Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 23/03

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Derkach, Volodymyr; Trunk, Carsten

Januar 2023

URN: urn:nbn:de:gbv:ilm1-2023200049

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
https://www.tu-ilmenau.de/mathematik/

$\mathcal{PT}$-SYMMETRIC COUPLINGS OF DUAL PAIRS

VOLODYMYR DERKACH AND CARSTEN TRUNK

Abstract. We apply the boundary triple technique to construct a coupling (A, B) of two dual pairs (A_+, B_+) and (A_-, B_-) relative to some boundary triples. The notion of a real dual pair with respect to the time reversal operator $\mathcal{T}$ is introduced and it is shown that the coupling of two real dual pairs corresponding to real boundary triples is also real. If the operator $\mathcal{PT}$ intertwines the dual pairs (A_+, B_+) and (A_-, B_-) for some parity operator $\mathcal{P}$, then it is shown that there exists a coupling (A, B) of two dual pairs (A_+, B_+) and (A_-, B_-) such that the operator A is $\mathcal{PT}$-symmetric and $\mathcal{P}$-symmetric in the Krein space $(\mathfrak{H}, \langle\cdot,\cdot\rangle)$ with the fundamental symmetry $\mathcal{P}$. As the main result we describe proper extensions of A which are $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint.

We apply this result to interpret the $\mathcal{PT}$-symmetric Hamiltonian considered in Bender & Boettcher (1998) as a member of a family of $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of the corresponding minimal operator.

Keywords: dual pair; boundary triple; coupling; $\mathcal{PT}$-symmetric operator; non-Hermitian Hamiltonian

1. Introduction

In the seminal paper by Bender and Boettcher [6] a new view at Quantum Mechanics was proposed which adopts all its axioms except the one that restricts the Hamiltonian to be Hermitian, relaxing it to the assumption that the Hamiltonian is $\mathcal{PT}$-symmetric. Here, $\mathcal{P}$ is parity and $\mathcal{T}$ is time reversal. Since 1998, $\mathcal{PT}$-symmetric Hamiltonians have been analyzed intensively by many authors. In [24] $\mathcal{PT}$-symmetry was embedded into a more general mathematical framework: pseudo-Hermiticity or, what is the same, self-adjoint operators in Krein spaces, ([20], [4], [17], [21]) For a general introduction into $\mathcal{PT}$-symmetric Quantum Mechanics we refer to the overview paper of Mostafazadeh [26] and to the books of Moiseyev [23] and Bender [5].

A prominent class consists of the $\mathcal{PT}$-symmetric Hamiltonians

$$H := \frac{1}{2}p^2 - (iz)^{N+2},$$

where N is a positive integer [8]. The associated eigenvalue problem is defined on a contour Γ in the complex plane which is contained in a specific area in the complex plane, the so-called Stokes wedges, see [6],

$$-y''(z) - (iz)^{N+2}y(z) = \lambda y(z), \quad z \in \Gamma, \tag{1.1}$$

where $\lambda \in \mathbb{C}$ is the eigenvalue parameter. Recall ([7]) that a *Stokes wedge* S_k, $k = 0, \dots, N+3$, is an open sector in the plane with vertex zero,

$$S_k := \left\{ z \in \mathbb{C} \,\middle|\, -\frac{N+2}{2N+8}\pi + \frac{2k-2}{4+N}\pi < \arg(z) < -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi \right\}.$$

The boundary of S_k is called *Stokes lines* and consists of two rays through the origin. $\mathcal{PT}$-symmetry forces Γ to lie in two Stokes wedges, which are symmetric with respect to the imaginary axis.

In [25] the problem was mapped back to the real axis using a real parametrization. In [7] and in [18] this approach was extended to different parameterizations and contours. For simplicity, we choose here Γ to be a wedge-shaped contour,

$$\Gamma := \{xe^{i\phi \operatorname{sgn} x} |\, x \in \mathbb{R}\}, \tag{1.2}$$

for some angle $\phi \in (-\pi/2, \pi/2)$. Let $z : \mathbb{R} \to \mathbb{C}$ parameterize Γ via $z(x) := xe^{i\phi \operatorname{sgn} x}$. Then y solves (1.1) for $z \neq 0$ if and only if the pair of functions u_+, u_- given by $u_\pm(x) := y(z(x))$, $x \in \mathbb{R}_\pm$, solve

$$\mathfrak{a}_-[u_-] = \lambda u_-, \quad x \in \mathbb{R}_-, \qquad \mathfrak{a}_+[u_+] = \lambda u_+, \quad x \in \mathbb{R}_+, \tag{1.3}$$

where the differential expressions $\mathfrak{a}_\pm$ are given by

$$\mathfrak{a}_\pm[u_\pm] = -e^{\mp 2i\phi} u_\pm'' - (ix)^{N+2} e^{\pm i(N+2)\phi} u_\pm. \tag{1.4}$$

In what follows we assume that Γ lies in Stokes wedges and then by Leben & Trunk [21] the differential expressions $\mathfrak{a}_\pm$ are in the limit-point cases at $\pm\infty$ according to the classification in Brown et al. [11] (which is a refinement of the classification in Sims [27]). We mention, that the limit-circle case can be treated in a similar way as in [3],[4].

The theory of $\mathcal{PT}$-symmetry claims that the main object, the Hamiltonian, commutes under the joint action of the parity $\mathcal{P}$ and the time reversal $\mathcal{T}$,

$$(\mathcal{P}f)(x) := f(-x), \qquad (\mathcal{T}f)(x) := \overline{f(x)}. \tag{1.5}$$

The time reversal $\mathcal{T}$ applied to $\mathfrak{a}_\pm$ gives rise to new differential expressions $\mathfrak{b}_\pm = \mathcal{T}\mathfrak{a}_\pm\mathcal{T}$ defined on $\mathbb{R}_\pm$ of the form

$$\mathfrak{b}_\pm[v_\pm] = -e^{\pm 2i\phi} v_\pm'' - (-ix)^{N+2} e^{\mp i(N+2)\phi} v_\pm. \tag{1.6}$$

In Section 7 we introduce the minimal operators $A_\pm$ and $B_\pm$ associated with $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ in $L^2(\mathbb{R}_\pm)$. Due to their (in general) non-real coefficients, the operators $A_\pm$ and $B_\pm$ are neither selfadjoint nor symmetric. But they satisfy

$$\langle A_\pm f, g\rangle_\pm = \langle f, B_\pm g\rangle_\pm \tag{1.7}$$

for all $f \in \operatorname{dom} A_\pm$ and $g \in \operatorname{dom} B_\pm$. Here $\langle\cdot,\cdot\rangle_\pm$ stands for the usual inner products in the Hilbert spaces $L^2(\mathbb{R}_\pm)$. Condition (1.7) shows that the pairs (A_+, B_+) and (A_-, B_-) form *dual pairs* (see Section 2 for details). An extension theory for dual pairs based on the boundary triples technique was developed by Malamud and Mogilevskii in [22]. This is a generalization of the boundary triple approach to the extension theory of symmetric operators which was elaborated by Kochubei [19], M. & V. Gorbachuk [16], Derkach & Malamud [14] and others.

Following this approach we construct in Theorem 7.1 boundary triples for dual pairs (A_+, B_+) and (A_-, B_-). As our interest is focused on the Hamiltonian in $L^2(\mathbb{R})$ and not on the differential expressions $\mathfrak{a}_\pm$, $\mathfrak{b}_\pm$, which are defined on the semi-axes, we extend the coupling method for symmetric operators from [13] to the case of dual pairs and create a new dual pair (A, B) of operators defined on $\mathbb{R}$. This dual pair (A, B) is called the *coupling of the dual pairs* (A_+, B_+) and (A_-, B_-) (see Theorem 3.1 below).

We show that the operator $\mathcal{PT}$ intertwines the dual pairs (A_+, B_+), and (A_-, B_-), i.e.

$$\mathcal{PT}A_+ = A_-\mathcal{PT}, \qquad \mathcal{PT}B_+ = B_-\mathcal{PT}.$$

Due to our construction of the coupling these relations imply that the operator A is $\mathcal{PT}$*-symmetric*, i.e.

$$\mathcal{PT}A = A\mathcal{PT}.$$

Moreover, the operator A turns out to be $\mathcal{P}$-symmetric in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$ with the fundamental symmetry $\mathcal{P}$ in $\mathfrak{H} = L^2(\mathbb{R})$. In Trunk & Leben [21] it was shown that the extension H_0 of A defined as a restriction of the adjoint A^+ to the domain

$$\operatorname{dom} H_0 = \left\{u_+ \oplus u_- \in \operatorname{dom} A^+ \,|\, u_+(0) - u_-(0) = e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) = 0\right\}$$

is a $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint operator in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$. Here A^+ stands for the adjoint with respect to the Krein space inner product $[\cdot,\cdot]$. In Theorem 7.2 we find a one-parameter family $\{H_\alpha\}_{\alpha\in\mathbb{R}}$ of $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A in the Krein space $(\mathfrak{H}, [\cdot,\cdot])$. with domain

$$\operatorname{dom} H_\alpha = \left\{u_+ \oplus u_- \in \operatorname{dom} A^+ \,|\, u_+(0) - u_-(0) = 0,\, e^{-2i\phi}u_+'(0) - e^{2i\phi}u_-'(0) = \alpha u_+(0)\right\}.$$

The result of Theorem 7.2 is based on the abstract construction of the coupling (A, B) of two dual pairs (A_+, B_+) and (A_-, B_-) in Theorem 3.1 and the description of all $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A given in Theorem 5.5.

Summing up, the results presented here promote the use of boundary triple techniques for dual pairs and techniques from Sturm–Liouville theory for complex potentials in the study of $\mathcal{PT}$-symmetric Quantum Mechanics. This is in a line with [21]. The here presented approach via dual pairs is the tool to deal with the various symmetries $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{PT}$. It is the aim of this paper to recall those techniques and, hence, provide a mathematically sound setting of the (nowadays) classical Bender–Boettcher-theory.

Notation. By $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) we denote the set of all positive (resp. negative) reals. For $z \in \mathbb{C}$, $\overline{z}$ denotes the complex conjugate of z and for a subset $U \subset \mathbb{C}$ we set $\overline{U} := \{\overline{z} |\, z \in U\}$.

All operators in this paper are densely defined linear operators in some Hilbert spaces. For such an operator T, we use the common notation $\operatorname{dom} T$, $\operatorname{ran} T$ and $\ker T$ for the domain, the range and the null-space, respectively, of T. Moreover, as usual, $\rho(T)$, $\sigma(T)$ and $\sigma_p(T)$ stands for the resolvent set, the spectrum and the point spectrum, respectively, of T. The inner product in a Hilbert space is usually denoted by $\langle\cdot,\cdot\rangle$ and the adjoint of the operator T by T^*. The set of all bounded and everywhere defined operators in a Hilbert space $\mathfrak{H}$ is denoted by $\mathcal{L}(\mathfrak{H})$.

2. Preliminaries: Dual pairs of linear operators and Weyl functions

In this section we remind known facts about dual pairs of linear operators, their boundary triples and the corresponding Weyl functions from [22].

Definition 2.1. *A pair (A, B) of densely defined closed linear operators A and B in a Hilbert space $\mathfrak{H}$ with inner product $\langle\cdot,\cdot\rangle$ is called a* dual pair*, if*

$$\langle Af, g\rangle - \langle f, Bg\rangle = 0 \quad \textit{for all} \quad f \in \operatorname{dom} A, \quad g \in \operatorname{dom} B. \tag{2.1}$$

The equality (2.1) means that

$$A \subseteq B^*, \quad B \subseteq A^*. \tag{2.2}$$

Clearly, if (A, B) is a dual pair, then (B, A) is also a dual pair.

Definition 2.2. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be auxiliary Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$, respectively. Let*

$$\Gamma^B = \begin{pmatrix} \Gamma_1^B \\ \Gamma_2^B \end{pmatrix} : \mathrm{dom}\, B^* \to \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}, \quad \Gamma^A = \begin{pmatrix} \Gamma_1^A \\ \Gamma_2^A \end{pmatrix} : \mathrm{dom}\, A^* \to \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix} \tag{2.3}$$

be linear operators. The triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ is called a boundary triple for the dual pair (A, B)*, if:*

1. *the mappings Γ^B and Γ^A in (2.3) are surjective;*
2. *the following identity holds for every $f \in \mathrm{dom}\, B^*$, $g \in \mathrm{dom}\, A^*$*

$$\langle B^* f, g \rangle - \langle f, A^* g \rangle = \langle \Gamma_1^B f, \Gamma_1^A g \rangle_{\mathcal{H}_1} - \langle \Gamma_2^B f, \Gamma_2^A g \rangle_{\mathcal{H}_2}, \tag{2.4}$$

It is easily seen that if a triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ is a boundary triple for a dual pair (A, B), then the following identity holds

$$\langle A^* g, f \rangle - \langle g, B^* f \rangle = \langle \Gamma_2^A g, \Gamma_2^B f \rangle_{\mathcal{H}_2} - \langle \Gamma_1^A g, \Gamma_1^B f \rangle_{\mathcal{H}_1}, \quad f \in \mathrm{dom}\, B^*,\ g \in \mathrm{dom}\, A^*$$

and hence the triple

$$(\mathcal{H}_2 \times \mathcal{H}_1, (\Gamma^B)^T, (\Gamma^A)^T) := \left(\mathcal{H}_2 \times \mathcal{H}_1, \begin{pmatrix} \Gamma_2^B \\ \Gamma_1^B \end{pmatrix} \begin{pmatrix} \Gamma_2^A \\ \Gamma_1^A \end{pmatrix} \right) \tag{2.5}$$

is a boundary triple for the dual pair (B, A). The boundary triple (2.5) is called *transposed* with respect to the boundary triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$. Moreover, it follows from (2.4), $A^{**} = A$ and $B^{**} = B$ that

$$\mathrm{dom}\, A = \{ f \in \mathrm{dom}\, B^* |\, \Gamma^B f = 0 \} \quad \text{and} \quad \mathrm{dom}\, B = \{ f \in \mathrm{dom}\, A^* |\, \Gamma^A f = 0 \}.$$

A linear operator $\widetilde{A}$ is called a proper extension of A and is written $\widetilde{A} \in \mathrm{Ext}(A, B)$, if

$$A \subset \widetilde{A} \subset B^* \quad \text{and} \quad A \neq \widetilde{A}. \tag{2.6}$$

Define the proper extensions $A_1, A_2 \in \mathrm{Ext}(A, B)$ of A as restrictions of B^* to the sets

$$\mathrm{dom}\, A_1 = \{ f \in \mathrm{dom}\, B^* |\, \Gamma_1^B f = 0 \}, \quad \text{and} \quad \mathrm{dom}\, A_2 = \{ f \in \mathrm{dom}\, B^* |\, \Gamma_2^B f = 0 \}. \tag{2.7}$$

Similarly, the proper extensions $B_1, B_2 \in \mathrm{Ext}(B, A)$ of B are defined as restrictions of A^*,

$$\mathrm{dom}\, B_1 = \{ g \in \mathrm{dom}\, A^* |\, \Gamma_1^A g = 0 \} \quad \text{and} \quad \mathrm{dom}\, B_2 = \{ g \in \mathrm{dom}\, A^* |\, \Gamma_2^A g = 0 \}. \tag{2.8}$$

In the following lemma we collect some statements from [22] and provide short proofs in the present notations for the convenience of the reader.

