,, \tag{5.3.26b}$$

which, in turn, implies

$$\mathrm{e}^{\Delta\alpha p^a J_a} = g_{\mathrm{L}}(s_{\mathrm{f}})C_{\mathrm{L}}^{-1}(s_{\mathrm{f}})g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})g_{\mathrm{L}}(s_{\mathrm{i}})C_{\mathrm{L}}(s_{\mathrm{i}})g_{\mathrm{L}}^{-1}(s_{\mathrm{i}}) \tag{5.3.27}$$

$$= \exp\left(\frac{ar_0}{2}g_{\mathrm{L}}(s_{\mathrm{f}})J_{-1}g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})\right)\exp\left(-\frac{ar_0}{2}g_{\mathrm{L}}(s_{\mathrm{i}})J_{-1}g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})\right). \tag{5.3.28}$$

Minkowski

The gauge transformations in the case of Minkowski spacetime can be decomposed into translations and Lorentz transformations as

$$g_{\mathrm{T}} = \mathrm{e}^{u(P_1+\frac{1}{4}P_{-1})}\exp\left(\frac{r}{2}g_{\mathrm{L}}P_{-1}g_{\mathrm{L}}^{-1}\right), \tag{5.3.29a}$$

$$g_{\mathrm{L}} = \mathrm{e}^{\phi(J_1+\frac{1}{4}J_{-1})}\,, \tag{5.3.29b}$$

and one may immediately solve

$$\Delta\xi_{\mathrm{T}} = \Delta u\left(P_1 + \frac{1}{4}P_{-1}\right). \tag{5.3.30}$$

Equation (5.3.28) can be solved, either by choosing the fundamental matrix representation of $\mathfrak{isl}(2,\mathbb{R})$ or by repeated application of the Baker-Campbell-Hausdorff formula and the bilinear form: Given an arbitrary element of the form $\xi = \xi^m J_m$ and the bilinear form (4.2.10) with the abbreviation $\eta_{mn} = (-1)^m(1-m)!(1+m)!\delta_{m+n,0}$, call

$$X_{mn} \equiv \left\langle P_m, \mathrm{e}^{-\xi} J_n \mathrm{e}^{\xi} \right\rangle , \tag{5.3.31}$$

then one may show that

$$\cosh\left(\sqrt{\xi\cdot\xi}\right) = -\frac{1}{2}\left(1 - X_{00} + \frac{X_{1,-1}+X_{-1,1}}{2}\right) , \tag{5.3.32}$$

where $\xi\cdot\xi = \eta_{mn}\xi^m\xi^n$ and it is also possible to construct solutions for the components of ξ from the X_{mn}. The necessary adjoint expressions are

$$g_{\mathrm{L}}(s_{\mathrm{i/f}})J_{-1}g_{\mathrm{L}}^{-1}(s_{\mathrm{i/f}}) = 2\left((1-\cos(\phi_{\mathrm{i/f}}))J_1 + \sin(\phi_{\mathrm{i/f}})J_0 + \frac{1+\cos(\phi_{\mathrm{i/f}})}{4}J_{-1}\right) , \tag{5.3.33a}$$

$$g_{\mathrm{L}}(s_{\mathrm{i/f}})P_{-1}g_{\mathrm{L}}^{-1}(s_{\mathrm{i/f}}) = 2\left((1-\cos(\phi_{\mathrm{i/f}}))P_1 + \sin(\phi_{\mathrm{i/f}})P_0 + \frac{1+\cos(\phi_{\mathrm{i/f}})}{4}P_{-1}\right) \tag{5.3.33b}$$

and at this point it is important to note that one actually has to take the euclidean version of the theory, in which the normalisation conditions for the momenta come with an additional minus sign, thus setting $p^a p_a = 1$. Then one arrives at

$$\cosh\left(\frac{\Delta\alpha}{2}\right) = 1 + \frac{a^2 r_0^2}{2}\sin^2\frac{\Delta\phi}{2} \tag{5.3.34}$$

as well as an explicit solution for the components of $\mathcal{P}$. Putting these results back into the expression (5.3.21) for the on-shell action, keeping in mind the euclideanisation of the theory (i.e. $\mathcal{S}\mapsto-\mathcal{S}$ and taking a positive bilinear form) and choosing appropriate signs where necessary, one arrives at

$$S_{\text{on-shell}} = \frac{2\mathcal{S}}{\mathcal{M}}\operatorname{arcosh}\left(1 + \frac{a^2r_0^2}{2}\sin^2\frac{\Delta\phi}{2}\right) + \frac{a^2r_0^2\mathcal{M}}{4}\frac{\Delta u \sin\Delta\phi}{\sqrt{\left(1+\frac{a^2r_0^2}{2}\sin^2\frac{\Delta\phi}{2}\right)^2 - 1}} . \tag{5.3.35}$$

Finally, to attach the endpoints to the boundary, we have to send $r_0\to\infty$, for which the on-shell value becomes

$$S_{\text{on-shell}} \xrightarrow{ar_0\gg 1} \frac{4\mathcal{S}}{\mathcal{M}}\ln\left(ar_0\sin\frac{\Delta\phi}{2}\right) + \mathcal{M}\Delta u\cot\frac{\Delta\phi}{2} . \tag{5.3.36}$$

This precisely matches the entanglement entropy of the dual field theory [60, 114] when the central charges of the asymptotic symmetry algebra (see (3.3.3)) are identified with the parametrisations of the Casimir elements like $c_{\mathrm{M}} = 12\mathcal{M}$ and $c_{\mathrm{L}} = 24\mathcal{S}/\mathcal{M}$.

Though this result is in general not new to the literature, there are some novel and interesting aspects of the derivation here presented. First of all, the origin of the action of the topological probe has been made apparent, at least in so far that the starting point here is an action of a massive, spinning particle on the group manifold $ISO(2,1)$ that has been identified in the literature before. The form of the covariant derivatives (5.3.8) is closely related to the coupling equation for massive fields introduced in section 5.2. All in all, the invariance properties of the probe action and the manifestation of the non-semi-simplicity of the underlying symmetry appear to be more transparent in the present set-up.

Secondly, the role of the deformed bilinear form (4.2.10) should be clearer in the Wilson-line set-up, since it at least provides a handy tool to lift quantities appearing in the action to the algebra level, for example the constraints on the normalisation of momenta. Any calculation in this section can as well be carried out without the usage of that particular deformation and, yet, it is the feeling of the author that the deformed bilinear form better reflects the semi-direct sum structure of the underlying symmetry algebra and that it may prove helpful in future applications.

Lastly, and most importantly, the result obtained for the entanglement entropy brings about a remarkable conclusion: though classical Einstein gravity possesses an asymptotic symmetry with one vanishing central charge $c_{\mathrm{L}} = 0$, the presence of a massive, *spinning* probe apparently enables one to detect a non-vanishing central charge $c_{\mathrm{L}} \neq 0$. Usually, a non-vanishing second central charge is associated to chiral gravity, a deformation of Einstein gravity by an additional Lorentz-Chern-Simons term [195, 196], which describes topologically massive gravity [182, 183, 185]. This is, however, not the case in the present calculation, where a standard Minkowski background was considered.

Flat-Space Cosmologies

Turning to the more general case of flat-space cosmologies, i.e. asymptotically flat solutions (3.1.20) with constant $M > 0$ and $N \neq 0$, the corresponding gauge transformation decomposed into translations and Lorentz transformations reads

$$\begin{aligned} g_{\mathrm{T}} = \mathrm{e}^{\xi_{\mathrm{T}}} = \exp\Bigg[&\left(u + \frac{N}{M}\phi\right)\left(P_1 - \frac{M}{4}P_{-1}\right) + \frac{r}{2} g_{\mathrm{L}} P_{-1} g_{\mathrm{L}}^{-1} \\ &- \frac{2N}{M}\cosh\left(\frac{\sqrt{M}}{2}\phi\right)\left(\frac{\sinh\left(\frac{\sqrt{M}}{2}\phi\right)}{\sqrt{M}}P_1 \right. \\ &\left. - \cosh\left(\frac{\sqrt{M}}{2}\phi\right)P_0 + \frac{M}{4}\frac{\sinh\left(\frac{\sqrt{M}}{2}\phi\right)}{\sqrt{M}}P_{-1}\right)\Bigg], \end{aligned} \tag{5.3.37a}$$

$$g_{\mathrm{L}} = \mathrm{e}^{\xi_{\mathrm{L}}} = \exp\left[\phi\left(J_1 - \frac{M}{4}J_{-1}\right)\right]. \tag{5.3.37b}$$

The on-shell action is still given by (5.3.21) and the boundary conditions remain unchanged. Therefore, one may still use (5.3.25) to extract $\Delta\xi_{\mathrm{T}}$, which gives

$$\begin{aligned}\Delta\xi_{\mathrm{T}} = &\left(\Delta u + \frac{N}{M}\Delta\phi\right)\left(P_1 - \frac{M}{4}P_{-1}\right) + \frac{2N}{M}\Bigg[\left(\cosh^2\left(\frac{\sqrt{M}\phi_{\mathrm{f}}}{2}\right) - \cosh^2\left(\frac{\sqrt{M}\phi_{\mathrm{i}}}{2}\right)\right)P_0 \\ &- \frac{\sinh\left(\frac{\sqrt{M}\phi_{\mathrm{f}}}{2}\right)\cosh\left(\frac{\sqrt{M}\phi_{\mathrm{f}}}{2}\right) - \sinh\left(\frac{\sqrt{M}\phi_{\mathrm{i}}}{2}\right)\cosh\left(\frac{\sqrt{M}\phi_{\mathrm{i}}}{2}\right)}{\sqrt{M}}\left(P_1 + \frac{M}{4}P_{-1}\right)\Bigg] \,.\end{aligned} \tag{5.3.38}$$

Similarly, $\Delta\alpha$ as well as the solution for the components of $\mathcal{P}$ may still be extracted from (5.3.28), which results in

$$\cosh\left(\frac{\Delta\alpha}{2}\right) = 1 + \frac{a^2 r_0^2}{2M}\sinh^2\left(\frac{\sqrt{M}}{2}\Delta\phi\right). \tag{5.3.39}$$

The result for the on-shell action (taking care of the appropriate signs due to euclideanisation) reads

$$\begin{aligned}S_{\text{on-shell}} = &\frac{2\mathcal{S}}{\mathscr{M}}\operatorname{arcosh}\left(1 + \frac{a^2 r_0^2}{2M}\sinh^2\left(\frac{\sqrt{M}}{2}\Delta\phi\right)\right) \\ &+ \frac{a^2 r_0^2 \mathscr{M}}{2\sqrt{M}}\frac{\left(\Delta u + \frac{N}{M}\Delta\phi\right)\sinh\left(\frac{\sqrt{M}}{2}\Delta\phi\right)\cosh\left(\frac{\sqrt{M}}{2}\Delta\phi\right)}{\sqrt{\left(1 + \frac{a^2 r_0^2}{2M}\sinh^2\left(\frac{\sqrt{M}}{2}\Delta\phi\right)\right)^2 - 1}} \\ &- \frac{a^2 r_0^2 N\mathscr{M}}{M^2}\frac{\sinh^2\left(\frac{\sqrt{M}}{2}\Delta\phi\right)}{\sqrt{\left(1 + \frac{a^2 r_0^2}{2M}\sinh^2\left(\frac{\sqrt{M}}{2}\Delta\phi\right)\right)^2 - 1}} \,.\end{aligned} \tag{5.3.40}$$

The behaviour of this expression for $r_0 \to \infty$ then is

$$\begin{aligned}S_{\text{on-shell}} \overset{a r_0 \gg 1}{\longrightarrow} &\frac{c_{\mathrm{L}}}{6}\ln\left(\frac{a r_0}{\sqrt{M}}\sinh\left(\frac{\sqrt{M}}{2}\Delta\phi\right)\right) \\ &+ \frac{c_{\mathrm{M}}}{12}\left(\sqrt{M}\left(\Delta u + \frac{N}{M}\Delta\phi\right)\coth\left(\frac{\sqrt{M}}{2}\Delta\phi\right) - \frac{2N}{M}\right),\end{aligned} \tag{5.3.41}$$

where the same identification of Casimir elements and central charges as in the Minkowski case, $c_{\mathrm{M}} = 12\mathscr{M}$ and $c_{\mathrm{L}} = 24\mathcal{S}/\mathscr{M}$, was made. This is precisely the result for the entanglement entropy of the dual BMS field theory at finite temperature as obtained in the literature [114]. The Minkowski case $M = -1$, $N = 0$ is naturally included.

This is the final illustration of the applicability of the Wilson-line prescription as constructed in this work.

5.3.3 Generalisation to Higher-Spin Gravity

Given the Wilson-line prescription of the previous subsection and the theory of higher-spin gravity obtained in section 5.1, it is a natural question how to generalise the above results to higher-spin gauge backgrounds.

On Group Elements from Exponentiation

To appropriately generalise the previous considerations to the higher-spin case, great care is to be taken with respect to the group elements appearing in the formalism, both in the form of the probe fields C and in form of finite gauge transformations, which are needed to perform calculations in nothingness gauge.

First of all, the notion of a group of finite higher-spin transformations might be cumbersome, since the Lie algebra is infinite-dimensional. However, in the case of AdS and $\mathfrak{hs}(\lambda)$ it is actually possible to consider an infinite-dimensional topological group $HS(\lambda)$ from exponentiation of Lie-algebra elements, at least in a certain parameter range of λ [253]. Accordingly, I will in the following assume that the exponentiation of elements of the Lie algebra $\mathfrak{ihs}(\mathscr{M}^2, \mathcal{S})$ can be made sense of and loosely refer to the set of exponentiated Lie-algebra elements as the group $IHS(\mathscr{M}^2, \mathcal{S})$.

The main problem actually lies in the splitting of group elements $g \in IHS(\mathscr{M}^2, \mathcal{S})$ into higher-spin translations and higher-spin Lorentz transformations, since the Lorentz-like generators do not form a Lie-subalgebra;[6] recall the commutation relations (4.2.1). Suppose one defines a Lie-algebra element and its exponent

$$\xi_{\mathrm{L}} = \sum_{s=2}^{\infty} \sum_{|m| \le s-1} \xi_{\mathrm{L}}^{(s,m)} \mathbf{J}_m^s , \qquad g_{\mathrm{L}} = \mathrm{e}^{\xi_{\mathrm{L}}} . \tag{5.3.42}$$

Then Lie-algebra valued objects built from the group element, such as the Cartan-Maurer element $g^{-1}\,\mathrm{d}g$, will be composed both of Lorentz-like generators $\mathbf{J}_m^s$ and translation-like generators $\mathbf{P}_m^s$. To re-instate a distinction between both, one has to sort out factors of $\mathcal{S}$, which appear in the commutation relation (4.2.1a). Accordingly, if an $IHS(\mathscr{M}^2, \mathcal{S})$-transformation $g = g_{\mathrm{T}} g_{\mathrm{L}}$ is applied, the transformation behaviour of the gauge fields (3.1.23) has to be modified to

$$\omega \mapsto \left(g_{\mathrm{L}}^{-1} \omega g_{\mathrm{L}} + g_{\mathrm{L}}^{-1}\,\mathrm{d}g_{\mathrm{L}}\right)_{\mathcal{S}=0} , \tag{5.3.43a}$$

$$e \mapsto g_{\mathrm{L}}^{-1} \left(e + [\omega, \xi_{\mathrm{T}}] + \mathrm{d}\xi_{\mathrm{T}}\right) g_{\mathrm{L}} + \mathcal{S}\partial_{\mathcal{S}} \left(g_{\mathrm{L}}^{-1} \omega g_{\mathrm{L}} + g_{\mathrm{L}}^{-1}\,\mathrm{d}g_{\mathrm{L}}\right) . \tag{5.3.43b}$$

Note the important point that the split into translational and Lorentz elements is still com-

[6] From a general perspective it is clear that a decomposition of group elements only works for closed subgroups [254].

patible with composition of finite transformations, since

$$g = g^{(1)} g^{(2)} = g_{\mathrm{T}}^{(1)} g_{\mathrm{L}}^{(1)} g_{\mathrm{T}}^{(2)} g_{\mathrm{L}}^{(2)} = g_{\mathrm{T}}^{(1)} g_{\mathrm{L}}^{(1)} g_{\mathrm{T}}^{(2)} \left(g_{\mathrm{L}}^{(1)}\right)^{-1} g_{\mathrm{L}}^{(1)} g_{\mathrm{L}}^{(2)} = g_{\mathrm{T}} g_{\mathrm{L}} . \tag{5.3.44}$$

What does not appear to be possible anymore, unfortunately, is the split of the probe field $C = C_{\mathrm{T}} C_{\mathrm{L}}$, because this factorisation is not compatible with the transformation behaviour $C \mapsto g_{\mathrm{L}}^{-1} g_{\mathrm{T}}^{-1} C g_{\mathrm{L}}$.

Within the present work, no apparent way around this issue could be found. Therefore, as an attempt to at least extract part of the information about higher-spin entanglement entropy (of a suitable, yet unknown higher-spin field-theory dual), I will restrict the background higher-spin fields by setting $Z^{(s)} = 0$ for $s \geq 3$. That way, the Lorentz part of the gauge transformation to nothingness gauge will contain only spin-2 generators.

Higher-Spin Probe Action

It is more or less straightforward to promote the momenta $\mathcal{P}$ and $\mathcal{J}$ to be elements of the Lie algebra $\mathfrak{ihs}(\mathcal{M}^2, \mathcal{S})$,

$$\mathcal{P} = \sum_{s=2}^{\infty} \sum_{|m| \leq s-1} p^{(s,m)} \mathbf{P}_m^s , \qquad \mathcal{J} = \sum_{s=2}^{\infty} \sum_{|m| \leq s-1} j^{(s,m)} \mathbf{P}_m^s , \tag{5.3.45}$$

and to find the higher-order Casimir elements, which should be used to fix the normalisation of higher powers of these momenta, using an infinite number of Lagrange multipliers $\lambda_{\mathrm{P}}^{(s)}$ and $\lambda_{\mathrm{J}}^{(s)}$. The higher-order Casimir elements are $\mathcal{M}^{2(s-1)}$ and $\mathcal{S}\mathcal{M}^{2(s-2)}$. Then a possible higher-spin generalisation of the action (5.3.10) is

$$\begin{aligned} S = \frac{1}{\mathcal{M}} \int \mathrm{d}s \, \Big(& \left\langle \mathcal{P} , (\mathrm{D}_s C_{\mathrm{T}}) \, C_{\mathrm{T}}^{-1} + \left[\xi_{\mathrm{T}} , (\nabla_s C_{\mathrm{L}}) \, C_{\mathrm{L}}^{-1} \right] \right\rangle - \frac{\mathcal{M}^2}{\mathcal{S}} \left\langle \mathcal{J} , (\nabla_s C_{\mathrm{L}}) C_{\mathrm{L}}^{-1} \right\rangle \\ & + \sum_{s=2}^{\infty} \lambda_{\mathrm{P}}^{(s)} \left(\left\langle \mathcal{P}^{\star(2s-3)} , \mathcal{P} \right\rangle + \mathcal{M}^{2(s-1)} \right) \\ & + \sum_{s=2}^{\infty} \lambda_{\mathrm{J}}^{(s)} \left(\left\langle \mathcal{P}^{\star(2s-3)} , \mathcal{J} \right\rangle - \mathcal{S}\mathcal{M}^{2(s-2)} \right) \Big) . \end{aligned} \tag{5.3.46}$$

Here $\mathcal{P}^{\star n}$ refers to the nth power of $\mathcal{P}$ with respect to the star product (4.1.12). Now the bilinear, non-ad-invariant form is given by (4.2.14). In what follows I will make use of the function $B(.,.)$ introduced in (4.2.15) in connection to skew symmetry of the bilinear form and introduce a similar function $B(.)$ with a single argument that simply exchanges translational generators with Lorentz-like generators, in the sense

$$B\left(\sum_{s=2}^{\infty} \sum_{|m| \leq s-1} \xi^{(s,m)} \mathbf{P}_m^s \right) := \sum_{s=2}^{\infty} \sum_{|m| \leq s-1} \xi^{(s,m)} \mathbf{J}_m^s . \tag{5.3.47}$$

Variation with respect to $\mathcal{J}$ and $\mathcal{P}$ gives equations of motion

$$(\nabla_s C_\mathrm{L})\, C_\mathrm{L}^{-1} = \sum_{s=2}^{\infty} \lambda_\mathrm{J}^{(s)} B\left(\mathcal{P}^{\star(2s-3)}\right), \tag{5.3.48a}$$

$$(\mathrm{D}_s C_\mathrm{T})\, C_\mathrm{T}^{-1} = \sum_{s=2}^{\infty} \Big(\lambda_\mathrm{J}^{(s)} B\left(\mathcal{P}^{\star(2s-3)}, \xi_\mathrm{T}\right) - 2(s - {}^{3}\!/\!{}_{2})\lambda_\mathrm{J}^{(s)} \mathcal{P}^{\star(2s-4)} \star \mathcal{J} \\ -2(s-1)\lambda_\mathrm{P}^{(s)} \mathcal{P}^{\star(2s-3)}\Big)\,. \tag{5.3.48b}$$

Variation with respect to C_L and C_T results in

$$\dot{\mathcal{J}} = [\mathcal{J}, \omega_s] - \frac{\mathcal{S}}{\mathcal{M}^2} B\left(\mathcal{P}, e_s\right) + \sum_{s=2}^{\infty} \lambda_\mathrm{J}^{(s)} \left(B\left(\mathcal{P}^{\star(2s-3)}, \mathcal{J}\right) - 2(s - {}^{3}\!/\!{}_{2}) B\left(\mathcal{P}, \mathcal{P}^{\star(2s-4)} \star \mathcal{J}\right)\right), \tag{5.3.49a}$$

$$\dot{\mathcal{P}} = [\mathcal{P}, \omega_s]\,. \tag{5.3.49b}$$

Then the on-shell action takes the form

$$\boxed{S_\text{on-shell} = \frac{2}{\mathcal{M}} \sum_{s=2}^{\infty} (s-1) \mathcal{M}^{2(s-2)} \int \mathrm{d}s \left(\mathcal{M}^2 \lambda_\mathrm{P}^{(s)} - \mathcal{S}\lambda_\mathrm{J}^{(s)}\right).} \tag{5.3.50}$$

Discussion of Further Steps

For constant charges $Z^{(s)}$, $W^{(s)}$ and the gauge transformation to nothingness gauge for the higher-spin background fields (5.1.3) is given by

$$g = \mathrm{e}^{u a_u}\, \mathrm{e}^{\phi a_\phi}\, \mathrm{e}^{\frac{r}{2} P_{-1}}, \tag{5.3.51}$$

where

$$a_u = P_1 - \frac{1}{4} \sum_{s=2}^{\infty} Z^{(s)} \mathbf{P}^s_{-s+1}, \qquad a_\phi = J_1 - \frac{1}{4} \sum_{s=2}^{\infty} Z^{(s)} \mathbf{J}^s_{-s+1} - \frac{1}{2} \sum_{s=2}^{\infty} W^{(s)} \mathbf{P}^s_{-s+1}, \tag{5.3.52}$$

which needs to be split into a purely translational and a purely Lorentz-like part, $g = g_\mathrm{T} g_\mathrm{L}$. As already mentioned above, it will not be possible to include higher-spin Lorentz generators $\mathbf{J}^s_m$ for $s \geq 3$ in the present formalism, since such transformations are not compatible with the splitting of the probe field. One should therefore start with $Z^{(s)} = M\delta_{s,2}$.

