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The socle of the center of a group algebra

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Abstract

Let A be a finite-dimensional algebra over an algebraically closed field F. We consider the socle $\mathrm{soc}(Z(A))$ of its center $Z(A)$, which is known to be an ideal of $Z(A)$. The principal question treated in this thesis is under which conditions $\mathrm{soc}(Z(A))$ is even an ideal in the entire algebra A. Our main focus lies on the case that A is the group algebra FG of a finite group G over F. Here, it suffices to consider fields of characteristic $p > 0$.

In the first main result, we classify the p-groups G for which $\mathrm{soc}(ZFG)$ is an ideal in FG. In case that p is an odd prime number, these are exactly the p-groups of nilpotency class at most two. For $p = 2$, we obtain a characterization in terms of the conjugacy class structure of the group. Here, in contrast to the case of odd characteristic, the group G can have arbitrary nilpotency class.

However, the main focus of this thesis lies on the structural analysis of arbitrary finite groups G for which $\mathrm{soc}(ZFG)$ is an ideal in FG. Our first fundamental observation is that G has a normal Sylow p-subgroup P, which contains the derived subgroup G'. Then we study the quotient group G/P' and determine a decomposition of G'/P' as a direct product. Subsequently, we decompose G into a central product, which reduces our investigation to the case $P = G'$. In case that one term in the decomposition of G'/P' vanishes, we give a complete characterization of the groups G for which $\mathrm{soc}(ZFG)$ is an ideal of FG. Making use of these results, we state a conjecture on the structure of G in the general setting, which we prove in a special case.

We also investigate the main problem for symmetric algebras, with a particular focus on symmetric local algebras. There, we concentrate on two aspects: Firstly, we show that the minimal dimension of a symmetric local algebra A in which the Jacobson radical $J(Z(A))$ of $Z(A)$ is not an ideal is twelve, and that the minimal dimension of such an algebra A in which the socle $\mathrm{soc}(Z(A))$ is not an ideal is 17, 18, 19 or 20. Secondly, these properties are investigated for quantum complete intersection algebras as well as trivial extension algebras.

Zusammenfassung

Sei A eine endlich-dimensionale Algebra über einem algebraisch abgeschlossenen Körper F. Bekanntermaßen ist der Sockel $\mathrm{soc}(Z(A))$ des Zentrums $Z(A)$ von A ein Ideal in $Z(A)$. In dieser Arbeit untersuchen wir, unter welchen Bedingungen $\mathrm{soc}(Z(A))$ sogar ein Ideal in der gesamten Algebra A ist. Der Fokus liegt dabei auf Gruppenalgebren FG von endlichen Gruppen G über F. Hierbei reicht es aus, Körper mit positiver Charakteristik p zu betrachten.

In unserem ersten Hauptresultat klassifizieren wir die p-Gruppen G, für die $\mathrm{soc}(ZFG)$ ein Ideal in FG ist. Falls p eine ungerade Primzahl ist, sind dies genau die p-Gruppen der Nilpotenzklasse höchstens zwei. Für $p = 2$ erhalten wir eine Charakterisierung anhand der Konjugationsklassenstruktur von G. Im Unterschied zum Fall ungerader Charakteristik kann die Nilpotenzklasse von G hier beliebig große Werte annehmen.

Der Fokus dieser Arbeit liegt jedoch auf der strukturellen Untersuchung beliebiger endlicher Gruppen G, für die $\mathrm{soc}(ZFG)$ ein Ideal in FG ist. Wir zeigen zunächst, dass die Gruppe G in diesem Fall eine normale p-Sylowgruppe P besitzt, die die Kommutatoruntergruppe G' enthält. Danach untersuchen wir die Struktur der Faktorgruppe G/P' und leiten eine Zerlegung von G'/P' als direktes Produkt von Untergruppen her. Im Anschluss finden wir eine Zerlegung von G als Zentralprodukt gewisser Untergruppen, die es uns im Folgenden ermöglicht, unsere Untersuchung auf den Spezialfall $P = G'$ zu beschränken. Im Fall, dass ein bestimmter Faktor in der Zerlegung von G'/P' verschwindet, leiten wir eine vollständige Klassifikation der Gruppen G, für die $\mathrm{soc}(ZFG)$ ein Ideal in FG ist, her. Mithilfe dieser Resultate formulieren wir eine Vermutung über die Struktur von G im Allgemeinen, die wir in einem Spezialfall beweisen.

Das zentrale Problem dieser Arbeit wird außerdem für symmetrische Algebren, insbesondere für symmetrische, lokale Algebren, untersucht. Dabei liegt der Fokus auf zwei Aspekten: Zum einen zeigen wir, dass die minimale Dimension einer symmetrischen, lokalen Algebra A, in der das Jacobson Radikal $J(Z(A))$ des Zentrums kein Ideal ist, zwölf ist, und dass die minimale Dimension einer solchen Algebra, in der $\mathrm{soc}(Z(A))$ kein Ideal ist, 17, 18, 19 oder 20 beträgt. Zum anderen betrachten wir diese Eigenschaften für „quantum complete intersection algebras“ sowie triviale Erweiterungen von lokalen Algebren.

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Contents

Introduction

Group algebras are the fundamental objects studied in the representation theory of finite groups: Instead of directly investigating the finite-dimensional representations of a finite group G over a field F, one usually studies modules over the group algebra FG of G over F. This is possible since there exists a one-to-one correspondence between FG-modules and representations of G over F. Since the group elements form a vector space basis for the corresponding group algebra, the group G completely determines the structure of FG. Naturally, this raises the question to which extent the converse holds, that is, which structural aspects of the group G can be recovered from FG. In other words, we would like to know which information is preserved in the transition from the group to its group algebra.

A famous example for this kind of questions is the isomorphism problem: In the original version, posed by Higman in his thesis [22], the question was whether two finite groups G and H are isomorphic provided that they have isomorphic integral group rings $\mathbb{Z}G$ and $\mathbb{Z}H$. Although there exist some affirmative results for special classes of groups (for instance, for p-groups [46]), the general problem was answered negatively by Hertweck [21] in 2001. Many variants of this problem were considered. We mention Brauer [6] who asked whether two finite groups G and H are isomorphic if their group algebras FG and FH are isomorphic for every field F. A counterexample to this conjecture was given by Dade [14]. Another variant is the modular isomorphism problem which states whether two p-groups G and H are isomorphic if their group algebras over the field with p elements are isomorphic. Also this question was recently answered negatively [18].

Since group algebras form a special class of symmetric algebras, it is useful to investigate the latter in greater detail. This approach proved to be an efficient tool in representation theory, for instance in the understanding of certain blocks of finite groups. Külshammer's series of papers on the group algebra viewed as a symmetric algebra (see [31], [32], [33], [35]) is one of the first sources in which abstract symmetric algebras are explicitly studied in order to obtain information on group algebras and related objects. This strategy was subsequently adopted by various authors. For instance in [27], Broué's abelian defect group conjecture ([8, Question 6.2]) is proven in a special case by analyzing symmetric local algebras of dimension nine with a prescribed isomorphism type of the center. Landrock [38] investigates the Morita equivalence classes of certain blocks of finite-dimensional group algebras by classifying symmetric local algebras of a certain type. Similar techniques are for example also used in [29] and [40].

Here, we focus on properties of symmetric algebras originating from ideals in their center. For group algebras, this is motivated by the fact that there exist many connections between aspects of the representation theory of the group and the structure of certain ideals in the center of the corresponding group algebra. For instance in [25] and [7], the investigation of such ideals gives rise to a method to detect odd diagonal entries in the Cartan matrix of a finite-dimensional group algebra in characteristic two. More specifically, this thesis is mostly concerned with the question under which conditions an ideal I of $Z(A)$ is even an ideal in the entire algebra A. This is then denoted by $I \trianglelefteq A$. Our main focus lies on the so-called socle $\mathrm{soc}(Z(A))$, which is the annihilator of the Jacobson radical $J(Z(A))$ and of fundamental interest in representation theory.

Main Problem. *For which finite-dimensional F-algebras A is the socle* $\mathrm{soc}(Z(A))$ *of $Z(A)$ an ideal of A?*

To set the stage, we also deal with certain aspects of the analogous problem for the Jacobson radical itself.

Question. *For which finite-dimensional F-algebras A is the Jacobson radical $J(Z(A))$ of $Z(A)$ an ideal of A?*

The latter question has been investigated for a long time, especially for group algebras over an algebraically closed field F of positive characteristic p. In 1969, Clarke [12] provided a characterization of the finite groups G that are p-solvable and satisfy $J(ZFG) \trianglelefteq FG$. In 1978, Koshitani [28] showed that the assumption on the p-solvability of G is in fact redundant, thus completing the classification. In 2020, Külshammer [37] dealt with blocks of finite group algebras and gave some approaches for general finite-dimensional symmetric algebras. Furthermore, Landrock [39] proved that every symmetric local F-algebra A of dimension at most ten satisfies $J(Z(A)) \trianglelefteq A$. It should be mentioned that especially for group algebras, there exist various prequels and variants of this question (for a collection of related problems, see [37]).

In contrast, results related to the corresponding problem for the socle $\mathrm{soc}(Z(A))$ are rare. In the case that A is a symmetric local algebra, a criterion which is equivalent to $\mathrm{soc}(Z(A)) \trianglelefteq A$ is mentioned in [38, Lemma 1.1.15] and [39, Lemma 3.1]. It is generalized to arbitrary finite-dimensional F-algebras in [37, Lemma 2.1]. Apart from these sources, we are not aware of any other results concerning this question.

In this thesis, we are mainly interested in treating this problem in the case that A is a symmetric algebra, or, as a special case, in the situation where A is the group algebra FG of a finite group G over the field F. This situation is of particular interest since the property $\mathrm{soc}(ZFG) \trianglelefteq FG$ is an invariant of the group algebra. That is, it only depends on the structure of the algebra FG itself and it can be verified without any information on the underlying group G. Such invariants play a crucial role in the treatment of the isomorphism problems that we mentioned earlier. Moreover, from a practical point of view, the investigation of the special case of group algebras has the additional advantage that the group carries a lot of structural information. It turns out that our problem can be expressed completely in terms of group-theoretic notions, which allows us to make use of the rich theoretical background developed in this area.

This thesis is structured in the following way: The first chapter is devoted to introducing the main problem in its most general setting. We begin with some universal results on arbitrary finite-dimensional F-algebras. In particular, we introduce the structures investigated in this thesis and present an equivalent condition for our main question which is stated in [37]. With this, we examine our problem for tensor products of algebras. In the next step, we restrict ourselves to symmetric algebras. Apart from collecting some background results on this topic, the main objective of this part is to transfer the property $\mathrm{soc}(Z(A)) \trianglelefteq A$ to certain quotient algebras of A.

The second chapter is concerned with the treatment of group algebras. We rephrase the problem in terms of group-theoretic notions and derive some first results on the structure of the finite groups G which satisfy $\mathrm{soc}(ZFG) \trianglelefteq FG$ for a fixed algebraically closed field F. In this situation, we may restrict ourselves to fields of positive characteristic p. By investigating the Reynolds ideal of FG, we find a decomposition of G as a certain semidirect product. This allows us to determine a basis for the Jacobson radical $J(ZFG)$, which we will use throughout the entire thesis. Another important aim of this chapter is to establish a connection between our main problem for the group algebra FG and the corresponding question for the group algebra of a quotient group G/N. We conclude this chapter with the investigation of groups which have an abelian Sylow p-subgroup. This will form the starting point for further derivations.

In the third chapter, we study groups of prime power order. Although this is a special case of our problem, the treatment of p-groups is particularly insightful and forms a basis for the further results in this thesis. After some general observations concerning p-groups of small nilpotency class and the behavior of the problem with respect to isoclinism, we distinguish the cases $p \geq 3$ and $p = 2$. We obtain the following characterization of the p-groups G which satisfy $\mathrm{soc}(ZFG) \trianglelefteq FG$:

Theorem. *Let F be an algebraically closed field of characteristic $p > 0$ and let G be a finite p-group. Then $\mathrm{soc}(ZFG)$ is an ideal in FG if and only if one of the following statements holds:*

(i) The nilpotency class of G is at most two.

(ii) We have $p = 2$ and $G' \subseteq Y(G)Z(G)$, where G' and $Z(G)$ denote the derived subgroup and the center of G, respectively, and $Y(G)$ is the subgroup generated by all elements fg^{-1} for which the set $\{f, g\}$ is a G-conjugacy class of length two.

In particular, we find examples of 2-groups G of arbitrary nilpotency class for which $\mathrm{soc}(ZFG)$ is an ideal in FG. This demonstrates the substantial difference in the behavior of p-groups with respect to this problem, depending on whether the characteristic is odd or even. Moreover, this theorem has consequences in view of the isomorphism problems that we mentioned at the beginning: It implies that one can distinguish p-groups of nilpotency class at most two from p-groups of class at least three by examining the socle of the center of their group algebra if p is odd.

In the fourth chapter, we treat our main problem for arbitrary finite groups. It turns out that if G is a finite group which satisfies $\mathrm{soc}(ZFG) \trianglelefteq FG$, then G is of the form $G = P \rtimes H$, where P is the unique Sylow p-subgroup of G and H is an abelian Hall p'-subgroup. In order to exploit our results on groups with an abelian Sylow p-subgroup from the second chapter, we first describe the structure of the quotient group G/P' before examining the structure of G itself. After that, we distinguish the cases $C_{G'}(P) \subseteq P'$ and $C_{G'}(P) \nsubseteq P'$. In the first case, we can classify the groups G which satisfy $\mathrm{soc}(ZFG) \trianglelefteq FG$:

Theorem. *Let F be an algebraically closed field of characteristic $p > 0$ and let G be a finite group of the form $G = P \rtimes H$ with a Sylow p-subgroup P and an abelian p'-group H. Moreover, we assume $C_{G'}(P) \subseteq P'$ and $O_{p'}(G) = 1$, where $O_{p'}(G)$ denotes the p'-core of G. Then $\operatorname{soc}(ZFG)$ is an ideal of FG if and only if there exist normal subgroups $K, Q_1, \ldots, Q_n$ of G for some $n \in \mathbb{N}_0$ such that*

$$G = K * Q_1 * \ldots * Q_n$$

is a central product and the following hold:

(i) K *is a p-group with* $\operatorname{soc}(ZFK) \trianglelefteq FK$,

(ii) $Q_i = Q_i' \rtimes H_i$ *is a semidirect product of the derived subgroup Q_i', which is a p-group of nilpotency class exactly two, and a cyclic subgroup H_i of order $|Q_i'/Q_i''| - 1$ such that* $Q_i/Q_i'' \cong \operatorname{AGL}(1, |Q_i'/Q_i''|)$ *holds,*

(iii) the centralizer $C_{H_i}(Q_i'')$ of Q_i'' in H_i is nontrivial,

(iv) $Q_i'/Z(Q_i)$ *is a Camina group.*

If $C_{G'}(P)$ is not contained in P', we formulate a conjecture on the structure of H and its action on P, which we prove in a special case. Conversely, if H is of the presumed form, then the group G can be decomposed into a certain direct product to which our previous results can be applied in order to determine whether $\operatorname{soc}(ZFG)$ is an ideal of FG.

In the last chapter, we move to the investigation of symmetric local algebras over an algebraically closed field F of arbitrary characteristic. The aim of this chapter is to provide examples of minimal dimension in which the Jacobson radical or the socle of the center are not ideals, respectively. To this end, we first consider two special classes of symmetric local algebras, namely quantum complete intersection algebras and trivial extension algebras, before moving to the investigation of general symmetric local algebras. In [39, Theorem 3.2], Landrock proved that every symmetric local F-algebra A of dimension at most ten satisfies $J(Z(A)) \trianglelefteq A$. Here, we refine his result in the following way:

Theorem. *Let F be an algebraically closed field. Every symmetric local F-algebra A of dimension $\dim A \leq 11$ satisfies $J(Z(A)) \trianglelefteq A$ and there exists a symmetric local F-algebra A of dimension twelve in which $J(Z(A))$ is not an ideal.*

We conclude this thesis with the analogous investigation for the socle. In this case, we show the following:

Theorem. *Let F be an algebraically closed field. Every symmetric local F-algebra A of dimension at most 16 satisfies $\operatorname{soc}(Z(A)) \trianglelefteq A$.*

Moreover, using trivial extension algebras, we are able to provide an example of a symmetric local F-algebra A of dimension 20 with $\operatorname{soc}(Z(A)) \ntrianglelefteq A$, which yields an upper bound for the minimal dimension of such an algebra. Additionally, we show that for the class of trivial extension algebras, this bound is optimal.

Chapter 1

General Results

In this chapter, we introduce our notation and present the problems examined in this thesis as well as some first results in a general context. We begin with some observations for finite-dimensional algebras in Section 1.1 before studying the special case of symmetric algebras in Section 1.2.

1.1 The problem for finite-dimensional algebras

In this section, we discuss our main problem in the most general setting, that is, for arbitrary finite-dimensional algebras over an algebraically closed field. We begin by introducing the principal structures investigated in this thesis and give some first examples in Section 1.1.1. Subsequently, we state an equivalent condition for our main problem which will be used throughout this thesis (see Section 1.1.2). In the third part of this section, we move to a first universally applicable result by investigating the problem for the tensor product of two algebras (see Section 1.1.3).

Throughout, F is assumed to be an algebraically closed field and A denotes a finite-dimensional F-algebra. Note that the left and right ideals as well as the subalgebras of A are F-vector spaces. We write $F\{a_1, \dots, a_n\}$ for the F-vector space spanned by the elements $a_1, \dots, a_n \in A$. All F-algebras occurring in this thesis are assumed to be unitary.

1.1.1 Jacobson radical, socle and Reynolds ideal

The aim of this section is to define the structures studied in this thesis and to discuss some of their properties. As announced in the introduction, our main problem is the following:

Question 1.1. *Under which conditions is the socle* $\operatorname{soc}(Z(A))$ *of the center* $Z(A)$ *an ideal of* A*?*

Moreover, it sometimes makes sense to study the following related question:

Question 1.2. *Under which conditions is the Jacobson radical* $J(Z(A))$ *of* $Z(A)$ *an ideal of* A*?*

We now introduce the notions arising in the above questions. We begin with the *Jacobson radical* $J(A)$ of A, which is one of the most important subspaces of an algebra and plays a prominent role in many ring-theoretic results. It is defined as the intersection of the annihilators of all simple left A-modules, that is, we have

$$J(A) = \{a \in A \colon aS = 0 \text{ for every simple left } A\text{-module } S\}.$$

In the following lemma, we collect some properties of $J(A)$. Note that in this thesis, an *ideal* I of A, denoted by $I \trianglelefteq A$, is meant to be a two-sided ideal of A. The ideal I is called *nilpotent* if $I^n = 0$ holds for some $n \in \mathbb{N}$. With this, we have the following identities:

Lemma 1.3.

(i) $J(A)$ is the intersection of the annihilators of all simple right A-modules.

(ii) $J(A)$ is the intersection of all maximal left ideals of A.

(iii) $J(A)$ is the intersection of all maximal right ideals of A.

(iv) $J(A)$ is a nilpotent ideal of A, and any nilpotent ideal of A is contained in $J(A)$.

Proof. The proof of (i) – (iii) is given in [41, Theorem 1.10.6], the proof of (iv) can be found in [47, Corollary I.3.4 and Lemma I.3.5]. □

As customary, let

$$Z(A) = \{z \in A\colon az = za \text{ for all } a \in A\}$$

denote the center of A. Note that $Z(A)$ is a subalgebra of A (see [47, page 368]) and hence it makes sense to consider its Jacobson radical $J(Z(A))$, which is an ideal in $Z(A)$ by the preceding lemma. In particular, it is closed under addition, so Question 1.2 is equivalent to asking whether $A \cdot J(Z(A)) \subseteq J(Z(A))$ holds. By [41, Theorem 1.10.8], we have

$$J(Z(A)) = J(A) \cap Z(A). \tag{1.1}$$

As explained at the beginning, the main focus of this thesis lies on the socle of the center of an algebra instead of its Jacobson radical, although these notions are related. By $\operatorname{soc}(A)$, we denote the *socle* of the regular left A-module, that is, the sum of all simple left ideals of A. In the literature, this is usually referred to as the *left socle* of A. Analogously, one can define the *right socle* $\operatorname{rsoc}(A)$ of A as the socle of the regular right A-module. Although both the left and the right socle are ideals of A, they do not necessarily coincide. For the majority of the algebras studied in this thesis, however, the left and the right socle agree, so the ambiguity in the definition of $\operatorname{soc}(A)$ does not make a difference. In particular, this holds for the socle of the center of A. Again, note that $\operatorname{soc}(Z(A))$ is an ideal in A if and only if $A \cdot \operatorname{soc}(Z(A)) \subseteq \operatorname{soc}(Z(A))$ holds.

We nearly exclusively use an alternative characterization of the socle in terms of annihilators, which is relatively amenable to computation. For a subset $X \subseteq A$, we define the *left annihilator* of X in A by

$$\operatorname{lAnn}_A(X) = \{a \in A \mid ax = 0 \text{ for all } x \in X\}.$$

Similarly, the right annihilator of X in A is defined by

$$\operatorname{rAnn}_A(X) = \{a \in A \mid xa = 0 \text{ for all } x \in X\}.$$

If the two sets coincide (for instance, if A is commutative), we simply write

$$\operatorname{Ann}_A(X) \coloneqq \operatorname{lAnn}_A(X) = \operatorname{rAnn}_A(X).$$

By [41, Theorem 1.10.22], the socles of A and $Z(A)$ can be expressed as the (right) annihilators of the corresponding Jacobson radicals, that is, we have

$$\mathrm{soc}(A) = \mathrm{rAnn}_A(J(A)) \text{ and } \mathrm{soc}(Z(A)) = \mathrm{Ann}_{Z(A)}(J(Z(A))). \tag{1.2}$$

In this sense, the notion of the socle is therefore dual to that of the Jacobson radical.

The third important subspace of A which we investigate is the *Reynolds ideal* $R(A)$. It is defined by $R(A) := Z(A) \cap \mathrm{soc}(A)$, where $\mathrm{soc}(A)$ denotes the (left) socle of A as before. Again, this ambiguity in the definition does not play a role since we have

$$R(A) = \{z \in Z(A) \colon J(A) \cdot z = 0\} = \{z \in Z(A) \colon z \cdot J(A) = 0\} = Z(A) \cap \mathrm{rsoc}(A).$$

Note that $R(A)$ is contained in $\mathrm{soc}(Z(A))$ since it annihilates $J(Z(A)) = J(A) \cap Z(A)$. Analogously to $J(Z(A))$ and $\mathrm{soc}(Z(A))$, the Reynolds ideal $R(A)$ is an ideal of $Z(A)$. In order to answer Questions 1.1 and 1.2, it is useful to consider the corresponding problem for the Reynolds ideal:

Question 1.4. *Under which conditions is the Reynolds ideal $R(A)$ an ideal of A?*

The relation between Questions 1.1, 1.2 and 1.4 will be investigated in the next section. In the remainder of this part, we treat the first two problems in special cases, beginning with some concrete examples:

Example 1.5.

(i) If A is a commutative algebra, we have $Z(A) = A$ and hence both $J(Z(A)) = J(A)$ and $\mathrm{soc}(Z(A)) = \mathrm{soc}(A)$ are ideals of A.

(ii) Let $A = F[X]/\langle X^2 \rangle$ be a quotient of the polynomial ring $F[X]$ and set x to be the image of X in A. Since x is nilpotent, we have $\langle x \rangle \subseteq J(A)$. On the other hand, $A/\langle x \rangle \cong F$ is simple and hence we obtain $J(A) = \langle x \rangle$. Moreover, note that $J(A)^2 = 0$ holds, which yields $J(A) \subseteq \mathrm{soc}(A)$ and hence $J(A) = \mathrm{soc}(A)$. Now consider the matrix algebra $M := \mathrm{Mat}_2(A)$ of 2×2-matrices with entries in A. Its center

$$Z(M) = \{a \cdot \mathbb{1} \colon a \in A\} \cong A$$

consists of scalar multiples of the identity matrix. By the above, we have $\mathrm{soc}(Z(M)) = J(Z(M)) \cong J(A)$. Clearly, $J(Z(M))$ is not closed under multiplication with elements of M: For instance, we take the matrix $x \cdot \mathbb{1} \in J(Z(M))$ and consider the product

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \notin J(Z(M)).$$

This shows that $J(Z(M)) = \mathrm{soc}(Z(M))$ is not an ideal of M. ◁

A second class of examples for which Questions 1.1 and 1.2 can be easily answered are semisimple algebras. Recall that a finite-dimensional F-algebra A is called *semisimple* if $J(A) = 0$ holds. This is equivalent to A being isomorphic to a direct product of simple F-algebras (see [47, Corollary I.6.6]).