Lemma 2.3. *Let us consider $\mathrm{dom}\, A^*$ and $\mathrm{dom}\, B^*$ as Hilbert spaces with the graph norms*

$$(\|g\|^2 + \|A^* g\|^2)^{1/2}, \quad (\|f\|^2 + \|B^* f\|^2)^{1/2}, \quad g \in \mathrm{dom}\, A^*, \quad f \in \mathrm{dom}\, B^*$$

and let $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$ be a boundary triple for the dual pair (A, B). Then

(i) *The operators $\Gamma^A : \mathrm{dom}\, A^* \to \mathcal{H}_1 \times \mathcal{H}_2$ and $\Gamma^B : \mathrm{dom}\, B^* \to \mathcal{H}_1 \times \mathcal{H}_2$ are bounded.*

(ii) *$B_1 = A_2^*$. Hence $\rho(B_1) = \overline{\rho(A_2)}$ and (A_2, B_1) is a dual pair.*

(iii) *$B_2 = A_1^*$. Hence $\rho(B_2) = \overline{\rho(A_1)}$ and (A_1, B_2) is a dual pair.*

(iv) *For every $z \in \rho(A_j)$, $j = 1, 2$, the following direct decomposition holds*

$$\operatorname{dom} B^* = \operatorname{dom} A_j \dotplus \mathfrak{N}_z(B^*), \quad \textit{where} \quad \mathfrak{N}_z(B^*) = \ker (B^* - zI), \tag{2.9}$$

and the mapping $\Gamma_j^B|_{\mathfrak{N}_z(B^*)} : \mathfrak{N}_z(B^*) \to \mathcal{H}_j$, $j = 1, 2$, is boundedly invertible.

(v) *For every $z \in \rho(B_j)$, $j = 1, 2$, the following direct decomposition holds*

$$\operatorname{dom} A^* = \operatorname{dom} B_j \dotplus \mathfrak{N}_z(A^*), \quad \textit{where} \quad \mathfrak{N}_z(A^*) = \ker (A^* - zI), \tag{2.10}$$

and the mapping $\Gamma_j^A|_{\mathfrak{N}_z(A^*)} : \mathfrak{N}_z(A^*) \to \mathcal{H}_j$, $j = 1, 2$, is boundedly invertible.

Proof. (i) Notice first that the operator $\Gamma^A : \operatorname{dom} A^* \to \mathcal{H}_1 \times \mathcal{H}^2$ is closable. Indeed, assume that

$$g_n \in \operatorname{dom} A^*, \quad g_n \to 0, \quad A^* g_n \to 0, \quad \Gamma^A g_n \to \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Since $\operatorname{ran} \Gamma^B = \mathcal{H}_1 \times \mathcal{H}_2$ there exists $f \in \operatorname{dom} B^*$ such that $\Gamma^B f = \begin{pmatrix} h_1 \\ -h_2 \end{pmatrix}$. Considering (2.4) with $g := g_n$ and taking the limit as $n \to \infty$ we obtain

$$0 = \|h_1\|^2 + \|h_2\|^2 \Rightarrow h_1 = h_2 = 0.$$

Hence the operator $\Gamma^A : \operatorname{dom} A^* \to \mathcal{H}_1 \times \mathcal{H}^2$ is closable and by the Closed Graph Theorem it is bounded.

(ii) The inclusion $B_1 \subseteq A_2^*$ follows from (2.4). Let $g \in \operatorname{dom} A_2^*$. Then for every $f \in \operatorname{dom} A_2$ one obtains from (2.4)

$$0 = \langle A_2^* g, f \rangle - \langle g, A_2 f \rangle = \langle \Gamma_1^A g, \Gamma_1^B f \rangle$$

Since $\Gamma_1^B(\operatorname{dom} A_2) = \mathcal{H}_1$ this implies $\Gamma_1^A g = 0$ and hence $g \in \operatorname{dom} B_1$. This proves (ii).

The proof of (iii) is similar.

(iv) Let $z \in \rho(A_j)$, $j = 1, 2$. Then for every $f \in \operatorname{dom} B^*$ there exists $f_j \in \operatorname{dom} A_j$ such that

$$(B^* - zI) f = (A_j - zI) f_j \Rightarrow f_z := f - f_j \in \mathfrak{N}_z(B^*)$$

and f admits the decomposition $f = f_j + f_z$. Clearly, the decomposition (2.9) is direct, since $\operatorname{dom} A_j \cap \mathfrak{N}_z(B^*) = \{0\}$ for $z \in \rho(A_j)$. It follows from (2.9) and Definition 2.2 that for $z \in \rho(A_j)$ the mapping $\Gamma_j^B|_{\mathfrak{N}_z(B^*)} : \mathfrak{N}_z(B^*) \to \mathcal{H}_j$ is surjective and hence boundedly invertible.

The proof of (v) is similar. □

Definition 2.4. *The operator functions*

$$\gamma(z) = (\Gamma_2^B|_{\mathfrak{N}_z(B^*)})^{-1} \textit{ and } M(z) := \Gamma_1^B (\Gamma_2^B|_{\mathfrak{N}_z(B^*)})^{-1}, \quad z \in \rho(A_2) \tag{2.11}$$

*are called the γ-*field *and* the Weyl function, *respectively, of the dual pair (A, B), corresponding to the boundary triple $(\mathcal{H}_1 \times \mathcal{H}_2, \Gamma^A, \Gamma^B)$.*

Clearly, the operator functions

$$\gamma^T(z) = (\Gamma_1^A|_{\mathfrak{N}_z(A^*)})^{-1} \text{ and } M^T(z) := \Gamma_2^A (\Gamma_1^A|_{\mathfrak{N}_z(A^*)})^{-1}, \quad z \in \rho(B_1) \tag{2.12}$$

are the $\gamma-$field and the Weyl function, respectively, of the dual pair (B, A), corresponding to the transposed boundary triple $(\mathcal{H}_2 \times \mathcal{H}_1, (\Gamma^B)^T, (\Gamma^A)^T)$.

Lemma 2.5. *Let* $(\mathcal{H}_1\times\mathcal{H}_2,\Gamma^A,\Gamma^B)$ *be a boundary triple for the dual pair* (A,B) *and let* M, M^T, γ *and* γ^T *be defined by* (2.11) *and* (2.12). *Then the following statements hold*

(i) *The Weyl functions* M *and* M^T *satisfy the relations*

$$M^T(\bar z)=M(z)^*,\quad z\in\rho(A_2)=\overline{\rho(B_1)}. \tag{2.13}$$

$$M(z)-M^T(\zeta)^*=(z-\overline{\zeta})\gamma^T(\zeta)^*\gamma(z),\quad z\in\rho(A_2),\,\zeta\in\rho(B_1). \tag{2.14}$$

(ii) *The following identities hold*

$$\Gamma_1^B(A_2-zI)^{-1}=\gamma^T(\bar z)^*,\quad z\in\rho(A_2); \tag{2.15}$$

$$\Gamma_2^A(B_1-zI)^{-1}=\gamma(\bar z)^*,\quad z\in\rho(B_1). \tag{2.16}$$

Proof. (i) It follows from (2.4) with $f=\gamma(z)u\in\mathfrak{N}_z(B^*)$, $z\in\rho(A_2)$, $u\in\mathcal{H}_2$ and $g=\gamma^T(\zeta)v\in\mathfrak{N}_\zeta(A^*)$, $\zeta\in\rho(B_1)$, $v\in\mathcal{H}_1$ that

$$(z-\bar\zeta)\,\langle\gamma(z)u,\gamma^T(\zeta)v\rangle=\langle\,M(z)u,v\rangle_{\mathcal{H}_1}-\langle\,u,M^T(\zeta)v\rangle_{\mathcal{H}_2}=\langle(M(z)-M^T(\zeta)^*)u,v\rangle_{\mathcal{H}_1}.$$

This proves (2.14) and (2.13) when setting $\zeta=\bar z$.

(ii) Let us set in (2.4)

$$f=(A_2-zI)^{-1}h,\quad g=\gamma^T(\bar z)v\in\mathfrak{N}_{\bar z}(A^*),\quad h\in\mathfrak{H},\quad v\in\mathcal{H}_1,\quad z\in\rho(A_2).$$

Since $\Gamma_2^Bf=0$ one obtains from (2.4)

$$\langle\,A_2(A_2-zI)^{-1}h,\gamma^T(\bar z)v\rangle-\langle(A_2-zI)^{-1}h,\bar z\gamma^T(\bar z)v\rangle=\langle\,\Gamma_1^B(A_2-zI)^{-1}h,v\rangle_{\mathcal{H}_1}$$

and hence

$$\langle\,\Gamma_1^B(A_2-zI)^{-1}h,v\rangle_{\mathcal{H}_1}=\langle\,\gamma^T(\bar z)^*h,v\rangle_{\mathcal{H}_1}.$$

This proves (2.15). The proof of (2.16) is similar. □

In order to describe all proper extension $\widetilde{A}\in\mathrm{Ext}(A,B)$ of a dual pair (A,B), we are using the notion of a linear relation. Recall [1] that a linear relation Θ from $\mathcal{H}_1$ to $\mathcal{H}_2$ is understood as a linear subspace $\Theta\subset\mathcal{H}_1\times\mathcal{H}_2$. For a linear relation Θ the symbols $\mathrm{dom}\,\Theta$, $\ker\Theta$, $\mathrm{ran}\,\Theta$ and $\rho(\Theta)$ stand for the domain, kernel, range and the resolvent set, respectively, [1]. The adjoint Θ^* is the closed linear relation from $\mathfrak{H}_2$ to $\mathfrak{H}_1$ defined by

$$\Theta^*=\left\{\begin{pmatrix}v_2\\v_1\end{pmatrix}\in\mathfrak{H}_2\times\mathfrak{H}_1\,|\,(v_1,u_1)_{\mathfrak{H}_1}=(v_2,u_2)_{\mathfrak{H}_2},\ \begin{pmatrix}u_1\\u_2\end{pmatrix}\in\Theta\right\}. \tag{2.17}$$

If $\mathcal{H}_1=\mathcal{H}_2$ then the linear relation Θ is called selfadjoint if $\Theta=\Theta^*$.

In what follows we suppose that $\mathcal{H}_1=\mathcal{H}_2=\mathcal{H}:=\mathbb{C}^d$, $d\in\mathbb{N}$. Every linear relation in $\mathcal{H}$ of rank d can be defined by the equality (see [13])

$$\Theta=\ker\begin{pmatrix}C & D\end{pmatrix}=\left\{\begin{pmatrix}u_1\\u_2\end{pmatrix}|\,Cu_1+Du_2=0\right\}. \tag{2.18}$$

where $C,D\in\mathbb{C}^{d\times d}$ are two $d\times d$-matrices, such that

$$\det(CC^*+DD^*)\neq 0. \tag{2.19}$$

The adjoint linear relation Θ^* takes the form

$$\Theta^*=\mathrm{ran}\begin{pmatrix}D^*\\-C^*\end{pmatrix}=\left\{\begin{pmatrix}D^*v\\-C^*v\end{pmatrix}|\,v\in\mathbb{C}^d\right\} \tag{2.20}$$

and therefore the linear relation Θ is self-adjoint if and only if

$$CD^* - DC^* = 0. \tag{2.21}$$

For a linear relation Θ in $\mathcal{H}$ denote by A_Θ the proper extension of A defined on

$$\operatorname{dom} A_\Theta = \{f \in \operatorname{dom} B^* \,\big|\, \Gamma^B f \in \Theta\} \tag{2.22}$$

as the restriction of B^*: $A_\Theta = B^* \restriction_{\operatorname{dom} A_\Theta} \in \operatorname{Ext}(A, B)$. The following statement describes spectral properties of A_Θ, cf. [22, Proposition 5.2 and Theorem 5.5].

Lemma 2.6. *Let (A, B) be a dual pair in a Hilbert space $\mathfrak{H}$, let $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ be a boundary triple for the dual pair (A, B), $\mathcal{H} = \mathbb{C}^d$, let M be the corresponding Weyl function, let Θ be a linear relation in $\mathcal{H}$ defined by (2.18), (2.19) and $z \in \rho(A_2)$. Then the following statements hold.*

(i) *$A_\Theta^* \in \operatorname{Ext}(B, A)$ is the restriction of A^* to*

$$\operatorname{dom} A_\Theta^* = \{g \in \operatorname{dom} A^* \,\big|\, (\Gamma^A)^T g \in \Theta^*\}. \tag{2.23}$$

(ii) *$z \in \sigma_p(A_\Theta) \Longleftrightarrow 0 \in \sigma_p(CM(z) + D)$ and for such z*

$$\ker (A_\Theta - zI) = \gamma(z) \ker (CM(z) + D).$$

(iii) *$z \in \rho(A_\Theta) \Longleftrightarrow 0 \in \rho(CM(z) + D)$ and for such z the resolvent $(A_\Theta - zI)^{-1}$ takes the form*

$$(A_\Theta - zI)^{-1} = (A_2 - zI)^{-1} - \gamma(z)(CM(z) + D)^{-1} C \gamma^T(\bar{z})^*, \quad z \in \rho(A_\Theta) \cap \rho(A_2). \tag{2.24}$$

The formula (2.24) is obtained by comparing [22, (5.14),(5.20)] with (2.18).