The next step would then be to find a decomposition of the group element $\mathrm{e}^{\phi a_\phi}$, which is expected to be of the form

$$\mathrm{e}^{\phi a_\phi} = \mathrm{e}^{\xi_\mathrm{T}}\, g_\mathrm{L}, \qquad g_\mathrm{L} = \mathrm{e}^{\phi\left(J_1 + \frac{M}{4} J_{-1}\right)}. \tag{5.3.53}$$

Within this work ξ_T could not be determined; it is however suspected to consist of hypergeometric functions.

Further, in nothingness gauge, $\omega_s = 0 = e_s$, the equations of motion read

$$\dot{\mathcal{J}} = \sum_{s=2}^{\infty} \lambda_{\mathrm{J}}^{(s)} \left(B\left(\mathcal{P}^{\star(2s-3)}, \mathcal{J}\right) - 2(s - 3/2) B\left(\mathcal{P}, \mathcal{P}^{\star(2s-4)} \star \mathcal{J}\right)\right), \qquad \dot{\mathcal{P}} = 0, \tag{5.3.54}$$

as well as

$$\dot{C}_{\mathrm{L}} C_{\mathrm{L}}^{-1} = \sum_{s=2}^{\infty} \lambda_{\mathrm{J}}^{(s)} B\left(\mathcal{P}^{\star(2s-3)}\right), \tag{5.3.55a}$$

$$\dot{\xi}_{\mathrm{T}} = \sum_{s=2}^{\infty} \Big(\lambda_{\mathrm{J}}^{(s)} B\left(\mathcal{P}^{\star(2s-3)}, \xi_{\mathrm{T}}\right) - 2(s - 3/2) \lambda_{\mathrm{J}}^{(s)} \mathcal{P}^{\star(2s-4)} \star \mathcal{J} \\ -2(s-1)\lambda_{\mathrm{P}}^{(s)} \mathcal{P}^{\star(2s-3)} \Big). \tag{5.3.55b}$$

Equation (5.3.55a) can immediately be integrated,

$$C_{\mathrm{L}} = c_0 \exp\left(\sum_{s=2}^{\infty} \alpha^{(s)} B\left(\mathcal{P}^{\star(2s-3)}\right)\right), \qquad \dot{\alpha}^{(s)} = \lambda_{\mathrm{J}}^{(s)}, \tag{5.3.56}$$

and equation (5.3.55b) can be integrated after applying $\langle \mathcal{P}, . \rangle$. All together, this enables one to write the on-shell action as

$$\mathscr{M} S_{\text{on-shell}} = \langle \mathcal{P}, \Delta\xi_{\mathrm{T}} \rangle - \mathcal{S} \sum_{s=2}^{\infty} \mathscr{M}^{2(s-2)} \Delta\alpha^{(s)}. \tag{5.3.57}$$

Keeping the boundary conditions known from the spin-2 case, the extraction of $\Delta\xi_{\mathrm{T}}$ should be straightforward, while the equation for $\Delta\alpha^{(s)}$ and $\mathcal{P}$ remains in implicit form,

$$\exp\left(\sum_{s=2}^{\infty} \Delta\alpha^{(s)} B\left(\mathcal{P}^{\star(2s-3)}\right)\right) \\ = \exp\left(\frac{a r_0}{2} g_{\mathrm{L}}(s_{\mathrm{f}}) J_{-1} g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})\right) \exp\left(-\frac{a r_0}{2} g_{\mathrm{L}}(s_{\mathrm{i}}) J_{-1} g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})\right). \tag{5.3.58}$$

Determining $\Delta\alpha^{(s)}$ and $\mathcal{P}$ is yet another challenge. One starting point may be the fact that powers of an algebra element commute with a single factor even after the respective generators are exchanged by the function $B(.)$, namely

$$\left[B\left(\mathcal{P}^{\star\sigma}\right), \mathcal{P}\right] = 0. \tag{5.3.59}$$

From this one can write

$$\mathcal{P} = \exp\left(\frac{a r_0}{2} g_{\mathrm{L}}(s_{\mathrm{f}}) J_{-1} g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})\right) \exp\left(-\frac{a r_0}{2} g_{\mathrm{L}}(s_{\mathrm{i}}) J_{-1} g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})\right) \mathcal{P} \\ \exp\left(\frac{a r_0}{2} g_{\mathrm{L}}(s_{\mathrm{i}}) J_{-1} g_{\mathrm{L}}^{-1}(s_{\mathrm{i}})\right) \exp\left(-\frac{a r_0}{2} g_{\mathrm{L}}(s_{\mathrm{f}}) J_{-1} g_{\mathrm{L}}^{-1}(s_{\mathrm{f}})\right), \tag{5.3.60}$$

this way getting rid of the higher powers of $\mathcal{P}$.

Anyway, apart from the particular technicalities discussed in the previous paragraphs, it remains an open issue to get control over the exponentiation of elements of $\mathfrak{ihs}(\mathscr{M}^2, \mathcal{S})$ and analyse properties of the so-defined group structure. Also the question should be kept in mind whether such an object even makes sense in the first place. These problems could not be tackled in the present work but will be content of future investigations.

Chapter 6

On Oscillator Representations

Previous chapters have focused on open questions in flat-space holography from a gravitational point of view, both in the case of Einstein and higher-spin gravity. In this chapter I will slightly shift perspective towards the boundary theory, which in the case of three-dimensional, asymptotically flat spacetimes is presumed to be a two-dimensional BMS-invariant field theory, also known as Carrollian filed theory.

Carrollian field theories present themselves as a most peculiar subject of study. Arising from an ultra-relativistic limit of conformal field theories, i.e. a limit in which the speed of light is taken to zero, such that the light-cone closes up, these theories exhibit an unfamiliar ultra-local behaviour. Accordingly, usual methods known from CFT need to be carefully revised.

In the following sections I will focus on representation-theoretical aspects of flat-space holography, i.e. on representations of the Poincaré algebra $\mathfrak{iso}(2,1)$ as well as the $\mathfrak{bms}_3$ algebra. Representations of Poincaré are well known and classified [255–257]; the representation theory of $\mathfrak{bms}_3$ has been discussed to some extend in the literature [258–261].

This chapter contains results of [2].

6.1 An Oscillator Representation of BMS

The success of the AdS/CFT correspondence rests on a variety of examples, in which observables, such as entanglement entropy or general correlation functions, can be accessed both from the CFT side and the AdS side of the duality and shown to be in agreement. A selection of such examples has been presented in section 3.3 of the Foundations. Here the focus will be laid on the efficient calculation of conformal blocks (in AdS/CFT) or $\mathfrak{bms}_3$-blocks (in flat/Carroll) in the semi-classical limit, which necessarily requires a particular realisation of the respective symmetry algebra, known as oscillator construction.

6.1.1 Pre-Consideration: Virasoro Oscillators

Let me briefly review some aspects of the representation theory of the conformal algebra (3.3.1). I will present a particular realisation of its highest-weight representation in terms of so-called oscillator variables, here concentrating on only one of the two Virasoro sectors.

The great advantage of the oscillator representation lies in its applicability in the computation of conformal blocks, which is usually an involved task to perform.

The Global Case

To warm up, consider the global part of the Virasoro algebra, $\mathfrak{sl}(2,\mathbb{R})$, spanned by generators L_m with $m \in \{0, \pm 1\}$ and commutation relations given in (3.1.12). A highest-weight state $|h\rangle$ may be defined as an L_0-eigenstate that is annihilated by L_1,

$$L_1 |h\rangle = 0\,, \qquad\qquad L_0 |h\rangle = h|h\rangle\,. \tag{6.1.1}$$

Then a representation space is given by states $|n\rangle$ that are obtained by repeated application of the generator L_{-1},

$$|n\rangle \sim (L_{-1})^n |h\rangle\,, \tag{6.1.2}$$

called descendant states. These states span a module, on which a hermitian product can uniquely be defined, such that $\langle m|n\rangle = \delta_{m,n}$.

An alternative realisation of such a highest-weight representation can be given in terms of the space of holomorphic functions on the complex disk $\mathbb{D}$, on which the $\mathfrak{sl}(2,\mathbb{R})$-generators take the form of differential operators [262],

$$\ell_m = u^{1-m}\partial_u + (1-m)hu^{-m}\,. \tag{6.1.3}$$

The highest-weight state $|h\rangle$ is apparently mapped to the unit function $f_h(u) = \mathbf{1}$, which fulfils an eigenvalue equation with ℓ_0 and is annihilated by ℓ_1. Unitarity of the representation is tied to the definition of an appropriate inner product, which can be defined as

$$(f,g) = \int_{\mathbb{D}} [\mathrm{d}^2 u]\, \overline{f(u)} g(u) \tag{6.1.4}$$

with measure

$$[\mathrm{d}^2 u] = \frac{2h-1}{2\pi} \frac{\mathrm{d}^2 u}{(1-u\bar{u})^{2(1-h)}}\,. \tag{6.1.5}$$

This turns the function space under consideration into a weighted Bergman space [263]. A useful property of this construction is the orthogonality of monomials of the oscillator variable with respect to the inner product, namely

$$\left(u^m, u^n\right) = \frac{m!}{(2h)^{\overline{m}}}\delta_{m,n}, \tag{6.1.6}$$

where $a^{\overline{n}} = a(a+1)\dots(a+n-1)$ denotes the rising factorial. An analogous construction will in the following be reviewed for the case of the Virasoro algebra.

The Virasoro Case

It is rather straightforward to build a representation of the Virasoro algebra given by the commutation relations (3.3.1a) by constructing a highest-weight module, basically the same way as in the case of $\mathfrak{sl}(2,\mathbb{R})$. Consider a highest-weight (primary) state $|h\rangle$, which is defined as being annihilated by positive-mode generators and being an eigenstate of L_0,

$$L_m|h\rangle = 0, \qquad L_0|h\rangle = h|h\rangle, \qquad m \geq 1. \tag{6.1.7}$$

Then the complete representation space, called Verma module, is spanned by descendant states, which are obtained through repeated application of negative-mode generators,

$$|(m_1,\dots,m_k);h\rangle := \left(\prod_{i=1}^{k} L_{-m_i}\right)|h\rangle \tag{6.1.8}$$

with $m_1 \geq \dots \geq m_k \geq 1$. If both the central charge c and the eigenvalue h are greater than zero, one may define a hermitian product on the Verma module, such that the representation so defined is unitary and irreducible.

As in the global case above, we will turn to an alternative, particularly useful implementation of a highest-weight representation in terms of so-called oscillators [262, 264, 265]. Here the representation space is a function space, spanned by monomials of an infinite set of complex variables $u_n \in \mathbb{C}$, $n \in \mathbb{N}$. The Virasoro generators take on the form of differential operators

$$\ell_0 = h + \sum_{k=1}^{\infty} k u_k \partial_{u_k}, \tag{6.1.9a}$$

$$\ell_m = \sum_{k=1}^{\infty} k u_k \partial_{u_{k+m}} - \frac{1}{4}\sum_{k=1}^{m-1} \partial_{u_k}\partial_{u_{m-k}} + \left(\mu m + \mathrm{i}\lambda\right)\partial_{u_m}, \tag{6.1.9b}$$

$$\ell_{-m} = \sum_{k=1}^{\infty} (k+m) u_{k+m}\partial_{u_k} - \sum_{k=1}^{m-1} k(m-k) u_k u_{m-k} + 2m\left(\mu m - \mathrm{i}\lambda\right) u_m, \tag{6.1.9c}$$

where $m \geq 1$. Central charge and conformal weight are encoded in terms of the constants

$\lambda, \mu \in \mathbb{R}$ via

$$c = 1 + 24\mu^2 \qquad\qquad h = \lambda^2 + \mu^2 . \tag{6.1.10}$$

Let u denote the complete set of oscillator variables $\{u_n \,|\, n \in \mathbb{N}\}$ and let $|p\rangle$ denote a generic state of the Verma module. Apparently, the highest-weight state is now mapped to the unit function $f_h(u) = \mathbf{1}$, since $\ell_m \cdot f_h(u) = 0$ for all $m \geq 1$ and $\ell_0 \cdot f_h(u) = h f_h(u)$. The remaining states $|p\rangle$ of the Verma module are related to functions on $\mathbb{C}^\infty$ through generalised coherent states $|u\rangle$,

$$f_p(u) = \langle u | p \rangle . \tag{6.1.11}$$

The advantage of that formalism lies in the simplicity of the hermitian product that can be defined on the function space, which is given as the integral

$$\big(f_p, f_q\big) = \int_{\mathbb{C}^\infty} [\mathrm{d}^2 u]_h \, \overline{f_p(u)} f_q(u) \tag{6.1.12}$$

with the measure

$$[\mathrm{d}^2 u]_h = \prod_{n=1}^{\infty} \mathrm{d}^2 u_n \frac{2n}{\pi} \, \mathrm{e}^{-2n u_n \bar{u}_n} , \tag{6.1.13}$$

where $\mathrm{d}^2 u_n = \mathrm{d}u_n \, \mathrm{d}\bar{u}_n$. This follows directly from the fact that (generalised) coherent states provide a resolution of unity; see [266–268] for introductory material on coherent states. With respect to the inner product so defined, monomials of oscillator variables are orthogonal in the sense that

$$\big(u_1^{m_1} u_2^{m_2} \ldots , u_1^{n_1} u_2^{n_2} \ldots\big) = \prod_{k=1}^{\infty} \frac{m_k!}{(2k)^{m_k}} \delta_{m_k, n_k} . \tag{6.1.14}$$

It is precisely this orthogonality relation that can be of great use in CFT computations. For instance, it was used in [262] in the context of quantum thermalisation and in [265] to prove the exponentiation of conformal blocks.

6.1.2 BMS Oscillators

Given the necessity of the oscillator representation to compute conformal blocks (at least in the semi-classical limit), it is desirable to be in possession of a similar construction for the algebra $\mathfrak{bms}_3$. In the following I will present such an oscillator construction for the case of a highest-weight representation, first appeared in [2].

A highest-weight representation of $\mathfrak{bms}_3$ is defined starting from a primary state $|\Delta, \xi\rangle$ that satisfies the eigenvalue equations

$$L_0\,|\Delta,\xi\rangle = \Delta\,|\Delta,\xi\rangle\;, \qquad\qquad M_0\,|\Delta,\xi\rangle = \xi\,|\Delta,\xi\rangle\;, \tag{6.1.15}$$

where the L_0-eigenvalue Δ is called the scaling dimension and the M_0-eigenvalue ξ is called rapidity. The highest-weight state is defined to be annihilated by all positive-mode generators,

$$L_m\,|\Delta,\xi\rangle = 0\,, \qquad\qquad M_m\,|\Delta,\xi\rangle = 0\,, \qquad\qquad m \geq 1\,. \tag{6.1.16}$$

Then the corresponding Verma module is spanned by all states produced through the action of ordered products of negative-mode generators like

$$|(m_1,\ldots,m_k),(n_1,\ldots,n_l);\Delta,\xi\rangle = \left(\prod_{i=1}^{k} L_{-m_i}\right)\left(\prod_{j=1}^{l} M_{-n_j}\right)|\Delta,\xi\rangle\;, \tag{6.1.17}$$

where $m_1 \geq \cdots \geq m_k \geq 1$ and $n_1 \geq \cdots \geq n_l \geq 1$. A hermitian product is uniquely defined on the Verma module, though it is not positive semi-definite, such that the representation cannot be unitary, as expected from a highest-weight representation of $\mathfrak{bms}_3$ (or, equivalently, a non-relativistic limit). An exception is the special case $c_{\mathrm{M}} = 0$ and $\xi = 0$, in which a quotient can be taken that transfers the representation to a highest-weight representation of the Virasoro algebra [269].

I will now introduce complex oscillator variables $v_m^{(1)}, v_m^{(2)} \in \mathbb{C}$ with $m \in \mathbb{N}$. Monomials of these variables span a function space, on which the $\mathfrak{bms}_3$-generators act. The following form of the generators is proposed:

$$\ell_0 = \Delta + \sum_{k=1}^{\infty} k\left(v_k^{(1)}\partial_{v_k^{(1)}} + v_k^{(2)}\partial_{v_k^{(2)}}\right)\,, \tag{6.1.18a}$$

$$\ell_m = \sum_{k=1}^{\infty} k\left(v_k^{(1)}\partial_{v_{m+k}^{(1)}} + v_k^{(2)}\partial_{v_{m+k}^{(2)}}\right) - \frac{1}{4}\sum_{k=1}^{m-1}\partial_{v_k^{(1)}}\partial_{v_{m-k}^{(2)}} + A_m\partial_{v_m^{(1)}} + B_m\partial_{v_m^{(2)}}\,, \tag{6.1.18b}$$

$$\begin{aligned}\ell_{-m} = {} & \sum_{k=1}^{\infty}(m+k)\left(v_{m+k}^{(1)}\partial_{v_k^{(1)}} + v_{m+k}^{(2)}\partial_{v_k^{(2)}}\right) - 4\sum_{k=1}^{m-1}k(m-k)\,v_k^{(1)}v_{m-k}^{(2)} \\ & + 4m\left(\hat{B}_m v_m^{(1)} + \hat{A}_m v_m^{(2)}\right)\,,\end{aligned} \tag{6.1.18c}$$

for super-rotations and

$$m_0 = \xi + \sum_{k=1}^{\infty} k v_k^{(1)} \partial_{v_k^{(2)}} , \tag{6.1.19a}$$

$$m_m = \sum_{k=1}^{\infty} k v_k^{(1)} \partial_{v_{m+k}^{(2)}} - \frac{1}{8} \sum_{k=1}^{m-1} \partial_{v_{m-k}^{(2)}} \partial_{v_k^{(2)}} + A_m \partial_{v_m^{(2)}} , \tag{6.1.19b}$$

$$m_{-m} = \sum_{k=1}^{\infty} (m+k) v_{m+k}^{(1)} \partial_{v_k^{(2)}} - 2 \sum_{k=1}^{m-1} k(m-k) v_{m-k}^{(1)} v_k^{(1)} + 4m\hat{A}_m v_m^{(1)} , \tag{6.1.19c}$$

for super-translations. In both cases it is $m \geq 1$. One may verify that these operators fulfil the commutation relations (3.3.3). The following abbreviations were used:

$$A_m \equiv -\frac{\mathrm{i}}{2}\sqrt{2\xi - \frac{c_\mathrm{M}}{12}} - m\sqrt{\frac{c_\mathrm{M}}{48}} , \qquad B_m \equiv \mathrm{i}\,\frac{c_\mathrm{L} - 2 - 24\Delta}{48\sqrt{2\xi - \frac{c_\mathrm{M}}{12}}} - m\frac{c_\mathrm{L} - 2}{48\sqrt{\frac{c_\mathrm{M}}{12}}} , \tag{6.1.20a}$$

$$\hat{A}_m \equiv \frac{\mathrm{i}}{2}\sqrt{2\xi - \frac{c_\mathrm{M}}{12}} - m\sqrt{\frac{c_\mathrm{M}}{48}} , \qquad \hat{B}_m \equiv -\mathrm{i}\,\frac{c_\mathrm{L} - 2 - 24\Delta}{48\sqrt{2\xi - \frac{c_\mathrm{M}}{12}}} - m\frac{c_\mathrm{L} - 2}{48\sqrt{\frac{c_\mathrm{M}}{12}}} , \tag{6.1.20b}$$

as well as the assumption $\xi \geq c_\mathrm{M}/24$ (otherwise, an analytic continuation is necessary). The oscillator representation presented here can be derived by a non-relativistic limit from two copies of Virasoro oscillators (6.1.9). Call the complex variables associated to the two copies $u_m^{(1)}$ and $u_m^{(2)}$, the conformal weights $h^{(1)}$ and $h^{(2)}$ and the central charges $c^{(1)}$ and $c^{(2)}$. Then the contraction[1] (3.2.16) can be preformed using the scalings

$$u_m^{(1)} = \frac{1}{\sqrt{\varepsilon}} v_m^{(1)} + \sqrt{\varepsilon} v_m^{(2)} , \qquad \Delta = h^{(1)} + h^{(2)} , \qquad c_\mathrm{L} = c^{(1)} + c^{(2)} , \tag{6.1.21a}$$

$$u_m^{(2)} = \pm\mathrm{i}\left(\frac{1}{\sqrt{\varepsilon}} v_m^{(1)} - \sqrt{\varepsilon} v_m^{(2)}\right) , \qquad \xi = \varepsilon\left(h^{(1)} - h^{(2)}\right) , \qquad c_\mathrm{M} = \varepsilon\left(c^{(1)} - c^{(2)}\right) . \tag{6.1.21b}$$

Call a generic state of the Verma module $|p\rangle$ and use v as an abbreviation for the complete set of oscillator variables $\left\{ v_m^{(1)}, v_m^{(2)} \,\middle|\, m \in \mathbb{N} \right\}$. Then the mapping of states in the Verma module to functions on $\mathbb{C}^\infty$ is given in terms of generalised coherent states $|v\rangle$ as

$$f_p(v) = \langle v | p \rangle . \tag{6.1.22}$$

Clearly, the highest-weight state is mapped to the unit function,

$$f_{\Delta,\xi}(v) = \langle v | \Delta, \xi \rangle = \mathbf{1} , \tag{6.1.23}$$

[1] Alternatively, the representation can be derived from a non-relativistic limit of a linear dilaton-like theory, as shown in appendix A of [2]; see also appendix A.2.1 in [262].

since it automatically fulfils

$$\ell_0 \cdot \mathbf{1} = \Delta\,, \qquad m_0 \cdot \mathbf{1} = \xi\,, \tag{6.1.24a}$$

$$\ell_m \cdot \mathbf{1} = 0\,, \qquad m_m \cdot \mathbf{1} = 0\,, \tag{6.1.24b}$$

for $m \geq 1$. A basis of the function space is then given by the polynomials obtained from repeated application of negative-mode generators,

$$|(m_1,\dots,m_k),(n_1,\dots,n_l);\Delta,\xi\rangle \qquad \leftrightarrow \qquad \left(\prod_{i=1}^{k} \ell_{-m_i}\right)\left(\prod_{j=1}^{l} m_{-n_j}\right)\cdot \mathbf{1}\,. \tag{6.1.25}$$

Furthermore, one has to specify a hermitian product on the function space. It is possible to define

$$\left(f_p\,,f_q\right) = \int\limits_{\mathbb{C}^\infty} \left[\mathrm{d}^2 v\right]_{\Delta,\xi}\,\overline{f_p(v)}\,f_q(v)\,, \tag{6.1.26}$$

with the measure

$$\left[\mathrm{d}^2 v\right]_{\Delta,\xi} = \prod_{n=1}^{\infty} 16 n^2 \exp\left[-4n\left(v_n^{(1)}\,\bar{v}_n^{(2)} + v_n^{(2)}\,\bar{v}_n^{(1)}\right)\right]\,\mathrm{d}^2 v_n^{(1)}\,\mathrm{d}^2 v_n^{(2)}\,, \tag{6.1.27}$$

where $\mathrm{d}^2 v_n^{(1/2)} = \mathrm{d}v_n^{(1/2)}\,\mathrm{d}\bar{v}_n^{(1/2)}$. The product (6.1.26) follows directly from the inner product on the Verma module, given that the generalised coherent states $|v\rangle$ provide a resolution of identity (and given the measure with respect to which they do).