Remark 1.6. Let A be a finite-dimensional semisimple F-algebra. Then also the subalgebra $Z(A)$ is semisimple since $J(Z(A)) = J(A) \cap Z(A) = 0$ holds by (1.1). In particular, $J(Z(A))$ is an ideal of A. Moreover, we obtain $\operatorname{soc}(Z(A)) = Z(A)$ and hence $\operatorname{soc}(Z(A))$ is an ideal of A if and only if A is commutative. ◁

We have now answered Questions 1.1 and 1.2 in some example cases. Before beginning a systematic analysis of these problems for various classes of algebras, however, we need a better criterion to determine whether a given subspace $I \subseteq Z(A)$ is an ideal of A. This leads us to the definition of commutator spaces.

1.1.2 Commutator space

As before, let A be a finite-dimensional F-algebra. First, we derive a criterion to determine whether $J(Z(A))$ or $\operatorname{soc}(Z(A))$ are ideals of A. It is applied in the second part of this section in order to study the relation between Questions 1.1, 1.2 and 1.4.

For elements $a_1, a_2 \in A$, we define their *commutator* $[a_1, a_2] := a_1a_2 - a_2a_1$. For two F-subspaces A_1, A_2 of A, we set

$$[A_1, A_2] := F\{[a_1, a_2] \colon a_1 \in A_1,\ a_2 \in A_2\}.$$

In particular, $K(A) := [A, A]$ denotes the *commutator subspace* of A. Note that $K(A)$ is a $Z(A)$-submodule, but not necessarily an ideal of A. In the following, we collect some useful properties of commutator spaces. We begin with a result on the *commutator ideal* of A, that is, the ideal of A generated by $K(A)$:

Lemma 1.7. *We have $A \cdot K(A) = K(A) \cdot A$, and this is the smallest ideal I of A such that A/I is commutative.*

Proof. For arbitrary elements $a, b, c \in A$, we obtain

$$a[b, c] = a(bc - cb) = abc - bac + bac - acb = [a, b]c + [b, ac] \in K(A) \cdot A$$

and hence we have $A \cdot K(A) \subseteq K(A) \cdot A$ by linearity. The other inclusion follows by symmetry. Let I be an ideal of A such that the quotient algebra A/I is commutative. This implies $0 = [a + I, b + I] = [a, b] + I$ and hence $[a, b] \in I$ for all $a, b \in A$. Hence we have $K(A) \subseteq I$, which yields $A \cdot K(A) \subseteq I$ since I is an ideal of A. On the other hand, the algebra $A/A \cdot K(A)$ is commutative, so the claim follows. □

A main step in the derivation in the subsequent chapters will be the transition to quotient algebras. Their commutator spaces are simply the images of $K(A)$:

Lemma 1.8 ([36, Equation (3)])**.** *For any ideal I of A, we have $K(A/I) = K(A) + I/I$.*

The following criterion relates the problems given in Questions 1.1 and 1.2 to the question under which conditions an ideal of A which is contained in $Z(A)$ annihilates the commutator space $K(A)$:

Lemma 1.9 ([37, Lemma 2.1])**.**

(i) If $I \subseteq Z(A)$ is an ideal of A, then $I \cdot K(A) = 0$ holds.

(ii) $J(Z(A))$ *is an ideal of* A *if and only if* $J(Z(A)) \cdot K(A) = 0$ *holds.*

(iii) $\operatorname{soc}(Z(A))$ *is an ideal of* A *if and only if* $\operatorname{soc}(Z(A)) \cdot K(A) = 0$ *holds.*

Proof.

(i) Let $I \subseteq Z(A)$ be an ideal of A. For $z \in I$ and arbitrary elements $a, b \in A$, we have $za \in I \subseteq Z(A)$, which yields $z[a,b] = z(ab - ba) = za \cdot b - b \cdot za = za \cdot b - za \cdot b = 0$. This implies $I \cdot K(A) = 0$ by linearity.

(ii) If $J(Z(A))$ is an ideal of A, then $J(Z(A)) \cdot K(A) = 0$ follows by (i). Now assume that $J(Z(A)) \cdot K(A) = 0$ holds and consider elements $j \in J(Z(A))$ and $a \in A$. For any $b \in A$, we obtain $j[a,b] = j(ab - ba) = 0$, which yields $ja \cdot b = jba = b \cdot ja$ due to $j \in Z(A)$. This implies $aj = ja \in Z(A) \cap J(A) = J(Z(A))$, which shows that $J(Z(A))$ is an ideal of A.

(iii) If $\operatorname{soc}(Z(A))$ is an ideal of A, then $\operatorname{soc}(Z(A))$ annihilates $K(A)$ by (i). Now assume that $\operatorname{soc}(Z(A)) \cdot K(A) = 0$ holds. For $s \in \operatorname{soc}(Z(A))$ and elements $a, b \in A$, we have $0 = s(ab - ba) = sa \cdot b - b \cdot sa$, which yields $sa = as \in Z(A)$. For $j \in J(Z(A))$, we obtain $j \cdot sa = js \cdot a = 0$ and hence $sa \in \operatorname{Ann}_{Z(A)}(J(Z(A))) = \operatorname{soc}(Z(A))$ follows. This shows that $\operatorname{soc}(Z(A))$ is an ideal of A. □

Remark 1.10. More precisely, for a single element $z \in J(Z(A))$, the proof of the preceding lemma shows that $Az \subseteq Z(A)$ is equivalent to $K(A) \cdot z = 0$. ◁

As a first example, we apply the criterion given in Lemma 1.9 to a special class of basic algebras. Recall that A is called *basic* if $A/J(A)$ is a direct product of division algebras. Since we assume F to be algebraically closed, each of these division algebras is isomorphic to F. Hence A being basic is equivalent to $A/J(A)$ being commutative in our situation.

Example 1.11. Let A be a basic F-algebra with $K(A) \subseteq Z(A)$. Since $A/J(A)$ is commutative, this yields $K(A) \subseteq J(A) \cap Z(A) = J(Z(A))$ by Lemma 1.7 and (1.1). Then we obtain

$$\operatorname{soc}(Z(A)) \cdot K(A) \subseteq \operatorname{soc}(Z(A)) \cdot J(Z(A)) = 0$$

and hence $\operatorname{soc}(Z(A))$ is an ideal of A by Lemma 1.9. ◁

In the remainder of this section, we study the relation between Questions 1.1, 1.2 and 1.4. The next result shows that the condition $R(A) \trianglelefteq A$ is weaker than $\operatorname{soc}(Z(A)) \trianglelefteq A$:

Lemma 1.12. *If* $\operatorname{soc}(Z(A)) \trianglelefteq A$ *holds, then also* $R(A) \trianglelefteq A$ *follows.*

Proof. Let $\operatorname{soc}(Z(A))$ be an ideal of A. Clearly, $R(A)$ is a subspace of A contained in $\operatorname{soc}(Z(A))$. For $a \in A$ and $r \in R(A)$, we obtain $ra \in \operatorname{soc}(A)$ since $\operatorname{soc}(A) \trianglelefteq A$ holds, and $ra \in \operatorname{soc}(Z(A)) \subseteq Z(A)$ since $\operatorname{soc}(Z(A))$ is an ideal of A. Hence $ar = ra$ is contained in $\operatorname{soc}(A) \cap Z(A) = R(A)$, which shows that $R(A)$ is an ideal of A. □

In contrast, there is no immediate relation between Questions 1.1 and 1.2 in general, as the following example demonstrates:

Example 1.13.

(i) Assume that F is a field of characteristic $p = 3$ and let $A = F\langle X_1, X_2, X_3\rangle$ be the free algebra in variables X_1, X_2, X_3. We consider the quotient algebra

$$Q := A/\langle X_i^3, X_iX_j + X_jX_i\rangle_{i,j=1,2,3,\ i\neq j}.$$

This is an example of a quantum complete intersection algebra, which we investigate in Section 5.2. One can show that the set $\{x_1^{r_1}x_2^{r_2}x_3^{r_3} : r_1, r_2, r_3 \in \{0,1,2\}\}$ forms an F-basis of Q, where x_i denotes the image of X_i in Q for $i = 1, 2, 3$. With this, a short direct computation shows

$$\operatorname{soc}(Z(Q)) = F\{x_1x_2^2x_3^2, x_1^2x_2x_3^2, x_1^2x_2^2x_3, x_1^2x_2^2x_3^2\} \trianglelefteq Q.$$

Moreover, it is easily verified that $x_1^2 \in J(Z(Q))$ holds. However, the element $x_1^2x_2$ is not contained in $Z(Q)$ since we have $(x_1^2x_2)x_3 = -x_3(x_1^2x_2) \neq x_3(x_1^2x_2)$, and hence $J(Z(Q))$ is not an ideal of Q. Summarizing, Q is an example of an algebra in which $\operatorname{soc}(Z(Q))$ is an ideal, whereas $J(Z(Q))$ is not.

(ii) Now let F be an arbitrary algebraically closed field and consider a non-commutative semisimple F-algebra M. For instance, let $M = \operatorname{Mat}_n(F)$ for $n > 1$ be the matrix algebra of $n \times n$-matrices over F. By Remark 1.6, we obtain $J(Z(M)) \trianglelefteq M$ and $\operatorname{soc}(Z(M)) \not\trianglelefteq M$. ◁

We now introduce a class of algebras for which the condition $J(Z(A)) \trianglelefteq A$ implies $\operatorname{soc}(Z(A)) \trianglelefteq A$. Recall that the algebra A is called *local* if $A/J(A)$ is one-dimensional. In this case, every element in $A\backslash J(A)$ is invertible (see [47, Lemma I.3.8]). Note that every local algebra is basic. Moreover, the center of a local algebra is local as well.

Remark 1.14. Let A be a local F-algebra and assume $\dim A \geq 2$, that is, we have $J(A) \neq 0$. If $n \in \mathbb{N}$ is minimal with $J(A)^n = 0$, then $J(A)^{n-1}$ is contained in $J(Z(A))$, which implies $J(Z(A)) \neq 0$. It follows that $\operatorname{soc}(Z(A))$ is a proper (nilpotent) ideal of $Z(A)$, which yields $\operatorname{soc}(Z(A)) \subseteq J(Z(A))$. ◁

This implies:

Lemma 1.15. *Let A be a local F-algebra. If $J(Z(A))$ is an ideal of A, then also $\operatorname{soc}(Z(A)) \trianglelefteq A$ follows.*

Proof. If $A \cong F$ holds, then the claim follows from Remark 1.6, so assume $\dim A \geq 2$ in the following. By Remark 1.14 and Lemma 1.9, we have

$$\operatorname{soc}(Z(A)) \cdot K(A) \subseteq J(Z(A)) \cdot K(A) = 0.$$

Hence $\operatorname{soc}(Z(A))$ is an ideal of A by Lemma 1.9. □

Summarizing, we obtain the following chain of implications for finite-dimensional local F-algebras:

$$(J(Z(A)) \trianglelefteq A) \Rightarrow (\operatorname{soc}(Z(A)) \trianglelefteq A) \Rightarrow (R(A) \trianglelefteq A).$$

The second implication holds for arbitrary finite-dimensional F-algebras (see Lemma 1.12).

1.1.3 Tensor products

After having collected some background information on Questions 1.1 and 1.2, we now present some first results. In this section, we consider the tensor product of two finite-dimensional (unitary) F-algebras A_1 and A_2. We prove that the socle of $Z(A_1 \otimes A_2)$ is an ideal in $A_1 \otimes A_2$ if and only if the socle of $Z(A_i)$ is an ideal in A_i for $i = 1, 2$, respectively. Later, this will be applied to the group algebras of direct products of subgroups. The corresponding statement for the Jacobson radical of the center does not hold.

All occurring tensor products are taken over the field F. It is well-known that the center of $A_1 \otimes A_2$ is given by $Z(A_1 \otimes A_2) = Z(A_1) \otimes Z(A_2)$. For the Jacobson radical, [41, Theorem 1.16.15] yields

$$J(A_1 \otimes A_2) = A_1 \otimes J(A_2) + J(A_1) \otimes A_2. \tag{1.3}$$

We first derive a corresponding formula for the socle of a tensor product of algebras.

Lemma 1.16. *We have* $\mathrm{soc}(A_1 \otimes A_2) = \mathrm{soc}(A_1) \otimes \mathrm{soc}(A_2)$.

Proof. The equality trivially holds if one of the algebras A_1, A_2 is zero. In the following, we therefore assume $A_1 \neq 0 \neq A_2$. We first show $\mathrm{soc}(A_1) \otimes A_2 = \mathrm{rAnn}_{A_1 \otimes A_2}(J(A_1) \otimes A_2)$. To this end, let $s_1 \in \mathrm{soc}(A_1)$, $a_2, a_2' \in A_2$ and $j_1 \in J(A_1)$. Then we have

$$(j_1 \otimes a_2') \cdot (s_1 \otimes a_2) = j_1 s_1 \otimes a_2' a_2 = 0 \otimes a_2' a_2 = 0,$$

so by linearity, $\mathrm{soc}(A_1) \otimes A_2$ annihilates $J(A_1) \otimes A_2$ from the right. For the other inclusion, we choose F-bases $\{v_1, \ldots, v_{n_1}\}$ of A_1 and $\{w_1, \ldots, w_{n_2}\}$ of A_2 for some $n_1, n_2 \in \mathbb{N}$. Then the elements in $\{v_i \otimes w_k \colon i = 1, \ldots, n_1,\ k = 1, \ldots, n_2\}$ form an F-basis of $A_1 \otimes A_2$. Now consider an element $x \in \mathrm{rAnn}(J(A_1) \otimes A_2)$ and write $x := \sum_{i,k} \lambda_{ik}(v_i \otimes w_k)$ with $\lambda_{ik} \in F$ ($i = 1, \ldots, n_1$, $k = 1, \ldots, n_2$). For any $j_1 \in J(A_1)$, we obtain

$$0 = (j_1 \otimes 1) \cdot x = \sum_{i,k} \lambda_{ik}(j_1 v_i \otimes w_k) = \sum_k \left(\sum_i \lambda_{ik} j_1 v_i \right) \otimes w_k.$$

For $k = 1, \ldots, n_2$, this yields $0 = \sum_i \lambda_{ik} j_1 v_i = j_1 \cdot \sum_i \lambda_{ik} v_i$. We obtain $\sum_i \lambda_{ik} v_i \in \mathrm{soc}(A_1)$, which implies $x \in \mathrm{soc}(A_1) \otimes A_2$. Analogously, we show

$$A_1 \otimes \mathrm{soc}(A_2) = \mathrm{rAnn}_{A_1 \otimes A_2}(A_1 \otimes J(A_2)).$$

With this, we obtain

$$\begin{aligned} \mathrm{soc}(A_1 \otimes A_2) &= \mathrm{rAnn}_{A_1 \otimes A_2}(J(A_1) \otimes A_2 + A_1 \otimes J(A_2)) \\ &= \mathrm{rAnn}_{A_1 \otimes A_2}(J(A_1) \otimes A_2) \cap \mathrm{rAnn}_{A_1 \otimes A_2}(A_1 \otimes J(A_2)) \\ &= (\mathrm{soc}(A_1) \otimes A_2) \cap (A_1 \otimes \mathrm{soc}(A_2)) \\ &= \mathrm{soc}(A_1) \otimes \mathrm{soc}(A_2). \end{aligned}$$

□

Lemma 1.17. *Assume that both A_1 and A_2 are nonzero. The socle* $\mathrm{soc}(Z(A_1 \otimes A_2))$ *is an ideal of $A_1 \otimes A_2$ if and only if* $\mathrm{soc}(Z(A_i))$ *is an ideal of A_i for $i = 1, 2$.*

Proof. Let $n_1, n_2 \in \mathbb{N}$ denote the dimensions of A_1 and A_2, respectively. Assume first that $\operatorname{soc}(Z(A_i)) \trianglelefteq A_i$ holds for $i = 1, 2$. By Lemma 1.16, we have $\operatorname{soc}(Z(A_1 \otimes A_2)) = \operatorname{soc}(Z(A_1)) \otimes \operatorname{soc}(Z(A_2))$ and

$$(a_1 \otimes a_2) \cdot (\operatorname{soc}(Z(A_1)) \otimes \operatorname{soc}(Z(A_2))) \subseteq \operatorname{soc}(Z(A_1)) \otimes \operatorname{soc}(Z(A_2))$$

holds for all $a_1, a_2 \in A$ by assumption. This shows that $\operatorname{soc}(Z(A_1 \otimes A_2))$ is an ideal of $A_1 \otimes A_2$.

Now assume conversely that $\operatorname{soc}(Z(A_1 \otimes A_2))$ is an ideal of $A_1 \otimes A_2$. We choose F-bases $\{v_1, \ldots, v_{n_1}\}$ of A_1 and $\{w_1, \ldots, w_{n_2}\}$ of A_2 such that $\{v_1, \ldots, v_{k_1}\}$ and $\{w_1, \ldots, w_{k_2}\}$ form a basis of $\operatorname{soc}(Z(A_1))$ and $\operatorname{soc}(Z(A_2))$ for some $k_i \in \{1, \ldots, n_i\}$ $(i = 1, 2)$, respectively. Note that $\operatorname{soc}(Z(A_i)) \neq 0$ holds since A_i is nonzero. The set $\{v_{j_1} \otimes w_{j_2} : 1 \leq j_i \leq k_i \text{ for } i = 1, 2\}$ forms an F-basis for $\operatorname{soc}(Z(A_1)) \otimes \operatorname{soc}(Z(A_2)) = \operatorname{soc}(Z(A_1 \otimes A_2))$.

We show that $a_1 v_i \in \operatorname{soc}(Z(A_1))$ holds for all $a_1 \in A_1$ and $i = 1, \ldots, k_1$. To this end, we set $a := a_1 \otimes 1 \in A_1 \otimes A_2$ and $v := v_i \otimes w_1$. Since $av \in \operatorname{soc}(Z(A_1 \otimes A_2))$ holds by assumption, there exist coefficients $\lambda_{rt} \in F$ for $1 \leq r \leq k_1$, $1 \leq t \leq k_2$ with

$$av = \sum_{r,t} \lambda_{rt} v_r \otimes w_t.$$

Expressing $a_1 v_i = \sum_{d=1}^{n_1} \mu_d v_d$ in terms of the basis of A_1 (with $\mu_1, \ldots, \mu_{n_1} \in F$) yields

$$av = (a_1 \otimes 1) \cdot (v_i \otimes w_1) = a_1 v_i \otimes w_1 = \left(\sum_{d=1}^{n_1} \mu_d v_d \right) \otimes w_1 = \sum_{d=1}^{n_1} \mu_d (v_d \otimes w_1).$$

By comparing the coefficients in the two expressions for av, we obtain $\mu_d = 0$ for $d > k_1$. This shows that $a_1 v_i$ is contained in $\operatorname{soc}(Z(A_1))$, which proves that this space is an ideal of A_1. For $\operatorname{soc}(Z(A_2))$, we proceed analogously. □

In the following, we may therefore restrict our investigation to individual factors of a tensor product of F-algebras.

1.2 Symmetric algebras

In this section, we consider the problem under which conditions $J(Z(A))$ or $\operatorname{soc}(Z(A))$ are ideals of A in the special situation where A is a symmetric F-algebra. This is of interest since symmetric algebras naturally arise in various contexts in representation theory. On the other hand, the requirement that A is symmetric provides a considerable amount of information on its structure.

Group algebras, which form a special class of symmetric algebras, will be extensively studied in the following chapters. Moreover, we investigate symmetric local algebras of small dimension in Chapter 5. Therefore, we only focus on two aspects of our main problem in this section: After introducing the necessary theoretical background, we investigate the transition to quotients of symmetric algebras in Section 1.2.2. In Section 1.2.3, we provide an answer to Question 1.4 in this special situation.

1.2.1 Theoretical background

In this part, we introduce symmetric algebras and discuss some of their properties.

Let A be a finite-dimensional F-algebra and consider a bilinear form $\beta\colon A \times A \to F$. Recall that β is called *non-degenerate* if $\beta(a,b) = 0$ for all $b \in A$ and a fixed element $a \in A$ implies $a = 0$. The bilinear form is called *associative* if $\beta(ab,c) = \beta(a,bc)$ holds for any $a,b,c \in A$. Finally, we say that β is *symmetric* if $\beta(a,b) = \beta(b,a)$ holds for all $a,b \in A$. Now the algebra A is called *symmetric* if it admits a non-degenerate associative symmetric bilinear form $\beta\colon A \times A \to F$. Instead of β, we sometimes consider the associated linear form $\lambda\colon A \to F,\ a \mapsto \beta(1,a)$. In this situation, the kernel of λ contains the commutator space $K(A)$, but no nonzero left or right ideal of A (see [47, Theorem IV.2.2]). Note that the class of finite-dimensional symmetric F-algebras is closed under Morita equivalence (see [47, Corollary IV.4.3]).

Example 1.18. Semisimple F-algebras form a prominent class of examples of symmetric algebras (see [47, Proposition IV.2.4 and Example IV.2.5]). Moreover, we will see in Section 2.1 that group algebras of finite groups over fields are symmetric. ◁

For a subspace X of the symmetric algebra A, we denote by $X^\perp$ its orthogonal space with respect to the bilinear form β. That is, we have

$$X^\perp = \{a \in A\colon \beta(a,x) = 0 \text{ for all } x \in X\}.$$

The following properties are well-known:

Lemma 1.19 ([36, Equations (28) – (32)])**.** *Let A be a symmetric F-algebra and consider subspaces X and Y of A. Then we have the following properties:*

(i) $\dim X + \dim X^\perp = \dim A$.

(ii) $(X^\perp)^\perp = X$.

(iii) $Y \subseteq X$ *implies* $X^\perp \subseteq Y^\perp$.

(iv) We have $(X \cap Y)^\perp = X^\perp + Y^\perp$ *and* $(X + Y)^\perp = X^\perp \cap Y^\perp$.

(v) For an ideal I of A, we have $I^\perp = \mathrm{lAnn}(I) = \mathrm{rAnn}(I)$. In particular, $I^\perp$ is an ideal of A as well.

By the last statement, it follows that the left and right socle of A, that is, the right and left annihilator of $J(A)$, coincide. Moreover, we obtain $J(A) = \mathrm{soc}(A)^\perp$. By [31, Lemma A], the orthogonal space of $K(A)$ is given by $K(A)^\perp = Z(A)$.

1.2.2 Quotients of symmetric algebras

Throughout, let A be a finite-dimensional symmetric F-algebra with symmetrizing linear form $\lambda\colon A \to F$. In this part, we consider various quotient algebras of A. Our main result is the observation that the properties $J(Z(A)) \trianglelefteq A$ and $\mathrm{soc}(Z(A)) \trianglelefteq A$ are inherited by symmetric quotient algebras of A.

We first prove a criterion for $J(Z(A)) \trianglelefteq A$ and $\mathrm{soc}(Z(A)) \trianglelefteq A$ in terms of the commutator spaces of certain quotient algebras:

Lemma 1.20.

(i) $J(Z(A)) \trianglelefteq A$ *holds if and only if* $K(\bar{A})$ *is an ideal of* $\bar{A} \coloneqq A/\operatorname{soc}(A)$.

(ii) $\operatorname{soc}(Z(A)) \trianglelefteq A$ *holds if and only if* $K(\bar{A})$ *is an ideal of* $\bar{A} \coloneqq A/A \cdot J(Z(A))$.