3. Coupling of dual pairs

Here we recall and extend some results for the coupling of two dual pairs and operators. Items (i)–(iii) in the following theorem are from [15]. For the sake of completeness, we also give a proof here.

Theorem 3.1. *Let (A_+, B_+) and (A_-, B_-) be dual pairs in Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$, respectively, let $(\mathcal{H}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ be a boundary triple for the dual pair $(A_\pm, B_\pm)$, $\mathcal{H} = \mathbb{C}^d$ and let $M_\pm$ be the corresponding Weyl function. Denote by A^* and B^* the restrictions of the operators $A_+^* \oplus A_-^*$ and $B_+^* \oplus B_-^*$, respectively, to the domains*

$$\operatorname{dom} A^* = \{g_+ \oplus g_- | g_\pm \in \operatorname{dom} A_\pm^*,\ \Gamma_1^{A_+} g_+ = \Gamma_1^{A_-} g_-\}; \tag{3.1}$$

$$\operatorname{dom} B^* = \{f_+ \oplus f_- | f_\pm \in \operatorname{dom} B_\pm^*,\ \Gamma_2^{B_+} f_+ = \Gamma_2^{B_-} f_-\}. \tag{3.2}$$

Then the following statements hold.

(i) *The operators $A := (A^*)^*$ and $B := (B^*)^*$ are restrictions of the operators B^* and A^*, respectively, to the domains*

$$\operatorname{dom} A = \{f_+ \oplus f_- | f_\pm \in \operatorname{dom} B_\pm^*,\ \Gamma_2^{B_+} f_+ = \Gamma_2^{B_-} f_- = \Gamma_1^{B_+} f_+ + \Gamma_1^{B_-} f_- = 0\}; \tag{3.3}$$

$$\operatorname{dom} B = \{g_+ \oplus g_- | g_\pm \in \operatorname{dom} A_\pm^*,\ \Gamma_1^{A_+} g_+ = \Gamma_1^{A_-} g_- = \Gamma_2^{A_+} g_+ + \Gamma_2^{A_-} g_- = 0\}, \tag{3.4}$$

and (A, B) is a dual pair in $\mathfrak{H}_+ \oplus \mathfrak{H}_-$.

(ii) *The triple* $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ *with* Γ^A *and* Γ^B *defined on* $g \in \operatorname{dom} A^*$ *and* $f \in \operatorname{dom} B^*$ *by the equalities*

$$\Gamma^A g = \begin{pmatrix} \Gamma_1^A g \\ \Gamma_2^A g \end{pmatrix} := \begin{pmatrix} \Gamma_1^{A+} g_+ \\ \Gamma_2^{A+} g_+ + \Gamma_2^{A-} g_- \end{pmatrix}, \quad \Gamma^B f = \begin{pmatrix} \Gamma_1^B f \\ \Gamma_2^B f \end{pmatrix} := \begin{pmatrix} \Gamma_1^{B+} f_+ + \Gamma_1^{B-} f_- \\ \Gamma_2^{B+} f_+ \end{pmatrix}, \tag{3.5}$$

is a boundary triple for (A, B).

(iii) *The operators* A_2, $A_{+,2}$, $A_{-,2}$ *and* B_1, $B_{+,1}$ *and* $B_{-,1}$ *defined by* (2.7) *and* (2.8), *respectively, are related by*

$$A_2 = A_{+,2} \oplus A_{-,2}, \quad \text{and} \quad B_1 = B_{+,1} \oplus B_{-,1}. \tag{3.6}$$

The Weyl function M *and the* γ*-field corresponding to the boundary triple* $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ *are given by*

$$M(z) = M_+(z) + M_-(z), \quad \gamma(z) = \begin{pmatrix} \gamma_+(z) \\ \gamma_-(z) \end{pmatrix}, \quad z \in \rho(A_2). \tag{3.7}$$

(iv) *The restrictions* A_1 *of* B^* *and* B_2 *of* A^* *defined in* (2.7) *and* (2.8) *have the domains*

$$\operatorname{dom} A_1 = \{f_+ \oplus f_- | f_\pm \in \operatorname{dom} B_\pm^*, \, \Gamma_2^{B+} f_+ - \Gamma_2^{B-} f_- = \Gamma_1^{B+} f_+ + \Gamma_1^{B-} f_- = 0\} \tag{3.8}$$

$$\operatorname{dom} B_2 = \{f_+ \oplus f_- | f_\pm \in \operatorname{dom} A_\pm^*, \, \Gamma_1^{A+} f_+ - \Gamma_1^{A-} f_- = \Gamma_2^{A+} f_+ + \Gamma_2^{A-} f_- = 0\}. \tag{3.9}$$

They are proper extensions of A *and* B, *respectively, i.e.* $A_1 \in \operatorname{Ext}(A, B)$, $B_2 \in \operatorname{Ext}(B, A)$, *and fulfill* $A_1 = B_2^*$.

(v) *If* $z \in \rho(A_2)$, *then*

$$z \in \rho(A_1) \Leftrightarrow 0 \in \rho(M_+(z) + M_-(z)) \tag{3.10}$$

and

$$(A_1 - zI)^{-1} = (A_2 - zI)^{-1} - \gamma(z)(M_+(z) + M_-(z))^{-1} \gamma^T(\bar{z})^*, \quad z \in \rho(A_1) \cap \rho(A_2). \tag{3.11}$$

Proof. (i)&(ii) Let $f = f_+ \oplus f_- \in \operatorname{dom}(B_+^* \oplus B_-^*)$, $g = g_+ \oplus g_- \in \operatorname{dom}(A_+^* \oplus A_-^*)$. Then it follows from the equalities

$$\begin{aligned} \langle B_+^* f_+, g_+ \rangle - \langle f_+, A_+^* g_+ \rangle &= \langle \Gamma_1^{B+} f_+, \Gamma_1^{A+} g_+ \rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+ \rangle_{\mathcal{H}_2}, \\ \langle B_-^* f_-, g_- \rangle - \langle f_-, A_-^* g_- \rangle &= \langle \Gamma_1^{B-} f_-, \Gamma_1^{A-} g_- \rangle_{\mathcal{H}_1} - \langle \Gamma_2^{B-} f_-, \Gamma_2^{A-} g_- \rangle_{\mathcal{H}_2}. \end{aligned}$$

that

$$\begin{aligned} \langle (B_+^* \oplus B_-^*) f, g \rangle - \langle f, (A_+^* \oplus A_-^*) g \rangle &= \langle \Gamma_1^{B+} f_+, \Gamma_1^{A+} g_+ \rangle_{\mathcal{H}} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+ \rangle_{\mathcal{H}} \\ &\quad + \langle \Gamma_1^{B-} f_-, \Gamma_1^{A-} g_- \rangle_{\mathcal{H}} - \langle \Gamma_2^{B-} f_-, \Gamma_2^{A-} g_- \rangle_{\mathcal{H}}. \end{aligned} \tag{3.12}$$

The equality (3.3) follows from (3.12) since the mappings $\Gamma^{A\pm} : \operatorname{dom} A_\pm^* \to \mathcal{H}^2$ are surjective. Similarly, (3.4) follows from (3.12) since the mappings $\Gamma^{B\pm} : \operatorname{dom} B_\pm^* \to \mathcal{H}^2$ are surjective. Next, for $f \in \operatorname{dom} B^*$, $g \in \operatorname{dom} A^*$ the equation (3.12) takes the form

$$\langle B^* f, g \rangle - \langle f, A^* g \rangle = \langle \Gamma_1^{B+} f_+ + \Gamma_1^{B-} f_-, \Gamma_1^{A+} g_+ \rangle_{\mathcal{H}} - \langle \Gamma_2^{B+} f_+, \Gamma_2^{A+} g_+ + \Gamma_2^{A-} g_- \rangle_{\mathcal{H}}. \tag{3.13}$$

This proves that (A, B) is a dual pair in $\mathfrak{H}_+ \oplus \mathfrak{H}_-$ and that (ii) holds.

(iii) From (3.5) we obtain $\operatorname{dom} A_2 = \ker \Gamma_2^B = \ker \Gamma_2^{B+}$ and, as $A_2 \subset B^*$, by (3.2) formula (3.6) follows. Moreover, (3.2) and (3.5) imply that the $\gamma-$field of (A, B) corresponding to the boundary triple $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ takes the form

$$\gamma(z) = \begin{pmatrix} \gamma_+(z) \\ \gamma_-(z) \end{pmatrix} \tag{3.14}$$

where $\gamma_\pm(z)$ are the $\gamma-$fields of $(A_\pm, B_\pm)$ corresponding to the boundary triples $(\mathcal{H}^2, \Gamma^{A\pm}, \Gamma^{B\pm})$. Now formula (3.7) follows from the definition of the Weyl function, see Definition 2.4.

(iv)&(v) The extension A_1 of A given by (3.8) coincides with the extension A_Θ of A defined by (2.22) with $\Theta = \ker \begin{pmatrix} C & D \end{pmatrix} = \ker \begin{pmatrix} I_{\mathcal{H}} & 0 \end{pmatrix}$. Then the equivalence (3.10) follows from Lemma 2.6 (iii) and formula (3.11) follows from (2.24). □

Definition 3.2. (1) *The dual pair (A, B) given by (3.3), (3.4) is called the* coupling *of the dual pairs (A_+, B_+) and (A_-, B_-) relative to the triples $(\mathcal{H}^2, \Gamma^{A+}, \Gamma^{B+})$ and $(\mathcal{H}^2, \Gamma^{A-}, \Gamma^{B-})$.*

(2) *The operator A_1 from (3.8) is called the* coupling *of the operators A_+ and A_- relative to the triples $(\mathcal{H}^2, \Gamma^{A+}, \Gamma^{B+})$, $(\mathcal{H}^2, \Gamma^{A-}, \Gamma^{B-})$ and the operator $A_2 = A_{+,2} \oplus A_{-,2}$ from (3.6) is called the* decoupled operator.

According to this definition the operator B_2 from (3.9) is the coupling of the operators B_+ and B_- relative to the triples $(\mathcal{H}^2, (\Gamma^{B+})^T, (\Gamma^{A+})^T)$, $(\mathcal{H}^2, (\Gamma^{B-})^T, (\Gamma^{A-})^T)$, see also (2.5). Since the operators A_1 and B_2 are connected by the formula $A_1 = B_2^*$, in what follows we will be interested only in the coupling A_1.

4. Real dual pairs and real boundary triples

Let $\mathcal{T}$ be a conjugation (time reversal) operator in $\mathfrak{H}$, i.e. $\mathcal{T}$ is antilinear, $\mathcal{T}^2 = I_{\mathfrak{H}}$ and

$$(\mathcal{T} f, \mathcal{T} g)_{\mathfrak{H}} = (g, f)_{\mathfrak{H}} \quad \text{for all} \quad f, g \in \mathfrak{H}. \tag{4.1}$$

In what follows, besides the conjugation $\mathcal{T}$ in $\mathfrak{H}$, we will also consider a conjugation in $\mathcal{H}$, which will be denoted by $j_{\mathcal{H}}$.

Definition 4.1. *A dual pair (A, B) in $\mathfrak{H}$ is called $\mathcal{T}$-*real*, if*

$$\mathcal{T} \operatorname{dom} A = \operatorname{dom} B \quad \textit{and} \quad \mathcal{T} A = B \mathcal{T}. \tag{4.2}$$

*A boundary triple $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ for a $\mathcal{T}$-real pair (A, B) is called $(j_{\mathcal{H}}, \mathcal{T})$-*real*, if*

$$j_{\mathcal{H}} \Gamma_1^B = \Gamma_2^A \mathcal{T}, \quad j_{\mathcal{H}} \Gamma_2^B = \Gamma_1^A \mathcal{T}. \tag{4.3}$$

Observe that the conditions (4.2) are clearly equivalent to

$$\mathcal{T} \operatorname{dom} A^* = \operatorname{dom} B^* \quad \text{and} \quad \mathcal{T} A^* = B^* \mathcal{T}. \tag{4.4}$$

Lemma 4.2. *Let (A, B) be a $\mathcal{T}$-real dual pair and let $(\mathcal{H}^2, \Gamma^A, \Gamma^B)$ be a $(j_{\mathcal{H}}, \mathcal{T})$-real boundary triple for (A, B). Then the corresponding Weyl function $M(z)$ satisfies the condition*

$$M(z) = j_{\mathcal{H}} M(z)^* j_{\mathcal{H}}, \quad z \in \rho(A_2). \tag{4.5}$$

Proof. Notice that for $f_{\bar z}\in\mathfrak{N}_{\bar z}(A^*)$ one has

$$\mathcal{T}f_{\bar z}\in\mathfrak{N}_z(B^*).$$

Indeed, for every $h\in\operatorname{dom}B$ one gets by (4.2) $\mathcal{T}h\in\operatorname{dom}A$ and hence

$$\langle\mathcal{T}f_{\bar z},(B-\bar z)h\rangle=\langle\mathcal{T}(B-\bar z)h,f_{\bar z}\rangle=\langle(A-z)\mathcal{T}h,f_{\bar z}\rangle=0.$$

Moreover, by a similar argument, we have also $\mathcal{T}\mathfrak{N}_z(B^*)\subseteq\mathfrak{N}_{\bar z}(A^*)$ and therefore

$$\mathcal{T}\mathfrak{N}_{\bar z}(A^*)=\mathfrak{N}_z(B^*).$$

Next by (4.3), (2.11) and by (2.12) one obtains for $z\in\rho(A_2)$

$$\begin{aligned}M(z)\Gamma_2^B\mathcal{T}f_{\bar z}&=\Gamma_1^B\mathcal{T}f_{\bar z}=j_{\mathcal{H}}\Gamma_2^Af_{\bar z}\\&=j_{\mathcal{H}}M^T(\bar z)\Gamma_1^Af_{\bar z}=j_{\mathcal{H}}M^T(\bar z)j_{\mathcal{H}}\Gamma_2^B\mathcal{T}f_{\bar z}\end{aligned}\tag{4.6}$$

Since $\Gamma_2^B\mathcal{T}\mathfrak{N}_{\bar z}(A^*)=\mathcal{H}$ for $z\in\rho(A_2)=\overline{\rho(B_1)}$, this implies

$$M(z)=j_{\mathcal{H}}M^T(\bar z)j_{\mathcal{H}}.$$

By (2.13) one obtains (4.5). □

In what follows we consider a Hilbert space $\mathfrak{H}$ decomposed into an orthogonal sum

$$\mathfrak{H}=\mathfrak{H}_+\oplus\mathfrak{H}_-\tag{4.7}$$

of two subspaces $\mathfrak{H}_\pm$ with conjugations $\mathcal{T}_\pm\in\mathcal{L}(\mathfrak{H}_\pm)$. Then the orthogonal sum

$$\mathcal{T}=\mathcal{T}_+\oplus\mathcal{T}_-\tag{4.8}$$

is a conjugation in $\mathfrak{H}$.