Unfortunately, the construction at hand does not provide an equally strong orthogonality relation between monomials of oscillators as (6.1.14) does. From the remarks on complex integration given in Appendix B.2, it follows that

$$\left(\left(v_m^{(1)}\right)^a\left(v_m^{(2)}\right)^b,\left(v_m^{(1)}\right)^c\left(v_m^{(2)}\right)^d\right) = \frac{a!\,b!}{(4m)^{a+b}}\delta_{a,d}\delta_{b,c}\,. \tag{6.1.28}$$

Nevertheless, the identity (6.1.28) turns out to be useful, for example in performing an important sanity check of the oscillator construction, which consists in the calculation of the Gram matrix. For its lowest-order entries one obtains

$$(\ell_{-1}\cdot\mathbf{1}\,,\ell_{-1}\cdot\mathbf{1}) = 2\Delta\,, \qquad (\ell_{-1}\cdot\mathbf{1}\,,m_{-1}\cdot\mathbf{1}) = 2\xi\,, \tag{6.1.29a}$$

$$(m_{-1}\cdot\mathbf{1}\,,\ell_{-1}\cdot\mathbf{1}) = 2\xi\,, \qquad (m_{-1}\cdot\mathbf{1}\,,m_{-1}\cdot\mathbf{1}) = 0\,, \tag{6.1.29b}$$

which indeed reproduce the results of [270]. The same holds true for the second-order entries of the Gram matrix, which shall not be displayed here.

After all, note that the $\mathfrak{bms}_3$ module is generically not unitary, as can already be seen from

(6.1.29): at least one of the eigenvalues of the Gram matrix is negative, as long as $\xi \neq 0$. This renders the Hermitian product negative definite (or indefinite) and, thus, the representation non-unitary.

Despite the highest-weight and, accordingly, non-unitary character of the oscillator representation here presented, it could successfully be employed in [2] for the calculation of perturbatively heavy and heavy-light $\mathfrak{bms}_3$-vacuum blocks (generalising the calculation of [221]) and to prove the exponentiation of $\mathfrak{bms}_3$-blocks in the semi-classical limit.

6.2 Poincaré Oscillators and Coherent States

In the previous subsection I presented a practical but non-unitary oscillator representation of the algebra $\mathfrak{bms}_3$. This section is devoted to some initial steps towards a unitary version, preferably derived from first principles, without having to rely on limiting procedures and educated guesses.

Although it could not be achieved within the present project to arrive at such a representation, I will give some information on unitary representations and coherent states in the Poincaré case. Since some of the features of the $\mathfrak{bms}_3$ algebra are also present in $\mathfrak{iso}(2,1)$, especially the non-semisimple structure of a semi-direct sum algebra, it is expected that a lot can be learned from this simpler case already.

6.2.1 Induced Representation of Poincaré – Ad-Hoc Construction

Let me briefly show a simple implementation of an induced representation and some particular technical problems it poses, mostly following the remarks made in [260, 261].

The idea is to induce a representation from the subalgebra of $\mathfrak{isl}(2,\mathbb{R})$ spanned by generators $\{J_0, P_m\}$. Define a rest-frame state $|M,s\rangle$ as an eigenstate of J_0 and P_0,

$$J_0\,|M,s\rangle = s\,|M,s\rangle\,, \qquad\qquad P_0\,|M,s\rangle = M\,|M,s\rangle\,, \tag{6.2.1}$$

that is annihilated by all remaining translations, $P_{\pm1}\,|M,s\rangle = 0$. Then a module of $\mathfrak{isl}(2,\mathbb{R})$ is spanned by the boosted states

$$|m,n\rangle = c_{mn}^{-1}(J_1)^m(J_{-1})^n\,|M,s\rangle\,, \tag{6.2.2}$$

with an appropriate normalisation constant c_{mn}. However, it is not possible to define a unique inner product on this module in the usual way; the only relation that is implied directly by the action of the generators on $|M,s\rangle$ and the commutation relations is

$$\big\langle M,s\big|(J_1)^m(J_{-1})^n\big|M,s\big\rangle = \big\langle M,s\big|(J_1)^m(J_{-1})^m\big|M,s\big\rangle\,\delta_{mn}\,. \tag{6.2.3}$$

The object $\alpha_m \equiv \langle M, s|(J_1)^m (J_{-1})^m|M, s\rangle$ remains undetermined by the general theory.

Moreover, starting from highest-weight representations of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ and performing an ultra-relativistic contraction (3.2.17), the scalar product $\langle m', n'|m, n\rangle$ diverges in the limit [261], which can be seen as a pre-courser to the delta-function normalisability in a plane-wave basis of single-particle representations (see [271] for the standard introduction to induced representations).

6.2.2 Induced Representation of Poincaré from Coadjoint Orbits

In subsection 6.1.1 I presented an oscillator description of the global symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ and it is shown in appendix D how the particular form of the generators (6.1.3) can be found from a consideration of discrete-series representations of $SL(2,\mathbb{R}) \simeq SU(1,1)$, which play a role in the construction of coherent states.

The purpose of this subsection is twofold: First, it is hoped that the construction of (generalised) coherent states leads to a similar, unitary oscillator representation for $\mathfrak{isl}(2,\mathbb{R})$ on some complex function space, as it is the case for $\mathfrak{sl}(2,\mathbb{R})$. This should be seen as a first step towards an induced oscillator representation of $\mathfrak{bms}_3$. Second, being in possession of explicit expressions for coherent states of the Poincaré group could provide a practical calculational tool; we have seen in the previous subsection how the simplest possible way to write down an induced algebra representation already brings with it some difficulties, presumably due to the delta-function normalisability of the usual plane-wave states. Coherent states, on the other side, are by definition equipped with useful properties, such as a resolution of identity. For introductory material on (generalised) coherent states see [266–268].

In the following I will closely follow the expositions in [272, 273] (see also [274]), but clarify many of the steps by giving explicit expressions.

Matrix Representations

Starting with the Poincaré algebra in the $\mathfrak{iso}(2,1)$-basis (3.1.17), a matrix representation of the generators J_a and P_a can be given as

$$J_a = \begin{pmatrix} L_a & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_a = \begin{pmatrix} \mathbb{O} & e_a \\ 0 & 0 \end{pmatrix}, \tag{6.2.4}$$

where L_a are (3×3)-matrices that form a representation of the Lorentz algebra $\mathfrak{so}(2,1)$ and (e_a) are three-vectors with entries $(e_a)_i = \delta_{a,i}$, see B.3 of the Appendix. Any element $X \in \mathfrak{iso}(2,1)$ can be expanded like $X = \alpha^a J_a + \beta^a P_a$ and, thus, is characterised by a six-component vector $X \mathrel{\hat{=}} \begin{pmatrix} \alpha & \beta \end{pmatrix}^\top$.

A matrix representation of the dual Lie algebra $\mathfrak{iso}^*(2,1)$ may be defined through

$$J_a^* := \frac{1}{2} J_a^\top = \frac{1}{2}\begin{pmatrix} L_a^\top & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_a^* := P_a^\top = \begin{pmatrix} \mathbb{O} & 0 \\ e_a^\top & 0 \end{pmatrix}, \tag{6.2.5}$$

where the pairing $(.,.): \mathfrak{iso}^*(2,1) \times \mathfrak{iso}(2,1) \to \mathbb{R}$ for any $X^* \in \mathfrak{iso}(2,1)^*$, $X \in \mathfrak{iso}(2,1)$ is given by $(X^*, X) := \mathrm{tr}(X^* X)$, such that

$$\left(J_a^*, J_b\right) = \delta_{ab}, \qquad \left(J_a^*, P_b\right) = 0, \qquad \left(P_a^*, P_b\right) = \delta_{ab}. \tag{6.2.6}$$

Since any element $X^* \in \mathfrak{iso}^*(2,1)$ can be expanded like $X^* = (\alpha^*)^a J_a^* + (\beta^*)^a P_a^*$, one can associate a six-component vector to it, $X^* \triangleq \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$.

Elements g of the Poincaré group can generally be written as (4×4)-matrices

$$g = (\Lambda, v) = \begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix}, \qquad \Lambda \in SO(2,1), \quad v \in \mathbb{R}^{2,1}, \tag{6.2.7}$$

and be decomposed into a translation and a Lorentz part like

$$\begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Lambda & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.2.8}$$

The one-parameter subgroup obtained by exponentiation is given in terms of

$$\Lambda_0(\alpha) \equiv \mathrm{e}^{\alpha L_0}, \qquad v_0 = \begin{pmatrix} -t & 0 & 0 \end{pmatrix}^\top, \tag{6.2.9a}$$

$$\Lambda_1(\beta) \equiv \mathrm{e}^{\beta L_1}, \qquad v_1 = \begin{pmatrix} 0 & x & 0 \end{pmatrix}^\top, \tag{6.2.9b}$$

$$\Lambda_2(\gamma) \equiv \mathrm{e}^{\gamma L_2}, \qquad v_2 = \begin{pmatrix} 0 & 0 & y \end{pmatrix}^\top, \tag{6.2.9c}$$

such that we can decompose $\Lambda(\alpha,\beta,\gamma) = \Lambda_2(\gamma)\Lambda_1(\beta)\Lambda_0(\alpha)$ and $v(t,x,y) = v_0(t) + v_1(x) + v_2(y)$; in other words, we parametrise any group element $g \in ISO(2,1)$ by $(\alpha,\beta,\gamma;t,x,y)$.

Hyperbolic Orbit

For the later purpose of writing down an induced representation, we will need an action of the Lorentz subgroup $SO(2,1)$ on the translation part of the dual Lie algebra $\left(\mathbb{R}^{2,1}\right)^* \simeq \mathbb{R}^{2,1}$. Call a generic element of this vector space $K = k^a P_a^* \in \left(\mathbb{R}^{2,1}\right)^*$, which is a (4×4)-matrix that can be represented by the three-vector k, then such an action can be given by right multiplication of the transposed group element,

$$K \mapsto k \cdot (\Lambda, 0)^\top = \begin{pmatrix} 0 & 0 \\ (\Lambda k)^\top & 0 \end{pmatrix}, \qquad \text{or} \qquad k \mapsto \Lambda k. \tag{6.2.10}$$

Now, choosing an initial vector $k_0 = \begin{pmatrix} m & 0 & 0 \end{pmatrix}^\top$, with $m > 0$, the action of a Lorentz transformation $\Lambda(\alpha,\beta,\gamma)$ yields

$$\Lambda(\alpha,\beta,\gamma)\,k_0 = m\begin{pmatrix} \cosh\beta\cosh\gamma \\ \cosh\beta\sinh\gamma \\ -\sinh\beta \end{pmatrix} \equiv m\begin{pmatrix} q_0 \\ -q_1 \\ -q_2 \end{pmatrix}, \tag{6.2.11}$$

where I introduced coordinates q_a with $q_0 \geq 1$ that apparently fulfil $q_0^2 - q_1^2 - q_2^2 = 1$. In other words, the coordinates q_a parameterise the upper sheet of a hyperboloid and the orbit of the vector k_0 under the group action of $SO(2,1)$ is given by

$$\mathcal{O}_{k_0} = \left\{ m\begin{pmatrix} q_0 \\ -q_1 \\ -q_2 \end{pmatrix} \middle|\; q_0^2 - q_1^2 - q_2^2 = 1\,,\;\; q_0 \geq 1 \right\}. \tag{6.2.12}$$

This orbit will be of interest for the massive, induced representation.

(Co)Adjoint Action and Coadjoint Orbit

It is furthermore necessary to define a coadjoint action of the group $ISO(2,1)$ on the dual algebra $\mathfrak{iso}^*(2,1)$. The coadjoint group action with respect to a group element g, written Ad_g^*, is defined in terms of the adjoint group action Ad_g on the Lie algebra $\mathfrak{iso}(2,1)$ through the pairing $(.,.)$ introduced above, via

$$\left(\mathrm{Ad}_g^*(X^*)\,,X\right) := \left(X^*, \mathrm{Ad}_{g^{-1}}(X)\right), \qquad X^* \in \mathfrak{iso}^*(2,1)\,,\;\; X \in \mathfrak{iso}(2,1)\,. \tag{6.2.13}$$

The adjoint action, in turn, is simply given by $\mathrm{Ad}_g(X) = gXg^{-1}$, which can be expressed in terms of the vector-presentation of X through matrix multiplication

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \overset{\mathrm{Ad}_g}{\longmapsto} M(\Lambda,v)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{6.2.14}$$

where

$$M(\Lambda,v) = \begin{pmatrix} \Lambda & 0 \\ -(Jv)\cdot\Lambda & \Lambda \end{pmatrix}, \qquad \text{with }\; (Jv) \equiv \begin{pmatrix} | & | & | \\ L_0 v & L_1 v & L_2 v \\ | & | & | \end{pmatrix}. \tag{6.2.15}$$

Then the coadjoint group action can be written in terms of the same matrix,

$$\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \overset{\mathrm{Ad}_g^*}{\longmapsto} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} M(\Lambda,v)^{-1}\,. \tag{6.2.16}$$

Consider again the initial vector k_0 or, in terms of the complete dual Lie algebra, $K_0 = \begin{pmatrix} 0 & k_0 \end{pmatrix}$ and the coadjoint action of a generic group element (Λ, v) on this vector. It is

$$K_0 \overset{\mathrm{Ad}^*_g}{\longmapsto} m \begin{pmatrix} q_2 v_1 - q_1 v_2 \\ -(q_2 v_0 + q_0 v_2) \\ q_0 v_1 + q_1 v_0 \\ q_0 \\ q_1 \\ q_2 \end{pmatrix}^\top \equiv m \begin{pmatrix} p \\ q \end{pmatrix}^\top , \tag{6.2.17}$$

where I collected the coordinates q_a in the vector q and introduced coordinates p_a, collected in a vector p. Note that one can write $p = (Jv)^\top q$. The new coordinates fulfil the equation

$$q_0 p_0 - q_1 p_1 - q_2 p_2 = 0 , \tag{6.2.18}$$

that is, they parametrise a plane attached to the upper sheet of the hyperboloid at the point q. In other words, the orbit of the element K_0 under the coadjoint action of $ISO(2,1)$ is given as

$$\mathcal{O}^*_{K_0} = \left\{ m \begin{pmatrix} p & q \end{pmatrix} \middle| \eta^{ab} q_a q_b = -1 , \ \eta^{ab} q_a p_b = 0 , \ q_0 \geq 1 \right\} . \tag{6.2.19}$$

Induced Representation

We are now ready to define an induced representation [254] of $ISO(2,1)$, starting with the stabiliser subgroup $S_0 \in SO(2,1)$ of the orbit $\mathcal{O}_{k_0}$, which is apparently given by rotations $\Lambda(\theta)$. For this subgroup it is easy to write down a one-dimensional representation[2]

$$L(s) = \mathrm{e}^{\mathrm{i}n\theta(s)} , \qquad s \in S_0 , \ n \in \mathbb{Z} . \tag{6.2.20}$$

From here on it is simply possible to induce a representation of $S_0 \ltimes \mathbb{R}^{2,1}$ through the character $\chi(v) = \mathrm{e}^{-\mathrm{i}(k_0, v)}$ of the vector space $\mathbb{R}^{2,1}$, namely

$$(\chi L)(s, v) = \mathrm{e}^{-\mathrm{i}(k_0, v)} L(s) . \tag{6.2.21}$$

Note that the vector spaces $\mathbb{R}^{2,1}$ belonging to the Lie algebra and belonging to the Lie group can be identified, such that the pairing $(.,.)$ can be applied to either of them.

The last step in obtaining an induced representation for the whole inhomogeneous group is to perform a coset decomposition of its elements. To this end, define a so-called global Borel section Φ as a map from the orbit $\mathcal{O}_{k_0}$ to the Lorentz subgroup, $\Phi : \mathcal{O}_{k_0} \to SO(2,1)$ with

[2] Any irreducible unitary representation of an abelian group is one-dimensional and of exponential form [275].

the properties

$$\Phi(k_0) = \mathbb{1}\,, \qquad\qquad \Phi(k)k_0 = k\,. \tag{6.2.22}$$

This provides a unique decomposition of any element $\Lambda \in SO(2,1)$ of the form $\Lambda = \Phi(k)s_0$ with $k \in \mathscr{O}_{k_0}$ and $s_0 \in S_0$, and therefore also a unique decomposition of any element $(\Lambda, v) \in ISO(2,1)$ of the form

$$\begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Phi(k) & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} s_0 & \Phi(k)^{-1}v \\ 0 & 1 \end{pmatrix}}_{\in S_0 \ltimes \mathbb{R}^{2,1}} . \tag{6.2.23}$$

Finally, the action of a generic group element on an element of the Borel section can be decomposed like

$$\begin{pmatrix} \Lambda & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Phi(k) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Phi(\Lambda k) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h_0(\Lambda,k) & \Phi(k)^{-1}v \\ 0 & 1 \end{pmatrix}, \tag{6.2.24}$$

where k is independent of Λ; to be more precise one could write $k(\tilde{\Lambda})$ to indicate that k is associated to the decomposition of a different Lorentz transformation. The element $h_0(\Lambda, k) \equiv \Phi(\Lambda k)^{-1}\Lambda\Phi(k)$ is a pure rotation and can be seen as a map $h_0 : SO(2,1) \times \mathscr{O}_{k_0} \to S_0$. The second factor in (6.2.24), the element $h((\Lambda, v), k) \equiv (h_0(\Lambda, k), \Phi(k)^{-1}v)$ is a cocycle. One may switch the notation from k to coordinates q, carefully minding the appropriate signs. The last step consists in calculating the angle θ, to which the element $h_0(\Lambda^{-1}, q) = \Lambda_0(\theta)$ is associated (following the exhibitions in [267] for semi-direct product groups).

Taking everything together, one arrives at an induced representation $U(\Lambda, v)$ of the Poincaré group on the Hilbert space $\mathscr{H} = L^2(\mathscr{O}_{k_0}, \mathrm{d}\mu)$ of the form

$$\boxed{\big(U(\Lambda, v)\phi\big)(q) = \mathrm{e}^{-\mathrm{i}mq\cdot v}\, \mathrm{e}^{-\mathrm{i}n\theta(\Lambda,q)}\, \phi\big(\Lambda^{-1}q\big)\,,} \tag{6.2.25}$$

where the Lorentz transformation is parametrised by three angles as before, $\Lambda = \Lambda(\alpha, \beta, \gamma)$, the scalar product is given as $q \cdot v \equiv \eta^{ab}q_a v_b = -q_0v_0 + q_1v_1 + q_2v_2$ and $\theta(\Lambda, q)$ is given through

$$\tan\theta(\Lambda, q) = \frac{\vartheta_0(\alpha,\beta,\gamma)q_0 - \vartheta_1(\alpha,\beta,\gamma)q_1}{\vartheta_0(\alpha,\beta,\gamma)q_1q_2 - \vartheta_1(\alpha,\beta,\gamma)q_0q_2 - \cosh\alpha\,\cosh\beta\big(1+q_2^2\big)} \tag{6.2.26}$$

with

$$\vartheta_0(\alpha,\beta,\gamma) \equiv \sin\alpha\,\cosh\gamma + \cos\alpha\,\sinh\beta\,\sinh\gamma\,, \tag{6.2.27a}$$

$$\vartheta_1(\alpha,\beta,\gamma) \equiv \sin\alpha\,\sinh\gamma + \cos\alpha\,\sinh\beta\,\cosh\gamma\,. \tag{6.2.27b}$$

The invariant measure on the hyperboloid $\mathbb{H}^+$ is given as

$$\mathrm{d}\mu(q) = \frac{\mathrm{d}q_1 \wedge \mathrm{d}q_2}{q_0}\,. \tag{6.2.28}$$

This representation is unitary but not square-integrable, that is, given a scalar product

$$\langle\phi|\psi\rangle := \int_{\mathbb{H}^+} \mathrm{d}\mu(q)\, \overline{\phi(q)}\psi(q)\,, \tag{6.2.29}$$

the integral

$$\int_{\mathcal{O}^*_{K_0}} \mathrm{d}\nu(g)\, \left|\langle U(g)\phi|\psi\rangle\right|^2\,, \tag{6.2.30}$$

with measure on the coadjoint orbit $\mathcal{O}^*_{K_0}$, running over the whole group is generically not finite. This is due to the infinite area of the translational plane.

Oscillators on the Poincaré Disk

The induced representation (6.2.25) can be expanded around the unit element $(\mathbb{1}, 0)$ to first order, which provides a representation of the Lie algebra in terms of differential operators acting on functions on the hyperboloid. From now on, I will write $n \mapsto s$. The generators so obtained read

$$j_0 = q_2\partial_{q_1} - q_1\partial_{q_2} + \mathrm{i}s\frac{q_0}{1+q_2^2}\,, \qquad p_0 = \mathrm{i}mq_0\,, \tag{6.2.31a}$$

$$j_1 = q_0\partial_{q_2} - \mathrm{i}s\frac{q_1}{1+q_2^2}\,, \qquad p_1 = -\mathrm{i}mq_1\,, \tag{6.2.31b}$$

$$j_2 = -q_0\partial_{q_1}\,, \qquad p_2 = -\mathrm{i}mq_2\,. \tag{6.2.31c}$$

Since the validity of the $\mathfrak{iso}(2,1)$-commutation relations are not concerned with the field the parameter s is allowed to take values in, one may at this point analytically continue to $s \in \mathbb{R}$.