Proof. By Lemma 1.19, $J(Z(A))$ is an ideal of A if and only if $J(Z(A))^{\perp}$ is. Note that

$$J(Z(A))^{\perp} = (J(A) \cap Z(A))^{\perp} = J(A)^{\perp} + Z(A)^{\perp} = \operatorname{soc}(A) + K(A)$$

follows by (1.1) and Lemma 1.19. We therefore have $A \cdot J(Z(A))^{\perp} \cdot A \subseteq J(Z(A))^{\perp}$ if and only if $A \cdot K(A) \cdot A \subseteq K(A) + \operatorname{soc}(A)$ holds since $\operatorname{soc}(A)$ is an ideal of A. Setting $\bar{A} \coloneqq A/\operatorname{soc}(A)$, this is equivalent to $K(\bar{A}) = K(A) + \operatorname{soc}(A)/\operatorname{soc}(A)$ being an ideal of $\bar{A}$ (see Lemma 1.8). Similarly, $\operatorname{soc}(Z(A))$ is an ideal of A if and only if $\operatorname{soc}(Z(A))^{\perp}$ is. Note that

$$\operatorname{Ann}(J(Z(A))) = \operatorname{Ann}(A \cdot J(Z(A))) = (A \cdot J(Z(A)))^{\perp}$$

follows by Lemma 1.19 (v) and hence we have

$$\operatorname{soc}(Z(A)) = Z(A) \cap \operatorname{Ann}(J(Z(A)) = Z(A) \cap (A \cdot J(Z(A)))^{\perp}.$$

This implies

$$\operatorname{soc}(Z(A))^{\perp} = Z(A)^{\perp} + A \cdot J(Z(A)) = K(A) + A \cdot J(Z(A)).$$

Hence $\operatorname{soc}(Z(A))$ is an ideal of A if and only if $K(A) + A \cdot J(Z(A))$ is, which is again equivalent to $K(\bar{A})$ being an ideal of $\bar{A} \coloneqq A/A \cdot J(Z(A))$. □

Remark 1.21. By Lemma 1.19 (v), $K(A)$ is an ideal of A if and only if $K(A)^{\perp} = Z(A)$ is. Since A is unitary, the latter is the case if and only if A is commutative, that is, if we have $K(A) = 0$. If $J(Z(A))$ or $\operatorname{soc}(Z(A))$ are ideals of A, then the corresponding algebras $\bar{A}$ defined in the previous result are therefore either commutative or non-symmetric. ◁

From now on, we study quotient algebras of A which are again symmetric. We emphasize that this is an additional condition which is not satisfied for arbitrary quotients of A. In fact, our first result shows that requiring the quotient algebra A/I to be symmetric forces the ideal I to be of a specific shape.

Lemma 1.22. *Let* I *be an ideal of* A *such that* A/I *is symmetric with corresponding linear form* $\bar{\lambda}$. *Then there exists an element* $z \in Z(A)$ *with* $I = (Az)^{\perp}$ *such that* $\bar{\lambda}(a+I) = \lambda(az)$ *holds for all* $a \in A$. *Conversely, for any* $z \in Z(A)$, *the algebra* $A/(Az)^{\perp}$ *is symmetric with respect to a linear form* $\bar{\lambda}$ *of the above form.*

Proof. The proof is given in [36, pages 429 – 430]. □

With this characterization, we can simplify the criterion for $J(Z(A)) \trianglelefteq A$ in case that A is a symmetric local algebra.

Lemma 1.23. *Let* A *be a symmetric local algebra. Then* $J(Z(A)) \trianglelefteq A$ *holds if and only if for all ideals* $0 \neq I \trianglelefteq A$ *such that* A/I *is symmetric, it follows that* A/I *is commutative.*

Proof. First assume $J(Z(A)) \trianglelefteq A$ and let I be a nonzero ideal of A such that the quotient algebra A/I is symmetric. The ideal $I^\perp$ is contained in $J(A)$, which yields $\operatorname{soc}(A) \cdot I^\perp = 0$ and hence $\operatorname{soc}(A) \subseteq (I^\perp)^\perp = I$. The algebra A/I can therefore be interpreted as a quotient of $\bar{A} := A/\operatorname{soc}(A)$. By Lemma 1.20, we have $K(\bar{A}) \trianglelefteq \bar{A}$ and hence $K(A/I) \trianglelefteq A/I$ follows since $K(A/I)$ is the image of $K(\bar{A})$ under the quotient map. Since A/I is symmetric, this is only possible if $K(A/I) = 0$ holds (see Remark 1.21), so if A/I is commutative.

Conversely, assume that $K(A/I) = 0$ holds for every ideal $0 \neq I \trianglelefteq A$ for which the quotient algebra A/I is symmetric. By Lemma 1.22, such an ideal is of the form $(Az)^\perp$ with $z \in J(Z(A))$. Lemma 1.8 then yields

$$K(A) \subseteq \bigcap_{z \in J(Z(A))} (Az)^\perp = \left(\sum_{z \in J(Z(A))} Az \right)^\perp = (A \cdot J(Z(A)))^\perp$$

and hence we obtain $J(Z(A)) \cdot K(A) \subseteq J(Z(A)) \cdot (A \cdot J(Z(A)))^\perp = 0$. By Lemma 1.9, this implies that $J(Z(A))$ is an ideal of A. □

Now we return to the assumption that A is a (not necessarily local) symmetric algebra. In the following, we consider an ideal I of A for which the quotient algebra $\bar{A} := A/I$ is symmetric with a corresponding bilinear form $\bar{\beta}\colon \bar{A} \times \bar{A} \to F$ and symmetrizing linear form $\bar{\lambda}\colon \bar{A} \to F$. In the remaining part of this section, we prove that the properties $J(Z(A)) \trianglelefteq A$ and $\operatorname{soc}(Z(A)) \trianglelefteq A$ are inherited by the quotient algebra $\bar{A}$.

By Lemma 1.22, the ideal I is of the form $(Az)^\perp$ for some $z \in Z(A)$. Recall that the map $\bar{\lambda}$ is then given by $\bar{\lambda}(a + I) = \lambda(az)$ for all $a \in A$. In the following, we consider the canonical projection $\nu\colon A \to \bar{A},\ a \mapsto \bar{a} := a + I$ and its adjoint map $\nu^*\colon \bar{A} \to A$, which is defined by requiring $\beta(\nu^*(\bar{x}), y) = \bar{\beta}(\bar{x}, \nu(y))$ for all $x, y \in A$. This is equivalent to

$$\lambda\big(\nu^*(\bar{x}) \cdot y\big) = \bar{\lambda}(\bar{x} \cdot \bar{y}) = \lambda(xyz) = \lambda(xzy)$$

for all $x, y \in A$. Note that the right ideal $\big(\nu^*(\bar{x}) - xz\big)A$ is contained in the kernel of λ for all $x \in A$, which yields $\nu^*(\bar{x}) = xz$. For all $x, y \in A$, we then obtain

$$\nu^*(\bar{x}) \cdot y = xzy = xyz = \nu^*(\bar{x} \cdot \bar{y}) \tag{1.4}$$

and similarly

$$x \cdot \nu^*(\bar{y}) = xyz = \nu^*(\bar{x} \cdot \bar{y}). \tag{1.5}$$

Note that since ν is surjective, the adjoint map ν^* is injective.

Lemma 1.24. *The map ν^* has the following properties:*

(i) $\nu^*\big(Z(\bar{A})\big) = Z(A) \cap \operatorname{Im}(\nu^*)$.

(ii) $\nu^*\big(J(Z(\bar{A}))\big) \subseteq J(Z(A))$.

(iii) $\nu^*\big(\operatorname{soc}(Z(\bar{A}))\big) \subseteq \operatorname{soc}(Z(A))$.

Proof.

(i) Consider an element $a \in A$ with $\bar{a} \in Z(\bar{A})$ and let $b \in A$ be an arbitrary element. Since $\bar{a}$ and $\bar{b}$ commute, we obtain $ab - ba \in I$, which yields $0 = (ab-ba)z = azb - baz$. Since b was arbitrary, this shows $\nu^*(\bar{a}) = az \in Z(A) \cap \mathrm{Im}(\nu^*)$. For the other inclusion, consider an element $\bar{a} \in \bar{A}$ with $\nu^*(\bar{a}) \in Z(A)$. For every $x \in A$, (1.4) and (1.5) yield

$$\nu^*(\bar{x} \cdot \bar{a}) = x \cdot \nu^*(\bar{a}) = \nu^*(\bar{a}) \cdot x = \nu^*(\bar{a} \cdot \bar{x})$$

and hence we have $\bar{x} \cdot \bar{a} = \bar{a} \cdot \bar{x}$ since ν^* is injective. This shows $\bar{a} \in Z(\bar{A})$, which proves the equality.

(ii) Consider an element $u \in A$ with $\bar{u} \in J(Z(\bar{A}))$. By (i), we have $\nu^*(\bar{u}) \in Z(A)$. Moreover, $\bar{u}$ is nilpotent, so there exists some $n \in \mathbb{N}$ with $\bar{u}^n = 0$, which is equivalent to $u^n \in (Az)^\perp$. This yields $u^n \cdot z = 0$ and hence $\nu^*(\bar{u})^n = (uz)^n = u^n z^n = 0$, so $\nu^*(\bar{u})$ is contained in $J(A)$. By (1.1), we obtain $\nu^*(\bar{u}) \in J(A) \cap Z(A) = J(Z(A))$.

(iii) Let $s \in A$ with $\bar{s} \in \mathrm{soc}(Z(\bar{A}))$ and consider an element $u \in J(Z(A)) = J(A) \cap Z(A)$. We have $\bar{u} \in J(\bar{A}) \cap Z(\bar{A}) = J(Z(\bar{A}))$ by [41, Theorem 1.10.12]. Then (1.4) yields $\nu^*(\bar{s}) \cdot u = \nu^*(\bar{s} \cdot \bar{u}) = 0$, which shows $\nu^*(\bar{s}) \in \mathrm{soc}(Z(A))$. □

We now prove that the properties $J(Z(A)) \trianglelefteq A$ and $\mathrm{soc}(Z(A)) \trianglelefteq A$ are inherited by symmetric quotient algebras of A:

Lemma 1.25. *Let A be a symmetric algebra and consider an ideal I of A for which the quotient algebra $\bar{A} := A/I$ is symmetric.*

(i) If $J(Z(A)) \trianglelefteq A$ holds, then $J(Z(\bar{A}))$ is an ideal of $\bar{A}$.

(ii) If $\mathrm{soc}(Z(A)) \trianglelefteq A$ holds, then $\mathrm{soc}(Z(\bar{A}))$ is an ideal of $\bar{A}$.

Proof. By Lemma 1.9, the condition $J(Z(A)) \trianglelefteq A$ translates to $J(Z(A)) \cdot K(A) = 0$. Let $u, k \in A$ be elements with $\bar{u} \in J(Z(\bar{A}))$ and $\bar{k} \in K(\bar{A})$. By Lemma 1.8, we may assume $k \in K(A)$. The preceding lemma yields $\nu^*(\bar{u}) \in J(Z(A))$ and hence $\nu^*(\bar{u} \cdot \bar{k}) = \nu^*(\bar{u}) \cdot k = 0$ follows by (1.4). Since ν^* is injective, this yields $\bar{u} \cdot \bar{k} = 0$. We obtain $J(Z(\bar{A})) \cdot K(\bar{A}) = 0$, which implies $J(Z(\bar{A})) \trianglelefteq \bar{A}$ again by Lemma 1.9. The second statement can be proven in a similar way. □

Remark 1.26. Later in this thesis (in Example 5.29), we will see an example of a symmetric local algebra A which satisfies $\mathrm{soc}(Z(A)) \trianglelefteq A$ and an ideal $I \trianglelefteq A$ such that $\mathrm{soc}(Z(A/I))$ is not an ideal in A/I. Hence the assumption that the quotient algebra $\bar{A}$ is symmetric cannot be omitted in Lemma 1.25. ◁

At the end of this part, we make an observation which will be useful in the context of group algebras:

Remark 1.27. For any $a \in \bar{A}$, Lemma 1.24 yields the following equivalence:

$$\begin{aligned} \nu^*(a) \in \mathrm{soc}(Z(A)) &\Leftrightarrow a \in Z(\bar{A}) \text{ and } 0 = \nu^*(a) \cdot J(Z(A)) = \nu^*(a \cdot \nu(J(Z(A))) \\ &\Leftrightarrow a \in Z(\bar{A}) \text{ and } 0 = a \cdot \nu(J(Z(A))). \end{aligned}$$

In the last step, we used that the adjoint map ν^* is injective. ◁

1.2.3 Reynolds ideal in symmetric algebras

The following lemma answers the question under which conditions the Reynolds ideal $R(A)$ is an ideal of A in the case where A is a symmetric algebra (see Question 1.4).

Lemma 1.28. *Let A be a finite-dimensional symmetric F-algebra. Then $R(A)$ is an ideal of A if and only if A is a basic F-algebra. In this case, we have $R(A) = \operatorname{soc}(A)$.*

Proof. Assume that $R(A)$ is an ideal of A. By [37, Remark 3.1], we obtain $R(A) = A \cdot R(A) = \operatorname{soc}(A)$ and A is basic in this case. Conversely, assume that A is a basic F-algebra, which is equivalent to $A/J(A)$ being commutative by [47, Proposition II.6.19]. In particular, we have $K(A) \subseteq J(A)$ and hence $\operatorname{soc}(A) = J(A)^{\perp} \subseteq K(A)^{\perp} = Z(A)$ by Lemma 1.19. This implies that $R(A) = \operatorname{soc}(A) \cap Z(A) = \operatorname{soc}(A)$ is an ideal of A. □

At this point, it becomes clear that the property $\operatorname{soc}(Z(A)) \trianglelefteq A$ is not preserved under Morita equivalence:

Example 1.29. Let $A = F$ and let $M := \operatorname{Mat}_n(A)$ for some $n > 1$ be a matrix algebra with entries in A. Then $\operatorname{soc}(Z(A)) \trianglelefteq A$ holds since A is commutative. By [47, Lemma II.6.13], the algebras A and M are Morita equivalent, but M is not basic. By the preceding result, this implies $R(M) \ntrianglelefteq M$ and hence $\operatorname{soc}(Z(M)) \ntrianglelefteq M$ follows by Lemma 1.12. ◁

Chapter 2

General results on group algebras

We now investigate our main problem for the case of group algebras. To this end, we first introduce the necessary background on groups as well as on group algebras and rephrase the equivalent condition stated in Lemma 1.9 in this setting. In Section 2.3, we proceed with some first general results on finite groups G satisfying $\mathrm{soc}(ZFG) \trianglelefteq FG$. By investigating the Reynolds ideal of FG in Section 2.4.1, we find that these groups have a special structure, which allows us to determine a basis for the Jacobson radical $J(ZFG)$. In Section 2.5, we focus on the transition to quotient algebras and examine central products. Both concepts will be crucial for our further derivation. We conclude this part by studying the special case that G has an abelian Sylow p-subgroup in Section 2.6.

2.1 Groups and group algebras

Here, we collect some group-theoretic results which will be needed in the following chapters. We assume familiarity with the basic group-theoretic concepts and focus on the results we frequently need later on. In the second part of this section, we introduce group algebras and related notions. Most of the group-theoretical results presented in the following can be found in standard textbooks on group theory, for example [20], [24] and [30], those concerning group algebras and their blocks are contained [41], [44] and [45]. All occurring groups are assumed to be finite.

For a group G and two elements $x, y \in G$, we define the *commutator* $[x, y]$ of x and y as

$$[x, y] := xyx^{-1}y^{-1}.$$

With this convention, the following identities hold for all $x, y, z \in G$:

$$[xy, z] = x[y, z]x^{-1} \cdot [x, z]$$
$$[x, yz] = [x, y] \cdot y[x, z]y^{-1}.$$

Note that we defined two notions of commutators, which are both denoted by $[\,.\,,\,.\,]$: Additionally to the commutator of two group elements introduced above, we defined the commutator of two elements of an algebra in Section 1.1.2. Usually, it will become clear from the context which notion is meant. As customary, we set $[A, B] := \langle [a, b] \colon a \in A,\ b \in B\rangle$ for subgroups A and B of G and write $G' = [G, G]$ for the derived subgroup of G. The nilpotency class of a nilpotent group G is denoted by $c(G)$. Recall that every p-group is nilpotent. The group G is called *metabelian* if G' is abelian, that is, if the second derived subgroup $G'' := [G', G']$ is trivial. Moreover, we set $Z(G)$ to be the center of G and denote the centralizer of a subset $M \subseteq G$ in G by $C_G(M)$.

By $\mathrm{Cl}(G)$, we denote the set of conjugacy classes of G. For $g \in G$, we write $[g]_G$ for the conjugacy class of g in G, which is sometimes also called the *G-conjugacy class* of g, and omit the index if it is clear from the context to which group we refer. We write $g \sim h$ if $g, h \in G$ are conjugate elements. For any $g \in G$, we have

$$[g] = \{aga^{-1} \colon a \in G\} = \{aga^{-1}g^{-1}g \colon a \in G\} = \{[a, g]g \colon a \in G\} \subseteq G'g = gG'.$$

We define

$$U_g := \{[a, g] \colon a \in G\} \subseteq G' \tag{2.1}$$

to be the set of commutators of g with elements in G. In this notation, we have $[g] = U_g \cdot g$. The group G is called a *Camina group* if $[g] = gG'$, or equivalently $U_g = G'$, holds for all elements $g \in G \backslash G'$.

As customary, the set of prime numbers is denoted by $\mathbb{P}$. For $p \in \mathbb{P}$, we write $\mathrm{Syl}_p(G)$ for the set of Sylow p-subgroups of G. For a subset $\pi \subseteq \mathbb{P}$, $O_\pi(G)$ denotes the *π-core* of G, that is, the product of all normal π-subgroups of G. Note that $O_\pi(G)$ itself is a π-subgroup of G. As usual, we set $\pi' := \mathbb{P} \backslash \pi$. In the special case that $\pi = \{p\}$ consists of a single prime number $p \in \mathbb{P}$, we write $O_p(G) := O_{\{p\}}(G)$ as well as $O_{p'}(G) := O_{\{p\}'}(G)$ and call these groups the *p-core* and *p'-core* of G, respectively. Furthermore, we will encounter the *p', p-core* $O_{p',p}(G)$ of G, which is defined by the identity

$$O_{p',p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G)).$$

We frequently use the following special case of [30, Theorem 6.4.3]:

Theorem 2.1. *Let $p \in \mathbb{P}$ be a prime number and let G be a finite solvable group with $O_{p'}(G) = 1$. Then we have*

$$C_G(O_p(G)) \subseteq O_p(G).$$

In particular, in case that G has a normal Sylow p-subgroup P, we obtain $C_G(P) = Z(P)$.

Let $p \in \mathbb{P}$ be a prime number. An element $g \in G$ is called a *p-element* if its order is a power of p and a *p'-element* if its order is coprime to p. We denote the set of p-elements by G_p. Recall that every element $g \in G$ can be decomposed in the form $g = g_p \cdot g_{p'}$ with a p-element g_p and a p'-element $g_{p'}$ satisfying $g_p \cdot g_{p'} = g_{p'} \cdot g_p$, and this decomposition is unique. We say that $g, h \in G$ lie in the same *p'-section* of G if $g_{p'}$ is conjugate to $h_{p'}$. For an element $g \in G$, we denote its corresponding p'-section by S_g. In particular, we see that $G_p = S_1$ is a p'-section of G, which consists of all elements with trivial p'-part.

Recall that the *Frattini subgroup* $\Phi(G)$ of G is defined as the intersection of all maximal subgroups of G. For $G = 1$, we set $\Phi(G) = 1$ since G has no maximal subgroups in this case. If P is a p-group for some prime number $p \in \mathbb{P}$, then its Frattini subgroup is given by $\Phi(P) = P' \cdot P^p$, where $P^p = \langle u^p \colon u \in P \rangle$ is generated by the p-th powers of the elements in P. In particular, $P/\Phi(P)$ is elementary abelian, and $\Phi(P)$ is the smallest normal subgroup of P with an elementary abelian quotient (see [24, Satz III.3.14]).

As customary, $\mathrm{Aut}(G)$ denotes the automorphism group of G. A p'-automorphism of a p-group P is an element of $\mathrm{Aut}(P)$ of order coprime to p. With this, we can state the following important theorem, which is due to Thompson:

Theorem 2.2 ([20, Theorem 5.3.11]). *In every p-group P, there exists a characteristic subgroup C with the following properties:*

(i) $c(C) \leq 2$ *and* $C/Z(C)$ *is elementary abelian.*

(ii) $[P, C] \subseteq Z(C)$.

(iii) $C_P(C) = Z(C)$.

(iv) Every nontrivial p'-automorphism of P induces a nontrivial automorphism on C.

A group C satisfying these conditions is called a *critical subgroup* of G.

The study of the p'-automorphisms of certain p-groups forms a main ingredient of this thesis since the existence of such an automorphism inflicts a series of conditions on the structure of the p-group as well as the action of the automorphism. The following result is due to Burnside:

Theorem 2.3 ([20, Theorem 5.1.4]). *Consider a p'-automorphism α of a p-group P which induces the identity on $P/\Phi(P)$. Then α is the identity automorphism of P.*

The action of a p'-group on a p-group gives rise to the following decomposition, which we frequently use throughout this thesis:

Theorem 2.4 ([20, Theorem 5.2.3 and 5.3.5]). *Let H be a p'-group of automorphisms of the p-group P. Then we have $P = C_P(H)[P, H]$. If P is abelian, then $P = C_P(H) \times [P, H]$ holds.*

The following result allows us to relate the decomposition of P given in Theorem 2.4 to that of a quotient group:

Theorem 2.5 ([20, Theorem 5.3.15]). *Let H be a p'-group of automorphisms of a p-group P and let N be an H-invariant normal subgroup of P. Then $C_{P/N}(H)$ is the image of $C_P(H)$ in P/N.*

A group G is called a *Frobenius group* if it has a nontrivial proper subgroup H such that $H \cap gHg^{-1} = 1$ holds for all $g \in G \backslash H$. In this case, there exists a normal subgroup $K \trianglelefteq G$, the *Frobenius kernel*, such that $G \cong K \rtimes H$ holds. Furthermore, every nontrivial element of H induces an automorphism of K by conjugation which only fixes the identity element of K (see [20, Theorem 2.7.6]). The group H is called a *Frobenius complement*. It is unique up to conjugation in G whereas the Frobenius kernel is uniquely determined as the *Fitting subgroup*, that is, the product of all nilpotent subgroups of G.

We now introduce a subgroup which will play a central role in the classification of the finite 2-groups P for which $\mathrm{soc}(ZFP) \trianglelefteq FP$ holds. Let P be a finite 2-group and consider a conjugacy class $C = \{f, g\}$ of length two of P. Every inner automorphism of P either fixes both f and g, or it interchanges the two elements. For $c := gf^{-1} \in P'$, this yields $C_P(f) = C_P(g) \subseteq C_P(c)$. For any $h \in P \setminus C_P(f)$, we have

$$hch^{-1} = hgf^{-1}h^{-1} = fg^{-1} = c^{-1}.$$

This shows that the subgroup $Y_C := \langle c \rangle \subseteq P'$ is normal in P. In the following, we consider the group

$$Y(P) := \langle Y_C : C \in \mathrm{Cl}(G),\ |C| = 2 \rangle. \tag{2.2}$$

Note that $Y(P)$ is characteristic in P. Moreover, we obtain the following:

Lemma 2.6. *Let P be a finite 2-group. Then the subgroup $Y(P)$ is contained in $Z(\Phi(P))$. In particular, $Y(P)$ is abelian.*

Proof. Note that $Y(P) \subseteq P'$ is contained in the Frattini subgroup $\Phi(P)$. For any conjugacy class $C = \{f, g\}$ of length two, $C_P(f)$ is a maximal subgroup of P. By the above observation, this implies $\Phi(P) \subseteq C_P(f) \subseteq C_P(gf^{-1})$ and hence $\Phi(P)$ centralizes Y_C. Since this argument is valid for all conjugacy classes of length two, the Frattini subgroup $\Phi(P)$ centralizes $Y(P)$. Conversely, $Y(P)$ is contained in $Z(\Phi(P))$, which is abelian. □

Now we move to the concept of central products. Let G_1, G_2, Z_1, Z_2 be finite groups with $Z_i \subseteq Z(G_i)$ for $i = 1, 2$ such that there is an isomorphism $\varphi\colon Z_1 \to Z_2$. Then there exists a group G of the form $G = G_1G_2$ with $Z_1 = G_1 \cap G_2 \subseteq Z(G)$ (identifying Z_1 and Z_2 at this point) such that G_1 centralizes G_2 (see [20, Theorem 2.5.3]). We call G the *central product* of G_1 and G_2 and write $G = G_1 * G_2$. This group can also be viewed as a quotient of $G_1 \times G_2$ by a certain central subgroup. Note that G_1 and G_2 are normal subgroups of G and that its derived subgroup is given by $G' = G_1' \cdot G_2'$. Moreover, for an element $c = c_1c_2$ with $c_1 \in G_1$ and $c_2 \in G_2$, the conjugacy class $[c]$ decomposes as $[c] = [c_1] \cdot [c_2]$ since for any $g_1 \in G_1$ and $g_2 \in G_2$, it follows that

$$(g_1g_2) \cdot c \cdot (g_1g_2)^{-1} = g_1c_1g_1^{-1} \cdot g_2c_2g_2^{-1}.$$

Note that the factors c_1 and c_2 are unique up to multiplication with elements of Z_1. In particular, the lengths of the classes $C_1 := [c_1]$ and $C_2 := [c_2]$ are well-defined. More precisely, we observe that the group G_1 acts on C_1C_2 by conjugation. The orbits are of the form C_1c_2 with elements $c_2 \in G_2$. In particular, two sets of this form are either equal or disjoint. Furthermore, the group G_2 acts transitively on the set of these orbits $R = \{C_1c_2 \colon c_2 \in C_2\}$ by conjugation. This yields $|R| = |G_2 \colon N_{G_2}(C_1c_2)|$ for any $c_2 \in C_2$ and we obtain $|C_1C_2| = |R| \cdot |C_1|$. A similar argument holds for C_2.