Theorem 4.3. *Let $\mathfrak{H}$ be a Hilbert space with a conjugation $\mathcal{T}$ such that (4.7) and (4.8) hold. Moreover, let $(A_\pm,B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in Hilbert spaces $\mathfrak{H}_\pm$. Finally, with $j_{\mathcal{H}}$ a conjugation in $\mathcal{H}$, let $(\mathcal{H}^2,\Gamma^{A_\pm},\Gamma^{B_\pm})$ be $(j_{\mathcal{H}},\mathcal{T}_\pm)$-real boundary triples for $(A_\pm,B_\pm)$, and let*

$$A_0:=A_+\oplus A_-,\qquad B_0:=B_+\oplus B_-.\tag{4.9}$$

Then the following statements hold.

(i) *The dual pair (A_0,B_0) is $\mathcal{T}$-real and the boundary triple $((\mathcal{H}\oplus\mathcal{H})^2,\Gamma^{A_0},\Gamma^{B_0})$ with*

$$\Gamma^{A_0}:=\begin{pmatrix}\Gamma_1^{A_0}\\\Gamma_2^{A_0}\end{pmatrix}:=\begin{pmatrix}\Gamma_1^{A_+}\oplus\Gamma_1^{A_-}\\\Gamma_2^{A_+}\oplus\Gamma_2^{A_-}\end{pmatrix},\quad\Gamma^{B_0}:=\begin{pmatrix}\Gamma_1^{B_0}\\\Gamma_2^{B_0}\end{pmatrix}:=\begin{pmatrix}\Gamma_1^{B_+}\oplus\Gamma_1^{B_-}\\\Gamma_2^{B_+}\oplus\Gamma_2^{B_-}\end{pmatrix}\tag{4.10}$$

is $(j_{\mathcal{H}\oplus\mathcal{H}},\mathcal{T})$-real, where $j_{\mathcal{H}\oplus\mathcal{H}}:=j_{\mathcal{H}}\oplus j_{\mathcal{H}}$.

(ii) *The coupling (A,B) of the dual pairs (A_+,B_+) and (A_-,B_-) constructed in (3.3), (3.4) is $\mathcal{T}$-real.*

(iii) *The boundary triple $(\mathcal{H}^2,\Gamma^A,\Gamma^B)$ from Theorem 3.1 is $(j_{\mathcal{H}},\mathcal{T})$-real.*

(iv) *The dual pairs (A_1,B_2) and (A_2,B_1) are $\mathcal{T}$-real.*

Proof. Since the dual pairs $(A_\pm,B_\pm)$ are $\mathcal{T}_\pm$-real one has

$$\mathcal{T}_+A_+=B_+\mathcal{T}_+,\quad\mathcal{T}_-A_-=B_-\mathcal{T}_-.\tag{4.11}$$

Since the boundary triples $(\mathcal{H}\oplus\mathcal{H},\Gamma^{B_\pm},\Gamma^{A_\pm})$ are $(j_{\mathcal{H}},\mathcal{T})$-real one has

$$j_{\mathcal{H}}\Gamma_1^{B+}f_+=\Gamma_2^{A+}\mathcal{T}_+f_+,\qquad j_{\mathcal{H}}\Gamma_1^{B-}f_-=\Gamma_2^{A-}\mathcal{T}_-f_-,\qquad f_\pm\in\operatorname{dom}B_\pm^*. \tag{4.12}$$
$$j_{\mathcal{H}}\Gamma_2^{B+}f_+=\Gamma_1^{A+}\mathcal{T}_+f_+,\qquad j_{\mathcal{H}}\Gamma_2^{B-}f_-=\Gamma_1^{A-}\mathcal{T}_-f_-,$$

(i) From (4.12) we see $j_{\mathcal{H}\oplus\mathcal{H}}\Gamma_1^{B_0}=\Gamma_2^{A_0}\mathcal{T}$ and $j_{\mathcal{H}\oplus\mathcal{H}}\Gamma_2^{B_0}=\Gamma_1^{A_0}\mathcal{T}$. This shows the $\mathcal{T}$-realness of the dual pair (A_0,B_0) and the $(j_{\mathcal{H}\oplus\mathcal{H}},\mathcal{T})$-realness of the boundary triple $((\mathcal{H}\oplus\mathcal{H})^2,\Gamma^{A_0},\Gamma^{B_0})$ follows from (4.11) and (4.12).

(ii) It follows from (4.11) and (4.4) that

$$\mathcal{T}_\pm\operatorname{dom}B_\pm^*=\operatorname{dom}A_\pm^*\quad\text{and}\quad A_+^*\mathcal{T}_+=\mathcal{T}_+B_+^*,\quad A_-^*\mathcal{T}_-=\mathcal{T}_-B_-^* \tag{4.13}$$

Hence

$$(A_+^*\oplus A_-^*)\mathcal{T}=\mathcal{T}(B_+^*\oplus B_-^*).$$

Let $f=f_+\oplus f_-\in\operatorname{dom}B^*$, $f_\pm\in\operatorname{dom}B_\pm^*$. Then, by (3.2),

$$\Gamma_2^{B+}f_+=\Gamma_2^{B-}f_-. \tag{4.14}$$

In view of (4.12) the condition (4.14) takes the form

$$j_{\mathcal{H}}\Gamma_1^{A+}\mathcal{T}_+f_+=\Gamma_2^{B+}f_+=\Gamma_2^{B-}f_-=j_{\mathcal{H}}\Gamma_1^{A-}\mathcal{T}_-f_-$$

and, by (3.1), $\mathcal{T}f=\mathcal{T}_+f_+\oplus\mathcal{T}_-f_-\in\operatorname{dom}A^*$. Therefore,

$$A^*\mathcal{T}f=(A_+^*\mathcal{T}_+f_+)\oplus(A_-^*\mathcal{T}_-f_-)=\mathcal{T}(B_+^*f_+\oplus B_-^*f_-)=\mathcal{T}B^*f$$

and thus (A,B) is a $\mathcal{T}$-real dual pair in $\mathfrak{H}$.

(iii) Because of (ii) for $f\in\operatorname{dom}B^*$ we have $\mathcal{T}f\in\operatorname{dom}A^*$. It follows from (3.5) and (4.12) that

$$\Gamma_1^A\mathcal{T}f=\Gamma_1^{A+}\mathcal{T}_+f_+=j_{\mathcal{H}}\Gamma_2^{B+}f_+=j_{\mathcal{H}}\Gamma_2^Bf,$$

and

$$\Gamma_2^A\mathcal{T}f=\Gamma_2^{A+}\mathcal{T}_+f_++\Gamma_2^{A-}\mathcal{T}_-f_-=j_{\mathcal{H}}(\Gamma_1^{B+}f_++\Gamma_1^{B-}f_-)=j_{\mathcal{H}}\Gamma_1^Bf.$$

Thus the boundary triple $(\mathcal{H}^2,\Gamma^A,\Gamma^B)$ is real.

(iv) Assume that $f\in\operatorname{dom}A_1=\{f\in\operatorname{dom}B^*|\,\Gamma_1^Bf=0\}$. Then, due to (4.3), $\Gamma_2^A\mathcal{T}f=j_{\mathcal{H}}\Gamma_1^Bf=0$ and hence $\mathcal{T}f\in\operatorname{dom}B_2$. By (4.4),

$$\mathcal{T}\operatorname{dom}A_1=\operatorname{dom}B_2\quad\text{and}\quad\mathcal{T}B_2=A_1\mathcal{T}. \tag{4.15}$$

Similarly, is shown that

$$\mathcal{T}\operatorname{dom}A_2=\operatorname{dom}B_1\quad\text{and}\quad\mathcal{T}B_1=A_2\mathcal{T}. \tag{4.16}$$

□

5. Parity and $\mathcal{PT}$-symmetric operators

Definition 5.1. *Let $\mathfrak{H}_\pm$ be Hilbert spaces and $\mathfrak{H}=\mathfrak{H}_+\oplus\mathfrak{H}_-$. A linear operator $\mathcal{P}\in\mathcal{L}(\mathfrak{H})$ will be called an* abstract parity operator*, if*

$$\mathcal{P}=\mathcal{P}^*,\quad\mathcal{P}^2=I_{\mathfrak{H}}\quad\textit{and}\quad\mathcal{P}\mathfrak{H}_\pm=\mathfrak{H}_\mp. \tag{5.1}$$

Now consider a Hilbert space $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ with a parity operator $\mathcal{P}$ and a conjugation $\mathcal{T} \in \mathcal{L}(\mathfrak{H})$, such that

$$\mathcal{T}\mathcal{P} = \mathcal{P}\mathcal{T} \quad \text{and} \quad \mathcal{T}\mathfrak{H}_\pm = \mathfrak{H}_\pm. \tag{5.2}$$

The conditions (5.2) mean that the operator $\mathcal{T}$ admits the representation as an orthogonal sum $\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$ of two conjugations $\mathcal{T}_+$ and $\mathcal{T}_-$ in the Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$, respectively. In what follows we identify the spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$ with the subspaces $\mathfrak{H}_+ \oplus \{0\}$ and $\{0\} \oplus \mathfrak{H}_-$ of $\mathfrak{H}$, and hence we will set

$$\mathcal{T} f_+ := \mathcal{T}\begin{pmatrix} f_+ \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{T}_+ f_+ \\ 0 \end{pmatrix}, \quad \mathcal{T} f_- := \mathcal{T}\begin{pmatrix} 0 \\ f_- \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{T}_- f_- \end{pmatrix} \quad \text{for all} \quad f_\pm \in \mathfrak{H}_\pm.$$

Similarly, we set

$$\mathcal{P} f_+ := \mathcal{P}\begin{pmatrix} f_+ \\ 0 \end{pmatrix}, \quad \mathcal{P} f_- := \mathcal{P}\begin{pmatrix} 0 \\ f_- \end{pmatrix} \quad \text{for all} \quad f_\pm \in \mathfrak{H}_\pm \tag{5.3}$$

The operator $\mathcal{P}$ maps the subspace $\mathfrak{H}_+$ onto $\mathfrak{H}_-$, and vice versa. It acts by the formula

$$\mathcal{P}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} \mathcal{P} f_- \\ \mathcal{P} f_+ \end{pmatrix}, \quad f_\pm \in \mathfrak{H}_\pm. \tag{5.4}$$

We will say that the parity $\mathcal{P}$ *intertwines* two given dual pairs $(A_\pm, B_\pm)$ in two Hilbert spaces $\mathfrak{H}_\pm$, if

$$\mathcal{P} A_+ = B_- \mathcal{P}, \quad \mathcal{P} B_+ = A_- \mathcal{P}. \tag{5.5}$$

Lemma 5.2. *Let $\mathcal{P}$ be a parity operator in $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ and let $\mathcal{T}$ be a conjugation in $\mathfrak{H}$, such that (5.2) holds. Let $(A_\pm, B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in Hilbert spaces $\mathfrak{H}_\pm$ intertwined by the parity $\mathcal{P}$. With $j_{\mathcal{H}}$ a conjugation in $\mathcal{H}$ let $(\mathcal{H}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ be $(j_{\mathcal{H}}, \mathcal{T})$-real boundary triples for $(A_\pm, B_\pm)$, such that*

$$\begin{pmatrix} \Gamma_1^{B_+} \\ \Gamma_2^{B_+} \end{pmatrix} f_+ = \begin{pmatrix} \Gamma_2^{A_-} \\ \Gamma_1^{A_-} \end{pmatrix} \mathcal{P} f_+, \quad \begin{pmatrix} \Gamma_1^{B_-} \\ \Gamma_2^{B_-} \end{pmatrix} f_- = \begin{pmatrix} \Gamma_2^{A_+} \\ \Gamma_1^{A_+} \end{pmatrix} \mathcal{P} f_-, \quad f_\pm \in \operatorname{dom} B_\pm^*, \tag{5.6}$$

let $M_\pm$ be the Weyl functions of the operators $A_\pm$ corresponding to the boundary triples $(\mathcal{H}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ and let $A_{\pm,j}$ be the restrictions of $B_\pm^*$ to the sets (cf. (2.7))

$$\operatorname{dom} A_{\pm,j} = \ker \Gamma_j^{B_\pm}, \quad j = 1, 2. \tag{5.7}$$

Then the following statements hold.