Dealing with functions on the hyperboloid can be tedious and it therefore seems advantageous to project the hyperboloid to the Poincaré disk $\mathbb{D}$ of unit radius in the complex plane (see [276, 277] for general introductions). A projection to complex variables u and $v \equiv \bar{u}$ on the disk, $uv < 1$, can be given as

$$q_0 = \frac{1+uv}{1-uv}\,, \qquad q_1 = \frac{u+v}{1-uv}\,, \qquad q_2 = -\mathrm{i}\frac{u-v}{1-uv}\,. \tag{6.2.32}$$

The generators (6.2.31) then take the form

$$j_0 = -\mathrm{i}\left(u\partial_u - v\partial_v - s\frac{(1-uv)(1+uv)}{(1-u^2)(1-v^2)}\right), \qquad p_0 = \mathrm{i}m\frac{1+uv}{1-uv}, \tag{6.2.33a}$$

$$j_1 = \frac{\mathrm{i}}{2}\left(\left(1+u^2\right)\partial_u - \left(1+v^2\right)\partial_v - 2s\frac{(1-uv)(u+v)}{(1-u^2)(1-v^2)}\right), \qquad p_1 = -\mathrm{i}m\frac{u+v}{1-uv}, \tag{6.2.33b}$$

$$j_2 = -\frac{1}{2}\left(\left(1-u^2\right)\partial_u + \left(1-v^2\right)\partial_v\right), \qquad p_2 = -m\frac{u-v}{1-uv}. \tag{6.2.33c}$$

The measure $\mathrm{d}\mu(q)$ transforms into

$$\mathrm{d}\mu(u,v) = -\frac{2\mathrm{i}}{(1-uv)^2}\,\mathrm{d}u\wedge\mathrm{d}v. \tag{6.2.34}$$

Finally, one may switch to an $\mathfrak{isl}(2,\mathbb{R})$-basis, for which the complex linear combinations $j_{\pm1} = \mathrm{i}(j_1 \pm \mathrm{i}\, j_2)$, $j_0 = -\mathrm{i}\, j_0$ appear to be appropriate. This results in the following form of Lie-algebra generators that I will refer to as induced oscillator representation of $\mathfrak{isl}(2,\mathbb{R})$:

$$j_1 = u^2\partial_u - \partial_v - s\frac{(1-uv)(u+v)}{(1-u^2)(1-v^2)}, \qquad p_1 = -m\frac{2u}{1-uv}, \tag{6.2.35a}$$

$$j_0 = -u\partial_u + v\partial_v + s\frac{(1-uv)(1+uv)}{(1-u^2)(1-v^2)}, \qquad p_0 = m\frac{1+uv}{1-uv}, \tag{6.2.35b}$$

$$j_{-1} = \partial_u - v^2\partial_v - s\frac{(1-uv)(u+v)}{(1-u^2)(1-v^2)}, \qquad p_{-1} = -m\frac{2v}{1-uv}. \tag{6.2.35c}$$

The generators act on functions $f(u,v) \in L^2(\mathbb{D},\mathrm{d}\mu)$. A class of eigenfunctions of j_0 with eigenvalue s is given by

$$f_0(u,v) = c(uv)\sqrt{\frac{1-v^2}{1-u^2}}^{\,s}, \tag{6.2.36}$$

where $c(uv)$ is an arbitrary function of the absolute value uv. An eigenvalue equation with p_0 is fulfilled only in the rest frame $u = v = 0$. The role of m and s as parameters of a massive particle representation can be verified by evaluation of the action of Casimir elements,

$$\left(p_0p_0 - p_1p_{-1}\right)\cdot f(u,v) = m^2 f(u,v), \tag{6.2.37a}$$

$$\left(j_0p_0 - \frac{1}{2}j_1p_{-1} - \frac{1}{2}j_{-1}p_1\right)\cdot f(u,v) = msf(u,v). \tag{6.2.37b}$$

A complete study of the properties of the generators on the function space given here, the module spanned from eigenfunctions $f_0(u,v)$ and the question of integrability of this representation – which may be achieved by introduction of a weighted measure – is beyond the scope of the work here presented and will be postponed to future investigations.

As a concluding remark, note that the representation (6.2.25) can be re-written with the help of the operators (6.2.33) in operator form as

$$\big(U(\Lambda, v)\phi\big)(q) = \mathrm{e}^{v\cdot p}\,\mathrm{e}^{\gamma j_2}\,\mathrm{e}^{\beta j_1}\,\mathrm{e}^{\alpha j_0}\cdot\phi(q)\,, \tag{6.2.38}$$

where now the scalar product is euclidean, $v \cdot p = v_0 p_0 + v_1 p_1 + v_2 p_2$.

6.2.3 Coherent States of Poincaré

To conclude this quick excursion into particular aspects of the representation theory of the Poincaré group in three dimensions, let me provide a definition of coherent states, still following and explicating the construction given in [272, 273].

Principal Section

The first step is to cure the lack of integrability of the unitary irreducible representation constructed in the previous subsection. To do so, one may introduce another section, here the so-called principle section.

Consider the stabiliser subgroup $S_0^* \subset ISO(2,1)$ of the element K_0 under coadjoint action of the group, which, as is apparent from (6.2.19), consists of proper rotations and time translations. Then there exists a unique decomposition of generic group elements (Λ, v) into a product of a section $\sigma(p,q)$ with an element of the stabiliser,

$$\begin{pmatrix}\Lambda & v\\ 0 & 1\end{pmatrix} = \underbrace{\begin{pmatrix}\Phi(q) & \Phi(q)p\\ 0 & 1\end{pmatrix}}_{=:\sigma(p,q)}\underbrace{\begin{pmatrix}s(\Lambda) & \tau(\Lambda,v)\\ 0 & 1\end{pmatrix}}_{\in S_0^*}, \qquad \tau = \begin{pmatrix}a\\ 0\\ 0\end{pmatrix}. \tag{6.2.39}$$

The defining conditions on the principal section $\sigma(p,q)$ are analogous to those on the Borel section, (6.2.22), namely (note that K_0 is represented by the coordinate $q = k_0/m$)

$$\sigma\left(0, \frac{k_0}{m}\right) = \mathbb{1}\,, \qquad \mathrm{Ad}^*_{\sigma(p,q)}(K_0) = \begin{pmatrix}p & q\end{pmatrix}. \tag{6.2.40}$$

The decomposition of the Lorentz transformation is then the same as in the case of the Borel section introduced above, i.e. the element $\Phi(q)$ is the same. One has to solve $\Phi(q)(\tau + p) = v$ for τ and p.

Towards Coherent States

It is now possible to define coherent states from the UIR (6.2.25) by applying it in the principle section to some reference state η_0, i.e. the representation is applied with elements of the form $(\Lambda, v) = (\Phi(q), \Phi(q)p)$, where $\Phi(q)$ and p are the solutions obtained from the principal-section decomposition above. It is practical to first project the representation space to the complex disk, such that it is of the form

$$\big(U(\Lambda, v)\phi\big)(u,v) = \exp\left[\frac{im\,(v_0(1+uv)-v_1(u+v)+\mathrm{i}\,v_2(u-v))}{1-uv}\right]\mathrm{e}^{-\mathrm{i}n\theta(\Lambda;u,v)}\,\phi(\tilde{u},\tilde{v})\,, \quad (6.2.41)$$

where the tildes indicate the action of the Lorentz transformation on the disk. Then, in the principal section, the components v_a are obtained to be

$$v_0 = p_0 q_0 - \frac{p_1 q_1 + p_2 q_0 q_2}{\sqrt{1+q_2^2}}\,, \quad (6.2.42a)$$

$$v_1 = -p_0 q_1 + \frac{p_1 q_0 + p_2 q_1 q_2}{\sqrt{1+q_2^2}}\,, \quad (6.2.42b)$$

$$v_2 = -p_0 q_2 + p_2\sqrt{1+q_2^2}\,, \quad (6.2.42c)$$

while the angle $\theta(q;u,v)$ is given by

$$\tan\theta(q;u,v) = \frac{q_2\big(q_0(u+v)+q_1(1+uv)\big)(1-uv)}{(1+q_2^2)(1-u^2)(1-v^2)-\mathrm{i}\,q_2\big(q_0(1+uv)+q_1(u+v)\big)(u-v)}\,, \quad (6.2.43)$$

and the action on the disk variables u, v as

$$\tilde{u}(q;u,v) = \frac{(q_1+\mathrm{i}\,q_0q_2)(1+uv)+(q_0+\mathrm{i}\,q_1q_2)(u+v)+(1+q_2^2)(u-v)}{\big(q_0(1+uv)+1-uv+q_1(u+v)-\mathrm{i}\,q_2(u-v)\big)\sqrt{1+q_2^2}}\,, \quad (6.2.44)$$

and $\tilde{v} = \bar{\tilde{u}}$. In terms of these quantities, the generalised coherent states, which are functions of u, v and are parametrised by p, q, read (up to normalisation)

$$\eta_{(p,q)}(u,v) = \exp\left[\frac{im\,(v_0(1+uv)-v_1(u+v)+\mathrm{i}v_2(u-v))}{1-uv}\right]\mathrm{e}^{-\mathrm{i}n\theta(q;u,v)}\,\eta_0(\tilde{u},\tilde{v})\,. \quad (6.2.45)$$

The absolute value squared of the reference state is required to be invariant under rotation, which implies that it is a function of the combination $(1-uv)/(1+uv)$, only.

An obvious attempt to simplify these results lies in an additional projection of the hyperboloid variables p and q to the complex disk, through

$$q_0 = \frac{1+z\bar{z}}{1-z\bar{z}}\,, \qquad q_1 = \frac{z+\bar{z}}{1-z\bar{z}}\,, \qquad q_2 = -\mathrm{i}\,\frac{z-\bar{z}}{1-z\bar{z}}\,. \quad (6.2.46)$$

Though this indeed significantly shortens the expressions for $\tilde{u}$ and $\tilde{v}$,

$$\tilde{u} = \frac{(1-\bar{z}^2)(u+z)}{(1+\bar{z}u)\sqrt{(1-z^2)(1-\bar{z}^2)}}\,, \quad (6.2.47)$$

the other building blocks of the coherent states remain involved (the introduction of complex coordinates for p on the plane did not improve the situation, either). The most compact form that could be achieved so far is in terms of the Lie-algebra generators given above; in the $\mathfrak{isl}(2,\mathbb{R})$-basis j_0 acts as a pure phase and can be neglected, if the reference state η_0 is assumed to be a j_0-eigenstate. Then

$$\eta_{(p,q)}(u,v) = \mathrm{e}^{v(p,q)\cdot\hat{p}} \exp\left[-\frac{1}{2\sqrt{z\bar{z}}}\ln\left(\frac{1+\sqrt{z\bar{z}}}{1-\sqrt{z\bar{z}}}\right)\left(\bar{z}j_1 - zj_{-1}\right)\right]\cdot\eta_0(u,v)\,, \tag{6.2.48}$$

where the collection of momentum operators is denoted with a hat to avoid confusion with the coherent-state parameter p, and q is projected on the disk with coordinates z, $\bar{z}$. Anyway, it is expected that a more compact presentation of the coherent states here constructed is possible through a sophisticated choice of coordinates, albeit no such form could be found in the course of the present work.

As final note, keep in mind that (generalised) coherent states are constructed to provide a resolution of unity, which in the case at hand should provide an identity (up to normalisation)

$$\int\limits_{\mathbb{C}\times\mathbb{D}} \mathrm{d}\nu(p,q)\ \left\langle\phi\middle|\eta_{(p,q)}\right\rangle\left\langle\eta_{(p,q)}\middle|\psi\right\rangle = \langle\phi|\psi\rangle\,, \tag{6.2.49}$$

with an appropriate measure $\mathrm{d}\nu(p,q)$ on the coadjoint orbit.

Chapter 7

Summary and Outlook

This chapter provides a comprehensive summary of the results that are presented in this thesis. In a first part, I list the most important findings and discuss a variety of immediate further steps to take, as well as their relevance within the current state of the art.

In a second part I intend to give a broader outlook on future research related to the objectives of this work, particularly in respect to flat-space holography and higher-spin theory.

7.1 Summary and Discussion

The results obtained in this thesis concern the physics of three-dimensional Einstein gravity and its higher-spin generalisation, as well as some aspects of the representation theory of asymptotic and global symmetries of flat spacetimes. In the following, I will give an account of the main findings, grouped by the different topics.

Algebraic Foundations of Higher-Spin Gravity in Three Dimensions

This work applied an algebraic formulation of higher-spin gravity to three-dimensional, asymptotically flat spacetimes. Though there have been results on spin-three deformations and infinite-spin settings from contractions of algebras before, there was no independent construction, purely building on first principles in flat space. It is one objective of this thesis to put flat-space higher-spin gravity on a somewhat stronger footing. In particular, the following steps were taken.

- A higher-spin Lie algebra was constructed from a quotient of the universal enveloping algebra of the three-dimensional Poincaré algebra and proposed as flat-space analogue to $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\lambda)$, capable of furnishing a theory of higher-spin gravity. Its Lie brackets were derived and it was equipped with a suitable bilinear form.
- The novel Lie algebra allowed the definition of a theory of higher-spin gravity by introduction of infinite towers of charges $Z^{(s)}(\phi)$ and $W^{(s)}(u,\phi)$, $s \geq 2$, that generalise

the classical quantities mass aspect and angular momentum aspect.

- On the algebraic level, the role of the vanishing-cosmological-constant limit was clarified. In particular, it was stressed that the starting point of a contraction from AdS should be a larger algebra than $\mathfrak{hs}(\lambda) \oplus \mathfrak{hs}(\lambda)$, namely one containing mixed terms of both sectors, as well.

- The necessary ingredients for a generalisation of the algebraic results to the case of $\mathscr{N} = 1$ supersymmetry were presented.

A few more comments on the above items are in order. Since the constructed higher-spin Lie algebra does not allow for finite-spin truncations, there is no direct link to the spin-three considerations that were made in the literature before. This should possibly be seen as a feature because it is actually known that an infinite tower of fields can defuse certain no-go theorems on higher-spin theories.

In the course of finding a non-degenerate bilinear form on the Lie algebra, the question of ad-invariance was brought up and discussed in detail in the case of the Poincaré algebra. Due to the semi-direct sum structure of the symmetry algebra, it is speculated (and brought to an application in the context of Wilson lines) that a weaker form of invariance, known in the literature as skew-symmetry, should be considered.

It should be interesting to uncover the asymptotic symmetry algebra implied by the boundary conditions that were presented here, in particular to see if any known (non-linear) $\mathscr{W}$-algebra will make an appearance. Steps towards a quantisation of the higher-spin Chern-Simons theory treated as constrained Hamiltonian system are collected in Appendix C.

Matter Fields in Asymptotically Flat Spacetimes

Motivated, firstly, as an attempt to introduce propagating degrees of freedom to the otherwise purely topological theory of three-dimensional gravity and, secondly, by the working example of minimal-model holography in AdS/CFT, a coupling to massive matter fields was introduced. The most important features are the following.

- The universal-enveloping-algebra construction performed in this thesis naturally results in an associative algebra, which can be equipped with a star product (in a highest-weight basis). It was proposed that this product is the necessary ingredient to mediate a (linear) coupling of matter fields to gravity in the Chern-Simons formalism. A set of product rules was derived in closed form.

- A linear coupling equation of matter fields embodied by a master field C was proposed. It takes the form of a covariant constancy condition, formulated on the associative algebra, $\mathrm{d}C + [\omega, C]_\star + e \star C = 0$. Its consistency and invariance under finite Poincaré gauge transformations was shown.

- The matter-coupling equation was shown to neatly capture the dynamics of a massive scalar field in the sense of an unfolded version of the Klein-Gordon equation in classical background geometries, posing a non-trivial sanity check both of the coupling equation and the algebraic construction. The parametrisation of the mass Casimir element is to be identified with the mass of the Klein-Gordon field. Moreover, a generalisation of the Klein-Gordon equation to backgrounds deformed by the higher-spin charges $Z^{(s)}(\phi)$ and $W^{(s)}(u,\phi)$ was derived.

- Finally, utilising the associative algebra $\mathfrak{ihs}(\mathscr{M}^2, \mathbb{S})$ to its full extend, the matter-coupling equation was shown to pose an unfolded equation of an infinite set of massive, higher-spin Fierz-Pauli fields propagating on a classical background geometry, all of the same mass given in terms of the mass Casimir element.

Immediate further steps concerning technical aspects are apparent: It would be desirable to have full control over the associative algebra structure at hand, which would, for instance, allow the coupling of massive higher-spin degrees of freedom to higher-spin backgrounds. Certainly, a derivation of closed-form product rules will be possible, if only a more convenient basis was defined. Furthermore, the connection to other unfolded formulations, such as [278], should be established.

From a higher-spin perspective, the findings here can be seen as the first necessary step to study a flat-space version of Vasiliev theory of fully interacting higher spins. Since such a theory is not at our disposal in flat spacetimes, a bottom-up approach starting from the linearised equations may be promising. Accordingly, a most interesting question concerns the back-reaction of matter fields to the spacetime-geometry, as well as interaction of these fields amongst each other.[1] An important next step will be the attempt to consistently introduce interaction terms to lowest order.

An immediate generalisation of the matter-coupling formalism is by the incorporation of supersymmetry. If the master field takes values in a supersymmetric version of the associative algebra (see 4.4), the coupling equation should turn into a set of unfolded equations for both bosonic and fermionic massive higher-spin fields.

It is furthermore expected that the same construction of unfolded equations of the Fierz-Pauli system is possible in the case of asymptotically AdS spacetimes. The necessary ingredient would be the larger higher-spin algebra discussed in section 4.3, which is obtained from the universal enveloping algebra of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, thus containing mixed products of generators from both sectors. It would be interesting to see how this fits in with Vasiliev higher-spin theory.

An application of the technique developed here to higher-dimensional cases, especially the four-dimensional case, could give valuable insights into the behaviour of higher spins in

[1] Note that the consistent interaction of massive higher-spin fields is an interesting open question on its own, for recent progress see [279].

more realistic, real-world settings. Though gravity is topological only in three dimensions, a universal-enveloping-algebra construction is still accessible in higher dimensions.

Revised Wilson-Line Prescription in Einstein and Higher-Spin Gravity

This thesis revisited the application of Wilson lines as topological probes to calculate the entanglement entropy of a boundary field theory. Though there have been results on flat-space Wilson lines in the case of Einstein and spin-three gravity before, the present work provides a more transparent derivation of the respective probe action.

- The action of a massive, spinning particle on the Poincaré group manifold was coupled to gravity in the Chern-Simons formalism. This action naturally corresponds to a unitary representation of the symmetry group. Previously known results for the entanglement entropy of BMS field theories were recovered, at the same time clarifying the usage of the non-invariant but skew-symmetric bilinear form encountered earlier and the emergence of the second central charge from the spin of the particle probe.
- Part of the construction was generalised to the case of higher-spin gravity, though a concluding calculation of higher-spin entanglement entropy was not possible, due to fundamental issues with the exponentiation of Lie-algebra generators.

In general, the holographic calculation of bi-local field-theory observables other than entanglement entropy and of higher-point correlation functions may be performed using the Wilson-line approach. It is however advisable to first broaden the current understanding of the representation theory behind this formalism, in particular of its Hilbert-space interpretation, such as it is presented in [115] for the AdS case.

Flat-space holography often only treats one of the asymptotic regions at a time, either past or future null infinity. An interesting object to study in Minkowski spacetime would be a Wilson line that originates at past null infinity $\mathscr{I}^-$ and connects to one that ends on future null infinity $\mathscr{I}^+$, thus calculating two-point functions of a Carrollian field theory living on the complete boundary $\mathscr{I}^+ \cup i^0 \cup \mathscr{I}^-$. Note that it is well known that both light-like infinities are related by an anti-podal matching [136, 280–283], see also [284] for recent developments concerning the role of (a neighbourhood of) spatial infinity in 3+1 dimensions.

Wilson lines that back-react to the geometry will derive so-called Rényi entropies, which are interesting field-theory observables on their own. Finally, the calculation of quantum corrections along the lines of [285] could be of interest.

On Oscillator Representations of Poincaré and BMS

Representations of the $\mathfrak{bms}_3$-algebra (respectively the BMS group), particularly unitary representations, are not yet well understood with respect to their field-theoretic implementation and we are currently lacking efficient tools to perform calculations in BMS-invariant field theories. Indeed, many of the results on two-dimensional BMS field theories obtained in the literature are actually derived for Galilean field theories, which are the non-relativistic counterparts of Carrollian field theories, and rely on non-unitary highest-weight representations. It is only through a one-to-one correspondence between these theories, which is accidental in the two-dimensional case, that the results agree in both kinds of theory.

One part of this thesis deals with the representation theory of $\mathfrak{bms}_3$ and, as a precursor, representations and coherent states of Poincaré.

- Inspired by a so-called oscillator representation of the Virasoro algebra (which allows calculation of conformal blocks in the semi-classical limit), a similar representation was constructed for the case of $\mathfrak{bms}_3$, in the form of a highest-weight representation (rendering it intrinsically non-unitary). Algebra generators are given in terms of differential operators acting on functions on $\mathbb{C}^\infty$ and an appropriate measure was defined. This realisation allows the calculation of $\mathfrak{bms}_3$-blocks as well as the proof of their exponentiation (both in a semi-classical limit).

- As a step towards a unitary oscillator construction of the $\mathfrak{bms}_3$-algebra, an induced representation of the Poincaré algebra was given in oscillator form. In analogy to the $\mathfrak{sl}(2,\mathbb{R})$-case the algebra generators are realised in terms of differential operators acting on functions on the complex unit disk $\mathbb{D}$.

- An emphasis was laid on the existence of generalised coherent states of the Poincaré group. First steps were taken towards an application of coherent states as an alternative to the usual approach to induced representations by explicating their construction and relating them to complex variables on the unit disk.

Immediate steps following the considerations presented here include, firstly, efforts towards a better handling of coherent states of the Poincaré group and, secondly, the construction of said states in the case of the BMS group. This track of thought should produce helpful calculational tools, eventually simplifying the investigation on Carrollian field theories with a focus on the semi-classical limit.

7.2 Outlook

To conclude the rather qualitative discussions of this chapter it seems worthwhile to step back a bit and take a broader view on the application of the holographic principle to flat spacetimes.