At the end of our review of group-theoretic results, we introduce the notion of isoclinism, which will play an important role in our treatment of p-groups. Two finite groups G_1 and G_2 are *isoclinic* if there exist isomorphisms $\varphi\colon G_1' \to G_2'$ and $\beta\colon G_1/Z(G_1) \to G_2/Z(G_2)$ such that $\beta(a_1Z(G_1)) = a_2Z(G_2)$ and $\beta(b_1Z(G_1)) = b_2Z(G_2)$ for some $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$ implies $\varphi([a_1, b_1]) = [a_2, b_2]$. A group G is called a *stem group* if $Z(G)$ is contained in G'. For any finite group G, there exists a stem group H such that G and H are isoclinic (see [5, Proposition 2.6]). Moreover, we have $|H| \leq |G|$ in this case (see [5, page 134]).

Now we move to the investigation of group algebras. Recall that the group algebra FG of a finite group G over an algebraically closed field F is given as the set

$$FG = \left\{ \sum_{g \in G} a_g g \colon a_g \in F \text{ for all } g \in G \right\},$$

together with a "component-wise" addition and scalar multiplication

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$
$$\lambda \cdot \sum_{g \in G} a_g g = \sum_{g \in G} (\lambda a_g) g$$

for all $\lambda, a_g, b_g \in F$ $(g \in G)$, and a multiplicative structure induced by the group multiplication:

$$\left(\sum_{g \in G} a_g g \right) \cdot \left(\sum_{g \in G} b_g g \right) = \sum_{h \in G} \left(\sum_{\substack{s,t \in G \\ st=h}} a_s b_t \right) h.$$

In particular, FG is an F-vector space with basis G. For a subset $X \subseteq G$, we define

$$X^+ := \sum_{x \in X} x \in FG.$$

Mostly, this will be used in case that X is a conjugacy class of G. The center and the commutator space of FG are explicitly given by

$$ZFG = \left\{ \sum_{C \in \mathrm{Cl}(G)} a_C \cdot C^+ \colon a_C \in F \text{ for all } C \in \mathrm{Cl}(G) \right\}$$

and

$$K(FG) = \left\{ \sum_{g \in G} a_g g \colon \sum_{g \in C} a_g = 0 \text{ for all } C \in \mathrm{Cl}(G) \right\},$$

respectively (see [41, Theorem 1.5.1 and Proposition 1.5.4]). As mentioned in Example 1.18, group algebras form an important subclass of symmetric algebras. The usual choice for the symmetrizing form is the map

$$\lambda \colon FG \to F, \ \sum_{g \in G} a_g g \mapsto a_1 \tag{2.3}$$

(see [41, Theorem 2.11.2]). Furthermore, recall that by Maschke's theorem (see [47, Theorem I.6.18]), the group algebra FG is semisimple if and only if $\mathrm{char}(F) = 0$ holds or $\mathrm{char}(F)$ is a prime number not dividing the order of G. Since semisimple algebras were already considered in Remark 1.6, we usually assume that F is of positive characteristic.

For a subgroup H of G, we consider the *augmentation map*

$$\epsilon\colon FH \to F,\ \sum_{h\in H} a_h h \mapsto \sum_{h\in H} a_h,$$

which is a homomorphism of F-algebras. Moreover, we will encounter the *augmentation ideal*

$$\omega(FH) := \operatorname{Ker}(\epsilon) = \left\{ \sum_{h\in H} a_h h\colon \sum_{h\in H} a_h = 0 \right\}.$$

A basis of $\omega(FH)$ is given by the set $\{1-h\colon 1 \neq h \in H\}$. In the special case where F is a field of characteristic p and H is a finite p-group, [41, Theorem 1.11.1] yields

$$\omega(FH) = J(FH). \tag{2.4}$$

In our derivation, we will also need the structure of the right annihilator of $\omega(FH)$ in FG, which is given by

$$\operatorname{rAnn}_{FG}(\omega(FH)) = H^+ \cdot FG \tag{2.5}$$

(see [45, Lemma 3.1.2]). For a normal subgroup $N \trianglelefteq G$, we consider the canonical projection map

$$\nu_N\colon FG \to F[G/N],\ \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g \cdot gN. \tag{2.6}$$

Note that ν_N is a homomorphism of F-algebras. This map, together with its adjoint which we introduce later, will play an important role in order to relate Question 1.1 to the corresponding problem for certain quotient groups. We obtain the following relation between the map ν_N and the augmentation ideal of FN (see [41, Proposition 1.6.4]):

$$\operatorname{Ker}(\nu_N) = \omega(FN) \cdot FG = FG \cdot \omega(FN). \tag{2.7}$$

Later, we encounter the situation that F is a field of characteristic $p > 0$ and N is a normal Sylow p-subgroup of G. By [41, Theorem 1.11.10], the previous formula then simplifies to

$$\operatorname{Ker}(\nu_N) = \omega(FN) \cdot FG = J(FG). \tag{2.8}$$

At the end of this part, we briefly review the concept of blocks of a group algebra. To this end, let F be an algebraically closed field of characteristic $p > 0$. Then the group algebra FG can be decomposed uniquely into a direct sum $FG = B_1 \oplus \ldots \oplus B_n$ of indecomposable (two-sided) ideals, the *p-blocks* of FG (see [1, Lemma 1.8.2]). Each of them is an F-algebra in its own right. To each block B, one associates a conjugacy class of p-subgroups of G, the *defect groups* of B. We say that B is of *defect* $d \in \mathbb{N}_0$ if its defect groups have order p^d. The block B is called of *full defect* if its defect groups are Sylow p-subgroups of G. For every simple FG-module M, there exists a unique block B of FG which satisfies $BM \neq 0$. We say that the module M *lies in* B. The block containing the trivial FG-module is called the *principal block* of FG. It has full defect (see [44, page 119]).

2.2 Condition for group algebras

Throughout, let F be an algebraically closed field and let G be a finite group. By Lemma 1.9, the condition $\operatorname{soc}(ZFG) \trianglelefteq FG$ is equivalent to $\operatorname{soc}(ZFG) \cdot K(FG) = 0$. In this section, we further adapt this statement to our group-theoretic setting. We begin with a result on the commutator ideal $K(FG) \cdot FG = FG \cdot K(FG)$ (see Lemma 1.7):

Proposition 2.7. *The commutator ideal is given by*

$$FG \cdot K(FG) = \operatorname{Ker}(\nu_{G'}) = FG \cdot \omega(FG'). \tag{2.9}$$

Proof. Note that the right equality in (2.9) is the statement of (2.7), applied to $N = G'$. Now consider the left equality. Since $FG/\operatorname{Ker}(\nu_{G'}) \cong F[G/G']$ is commutative, we obtain $FG \cdot K(FG) \subseteq \operatorname{Ker}(\nu_{G'})$ by Lemma 1.7. For the other inclusion, we first prove that $1 - g \in FG \cdot K(FG)$ holds for all $g \in G'$. If $g = [a, b]$ holds for some $a, b \in G$, we obtain

$$1 - g = 1 - [a, b] = ab(b^{-1}a^{-1} - a^{-1}b^{-1}) \in FG \cdot K(FG).$$

An arbitrary element $g \in G'$ can be expressed as a product $g = x_1 \cdots x_n$ of commutators $x_1, \ldots, x_n$. We have

$$1 - x_1x_2 = x_1(1 - x_2) + (1 - x_1) \in FG \cdot K(FG)$$

and hence $1 - g \in FG \cdot K(FG)$ follows by induction. This shows $\omega(FG') \subseteq FG \cdot K(FG)$ and hence $\operatorname{Ker}(\nu_{G'}) = FG \cdot \omega(FG') \subseteq FG \cdot K(FG)$ follows. □

The following characterization of the property $\operatorname{soc}(ZFG) \trianglelefteq FG$ for group algebras will be used throughout the entire thesis:

Lemma 2.8. *The socle* $\operatorname{soc}(ZFG)$ *is an ideal of* FG *if and only if we have*

$$\operatorname{soc}(ZFG) \subseteq (G')^+ \cdot FG.$$

Proof. By Lemma 1.9, we have $\operatorname{soc}(ZFG) \trianglelefteq FG$ if and only if $K(FG) \cdot \operatorname{soc}(ZFG) = 0$ holds, which is equivalent to $FG \cdot K(FG) \cdot \operatorname{soc}(ZFG) = 0$. That is, we have

$$\operatorname{soc}(ZFG) \subseteq \operatorname{Ann}_{FG}(FG \cdot K(FG)) = \operatorname{Ann}_{FG}(FG \cdot \omega(FG'))$$

by Proposition 2.7. Note that the left and the right annihilator of $FG \cdot K(FG)$ in FG coincide by Lemma 1.19 (v). For any $x \in FG$, the condition $\omega(FG') \cdot x = 0$ is equivalent to $FG \cdot \omega(FG') \cdot x = 0$. Therefore, $\operatorname{soc}(ZFG)$ is an ideal of FG if and only if we have

$$\operatorname{soc}(ZFG) \subseteq \operatorname{Ann}_{FG}(FG \cdot \omega(FG')) = \operatorname{rAnn}_{FG}(\omega(FG')) = (G')^+ \cdot FG$$

(see (2.5)), which finishes the proof. □

Remark 2.9. Explicitly, the set $(G')^+ \cdot FG$ is given by

$$(G')^+ \cdot FG = \left\{ \sum_{g \in G} a_g g \in FG \colon a_{g_1} = a_{g_2} \text{ if } g_1^{-1}g_2 \in G' \text{ holds} \right\}.$$

In other words, every element in $(G')^+ \cdot FG$ has constant coefficients on the cosets of G' in G. For instance, if there exists a normal subgroup $N \trianglelefteq G$ with $N^+ \in \operatorname{soc}(ZFG)$, then $\operatorname{soc}(ZFG) \trianglelefteq FG$ implies $G' \subseteq N$. ◁

This allows us to decide whether $\operatorname{soc}(ZFG)$ is an ideal of FG by examining the coefficients of the individual elements in $\operatorname{soc}(ZFG)$ instead of determining the structure of this space. For instance, in order to show that $\operatorname{soc}(ZFG)$ is not an ideal of FG, we can construct an element in ZFG with non-constant coefficients on a certain coset of G' and show directly that this element annihilates a basis of $J(ZFG)$.

2.3 Some special elements of FG

Let F be an algebraically closed field and let G be a finite group. From now on, we assume that F is of characteristic $p > 0$ since otherwise, FG is semisimple and this case was already treated in Remark 1.6. We study the case that G has a nontrivial normal p-subgroup N, which will be automatically ensured later on. From this, we construct a subgroup M of G such that M^+ annihilates a large subset of the conjugacy class sums of G. In the following, this result will be applied in different circumstances in order to construct interesting elements in $\operatorname{soc}(ZFG)$. We begin with the following simple observation, which will be used frequently.

Remark 2.10. Consider a normal subgroup L of G and let $C \in \operatorname{Cl}(G)$. For any $c_1, c_2 \in C$, there exists an element $x \in G$ with $xc_1x^{-1} = c_2$. Conjugation with x permutes the conjugacy class C and maps $C \cap c_1L$ to $C \cap c_2L$. This shows that the cosets c_1L and c_2L contain the same number of elements in C. ◁

We now move to the construction of the subgroup M. To this end, we first consider the special case that the normal p-subgroup N is abelian.

Proposition 2.11. *Assume that N is an abelian normal p-subgroup of G. We set*

$$M := \{x \in [N, G] \colon x^p = 1\}.$$

If $C \in \operatorname{Cl}(G)$ is a conjugacy class with $C \not\subseteq C_G(N)$, we have $\nu_M(C^+) = 0$. In particular, this yields $M^+ \cdot C^+ = 0$.

Proof. Note that M is a normal subgroup of G. The p-group N acts on C by conjugation. Let B be an orbit of this action and consider an element $b \in B$. Since C is not contained in the normal subgroup $C_G(N)$, we obtain $N \not\subseteq C_G(b)$, which yields $|B| = |N : C_N(b)| \neq 1$. Set $X := \langle N, B\rangle = \langle N, b\rangle$. For $n_1, n_2 \in N$, we have $[n_1n_2, b] = [n_1, b] \cdot [n_2, b]$ since N is abelian. Hence the map $f \colon N \to N,\ n \mapsto [n, b]$ is a group endomorphism with image X' and kernel $C_N(b)$. This yields $|B| = |N \colon C_N(b)| = |X'|$, so in particular, $|X'|$ is a nontrivial power of p.

Now we consider the quotient group $\bar{G} := G/M$. Again, the group $\bar{N} := N/M$ acts on the image $\bar{C} \in \operatorname{Cl}(\bar{G})$ of C in $\bar{G}$ by conjugation. Let $\bar{B}$ denote the image of B under the canonical projection. Note that this is an orbit of the action of $\bar{N}$ on $\bar{C}$. Set $\bar{b} := bM$ and $\bar{X} := X/M$. By carrying out the same argument as before, we obtain

$$|\bar{B}| = |\bar{N} : C_{\bar{N}}(\bar{b})| = |\bar{X}'| = |X' : X' \cap M|.$$

Since $X' \subseteq [N, G]$ is a nontrivial p-group, it contains an element of order p. It follows that $|X' \cap M|$ is divisible by p. With this, we obtain

$$\nu_M(B^+) = \frac{|B|}{|\bar{B}|} \cdot \bar{B}^+ = |X' \cap M| \cdot \bar{B}^+ = 0.$$

Since B is an arbitrary orbit of the action of N on C, this yields $\nu_M(C^+) = 0$, which implies $M^+ \cdot C^+ = 0$. □

Now we consider the general case, that is, we drop the assumption that N is abelian.

Lemma 2.12. *Let N be a normal p-subgroup of G. We set*

$$M := \{x \in [N, G] \colon x^p \in N'\}$$

and let $C \in \mathrm{Cl}(G)$ be a conjugacy class with $C \not\subseteq C_G(N)$. Then we have $\nu_M(C^+) = 0$, which implies $M^+ \cdot C^+ = 0$.

Proof. Again, the p-group N acts on C by conjugation. Let B be an orbit of this action and consider an element $b \in B$. As before, $|B| = |N : C_N(b)|$ is divisible by p. We now go over to the quotient group $\bar{G} := G/N'$ and set $\bar{b} := bN' \in \bar{G}$. Note that the image $\bar{N} := N/N'$ is abelian. First assume $C_{\bar{N}}(\bar{b}) = \bar{N}$. This means that for any $n \in N$, one has $[n, b] \in N'$, which implies

$$\nu_{N'}(B^+) = |B| \cdot \bar{b} = 0.$$

Since $N' \subseteq M$ holds, this yields $\nu_M(B^+) = 0$. Now assume that the centralizer $C_{\bar{N}}(\bar{b})$ is a proper subgroup of $\bar{N}$. In particular, we have $\bar{C} \not\subseteq C_{\bar{G}}(\bar{N})$, where $\bar{C}$ denotes the image of C in $\bar{G}$. The preceding lemma then yields $\nu_{\bar{M}}(\bar{C}^+) = 0$ for

$$\bar{M} := \{x \in [\bar{N}, \bar{G}] \colon x^p = 1\} = M/N'.$$

Moreover, we have $\nu_{N'}(C^+) = \frac{|C|}{|\bar{C}|} \cdot \bar{C}^+$ and hence

$$\nu_{\bar{M}}(\nu_{N'}(C^+)) = \frac{|C|}{|\bar{C}|} \cdot \nu_{\bar{M}}(\bar{C}^+) = 0.$$

This yields $\nu_M(C^+) = 0$ since we have $\bar{G}/\bar{M} = (G/N')/(M/N') \cong G/M$ and hence the map $\nu_{\bar{M}} \circ \nu_{N'}$ can be identified with ν_M. □

The following special case will arise frequently:

Corollary 2.13. *Let N be a normal p-subgroup of G and consider an element $g \in G$ with $g \notin C_G(N)$. In this case, we have $\nu_N([g]^+) = 0$. In particular, $|[g] \cap gN|$ is divisible by p.*

Proof. By the preceding lemma, we have $\nu_M([g]^+) = 0$ for $M := \{x \in [N, G] \colon x^p \in N'\}$, which yields $\nu_N([g]^+) = 0$ since N contains M. By Remark 2.10, $|[g] \cap gN|$ is divisible by p. □

In this section, we did not impose any conditions on the structure of G. In the next section, we will see that the condition $\mathrm{soc}(ZFG) \trianglelefteq FG$ forces the group to be of a particular form, for which we refine the above results in Section 2.4.3.

2.4 Structure of G

Throughout, G denotes a finite group and F is an algebraically closed field of characteristic $p > 0$. In this section, we make some fundamental observations on the structure of the finite groups G for which $\operatorname{soc}(ZFG) \trianglelefteq FG$ holds. In Section 2.4.1, we study the Reynolds ideal $R(FG)$. Our main result is that $R(FG)$ is an ideal in FG if and only if G' is contained in the p-core of G. In the subsequent parts, we exploit this structure. First, we investigate the transition to the p-blocks of FG. In Section 2.4.3, we describe an F-basis for $J(ZFG)$ which we use throughout this thesis. We conclude this part by showing that FG is an H-graded algebra and by refining the results obtained in Section 2.3.

2.4.1 Reynolds ideal

In this section, we investigate the Reynolds ideal $R(FG)$ and answer the question under which conditions it is an ideal of FG (see Question 1.4). Recall that the latter is a necessary condition for $\operatorname{soc}(ZFG) \trianglelefteq FG$ by Lemma 1.12. This will lead to a decomposition of G into a semidirect product, which will be used throughout this entire thesis.

By [36, Equation (39)], the Reynolds ideal of FG is explicitly given as

$$R(FG) = F\left\{S^+ \colon S\ p'\text{-section of } G\right\}. \tag{2.10}$$

In particular, we have $G_p^+ \in R(FG)$, where G_p denotes the set of p-elements of G.

Remark 2.14. We obtain the following chain of inclusions:

$$R(FG) \subseteq \operatorname{soc}(ZFG) \subseteq O_p(Z(G))^+ \cdot FG.$$

Note that $R(FG)$ is contained in $\operatorname{soc}(ZFG)$ by definition. In order to show the second inclusion, we consider an element $1 \neq z \in O_p(Z(G))$. Since $(z-1)^{p^n} = z^{p^n} - 1 = 0$ holds for some sufficiently large $n \in \mathbb{N}$, the element $z-1 \in ZFG$ is nilpotent and hence contained in $J(ZFG)$. For any element $y = \sum_{g\in G} a_g g \in \operatorname{soc}(ZFG)$, this yields $y \cdot (z-1) = 0$, which translates to $a_{gz} = a_g$ for all $g \in G$. Since this holds for every $1 \neq z \in O_p(Z(G))$, we obtain $\operatorname{soc}(ZFG) \subseteq O_p(Z(G))^+ \cdot FG$ as claimed. ◁

The following theorem, which answers Question 1.4 in this case, forms the basis for all further derivations concerning group algebras.

Theorem 2.15. *Let G be a finite group. Then the following are equivalent:*

(i) $R(FG)$ is an ideal of FG.

(ii) $G' \subseteq O_p(G)$.

(iii) $G = P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H.

In this case, we have $R(FG) = O_p(G)^+ \cdot FG$.

Proof. We first show the equivalence of (ii) and (iii). Assume $G' \subseteq O_p(G)$. For any Sylow p-subgroup P of G, we obtain $G' \subseteq O_p(G) \subseteq P$, so P is a normal subgroup of G. By the Schur-Zassenhaus theorem (see [30, Theorem 6.2.1]), there exists a Hall p'-subgroup

H of G and we have $G = P \rtimes H$. Moreover, $H \cong G/P$ is abelian since P contains the derived subgroup G'. Conversely, assume that $G = P \rtimes H$ is a group of the form given in (iii). Then P is a normal subgroup of G, which implies $P = O_p(G)$. Since H is abelian, we obtain $G' \subseteq P$.

Now we show the equivalence of the properties (i) and (ii). Recall that $R(FG) \trianglelefteq FG$ is equivalent to $R(FG)^\perp \trianglelefteq FG$, where $R(FG)^\perp$ denotes the orthogonal space of $R(FG)$ with respect to a symmetrizing linear form of FG (see Section 1.2 and (2.3)). By Lemma 1.19, we have

$$R(FG)^\perp = (ZFG \cap \operatorname{soc}(FG))^\perp = K(FG) + J(FG)$$

and this space is an ideal of FG if and only if $K(FG/J(FG)) = K(FG)+J(FG)/J(FG)$ is an ideal in $FG/J(FG)$ (see Lemma 1.8). Since the latter algebra is semisimple and hence symmetric (see Example 1.18), this is the case if and only if $FG/J(FG)$ is commutative by Remark 1.21. This in turn is equivalent to $\omega(FG') \cdot FG = K(FG) \cdot FG \subseteq J(FG)$ (see Proposition 2.7 and Lemma 1.7).

Now if $G' \subseteq O_p(G)$ holds, then G' is a p-group and we obtain $\omega(FG') = J(FG') \subseteq J(FG)$ (see (2.4)), so $\omega(FG') \cdot FG \subseteq J(FG)$ also holds. By the above, $R(FG)$ is an ideal of FG. Conversely, assume that this holds, so $\omega(FG') \cdot FG$ is contained in $J(FG)$. In particular, the element $g - 1 \in \omega(FG')$ is nilpotent for any $g \in G'$. Hence there exists $n \in \mathbb{N}$ with $0 = (g-1)^{p^n} = g^{p^n} - 1$, so g is a p-element. This implies that G' is a p-group and hence contained in $O_p(G)$, which establishes the equivalence of (i) and (ii).

If G is of the form given in (iii), then the p'-sections of G are of the form hP for $h \in H$. Since $R(FG)$ is spanned by their sums, we obtain $R(FG) = P^+ \cdot FG = O_p(G)^+ \cdot FG$ (see (2.10)). □

By Lemma 1.12, $R(FG) \trianglelefteq FG$ is a necessary condition for $\operatorname{soc}(ZFG) \trianglelefteq FG$ and hence the previous theorem has the following fundamental consequence for the structure of the finite groups G which satisfy $\operatorname{soc}(ZFG) \trianglelefteq FG$:

Corollary 2.16. *If* $\operatorname{soc}(ZFG)$ *is an ideal of* FG*, then* G *is of the form* $P \rtimes H$ *with* $P \in \operatorname{Syl}_p(G)$ *and an abelian* p'*-group* H*. In particular,* G *is solvable.*

Note that in this situation, the Sylow p-subgroup P is characteristic in G since any automorphism of G preserves the orders of the group elements. This implies that the derived subgroups P', P'' etc. are characteristic in G as well. Moreover, we make the following observation on the structure of the conjugacy classes of elements in H:

Remark 2.17. Let G be a finite group of the form $P \rtimes H$ as in Theorem 2.15 (iii) and consider an element $h \in H$. For $h' \in [h]$, we write $h' = ch$ with $c \in U_h \subseteq G'$ (see (2.1)). If h and c commute, then c is the p-part of h'. Since h and $h' = ch$ are conjugate elements, also their p-parts are conjugate, which yields $c = 1$. This shows that h does not commute with any nontrivial element of U_h. ◁

This property will be frequently used in our investigation of group algebras.