(i) $\mathcal{P}\operatorname{dom} A_+^* = \operatorname{dom} B_-^*$, $\mathcal{P}\operatorname{dom} B_+^* = \operatorname{dom} A_-^*$ *and*

$$\mathcal{P} A_+^* = B_-^* \mathcal{P}, \quad \mathcal{P} B_+^* = A_-^* \mathcal{P}. \tag{5.8}$$

(ii) $\mathcal{P}\mathcal{T}\operatorname{dom} A_+ = \operatorname{dom} A_-$, $\mathcal{P}\mathcal{T}\operatorname{dom} B_+ = \operatorname{dom} B_-$ *and*

$$\mathcal{P}\mathcal{T} A_+ = A_- \mathcal{P}\mathcal{T}, \quad \mathcal{P}\mathcal{T} B_+ = B_- \mathcal{P}\mathcal{T}. \tag{5.9}$$

(iii) $\mathcal{P}\mathcal{T}\operatorname{dom} A_+^* = \operatorname{dom} A_-^*$, $\mathcal{P}\mathcal{T}\operatorname{dom} B_+^* = \operatorname{dom} B_-^*$ *and*

$$\mathcal{P}\mathcal{T} A_+^* = A_-^* \mathcal{P}\mathcal{T}, \quad \mathcal{P}\mathcal{T} B_+^* = B_-^* \mathcal{P}\mathcal{T}, \quad \mathcal{P}\mathcal{T} A_{+,j} = A_{-,j} \mathcal{P}\mathcal{T}, \quad j = 1, 2. \tag{5.10}$$

(iv) *The Weyl functions M_+ and M_- of the operators A_+ and A_- are related by*

$$M_+(z) = M_-(\bar{z})^*, \quad z \in \rho(A_{+,2}) = \overline{\rho(A_{-,2})}. \tag{5.11}$$

Proof. (i) Applying $\mathcal{P}$ to the left and right hand sides of the equalities in (5.5) and using the identity $\mathcal{P}^2 = I$ yields that $A_+\mathcal{P} = \mathcal{P}B_-$ and $B_+\mathcal{P} = \mathcal{P}A_-$. From these identities the statement (i) is immediate.

(ii) Since the dual pairs $(A_\pm, B_\pm)$ are real with respect to $\mathcal{T}_\pm$ one has (4.11). Let $f_+ \in \operatorname{dom} A_+$. Then by (4.11) $\mathcal{T}_+ f_+ \in \operatorname{dom} B_+$ and $B_+\mathcal{T}_+ f_+ = \mathcal{T}_+ A_+ f_+$. By (5.5)

$$\mathcal{P}\mathcal{T}_+ f_+ \in \operatorname{dom} A_- \quad \text{and} \quad A_-\mathcal{P}\mathcal{T}_+ f_+ = \mathcal{P}B_+\mathcal{T}_+ f_+ = \mathcal{P}\mathcal{T}_+ A_+ f_+.$$

The proofs of the inclusion $\operatorname{dom} A_- \subseteq \mathcal{P}\mathcal{T}\operatorname{dom} A_+$ and of the second equality in (5.9) are similar.

(iii) The equalities $\mathcal{P}\mathcal{T}A_+^* = A_-^*\mathcal{P}\mathcal{T}$, $\mathcal{P}\mathcal{T}B_+^* = B_-^*\mathcal{P}\mathcal{T}$ follow from (4.13) and item (i). These equalities imply (5.10) since by (5.6) and (4.12) we have for $f_+ \in \operatorname{dom} B_+^*$, $g_+ \in \operatorname{dom} A_+^*$

$$\begin{aligned} \Gamma_1^{B_-}(\mathcal{P}\mathcal{T}f_+) &= \Gamma_2^{A_+}\mathcal{T}f_+ = \jmath_{\mathcal{H}}\Gamma_1^{B_+}f_+, \qquad & \Gamma_1^{A_-}(\mathcal{P}\mathcal{T}g_+) &= \Gamma_2^{B_+}\mathcal{T}g_+ = \jmath_{\mathcal{H}}\Gamma_1^{A_+}g_+, \\ \Gamma_2^{B_-}(\mathcal{P}\mathcal{T}f_+) &= \Gamma_1^{A_+}\mathcal{T}f_+ = \jmath_{\mathcal{H}}\Gamma_2^{B_+}f_+, \qquad & \Gamma_2^{A_-}(\mathcal{P}\mathcal{T}g_+) &= \Gamma_1^{B_+}\mathcal{T}g_+ = \jmath_{\mathcal{H}}\Gamma_2^{A_+}g_+. \end{aligned} \tag{5.12}$$

(iv) In particular, we obtain due to (iii) that

$$\rho(A_{+,1}) = \overline{\rho(A_{-,1})}, \quad \rho(A_{+,2}) = \overline{\rho(A_{-,2})}. \tag{5.13}$$

For every $z \in \rho(A_{+,2})$ and $f_z^+ \in \mathfrak{N}_z(B_+^*)$ we get from (iii) $f_{\bar z}^- := \mathcal{P}\mathcal{T}f_z^+ \in \mathfrak{N}_{\bar z}(B_-^*)$, where we used $\mathcal{P}\mathcal{T}zf_z^+ = \bar z\mathcal{P}\mathcal{T}f_z^+$ which follows from (4.1). By Definition 2.4 the Weyl functions M_+ and M_- satisfy the equalities

$$M_+(z)\Gamma_2^{B_+}f_z^+ = \Gamma_1^{B_+}f_z^+, \quad M_-(\bar z)\Gamma_2^{B_-}f_{\bar z}^- = \Gamma_1^{B_-}f_{\bar z}^-.$$

By (5.12)

$$\begin{aligned} \Gamma_2^{B_-}f_{\bar z}^- &= \Gamma_2^{B_-}(\mathcal{P}\mathcal{T}f_z^+) = \Gamma_1^{A_+}\mathcal{T}f_z^+ = \jmath_{\mathcal{H}}\Gamma_2^{B_+}f_z^+, \\ \Gamma_1^{B_-}f_{\bar z}^- &= \Gamma_1^{B_-}(\mathcal{P}\mathcal{T}f_z^+) = \Gamma_2^{A_+}\mathcal{T}f_z^+ = \jmath_{\mathcal{H}}\Gamma_1^{B_+}f_z^+. \end{aligned}$$

Hence

$$M_-(\bar z)(\jmath_{\mathcal{H}}\Gamma_2^{B_+}f_z^+) = M_-(\bar z)(\Gamma_2^{B_-}f_{\bar z}^-) = \Gamma_1^{B_-}f_{\bar z}^- = \jmath_{\mathcal{H}}\Gamma_1^{B_+}f_z^+ = \jmath_{\mathcal{H}}M_+(z)\Gamma_2^{B_+}f_z^+$$

and thus $M_-(\bar z) = \jmath_{\mathcal{H}}M_+(z)\jmath_{\mathcal{H}}$. In view of (4.5) this proves (iv). □

Definition 5.3. *A closed linear operator A in $\mathfrak{H}$ is said to be $\mathcal{P}\mathcal{T}$-*symmetric *if for all $f \in \operatorname{dom} A$ we have*

$$\mathcal{P}\mathcal{T}f \in \operatorname{dom} A \quad \text{and} \quad \mathcal{P}\mathcal{T}Af = A\mathcal{P}\mathcal{T}f.$$

*A dual pair (A, B) is said to be $\mathcal{P}\mathcal{T}$-*symmetric*, if both A and B are $\mathcal{P}\mathcal{T}$-*symmetric.

Theorem 5.4. *Let $\mathcal{P}$ be a parity operator in $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, let $\mathcal{T}$ be a conjugation in $\mathfrak{H}$, such that (5.2) holds and let $(A_\pm, B_\pm)$ be $\mathcal{T}_\pm$-real dual pairs in the Hilbert spaces $\mathfrak{H}_\pm$ intertwined by the parity $\mathcal{P}$. With $\jmath_{\mathcal{H}}$ a conjugation in $\mathcal{H} = \mathbb{C}^d$ let $(\mathcal{H}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ be a $(\jmath_{\mathcal{H}}, \mathcal{T})$-real boundary triples for $(A_\pm, B_\pm)$, such that (5.6) holds. Moreover, let the dual pair (A_0, B_0) as in (4.9) and let (A, B) be the coupling of the dual pairs (A_+, B_+) and (A_-, B_-) relative to the triples $(\mathcal{H}^2, \Gamma^{A_+}, \Gamma^{B_+})$ and $(\mathcal{H}^2, \Gamma^{A_-}, \Gamma^{B_-})$, cf. (3.3), (3.4). Then the following holds.*

(i) *The dual pair (A_0, B_0) is $\mathcal{P}\mathcal{T}$-symmetric. Moreover, the operators A_0^*, B_0^* are $\mathcal{P}\mathcal{T}$-symmetric.*

(ii) *The dual pair* (A, B) *is* $\mathcal{PT}$*-symmetric. Moreover, the operators* A^*, B^* *are* $\mathcal{PT}$*-symmetric.*

(iii) *The dual pair* (A_1, B_2) *is* $\mathcal{PT}$*-symmetric.*

Proof. (i) Notice first that by Lemma 5.2 (i)

$$\mathcal{P}\mathrm{dom}\, B_0^* = \mathcal{P}\mathrm{dom}\,(B_+^* \oplus B_-^*) = \mathrm{dom}\,(A_-^* \oplus A_+^*) = \mathrm{dom}\, A_0^*. \tag{5.14}$$

We obtain by (5.4) and by (5.8) for $f = \begin{pmatrix} f_+ & f_- \end{pmatrix}^T$, $f_\pm \in \mathrm{dom}\, B_\pm^*$

$$A_0^*\mathcal{P}f = \begin{pmatrix} A_+^* & 0 \\ 0 & A_-^* \end{pmatrix} \mathcal{P} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} A_+^*\mathcal{P}f_- \\ A_-^*\mathcal{P}f_+ \end{pmatrix} = \begin{pmatrix} \mathcal{P}B_-^*f_- \\ \mathcal{P}B_+^*f_+ \end{pmatrix} = \mathcal{P}B_0^*f. \tag{5.15}$$

It follows that

$$\mathcal{P}\mathrm{dom}\, B_0^* = \mathrm{dom}\, A_0^* \quad \text{and} \quad A_0^*\mathcal{P} = \mathcal{P}B_0^*. \tag{5.16}$$

Next, by (4.13) $\mathcal{PT}\mathrm{dom}\, A_0^* = \mathrm{dom}\, A_0^*$ and

$$A_0^*(\mathcal{PT})f = A_0^*\mathcal{P}\begin{pmatrix} \mathcal{T}_+f_+ \\ \mathcal{T}_-f_- \end{pmatrix} = \mathcal{P}B_0^*\begin{pmatrix} \mathcal{T}_+f_+ \\ \mathcal{T}_-f_- \end{pmatrix} = (\mathcal{PT})A_0^*f \tag{5.17}$$

and the operator A_0^* is $\mathcal{PT}$-symmetric. By (5.9), $A_0(\mathcal{PT}) = (\mathcal{PT})A_0$, i.e. the operator A_0 is $\mathcal{PT}$-symmetric. The proofs of the equalities $B_0(\mathcal{PT}) = (\mathcal{PT})B_0$ and $B_0^*(\mathcal{PT}) = (\mathcal{PT})B_0^*$ are similar.

(ii) Due to (3.1) for $g = g_+ \oplus g_- \in \mathrm{dom}\, A^*$ we have $\Gamma_1^{A+}g_+ = \Gamma_1^{A-}g_-$ and hence, by (5.12), the following equalities holds

$$\Gamma_1^{A+}(\mathcal{PT})g_- = \jmath_{\mathcal{H}}\Gamma_1^{A-}g_- = \jmath_{\mathcal{H}}\Gamma_1^{A+}g_+ = \Gamma_1^{A-}(\mathcal{PT})g_+. \tag{5.18}$$

Therefore, $(\mathcal{PT})g = (\mathcal{PT})g_- \oplus (\mathcal{PT})g_+ \in \mathrm{dom}\, A^*$ and, by (5.17),

$$A^*(\mathcal{PT})g = A_0^*(\mathcal{PT})g = (\mathcal{PT})A_0^*g = (\mathcal{PT})A^*g. \tag{5.19}$$

Since the operator $\mathcal{PT} : \mathcal{H} \to \mathcal{H}$ is a bijection, we have $\mathcal{PT}\mathrm{dom}\, A^* = \mathrm{dom}\, A^*$ and

$$A^*(\mathcal{PT}) = (\mathcal{PT})A^*. \tag{5.20}$$

By straightforward calculations we derive from (5.20) that $A(\mathcal{PT}) = (\mathcal{PT})A$, i.e. the operator A is $\mathcal{PT}$-symmetric. Similarly, we can show that

$$\mathcal{PT}\mathrm{dom}\, B^* = \mathrm{dom}\, B^* \quad \text{and} \quad B^*(\mathcal{PT})f = (\mathcal{PT})B^*f \quad \text{for all} \quad f \in \mathrm{dom}\, B^*. \tag{5.21}$$

This implies the equality $B(\mathcal{PT}) = (\mathcal{PT})B$ and hence B and B^* are $\mathcal{PT}$-symmetric.

(iii) In view of (3.2), (3.3), (3.4) and (5.6) for $f = f_+ \oplus f_- \in \mathrm{dom}\, B^*$ the following equivalences hold

$$f \in \mathrm{dom}\, A_1 \Leftrightarrow \Gamma_1^{B+}f_+ = -\Gamma_1^{B-}f_- \Leftrightarrow \Gamma_2^{A-}\mathcal{P}f_+ = -\Gamma_2^{A+}\mathcal{P}f_- \Leftrightarrow \mathcal{P}f \in \mathrm{dom}\, B_2.$$

Therefore, by (5.19) $B_2\mathcal{P} = \mathcal{P}A_1$ and hence, by Theorem 4.3

$$B_2(\mathcal{PT})f = \mathcal{P}A_1\mathcal{T}f = \mathcal{PT}B_2f$$

for all $f \in \mathrm{dom}\, B_2$. Since $A_1 = B_2^*$ this proves also that

$$A_1(\mathcal{PT}) = \mathcal{PT}A_1.$$

□

In the following theorem we consider a dual pair (A, B) from Theorem 5.4 and characterize $\mathcal{PT}$-symmetric proper extensions $\widetilde{A} \in \mathrm{Ext}(A, B)$ of A. Notice, that every proper extension $\widetilde{A}$ of A of rank d, (i.e. factor space $\mathrm{dom}\,\widetilde{A}/\mathrm{dom}\,A$ has dimension d) can be represented in the form (2.22): $\widetilde{A} = A_\Theta := B^* \restriction \mathrm{dom}\,A_\Theta$, where Θ is a linear relation in $\mathbb{C}^d$ with the kernel representation

$$\Theta = \ker \begin{pmatrix} C & D \end{pmatrix}, \quad C, D \in \mathbb{C}^{d\times d}, \quad \text{and (2.19) holds.} \tag{5.22}$$

Substitutions of (5.22) in (2.22) shows that $\mathrm{dom}\,A_\Theta$ has the form

$$\mathrm{dom}\,A_\Theta = \{f \in \mathrm{dom}\,B^* | C\Gamma_1^B f + D\Gamma_2^B f = 0\}. \tag{5.23}$$

Theorem 5.5. *Let in the assumptions of Theorem 5.4 the dual pair (A, B) and the boundary triple $(\mathbb{C}^d \times \mathbb{C}^d, \Gamma^A, \Gamma^B)$ be given by (3.3)–(3.5) and let $C, D \in \mathbb{C}^{d\times d}$ and let $A_\Theta \in Ext(A, B)$ be the proper extension of A defined by (5.23). Then A_Θ is $\mathcal{PT}$-symmetric if and only if $\Theta = \jmath_{\mathcal{H}}\Theta\jmath_{\mathcal{H}}$, i.e.* $\ker \begin{pmatrix} C\jmath_{\mathcal{H}} & D\jmath_{\mathcal{H}} \end{pmatrix} = \ker \begin{pmatrix} C & D \end{pmatrix}$.