With respect to latest developments it will be most interesting to uncover the connection between two manifestly different holographic approaches to flat spacetimes currently under investigation. While the picture painted so far in this work may be referred to as Carrollian holography, another approach, dubbed celestial holography [280, 286, 287], states a connection between (four-dimensional) flat space and two-dimensional conformal field theory through the behaviour of scattering amplitudes on the celestial sphere. Establishing a connection between both perspectives is object of currently conducted research [288, 289].

The unification of the different asymptotic regions of flat space, namely future and past light-like infinity as well as spatial infinity (see also the comments made above in the context of Wilson lines), may require the re-consideration of various definitions and calculations in adapted coordinates, starting with the definition of suitable boundary conditions to begin with. A promising approach to cover all asymptotic regions may be to work in double-null coordinates [290, 291].

From a more general perspective, there are (at least) two objectives in the context of flat-space holography that should be tackled in the near future. First, the search for an example of a duality involving flat spacetimes in which both sides are fully understood must be completed. It is clear that the working mechanisms of the holographic principle can best be understood if we are in possession of a variety of examples of different design, in which the theories on both sides of the duality can be solved. Here, the three-, respectively two-dimensional case should be of value, though Carrollian field theories are yet to be understood. An explicit realisation of such a duality could probably be established in the case of higher-spin gravity: $\mathrm{AdS}_3/\mathrm{CFT}_2$ teaches us that the dual field theories to AdS higher-spin gravity ($\mathscr{W}_N$- or $\mathscr{W}_\infty$-vector models) are in a certain sense simple. It is thus the task to first determine the asymptotic symmetries implied by flat-space higher-spin gravity and see if a dual field theory can be found that exhibits these symmetries as well as matching dynamics.

The second objective concerns the application of the holographic duality to draw conclusions about quantum gravity in flat spacetimes from Carrollian field theories. This also requires a much better understanding of Carrollian field theories, their universal features, and in particular their quantisation: How is such a quantisation performed? Do Carrollian quantum theories exist at all? What are the thermodynamic properties of Carrollian field theories? Such questions need to be answered to set the stage for an application of holography to quantum gravity in asymptotically flat space and to approach typical quantum-gravitational problems, such as the information-loss paradox of black holes or possible signatures of quantum gravity. Since the asymptotic symmetries of flat spacetimes are infinite-dimensional also in higher dimensions, which distinguishes them from the higher-dimensional cases of Virasoro, the respective field theories should be stronger constrained.

Concerning the status of higher-spin physics in flat space, the ultimate goal would certainly be the construction of a fully interacting flat-space model as a counterpart to Vasiliev theory if such a construction can exist at all. This would not only answer questions that are now nearly a century old but probably also give access to a string-theoretical embedding of higher spins in flat space.

Appendix A

Algebra Construction and Identities

Within this chapter I will describe the derivation of product rules and commutators of the algebra $\mathfrak{ihs}(\mathscr{M}^2, \mathcal{S})$ in its highest-weight basis. I will provide a couple of identities that may be of convenience for the reader who is eager to work with this algebra.

A.1 Relations in the Universal Enveloping Algebra

A first necessary step is to gain control over monomials of the UEA. Expressions that only involve generators J_m can be taken from [224]; the remaining ones are derived through inspection, trial and error, or solution of recurrence relations. One finds

$$(J_{-1})^a (J_0)^b = (J_0 - a)^b \, (J_{-1})^a = \sum_k (-1)^k a^k \binom{b}{k} (J_0)^{b-k} (J_{-1})^a \,, \tag{A.1.1a}$$

$$(J_0)^a (J_1)^b = (J_1)^b \sum_k (-1)^k b^k \binom{a}{k} (J_0)^{a-k} \,, \tag{A.1.1b}$$

$$(J_{-1})^a (J_1)^b = \sum_k k! \binom{a}{k} \binom{b}{k} (J_1)^{b-k} \, (-2J_0 + a + b - k - 1)^{\underline{k}} \, (J_{-1})^{a-k} \,, \tag{A.1.1c}$$

$$(P_{\pm 1})^a (J_0)^b = (J_0 \pm a)^b \, (P_{\pm 1})^a \,, \qquad , \tag{A.1.1d}$$

$$(P_0)^a (J_{-1})^b = \sum_k k! \binom{a}{k} \binom{b}{k} (J_{-1})^{b-k} (P_0)^{a-k} (P_{-1})^k \,, \tag{A.1.1e}$$

$$(P_0)^a (J_1)^b = \sum_k (-1)^k k! \binom{a}{k} \binom{b}{k} (J_1)^{b-k} (P_1)^k (P_0)^{a-k} \,, \tag{A.1.1f}$$

$$(P_{-1})^a (J_1)^b = \sum_{k,j} (-2)^{k-j} k! j! \binom{a}{k} \binom{b}{k} \binom{k}{j} \binom{b-k}{j} (J_1)^{b-k-j} (P_1)^j (P_0)^{k-j} (P_{-1})^{a-k} \,, \tag{A.1.1g}$$

$$(P_1)^a (J_{-1})^b = \sum_{k,j} 2^{k-j} k! j! \binom{a}{k} \binom{b}{k} \binom{k}{j} \binom{b-k}{j} (J_{-1})^{b-k-j} (P_1)^{a-k} (P_0)^{k-j} (P_{-1})^j \,. \tag{A.1.1h}$$

Similar expressions can be derived for reversed ordering, namely

$$(J_1)^a(J_{-1})^b=\sum_{k,j} k!\binom{a}{k}\binom{b}{k}(J_{-1})^{b-k}\,(2J_0+a+b-k-1)^{\underline{k}}\,(J_1)^{a-k}\,, \tag{A.1.2a}$$

$$(J_{-1})^a(P_1)^b=\sum_{k,j}(-2)^{k-j}k!j!\binom{a}{k}\binom{b}{k}\binom{k}{j}\binom{a-k}{j}(P_1)^{b-k}(P_0)^{k-j}(P_{-1})^j(J_{-1})^{a-k-j}, \tag{A.1.2b}$$

$$(J_1)^a(P_{-1})^b=\sum_{k,j}2^{k-j}k!j!\binom{a}{k}\binom{b}{k}\binom{k}{j}\binom{a-k}{j}(P_1)^j(P_0)^{k-j}(P_{-1})^{b-k}(J_1)^{a-k-j}\,. \tag{A.1.2c}$$

The falling factorial may further be expanded into powers of J_0 as

$$(2J_0+a+b-k-1)^{\underline{k}}=\sum_j 2^j\,(a+b-2k)^j_k\,(J_0)^{k-j}\,, \tag{A.1.3}$$

where $(a)^s_n := e^s_n(a,a+1,\dots,a+n-1)$, using the elementary symmetric function

$$e^s_n(x_1,\dots,x_n) := \sum_{1\le i_1<..<i_s\le n} x_{i_1}\dots x_{i_s}\,. \tag{A.1.4}$$

A.2 Derivation of Product Rules and Commutators

The derivation of product rules is utterly complicated. I will try and guide through the most important steps.

A.2.1 Derivation of Spin-s-Spin-2 Product Rules

Starting point is the power expansion of the definition of descendant generators through the adjoint action (4.1.9),

$$ {}^{l}_{\xi}\mathbf{Q}^s_m = \frac{(s+m-\xi-1)!}{(2s-2\xi-2)!}\sum_k(-1)^k\binom{s-1-\xi-m}{k}(J_{-1})^k\ {}^{l}_{\xi}\mathbf{Q}^s_{s-1-\xi}\,(J_{-1})^{s-1-\xi-m-k}. \tag{A.2.1}$$

It is from this expression, evaluated for shifted indices, that one may start and derive identities for various sums of the same form but contaminated with additional generators. Two rather simple examples are (for even ξ)

$$\begin{aligned}&\frac{(s+m-\xi-1)!}{(2s-2\xi-2)!}\sum_k(-1)^k\binom{s-1-\xi-m}{k}(J_{-1})^k(J_1)^{l-\xi}\mathscr{C}^{\frac{\xi}{2}}J_0(P_1)^{s-1-l}(J_{-1})^{s-1-\xi-m-k}\\ &=\frac{s-1-l}{s-\xi}\,{}^{l+1}_{\xi+1}\mathbf{Q}^{s+1}_m+\frac{2s-2\xi-1}{s+m-\xi}\,{}^{l+1}_{\xi}\mathbf{Q}^{s+1}_m+\frac{(l-\xi+1)^{\underline{2}}}{2(s-\xi)}\,{}^{l}_{\xi}\mathbf{Q}^s_m\,,\end{aligned} \tag{A.2.2a}$$

$$\frac{(s+m-\xi-1)!}{(2s-2\xi-2)!}\sum_k (-1)^k \binom{s-1-\xi-m}{k} (J_{-1})^k (J_1)^{l-\xi}\mathscr{C}^{\frac{\xi}{2}} P_0 (P_1)^{s-1-l}(J_{-1})^{s-1-\xi-m-k}$$
$$= -\frac{l-\xi}{s-\xi}\, {}^{\;l}_{\xi+1}\mathbf{Q}^{s+1}_m + \frac{2s-2\xi-1}{s+m-\xi}\, {}^{l}_{\xi}\mathbf{Q}^{s+1}_m + \frac{(l-\xi)^{\underline{2}}}{2(s-\xi)}\, {}^{l-1}_{\;\xi}\mathbf{Q}^{s}_m \,. \tag{A.2.2b}$$

To proceed, one needs to know how to commute translation generators with powers of the element $\mathscr{C}$. Inspecting a couple of lower-order examples, one is led to the ansatz

$$\left[P_1, \mathscr{C}^n\right] = \sum_{k=1}^{n} \mathscr{C}^{n-k}\left(\alpha^n_k (J_0 P_1 - J_1 P_0) + \beta^n_k P_1 - \gamma^n_k \mathcal{S} J_1\right), \tag{A.2.3}$$

and from there to recurrence relations for the unknown coefficients,

$$\alpha^{n+1}_k = \alpha^n_k + 2\beta^n_{k-1}, \qquad \beta^{n+1}_k = \beta^n_k + 2\beta^n_{k-1} + 2\alpha^n_k, \qquad \gamma^{n+1}_k = \gamma^n_k + 2\alpha^n_{k-1}, \tag{A.2.4}$$

where $\alpha^n_0 = 0$ as well as $\beta^n_0 = 1$. These can be decoupled and solved using the technique of generating functions (for help on “generatingfunctionology” consult the delightful introduction [292]). The result can then be generalised to arbitrary P_m and may be written as

$$P_{\pm 1}\mathscr{C}^n = \sum_{k=0}^{n} \mathscr{C}^{n-k}\left(\pm\alpha^n_k (J_0 P_{\pm 1} - J_{\pm 1} P_0) + \beta^n_k P_{\pm 1} - \gamma^n_k \mathcal{S} J_{\pm 1}\right), \tag{A.2.5a}$$

$$P_0\mathscr{C}^n = \sum_{k=0}^{n} \mathscr{C}^{n-k}\left(\alpha^n_k (J_0 P_0 - J_1 P_{-1} - \mathcal{S}) + \beta^n_k P_0 - \gamma^n_k \mathcal{S} J_0\right) \tag{A.2.5b}$$

with $\alpha^n_0 = 0 = \gamma^n_0$ and $\beta^n_0 = 1$ and

$$\alpha^n_k = \frac{2^k k}{(2k)!}\frac{(n+k-2)!}{(n-k)!}\Bigg((k-1)\,{}_2F_1\left[\begin{matrix}-(k-1), & -(n-k)\\ & -(n+k-2)\end{matrix}\,\middle|\, -1\right]$$
$$+(n+k-1)\,{}_2F_1\left[\begin{matrix}-k, & -(n+1-k)\\ & -(n+k-1)\end{matrix}\,\middle|\, -1\right]\Bigg), \tag{A.2.6a}$$

$$\beta^n_k = 2^k \binom{n+k}{2k}\,{}_2F_1\left[\begin{matrix}-k, & -(n-k)\\ & -(n+k)\end{matrix}\,\middle|\, -1\right], \tag{A.2.6b}$$

$$\gamma^n_k = \begin{cases} 2^k \frac{(3k-3)^{\overline{n-k}}}{(n-k)!}\,{}_3F_2\left[\begin{matrix}-(k-2), & -\frac{n-1-k}{2}, & -\frac{n-k}{2}\\ -\frac{n}{2}-k+2, & -\frac{n-1}{2}-k+2 & \end{matrix}\,\middle|\, 1\right], & k>1, \\ 0, & k=1. \end{cases} \tag{A.2.6c}$$

From these considerations one can construct expressions for ${}^{l}_{\xi}\mathbf{Q}^{s}_m \star P_n$ and $P_n \star {}^{l}_{\xi}\mathbf{Q}^{s}_m$, where a distinction of the cases of ξ being even or odd is necessary. The most complicated case is left multiplication wit P_n when ξ is odd; this however can be traced back to the other cases by using the identity

$$
\begin{aligned}
{}^{\;\;l}_{\xi+1}\mathbf{Q}^{s}_{m} = \frac{s-1-\xi}{2(l-\xi)}\left(\frac{m-s+\xi+2}{s+m-\xi-1}\,{}^{l}_{\xi}\mathbf{Q}^{s-1}_{m+1}\star P_{-1} - \frac{m+s-\xi-2}{m-s+\xi+1}\,{}^{l}_{\xi}\mathbf{Q}^{s-1}_{m-1}\star P_{1}\right) \\
+ \frac{l-\xi-1}{2}\,{}^{l-1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m} - \frac{m(s-\xi-1)(2s-2\xi-3)}{(l-\xi)(s+m-\xi-1)(s-m-\xi-1)}\,{}^{l}_{\xi}\mathbf{Q}^{s}_{m}\,.
\end{aligned} \tag{A.2.7}
$$

The explicit form of the product rules so obtained is given in the following subsection.

A.2.2 Explicit Form of Spin-s-Spin-2 Product Rules

The derivation sketched in the previous subsection results in a set of multiplication rules, where the coefficients α^n_k, β^n_k and γ^n_k are given in (A.2.6) and the functions

$$
N^s_1(m,n) := \frac{m-(s-1)n}{s^{\underline{2}}}\,, \tag{A.2.8a}
$$

$$
N^s_2(m,n) := \frac{m^2+(s-1)(2s-3)n^2-(2s-3)mn-(s-1)^2}{(s-1)^2(2s-1)(2s-3)}\,, \tag{A.2.8b}
$$

are introduced; the multiplication rules read

$$
\begin{aligned}
{}^{l}_{\xi}\mathbf{Q}^{s}_{m}\star J_n\big|_{\xi\text{ even}} = {}^{l+1}_{\;\;\xi}\mathbf{Q}^{s+1}_{m+n} + N^{s-\xi}_1(m,n)\Bigg[&(s-1-l)\,{}^{l+1}_{\xi+1}\mathbf{Q}^{s+1}_{m+n} - \frac{(l-\xi)(2s-1-l-\xi)-2(s-\xi)^{\underline{2}}}{2}\,{}^{l}_{\xi}\mathbf{Q}^{s}_{m+n}\Bigg] \\
- N^{s-\xi}_2(m,n)\Bigg[&(l-\xi)(2s-2-l-\xi)\,{}^{l+1}_{\xi+2}\mathbf{Q}^{s+1}_{m+n} + (l-\xi)(2s-2-l-\xi)(s-1-l)\,{}^{\;\;l}_{\xi+1}\mathbf{Q}^{s}_{m+n} \\
&-(s-1-l)^{\underline{2}}\mathcal{M}^2\,{}^{l+1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m+n} + (s-1-l)(2s-2l-3)\mathcal{S}\,{}^{l}_{\xi}\mathbf{Q}^{s-1}_{m+n} - \frac{(l-\xi)^{\underline{2}}(2s-1-l-\xi)^{\underline{2}}}{4}\,{}^{l-1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m+n}\Bigg]\,,
\end{aligned} \tag{A.2.9a}
$$

$$
\begin{aligned}
{}^{l}_{\xi}\mathbf{Q}^{s}_{m}\star J_n\big|_{\xi\text{odd}} = {}^{l+1}_{\;\;\xi}\mathbf{Q}^{s+1}_{m+n} + N^{s-\xi}_1(m,n)\Bigg[&(s-1-l)\,{}^{l+1}_{\xi+1}\mathbf{Q}^{s+1}_{m+n} - \frac{(l+1-\xi)(2s-2-l-\xi)-2(s-\xi)^{\underline{2}}}{2}\,{}^{l}_{\xi}\mathbf{Q}^{s}_{m+n} \\
&+(s-2-l)\mathcal{M}^2\,{}^{l+1}_{\xi-1}\mathbf{Q}^{s-1}_{m+n} - (2s-2l-3)\mathcal{S}\,{}^{\;\;l}_{\xi-1}\mathbf{Q}^{s-1}_{m+n}\Bigg] - N^{s-\xi}_2(m,n)\Bigg[(l-\xi)(2s-2-l-\xi)\,{}^{l+1}_{\xi+2}\mathbf{Q}^{s+1}_{m+n} \\
&+(l-\xi)(2s-2-l-\xi)(s-1-l)\,{}^{\;\;l}_{\xi+1}\mathbf{Q}^{s}_{m+n} - (s-2-l)^{\underline{2}}\mathcal{M}^2\,{}^{l+1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m+n} \\
&+(s-2-l)(2s-2l-3)\mathcal{S}\,{}^{l}_{\xi}\mathbf{Q}^{s-1}_{m+n} - \frac{(l+1-\xi)^{\underline{2}}(2s-2-l-\xi)^{\underline{2}}}{4}\,{}^{l-1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m+n} \\
&+(l+1-\xi)(s-2-l)(2s-l-\xi-3)\mathcal{M}^2\,{}^{\;\;l}_{\xi-1}\mathbf{Q}^{s-2}_{m+n} \\
&+(2s-2l-3)\Big(s(2\xi-1)-\xi(\xi+2)-l(2s-3)+l^2+1\Big)\mathcal{S}\,{}^{l-1}_{\xi-1}\mathbf{Q}^{s-2}_{m+n}\Bigg]\,,
\end{aligned} \tag{A.2.9b}
$$

$$
\begin{aligned}
J_n\star{}^{l}_{\xi}\mathbf{Q}^{s}_{m}\big|_{\xi\text{ even}} = {}^{l+1}_{\;\;\xi}\mathbf{Q}^{s+1}_{m+n} + N^{s-\xi}_1(m,n)\Bigg[&(s-1-l)\,{}^{l+1}_{\xi+1}\mathbf{Q}^{s+1}_{m+n} - \frac{(l-\xi)(2s-1-l-\xi)}{2}\,{}^{l}_{\xi}\mathbf{Q}^{s}_{m+n}\Bigg] \\
- N^{s-\xi}_2(m,n)\Bigg[&(l-\xi)(2s-2-l-\xi)\,{}^{l+1}_{\xi+2}\mathbf{Q}^{s+1}_{m+n} + (l-\xi)(2s-2-l-\xi)(s-1-l)\,{}^{\;\;l}_{\xi+1}\mathbf{Q}^{s}_{m+n} \\
&-(s-1-l)^{\underline{2}}\mathcal{M}^2\,{}^{l+1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m+n} + (s-1-l)(2s-2l-3)\mathcal{S}\,{}^{l}_{\xi}\mathbf{Q}^{s-1}_{m+n} - \frac{(l-\xi)^{\underline{2}}(2s-1-l-\xi)^{\underline{2}}}{4}\,{}^{l-1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m+n}\Bigg]\,,
\end{aligned} \tag{A.2.9c}
$$

$$
\begin{aligned}
J_n\star{}^{l}_{\xi}\mathbf{Q}^{s}_{m}\big|_{\xi\text{ odd}} = {}^{l+1}_{\;\;\xi}\mathbf{Q}^{s+1}_{m+n} + N^{s-\xi}_1(m,n)\Bigg[&(s-1-l)\,{}^{l+1}_{\xi+1}\mathbf{Q}^{s+1}_{m+n} - \frac{(l+1-\xi)(2s-2-l-\xi)}{2}\,{}^{l}_{\xi}\mathbf{Q}^{s}_{m+n} \\
&+(s-2-l)\mathcal{M}^2\,{}^{l+1}_{\xi-1}\mathbf{Q}^{s-1}_{m+n} - (2s-2l-3)\mathcal{S}\,{}^{\;\;l}_{\xi-1}\mathbf{Q}^{s-1}_{m+n}\Bigg] \\
- N^{s-\xi}_2(m,n)\Bigg[&(l-\xi)(2s-2-l-\xi)\,{}^{l+1}_{\xi+2}\mathbf{Q}^{s+1}_{m+n} + (l-\xi)(2s-2-l-\xi)(s-1-l)\,{}^{\;\;l}_{\xi+1}\mathbf{Q}^{s}_{m+n} \\
&-(s-2-l)^{\underline{2}}\mathcal{M}^2\,{}^{l+1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m+n} + (s-2-l)(2s-2l-3)\mathcal{S}\,{}^{l}_{\xi}\mathbf{Q}^{s-1}_{m+n} \\
&-\frac{(l+1-\xi)^{\underline{2}}(2s-2-l-\xi)^{\underline{2}}}{4}\,{}^{l-1}_{\;\;\xi}\mathbf{Q}^{s-1}_{m+n} + (l+1-\xi)(s-2-l)(2s-l-\xi-3)\mathcal{M}^2\,{}^{\;\;l}_{\xi-1}\mathbf{Q}^{s-2}_{m+n} \\
&+(2s-2l-3)\Big(s(2\xi-1)-\xi(\xi+2)-l(2s-3)+l^2+1\Big)\mathcal{S}\,{}^{l-1}_{\xi-1}\mathbf{Q}^{s-2}_{m+n}\Bigg]\,;
\end{aligned} \tag{A.2.9d}
$$

$$
\begin{aligned}
{}^{l}_{\xi}\mathbf{Q}^{s}_{m} \star P_n\big|_{\xi\text{ even}} = {}& {}^{l}_{\xi}\mathbf{Q}^{s+1}_{m+n} - N_1^{s-\xi}(m,n)(l-\xi)\left({}^{l}_{\xi+1}\mathbf{Q}^{s+1}_{m+n} - \frac{l-\xi-1}{2}\,{}^{l-1}_{\xi}\mathbf{Q}^{s}_{m+n}\right) + N_2^{s-\xi}(m,n)\Big((l-\xi)^{\underline{2}}\,{}^{l}_{\xi+2}\mathbf{Q}^{s+1}_{m+n}\\
&-\frac{(l-\xi)^{\underline{2}}(2l-2\xi-1)}{2}\,{}^{l-1}_{\xi+1}\mathbf{Q}^{s}_{m+n} - (s-1-l)(s-1+l-2\xi)\mathscr{M}^2\,{}^{l}_{\xi}\mathbf{Q}^{s-1}_{m+n}\\
&-(l-\xi)(2l-2\xi-1)\mathcal{S}\,{}^{l-1}_{\xi}\mathbf{Q}^{s-1}_{m+n} + \frac{(l-\xi)^{\underline{3}}(l-\xi-1)}{4}\,{}^{l-2}_{\xi}\mathbf{Q}^{s-1}_{m+n}\Big),
\end{aligned}
\tag{A.2.9e}
$$