2.4.2 Blocks of group algebras and the p'-core

Let G be a group of the form $P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H. In this section, we investigate the condition $\mathrm{soc}(Z(B)) \trianglelefteq B$ for a p-block B of FG (see Section 2.1). Many problems can be simplified by studying the p-blocks instead of the entire group algebra. For our problem, however, this turns out not to be the case since all p-blocks of FG are isomorphic in this situation. As a consequence, we may pass to the group algebra of $\bar{G} := G/O_{p'}(G)$ and hence restrict our investigation to groups with trivial p'-core in the following.

Remark 2.18. Let $FG = B_1 \oplus \ldots \oplus B_n$ be the decomposition of FG into its p-blocks $B_1, \ldots, B_n$. By [44, Corollary 3.12], we have

$$J(ZFG) = J(Z(B_1)) \oplus \ldots \oplus J(Z(B_n)).$$

Since $B_i \cdot B_j = 0$ holds for $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we also obtain

$$\mathrm{soc}(ZFG) = \mathrm{soc}(Z(B_1)) \oplus \ldots \oplus \mathrm{soc}(Z(B_n)).$$

In particular, we have $\mathrm{soc}(ZFG) \trianglelefteq FG$ if and only if $\mathrm{soc}(Z(B_i))$ is an ideal in B_i for $i = 1, \ldots, n$. Note that we have

$$R(FG) = R(B_1) \oplus \ldots \oplus R(B_n)$$

by [37, page 622]. Since $R(FG)$ is an ideal in FG by Theorem 2.15, this yields $R(B_i) \trianglelefteq B_i$ for $i = 1, \ldots, n$. By [37, Proposition 4.1], all blocks of FG are isomorphic to the principal block, which in turn is isomorphic to the principal block $\bar{B}_0$ of the group algebra $F\bar{G}$. In particular, we have $\mathrm{soc}(ZFG) \trianglelefteq FG$ if and only if $\mathrm{soc}(Z(\bar{B}_0)) \trianglelefteq \bar{B}_0$ holds. ◁

This has the following important consequence:

Lemma 2.19. *We have* $\mathrm{soc}(ZFG) \trianglelefteq FG$ *if and only if* $\mathrm{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ *holds.*

Proof. If $\mathrm{soc}(ZFG)$ is an ideal of FG, then $\mathrm{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ follows by Lemma 1.25 since $F\bar{G}$ can be viewed as a quotient algebra of FG (see (2.6)). Now assume that $\mathrm{soc}(ZF\bar{G})$ is an ideal of $F\bar{G}$. Note that $\bar{G}$ is of the form $\bar{G} = \bar{P} \rtimes \bar{H}$ with a normal Sylow p-subgroup $\bar{P}$ and an abelian p'-group $\bar{H}$. Furthermore, we have $O_{p'}(\bar{G}) = 1$ and hence Theorem 2.1 yields $C_{\bar{G}}(\bar{P}) \subseteq \bar{P}$. By [43, Corollary 7.3], the group algebra $F\bar{G}$ consists of a single block. By the preceding remark, the assumption $\mathrm{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ then implies $\mathrm{soc}(ZFG) \trianglelefteq FG$. □

By replacing G by the quotient group $G/O_{p'}(G)$, we may therefore restrict ourselves to groups with a trivial p'-core.

2.4.3 Basis for $J(ZFG)$

In this section, we assume that G is a finite group of the form $P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H (see Corollary 2.16). In this case, we determine an F-basis for $J(ZFG)$, which we use throughout this thesis. Since we need the results in the general setting later on, we do not require the p'-core of G to be trivial.

We begin by examining the conjugacy classes of G. To this end, we first analyze the structure of the centralizer $C_G(P)$.

Remark 2.20. Since $P = O_p(G) \subseteq O_{p',p}(G)$ holds, [20, Theorem 6.3.3] implies

$$C_G(P) \subseteq O_{p',p}(G) = O_{p'}(G)P = O_{p'}(G) \times P.$$

Conversely, note that $[P, O_{p'}(G)] \subseteq P \cap O_{p'}(G) = 1$ implies $O_{p'}(G) \subseteq C_G(P)$, which yields

$$C_G(P) = O_{p'}(G) \times Z(P).$$

Since H is an abelian group containing $O_{p'}(G)$, we have $H \subseteq C_G(O_{p'}(G))$. The above argument yields $P \subseteq C_G(O_{p'}(G))$, so we obtain $G = PH \subseteq C_G(O_{p'}(G))$ and hence $O_{p'}(G) \subseteq Z(G)$. Finally, note that for any $g \in G$, we obtain the equivalence

$$g \in C_G(P) \Leftrightarrow P \subseteq C_G(g) \Leftrightarrow p \nmid |[g]|, \tag{2.11}$$

since we have $|G| = |C_G(g)| \cdot |[g]|$. $\lhd$

As before, $\nu_P \colon FG \to F[G/P]$ denotes the canonical projection onto $F[G/P]$ (see (2.6)). By (2.8), its kernel is given by $\mathrm{Ker}(\nu_P) = J(FG)$. In the following, we distinguish two types of conjugacy classes of G, which are usually treated separately in the following:

Remark 2.21. Let $C \in \mathrm{Cl}(G)$ be a conjugacy class. Since G/P is abelian, we obtain $|\bar{C}| = 1$ for the image $\bar{C} \in \mathrm{Cl}(G/P)$ of C in G/P. Now two cases can occur:

- $|C|$ is divisible by p: In this case, we have $\nu_P(C^+) = |C| \cdot \bar{C}^+ = 0$, which yields $C^+ \in J(FG) \cap ZFG = J(ZFG)$.
- $|C|$ is not divisible by p: The equivalence in (2.11) then yields $C \subseteq C_G(P)$. By Remark 2.20, every element $c \in C$ can be expressed in the form sz with $s \in O_{p'}(G)$ and $z \in Z(P)$. Note that the elements s and z are uniquely determined since the intersection $O_{p'}(G) \cap Z(P)$ is trivial. Moreover, $s \in Z(G)$ (see Remark 2.20) yields $C = s[z]$. Observe that the element $C^+ - |C| \cdot s = s([z]^+ - |[z]| \cdot 1)$ is contained in $\mathrm{Ker}(\nu_P) \cap ZFG = J(ZFG)$. $\lhd$

According to this distinction, we now associate elements $b_C \in J(ZFG)$ to the conjugacy classes $C \in \mathrm{Cl}(G)$. We do not take the classes of elements in $O_{p'}(G)$ into account since the corresponding elements of $J(ZFG)$ are zero.

Definition 2.22. Let $C \in \mathrm{Cl}(G)$ be a conjugacy class with $C \not\subseteq O_{p'}(G)$. We set $b_C := C^+$ if p divides $|C|$. Otherwise, we write $C = s[z]$ with $s \in O_{p'}(G)$ and $1 \neq z \in Z(P)$ as in Remark 2.21 and set $b_C = s([z]^+ - |[z]| \cdot 1)$.

The key observation is that these elements form an F-basis for $J(ZFG)$.

Theorem 2.23. *Let G be a finite group of the form $G = P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H. Then an F-basis for $J(ZFG)$ is given by*

$$B := \left\{ b_C \colon C \in \mathrm{Cl}(G),\ C \not\subseteq O_{p'}(G) \right\},$$

where b_C denotes the element of $J(ZFG)$ corresponding to $C \in \mathrm{Cl}(G)$ by Definition 2.22.

Proof. Remark 2.21 shows that the elements in B are contained in $J(ZFG)$. Note that the elements in $B \cup O_{p'}(G)$ form an F-basis for ZFG. Since the algebra $FO_{p'}(G)$ is semisimple, the Jacobson radical $J(ZFG)$ is spanned by B. □

Remark 2.24. If $O_{p'}(G) = 1$ holds in the situation of Theorem 2.23, then the given basis of $J(ZFG)$ simplifies to

$$\left\{D^+ - |D| \cdot 1 \colon 1 \neq D \in \mathrm{Cl}(G),\ D \subseteq Z(P)\right\} \cup \left\{C^+ \colon C \in \mathrm{Cl}(G),\ p \text{ divides } |C|\right\}. \quad \triangleleft$$

Note that we have $\mathrm{soc}(ZFG) = \mathrm{Ann}_{ZFG}(J(ZFG)) = \mathrm{Ann}_{ZFG}(B)$, that is, it suffices to consider the multiplication with the elements of B. For $y \in FG$, it is often useful to restate the condition $y \cdot b_C = 0$ in terms of the coefficients of y:

Remark 2.25. For an element $y = \sum_{g \in G} a_g g \in FG$ and a subset $M \subseteq G$, we have

$$y \cdot M^+ = 0 \Leftrightarrow \sum_{m \in M} a_{wm^{-1}} = 0 \text{ for all } w \in G, \tag{2.12}$$

since the expression on the right hand side is the coefficient of w in the product $y \cdot M^+$. In particular, this will be applied in the case where $M = C \in \mathrm{Cl}(G)$ is a conjugacy class of length divisible by p. Similarly, for $C \in \mathrm{Cl}(G)$ of the form $C = sD$ with $s \in O_{p'}(G)$ and a G-conjugacy class $D \subseteq Z(P)$, we obtain

$$y \cdot b_C = 0 \Leftrightarrow y \cdot (D^+ - |D| \cdot 1) = 0 \Leftrightarrow \sum_{d \in D} a_{wd^{-1}} = |D| \cdot a_w \text{ for all } w \in G. \tag{2.13}$$

If y is contained in ZFG, setting $w = 1$ and $D = [z^{-1}]$ for some $z \in Z(P) \backslash \{1\}$ yields

$$|D| \cdot a_z = |D| \cdot a_1,$$

since the inverses of the elements in D are conjugate and hence the corresponding coefficients are equal. Since $|D|$ is not divisible by p (see (2.11)), this yields $a_z = a_1$. ◁

2.4.4 Applications

Again, we assume that G is a finite group of the form $P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H. In the following, we make use of the basis of $J(ZFG)$ determined in Section 2.4.3. First we show that the group algebra FG is an H-graded algebra, which allows us to restrict our investigation to elements which are homogeneous with respect to this grading. This considerably simplifies our further calculations. Afterwards, we use the results from Section 2.3 in order to construct interesting elements of $\mathrm{soc}(ZFG)$.

We begin by recalling some facts about graded algebras. Further information on this topic can be found in [42], for example.

Remark 2.26. Let M be a monoid. An F-algebra A is called *M-graded* if there exists a decomposition into F-vector spaces

$$A = \bigoplus_{m \in M} A_m$$

such that $A_mA_n \coloneqq F\{a_ma_n : a_m \in A_m, a_n \in A_n\}$ is contained in A_{mn} for all $m, n \in M$. An element is called *homogeneous* if it is contained in A_m for some $m \in M$. A subspace V spanned by homogeneous elements is called an *M-graded subspace*. This is equivalent to V being of the form $\bigoplus_{m\in M}(A_m \cap V)$. An *$M$-graded subalgebra* B of A is a subalgebra that is an M-graded subspace. Note that B can be viewed as an M-graded algebra with components $B_m \coloneqq B \cap A_m$ for all $m \in M$. Similarly, we define *M-graded ideals* of A. ◃

In our situation, the group algebra FG is H-graded in a natural way:

Remark 2.27. For $h \in H$, we consider the F-subspace $FP \cdot h = FhP$ of FG. We have

$$FG = \bigoplus_{h\in H} FhP$$

since any element $g \in G$ can be expressed as a product hu of elements $h \in H$ and $u \in P$ in a unique way. Moreover, $Fh_1P \cdot Fh_2P \subseteq Fh_1h_2P$ holds for all $h_1, h_2 \in H$. In this way, FG becomes an H-graded algebra. Since the conjugacy class sums of G, which form a basis for ZFG, are homogeneous with respect to this grading, it follows that ZFG is an H-graded subalgebra of FG. Note that the basis of $J(ZFG)$ given in Theorem 2.23 also consists of homogeneous elements, so $J(ZFG)$ is a homogeneous subspace of FG. In particular, we have

$$J(ZFG) = \bigoplus_{h\in H} (FhP \cap J(ZFG))\,.$$

It follows that the annihilator $\operatorname{soc}(ZFG) = \operatorname{Ann}_{ZFG}(J(ZFG))$ is a homogeneous subspace of FG as well, which implies

$$\operatorname{soc}(ZFG) = \bigoplus_{h\in H} (FhP \cap \operatorname{soc}(ZFG))\,.$$

In order to answer the question whether $\operatorname{soc}(ZFG)$ is an ideal of FG, it therefore suffices to check whether $FG \cdot y \subseteq ZFG$ holds for all homogeneous elements $y \in \operatorname{soc}(ZFG)$. ◃

At the end of this part, we apply Lemma 2.12 in order to construct elements in $\operatorname{soc}(ZFG)$ that arise from self-centralizing normal p-subgroups of G. By Lemma 2.19, we may assume that G has a trivial p'-core.

Lemma 2.28. *Assume $O_{p'}(G) = 1$ and let $N \subseteq P$ be a normal subgroup of G which satisfies $C_P(N) \subseteq N$. Then we have $N^+ \in \operatorname{soc}(ZFG)$.*

Proof. We first show that $C_G(N) = C_P(N)$ holds. Note that $C_P(N) = P \cap C_G(N)$ is a normal Sylow p-subgroup of $C_G(N)$ and hence by the Schur-Zassenhaus theorem, there exists a p'-subgroup V with $C_G(N) = C_P(N)V$. Since V centralizes the p-group $C_P(N) \subseteq N$, we even obtain $C_G(N) = C_P(N) \times V$. Hence V is a normal subgroup of $C_G(N)$, which yields $V \subseteq O_{p'}(C_G(N)) \subseteq O_{p'}(G) = 1$. With this, $C_G(N) = C_P(N)$ follows as claimed.

Now consider a conjugacy class $1 \neq C \in \operatorname{Cl}(G)$ and first assume $C \subseteq C_P(N) \subseteq N$. If p divides $|C|$, we obtain $\nu_N(b_C) = \nu_N(C^+) = 0$. Similarly, if $|C|$ is not divisible by p, we have $\nu_N(b_C) = \nu_N(C^+ - |C| \cdot 1) = 0$. Now let $C \not\subseteq C_P(N) = C_G(N)$. By (2.11),

p divides $|C|$ and hence $\nu_N(b_C) = \nu_N(C^+) = 0$ follows by Corollary 2.13. Summarizing, we obtain $\nu_N(b_C) = 0$ and hence $N^+ \cdot b_C = 0$ for all $1 \neq C \in \mathrm{Cl}(G)$, which implies $N^+ \in \mathrm{soc}(ZFG)$ by Theorem 2.23. □

Later on, we also need the following variant of the previous lemma:

Corollary 2.29. *Assume $O_{p'}(G) = 1$ and let $M \subseteq P$ be a normal subgroup of G. For $N := C_P(M)M$, we obtain $N^+ \in \mathrm{soc}(ZFG)$.*

Proof. Note that since $C_P(N) \subseteq C_P(M) \subseteq N$ follows in this case, the previous lemma yields $N^+ \in \mathrm{soc}(ZFG)$. □

2.5 Relation to smaller groups

Let F be an algebraically closed field of characteristic $p > 0$. In order to characterize the finite groups G for which $\mathrm{soc}(ZFG)$ is an ideal in FG, it is convenient to address this question for certain smaller groups. In particular, one might wonder whether this property is inherited by the group algebra of a quotient group or a subgroup of G. We have already seen that the transition to symmetric quotient algebras is possible in a more general context (see Lemma 1.25). In Section 2.5.1, we study this phenomenon for group algebras in greater detail.

In contrast, the condition $\mathrm{soc}(ZFG) \trianglelefteq FG$ does in general not imply $\mathrm{soc}(ZFH) \trianglelefteq FH$ for every subgroup H of G. We will later present a counterexample (see Example 3.22). Here, we focus on the particular situation that $G = G_1 * G_2$ is a central product of two subgroups G_1 and G_2. We show that in this case, $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds if and only if both FG_1 and FG_2 have the corresponding property (see Section 2.5.2).

2.5.1 Quotient groups

Throughout this section, let G be a finite group and consider a normal subgroup $N \trianglelefteq G$ with corresponding quotient group $\bar{G} := G/N$. In this part, we study the transition to group algebra $F\bar{G}$ in greater detail.

If $\mathrm{soc}(ZFG)$ is an ideal of FG, then Lemma 1.25 yields $\mathrm{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ since $F\bar{G}$ is isomorphic to the quotient algebra $FG/\omega(FN) \cdot FG$ (see (2.7)). In this section, we refine this result by deriving a correspondence between the set $N^+ \cdot FG \cap \mathrm{soc}(ZFG)$ and the elements in $\mathrm{Ann}_{ZF\bar{G}}(S)$, where S is a subset of the basis elements of $J(ZF\bar{G})$ introduced in Definition 2.22. A key observation is that this bijection maps elements of $(G')^+ \cdot FG$ to elements in $(\bar{G}')^+ \cdot F\bar{G}$ and vice versa. By Lemma 2.8, this allows us to transfer the question whether $\mathrm{soc}(ZFG)$ is an ideal of FG to an investigation of certain annihilators in the group algebra $F\bar{G}$.

We consider the map

$$\Lambda_N \colon F\bar{G} \to FG, \ \sum_{gN \in \bar{G}} a_{gN} \cdot gN \mapsto \sum_{g \in G} a_{gN} \cdot g.$$

By identifying $F\bar{G}$ and $FG/\omega(FN)\cdot FG$, we see that Λ_N is the adjoint map ν_N^* (see Section 1.2) of the canonical projection $\nu_N\colon FG\to F\bar{G}$ with respect to the symmetrizing linear form given in (2.3). We collect some properties of ν_N^*:

Lemma 2.30.

(i) For $x\in G$ and $a\in F\bar{G}$, we have $x\cdot\nu_N^*(a)=\nu_N^*(xN\cdot a)$ and $\nu_N^*(a)\cdot x=\nu_N^*(a\cdot xN)$.

(ii) The map ν_N^* is linear and injective. Its image is given by $N^+\cdot FG=FG\cdot N^+$.

(iii) We have $\nu_N^*(ZF\bar{G})=ZFG\cap\operatorname{Im}(\nu_N^*)$.

(iv) For $a\in F\bar{G}$, the condition $a\in(\bar{G}')^+\cdot F\bar{G}$ is equivalent to $\nu_N^*(a)\in(G')^+\cdot FG$.

Proof.

(i) This follows from (1.4) and (1.5).

(ii) The map ν_N^* is clearly linear and injective. Any $x\in\operatorname{Im}(\nu_N^*)$ has constant coefficients on cosets of N, so it is an element of $N^+\cdot FG=FG\cdot N^+$. Conversely, we have $N^+=\nu_N^*(N)\in\operatorname{Im}(\nu_N^*)$ and hence the claim follows by (i).

(iii) This is the statement of Lemma 1.24, rephrased in this context.

(iv) This follows from the fact that for any $x,y\in G$, we have $x^{-1}y\in NG'$ if and only if $\bar{x}^{-1}\bar{y}\in\bar{G}'$ holds, where $\bar{x},\bar{y}$ denote the images of x,y in $\bar{G}$. □

In the remainder of this section, we assume that G is of the form $P\rtimes H$ with $P\in\operatorname{Syl}_p(G)$ and an abelian Hall p'-subgroup H (see Corollary 2.16). Note that $\bar{G}=\bar{P}\rtimes\bar{H}$ has a similar structure with $\bar{P}:=PN/N\in\operatorname{Syl}_p(\bar{G})$ and the abelian p'-group $\bar{H}:=HN/N$. We first investigate the structure of the conjugacy classes in G and their images in $\bar{G}$.

Remark 2.31. We consider a conjugacy class $C\in\operatorname{Cl}(G)$ with $C\not\subseteq O_{p'}(G)N$ and set $\bar{C}\in\operatorname{Cl}(\bar{G})$ to be the image of C in $\bar{G}$. By Remark 2.10, every element in $\bar{C}$ has $k:=|C|/|\bar{C}|$ preimages in C, so we obtain $\nu_N(C^+)=k\cdot\bar{C}^+$. In particular, we have $C^+\in\operatorname{Ker}(\nu_N)$ if and only if k is divisible by p. ◁

This gives rise to the following distinction:

Definition 2.32. Let $C\in\operatorname{Cl}(G)$ be a conjugacy class with $C\not\subseteq O_{p'}(G)N$ and denote by $\bar{C}\in\operatorname{Cl}(\bar{G})$ the image of C in $\bar{G}$. Then the following cases can arise:

(i) $|C|/|\bar{C}|$ is not divisible by p: We denote the set of these conjugacy classes by $\operatorname{Cl}_{p',N}(G)$ and set
$$\operatorname{Cl}^+_{p',N}(G):=\left\{b_C\colon C\in\operatorname{Cl}_{p',N}(G)\right\}$$
to be the set of corresponding basis elements of $J(ZFG)$ (see Definition 2.22).

(ii) p divides $|C|/|\bar{C}|$: The set of these classes will be denoted by $\operatorname{Cl}_{p,N}(G)$. As in the first case, we set
$$\operatorname{Cl}^+_{p,N}(G):=\{b_C\colon C\in\operatorname{Cl}_{p,N}(G)\}\,.$$

In the following, C is always assumed to be a conjugacy class of G and $\bar{C}\in\operatorname{Cl}(\bar{G})$ denotes its image in $\bar{G}$.

Lemma 2.33. *For $C \in \mathrm{Cl}_{p',N}(G)$, we have $\bar{C} \nsubseteq O_{p'}(\bar{G})$.*

Proof. Let $C \in \mathrm{Cl}_{p',N}(G)$ and assume $\bar{C} \subseteq O_{p'}(\bar{G})$. By Remark 2.20, $O_{p'}(\bar{G})$ is contained in $Z(\bar{G})$ and hence we have $|\bar{C}| = 1$. The assumption $C \in \mathrm{Cl}_{p',N}(G)$ implies that $|C|$ is not divisible by p, so by Remark 2.21, there exist elements $s \in O_{p'}(G)$ and $z \in Z(P)$ with $C = [sz]$. Since we have $szN \in O_{p'}(\bar{G})$ as well as $sN \in O_{p'}(G)N/N \subseteq O_{p'}(\bar{G})$, we obtain $zN \in O_{p'}(\bar{G})$ and hence $z \in N$ since z is a p-element. This implies $C \subseteq O_{p'}(G)N$, which contradicts the assumption $C \in \mathrm{Cl}_{p',N}(G)$. □

In particular, for every conjugacy class $C \in \mathrm{Cl}_{p',N}(G)$, the basis element $b_{\bar{C}}$ of $J(ZF\bar{G})$ corresponding to $\bar{C}$ (see Definition 2.22) is well-defined and hence the following definition makes sense:

Definition 2.34. By $\overline{\mathrm{Cl}}_{p',N}(G) \subseteq \mathrm{Cl}(\bar{G})$, we denote the set of images of the conjugacy classes in $\mathrm{Cl}_{p',N}(G)$. As before, we set

$$\overline{\mathrm{Cl}}^{+}_{p',N}(G) := \left\{ b_{\bar{C}} \colon \bar{C} \in \overline{\mathrm{Cl}}_{p',N}(G) \right\},$$

where $b_{\bar{C}}$ denotes the basis element of $J(ZF\bar{G})$ corresponding to $\bar{C}$ (see Definition 2.22).

Remark 2.35. For a conjugacy class $\bar{C} \in \mathrm{Cl}(\bar{G})$, it is possible that there exist preimages $C_1, C_2 \in \mathrm{Cl}(G)$ of $\bar{C}$ with $C_1 \in \mathrm{Cl}_{p',N}(G)$ and $C_2 \in \mathrm{Cl}_{p,N}(G)$: An example for $p = 2$ is given by the 2-group

$$G = \langle r, s, t \mid r^8 = s^2 = t^2 = [r,t] = [s,t] = 1,\ srs = r^{-1} \rangle \cong D_{16} \times C_2.$$

Set $C_1 := [t]$ and $C_2 := [tr^2]$. Since $t \in Z(G)$ holds, we have $|C_1| = 1$. One easily verifies that $C_2 = \{tr^2, tr^6\}$ is a class of length two. Now consider the normal subgroup $G' = \langle r^2 \rangle$. In G/G', the images of C_1 and C_2 coincide, which yields $C_1 \in \mathrm{Cl}_{2',G'}(G)$ and $C_2 \in \mathrm{Cl}_{2,G'}(G)$. ◁

Our aim is to establish a correspondence between the set $\mathrm{soc}(ZFG) \cap \mathrm{Im}(\nu_N^*)$ and the annihilator of $\overline{\mathrm{Cl}}^{+}_{p',N}(G)$ in $ZF\bar{G}$. As a first step, we observe the following:

Remark 2.36. Let $a \in ZF\bar{G}$. By Remark 1.27, we have $\nu_N^*(a) \in \mathrm{soc}(ZFG)$ if and only if $a \cdot \nu_N(J(ZFG)) = 0$ holds. In order to check whether $\nu_N^*(a)$ is contained in $\mathrm{soc}(ZFG)$, it therefore suffices to consider the product of a with the images of those basis elements of $J(ZFG)$ which are not contained in the kernel of ν_N. ◁

In the following, we therefore focus on the basis elements of $J(ZFG)$ corresponding to conjugacy classes in $\mathrm{Cl}_{p',N}(G)$. For these, we obtain an equivalent annihilation condition in the group algebra $F\bar{G}$.