Proof. By Theorems 4.3 and 5.4 the dual pair (A, B) is $\mathcal{T}$-real and $\mathcal{PT}$-symmetric. It follows from (5.21) that $\mathcal{PT}\mathrm{dom}\,B^* = \mathrm{dom}\,B^*$. By (5.12) and (3.2), we obtain for $f \in \mathrm{dom}\,B^*$

$$\begin{pmatrix} \Gamma_1^B(\mathcal{PT}f) \\ \Gamma_2^B(\mathcal{PT}f) \end{pmatrix} = \begin{pmatrix} \Gamma_1^{B-}\mathcal{PT}f_+ + \Gamma_1^{B+}\mathcal{PT}f_- \\ \Gamma_2^{B+}\mathcal{PT}f_- \end{pmatrix} = \begin{pmatrix} \jmath_{\mathcal{H}}\Gamma_1^{B+}f_+ + \jmath_{\mathcal{H}}\Gamma_1^{B-}f_- \\ \jmath_{\mathcal{H}}\Gamma_2^{B+}f_+ \end{pmatrix} = \begin{pmatrix} \jmath_{\mathcal{H}}\Gamma_1^B f \\ \jmath_{\mathcal{H}}\Gamma_2^B f \end{pmatrix}$$

Therefore,

$$\Gamma^B\mathcal{PT}f \in \Theta = \ker \begin{pmatrix} C & D \end{pmatrix} \iff \Gamma^B f \in \ker \jmath_{\mathcal{H}}\Theta\jmath_{\mathcal{H}} = \ker \begin{pmatrix} C\jmath_{\mathcal{H}} & D\jmath_{\mathcal{H}} \end{pmatrix}. \tag{5.24}$$

Now it follows from (5.21) that for $f \in \mathrm{dom}\,A_\Theta$

$$A_\Theta(\mathcal{PT})f = B^*(\mathcal{PT})f = (\mathcal{PT})B^*f = (\mathcal{PT})A_\Theta f.$$

This proves the claim. ∎

Remark 5.6. *In the special case of Theorem 5.5 when $d = 1$ and $\jmath_{\mathcal{H}}$ is the standard complex conjugation in $\mathbb{C}$, the set of all $\mathcal{PT}$-symmetric extensions of A other than the decoupled operator $A_{+,2} \oplus A_{-,2}$ is parameterized by a real parameter $\alpha \in \mathbb{R}$ via*

$$A_\alpha = B^* \restriction_{\mathrm{dom}\,A_\alpha}, \quad \mathrm{dom}\,A_\alpha = \left\{u \in \mathrm{dom}\,B^* |\, \Gamma_1^B u = \alpha\Gamma_2^B u\right\}. \tag{5.25}$$

Note that the decoupled operator $A_{+,2} \oplus A_{-,2}$ is also $\mathcal{PT}$-symmetric which follows from Theorem 5.5 with $C = 0$ and $D = 1$.

Next we present a description of $\mathcal{PT}$-symmetric proper extensions of the $\mathcal{PT}$-symmetric operator $A_0 = A_+ \oplus A_-$, see Theorem 4.3.

Theorem 5.7. *Let in the assumptions of Theorem 5.4 the dual pair (A_0, B_0) and the boundary triple $(\mathbb{C}^{2d} \times \mathbb{C}^{2d}, \Gamma^{A_0}, \Gamma^{B_0})$ be given by (4.9), (4.10) let $\widehat{C}, \widehat{D}$ be $2d \times 2d$-matrices such that (2.19) holds, and let $\widehat{\Theta}$ be a linear relation in $\mathcal{H}^2$, $\mathcal{H} := \mathbb{C}^{2d}$ with the kernel representation $\widehat{\Theta} = \ker \begin{pmatrix} \widehat{C} & \widehat{D} \end{pmatrix}$. Then the extension $(A_0)_{\widehat{\Theta}} = B_0^* \restriction \mathrm{dom}\,(A_0)_{\widehat{\Theta}} \in \mathrm{Ext}(A, B)$ of A_0 with*

$$\mathrm{dom}\,(A_0)_{\widehat{\Theta}} = \{f \in \mathrm{dom}\,B_0^* |\, \widehat{C}\Gamma_1^{B_0} f + \widehat{D}\Gamma_2^{B_0} f = 0\} \tag{5.26}$$

is $\mathcal{PT}$-symmetric if and only if $\widehat{\Theta} = \widehat{\jmath}_{\mathcal{H}}\widehat{\Theta}\widehat{\jmath}_{\mathcal{H}}$, i.e.

$$\ker\begin{pmatrix}\widehat{C}\widehat{\jmath}_{\mathcal{H}} & \widehat{D}\widehat{\jmath}_{\mathcal{H}}\end{pmatrix} = \ker\begin{pmatrix}\widehat{C} & \widehat{D}\end{pmatrix} \quad \textit{where} \quad \widehat{\jmath}_{\mathcal{H}} := \begin{pmatrix}0 & \jmath_{\mathcal{H}}\\ \jmath_{\mathcal{H}} & 0\end{pmatrix}. \tag{5.27}$$

Proof. Recall that $\Gamma_1^{B_0}$ and $\Gamma_2^{B_0}$ are given by formulas (4.10).

The invariance of $\operatorname{dom} B_0^*$ with respect to $\mathcal{PT}$ is shown in Theorem 5.4 (i). By (5.12) we obtain for $f = f_+ \oplus f_-, g = g_+ \oplus g_- \in \operatorname{dom} B_0^*$

$$\Gamma_1^{B_0}(\mathcal{PT}f) = \begin{pmatrix}\Gamma_1^{B_+}\mathcal{PT}f_-\\ \Gamma_1^{B_-}\mathcal{PT}f_+\end{pmatrix} = \begin{pmatrix}\jmath_{\mathcal{H}}\Gamma_1^{B_-}f_-\\ \jmath_{\mathcal{H}}\Gamma_1^{B_+}f_+\end{pmatrix} = \widehat{\jmath}_{\mathcal{H}}\begin{pmatrix}\Gamma_1^{B_+}f_+\\ \Gamma_1^{B_-}f_-\end{pmatrix} = \widehat{\jmath}_{\mathcal{H}}\Gamma_1^{B_0}f$$

and

$$\Gamma_2^{B_0}(\mathcal{PT}f) = \begin{pmatrix}\Gamma_2^{B_+}\mathcal{PT}f_-\\ \Gamma_2^{B_-}\mathcal{PT}f_+\end{pmatrix} = \begin{pmatrix}\jmath_{\mathcal{H}}\Gamma_2^{B_-}f_-\\ \jmath_{\mathcal{H}}\Gamma_2^{B_+}f_+\end{pmatrix} = \widehat{\jmath}_{\mathcal{H}}\begin{pmatrix}\Gamma_2^{B_+}f_+\\ \Gamma_2^{B_-}f_-\end{pmatrix} = \widehat{\jmath}_{\mathcal{H}}\Gamma_2^{B_0}f.$$

Therefore, the condition $\mathcal{PT}f \in \operatorname{dom}(A_0)_{\widehat{\Theta}}$ is equivalent to

$$\begin{pmatrix}\widehat{C}\widehat{\jmath}_{\mathcal{H}} & \widehat{D}\widehat{\jmath}_{\mathcal{H}}\end{pmatrix}\begin{pmatrix}\Gamma_1^0 f_+\\ \Gamma_2^0 f_-\end{pmatrix} = 0. \tag{5.28}$$

Comparison of (5.26) and (5.28) yields (5.27). □

6. $\mathcal{P}$-symmetric and $\mathcal{P}$-selfadjoint extensions of the operator A_0

6.1. **Boundary triples for $\mathcal{P}$-symmetric operators.** In this section we describe $\mathcal{P}$-symmetric and $\mathcal{P}$-selfadjoint extensions of the operators A_0 and A and compare them with the descriptions of $\mathcal{PT}$-symmetric extensions of A_0 and A from the previous section.

In order to define $\mathcal{P}$-symmetry it is convenient to consider the abstract parity $\mathcal{P}$ from Section 5 as a Gramian which leads to the theory of Krein spaces. Krein spaces are Hilbert spaces equipped with an additional inner product. This additional inner product is a symmetric, non-degenerated sesquilinear form, which is in general not definite. Or, what is the same, the additional inner form is given via a selfadjoint, boundedly invertible Gramian, see [2, 10]. Here, $\mathcal{P}$ will be used as the Gramian. More precisely, let $(\mathfrak{H}_\pm, \langle\cdot,\cdot\rangle_{\mathfrak{H}_\pm})$ be two Hilbert spaces and let $\mathcal{P}$ be the abstract parity in $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. As usual, we define the Hilbert space scalar product in $\mathfrak{H}$ as the sum of the two scalar products in $\mathfrak{H}_\pm$,

$$\langle f, g\rangle_{\mathfrak{H}} := \langle f_+, g_+\rangle_{\mathfrak{H}_+} + \langle f_-, g_-\rangle_{\mathfrak{H}_-} \quad \text{for } f = f_+ \oplus f_-, g = g_+ \oplus g_- \in \mathfrak{H}_+ \oplus \mathfrak{H}_-.$$

Then the pair $(\mathfrak{H}, [\cdot,\cdot])$ with the inner product given by

$$[f, g] := \langle \mathcal{P}f, g\rangle_{\mathfrak{H}} \tag{6.1}$$

forms a Krein space. Recall, that a densely defined linear operator A in (a Krein space) $(\mathfrak{H}, [\cdot,\cdot])$ is called *$\mathcal{P}$-symmetric*, if it is symmetric with respect to the new (Krein space) inner product, namely

$$[Af, g] = [f, Ag] \quad \text{for all} \quad f, g \in \operatorname{dom} A.$$

Denote by A^+ the adjoint operator in $(\mathfrak{H}, [\cdot,\cdot])$, i.e. $A^+ = \mathcal{P}A^*\mathcal{P}$. For a $\mathcal{P}$-symmetric operator A one has $A \subseteq A^+$. The operator A is called *$\mathcal{P}$-selfadjoint* if $A = A^+$.

In what follows we will apply the two notions of $\mathcal{P}$-symmetry and $\mathcal{PT}$-symmetry to extensions of the operators A and A_0, defined in Theorem 3.1 and in Theorem 4.3, respectively. The coupling A_1 and the decoupled A_2, cf. Definition 3.2, are examples of operators which are simultaneously $\mathcal{P}$-selfadjoint and $\mathcal{PT}$-symmetric. We have the following chain of operators

$$A_0 = A_+ \oplus A_- \subset A \subset A_j \subset B^* \subset B_0^* = B_+^* \oplus B_-^* \quad j = 1, 2.$$

Note that Definition 2.1 of boundary triples is made for dual pairs. The following definition of a boundary triple for a single operator is from [12]. In the case of a Hilbert space symmetric operator it was introduced in [19], see also [16]. We will use it to describe the extensions of the $\mathcal{P}$-symmetric operators A and A_0.

Definition 6.1. *Let $\mathcal{H}$ be an auxiliary Hilbert space and let Γ_1, Γ_2 be linear operators from* dom A^+ *to $\mathcal{H}$. The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ is called a* boundary triple *for the $\mathcal{P}$-symmetric operator A, if:*

(1) *the mapping $\Gamma := \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}$ from* dom A^+ *to $\mathcal{H}^2$ is surjective;*
(2) *the following identity holds for every $f, g \in$* dom A^+

$$[A^+ f, g] - [f, A^+ g] = \langle \Gamma_1 f, \Gamma_2 g \rangle_{\mathcal{H}} - \langle \Gamma_2 f, \Gamma_1 g \rangle_{\mathcal{H}}.$$

Remind [12], that given a boundary triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ for the $\mathcal{P}$-symmetric operator A, the set of all $\mathcal{P}$-selfadjoint extensions $\widehat{A}$ of A can be parametrized by the formula

$$\widehat{A} = A^+ \upharpoonright_{\text{dom}\, \widehat{A}}, \quad \text{dom}\, \widehat{A} = \{f \in \text{dom}\, A^+ | \, \Gamma f \in \Phi\}, \tag{6.2}$$

where Φ ranges over the set of all selfadjoint relations in $\mathcal{H}$.

In the following theorem we characterize $\mathcal{P}$-selfadjoint extensions $\widehat{A}$ of the operator A_0 of the form (5.26).