$$
\begin{aligned}
{}^{l}_{\xi}\mathbf{Q}^{s}_{m} \star P_n\big|_{\xi\text{ odd}} = {}& {}^{l}_{\xi}\mathbf{Q}^{s+1}_{m+n} - N_1^{s-\xi}(m,n)\Big((l-\xi)\,{}^{l}_{\xi+1}\mathbf{Q}^{s+1}_{m+n} - \frac{(l+1-\xi)^{\underline{2}}}{2}\,{}^{l-1}_{\xi}\mathbf{Q}^{s}_{m+n} + \mathscr{M}^2(l+1-\xi)\,{}^{l}_{\xi-1}\mathbf{Q}^{s-1}_{m+n}\\
&-\mathcal{S}(2l-2\xi+1)\,{}^{l-1}_{\xi-1}\mathbf{Q}^{s-1}_{m+n}\Big) + N_2^{s-\xi}(m,n)\Big((l-\xi)^{\underline{2}}\,{}^{l}_{\xi+2}\mathbf{Q}^{s+1}_{m+n} - \frac{(l-\xi)^{\underline{2}}(2l-2\xi-1)}{2}\,{}^{l-1}_{\xi+1}\mathbf{Q}^{s}_{m+n}\\
&-(s-2-l)(s+l-2\xi)\mathscr{M}^2\,{}^{l}_{\xi}\mathbf{Q}^{s-1}_{m+n} - (l-\xi)(2l-2\xi+1)\mathcal{S}\,{}^{l-1}_{\xi}\mathbf{Q}^{s-1}_{m+n} + \frac{(l+1-\xi)^{\underline{3}}(l-\xi)}{4}\,{}^{l-2}_{\xi}\mathbf{Q}^{s-1}_{m+n}\\
&-\mathscr{M}^2\,\frac{(l+1-\xi)^{\underline{2}}(2l-2\xi+1)}{2}\,{}^{l-1}_{\xi-1}\mathbf{Q}^{s-2}_{m+n} + \mathcal{S}(l-\xi)(2(l-\xi)^2-1)\,{}^{l-2}_{\xi-1}\mathbf{Q}^{s-2}_{m+n}\Big),
\end{aligned}
\tag{A.2.9f}
$$

$$
\begin{aligned}
P_n \star {}^{l}_{\xi}\mathbf{Q}^{s}_{m}\big|_{\xi\text{ even}} = {}& \sum_{j=0}^{\frac{\xi}{2}}\Big\{\alpha_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi+1-2j}\mathbf{Q}^{s+2-2j}_{m+n} + \beta_j^{\frac{\xi}{2}}\,{}^{l-2j}_{\xi-2j}\mathbf{Q}^{s+1-2j}_{m+n} - \mathcal{S}\gamma_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi-2j}\mathbf{Q}^{s+1-2j}_{m+n}\\
&-N_1^{s-\xi}(m,n)\Big[(l-\xi)\alpha_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi+2-2j}\mathbf{Q}^{s+2-2j}_{m+n} + (s-1-l)\mathcal{S}\gamma_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi+1-2j}\mathbf{Q}^{s+1-2j}_{m+n}\\
&+\frac{l-\xi}{2}\left((2s-1-l-\xi)\alpha_j^{\frac{\xi}{2}} + 2\beta_j^{\frac{\xi}{2}}\right){}^{l-2j}_{\xi+1-2j}\mathbf{Q}^{s+1-2j}_{m+n} - (s-1-l)\mathscr{M}^2\alpha_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi-2j}\mathbf{Q}^{s-2j}_{m+n}\\
&+\mathcal{S}\left((s-1-2l+\xi)\alpha_j^{\frac{\xi}{2}} - \frac{(l-\xi)(2s-1-l-\xi)}{2}\gamma_j^{\frac{\xi}{2}}\right){}^{l-2j}_{\xi-2j}\mathbf{Q}^{s-2j}_{m+n}\\
&+\frac{(l-\xi)(2s+1-l-\xi)}{2}\beta_j^{\frac{\xi}{2}}\,{}^{l-1-2j}_{\xi-2j}\mathbf{Q}^{s-2j}_{m+n}\Big]\\
&+N_2^{s-\xi}(m,n)\Big[(l-\xi)^{\underline{2}}\,\alpha_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi+3-2j}\mathbf{Q}^{s+2-2j}_{m+n} + (l-\xi)(2s-2-l-\xi)\mathcal{S}\gamma_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi+2-2j}\mathbf{Q}^{s+1-2j}_{m+n}\\
&+\frac{(l-\xi)^{\underline{2}}}{2}\left((4s-2l-2\xi-1)\alpha_j^{\frac{\xi}{2}} + 2\beta_j^{\frac{\xi}{2}}\right){}^{l-2j}_{\xi+2-2j}\mathbf{Q}^{s+1-2j}_{m+n} + (s-1-l)^{\underline{2}}\mathscr{M}^2\alpha_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi+1-2j}\mathbf{Q}^{s-2j}_{m+n}\\
&+(s-1-l)(l-\xi)\mathcal{S}\left(2\alpha_j^{\frac{\xi}{2}} + (2s-2-l-\xi)\gamma_j^{\frac{\xi}{2}}\right){}^{l-2j}_{\xi+1-2j}\mathbf{Q}^{s-2j}_{m+n} + \frac{(l-\xi)^{\underline{2}}}{4}\Big((2s-1-l-\xi)^{\underline{2}}\alpha_j^{\frac{\xi}{2}}\\
&+2(4s-2l-2\xi-1)\beta_j^{\frac{\xi}{2}}\Big){}^{l-1-2j}_{\xi+1-2j}\mathbf{Q}^{s-2j}_{m+n} - (s-1-l)^{\underline{2}}\mathscr{M}^2\mathcal{S}\gamma_j^{\frac{\xi}{2}}\,{}^{l+1-2j}_{\xi-2j}\mathbf{Q}^{s-1-2j}_{m+n}\\
&-(s-1-l)\left(\mathscr{M}^2\left((l-\xi)(2s-2-l-\xi)\alpha_j^{\frac{\xi}{2}} + (s-1+l-2\xi)\beta_j^{\frac{\xi}{2}}\right) - (2s-2l-3)\mathcal{S}^2\gamma_j^{\frac{\xi}{2}}\right){}^{l-2j}_{\xi-2j}\mathbf{Q}^{s-1-2j}_{m+n}\\
&+\frac{l-\xi}{4}\mathcal{S}\Big(2(2(s+\xi)(2s-\xi)+4l^2-5\xi-l(10s-2\xi-5)-3)\alpha_j^{\frac{\xi}{2}} - 4(2l-2\xi-1)\beta_j^{\frac{\xi}{2}}\\
&-(l-1-\xi)(2s-1-l-\xi)^{\underline{2}}\,\gamma_j^{\frac{\xi}{2}}\Big){}^{l-1-2j}_{\xi-2j}\mathbf{Q}^{s-1-2j}_{m+n} + \frac{(l-\xi)^{\underline{2}}(2s+1-l-\xi)^{\underline{2}}}{4}\beta_j^{\frac{\xi}{2}}\,{}^{l-2-2j}_{\xi-2j}\mathbf{Q}^{s-1-2j}_{m+n}\Big]\Big\},
\end{aligned}
\tag{A.2.9g}
$$

$$
\begin{aligned}
P_n \star {}^{l}_{\xi}\mathbf{Q}^{s}_{m}\Big|_{\xi\,\text{odd}} = \sum_{j=0}^{\lfloor\frac{\xi}{2}\rfloor} \Bigg\{ & \alpha_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi+1-2j}\mathbf{Q}^{s+2-2j}_{m+n} + \beta_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l-2j}_{\xi-2j}\mathbf{Q}^{s+1-2j}_{m+n} - \mathcal{S}\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi-2j}\mathbf{Q}^{s+1-2j}_{m+n} + \mathscr{M}^2\alpha_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi-1-2j}\mathbf{Q}^{s-2j}_{m+n} \\
& - 2\mathcal{S}\alpha_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l-2j}_{\xi-1-2j}\mathbf{Q}^{s-2j}_{m+n} + \beta_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l-1-2j}_{\xi-1-2j}\mathbf{Q}^{s-2j}_{m+n} - N_1^{s-\xi}(m,n)\Bigg[(l-\xi)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi+2-2j}\mathbf{Q}^{s+2-2j}_{m+n} \\
& + (s-1-l)\mathcal{S}\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi+1-2j}\mathbf{Q}^{s+1-2j}_{m+n} + \frac{l-\xi}{2}\Bigg((2s+1-l-\xi)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} + 2\beta_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-2j}_{\xi+1-2j}\mathbf{Q}^{s+1-2j}_{m+n} \\
& - (s-2-l)\mathscr{M}^2\alpha_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi-2j}\mathbf{Q}^{s-2j}_{m+n} \\
& + \mathcal{S}\Bigg((s-2-2l+\xi)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} - \frac{(l+1-\xi)(2s-2-l-\xi)}{2}\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-2j}_{\xi-2j}\mathbf{Q}^{s-2j}_{m+n} \\
& + \frac{(l-\xi)(2s+1-l-\xi)}{2}\beta_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l-1-2j}_{\xi-2j}\mathbf{Q}^{s-2j}_{m+n} + (s-2-l)\mathscr{M}^2\mathcal{S}\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi-1-2j}\mathbf{Q}^{s-1-2j}_{m+n} \\
& + \Bigg(\frac{(l+1-\xi)(2s-2-l-\xi)}{2}\mathscr{M}^2\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} + (l+1-\xi)\mathscr{M}^2\beta_j^{\lfloor\frac{\xi}{2}\rfloor} - (2s-2l-3)\mathcal{S}^2\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-2j}_{\xi-1-2j}\mathbf{Q}^{s-1-2j}_{m+n} \\
& - \mathcal{S}\Bigg((l-\xi)(2s-1-l-\xi)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} + (2l-2\xi+1)\beta_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-1-2j}_{\xi-1-2j}\mathbf{Q}^{s-1-2j}_{m+n} \\
& + \frac{(l-\xi)(2s+1-l-\xi)}{2}\beta_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l-2-2j}_{\xi-1-2j}\mathbf{Q}^{s-1-2j}_{m+n}\Bigg] + N_2^{s-\xi}(m,n)\Bigg[(l-\xi)^{\underline{2}}\alpha_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi+3-2j}\mathbf{Q}^{s+2-2j}_{m+n} \\
& + (l-\xi)(2s-2-l-\xi)\mathcal{S}\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi+2-2j}\mathbf{Q}^{s+1-2j}_{m+n} + (l-\xi)^{\underline{2}}\Bigg((2s-l-\xi-{}^1\!/\!_2)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} + \beta_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-2j}_{\xi+2-2j}\mathbf{Q}^{s+1-2j}_{m+n} \\
& + \mathscr{M}^2\Big(s(s-3) - 2l(s+\xi-2) + \xi^{\underline{2}} + 2l^2 + 3\Big)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi+1-2j}\mathbf{Q}^{s-2j}_{m+n} \\
& + (l-\xi)\mathcal{S}\Bigg(2(s+\xi-2l-1)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} + (s-1-l)(2s-2-l-\xi)\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-2j}_{\xi+1-2j}\mathbf{Q}^{s-2j}_{m+n} \qquad \text{(A.2.9h)} \\
& + \frac{(l-\xi)^{\underline{2}}}{4}\Bigg((2s+1-l-\xi)^{\underline{2}}\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} + 2(4s+1-2l-2\xi)\beta_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-1-2j}_{\xi+1-2j}\mathbf{Q}^{s-2j}_{m+n} \\
& - (s-2-l)^{\underline{2}}\mathscr{M}^2\mathcal{S}\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi-2j}\mathbf{Q}^{s-1-2j}_{m+n} - (s-2-l)\Bigg((l+1-\xi)(2s-3-l-\xi)\mathscr{M}^2\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} \\
& + (s+l-2\xi)\mathscr{M}^2\beta_j^{\lfloor\frac{\xi}{2}\rfloor} - (2s-2l-3)\mathcal{S}^2\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-2j}_{\xi-2j}\mathbf{Q}^{s-1-2j}_{m+n} \\
& + \frac{l-\xi}{4}\mathcal{S}\Bigg(2(4s(s-2) + 4l^2 - 2\xi^2 - l(10s-2\xi-11) + 2s\xi - 3\xi + 5)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} - 4(2l-2\xi+1)\beta_j^{\lfloor\frac{\xi}{2}\rfloor} \\
& - (l+1-\xi)(2s-2-l-\xi)^{\underline{2}}\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-1-2j}_{\xi-2j}\mathbf{Q}^{s-1-2j}_{m+n} + \frac{(l-\xi)^{\underline{2}}(2s+1-l-\xi)^{\underline{2}}}{4}\beta_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l-2-2j}_{\xi-2j}\mathbf{Q}^{s-1-2j}_{m+n} \\
& + (s-2-l)^{\underline{2}}\mathscr{M}^4\alpha_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l+1-2j}_{\xi-1-2j}\mathbf{Q}^{s-2-2j}_{m+n} - (s-2-l)\mathscr{M}^2\mathcal{S}\Bigg(2(s+\xi-2l-3)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} \\
& - (l+1-\xi)(2s-l-\xi-3)\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-2j}_{\xi-1-2j}\mathbf{Q}^{s-2-2j}_{m+n} + \frac{1}{4}\Bigg(\Bigg((2s-\xi-2)^{\underline{2}}\xi^{\underline{2}}\mathscr{M}^2 - 2l^3(2s-3)\mathscr{M}^2 + l^4\mathscr{M}^2 \\
& + 4(2s-3)(2\xi-1)\mathcal{S}^2 + 16l^2\mathcal{S}^2 + l^2(4s^2 - 2\xi(\xi+2) + 2s(2\xi-7) + 11)\mathscr{M}^2 - 2l(8(s+\xi-2)\mathcal{S}^2 \\
& + (2s-3)(s(2\xi-1) - \xi(\xi+2) + 1)\mathscr{M}^2)\Bigg)\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} + 2\mathscr{M}^2\Big(2s^2 - l(2l^2+1) + l^2(4s+2\xi-3) \\
& - 2l(4s-\xi-3)\xi + s(4\xi^{\underline{2}}-2) - \xi(2\xi^2+\xi-3)\Big)\beta_j^{\lfloor\frac{\xi}{2}\rfloor} + 4\mathcal{S}^2(2s-2l-3)\,\Big(s(2\xi-1) - \xi(\xi+2) - l(2s-3) \\
& + l^2 + 1\Big)\gamma_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-1-2j}_{\xi-1-2j}\mathbf{Q}^{s-2-2j}_{m+n} - \frac{l-\xi}{2}\mathcal{S}\Bigg((l-\xi-1)(2s-1-l-\xi)^{\underline{2}}\alpha_j^{\lfloor\frac{\xi}{2}\rfloor} - 2(2s-1-2\xi \\
& - 2(l-\xi)(2s-l-\xi))\beta_j^{\lfloor\frac{\xi}{2}\rfloor}\Bigg)\, {}^{l-2-2j}_{\xi-1-2j}\mathbf{Q}^{s-2-2j}_{m+n} + \frac{(l-\xi)^{\underline{2}}(2s+1-l-\xi)^{\underline{2}}}{4}\beta_j^{\lfloor\frac{\xi}{2}\rfloor}\, {}^{l-3-2j}_{\xi-1-2j}\mathbf{Q}^{s-2-2j}_{m+n}\Bigg]\Bigg\}.
\end{aligned}
$$

Let me note the special case $\xi = 0$ for left multiplication,

$$\begin{aligned} P_n \star {}^{l}_{0}\mathbf{Q}^{s}_{m} = {}^{l}_{0}\mathbf{Q}^{s+1}_{m+n} &- l N^{s}_{1}(m,n)\left({}^{l}_{1}\mathbf{Q}^{s+1}_{m+n} + \frac{2s+1-l}{2}\,{}^{l-1}_{0}\mathbf{Q}^{s}_{m+n}\right) \\ &+ N^{s}_{2}(m,n)\left(l^{\underline{2}}\,{}^{l}_{2}\mathbf{Q}^{s+1}_{m+n} + \frac{l^{\underline{2}}(4s-2l-1)}{2}\,{}^{l-1}_{1}\mathbf{Q}^{s}_{m+n}\right. \\ &- (s-1-l)(s-1+l)\mathscr{M}^2\,{}^{l}_{0}\mathbf{Q}^{s-1}_{m+n} - l(2l-1)\mathcal{S}\,{}^{l-1}_{0}\mathbf{Q}^{s-1}_{m+n} \\ &\left.+\frac{l^{\underline{2}}(2s+1-l)^{\underline{2}}}{4}\,{}^{l-2}_{0}\mathbf{Q}^{s-1}_{m+n}\right). \end{aligned} \tag{A.2.10}$$

A.2.3 Further Identities

Some further identities may be derived, which are not necessarily needed in the main part of the present work but might be of interest.

A splitting of generators into $(l = s-1)$- and $(l = 0)$-factors can be performed using the commutation relations

$${}^{l}_{\xi}\mathbf{Q}^{s}_{m} \star (J_n)^k = \sum_{j=0}^{k}\binom{k}{j}(m-(s-1-\xi)n)^{\overline{k-j,n}}(J_n)^j \star {}^{l}_{\xi}\mathbf{Q}^{s}_{m+(k-j)n}\,, \tag{A.2.11a}$$

$$(J_n)^k \star {}^{l}_{\xi}\mathbf{Q}^{s}_{m} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(m-(s-1-\xi)n)^{\overline{k-j,n}}\,{}^{l}_{\xi}\mathbf{Q}^{s}_{m+(k-j)n} \star (J_n)^j\,, \tag{A.2.11b}$$

where $(a)^{\overline{k,n}}$ denotes the n-step rising factorial. The result is

$$\begin{aligned} {}^{l}_{\xi}\mathbf{Q}^{s}_{m}\Big|_{\xi\text{ even}} &= \frac{(s+m-\xi-1)!(s-m-\xi-1)!}{(2s-2\xi-2)!}\times \\ &\times\sum_{k}\binom{2l-2\xi}{k}\binom{2s-2l-2}{s-1-\xi-m-k}\,{}^{l}_{\xi}\mathbf{Q}^{l+1}_{l-\xi-k}\star{}^{0}_{0}\mathbf{Q}^{s-l}_{m-l+\xi+k}\,, \end{aligned} \tag{A.2.12a}$$

$$\begin{aligned} {}^{l}_{\xi}\mathbf{Q}^{s}_{m}\Big|_{\xi\text{ odd}} &= \frac{(s+m-\xi-1)!(s-m-\xi-1)!}{(2s-2\xi-2)!}\times \\ &\times\sum_{k}\binom{2l-2\xi+2}{k}\binom{2s-2l-4}{s-1-\xi-m-k}\,{}^{l}_{\xi}\mathbf{Q}^{l+2}_{l+1-\xi-k}\star{}^{0}_{0}\mathbf{Q}^{s-1-l}_{m-1-l+\xi+k}\,. \end{aligned} \tag{A.2.12b}$$