Lemma 2.37. *For a conjugacy class $C \in \mathrm{Cl}_{p',N}(G)$ with image $\bar{C} \in \mathrm{Cl}(\bar{G})$, we have*

$$\nu_N(b_C) = \frac{|C|}{|\bar{C}|} \cdot b_{\bar{C}}.$$

In particular, for $a \in F\bar{G}$, we obtain $a \cdot b_{\bar{C}} = 0$ if and only if $\nu_N^*(a) \cdot b_C = 0$ holds.

Proof. By Remark 2.31, we have $\nu_N(C^+) = k \cdot \bar{C}^+$ for $k := |C|/|\bar{C}|$. If C is of the form sD with $s \in O_{p'}(G)$ and a G-conjugacy class $D \subseteq Z(P)$, we have

$$\nu_N(b_C) = \nu_N(C^+ - |C| \cdot s) = k(\bar{C}^+ - |\bar{C}| \cdot \bar{s}) = k \cdot b_{\bar{C}},$$

since we have $\bar{C} = \bar{s}\bar{D}$, where $\bar{s} \in O_{p'}(\bar{G})$ and $\bar{D} \in \mathrm{Cl}(\bar{G})$ as well as $\bar{D} \subseteq Z(\bar{P})$ hold. Similarly, if p divides $|C|$, then we have $\nu_N(b_C) = \nu_N(C^+) = k \cdot \bar{C}^+ = k \cdot b_{\bar{C}}$ since p divides $|\bar{C}|$ as well. In both cases, we therefore obtain $\nu_N(b_C) = k \cdot b_{\bar{C}}$. This yields a chain of equivalences

$$\nu_N^*(a) \cdot b_C = 0 \Leftrightarrow \nu_N^*(a \cdot \nu_N(b_C)) = 0 \Leftrightarrow a \cdot \nu_N(b_C) = 0 \Leftrightarrow a \cdot k \cdot b_{\bar{C}} = 0 \Leftrightarrow a \cdot b_{\bar{C}} = 0.$$

In the first step, we used Lemma 2.30, in the second the injectivity of ν_N^* and in the last equality that k is invertible modulo p since $C \in \mathrm{Cl}_{p',N}(G)$ holds. □

With this, we obtain the following relation:

Lemma 2.38. *For any $a \in F\bar{G}$, there is an equivalence*

$$a \in \mathrm{Ann}_{ZF\bar{G}}\big(\overline{\mathrm{Cl}}^+_{p',N}(G)\big) \Leftrightarrow \nu_N^*(a) \in \mathrm{soc}(ZFG).$$

Proof. First consider an element $a \in \mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+_{p',N}(G))$. Since Lemma 2.30 yields $\nu_N^*(a) \in ZFG$, it remains to show that $\nu_N^*(a) \cdot b_C = 0$ holds for every conjugacy class $C \in \mathrm{Cl}(G)$ with $C \not\subseteq O_{p'}(G)$. If $C \in \mathrm{Cl}_{p,N}(G)$ holds, we have $b_C \in \mathrm{Ker}(\nu_N)$. Similarly, if $C \subseteq O_{p'}(G)N$ holds, then C is of the form $[sn] = s[n]$ with $s \in O_{p'}(G) \subseteq Z(G)$ and $n \in N\backslash\{1\}$, so we obtain $b_C = s \cdot b_{[n]} \in \mathrm{Ker}(\nu_N)$. By Remark 2.36, it therefore remains to consider the case $C \in \mathrm{Cl}_{p',N}(G)$. By assumption, we have $a \cdot b_{\bar{C}} = 0$ and this implies $\nu_N^*(a) \cdot b_C = 0$ by Lemma 2.37. Summarizing, this derivation shows $\nu_N^*(a) \in \mathrm{soc}(ZFG)$.

Conversely, let $a \in F\bar{G}$ be an element with $\nu_N^*(a) \in \mathrm{soc}(ZFG)$ and consider a conjugacy class $\bar{C} \in \overline{\mathrm{Cl}}_{p',N}(G)$. By definition, there exists a preimage $C \in \mathrm{Cl}_{p',N}(G)$ of $\bar{C}$. By assumption, we have $\nu_N^*(a) \cdot b_C = 0$ and Lemma 2.37 then yields $a \cdot b_{\bar{C}} = 0$. Since $a \in ZF\bar{G}$ holds by Lemma 2.30, we obtain $a \in \mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+_{p',N}(G))$ as claimed. □

The preceding lemma yields a bijective correspondence

$$\begin{aligned} \mathrm{Ann}_{ZF\bar{G}}\big(\overline{\mathrm{Cl}}^+_{p',N}(G)\big) &\longleftrightarrow N^+ \cdot FG \cap \mathrm{soc}(ZFG) \\ a &\longleftrightarrow \nu_N^*(a) \end{aligned}$$

Moreover, Lemma 2.30 (iv) shows that $a \in (\bar{G}')^+ \cdot F\bar{G}$ is equivalent to $\nu_N^*(a) \in (G')^+ \cdot FG$. In particular, we obtain the following necessary condition for $\mathrm{soc}(ZFG) \trianglelefteq FG$:

Theorem 2.39. *Let G be a finite group of the form $P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H and consider a normal subgroup $N \trianglelefteq G$. If $\mathrm{soc}(ZFG)$ is an ideal of FG, we have*

$$\mathrm{Ann}_{ZF\bar{G}}\big(\overline{\mathrm{Cl}}^+_{p',N}(G)\big) \subseteq (\bar{G}')^+ \cdot F\bar{G}. \tag{2.14}$$

In particular, if the set $\overline{\mathrm{Cl}}_{p',N}(G)$ is empty, then $\bar{G}$ is abelian.

Proof. Assume that $\operatorname{soc}(ZFG)$ is an ideal of FG. For any $a \in \operatorname{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+_{p',N}(G))$, Lemmas 2.38 and 2.8 yield

$$\nu_N^*(a) \in \operatorname{soc}(ZFG) \subseteq (G')^+ \cdot FG.$$

By Lemma 2.30 (iv), this implies $a \in (\bar{G}')^+ \cdot F\bar{G}$. If the set $\overline{\mathrm{Cl}}_{p',N}(G)$ is empty, we obtain

$$ZF\bar{G} = \operatorname{Ann}_{ZF\bar{G}}\big(\overline{\mathrm{Cl}}^+_{p',N}(G)\big) \subseteq (\bar{G}')^+ \cdot F\bar{G},$$

which implies $\bar{G}' = 1$, that is, $\bar{G}$ is abelian. □

Throughout this thesis, we obtain information on the structure of G by going over to a suitable quotient group G/N and determining the set of relevant basis elements $\overline{\mathrm{Cl}}^+_{p',N}(G)$. The inclusion given in (2.14) is then a necessary condition for $\operatorname{soc}(ZFG) \trianglelefteq FG$. As a first application of the above results, we give an alternative proof of the following special case of Lemma 1.25:

Corollary 2.40. *If* $\operatorname{soc}(ZFG)$ *is an ideal of* FG*, then also* $\operatorname{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ *holds.*

Proof. Assume that $\operatorname{soc}(ZFG)$ is an ideal of FG. Since $\overline{\mathrm{Cl}}^+_{p',N}(G)$ is a subset of the basis elements of $J(ZF\bar{G})$, Theorem 2.39 yields

$$\operatorname{soc}(ZF\bar{G}) = \operatorname{Ann}_{ZF\bar{G}} J(ZF\bar{G}) \subseteq \operatorname{Ann}_{ZF\bar{G}}\big(\overline{\mathrm{Cl}}^+_{p',N}(G)\big) \subseteq (\bar{G}')^+ \cdot F\bar{G}$$

and hence we have $\operatorname{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$ by Lemma 2.8. □

In general, the necessary condition given in (2.14) is much stronger than the statement of Corollary 2.40:

Remark 2.41. The set $\overline{\mathrm{Cl}}_{p',N}(G)$ can be much smaller than $\mathrm{Cl}(\bar{G})$ or even empty. In terms of annihilators, this means that $\operatorname{soc}(ZF\bar{G})$ can be a tiny subset of $\operatorname{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+_{p',N}(G))$. In particular, we frequently encounter the situation that $\operatorname{soc}(ZF\bar{G})$ is an ideal of $F\bar{G}$, which translates to $\operatorname{soc}(ZF\bar{G}) \subseteq (\bar{G}')^+ \cdot F\bar{G}$ by Lemma 2.8, but we have

$$\operatorname{Ann}_{ZF\bar{G}}\big(\overline{\mathrm{Cl}}^+_{p',N}(G)\big) \not\subseteq (\bar{G}')^+ \cdot F\bar{G},$$

which yields $\operatorname{soc}(ZFG) \ntrianglelefteq FG$ by Theorem 2.39. ◁

For the conjugacy classes of elements in H, we obtain the following characterization:

Lemma 2.42. *Let* $G = P \rtimes H$ *with* $P \in \mathrm{Syl}_p(G)$ *and an abelian* p'*-group* H *and consider a normal* p*-subgroup* $N \trianglelefteq G$*. For the conjugacy class* $C := [h]$ *of an element* $h \in H$ *with* $h \notin O_{p'}(G)$*, we have* $C \in \mathrm{Cl}_{p',N}(G)$ *if and only if* $h \in C_G(N)$ *holds.*

Proof. If h is not contained in $C_G(N)$, we have $\nu_N(C^+) = 0$ by Corollary 2.13 and hence $C \notin \mathrm{Cl}_{p',N}(G)$. Conversely, assume $h \in C_G(N)$. By Remark 2.17, we have $[g,h] \notin N$ for all $g \in G \backslash C_G(h)$ since h does not commute with its nontrivial commutators. This yields $|C \cap hN| = 1$ and hence $|C \cap aN| \leq 1$ for all $a \in G$ (see Remark 2.10). This yields $|C| = |\bar{C}|$ and hence $C \in \mathrm{Cl}_{p',N}(G)$ follows. □

By Remark 2.31, we therefore have $\nu_N(C^+) \neq 0$ if and only if $C \subseteq C_G(N)$ holds, that is, we obtain equivalence in Corollary 2.13.

2.5.2 Central products

After having inspected the transition to quotient groups in detail, we now consider our main problem in the special situation that $G = G_1 * G_2$ is a central product of subgroups G_1 and G_2. Remarkably, the property $\operatorname{soc}(ZFG) \trianglelefteq FG$ is inherited by the group algebras of the factors G_1 and G_2. Central products will arise naturally in the context of 2-groups as well as in several other occasions in our treatment of arbitrary finite groups.

Lemma 2.43. *Suppose that the finite group G is a central product of two subgroups G_1 and G_2. In this case, $\operatorname{soc}(ZFG) \trianglelefteq FG$ is equivalent to $\operatorname{soc}(ZFG_i) \trianglelefteq FG_i$ for $i = 1, 2$.*

Proof. If $\operatorname{soc}(ZFG_i) \trianglelefteq FG_i$ holds for $i = 1, 2$, then $\operatorname{soc}(ZFG) \trianglelefteq FG$ holds by Lemma 1.17 and Corollary 2.40 since G is isomorphic to a quotient group of the direct product $G_1 \times G_2$ and we have $F(G_1 \times G_2) \cong FG_1 \otimes_F FG_2$.

Conversely, assume that $\operatorname{soc}(ZFG)$ is an ideal of FG. By Corollary 2.16, G is of the form $P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H. In particular, $G_1', G_2' \subseteq P$ are p-groups. By Theorem 2.15, it follows that G_i is of the form $G_i = P_i \rtimes H_i$ with $P_i \in \operatorname{Syl}_p(G_i)$ and an abelian p'-group H_i $(i = 1, 2)$. In particular, there exist bases for $J(ZFG)$ and $J(ZFG_i)$ of the form given in Theorem 2.23.

First assume $O_{p'}(G) = 1$. This implies $O_{p'}(G_i) = O_{p'}(G) \cap G_i = 1$ for $i = 1, 2$. In this case, the intersection $Z \coloneqq G_1 \cap G_2 \subseteq Z(G)$ is a p-group (see Theorem 2.1). We consider an element

$$x_1 \coloneqq \sum_{g_1 \in G_1} a_{g_1} g_1 \in \operatorname{soc}(ZFG_1).$$

Our aim is to show $x_1 \in (G_1')^+ \cdot FG_1$. This is achieved by "extending" x_1 to an element $y \in \operatorname{soc}(ZFG)$. To this end, we decompose every element $g \in G$ in the form $g = g_1 g_2$ with $g_i \in G_i$ $(i = 1, 2)$ and set $b_g \coloneqq a_{g_1} \in F$. This is well-defined: Assume that $g_1 g_2 = g_1' g_2'$ holds for some $g_1' \in G_1$ and $g_2' \in G_2$. This implies $g_1' = g_1 z$ and $g_2' = g_2 z^{-1}$ for some $z \in Z$. Since the p-group Z is contained in $Z(G_1) \cap P_1$, we have $z - 1 \in J(ZFG_1)$ and hence $x_1 \cdot (z - 1) = 0$ holds, so $a_{g_1'} = a_{g_1 z} = a_{g_1}$ follows by Remark 2.25. In the following, we consider the element

$$y \coloneqq \sum_{g \in G} b_g g \in FG.$$

Note that $y \in FG \cdot G_2^+$ holds. First we show that y is contained in ZFG. To this end, let $g, h \in G$ be two conjugate elements and write $g = g_1 g_2$ and $h = h_1 h_2$ with $g_i, h_i \in G_i$ $(i = 1, 2)$. We can choose h_1 and h_2 in such a way that g_1 and h_1 are conjugate. Hence $b_g = a_{g_1} = a_{h_1} = b_h$ follows from the fact that x_1 has constant coefficients on the conjugacy classes of G_1.

We claim that y is contained in $\operatorname{soc}(ZFG)$. For any $C \in \operatorname{Cl}(G)$ with $C \not\subseteq O_{p'}(G)$, we need to show that $y \cdot b_C = 0$ holds for the basis element b_C of $J(ZFG)$ corresponding to C (see Definition 2.22). Write $C = C_1 \cdot C_2$ for some $C_i \in \operatorname{Cl}(G_i)$ for $i = 1, 2$. In Section 2.1, we have shown that there exists a set $R_2 \subseteq C_2$ such that C is the disjoint union of the sets $C_1 r$ with $r \in R_2$. Moreover, we have $|C| = |C_1| \cdot |R_2|$ and $|R_2|$ divides $|C_2|$. By interchanging the roles of C_1 and C_2, it follows similarly that C is a disjoint union of the sets rC_2 with $r \in R_1$ for a certain subset $R_1 \subseteq C_1$. In particular, $|C_1|$ and $|C_2|$ divide $|C|$.

If p divides $|C_2|$, then $|C|$ is divisible by p. Moreover, we then have $G_2^+ \cdot C_2^+ = 0$ and hence

$$y \cdot b_C = y \cdot C^+ = \sum_{r \in R_1} y \cdot (rC_2)^+ = 0,$$

since y is a multiple of G_2^+ in FG. Now assume that $|C_2|$ is not divisible by p. If p divides $|C_1|$, we have $x_1 \cdot C_1^+ = 0$. For any $t = t_1 t_2$ with $t_i \in G_i$ $(i = 1, 2)$, Remark 2.25 then yields

$$\sum_{c \in C} b_{tc^{-1}} = \sum_{c_1 \in C_1} \sum_{r \in R_2} a_{t_1 c_1^{-1}} = |R_2| \cdot \sum_{c_1 \in C_1} a_{t_1 c_1^{-1}} = 0.$$

Applying Remark 2.25 again, we obtain $y \cdot b_C = y \cdot C^+ = 0$.

Now assume that both $|C_1|$ and $|C_2|$ are not divisible by p. Since $|C|$ divides $|C_1| \cdot |C_2|$, also $|C|$ is not divisible by p, so we have $b_C = C^+ - |C| \cdot 1$. First suppose $C_1 = 1$. We then obtain $G_2^+ \cdot b_C = G_2^+ \cdot (C_2^+ - |C_2| \cdot 1) = 0$ and hence $y \cdot b_C = 0$ since y is a multiple of G_2^+. Now let $C_1 \neq 1$. By (2.11) together with Theorem 2.1, this yields $C_1 \subseteq Z(P_1)$ since we have $O_{p'}(G_1) = 1$, and hence $x_1 \cdot (C_1^+ - |C_1| \cdot 1) = 0$ follows from the assumption $x_1 \in \operatorname{soc}(ZFG_1)$. For any $t = t_1 t_2$ with $t_i \in G_i$ $(i = 1, 2)$, this yields

$$\sum_{c \in C} b_{tc^{-1}} = \sum_{c_1 \in C_1} \sum_{r \in R_2} a_{t_1 c_1^{-1}} = |R_2| \cdot \sum_{c_1 \in C_1} a_{t_1 c_1^{-1}} = |R_2| \cdot |C_1| \cdot a_{t_1} = |C| \cdot b_t$$

and hence $y \cdot b_C = 0$ follows again by Remark 2.25. In the third step in the above equation, we use the condition on x_1 together with Remark 2.25. Summarizing, the above derivation shows $y \in \operatorname{soc}(ZFG) \subseteq (G')^+ \cdot FG$. For two elements $g_1, h_1 \in G_1$ with $g_1 h_1^{-1} \in G_1' \subseteq G'$, we therefore obtain $a_{g_1} = b_{g_1} = b_{h_1} = a_{h_1}$. This yields $\operatorname{soc}(ZFG_1) \subseteq (G_1)^+ \cdot FG_1$, which implies $\operatorname{soc}(ZFG_1) \trianglelefteq FG_1$ by Lemma 2.8. Symmetry yields $\operatorname{soc}(ZFG_2) \trianglelefteq FG_2$, which proves the claim in the situation where G has a trivial p'-core.

Now we consider the general case, so we do not assume $O_{p'}(G) = 1$. Set $\bar{G} := G/O_{p'}(G)$. By Lemma 2.19, we obtain $\operatorname{soc}(ZF\bar{G}) \trianglelefteq F\bar{G}$, and $\bar{G}$ decomposes in the form $\bar{G}_1 * \bar{G}_2$, where $\bar{G}_i$ denotes the image of G_i in $\bar{G}$ for $i = 1, 2$. By the above argument, we obtain $\operatorname{soc}(ZF\bar{G}_i) \trianglelefteq F\bar{G}_i$ since $\bar{G}$ has a trivial p'-core. Since $O_{p'}(G_i) = O_{p'}(G) \cap G_i$ holds, we have $\bar{G}_i \cong G_i/O_{p'}(G) \cap G_i \cong G_i/O_{p'}(G_i)$ and hence $\operatorname{soc}(ZFG_i) \trianglelefteq FG_i$ follows by Lemma 2.19. □

Remark 2.44. In case that $G_1 \cap G_2 = 1$ holds, the group G is the direct product of the subgroups G_1 and G_2. In this case, the statement of the preceding lemma reduces to the known result that $\operatorname{soc}(ZF(G_1 \times G_2))$ is an ideal of $F(G_1 \times G_2)$ if and only if this holds for the respective group algebras FG_1 and FG_2 (see Lemma 1.17). For group algebras, Lemma 2.43 can therefore be viewed as a generalization of Lemma 1.17. ◁

2.6 Special case: abelian Sylow p-subgroups

Let F be an algebraically closed field of characteristic $p > 0$. Throughout, we assume that G is a finite group of the form $P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H. In this section, we additionally require P to be abelian, although this condition will be sometimes relaxed to $G' \subseteq Z(P)$ and other related properties. We study the conjugacy class structure of G and prove that $\operatorname{soc}(ZFG)$ is an ideal of FG in this situation. Later, we repeatedly return to this case by going over to the quotient group $D = G/P'$.

Recall that for any $g \in G$, we may write $[g] = U_g \cdot g$ with $U_g := \{[a, g] : a \in G\} \subseteq G'$ (see (2.1)). As before, the p-part and p'-part of g are denoted by g_p and $g_{p'}$, respectively. We begin by investigating the conjugacy class structure of G.

Lemma 2.45.

(i) Assume that P is abelian. For $g \in G$, we obtain $C := [g] = [g_p] \cdot [g_{p'}]$. Moreover, we have $[g_{p'}] = [h]$ for the element $h \in H$ with $h \in gP$.

(ii) Assume $G' \subseteq Z(P)$. For any $u \in P$ and $h \in C_G(H)$, we have $h[u] \subseteq [hu]$. In particular, U_h is a normal subgroup of G.

Proof.

(i) Since hP is a p'-section of G (see proof of Theorem 2.15), it contains a unique p'-conjugacy class, namely $[h]$, so we have $[g_{p'}] = [h]$. Since H is abelian, there exists an element $u \in P$ with $g_{p'} = uhu^{-1}$. This yields $g = uhg_pu^{-1}$ since P is abelian and hence we have $[g] = [hg_p]$. Without loss of generality, we therefore assume $g_{p'} = h \in H$. Let $x \in G$ and write $x = p_xh_x$ with $p_x \in P$ and $h_x \in H$. Then

$$xgx^{-1} = xhx^{-1} \cdot xg_px^{-1} = p_xhp_x^{-1} \cdot h_xg_ph_x^{-1}, \tag{2.15}$$

since both P and H are abelian. This yields

$$\begin{aligned} C &= \{p_xhp_x^{-1} : p_x \in P\} \cdot \{h_xg_ph_x^{-1} : h_x \in H\} \\ &= \{xhx^{-1} : x \in G\} \cdot \{yg_py^{-1} : y \in G\} \\ &= [h] \cdot [g_p]. \end{aligned}$$

(ii) By Theorem 2.4, we have

$$G = HP = HC_P(H)[P, H] = HC_P(H)G', \tag{2.16}$$

since $[P, H]$ is contained in G'. Hence for every $u' \in [u]$, there exists an element $h' \in HC_P(H)$ with $h'uh'^{-1} = u'$ since G' centralizes P. As $h' \in C_G(H)$ holds, we obtain $hu' = h'huh'^{-1} \in [hu]$. Since H is abelian, we have $U_h = \{[a, h] : a \in P\}$. Moreover, we obtain $[p_1p_2, h] = [p_1, h] \cdot [p_2, h]$ for all $p_1, p_2 \in P$ and hence U_h is a subgroup of G'. From the first part of this statement, it follows that U_h is a union of conjugacy classes and hence a normal subgroup of G. □

The following observation occasionally simplifies the verification that a given element $y \in ZFG$ is contained in $\operatorname{soc}(ZFG)$:

Corollary 2.46. *Assume that P is abelian and let $h \in H$. If an element $y \in ZFG$ annihilates $[h]^+$, then it annihilates C^+ for every conjugacy class $C \subseteq hP$.*

Proof. Let $C \subseteq hP$ be a conjugacy class and consider an element $g \in C$. The group P acts on C by conjugation and the orbits of this action are of the form $u[g_{p'}] = u[h]$ for elements $u \in P$ by Lemma 2.45. In particular, C is a disjoint union of sets of this form. Hence $y \cdot [h]^+$ implies $y \cdot C^+ = 0$. □

The main result of this part is the following:

Theorem 2.47. *If $G' \subseteq Z(P)$ holds, then* $\operatorname{soc}(ZFG)$ *is an ideal of* FG.