Theorem 6.2. *Let in the assumptions of Theorem 5.4 (A_0, B_0) be a $\mathcal{PT}$-symmetric dual pair, let $\widehat{C}, \widehat{D}$ be $2d \times 2d$-matrices such that (2.19) holds, and let $\widehat{\Theta} =$ ker $\begin{pmatrix} \widehat{C} & \widehat{D} \end{pmatrix}$ be a linear relation in $\mathcal{H}^2$, $\mathcal{H} := \mathbb{C}^d$ and let $(A_0)_{\widehat{\Theta}}$ be an extension of A_0 given by (5.26). Then the following hold.*

(i) *The operator A_0 is $\mathcal{P}$-symmetric and a boundary triple $(\mathbb{C}^{2d}, \Gamma_1^0, \Gamma_2^0)$ for A_0 can be chosen as*

$$\Gamma_1^0 f = \Gamma_1^{B_0}, \quad \Gamma_2^0 f = J_d \Gamma_2^{B_0}, \quad \textit{where } J_d := \begin{pmatrix} O_d & I_d \\ I_d & O_d \end{pmatrix}. \tag{6.3}$$

(ii) *The extension $(A_0)_{\widehat{\Theta}}$ of the operator A_0 given by (5.26) is $\mathcal{P}-$selfadjoint if and only if the linear relation $\Phi := J_d \widehat{\Theta} =$ ker $\begin{pmatrix} \widehat{C} & \widehat{D} J_d \end{pmatrix}$ is selfadjoint, i.e.*

$$\widehat{C} J_d \widehat{D}^* - \widehat{D} J_d \widehat{C}^* = 0. \tag{6.4}$$

Proof. (i) The $\mathcal{P}$-symmetry of A_0 follows from (5.16) and the fact that (A_0, B_0) forms a dual pair which imply

$$A_0^+ = \mathcal{P} A_0^* \mathcal{P} = B_0^* \supset A_0.$$

For $f=f_+\oplus f_-, g=g_+\oplus g_-\in\operatorname{dom}B_0^*$ we obtain with the help of (5.16), (3.12) and (5.6)

$$
\begin{aligned}
[A_0^+f,g]-[f,A_0^+g] &= \langle B_0^*f,\mathcal{P}g\rangle-\langle f,\mathcal{P}B_0^*g\rangle=\langle B_0^*f,\mathcal{P}g\rangle-\langle f,A_0^*\mathcal{P}g\rangle\\
&=\langle \Gamma_1^{B+}f_+,\Gamma_1^{A+}\mathcal{P}g_-\rangle_{\mathcal{H}}-\langle \Gamma_2^{B+}f_+,\Gamma_2^{A+}\mathcal{P}g_-\rangle_{\mathcal{H}}\\
&\quad+\langle \Gamma_1^{B-}f_-,\Gamma_1^{A-}\mathcal{P}g_+\rangle_{\mathcal{H}}-\langle \Gamma_2^{B-}f_-,\Gamma_2^{A-}\mathcal{P}g_+\rangle_{\mathcal{H}}\\
&=\langle \Gamma_1^{B+}f_+,\Gamma_2^{B-}g_-\rangle_{\mathcal{H}}-\langle \Gamma_2^{B+}f_+,\Gamma_1^{B-}g_-\rangle_{\mathcal{H}}\\
&\quad+\langle \Gamma_1^{B-}f_-,\Gamma_2^{B+}g_+\rangle_{\mathcal{H}}-\langle \Gamma_2^{B-}f_-,\Gamma_1^{B+}g_+\rangle_{\mathcal{H}}\\
&=\langle \Gamma_1^0f,\Gamma_2^0g\rangle_{\mathcal{H}^2}-\langle \Gamma_2^0f,\Gamma_1^0g\rangle_{\mathcal{H}^2}.
\end{aligned}
\tag{6.5}
$$

So the triple $(\mathcal{H},\Gamma_1^0,\Gamma_2^0)$ is a boundary triple for the $\mathcal{P}$-symmetric operator A_0.

(ii) The equivalence

$$(A_0)_{\widehat\Theta}=(A_0)_{\widehat\Theta}^+\iff\Phi=\Phi^*$$

is a general fact from [12]. The equivalence of the equality $\Phi=\Phi^*$ to (6.4) follows from (2.21). □

Remark 6.3. *It follows from (5.19), that $A\subseteq B^*=A^+$, and hence the operator A is $\mathcal{P}$-symmetric. A boundary triple $(\mathcal{H},\Gamma_1,\Gamma_2)$ for A^+ can be chosen as*

$$\mathcal{H}=\mathbb{C}^d,\quad \Gamma_1f=\Gamma_1^Bf:=\Gamma_1^{B+}f_++\Gamma_1^{B-}f_-,\quad \Gamma_2f=\Gamma_2^Bf:=\Gamma_2^{B+}f_+$$

for $f=f_+\oplus f_-\in\operatorname{dom}B^*$. For a linear relation Θ in $\mathbb{C}^d$ with the kernel representation $\Theta=\ker\begin{pmatrix}C & D\end{pmatrix}$, $C,D\in\mathbb{C}^{d\times d}$, the extension A_Θ of A of the form (5.23) is $\mathcal{P}$-selfadjoint if and only if $\Theta=\Theta^*$, i.e. (2.21) holds.

In the special case when $d=1$ the set of all $\mathcal{P}$-selfadjoint extensions of A different from the decoupled operator $A_{-,2}\oplus A_{+,2}$ is parametrized by a real parameter $\alpha\in\mathbb{R}$ via

$$A_\alpha=B^*\upharpoonright\operatorname{dom}A_\alpha,\quad \operatorname{dom}A_\alpha=\left\{u\in\operatorname{dom}B^*|\Gamma_1^Bu=\alpha\Gamma_2^Bu\right\}. \tag{6.6}$$

Moreover, the set of $\mathcal{PT}$-symmetric extensions of A coincides with the set of its $\mathcal{P}$-selfadjoint extensions.

According to [9] boundary condition (5.26) can be rewritten in the following form.

Theorem 6.4. *Let in the assumptions of Theorem 6.2 Θ be a linear relation in $\mathbb{C}^{2d}$ such that $(A_0)_{\widehat\Theta}$ is a $\mathcal{P}$-selfadjoint and $\mathcal{PT}$-symmetric extensions of A_0. Then*

(i) *There exist orthogonal projectors P_D and $P_R=I-P_D$ in $\mathbb{C}^{2d}$ and a selfadjoint matrix Λ in $\mathbb{C}^{2d}$ such that*

$$\ker\Lambda=P_D\mathcal{H},\quad \operatorname{ran}\Lambda\subseteq P_R\mathcal{H} \tag{6.7}$$

and $\operatorname{dom}(A_0)_{\widehat\Theta}$ is characterized as the set of $f\in\operatorname{dom}B_-^*\oplus B_+^*$ such that

$$P_D\Gamma_2^0f = 0, \tag{6.8}$$
$$P_R\Gamma_1^0f = \Lambda P_R\Gamma_2^0f. \tag{6.9}$$

(ii) *The extension $(A_0)_{\widehat\Theta}$ is $\mathcal{PT}$-symmetric, if and only if the matrices P_D and Λ commute with $\widehat{\mathfrak{J}}_{\mathcal{H}}$:*

$$P_D\widehat{\mathfrak{J}}_{\mathcal{H}}=\widehat{\mathfrak{J}}_{\mathcal{H}}P_D,\quad \Lambda\widehat{\mathfrak{J}}_{\mathcal{H}}=\widehat{\mathfrak{J}}_{\mathcal{H}}\Lambda. \tag{6.10}$$

The condition (6.8) gives the Dirichlet part and the condition (6.9) gives the Robin part of the boundary conditions. As distinct from (5.26) the conditions (6.8)–(6.9) are uniquely defined by the selfadjoint linear relation $\Phi := J_d\widehat{\Theta}$.

Proof. (i) By Theorem 6.2, the extension $(A_0)_{\widehat{\Theta}}$ of the operator A_0 is $\mathcal{P}-$selfadjoint if and only if the linear relation $\Phi = \ker\begin{pmatrix}\widehat{C} & \widehat{D}J_d\end{pmatrix}$ is selfadjoint. Let P_D be the orthogonal projector onto $\ker\Phi = \ker C$ and let $P_R := I_{2d} - P_D$. As it is known (see e.g. [9, Theorem 1.4.4]), the selfadjoint linear relation Φ^{-1} admits the representation

$$\Phi^{-1} = \operatorname{graph}\Lambda \dotplus \begin{pmatrix} 0 \\ P_D\mathcal{H}\end{pmatrix} = \left\{\begin{pmatrix} x_2 \\ \Lambda x_2 + x_1\end{pmatrix} : \begin{array}{l} x_1 \in P_D\mathcal{H} \\ x_2 \in P_R\mathcal{H}\end{array}\right\}, \tag{6.11}$$

where Λ is a selfadjoint matrix Λ in $\mathbb{C}^{2d}$ such that (6.7) holds. For every $f \in \operatorname{dom}(A_0)_{\widehat{\Theta}}$ we have the inclusion $\begin{pmatrix}\Gamma_1^0 f \\ \Gamma_2^0 f\end{pmatrix} \in \Phi$. In view of (6.11) and (6.3) there exist $x_1 \in P_D\mathcal{H}$ and $x_2 \in P_R\mathcal{H}$ such that

$$\begin{pmatrix}\Gamma_2^0 f \\ \Gamma_1^0 f\end{pmatrix} = \begin{pmatrix} x_2 \\ \Lambda x_2 + x_1\end{pmatrix}. \tag{6.12}$$

This yields the equations (6.8), (6.9).

(ii) If the extension $(A_0)_{\widehat{\Theta}}$ is $\mathcal{PT}$-symmetric, then, by Theorem 6.2, $\widehat{\Theta}$ is $\widehat{\jmath}_{\mathcal{H}}$-invariant and hence the linear relation $\Phi = J_d\widehat{\Theta}$ is also $\widehat{\jmath}_{\mathcal{H}}$-invariant. Therefore, the subspaces $P_D\mathcal{H} = \ker\Phi = \ker\widehat{\Theta}$ and $P_R\mathcal{H} = \operatorname{ran}\Phi$ are $\widehat{\jmath}_{\mathcal{H}}$-invariant and hence the orthoprojectors P_D and P_R commute with $\widehat{\jmath}_{\mathcal{H}}$:

$$P_D\widehat{\jmath}_{\mathcal{H}} = \widehat{\jmath}_{\mathcal{H}}P_D, \quad P_R\widehat{\jmath}_{\mathcal{H}} = \widehat{\jmath}_{\mathcal{H}}P_R. \tag{6.13}$$

Moreover, in this case for every $f \in \operatorname{dom}(A_0)_{\widehat{\Theta}}$ we obtain, by (6.9),

$$\begin{pmatrix}\Gamma_1^0 f \\ \Gamma_2^0 f\end{pmatrix} \in \Phi \quad \Longrightarrow \quad \begin{pmatrix}\widehat{\jmath}_{\mathcal{H}}\Gamma_1^0 f \\ \widehat{\jmath}_{\mathcal{H}}\Gamma_2^0 f\end{pmatrix} \in \Phi \quad \Longrightarrow \quad P_R\widehat{\jmath}_{\mathcal{H}}\Gamma_1^0 f = \Lambda P_R\widehat{\jmath}_{\mathcal{H}}\Gamma_2^0 f.$$

By (6.13), we obtain

$$\widehat{\jmath}_{\mathcal{H}}P_R\Gamma_1^0 f = \Lambda\widehat{\jmath}_{\mathcal{H}}P_R\Gamma_2^0 f.$$

Comparing this equality with (6.9), we obtain the equality

$$\widehat{\jmath}_{\mathcal{H}}\Lambda = \Lambda\widehat{\jmath}_{\mathcal{H}}. \tag{6.14}$$

Conversely, if (6.13) and (6.14) hold, then similar reasonings show that $\widehat{\Theta}$ is $\widehat{\jmath}_{\mathcal{H}}$-invariant. □

7. Non-Hermitian $\mathcal{PT}$-invariant Hamiltonians

7.1. **Dual pairs associated with the Bender-Boettcher Hamiltonian.** Here we return to the investigation of the non-Hermitian $\mathcal{PT}$-invariant Hamiltonians presented in the introduction, that is, we study equation (1.1) on the wedge shaped contour Γ, cf. (1.2). By substitution $z(x) := xe^{i\phi\operatorname{sgn} x}$ one obtains the two differential expressions given by (1.3) and (1.4). Assume that $\mathfrak{a}_\pm$ in (1.4) are in the limit point case at $\pm\infty$. As presented in Section 1, this is the case if and only if the angle ϕ of the wedge satisfies

$$\phi \neq -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi \quad \text{for } k = 0, \dots, \left[\frac{N+3}{2}\right]. \tag{7.1}$$

Then by [21, Lemma 1] the differential expressions $\mathfrak{b}_\pm$ in (1.6) are also in the limit point case at $\pm\infty$. Define the operators $A_\pm$ and $B_\pm$ associated with $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ in $L^2(\mathbb{R}_\pm)$ as

$$A_\pm f_\pm := \mathfrak{a}_\pm[f_\pm], \quad B_\pm g_\pm := \mathfrak{b}_\pm[g_\pm] \quad \text{for } f_\pm \in \operatorname{dom} A_\pm,\ g_\pm \in \operatorname{dom} B_\pm,$$

respectively, with the domains

$$\operatorname{dom} A_\pm := \{u_\pm \in L^2(\mathbb{R}_\pm) | \mathfrak{a}_\pm[u_\pm] \in L^2(\mathbb{R}_\pm),\ u'_\pm \in AC_{loc}(\mathbb{R}_\pm),\ u_\pm(0_\pm) = u'_\pm(0_\pm) = 0\},$$

$$\operatorname{dom} B_\pm := \{v_\pm \in L^2(\mathbb{R}_\pm) | \mathfrak{b}_\pm[v_\pm] \in L^2(\mathbb{R}_\pm),\ v'_\pm \in AC_{loc}(\mathbb{R}_\pm),\ v_\pm(0_\pm) = v'_\pm(0_\pm) = 0\}.$$