A.3 Product Rules and Commutators at $l = 0$ and $l = 1$

The more translation generators are involved, the easier becomes the calculation of product rules. Accordingly, it is possible to derive explicit expressions in the cases $l = 0$ and $l = 1$. Starting from the simplest case,

$${}^{0}_{0}\mathbf{Q}^{s}_{m} \star P_n = {}^{0}_{0}\mathbf{Q}^{s+1}_{m+n} - \frac{\mathcal{N}^{s2}_{2}(m,n)}{8(s-{}^{1}\!/_{2})^{\underline{2}}}\mathscr{M}^2\,{}^{0}_{0}\mathbf{Q}^{s-1}_{m+n}\,, \tag{A.3.1}$$

one may derive an expression for ${}^{0}_{0}\mathbf{Q}^{s}_{m} \star (P_{-1})^{\sigma}$ and replace

$$(s+m-1)^{\underline{2u}} = \frac{\mathcal{N}^{st}_{2u}(m,-(t-1))}{4^{u}(t-1)^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}}\,, \tag{A.3.2}$$

from which it is clear how to generalise to arbitrary modes in the factor on the right-hand side. This results in the multiplication rule (4.1.12) given in the main text. Since (A.2.12) allows to express higher-l generators through lower-l ones by

$$\begin{aligned}
{}^{1}_{0}\mathbf{Q}^{s}_{m} &= \frac{(s+m-1)^{\underline{2}}}{4(s-1)(s-{}^{3}\!/\!{}_{2})} J_{1} \star {}^{0}_{0}\mathbf{Q}^{s-1}_{m-1} + \frac{(s+m-1)(s-m-1)}{2(s-1)(s-{}^{3}\!/\!{}_{2})} J_{0} \star {}^{0}_{0}\mathbf{Q}^{s-1}_{m} \\
&\quad + \frac{(s-m-1)^{\underline{2}}}{4(s-1)(s-{}^{3}\!/\!{}_{2})} J_{-1} \star {}^{0}_{0}\mathbf{Q}^{s-1}_{m+1}\,,
\end{aligned} \tag{A.3.3a}$$

$${}^{1}_{1}\mathbf{Q}^{s}_{m} = -\frac{m+s-2}{2(s-2)} J_{1} \star {}^{0}_{0}\mathbf{Q}^{s-1}_{m-1} + \frac{m}{s-2} J_{0} \star {}^{0}_{0}\mathbf{Q}^{s-1}_{m} - \frac{m-s+2}{2(s-2)} J_{-1} \star {}^{0}_{0}\mathbf{Q}^{s-1}_{m+1}\,, \tag{A.3.3b}$$

one may bootstrap the structure constants of the $(l=1)$-slice of the algebra; here I summarise the results:

$${}^{0}_{0}\mathbf{Q}^{s}_{m} \star {}^{0}_{0}\mathbf{Q}^{t}_{n} = \sum_{u=0}^{\left\lfloor \frac{s+t-2}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u}u!} \frac{\mathcal{N}^{st}_{2u}(m,n)}{(s-{}^{3}\!/\!{}_{2})^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{3}\!/\!{}_{2})^{\underline{u}}}\, {}^{0}_{0}\mathbf{Q}^{s+t-1-2u}_{m+n}\,, \tag{A.3.4a}$$

$$\begin{aligned}
{}^{1}_{0}\mathbf{Q}^{s}_{m} \star {}^{0}_{0}\mathbf{Q}^{t}_{n} &= \frac{1}{s-1} \sum_{u=0}^{\left\lfloor \frac{s+t-3}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u}u!} \frac{((s+t-1)(s-1-2u)+u(2u+1))\,\mathcal{N}^{st}_{2u}(m,n)}{(s-{}^{3}\!/\!{}_{2})^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-2u-1)}\, {}^{1}_{0}\mathbf{Q}^{s+t-1-2u}_{m+n} \\
&\quad - \frac{1}{2(s-1)} \sum_{u=0}^{\left\lfloor \frac{s+t-4}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u}u!} \frac{\mathcal{N}^{st}_{2u+1}(m,n)}{(s-{}^{3}\!/\!{}_{2})^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{5}\!/\!{}_{2})^{\underline{u}}(s+t-2u-2)}\, {}^{1}_{1}\mathbf{Q}^{s+t-1-2u}_{m+n} \\
&\quad - \frac{2\mathcal{S}}{s-1} \sum_{u=0}^{\left\lfloor \frac{s+t-4}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u+2}u!} \frac{\mathcal{N}^{st}_{2u+2}(m,n)}{(s-{}^{3}\!/\!{}_{2})^{\underline{u+1}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{5}\!/\!{}_{2})^{\underline{u+1}}(s+t-2u-3)}\, {}^{0}_{0}\mathbf{Q}^{s+t-3-2u}_{m+n}\,,
\end{aligned} \tag{A.3.4b}$$

$$\begin{aligned}
{}^{1}_{1}\mathbf{Q}^{s}_{m} \star {}^{0}_{0}\mathbf{Q}^{t}_{n} &= \frac{2}{s-2} \sum_{u=0}^{\left\lfloor \frac{s+t-3}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u-1}u!} \frac{u\mathcal{N}^{s-1,t}_{2u-1}(m,n)}{(s-{}^{5}\!/\!{}_{2})^{\underline{u-1}}(t-{}^{3}\!/\!{}_{2})^{\underline{u-1}}(s+t-u-{}^{5}\!/\!{}_{2})^{\underline{u-1}}(s+t-2u-1)}\, {}^{1}_{0}\mathbf{Q}^{s+t-1-2u}_{m+n} \\
&\quad + \frac{1}{s-2} \sum_{u=0}^{\left\lfloor \frac{s+t-4}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u}u!} \frac{\left(s^{2}+2u^{2}+s(t-2u-4)-2(t-1)(u+1)+3u+2\right)\mathcal{N}^{s-1,t}_{2u}(m,n)}{(s-{}^{5}\!/\!{}_{2})^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{5}\!/\!{}_{2})^{\underline{u}}(s+t-2u-2)}\, {}^{1}_{1}\mathbf{Q}^{s+t-1-2u}_{m+n} \\
&\quad + \frac{2\mathcal{S}}{s-2} \sum_{u=0}^{\left\lfloor \frac{s+t-4}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u+1}u!} \frac{\mathcal{N}^{s-1,t}_{2u+1}(m,n)}{(s-{}^{5}\!/\!{}_{2})^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{7}\!/\!{}_{2})^{\underline{u}}(s+t-2u-3)}\, {}^{0}_{0}\mathbf{Q}^{s+t-3-2u}_{m+n}\,,
\end{aligned} \tag{A.3.4c}$$

$$\left[{}^{1}_{0}\mathbf{Q}^{s}_{m}, {}^{0}_{0}\mathbf{Q}^{t}_{n} \right] = \frac{1}{2(s-1)} \sum_{u=0}^{\left\lfloor \frac{s+t-3}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u}u!} \frac{\mathcal{N}^{st}_{2u+1}(m,n)}{(s-{}^{3}\!/\!{}_{2})^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{5}\!/\!{}_{2})^{\underline{u}}}\, {}^{0}_{0}\mathbf{Q}^{s+t-2-2u}_{m+n}\,, \tag{A.3.4d}$$

$$\left[{}^{1}_{1}\mathbf{Q}^{s}_{m}, {}^{0}_{0}\mathbf{Q}^{t}_{n} \right] = \frac{1}{s-2} \sum_{u=0}^{\left\lfloor \frac{s+t-3}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u}u!} \frac{(2u(s-u-{}^{3}\!/\!{}_{2})-(s-2u-2)(t-1))\,\mathcal{N}^{s-1,t}_{2u}(m,n)}{(s-{}^{5}\!/\!{}_{2})^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{5}\!/\!{}_{2})^{\underline{u}}}\, {}^{0}_{0}\mathbf{Q}^{s+t-2-2u}_{m+n}\,, \tag{A.3.4e}$$

$$\begin{aligned}
\left[{}^{1}_{0}\mathbf{Q}^{s}_{m}, {}^{1}_{0}\mathbf{Q}^{t}_{n} \right] &= \frac{1}{2(s-1)(t-1)} \sum_{u=0}^{\left\lfloor \frac{s+t-4}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u}u!} \frac{(s+t-3-2u)\mathcal{N}^{st}_{2u+1}(m,n)}{(s-{}^{3}\!/\!{}_{2})^{\underline{u}}(t-{}^{3}\!/\!{}_{2})^{\underline{u}}(s+t-u-{}^{5}\!/\!{}_{2})^{\underline{u}}}\, {}^{1}_{0}\mathbf{Q}^{s+t-2-2u}_{m+n} \\
&\quad - \frac{\mathcal{S}}{(s-1)(t-1)} \sum_{u=0}^{\left\lfloor \frac{s+t-5}{2} \right\rfloor} \frac{(-1)^{u}\mathcal{M}^{2u}}{4^{2u+2}u!} \frac{\mathcal{N}^{st}_{2u+3}(m,n)}{(s-{}^{3}\!/\!{}_{2})^{\underline{u+1}}(t-{}^{3}\!/\!{}_{2})^{\underline{u+1}}(s+t-u-{}^{7}\!/\!{}_{2})^{\underline{u+1}}}\, {}^{0}_{0}\mathbf{Q}^{s+t-4-2u}_{m+n}\,.
\end{aligned} \tag{A.3.4f}$$

Appendix B

Supplementary Material and Conventions

This chapter provides some supplementary material and equations that could not be included into the main part for logistic reasons.

B.1 Unfolded Equations of Motion

The unfolded equations discussed in section 5.2 are given in the following, written out in components. First, the case of a massive scalar field propagating on a general higher-spin background leads to the rather involved set of equations

$$
\begin{aligned}
0 &= \partial_u c^s_m + c^{s-1}_{m-1} - \frac{(s-m+1)^{\underline{2}}\mathscr{M}^2}{4(s+{}^1\!/_2)^{\underline{2}}}\, c^{s+1}_{m-1} \\
&\quad - \sum_{s'=1}^{\infty} \sum_{\substack{m'=s+m-s' \\ s+s'+m+m' \text{ even}}}^{s'-1} \frac{(-1)^{\frac{s'-s+m'-m}{2}} (m'-m)^{\underline{\frac{s'-s+m'-m}{2}}} (s'+m'-1)^{\underline{s'-s+m'-m}} \mathscr{M}^{s'-s+m'-m} Z^{(m'-m+1)}}{2^{s'-s+m'-m+2} \left(\frac{s'-s+m'-m}{2}\right)! (s'-{}^3\!/_2)^{\underline{\frac{s'-s+m'-m}{2}}} \left(\frac{s'+s+m'-m-1}{2}\right)^{\underline{\frac{s'-s+m'-m}{2}}}}\, c^{s'}_{m'}\,, && \text{(B.1.1a)} \\
0 &= \partial_r c^s_m + \frac{1}{2} c^{s-1}_{m+1} - \frac{(s+m+1)^{\underline{2}}\mathscr{M}^2}{8(s+{}^1\!/_2)^{\underline{2}}}\, c^{s+1}_{m+1}\,, && \text{(B.1.1b)} \\
0 &= \partial_\phi c^s_m + (s-m) c^s_{m-1} + r c^{s-1}_m + \frac{r(s+m)(s-m)\mathscr{M}^2}{4(s+{}^1\!/_2)^{\underline{2}}}\, c^{s+1}_m \\
&\quad + \sum_{s'=1}^{\infty} \sum_{\substack{m'=s+m-s'+1 \\ s+s'+m+m' \text{ odd}}}^{s'-1} \frac{(-1)^{\frac{s'-s+m'-m-1}{2}} (m'-m)^{\underline{\frac{s'-s+m'-m+1}{2}}} (s'+m'-1)^{\underline{s'-s+m'-m}} \mathscr{M}^{s'-s+m'-m-1} Z^{(m'-m+1)}}{2^{s'-s+m'-m+1} \left(\frac{s'-s+m'-m-1}{2}\right)! (s'-{}^3\!/_2)^{\underline{\frac{s'-s+m'-m-1}{2}}} \left(\frac{s'+s+m'-m}{2}-1\right)^{\underline{\frac{s'-s+m'-m-1}{2}}}}\, c^{s'}_{m'} && \text{(B.1.1c)} \\
&\quad - \sum_{s'=1}^{\infty} \sum_{\substack{m'=s+m-s' \\ s+s'+m+m' \text{ even}}}^{s'-1} \frac{(-1)^{\frac{s'-s+m'-m}{2}} (m'-m)^{\underline{\frac{s'-s+m'-m}{2}}} (s'+m'-1)^{\underline{s'-s+m'-m}} \mathscr{M}^{s'-s+m'-m} W^{(m'-m+1)}}{2^{s'-s+m'-m+1} \left(\frac{s'-s+m'-m}{2}\right)! (s'-{}^3\!/_2)^{\underline{\frac{s'-s+m'-m}{2}}} \left(\frac{s'+s+m'-m-1}{2}\right)^{\underline{\frac{s'-s+m'-m}{2}}}}\, c^{s'}_{m'}\,.
\end{aligned}
$$

Calling $c \equiv c^1_0$, for $s=1$, $m=0$ these equations give

$$0 = \partial_u c - \frac{2\mathscr{M}^2}{3} c^2_{-1} + \frac{1}{4}\sum_{s=2}^{\infty} \frac{(-1)^s (s-1)!\,\mathscr{M}^{2(s-1)} Z^{(s)}}{(s-{}^1\!/_2)^{\underline{s-1}}}\, c^s_{s-1}\,, \tag{B.1.2a}$$

$$0 = \partial_r c - \frac{\mathscr{M}^2}{3} c^2_1\,, \tag{B.1.2b}$$

$$0 = \partial_\phi c + \frac{r\mathscr{M}^2}{3} c^2_0 + \frac{1}{2}\sum_{s=2}^{\infty} \frac{(-1)^s (s-1)!\,\mathscr{M}^{2(s-1)} W^{(s)}}{(s-{}^1\!/_2)^{\underline{s-1}}}\, c^s_{s-1} \tag{B.1.2c}$$

and higher derivatives with respect to the radial coordinate r can be written as

$$\partial_r^n c^s_{s-1} = \frac{(s+n-1)^{\underline{n}}\mathscr{M}^{2n}}{2^n(s+n-{}^1\!/_2)^{\underline{n}}} c^{s+n}_{s+n-1}\,, \qquad \partial_r^n c^s_{s-2} = \frac{(s+n-2)^{\underline{n}}\mathscr{M}^{2n}}{2^n(s+n-{}^1\!/_2)^{\underline{n}}} c^{s+n}_{s+n-2}\,. \tag{B.1.3}$$

Secondly, the case of massive higher-spin fields (of arbitrary spin) propagating in a classical (Einstein) background gives rise to the compact set of equations

$$\begin{aligned}
0 = {}& e^\mu_n \partial_\mu {}^l_\xi c^s_{m+n} + \frac{\delta_{n,0}}{r}\left((s-\xi-m)\,{}^l_\xi c^s_{m-1} + (s-\xi+m)\frac{M}{4}\,{}^l_\xi c^s_{m+1}\right) + {}^l_\xi c^{s-1}_m \\
&- \frac{2\mathscr{N}_1^{s-\xi,2}(m,n)}{(s-\xi)^{\underline{2}}}\Bigg[(l-\xi+1)\,{}^l_{\xi-1}c^{s-1}_m - \frac{(l-2\lfloor{}^\xi\!/_2\rfloor+1)^{\underline{2}}}{2}\,{}^{l+1}_\xi c^s_m \\
&+ (\xi-2\lfloor{}^{\xi-1}\!/_2\rfloor+1)\left((l-\xi)\mathscr{M}^2\,{}^l_{\xi+1}c^{s+1}_m - 2(l-\xi+{}^1\!/_2\mathcal{S}\,{}^{l+1}_{\xi+1}c^{s+1}_m\right)\Bigg] \\
&+ \frac{\mathscr{N}_2^{s-\xi+1,2}(m,n)}{2(s-\xi)^2(s-\xi+{}^1\!/_2)^{\underline{2}}}\Bigg[(l-\xi+2)^{\underline{2}}\,{}^l_{\xi-2}c^{s-1}_m - (l-\xi+2)^{\underline{2}}(l-\xi+{}^3\!/_2)\,{}^{l+1}_{\xi+1}c^s_m \\
&- \left((s-\xi)^2 - (l-2\lfloor{}^\xi\!/_2\rfloor)^2\right)\mathscr{M}^2\,{}^l_\xi c^{s+1}_m - 2(l-\xi+1)(l-2\lfloor{}^\xi\!/_2\rfloor+{}^1\!/_2)\mathcal{S}\,{}^{l+1}_\xi c^{s+1}_m \\
&+ \frac{(l-2\lfloor{}^\xi\!/_2\rfloor+1)(l-2\lfloor{}^\xi\!/_2\rfloor+2)^{\underline{3}}}{4}\,{}^{l+2}_\xi c^{s+1}_m \\
&- (\xi-2\lfloor{}^{\xi-1}\!/_2\rfloor+1)\Big((l-\xi+1)^{\underline{2}}(l-\xi+{}^1\!/_2)\mathscr{M}^2\,{}^{l+1}_{\xi+1}c^{s+2}_m \\
&-2(l-\xi+1)\left((l-\xi+1)^2-{}^1\!/_2\right)\mathcal{S}\,{}^{l+2}_{\xi+1}c^{s+2}_m\Big)\Bigg]\,.
\end{aligned} \tag{B.1.4}$$

In all of the above cases non-existing index combinations in ${}^l_\xi c^s_m$ or the charges are to be identified with zeros.

B.2 Complex Integration

In section 6.1 a measure is given to equip the oscillator representation with an inner product

$$(f,g) = \int_{\mathbb{C}^\infty} \left[\mathrm{d}^2 v\right]\, \overline{f(v)}\, g(v)\,, \tag{B.2.1}$$

where the measure is specified as

$$\left[\mathrm{d}^2 v\right] = \prod_{n=1}^{\infty} 16 n^2\, \mathrm{e}^{-4n\left(v_n^{(1)}\bar v_n^{(2)} + v_n^{(2)}\bar v_n^{(1)}\right)}\, \mathrm{d}^2 v_n^{(1)}\, \mathrm{d}^2 v_n^{(2)}\,. \tag{B.2.2}$$

It is normalised such that $(\mathbf{1},\mathbf{1}) = 1$. To evaluate such integrals it is advantageous to perform an analytic continuation $n \mapsto \mathrm{i}n$. It is then possible to utilise the complex delta distribution and its derivatives, which can be defined through

$$\int_{\mathbb{C}} \mathrm{d}v\,\mathrm{d}\bar{v}\; f(v,\bar{v})\; \partial_v^a \partial_{\bar{v}}^b \delta(v,\bar{v}) = (-1)^{a+b}\, \partial_v^a \partial_{\bar{v}}^b f(v,\bar{v})\Big|_{v=0,\,\bar{v}=0}\,, \tag{B.2.3}$$

for integers $a, b \in \mathbb{N}_0$. Then one may show how monomials of oscillator variables give rise to such derivatives of the complex delta distribution,

$$\int_{\mathbb{C}} \mathrm{d}w\,\mathrm{d}\bar{w}\; \bar{w}^a w^b\, \mathrm{e}^{\mathrm{i}\kappa(v\bar{w}+\bar{v}w)} = \left(-\frac{\mathrm{i}}{\kappa}\right)^{a+b+2} \partial_v^a \partial_{\bar{v}}^b \delta(v,\bar{v})\,, \tag{B.2.4}$$

for any $\kappa \in \mathbb{R}$ and $v \in \mathbb{C}$. This allows to determine the appropriate normalisation factor for the hermitian product as well as the (pseudo-)orthogonality relation

$$\left(\left(v_m^{(1)}\right)^a \left(v_m^{(2)}\right)^b, \left(v_m^{(1)}\right)^c \left(v_m^{(2)}\right)^d\right) = \frac{a!b!}{(4m)^{a+b}}\delta_{a,d}\delta_{b,c}\,. \tag{B.2.5}$$

B.3 Lie Algebras and Matrix Representations

This section lists various matrix representations of Lie algebras and Lie groups.

Spin-Three Algebra in AdS

The following matrices provide a representation of $\mathfrak{sl}(3,\mathbb{R})$:

$$W_2 = 2\sqrt{-\sigma}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{B.3.1a}$$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad W_1 = \sqrt{-\sigma}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \tag{B.3.1b}$$

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad W_0 = \frac{2}{3}\sqrt{-\sigma}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B.3.1c}$$

$$L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_{-1} = \sqrt{-\sigma}\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.3.1d}$$

$$W_{-2} = 2\sqrt{-\sigma}\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.3.1e}$$

Poincaré Algebra and Group

The Poicaré algebra in the $\mathfrak{iso}(2,1)$-basis (3.1.17) may be represented in terms of (4×4)-matrices

$$J_a = \begin{pmatrix} L_a & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_a = \begin{pmatrix} \mathbb{O} & e_a \\ 0 & 0 \end{pmatrix}, \tag{B.3.2}$$

where the (3×3)-matrices L_a furnish a matrix representation of the $\mathfrak{so}(2,1)$-subalgebra, given by

$$L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.3.3}$$

and the vectors e_a are given as

$$e_0 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{B.3.4}$$

Note that the trace cannot be used as bilinear form in a straightforward manner, since only $\operatorname{tr}(J_a J_b) = 2\eta_{ab}$ is non-vanishing. A non-degenerate form can be defined through the non-vanishing traces $\operatorname{tr}\left(J_a^\top J_b\right) = 2\delta_{ab}$ and $\operatorname{tr}\left(P_a^\top P_b\right) = \delta_{ab}$.

Exponentiation of the Lie-algebra generators provides a matrix representation of the Poincaré group $ISO(2,1)$,

$$\Lambda_0(\alpha) \equiv \mathrm{e}^{\alpha L_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \qquad v_0 = \begin{pmatrix} -t \\ 0 \\ 0 \end{pmatrix}, \tag{B.3.5a}$$

$$\Lambda_1(\beta) \equiv \mathrm{e}^{\beta L_1} = \begin{pmatrix} \cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & \cosh\beta \end{pmatrix}, \qquad v_1 = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}, \tag{B.3.5b}$$

$$\Lambda_2(\gamma) \equiv \mathrm{e}^{\gamma L_2} = \begin{pmatrix} \cosh\gamma & -\sinh\gamma & 0 \\ -\sinh\gamma & \cosh\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix}. \tag{B.3.5c}$$

Note that the inverse of an $ISO(2,1)$-element can be expressed in terms of the respective $SO(2,1)$-element as

$$g^{-1} = \begin{pmatrix} \Lambda^{-1} & -\Lambda^{-1} v \\ 0 & 1 \end{pmatrix}. \tag{B.3.6}$$

B.4 Metric Quantities

The metric (3.1.16) of the most general asymptotically flat spacetimes and its inverse read

$$\left(g_{\mu\nu}\right)=\begin{pmatrix} M(\phi) & -1 & N(u,\phi) \\ -1 & 0 & 0 \\ N(u,\phi) & 0 & r^2 \end{pmatrix}, \qquad \left(g^{\mu\nu}\right)=\begin{pmatrix} 0 & -1 & 0 \\ -1 & -M(\phi)+\frac{N(u,\phi)^2}{r^2} & \frac{N(u,\phi)}{r^2} \\ 0 & \frac{N(u,\phi)}{r^2} & \frac{1}{r^2} \end{pmatrix}, \tag{B.4.1}$$

The non-vanishing Christoffel symbols are

$$\begin{aligned} &\Gamma^u_{\phi\phi}=r\,, \quad \Gamma^r_{u\phi}=\Gamma^r_{\phi u}=-\frac{M'(\phi)}{2}\,, \quad \Gamma^r_{r\phi}=\Gamma^r_{\phi r}=\frac{N(u,\phi)}{r}\,, \\ &\Gamma^r_{\phi\phi}=rM(\phi)-\frac{N(u,\phi)^2}{r}-\partial_\phi N(u,\phi)\,, \quad \Gamma^\phi_{r\phi}=\Gamma^\phi_{\phi r}=\frac{1}{r}\,, \quad \Gamma^\phi_{\phi\phi}=-\frac{N(u,\phi)}{r}\,. \end{aligned} \tag{B.4.2}$$

In $\mathfrak{isl}(2,\mathbb{R})$-components we have for the spin connection $\omega_{mn}=-\varepsilon_{mnk}\omega^k$, where $\varepsilon_{-101}=1$. Accordingly, for asymptotically flat spacetimes

$$\omega_\phi{}^1{}_{-1}=0=\omega_\phi{}^{-1}{}_1\,, \quad \omega_\phi{}^1{}_0=1\,, \quad \omega_\phi{}^0{}_1=\frac{M(\phi)}{2}\,, \quad \omega_\phi{}^{-1}{}_0=\frac{M(\phi)}{4}\,, \quad \omega_\phi{}^0{}_{-1}=2\,. \tag{B.4.3}$$

Indices are moved using the form $\eta_{mn}=(-1)^m(1+m)!(1-m)!\delta_{m+n,0}$. The action of the spin connection in the $\mathfrak{isl}(2,\mathbb{R})$-basis for the relevant cases of ordered indices with at most one zero can be determined as

$\underline{(m_1,\ldots,m_{s-1})=\left((1)_{\frac{s-1+m}{2}},(-1)_{\frac{s-1-m}{2}}\right):}$

$$\left(\omega\cdot\phi\right)^{m_1\ldots m_{s-1}}=\frac{s-1+m}{2}\phi^{(1)_{\frac{s-3+m}{2}}0(-1)_{\frac{s-1-m}{2}}}+\frac{M}{4}\frac{s-1-m}{2}\phi^{(1)_{\frac{s-1+m}{2}}0(-1)_{\frac{s-3-m}{2}}}\,, \tag{B.4.4a}$$

$$\begin{aligned}\left(\omega\cdot\left(\omega\cdot\phi\right)\right)^{m_1\ldots m_{s-1}}=&(s-1+m)^{\underline{2}}\phi^{(1)_{\frac{s-3+m}{2}}(-1)_{\frac{s+1-m}{2}}}+\frac{M}{2}\left(s^{\underline{2}}-m^2\right)\phi^{(1)_{\frac{s-1+m}{2}}(-1)_{\frac{s-1-m}{2}}}\\ &+\left(\frac{M}{4}\right)^2(s-1-m)^{\underline{2}}\phi^{(1)_{\frac{s+1+m}{2}}(-1)_{\frac{s-3-m}{2}}}\,;\end{aligned} \tag{B.4.4b}$$

$\underline{(m_1,\ldots,m_{s-1})=\left((1)_{\frac{s-2+m}{2}},0,(-1)_{\frac{s-2-m}{2}}\right):}$

$$\left(\omega\cdot\phi\right)^{m_1\ldots m_{s-1}}=2(s-1+m)\phi^{(1)_{\frac{s-2+m}{2}}(-1)_{\frac{s-m}{2}}}+\frac{M}{2}(s-1-m)\phi^{(1)_{\frac{s+m}{2}}(-1)_{\frac{s-2-m}{2}}}\,, \tag{B.4.4c}$$

$$\begin{aligned}\left(\omega\cdot\left(\omega\cdot\phi\right)\right)^{m_1\ldots m_{s-1}}=&(s-1+m)^{\underline{2}}\phi^{(1)_{\frac{s-4+m}{2}}0(-1)_{\frac{s-m}{2}}}+\frac{M}{2}\left(s^{\underline{2}}-m^2\right)\phi^{(1)_{\frac{s-2+m}{2}}0(-1)_{\frac{s-2-m}{2}}}\\ &+\left(\frac{M}{4}\right)^2(s-1-m)^{\underline{2}}\phi^{(1)_{\frac{s+m}{2}}0(-1)_{\frac{s-4-m}{2}}}\,.\end{aligned} \tag{B.4.4d}$$

Let us also give the inverse components of the vielbein, $e^\mu_m = \eta_{mn} g^{\mu\nu} e^n_\nu$, which are

$$e = e^\mu_m P^m \partial_\mu = P^1 \partial_u + \left(\frac{M}{2} P^1 + \frac{N}{r} P_0 + 2P^{-1} \right) \partial_r + \frac{1}{r} P^0 \partial_\phi \,. \tag{B.4.5}$$

Accordingly,

$$e^\mu_1 \partial_\mu = \partial_u + \frac{M}{2} \partial_r \,, \qquad e^\mu_0 = \frac{1}{r} \left(N \partial_r + \partial_\phi \right) \,, \qquad e^\mu_{-1} \partial_\mu = 2\partial_r \,. \tag{B.4.6}$$

Appendix C

Asymptotic Analysis

Having fixed boundary conditions for a theory of higher-spin gravity it is in principle possible to determine the associated asymptotic symmetries. The approach I will take here is to treat the theory as a constraint Hamiltonian system and quantise it, following [84].