Proof. By replacing G by the quotient group $G/O_{p'}(G)$ whose Sylow p-subgroup is isomorphic to P, we may assume $O_{p'}(G) = 1$ (see Lemma 2.19). We consider an element $y = \sum_{g\in G} a_g g \in \operatorname{soc}(ZFG)$. Let $x \in G$ and write $x = ub$ with $u \in HC_P(H) \subseteq C_G(H)$ and $b \in G'$ (see (2.16)). For $b = 1$, we have $x = u$ and hence $a_x = a_u$, so assume $b \neq 1$. Observe that $m := |[b]|$ and p are coprime since b centralizes P (see (2.11)). By (2.13), applied to the conjugacy class $[b^{-1}]$, we obtain

$$m \cdot a_u = \sum_{b' \in [b]} a_{ub'} = m \cdot a_{ub},$$

because the elements in $u[b]$ are conjugate by Lemma 2.45 (ii). Since m is invertible modulo p, we obtain $a_x = a_{ub} = a_u$. This shows that $y \in (G')^+ \cdot FG$ holds and hence $\operatorname{soc}(ZFG)$ is an ideal of FG by Lemma 2.8. □

Remark 2.48. For an arbitrary finite group G, Theorem 2.15 yields

$$G' \subseteq O_p(G) \Rightarrow R(FG) \trianglelefteq FG,$$

where $R(FG)$ denotes the Reynolds ideal of FG. In particular, G is of the form $P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H in this case. With this, the statement of the previous theorem reads

$$G' \subseteq Z(O_p(G)) \Rightarrow \operatorname{soc}(ZFG) \trianglelefteq FG. \qquad \triangleleft$$

In the next chapter, we prove that if G is a p-group and the characteristic p is odd, then the condition $G' \subseteq Z(O_p(G)) = Z(G)$ is also necessary for $\operatorname{soc}(ZFG) \trianglelefteq FG$ to hold. The next example shows that this statement is not true for arbitrary finite groups, that is, $\operatorname{soc}(ZFG) \trianglelefteq FG$ does in general not imply $G' \subseteq Z(O_p(G))$:

Example 2.49. Let F be an algebraically closed field of characteristic $p = 3$ and consider the group $G = \texttt{SmallGroup}(216, 86)$ in the computer algebra system GAP [17]. It is of the form $G' \rtimes H$, where G' is the extraspecial group of order 27 and exponent three, and H is a cyclic subgroup of order eight, which permutes the eight nontrivial elements of G'/G'' transitively and acts on $G'' = Z(G')$ by inversion. In particular, $G' = O_3(G)$ is non-abelian. We now verify that $\operatorname{soc}(ZFG)$ is an ideal in FG.

The group G has eight $3'$-sections, namely the cosets of G' in G (see the proof of Theorem 2.15), which further decompose into conjugacy classes. One easily verifies that the $3'$-section G' decomposes into the three conjugacy classes $\{1\}$, $G''\backslash\{1\}$ and $G'\backslash G''$. For $1 \neq h \in H$, the $3'$-section hG' consists of a single conjugacy class for $\operatorname{ord}(h) = 8$ and of two conjugacy classes for $\operatorname{ord}(h) \in \{2, 4\}$.

By Remark 2.27, the socle $\mathrm{soc}(ZFG)$ is a H-graded module. We show that the homogeneous component $S_h := \mathrm{soc}(ZFG) \cap FhG'$ is one-dimensional for every $h \in H$ since this implies $S_h = F(hG')^+$ and hence $\mathrm{soc}(ZFG) \subseteq (G')^+ \cdot FG$. In the case $\mathrm{ord}(h) = 8$, we clearly have $\dim S_h = 1$. Now consider an element $h \in H$ of order $\mathrm{ord}(h) \in \{2, 4\}$. Since hG' consists of two conjugacy classes, S_h can be at most two-dimensional. If $\dim S_h = 2$ holds, then $[h]^+$ and $(hG')^+$ form a basis of S_h. Since the element h centralizes G'' and acts transitively on the nontrivial cosets in G'/G'', we obtain $(G'')^+ \cdot [h]^+ = (hG')^+ \neq 0$ by Remark 2.17, which is a contradiction to the assumption $[h]^+ \in \mathrm{soc}(ZFG)$. This implies $\dim S_h = 1$. Furthermore, this calculation shows that $(G'')^+$ is not contained in $\mathrm{soc}(ZFG)$, so by the same argument as before, we obtain $\dim S_1 = 1$. Summarizing, this yields $\mathrm{soc}(ZFG) \subseteq (G')^+ \cdot FG$ and hence $\mathrm{soc}(ZFG)$ is an ideal of FG (see Lemma 2.8). Since G' is non-abelian, this shows that the condition $G' \subseteq Z(O_p(G))$ stated in Theorem 2.47 is not necessary for $\mathrm{soc}(ZFG) \trianglelefteq FG$ to hold. $\triangleleft$

In characteristic two, the sufficient condition $G' \subseteq Z(P)$ can be relaxed to $G' \subseteq Y(P)Z(P)$, where $Y(P) \subseteq P'$ denotes the subgroup defined in (2.2). Similarly to the case of odd characteristic, this will be motivated by our results on 2-groups derived in Chapter 3, which we will also need for the proof. Here, we lay the foundation by proving the following:

Lemma 2.50. *Let $p = 2$. If $G' \subseteq Y(P)Z(P)$ holds, we have $G = C_P(H) * [Z(P), H]H$. In this case,* $\mathrm{soc}(ZFG) \trianglelefteq FG$ *is equivalent to* $\mathrm{soc}(ZFC_P(H)) \trianglelefteq FC_P(H)$.

Proof. Assume that $G' \subseteq Y(P)Z(P)$ holds. In particular, G' is abelian (see Lemma 2.6). By Theorem 2.4, this yields

$$P = C_P(H)[P, H] = C_P(H)G' = C_P(H)Y(P)Z(P). \tag{2.17}$$

We first show that P' is contained in $C_P(H)'$. By using the decomposition in (2.17), we obtain

$$P' \subseteq C_P(H)' \cdot [Y(P), C_P(H)]. \tag{2.18}$$

Observe that $C_P(H)'$ is a normal subgroup of P since G' is abelian. We consider the quotient group $\bar{P} := P/C_P(H)'$ and denote the image of any subgroup $S \subseteq P$ in $\bar{P}$ by $\bar{S}$. Then we have

$$\overline{Y(P)} \subseteq \bar{P}' = [\overline{Y(P)}, \overline{C_P(H)}] = [\overline{Y(P)}, \bar{P}].$$

In the second step, we use (2.18). The last equality follows from (2.17) together with the fact that $Y(P)$ centralizes G'. Since $\bar{P}$ is nilpotent, the above inclusion implies $\overline{Y(P)} = 1$, which translates to $Y(P) \subseteq C_P(H)'$. Hence $P' \subseteq C_P(H)'$ follows by (2.18). This implies that $C_P(H)$ is normal in P and since H centralizes $C_P(H)$, we even obtain $C_P(H) \trianglelefteq G$.

We claim that $G = C_P(H) * W$ holds for $W := H[Z(P), H]$. Note that $[P, H] = [Z(P), H]$ follows by (2.17) together with $Y(P) \subseteq C_P(H)$ and hence W is a normal subgroup of G. Moreover, we obtain $P = C_P(H)[Z(P), H]$, which implies that G is generated by $C_P(H)$ and W. Finally, observe that $C_P(H)$ centralizes $W \subseteq HZ(P)$. Since W has an abelian Sylow p-subgroup, we obtain $\mathrm{soc}(ZFW) \trianglelefteq FW$ by Theorem 2.47. By Lemma 2.43, the condition $\mathrm{soc}(ZFG) \trianglelefteq FG$ is therefore equivalent to $\mathrm{soc}(ZFC_P(H)) \trianglelefteq FC_P(H)$. □

In order to apply the results on p-groups which will be derived in the next chapter to the group $C_P(H)$, we will need the following consequence of the preceding proof:

Remark 2.51. In the situation of Lemma 2.50, we have $Y(P) \subseteq C_P(H)$ and hence

$$Y(P)Z(P) \cap C_P(H) = Y(P) \cdot (Z(P) \cap C_P(H)) = Y(P) \cdot Z(C_P(H))$$

by Dedekind's identity. We obtain $Y(P) = Y(C_P(H))$ since $P = C_P(H) * [Z(P), H]$ holds by Lemma 2.50 and $[Z(P), H]$ centralizes P. Since $G' \subseteq Y(P)Z(P)$ holds by assumption, this implies

$$C_P(H)' \subseteq G' \cap C_P(H) \subseteq Y(P) \cdot Z(C_P(H)) = Y(C_P(H)) \cdot Z(C_P(H)).$$

In the next section (see Corollary 3.17), we will see that $\operatorname{soc}(ZFC_P(H))$ is an ideal of $FC_P(H)$ in this situation. Lemma 2.50 then yields $\operatorname{soc}(ZFG) \trianglelefteq FG$. $\triangleleft$

Chapter 3

Groups of prime power order

Throughout, we assume that F is an algebraically closed field of characteristic $p > 0$. The principal aim of this section is to prove the following characterization of the p-groups G which satisfy $\operatorname{soc}(ZFG) \trianglelefteq FG$:

Theorem 3.1. *Let G be a finite p-group. Then $\operatorname{soc}(ZFG)$ is an ideal of FG if and only if one of the following statements holds:*

(i) $c(G) \leq 2$, *that is,* $G' \subseteq Z(G)$.

(ii) $p = 2$ *and* $G' \subseteq Y(G)Z(G)$, *where* $Y(G)$ *is the subgroup generated by all elements* gf^{-1} *for which* $\{f, g\}$ *is a conjugacy class of length two in* G.

In both cases, we have $\operatorname{soc}(ZFG) = (Z(G)G')^+ \cdot FG$.

This chapter is structured in the following way: In Section 3.1, we begin with some preliminary results on p-groups which satisfy $\operatorname{soc}(ZFG) \trianglelefteq FG$. Afterwards, we distinguish the cases $p \geq 3$ (see Section 3.2) and $p = 2$ (see Section 3.3).

3.1 Preliminary results

This section consists of general results concerning p-groups G for which $\operatorname{soc}(ZFG)$ is an ideal in FG. They hold independently of the characteristic p being odd or even. Throughout, we assume that G is a finite p-group. We first observe that G is metabelian if $\operatorname{soc}(ZFG) \trianglelefteq FG$ holds. Afterwards, we use an observation on the center of the group to determine the structure of $\operatorname{soc}(ZFG)$ and to treat groups of nilpotency class at most two. Finally, we show that the property $\operatorname{soc}(ZFG) \trianglelefteq FG$ is preserved under isoclinism.

The following result will be generalized to arbitrary finite groups in Section 4.2.2:

Lemma 3.2. *If* $\operatorname{soc}(ZFG)$ *is an ideal of* FG, *then* G *is metabelian.*

Proof. Let N be a maximal abelian normal subgroup of G. By [20, Lemma 5.3.12], we have $N = C_G(N)$. Applying Lemma 2.28 yields $N^+ \in \operatorname{soc}(ZFG)$ and hence $G' \subseteq N$ follows by Remark 2.9. This implies that G' is abelian. □

The next result, a simple observation on the coefficients of an element of $\operatorname{soc}(ZFG)$ on cosets of $Z(G)$, has remarkable consequences. For instance, it enables us to determine the structure of $\operatorname{soc}(ZFG)$ in case that this set is an ideal of FG and to show that $\operatorname{soc}(ZFG) \trianglelefteq FG$ holds if G is of nilpotency class at most two.

Remark 3.3. Since G is a p-group, we have $O_p(Z(G)) = Z(G)$ and hence Remark 2.14 translates to $\mathrm{soc}(ZFG) \subseteq Z(G)^+ \cdot FG$ in this situation. ◁

Lemma 3.4. *If* $\mathrm{soc}(ZFG)$ *is an ideal of* FG*, then* $\mathrm{soc}(ZFG) = (Z(G)G')^+ \cdot FG$ *holds.*

Proof. Assume $\mathrm{soc}(ZFG) \trianglelefteq FG$. Lemma 2.8 then yields $\mathrm{soc}(ZFG) \subseteq (G')^+ \cdot FG$. Together with Remark 3.3, this implies $\mathrm{soc}(ZFG) \subseteq (Z(G)G')^+ \cdot FG$. Now consider the basis of $J(ZFG)$ given in Remark 2.24. Note that $Z(G)^+$ annihilates all basis elements $b_{[z]} = z - 1$ with $1 \neq z \in Z(G)$. Since every conjugacy class of G is contained in a certain coset of G', it follows that $(G')^+$ annihilates $b_C = C^+$ for every $C \in \mathrm{Cl}(G)$ with $C \nsubseteq Z(G)$. By Remark 2.24, this implies $(Z(G)G')^+ \in \mathrm{soc}(ZFG)$, which yields $\mathrm{soc}(ZFG) = (Z(G)G')^+ \cdot FG$ as claimed. □

Lemma 3.5. *If* G *is of nilpotency class at most two, then* $\mathrm{soc}(ZFG) \trianglelefteq FG$ *holds.*

Proof. We have $G' \subseteq Z(G)$ and hence $\mathrm{soc}(ZFG) \subseteq Z(G)^+ \cdot FG \subseteq (G')^+ \cdot FG$ (see Remark 3.3). By Lemma 2.8, $\mathrm{soc}(ZFG)$ is an ideal of FG. □

Surprisingly, the converse of the last statement holds if p is an odd prime number (see Theorem 3.1). This will be proven in Section 3.2.

We now study the transition from FG to $F\bar{G} := F[G/Z(G)]$ in greater detail. To this end, we first recall some notions from the second chapter:

Remark 3.6. The set $\mathrm{Cl}_{p',Z(G)}(G)$ introduced in Definition 2.32 is explicitly given by

$$\mathrm{Cl}_{p',Z(G)}(G) = \{C \in \mathrm{Cl}(G) \colon C \nsubseteq Z(G),\, gz \notin C \text{ for all } g \in C \text{ and } 1 \neq z \in Z(G)\}\,.$$

As before, we denote by $\mathrm{Cl}^+_{p',Z(G)}(G)$ the corresponding basis elements of $J(ZFG)$ introduced in Definition 2.22, that is, we have

$$\mathrm{Cl}^+_{p',Z(G)}(G) = \left\{C^+ \colon C \in \mathrm{Cl}_{p',Z(G)}(G)\right\}.$$

Let $\overline{\mathrm{Cl}}_{p',Z(G)}(G)$ and $\overline{\mathrm{Cl}}^+ := \overline{\mathrm{Cl}}^+_{p',Z(G)}(G)$ denote the images of the classes in $\mathrm{Cl}_{p',Z(G)}(G)$ in $\bar{G}$ and the corresponding class sums in $F\bar{G}$, respectively (see Definition 2.34). ◁

The investigation of $F\bar{G}$ is particularly useful since the necessary condition stated in Theorem 2.39 is equivalent to $\mathrm{soc}(ZFG) \trianglelefteq FG$ in this case:

Lemma 3.7. *We have* $\mathrm{soc}(ZFG) \trianglelefteq FG$ *if and only if* $\mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+) \subseteq (\bar{G}')^+ \cdot F\bar{G}$ *holds.*

Proof. If $\mathrm{soc}(ZFG)$ is an ideal of FG, then $\mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+) \subseteq (\bar{G}')^+ \cdot F\bar{G}$ follows by Theorem 2.39. On the other hand, assume that the latter inclusion holds and let $x \in \mathrm{soc}(ZFG)$. Set $\alpha := \nu^*_{Z(G)} \colon \bar{G} \to G$ to be the map introduced in Section 2.5.1. By Remark 3.3, we then obtain $x \in Z(G)^+ \cdot FG = \mathrm{Im}(\alpha)$ and Lemma 2.30 (ii) implies

$$\alpha^{-1}(x) \in \mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+) \subseteq (\bar{G}')^+ \cdot F\bar{G}.$$

By Lemma 2.30, this yields $x \in (G')^+ \cdot FG$, so $\mathrm{soc}(ZFG) \trianglelefteq FG$ follows by Lemma 2.8. □

Instead of $\mathrm{soc}(ZFG)$, we may therefore examine the annihilator of the set $\overline{\mathrm{Cl}}^+$ in $ZF\bar{G}$. This raises the question whether $\mathrm{soc}(ZFG_1) \trianglelefteq FG_1$ is equivalent to $\mathrm{soc}(ZFG_2) \trianglelefteq FG_2$ for p-groups G_1 and G_2 with $G_1/Z(G_1) \cong G_2/Z(G_2)$. In order to see that this is not the case, we consider the following example:

Example 3.8. Let F be an algebraically closed field of characteristic $p = 2$. Consider the group $G_1 = \texttt{SmallGroup}(64, 146) \cong (C_8 \times C_2 \times C_2) \rtimes C_2$ in GAP [17], which has the following presentation:

$$\langle a, b, c, d \mid a^8 = b^2 = c^2 = d^2 = [a, c] = [b, c] = [a, b] = [c, d] = 1, dad = a^3, dbd = bc\rangle.$$

We have $G_1' = \langle a^2, c\rangle \cong C_4 \times C_2$ and $Z(G_1) = \langle a^4, c\rangle$, which yields $\bar{G}_1 \cong C_2 \times D_8$. A short computation shows that $\mathrm{soc}(ZFG_1)$ is an ideal of FG_1. As a second example, we consider the group $G_2 = \texttt{SmallGroup}(64, 151) \cong (Q_{16} \times C_2) \rtimes C_2$ with the presentation

$$\langle a, b, c, d \mid a^8 = b^2a^4 = c^2 = d^2 = [a, c] = [b, c] = [c, d] = 1,\ [a, b] = a^2,\ dad = a^5,\ dbd = bc\rangle.$$

A short computation shows $G_2' = \langle a^2, c\rangle \cong C_4 \times C_2$ and $Z(G_2) = \langle a^4, c\rangle$, which yields $\bar{G}_2 \cong C_2 \times D_8$. Furthermore, using Lemma 2.8, it is easy to verify that $\mathrm{soc}(ZFG_2)$ is not an ideal in FG_2. $\triangleleft$

This example demonstrates that even if G_1' and G_2' as well as $\bar{G}_1$ and $\bar{G}_2$ are isomorphic, the condition $\mathrm{soc}(ZFG_1) \trianglelefteq FG_1$ is not equivalent to $\mathrm{soc}(ZFG_2) \trianglelefteq FG_2$. We additionally need to require that the two isomorphisms are compatible with each other, which is naturally captured by the notion of isoclinism introduced in Section 2.1. In the following, we assume that G_1 and G_2 are isoclinic p-groups with respect to isomorphisms $\varphi\colon G_1' \to G_2'$ and $\beta\colon \bar{G}_1 \to \bar{G}_2$. We write $\pi_i\colon G_i \to \bar{G}_i$ for the canonical projection onto $\bar{G}_i$ and set $\mathrm{Cl}_i := \mathrm{Cl}_{p',Z(G_i)}(G_i)$ for $i = 1, 2$. The key observation is a bijection between the sets Cl_1 and Cl_2, which we establish in the next statement.

Lemma 3.9. *For $x_1 \in G_1$ and $x_2 \in G_2$ with $\beta(\pi_1(x_1)) = \pi_2(x_2)$, we have $[x_1] \in \mathrm{Cl}_1$ if and only if $[x_2] \in \mathrm{Cl}_2$ holds.*

Proof. Let $[x_2] \in \mathrm{Cl}_2$. In particular, we have $x_2 \notin Z(G_2)$, which yields $x_1 \notin Z(G_1)$. Assume $[x_1] \notin \mathrm{Cl}_1$, that is, there exists an element $k_1 \in G_1$ with $1 \neq [x_1, k_1] \in Z(G_1)$. Let $k_2 \in G_2$ with $\pi_2(k_2) = \beta(\pi_1(k_1))$. Then we have

$$1 = \beta\big([\pi_1(x_1), \pi_1(k_1)]\big) = \big[\beta(\pi_1(x_1)), \beta(\pi_1(k_1))\big] = \big[\pi_2(x_2), \pi_2(k_2)\big] = \pi_2\big([x_2, k_2]\big).$$

This implies $[x_2, k_2] \in Z(G_2)$, which yields $[x_2, k_2] = 1$ since $[x_2] \in \mathrm{Cl}_2$ holds. With this, we obtain $\varphi([x_1, k_1]) = [x_2, k_2] = 1$, which is a contradiction. The other implication follows by symmetry. $\square$

In particular, this shows that the sets $\overline{\mathrm{Cl}}_{p',Z(G_1)}(G_1)$ and $\overline{\mathrm{Cl}}_{p',Z(G_2)}(G_2)$ are in bijective correspondence under β. This naturally leads to the following equivalence:

Lemma 3.10. *We have $\mathrm{soc}(ZFG_1) \trianglelefteq FG_1$ if and only if $\mathrm{soc}(ZFG_2) \trianglelefteq FG_2$ holds.*

Proof. Extending the map β F-linearly gives rise to an isomorphism $\tilde{\beta}\colon F\bar{G}_1 \to F\bar{G}_2$ of F-algebras, which restricts to the respective centers. By the above, we have

$$\tilde{\beta}\big(\mathrm{Ann}_{ZF\bar{G}_1}\big(\overline{\mathrm{Cl}}^+_{p',Z(G_1)}(G_1)\big)\big) = \mathrm{Ann}_{ZF\bar{G}_2}\big(\overline{\mathrm{Cl}}^+_{p',Z(G_2)}(G_2)\big)$$

and hence $\operatorname{soc}(ZFG_1) \trianglelefteq FG_1$ is equivalent to $\operatorname{soc}(ZFG_2) \trianglelefteq FG_2$ by Lemma 3.7. $\square$

Since every p-group is isoclinic to a stem group, we may therefore restrict our investigation to p-groups G which satisfy $Z(G) \subseteq G'$.

We conclude this part with two results which are the main ingredients of the proof of Theorem 3.1 for $p \geq 3$. Nevertheless, they hold in arbitrary positive characteristic.

Lemma 3.11. *Let G be an elementary abelian group of order $p^n \geq 3$ for some $n \in \mathbb{N}$. Then we have $\prod_{g \in G} g = 1$.*

Proof. If p is odd, then the statement follows from the fact that every nontrivial element in G differs from its inverse and their product is the identity element. Now we assume $p = 2$. We write $G = H \,\dot{\cup}\, aH$ for some $a \in G$ with $a^2 = 1$ and a subgroup $H \cong C_2^{n-1}$. Since $|H| = 2^{n-1}$ is even, we obtain

$$\prod_{g \in G} g = \prod_{h \in H} h \cdot (ah) = a^{|H|} \cdot \prod_{h \in H} h^2 = 1. \qquad \square$$

Lemma 3.12. *Let G be a p-group of nilpotency class exactly two. There exists an element $y \in ZFG \backslash (G')^+ \cdot FG$ such that $y \cdot S^+ = 0$ holds for all subgroups $S \subseteq G'$ with $|S| \geq 3$.*

Proof. Since G' is an abelian p-group, there exist a nontrivial group homomorphism $\alpha \colon G' \to F$. We define an element $y := \sum_{g \in G} a_g g \in FG$ by setting $a_g := \alpha(g)$ for $g \in G'$ and $a_g = 0$ otherwise. Then y is contained in ZFG since every element in $G' \subseteq Z(G)$ forms a conjugacy class of its own. Moreover, we have

$$a_{g_1 g_2} = a_{g_1} + a_{g_2} \tag{3.1}$$

for all $g_1, g_2 \in G'$ since α is a group homomorphism. Now consider a subgroup $S \subseteq G'$ with $|S| \geq 3$. By Remark 2.25, we need to show that for all $w \in G$, we have

$$\sum_{s \in S} a_{ws^{-1}} = 0.$$

For $w \notin G'$, all occurring coefficients are zero, so assume $w \in G'$. In this case, we obtain

$$\sum_{s \in S} a_{ws^{-1}} = |S| \cdot a_w + \sum_{s \in S} a_{s^{-1}} = \sum_{s \in S} a_{s^{-1}}.$$

Let $\{s_1, \ldots, s_k\}$ be a set of representatives for the cosets of $S \cap \operatorname{Ker}(\alpha)$ in S. We obtain

$$\sum_{s \in S} a_{s^{-1}} = \sum_{s \in S} \alpha(s)^{-1} = \sum_{i=1}^{k} \sum_{g \in S \cap \operatorname{Ker}(\alpha)} \alpha(s_i g)^{-1} = |S \cap \operatorname{Ker}(\alpha)| \cdot \sum_{i=1}^{k} \alpha(s_i)^{-1}.$$

In case $S \cap \operatorname{Ker}(\alpha) \neq 1$ holds, this expression is zero. For $S \cap \operatorname{Ker}(\alpha) = 1$, the map α induces an isomorphism between S and $\alpha(S)$. Since $\alpha(S)$ is elementary abelian, so is S. Lemma 3.11 then yields

$$\sum_{s \in S} a_{s^{-1}} = a_{\prod_{s \in S} s^{-1}} = a_1 = 0. \qquad \square$$

3.2 Odd characteristic

Let F be an algebraically closed field of odd characteristic p. We now prove that Theorem 3.1 holds in this case. Recall that $\overline{\mathrm{Cl}}^+$ denotes the set of class sums $\overline{\mathrm{Cl}}^+_{p',Z(G')}(G)$.