These operators are called the minimal operators. It follows from [21, Proposition 1 and Theorem 3] that the (maximal) operators $A_\pm^*$ and $B_\pm^*$ are generated by differential expressions in $L^2(\mathbb{R}_\pm)$ where the roles of $\mathfrak{a}_\pm$ and $\mathfrak{b}_\pm$ are switched in the sense that the differential expressions $\mathfrak{a}_\pm$ are now related to $B_\pm^*$ and the differential expressions $\mathfrak{b}_\pm$ are related to $A_\pm^*$. More precisely,

$$B_\pm^* f_\pm := \mathfrak{a}_\pm[f_\pm], \quad A_\pm^* g_\pm := \mathfrak{b}_\pm[g_\pm] \quad \text{for } f_\pm \in \operatorname{dom} B_\pm^*,\ g_\pm \in \operatorname{dom} A_\pm^*,$$

with

$$\operatorname{dom} B_\pm^* := \{u_\pm \in L^2(\mathbb{R}_\pm) | \mathfrak{a}_\pm[u_\pm] \in L^2(\mathbb{R}_\pm),\ u'_\pm \in AC_{loc}(\mathbb{R}_\pm)\},$$

$$\operatorname{dom} A_\pm^* := \{v_\pm \in L^2(\mathbb{R}_\pm) | \mathfrak{b}_\pm[v_\pm] \in L^2(\mathbb{R}_\pm),\ v'_\pm \in AC_{loc}(\mathbb{R}_\pm)\}.$$

Lemma 7.1. *The pairs (A_-, B_-) and (A_+, B_+) are dual pairs. The triple $(\mathbb{C}^2, \Gamma^{A_+}, \Gamma^{B_+})$,*

$$\Gamma^{B_+} u_+ = \begin{pmatrix} e^{-2i\phi} u'_+(0) \\ u_+(0) \end{pmatrix}, \quad \Gamma^{A_+} v_+ = \begin{pmatrix} v_+(0) \\ e^{2i\phi} v'_+(0) \end{pmatrix}, \quad \begin{matrix} u_+ \in \operatorname{dom} B_+^*, \\ v_+ \in \operatorname{dom} A_+^*, \end{matrix} \tag{7.2}$$

is a boundary triple for the dual pair (A_+, B_+). The triple $(\mathbb{C}^2, \Gamma^{A_-}, \Gamma^{B_-})$,

$$\Gamma^{B_-} u_- = \begin{pmatrix} -e^{2i\phi} u'_-(0) \\ u_-(0) \end{pmatrix}, \quad \Gamma^{A_-} v_- = \begin{pmatrix} v_-(0) \\ -e^{-2i\phi} v'_-(0) \end{pmatrix}, \quad \begin{matrix} u_- \in \operatorname{dom} B_-^*, \\ v_- \in \operatorname{dom} A_-^*, \end{matrix} \tag{7.3}$$

is a boundary triple for the dual pair (A_-, B_-).

Proof. Integration by parts and [21, Proposition 1] show

$$\langle A_\pm u_\pm, v_\pm \rangle = \langle u_\pm, B_\pm v_\pm \rangle, \quad u_\pm \in \operatorname{dom} A_\pm, \quad v_\pm \in \operatorname{dom} B_\pm.$$

This proves the first statement. It follows from [21, Proposition 1] that for $u_+ \in \operatorname{dom} B_+^*$ and $v_+ \in \operatorname{dom} A_+^*$

$$\begin{aligned} \langle B_+^* u_+, v_+ \rangle - \langle u_+, A_+^* v_+ \rangle &= -e^{-2i\phi} \int_0^\infty u''_+(x) \overline{v_+(x)} dx + e^{-2i\phi} \int_0^\infty u_+(x) \overline{v''_+(x)} dx \\ &= e^{-2i\phi} (u'_+(0) \overline{v_+(0)} - u_+(0) \overline{v'_+(0)}). \end{aligned}$$

Hence, $(\mathbb{C}^2, \Gamma^{A_+}, \Gamma^{B_+})$ is a boundary triple for the dual pair (A_+, B_+). The statement for the triple (A_-, B_-) is shown in the same way. □

7.2. **$\mathcal{PT}$-symmetric extensions of the operator A.** The coupling (A, B) of the dual pairs $(A_\pm, B_\pm)$ relative to the boundary triples $(\mathbb{C}^2, \Gamma^{A_\pm}, \Gamma^{B_\pm})$ consists of two operators $A = (B_+^* \oplus B_-^*)|_{\mathrm{dom}\, A}$, $B = (A_+^* \oplus A_-^*)|_{\mathrm{dom}\, B}$ with the domains

$$\text{(7.4)}\quad \mathrm{dom}\, A = \{u_+ \oplus u_- \,|\, u_\pm \in \mathrm{dom}\, B_\pm^*,\ u_+(0) = u_-(0) = e^{-2i\phi} u_+'(0) - e^{2i\phi} u_-'(0) = 0\},$$

$$\text{(7.5)}\quad \mathrm{dom}\, B = \{u_+ \oplus u_- \,|\, u_\pm \in \mathrm{dom}\, A_\pm^*,\ u_+(0) = u_-(0) = e^{2i\phi} u_+'(0) - e^{-2i\phi} u_-'(0) = 0\},$$

see Theorem 3.1. Then the adjoints A^* and B^* are the restrictions of the operators $A_+^* \oplus A_-^*$ and $B_+^* \oplus B_-^*$, respectively, to the domains

$$\text{(7.6)}\qquad \mathrm{dom}\, A^* = \{u_+ \oplus u_- | u_\pm \in \mathrm{dom}\, A_\pm^*,\ u_+(0) = u_-(0)\};$$

$$\text{(7.7)}\qquad \mathrm{dom}\, B^* = \{u_+ \oplus u_- | u_\pm \in \mathrm{dom}\, B_\pm^*,\ u_+(0) = u_-(0)\}.$$

A boundary triple for (A, B) can be defined on $v \in \mathrm{dom}\, A^*$ and $u \in \mathrm{dom}\, B^*$ by

$$\text{(7.8)}\quad \Gamma^A v = \begin{pmatrix} \Gamma_1^A v \\ \Gamma_2^A v \end{pmatrix} := \begin{pmatrix} \Gamma_1^{A_+} v_+ \\ \Gamma_2^{A_+} v_+ + \Gamma_2^{A_-} v_- \end{pmatrix}, \quad \Gamma^B u = \begin{pmatrix} \Gamma_1^B u \\ \Gamma_2^B u \end{pmatrix} := \begin{pmatrix} \Gamma_1^{B_+} u_+ + \Gamma_1^{B_-} u_- \\ \Gamma_2^{B_+} u_+ \end{pmatrix},$$

We define the parity $\mathcal{P}$ and the time reversal $\mathcal{T}$ as in (1.5). It is easy to see that the parity $\mathcal{P}$ and the time reversal $\mathcal{T}$ satisfy (5.2). Due to Theorem 5.5 the operator A is $\mathcal{PT}$-symmetric and $\mathcal{P}$-symmetric in the Krein space $(L^2(\mathbb{R}) = L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+), [\cdot,\cdot])$ with the inner product

$$[\cdot,\cdot] = \langle \mathcal{P}\cdot, \cdot \rangle.$$

The (Krein space) adjoint A^+ of A coincides with the operator $B^* = (B_+^* \oplus B_-^*)|_{\mathrm{dom}\, B^*}$.

Application of Theorem 5.5 gives a one-parameter family $\{H_\alpha\}_{\alpha \in \mathbb{R}}$ of $\mathcal{PT}$-symmetric and $\mathcal{P}$-selfadjoint extensions of A in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$.

Theorem 7.2. *Let the angle ϕ satisfies (7.1) and let A be the coupling operator constructed in (7.4). Then the following statements are true.*

(i) *A boundary triple $(\mathbb{C}, \Gamma_1, \Gamma_2)$ for the $\mathcal{P}$-symmetric operator A is given by*

$$\Gamma_1 u = e^{-2i\phi} u_+'(0) - e^{2i\phi} u_-'(0), \quad \Gamma_2 u = u_+(0), \quad u = u_+ \oplus u_- \in \mathrm{dom}\, B^*.$$

(ii) *The extension H_α of the operator A defined as a restriction of B^* to the domain*

$$\text{(7.9)}\qquad \mathrm{dom}\, H_\alpha = \left\{u_+ \oplus u_- \in \mathrm{dom}\, B^* :\ e^{-2i\phi} u_+'(0) - e^{2i\phi} u_-'(0) = \alpha u_+(0)\right\}$$

is $\mathcal{P}$-selfadjoint if and only if $\alpha \in \mathbb{R}$.

(iii) *H_α is $\mathcal{PT}$-symmetric if and only if $\alpha \in \mathbb{R}$.*

Proof. By construction, the dual pairs (A_+, B_+) and (A_-, B_-) are $\mathcal{T}_\pm$-real and the parity operator $\mathcal{P}$ intertwines the operators A_+, B_- and A_-, B_+, that is, (5.5) holds. Moreover, the boundary triples $(\mathbb{C}^2, \Gamma^{A_+}, \Gamma^{B_+})$ and $(\mathbb{C}^2, \Gamma^{A_-}, \Gamma^{B_-})$ are also $(j_\mathbb{C}, \mathcal{T})$-real and satisfy the condition (5.6). Here $j_\mathbb{C}$ stands for the usual complex conjugation in $\mathbb{C}$. Hence, all assumptions in Theorem 5.5 are satisfied and the statements (i)-(iii) in Theorem 7.2 follows directly from Theorem 5.5. □

The boundary conditions in (7.9) are called "δ-type conditions" in the literature, see [9]. In [21] the extension H_α for the parameter value $\alpha = 0$ was considered, which is specified by "weighted Kirchhoff conditions":

$$\mathrm{dom}\, H_0 = \left\{u_+ \oplus u_- | u_\pm \in \mathrm{dom}\, B_\pm^*,\ u_+(0) - u_-(0) = e^{-2i\phi} u_+'(0) - e^{2i\phi} u_-'(0) = 0\right\}.$$

7.3. **$\mathcal{PT}$-symmetric extensions of A_0.** The family $(H_\alpha)_{\alpha\in\mathbb{R}}$, of extensions of A can be also treated via Theorem 6.2 as a family of extensions of the operator $A_0 = A_- \oplus A_+$. A boundary triple for the $\mathcal{P}$-symmetric operator A_0 can be defined on $u \in \operatorname{dom} B_0^*$ by the equalities, see (6.3)

$$\Gamma_1^0 u = \begin{pmatrix} \Gamma_1^{B_+} u_+ \\ \Gamma_1^{B_-} u_- \end{pmatrix} = \begin{pmatrix} e^{-2i\phi} u_+'(0) \\ -e^{2i\phi} u_-'(0) \end{pmatrix}, \ \Gamma_2^0 u = \begin{pmatrix} \Gamma_2^{B_-} u_- \\ \Gamma_2^{B_+} u_+ \end{pmatrix} = \begin{pmatrix} u_-(0) \\ u_+(0) \end{pmatrix}, \quad u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \in \operatorname{dom} B_0^*.$$

By Theorem 6.4, the set of $\mathcal{P}$-selfadjoint and $\mathcal{PT}$-symmetric extensions of A_0 is parametrized via (6.8)–(6.9) by pairs (P_D, Λ) where P_D is an orthogonal projector onto a subspace in $\mathbb{C}^{2d}$ and Λ is a 2×2-matrix which satisfy (6.7) and (6.10).

Let us discuss this formulas in our case when the boundary space is 2-dimensional.

Case 1. Let P_D be the projector on the whole $\mathbb{C}^2$, i.e. $P_D = I_{\mathbb{C}^2}$. Then $\Lambda = 0$ and boundary conditions (6.8)–(6.9) take the form of "Dirichlet boundary conditions":

$$u_+(0) = u_-(0) = 0.$$

The corresponding extension of A_0 is the decoupled operator $A_{+,2} \oplus A_{-,2}$.

Case 2. Let P_D be an 1-dimensional projector onto a $\widehat{\jmath}$-invariant subspace of $\mathbb{C}^2$. Every $\widehat{\jmath}$-invariant subspace $\mathcal{L}$ of $\mathbb{C}^2$ has the form $\mathcal{L} = \operatorname{span} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix}$, $\theta \in [0, 2\pi)$. In particular, for $\theta = \pi$ we have $P_D = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $P_R = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Condition (6.7) implies that Λ can be chosen as $\Lambda = \frac{1}{4}\begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix}$, $\alpha \in \mathbb{R}$, and boundary conditions (6.8)–(6.9) yield "δ-type conditions":

$$u_-(0) = u_+(0), \quad e^{-2i\phi} u_+'(0) - e^{2i\phi} u_-'(0) = \alpha u_+(0), \quad \alpha \in \mathbb{R}.$$

Case 3. Let the Dirichlet part in boundary conditions is missing, i.e. $P_D = O$. Then (6.7) is fulfilled automatically and $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$ is an arbitrary selfadjoint 2×2-matrix commuting with $\widehat{\jmath} = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}$. This leads to conditions

$$\lambda_{11} = \lambda_{22} \in \mathbb{R}, \quad \lambda_{21} = \overline{\lambda_{12}}. \tag{7.10}$$

In particular, for $\Lambda = 0$ we obtain the "Neumann boundary conditions":

$$u_+'(0) = u_-'(0) = 0.$$

Setting $\Lambda = \frac{1}{\beta}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\beta \in \mathbb{R} \setminus \{0\}$, we obtain the following family $\{T_\beta\}_{\beta \in \mathbb{R}\setminus\{0\}}$ of $\mathcal{PT}$-symmetric extensions of extension of A_0 specified by the boundary conditions

$$e^{-2i\phi} u_+'(0) = e^{2i\phi} u_-'(0) = \frac{1}{\beta}(u_+(0) - u_-(0)), \quad \beta \in \mathbb{R} \setminus \{0\},$$

which are analogues of "δ'-type conditions".

Acknowledgements
The research of Volodymyr Derkach was supported by the Ministry of Education and Science of Ukraine (project # 0121U109525) the German Research Foundation (DFG), grant TR 903/22-1 and a grant of the Volkswagen Foundation.

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Volodymyr Derkach: Institut für Mathematik, Technische Universität Ilmenau, PF 100565, 98684 Ilmenau, Germany and Department of Mathematics, Vasyl Stus Donetsk National University, Vinnytsya, Ukraine.
Email address: volodymyr.derkach@tu-ilmenau.de

Carsten Trunk: Institut für Mathematik, Technische Universität Ilmenau, PF 100565, 98684 Ilmenau, Germany.
Email address: carsten.trunk@tu-ilmenau.de