The strategy is the following: Given a Chern-Simons theory with underlying gauge algebra $\mathfrak{g}$, one expands the gauge-fixed field a into algebra generators, $a = a^A X_A$, $X_A \in \mathfrak{g}$, and its components into Fourier modes a_p^A, $p \in \mathbb{Z}$. Given the structure constants $[X_A, X_B] = f_{AB}^C X_C$ and a bilinear form $\langle X_A, X_B \rangle = \gamma_{AB}$ on the Lie algebra, one can write down the Poisson brackets [84, 127]

$$\left\{a_p^A, a_q^B\right\} = -f_C^{AB}\, a_{p+q}^C + \mathrm{i}\, p k_{\mathrm{CS}} \gamma^{AB} \delta_{p+q,0}\,, \tag{C.0.1}$$

where $f_C^{AB} = \gamma^{AD}\gamma^{BE}\gamma_{CF} f_{DE}^F$ are the inverse structure constants. Now, given a number of second-class constraints $\chi_\alpha \approx 0$, one defines the constraint matrix $C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}$ and from this the Dirac brackets

$$\{f, g\}_{\mathrm{DB}} = \{f, g\} - \{f, \chi_a\} \left(C^{-1}\right)^{\alpha\beta} \{\chi_\beta, g\} \tag{C.0.2}$$

for arbitrary phase-space functions f and g. These are the brackets that are to be replaced by the commutator, $\{.,.\}_{\mathrm{DB}} \mapsto -\mathrm{i}\,[.,.]$, in the sense of canonical quantisation.

C.1 Spin-2 Case

To clarify the quantisation procedure in the case of asymptotically flat spacetimes, I will briefly review the case of classical gravity, here using the deformed bilinear form (4.2.26) and its inverse (4.2.27), to see how the non-ad-invariance of the form affects the asymptotic analysis. Note that this forces us to use the components b_C^{AB} of the inverse bilinear B^{-1} instead of the structure constants in the Poisson bracket (C.0.1), in order to get brackets that satisfy the Jacobi identity. This is an ad-hoc fix; a rigorous analysis of the quantisation of constraint Hamiltonian systems in the case of a non ad-invariant bilinear form, however, is still to be carried out and beyond the scope of this thesis.

Since the u-component may be treated as a Lagrange multiplier in the Chern-Simons action, it suffices to consider the ϕ-component of the r-independent gauge field as a single field subject to second-class constraints. In an abuse of notation write $a \equiv a_\phi$ and expand $a = a^{(1,m)} J_m + a^{(0,m)} P_m = a^A X_A$, where I collectively denote the generators J_m and P_m by X_A with a multi-index A. Then an expansion into Fourier modes reads

$$a^{(l,m)}(u,\phi) = \frac{1}{k_{\mathrm{CS}}} \sum_{p\in\mathbb{Z}} a_p^{(l,m)}(u)\, \mathrm{e}^{-\mathrm{i}\, p\phi} \tag{C.1.1}$$

and the constraints written in terms of the modes are

$$\chi_p^{(1,1)} = a_p^{(1,1)} - k_{\mathrm{CS}}\delta_{p,0} \approx 0\,, \qquad \chi_p^{(0,1)} = a_p^{(0,1)} \approx 0\,, \tag{C.1.2a}$$

$$\chi_p^{(1,0)} = a_p^{(1,0)} \approx 0\,, \qquad \chi_p^{(0,0)} = a_p^{(0,0)} \approx 0\,. \tag{C.1.2b}$$

The constraint matrix takes the form

$$C = \left(C_{pq}^{(l,m)(k,n)}\right) = \begin{pmatrix} \left(\left\{\chi_p^{(0,m)}, \chi_q^{(0,n)}\right\}\right) & \left(\left\{\chi_p^{(0,m)}, \chi_q^{(1,n)}\right\}\right) \\ \left(\left\{\chi_p^{(1,m)}, \chi_q^{(0,n)}\right\}\right) & \left(\left\{\chi_p^{(1,m)}, \chi_q^{(1,n)}\right\}\right) \end{pmatrix} \tag{C.1.3}$$

Using (C.1.2) and the known expression for the Poisson brackets (C.0.1) (the inverse structure constants and the inverse bilinear form are noted down in (4.2.29) and (4.2.27)), one finds the constraint matrix to be

$$C = \frac{k_{\mathrm{CS}}}{\mathcal{S}} \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & 0 & \left(\delta_{p+q,0}\right) \\ & & -\left(\delta_{p+q,0}\right) & \mathrm{i}\left(p\delta_{p+q,0}\right) \\ \hline 0 & \left(\delta_{p+q,0}\right) & 0 & -\frac{\mathcal{M}^2}{\mathcal{S}}\left(\delta_{p+q,0}\right) \\ -\left(\delta_{p+q,0}\right) & \mathrm{i}\left(p\delta_{p+q,0}\right) & \frac{\mathcal{M}^2}{\mathcal{S}}\left(\delta_{p+q,0}\right) & -\frac{\mathrm{i}\mathcal{M}^2}{\mathcal{S}}\left(p\delta_{p+q,0}\right) \end{array}\right). \tag{C.1.4}$$

This matrix can be inverted due to its block structure and its inverse reads

$$C^{-1} = \frac{\mathcal{S}}{k_{\text{CS}}} \left(\begin{array}{cc|cc} -\frac{\mathrm{i}\mathcal{M}^2}{\mathcal{S}}\left(p\delta_{p+q,0}\right) & -\frac{\mathcal{M}^2}{\mathcal{S}}\left(\delta_{p+q,0}\right) & -\mathrm{i}\left(p\delta_{p+q,0}\right) & -\left(\delta_{p+q,0}\right) \\ \frac{\mathcal{M}^2}{\mathcal{S}}\left(\delta_{p+q,0}\right) & 0 & \left(\delta_{p+q,0}\right) & 0 \\ \hline -\mathrm{i}\left(p\delta_{p+q,0}\right) & -\left(\delta_{p+q,0}\right) & & \\ \left(\delta_{p+q,0}\right) & 0 & & \mathbf{0} \end{array}\right). \tag{C.1.5}$$

From this one can obtain the Dirac brackets of the modes of the charges, N_p and M_p, and, finally, after quantisation $\{.,.\}_{\text{DB}} \mapsto -\mathrm{i}[.,.]$ and re-scaling $\mathcal{S}N_m = -L_m$ one arrives at the algebra

$$[L_m, L_n] = (m-n)L_{m+n}, \tag{C.1.6a}$$

$$[L_m, M_n] = (m-n)M_{m+n} + 2k_{\text{CS}}m^3\delta_{m+n,0}, \tag{C.1.6b}$$

$$[M_m, M_n] = \frac{2\mathcal{M}^2}{\mathcal{S}^2}\left((m-n)M_{m+n} + 2k_{\text{CS}}m^3\delta_{m+n,0}\right), \tag{C.1.6c}$$

where $m, n \in \mathbb{Z}$. Obviously, setting $\mathcal{M}^2 \mapsto 0$ gives back the asymptotic symmetry algebra $\mathfrak{bms}_3$ with one central charge being zero, as expected for Einstein gravity.

We see that the usage of the deformed bilinear form (together with the replacement of the structure constants in the Poisson brackets above) does not result in the known asymptotic symmetry algebra presented in subsection 3.3.1 of the Foundations. We are thus well advised to proceed with the ad-invariant bilinear form in the higher-spin case.

C.2 Higher-Spin Case

The analysis in the case of higher-spin gravity is essentially the same as presented in the previous discussion of the Poincaré case. The constraints are given through

$$a = J_1 - \frac{1}{4}\sum_{s=2}^{\infty}\left(Z^{(s)}(\phi)\,\mathbf{J}^s_{-s+1} + 2W^{(s)}(u,\phi)\,\mathbf{P}^s_{-s+1}\right) \tag{C.2.1a}$$

$$= a^{(1,2,1)}J_1 + \sum_{s=2}^{\infty}\left(a^{(1,s,-s+1)}\,\mathbf{J}^s_{-s+1} + a^{(0,s,-s+1)}\,\mathbf{P}^s_{-s+1}\right), \tag{C.2.1b}$$

which expanded into modes,

$$a^{(l,s,m)}(u,\phi) = \frac{1}{k_{\text{CS}}}\sum_{p\in\mathbb{Z}} a_p^{(l,s,m)}(u)\,\mathrm{e}^{-\mathrm{i}p\phi}, \tag{C.2.2}$$

gives the second-class constraints

$$\chi_p^{(0,s,m)} = a_p^{(0,s,m)} \approx 0\,, \tag{C.2.3a}$$

$$\chi_p^{(1,s,m)} = a_p^{(1,s,m)} - k_{\mathrm{CS}}\delta_{s,2}\delta_{m,1}\delta_{p,0} \approx 0\,. \tag{C.2.3b}$$

The constraint matrix C again splits into four blocks,

$$C = \left(C_{pq}^{(l,s,m)(k,t,n)}\right) = \begin{pmatrix} \left(\left\{\chi_p^{(0,s,m)}, \chi_q^{(0,t,n)}\right\}\right) & \left(\left\{\chi_p^{(0,s,m)}, \chi_q^{(1,t,n)}\right\}\right) \\ \left(\left\{\chi_p^{(1,s,m)}, \chi_q^{(0,t,n)}\right\}\right) & \left(\left\{\chi_p^{(1,s,m)}, \chi_q^{(1,t,n)}\right\}\right) \end{pmatrix}. \tag{C.2.4}$$

Using the inverse structure constants (4.2.25) as well as the inverse bilinear form (4.2.21), the Poisson brackets of constraints can be derived to be

$$\begin{aligned} \left\{\chi_p^{(0,s,m)}, \chi_q^{(0,t,n)}\right\} \approx{}& \tilde{g}^{st}_{\frac{s+t+m+n-3}{2}}(m,n)\left(\frac{s+t+m+n-3}{2}\frac{\mathcal{S}}{\mathcal{M}^2}a_{p+q}^{(1,1-m-n,m+n)}\right. \\ &\left.-\frac{1}{2}a_{p+q}^{(0,1-m-n,m+n)}\right)\delta_{s+t+m+n\ \mathrm{odd}}\delta_{m+n<-|s-t|} \\ &+ k_{\mathrm{CS}}(s-2)\tilde{g}^{ss}_{s-2}(m,1-m)\frac{\mathcal{S}}{\mathcal{M}^2}\delta_{s,t}\delta_{m+n,1}\delta_{p+q,0} \\ &-2\,\mathrm{i}\,k_{\mathrm{CS}}(s-2)\gamma(s,m)p\delta_{s,t}\delta_{m+n,0}\delta_{p+q,0}\,, \end{aligned} \tag{C.2.5a}$$

$$\begin{aligned} \left\{\chi_p^{(0,s,m)}, \chi_q^{(1,t,n)}\right\} \approx{}& -\frac{\tilde{g}^{st}_{\frac{s+t+m+n-3}{2}}(m,n)}{2}\delta_{s+t+m+n\ \mathrm{odd}}\delta_{m+n<-|s-t|}a_{p+q}^{(1,1-m-n,m+n)} \\ &-\frac{k_{\mathrm{CS}}}{2}\tilde{g}^{ss}_{s-2}(m,1-m)\delta_{s,t}\delta_{m+n,1}\delta_{p+q,0} \\ &+\mathrm{i}\,k_{\mathrm{CS}}\frac{\mathcal{S}}{\mathcal{M}^2}\gamma(s,m)p\delta_{s,t}\delta_{m+n,0}\delta_{p+q,0}\,, \end{aligned} \tag{C.2.5b}$$

$$\left\{\chi_p^{(1,s,m)}, \chi_q^{(1,t,n)}\right\} \approx 0\,, \tag{C.2.5c}$$

Here I used the shorthand

$$\delta_{s\ \mathrm{odd}} \equiv \begin{cases} 1\,, & \text{if } s \text{ odd} \\ 0\,, & \text{if } s \text{ even} \end{cases} \qquad \text{as well as} \qquad \delta_{m<n} \equiv \begin{cases} 1\,, & \text{if } m<n \\ 0\,, & \text{if } m \geq n \end{cases}. \tag{C.2.6}$$

The lower-right block of C vanishes and we have to calculate the inverse of the matrix block $\mathscr{A} \equiv \left(\left\{\chi_p^{(0,s,m)}, \chi_q^{(1,t,n)}\right\}\right)$ to find C^{-1}. In particular, C is invertible if $\mathscr{A}$ is invertible.

The structure of the matrix $\mathscr{A}$ is as follows: It consists of matrix blocks

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}^{22} & \mathscr{A}^{23} & \cdots \\ \mathscr{A}^{32} & \mathscr{A}^{33} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{C.2.7}$$

where each block $\mathscr{A}^{st}$ in turn is a $(2s-2)\times(2t-2)$ block matrix consisting of matrix blocks $\mathscr{A}^{st}_{mn}$, where $-s+2\leq m\leq s-1$ and $-t+2\leq n\leq t-1$. These blocks are again infinite-dimensional, capturing the mode indices $p,q\in\mathbb{Z}$. Since the free mode functions $a_p^{(l,s,-s+1)}$ only appear for $m+n<-|s-t|$ and the contribution of the bilinear form and the J_1-constraint are only present on the anti-diagonal and the next-to anti-diagonal of $\mathscr{A}^{ss}$, respectively, the matrices $\mathscr{A}^{st}$ are of a special upper-left triangular form. Within these triangles, due to the condition "$s+t+m+n$ odd", there is a chessboard-like pattern of vanishing and non-vanishing entries.

To ensure invertibility of the matrix $\mathscr{A}$, one may find a (formal) expression for its determinant, first. The special triangular form of the matrix blocks may inspire the supposition that the determinant is given by the product of the anti-diagonal entries ($m+n=1$) of the diagonal blocks $\mathscr{A}^{ss}$, only. This conjecture can actually be proven by induction, assuming the matrix $\mathscr{A}$ to contain finitely many blocks, the largest being $\mathscr{A}^{NN}$, and then using the Schur determinant identity for matrix block decomposition to take the step $N\mapsto N+1$. One thus obtains the formal expression

$$\det(\mathscr{A})=\prod_{s=2}^{\infty}\prod_{m=-(s-2)}^{s-1}\prod_{p=-\infty}^{\infty}\frac{(-1)^{m+1}4^{s-1}(s-{}^{1}\!/_{2})^{\underline{s-1}}(s-{}^{3}\!/_{2})^{\underline{s-1}}k_{\mathrm{CS}}}{(s-1-m)!(s-2+m)!\mathscr{M}^{2(s-2)}\mathcal{S}}\,, \tag{C.2.8}$$

which is unequal to zero.

Call $\mathscr{A}^{-1}\equiv\mathscr{D}$ and all the sub-blocks respectively. Then the definition of the inverse reads

$$\sum_{u=2}^{\infty}\sum_{l=-u+2}^{u-1}\sum_{r\in\mathbb{Z}}\left(\mathscr{A}^{su}_{ml}\right)_{pr}\left(\mathscr{D}^{ut}_{ln}\right)_{rq}=\delta^{st}\delta_{mn}\delta_{pq}\,. \tag{C.2.9}$$

I will assume that the inverse matrix is of the respective inverse special triangular form, i.e. it has non-vanishing entries $\mathscr{D}^{st}_{mn}$ only if $m+n\geq 1$ in the case $s=t$ and if $m+n>|s-t|$ in the case $s\neq t$ (which could, in principle, be proven by induction). As a first step, one may consider the case $s=t$ and $m=n$, from which the anti-diagonal entries of diagonal blocks $\mathscr{D}^{ss}_{1-m,m}$ can be derived since it is

$$\mathscr{A}^{ss}_{m,1-m}\mathscr{D}^{ss}_{1-m,m}=\mathbb{1}\,. \tag{C.2.10}$$

However, one needs to solve (C.2.9) for all components of the inverse matrix $\mathscr{D}$ in order to proceed.

In the course of the present work it was unfortunately not possible to find a closed-form solution of equation (C.2.9), which can be read as a matrix-valued recurrence relation. It is however clear that the inverse will contain products of the lowest-weight gauge-field components, i.e. of the modes of functions $Z^{(s)}$ and $W^{(s)}$, of arbitrary length, which notably complicates the situation. The latter is of course expected since the asymptotic symmetry algebra should be (related to) some non-linear $\mathscr{W}$-algebra.

Appendix D

Discrete-Series Representation

In subsection 6.1.1 a so-called oscillator representation of the algebra $\mathfrak{sl}(2,\mathbb{R})$ was presented. Here, I will give the sketch of a derivation of the expressions (6.1.3) from the viewpoint of discrete-series representations. I will follow [266].

First, since $SL(2,\mathbb{R}) \simeq SU(1,1)$, one may focus on the latter. It is a non-compact, simple Lie group that admits a matrix representation of its elements g of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \text{with } |\alpha|^2 - |\beta|^2 = 1\,. \tag{D.0.1}$$

The maximal compact subgroup of $SU(1,1)$ is $K = SU(1) \times SU(1) \times U(1)$ and the quotient $D = SU(1,1)/K$ embodies a so-called bounded, symmetric domain and as such a hermitian symmetric space, whose automorphism group is again $SU(1,1)$. One thus has a natural group action of the latter on D, making it an equally natural space to study representations on.

Apparently, it is $D = \mathbb{D}$, the unit disk in the complex plane, and we may consider holomorphic functions $f(z)$ on it, on which the representation we are building will act. The group action on $\mathbb{D}$ is given as

$$z \to z \cdot g = \frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}} \tag{D.0.2}$$

and one may find expressions for an invariant measure $\mathrm{d}\mu(z)$ and a Jacobian $J_g(z)$ on the disk, given by

$$\mathrm{d}\mu(z) = \frac{\mathrm{i}}{2} \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1 - z\bar{z})^2}, \qquad J_g(z) = \frac{1}{\left(\beta z + \bar{\alpha}\right)^2}\,. \tag{D.0.3}$$

Now, for any natural or half-natural number k, excluding $1/2$, one may define a multiplier representation T_k as

$$\left(T^k(g)f\right)(z) := \left(J_g(z)\right)^k f(z\cdot g)\,. \tag{D.0.4}$$

To render this representation unitary it is necessary to enhance the measure given above by an additional weight,

$$\mathrm{d}\mu_k(z) := \frac{\mathrm{i}}{2}\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1-z\bar{z})^{2(1-k)}}\,, \tag{D.0.5}$$

and restrict the Hilbert space of the representation to functions that are square-integrable with respect to this measure. This function space is then a weighted Bergman space [263] and the representation (D.0.4) is called holomorphic discrete series of $SU(1,1)$.

Finally, considering the representation (D.0.4) for group elements obtained from the exponentiation of the $\mathfrak{su}(1,1)$-basis matrices

$$T_0 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\,, \qquad T_1 = \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\,, \qquad T_2 = -\frac{\mathrm{i}}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\,, \tag{D.0.6}$$

and expanding it around the unit element, one obtains the generators T_a in the form of differential operators acting on holomorphic functions,

$$t_0 = z\partial_z + k\,, \qquad t_1 = \frac{z^2+1}{2}\partial_z + kz\,, \qquad t_2 = \mathrm{i}\left(\frac{z^2-1}{2}\partial_z + kz\right)\,. \tag{D.0.7}$$

Changing to an $\mathfrak{sl}(2,\mathbb{R})$-basis by $l_{\pm1} = -(t_1 \mp \mathrm{i}\, t_2)$ and $l_0 = t_0$, one arrives at

$$l_1 = \partial_z\,, \qquad l_0 = z\partial_z + k\,, \qquad l_{-1} = z^2\partial_z + 2kz\,. \tag{D.0.8}$$

These are precisely the generators (6.1.3) of $\mathfrak{sl}(2,\mathbb{R})$ in the so-called oscillator representation.

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gemäß § 5, Satz 2, Nr. 3 PromO

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Jena, den 27. März 2023

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