Lemma 3.13. *Assume $p \geq 3$ and let G be a finite p-group. Then $\mathrm{soc}(ZFG)$ is an ideal of FG if and only if G has nilpotency class at most two.*

Proof. In Lemma 3.5, we have already proven that $\mathrm{soc}(ZFG)$ is an ideal of FG if G has nilpotency class at most two. For the converse implication, we proceed by induction on the nilpotency class $c(G)$. Note that the quotient group $\bar{G} := G/Z(G)$ has class $c(G) - 1$. First assume $c(G) = 3$. We apply Lemma 3.12 to $\bar{G}$ and consider the element $y \in ZF\bar{G}$ constructed therein. By Lemma 2.45 (ii), any conjugacy class $[h] \in \overline{\mathrm{Cl}}_{p',Z(G)}(G)$ is of the form $[h] = Sh$ for some nontrivial subgroup $S \subseteq \bar{G}'$. Since p is odd, we have $|S| \geq 3$ and hence $y \cdot [h]^+ = y \cdot (Sh)^+ = 0$ holds by construction of y. This shows $y \in \mathrm{Ann}_{ZF\bar{G}}(\overline{\mathrm{Cl}}^+)$. Since the coefficients of y are not constant on $\bar{G}'$ by construction, Lemma 3.7 implies $\mathrm{soc}(ZFG) \ntrianglelefteq FG$. Now assume $c(G) > 3$. Since $\bar{G}$ has nilpotency class $c(G)-1$, we obtain $\mathrm{soc}(ZF\bar{G}) \ntrianglelefteq F\bar{G}$ by induction, which implies $\mathrm{soc}(ZFG) \ntrianglelefteq FG$ by Corollary 2.40. □

Remark 3.14. In the preceding proof, the fact that p is an odd prime number is only used in the initial step of the induction. It ensures that the length of any noncentral conjugacy class of $\bar{G}$ is at least three and hence that the corresponding class sum is annihilated by the element y constructed in Lemma 3.12. This approach fails for $p = 2$ since Lemma 3.11 is no longer valid if the subgroup S in question has order two. ◁

This indicates that the conjugacy classes of length two play an fundamental role in the classification of the 2-groups G which satisfy $\mathrm{soc}(ZFG) \trianglelefteq FG$, as we will see in the next section.

3.3 Characteristic $p = 2$

In the following, let F be an algebraically closed field of characteristic two and assume that G is a finite 2-group. In the first part of this section, we prove Theorem 3.1 for this case, that is, we characterize the 2-groups G for which $\mathrm{soc}(ZFG)$ is an ideal in FG. In the second part, we present some example classes of 2-groups with this property and study the transition to certain subgroups of G.

3.3.1 Proof of Theorem 3.1

In this section, we prove Theorem 3.1 for the case $p = 2$ and state some consequences. It turns out that the group $Y(G)$ introduced in (2.2) plays a central role in this derivation. It is defined by

$$Y(G) := \langle Y_C : C \in \mathrm{Cl}(G),\ |C| = 2 \rangle, \tag{3.2}$$

where we set $Y_C := \langle gf^{-1} \rangle$ for any conjugacy class $C = \{f, g\}$ of length two. Recall that

$$C_G(f) = C_G(g) \subseteq C_G(gf^{-1}) \tag{3.3}$$

holds in this situation. We first note the following:

Lemma 3.15. *We have* $\mathrm{soc}(ZFG) \subseteq Y(G)^+ \cdot FG$.

Proof. Let $y = \sum_{g \in G} a_g g \in \mathrm{soc}(ZFG)$. For a conjugacy class $C = \{f, g\}$ of length two, we have $c := gf^{-1} \in Y(G)$ and the condition $y \cdot C^+ = 0$ yields $a_x = a_{xc^{-1}}$ for all $x \in G$ by Remark 2.25. By induction, this implies $a_x = a_{xc_1^{-1} \cdots c_n^{-1}}$ for every $x \in G$ and all elements $c_1, \ldots, c_n$ arising from G-conjugacy classes of length two as above. This shows that y has constant coefficients on the cosets of $Y(G)$ in G, that is, we obtain $y \in Y(G)^+ \cdot FG$. □

Now we complete the proof of Theorem 3.1.

Theorem 3.16. *Let G be a finite 2-group. Then $\mathrm{soc}(ZFG)$ is an ideal in FG if and only if $G' \subseteq Y(G)Z(G)$ holds.*

Proof. First assume that $\mathrm{soc}(ZFG)$ is an ideal of FG and suppose that $Y(G)Z(G)$ does not contain G'. Then $Y(G)Z(G) \cap G'$ is a proper subgroup of G'. By [24, Theorem III.7.2], there exists a normal subgroup $N \trianglelefteq G$ with $Y(G)Z(G) \cap G' \subseteq N \subseteq G'$ such that $|G' : N| = 2$ holds. We set $M := Y(G)Z(G)N$ and show that M^+ is contained in $\mathrm{soc}(ZFG)$.

Note that $M^+ \in ZFG$ holds because M is a normal subgroup of G. Now we show that M^+ annihilates the basis elements of $J(ZFG)$ given in Remark 2.24. For every element $z \in Z(G) \subseteq M$, we have $(1 + z) \cdot M^+ = 0$. For a G-conjugacy class $C = \{f, g\}$ of length two, we obtain $C^+ \cdot Y(G)^+ = fY(G)^+ + gY(G)^+ = 0$ since $gf^{-1} \in Y(G)$ holds, and hence also M^+ annihilates C^+. By Remark 2.10, every conjugacy class $C \in \mathrm{Cl}(G)$ with $|C| \geq 4$ contains an even number of elements in every coset of N since C is contained in a certain coset of G' and $|G' : N| = 2$ holds. This implies that C^+ is annihilated by N^+ and hence by M^+. Summarizing, we obtain $M^+ \in \mathrm{soc}(ZFG)$. Note that we have $M \cap G' = N \subsetneq G'$, which implies $M^+ \notin (G')^+ \cdot FG$. By Lemma 2.8, this yields $\mathrm{soc}(ZFG) \ntrianglelefteq FG$, which is a contradiction. Hence G' must be contained in $Y(G)Z(G)$.

Conversely, assume that $G' \subseteq Y(G)Z(G)$ holds. By Remark 3.3 and Lemma 3.15, we have

$$\mathrm{soc}(ZFG) \subseteq (Y(G)Z(G))^+ \cdot FG \subseteq (G')^+ \cdot FG$$

and hence $\mathrm{soc}(ZFG)$ is an ideal of FG by Lemma 2.8. □

Although the proof above requires $p = 2$, the statement of Theorem 3.16 remains valid if p is odd. In this case, the group $Y(G)$ is trivial, which reduces the given condition to $G' \subseteq Z(G)$. This is the characterization stated in Theorem 3.1 (i).

We note the following consequence of Theorem 3.16 for arbitrary finite groups over a field of characteristic two:

Corollary 3.17. *Let F be an algebraically closed field of characteristic $p = 2$ and consider an arbitrary finite group G with $G' \subseteq Y(O_2(G)) \cdot Z(O_2(G))$. Then $\mathrm{soc}(ZFG)$ is an ideal of FG.*

Proof. By Theorem 2.15, the group G is of the form $P \rtimes H$ with $P \in \mathrm{Syl}_2(G)$ and an abelian Hall $2'$-subgroup H. By Remark 2.51, we obtain

$$C_P(H)' \subseteq Y(C_P(H)) \cdot Z(C_P(H)).$$

Theorem 3.16 yields $\mathrm{soc}(ZFC_P(H)) \trianglelefteq FC_P(H)$, which implies $\mathrm{soc}(ZFG) \trianglelefteq FG$ by Lemma 2.50. □

Remark 3.18. Again, let F be an algebraically closed field of characteristic two. For an arbitrary finite group, the condition $G' \subseteq Y(O_2(G)) \cdot Z(O_2(G))$ is not necessarily satisfied if $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds. For instance, consider the group $G := \mathrm{SL}_2(\mathbb{F}_3)$, for which we will show in Example 4.5 that $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds. Since $G' = O_2(G)$ is of nilpotency class exactly two, we have $G' \nsubseteq Z(G') = Y(G')Z(G')$. Note that this phenomenon also occurred in odd characteristic (see Theorem 3.1 together with Theorem 2.47 and Example 2.49). ◁

After this brief discussion of arbitrary finite groups, we now return to the investigation of 2-groups and the consequences of Theorem 3.16. The following example shows that the set of 2-groups of nilpotency class at most two forms a proper subset of the 2-groups G which satisfy $\mathrm{soc}(ZFG) \trianglelefteq FG$. More precisely, there exist 2-groups of arbitrary nilpotency class with the latter property.

Example 3.19 (Groups of maximal class)**.** Let G be a 2-group of maximal class, that is, a dihedral, semidihedral or generalized quaternion group. Then $\mathrm{soc}(ZFG)$ is an ideal of FG. By Lemma 3.5, it suffices to show this for $|G| \geq 16$ since $c(G) \leq 2$ holds otherwise. For the dihedral group

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ srs = r^{-1} \rangle$$

with $n = 2^{k-1}$ for some $k \geq 4$, we have $D'_{2n} = \langle r^2 \rangle$. Then $[r] = \{r, r^{-1}\}$ yields $D'_{2n} = \langle r^2 \rangle = Y_{[r]} \subseteq Y(D_{2n})$. For the semidihedral group

$$\mathrm{SD}_{2^n} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}-1} \rangle$$

with $n \geq 4$, we obtain $\mathrm{SD}'_{2^n} = \langle r^2 \rangle \subseteq Y(\mathrm{SD}_{2^n})$ since we have $[r] = \{r, r^{2^{n-2}-1}\}$ and hence $Y_{[r]} = \langle r^{2^{n-2}-2} \rangle = \langle r^2 \rangle$. Finally, we consider the generalized quaternion group

$$Q_{2^n} = \langle h, k \mid h^{2^{n-2}} = k^2,\ h^{2^{n-1}} = 1,\ k^{-1}hk = h^{-1} \rangle$$

with $n \geq 4$. Again, we obtain $Q'_{2^n} = \langle h^2 \rangle = Y_{[h]} \subseteq Y(Q_{2^n})$. By Theorem 3.16, $\mathrm{soc}(ZFG)$ is an ideal of FG for $G \in \{D_{2n}, \mathrm{SD}_{2^n}, Q_{2^n}\}$. ◁

As a second application of Theorem 3.16, we classify the finite 2-groups G for which $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds and which have a cyclic derived subgroup. By Lemma 3.10, it suffices to consider stem groups, that is, we may assume $Z(G) \subseteq G'$.

Lemma 3.20. *Let G be a stem 2-group which satisfies $\mathrm{soc}(ZFG) \trianglelefteq FG$ and assume that $G' = \langle c \rangle$ is cyclic. We obtain the following classification:*

(i) For $|G'| = 1$, the group G is abelian, so $G = 1$ holds.

(ii) For $|G'| = 2$, we have $G \in \{D_8 * Q_8^{r-1}, Q_8^r\}$ for some $r \in \mathbb{N}$, where $Q_8^t := Q_8 * \ldots * Q_8$ for $t \in \mathbb{N}_0$ denotes the central product of t copies of Q_8.

(iii) For $|G'| > 2$, we either have $c(G) = 2$ or $G = G_1 * G_2$ is a central product, where G_1 is a dihedral, semidihedral or generalized quaternion group of order $4 \cdot |G'|$ and we have $G_2 \in \{1, D_8 * Q_8^{r-1}, Q_8^r\}$ for some $r \in \mathbb{N}$.

Proof.

(i) Clear, since G is a stem group by assumption.

(ii) Since $Z(G) \subseteq G'$ is nontrivial, we have $G' = Z(G)$. By [4, Lemma 5.2], the quotient group $G/G' = G/Z(G)$ is elementary abelian and it follows that G is extraspecial. Hence we obtain $G \cong Q_8^r$ or $G \cong D_8 * Q_8^{r-1}$ for some $r \in \mathbb{N}$ by [20, Theorem 5.5.2].

(iii) Assume that G has nilpotency class $c(G) > 2$, so c is not contained in $Z(G)$. By Theorem 3.16, we have $G' \subseteq Y(G)Z(G)$. On the other hand, $Y(G)$ and $Z(G)$ are contained in G', which yields equality. Since $Z(G)$ is a proper subgroup of the cyclic group G', we even obtain $G' = Y(G)$. Since $Y(G)$ is cyclic, there exists a conjugacy class $C = \{f, g\}$ of length two with $Y(G) = Y_C$, so we may assume $gf^{-1} = c$. By (3.3), we have $C_G(f) = C_G(c) = C_G(G')$. Note that this subgroup has index two in G. In the following, we fix an element $q \in G \backslash C_G(f)$ and consider the subgroups $G_1 := \langle f, q, G' \rangle$ and $G_2 := C_G(q) \cap C_G(G')$. Note that G_1 is a normal subgroup of G since it contains G'.

We claim that G is generated by G_1 and G_2. Since $G/C_G(G')$ is generated by $qC_G(G')$, it suffices to show $C_G(G') = G_1G_2 \cap C_G(G')$. To this end, we consider an element $v \in C_G(G')$ and assume $v \notin G_2$, that is, $v \notin C_G(q)$. This yields $[q, v] = c^\ell$ for some $\ell \in \mathbb{N}$. For $k := v^{-1} \cdot f^\ell \in C_G(G')$, we then have

$$[q, k] = [q, v]^{-1} \cdot [q, f]^\ell = c^{-\ell} \cdot c^\ell = 1$$

and hence $k \in C_G(q) \cap C_G(G') = G_2$ follows. This implies $v = f^\ell \cdot k^{-1} \in G_1G_2$ and the claim is proven. Furthermore, note that the generators of G_1 centralize G_2 since $G_2 \subseteq C_G(G') = C_G(f)$ holds, so $G = G_1 * G_2$ is a central product of G_1 and G_2. By Lemma 2.43, $\mathrm{soc}(ZFG_1)$ and $\mathrm{soc}(ZFG_2)$ are ideals in FG_1 and FG_2, respectively.

Now we determine the structure of the factors G_1 and G_2. We first show $|G_1 : G'| = 4$. To this end, we consider the element $f^2c \in G$. For any $h \in C_G(c) = C_G(f)$, we obtain $[h, f^2c] = 1$. For $h \in G \backslash C_G(f)$, we have $hch^{-1} = c^{-1}$ and hence $[h, c] = c^{-2}$. Since f and c centralize G', we obtain

$$[h, f^2c] = [h, f]^2 \cdot [h, c] = c^2 \cdot c^{-2} = 1.$$

This implies $f^2c \in Z(G) \subseteq G'$. Due to $|G : C_G(G')| = 2$, we have $q^2 \in C_G(G') = C_G(f)$. Then $q^2 \in Z(G_1) \subseteq Z(G) \subseteq G'$ follows from the fact that $qC_G(G')$ generates $G/C_G(G')$. Since the abelian group G_1/G' is generated by fG' and qG', this yields $|G_1/G'| = 4$. Note that $[q, f] = c$ implies $G_1' = \langle c \rangle = G'$. By [20, Theorem 5.4.5], G_1 is a dihedral, semidihedral or generalized quaternion group of order $4 \cdot |G'|$. Note that we have $G_2' \subseteq G' \cap C_G(q) \subseteq Z(G)$. Since $[c] = \{c, c^{-1}\}$ holds, we obtain $Z(G) = \langle c^{|G'|/2} \rangle$, which yields $|G_2'| \leq 2$. If $G_2' = 1$ holds, we have $G_2 = Z(G_2) \subseteq G'$, which implies $G = G_1$. Otherwise, we have $G_2' = Z(G) = Z(G_2)$, so G_2 is a stem group and hence it has the structure described in (ii). □

3.3.2 Transition to subgroups

Now we study the invariance of the property $\mathrm{soc}(ZFG) \trianglelefteq FG$ under the transition to certain subgroups of G. This is motivated by the following observation:

Remark 3.21. If p is odd and G is a p-group which satisfies $\mathrm{soc}(ZFG) \trianglelefteq FG$, then $c(G) \leq 2$ follows by Theorem 3.1. In particular, the nilpotency class of any subgroup $H \subseteq G$ is at most two, which implies $\mathrm{soc}(ZFH) \trianglelefteq FH$. In other words, the property $\mathrm{soc}(ZFG) \trianglelefteq FG$ is passed on to the group algebras of subgroups of G. ◁

In contrast, the following example illustrates that for $p = 2$, the analogous statement does not even hold for normal subgroups of G:

Example 3.22. Consider the group $G = \texttt{SmallGroup}(128, 544) \cong (C_8 \times C_2 \times C_2) \rtimes C_4$ of order 128 in GAP, which has the presentation

$$\left\langle a, b, c, d \;\middle|\; \begin{array}{l} a^8 = b^2 = c^2 = d^4 = [a, b] = [b, c] = [a, c] = 1, \\ [d, a] = a^2,\ [d, b] = a^4 c,\ [d, c] = a^4 \end{array} \right\rangle.$$

One easily verifies that $Y(G) = \langle a^2 \rangle$ and $Z(G) = \langle a^2 c \rangle$ hold. Therefore, the derived subgroup $G' = \langle a^2, c \rangle$ is contained in $Y(G)Z(G)$ and hence $\mathrm{soc}(ZFG)$ is an ideal of FG by Theorem 3.16. We now consider the subgroup $Q := \langle a^2, b, c, d \rangle$, which is normal in G since it has index two. A short computation shows $Y(Q) = \langle a^4 \rangle \subseteq \langle a^2 c \rangle = Z(Q)$ and hence the derived subgroup $Q' = \langle a^4, c \rangle$ is not contained in $Y(Q)Z(Q)$. By Theorem 3.16, this implies $\mathrm{soc}(ZFQ) \ntrianglelefteq FQ$. The reason for this phenomenon is that the majority of the conjugacy classes of G of size two is not contained in the subgroup Q. ◁

Now we consider centralizers $S := C_G(N)$ of normal subgroups N of G. For this particular class of normal subgroups of G, the question whether $\mathrm{soc}(ZFG) \trianglelefteq FG$ implies $\mathrm{soc}(ZFS) \trianglelefteq FS$ can be answered affirmatively. This investigation is motivated by the fact that in the study of group algebras, it is often useful to pass to centralizers or normalizers of p-subgroups. The following observation is the main ingredient for the transition to centralizers of normal subgroups:

Remark 3.23. Let G be a finite 2-group which satisfies $\mathrm{soc}(ZFG) \trianglelefteq FG$ and consider a normal subgroup $N \trianglelefteq G$. By Corollary 2.29, we have $G' \subseteq C_G(N)N$. For an element $c := gf^{-1}$ originating from a conjugacy class $C = \{f, g\}$ of length two, we obtain the stronger condition that $c \in C_G(N)$ or $c \in N$ holds. To see that this is the case, we assume $c \notin C_G(N)$, which yields $f \notin C_G(N)$ by (3.3). Hence there exists an element $n \in N$ with $[n, f] = c$, which implies $c \in N$. ◁

Lemma 3.24. *Let G be a 2-group which satisfies $\mathrm{soc}(ZFG) \trianglelefteq FG$. For every normal subgroup $N \trianglelefteq G$, we then have $\mathrm{soc}(ZFC_G(N)) \trianglelefteq FC_G(N)$.*

Proof. We set $D := C_G(N)$. We first show that $Y_C \subseteq Y(D)Z(D)N$ holds for all conjugacy classes $C = \{f, g\}$ of length two. To this end, set $c := gf^{-1}$. We may assume $c \notin N$, which yields $c \in D$ by the preceding remark, and $c \notin Z(D)$, since the claim is proven otherwise. This yields $c \notin Z(G)$ and hence $C_G(f) = C_G(c)$ follows by (3.3). In particular, this implies $f \in D \backslash Z(D)$. Observing that $[f]_D \subseteq C$ holds, this yields $[f]_D = C$. Hence $c \in Y(D)$ follows, which proves the claim. Since C is an arbitrary conjugacy class of length two, we obtain $Y(G) \subseteq Y(D)Z(D)N$. Furthermore, note that $Z(G)$ is contained in $Z(D)$. By Theorem 3.16, we therefore have

$$D' \subseteq G' \cap D \subseteq Y(G)Z(G) \cap D \subseteq Y(D)Z(D)N \cap D.$$

Applying Dedekind's identity and using that $N \cap D$ is contained in $Z(D)$, we obtain

$$Y(D)Z(D)N \cap D \subseteq Y(D)Z(D)(N \cap D) \subseteq Y(D)Z(D).$$

Summarizing, this shows $D' \subseteq Y(D)Z(D)$, which yields $\mathrm{soc}(ZFD) \trianglelefteq FD$ by Theorem 3.16. □

Remark 3.25. We do not know whether Lemma 3.24 holds without the assumption that N is a normal in G. Using GAP, we have verified this statement for the subgroups of the stem 2-groups of order at most 256. In contrast, $\mathrm{soc}(ZFG) \trianglelefteq FG$ does not imply $\mathrm{soc}(ZFN_G(U)) \trianglelefteq FN_G(U)$ for every subgroup U of G. As a counterexample, consider the group $G = \texttt{SmallGroup}(128, 544)$ in GAP, whose presentation is given in Example 3.22. We showed that $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds. Now consider the subgroup $U := \langle a^2b, c, d\rangle$ of G. Its normalizer $N_G(U)$ is the subgroup $Q = \langle a^2, b, c, d\rangle$, for which we have already shown that $\mathrm{soc}(ZFQ)$ is not an ideal of FQ (see Example 3.22). ◁

Chapter 4

Arbitrary finite groups

We fix an algebraically closed field F of characteristic $p \in \mathbb{P}$. The aim of this chapter is to analyze the structure of the finite groups G which satisfy $\operatorname{soc}(ZFG) \trianglelefteq FG$. In Theorem 2.47 and Corollary 3.17, we already stated sufficient conditions for this property, which generalized our classification result for p-groups (see Theorem 3.1). However, it was also demonstrated that these properties are in general not fulfilled. In this chapter, we therefore focus on deriving necessary conditions for $\operatorname{soc}(ZFG) \trianglelefteq FG$.

By Remark 1.6, we may assume that p divides the order of G. Furthermore, recall that $\operatorname{soc}(ZFG) \trianglelefteq FG$ implies that G is of the form $P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H (see Corollary 2.16). By Lemma 2.19, we may assume $O_{p'}(G) = 1$.

This chapter is organized as follows: First, we use our results on groups with an abelian Sylow p-subgroup (see Section 2.6) by going over to the group $D = G/P'$ whose structure we analyze in Section 4.1. It turns out that the conjugacy classes of certain elements in H are of particular relevance in this situation. We examine their structure in Section 4.2.1. With this, we derive a decomposition of the Sylow subgroup P in Section 4.2.2 which allows us to restrict to the case $P = G'$. Afterwards, we analyze the structure of G' in greater detail (see Section 4.2.3). Our results show that it is useful to distinguish the cases $C_{G'}(P) \subseteq P'$ and $C_{G'}(P) \nsubseteq P'$. In the first case, we can characterize the structure of G completely (see Section 4.3) whereas for the second, we give some approaches in special cases of the problem (see Section 4.4).

4.1 Structure of D

If not stated otherwise, we assume that G is a group with the following properties:

Hypothesis 4.1. Let G be a finite group with $O_{p'}(G) = 1$ satisfying $\operatorname{soc}(ZFG) \trianglelefteq FG$. We denote the (unique) Sylow p-subgroup of G by P and fix a Hall p'-subgroup H of G (see Theorem 2.15).

Recall that the group H is abelian in this situation. Since we also want to derive sufficient conditions for $\operatorname{soc}(ZFG) \trianglelefteq FG$, we sometimes replace this assumption by weaker prerequisites. This will then always be stated explicitly.

In this section, we examine the quotient group $D := G/P'$. First, we derive a decomposition of D' as a direct product which is frequently used throughout this chapter (see Section 4.1.1). We continue with a brief interlude concerning a special set of conjugacy classes (see Section 4.1.2). Using these results, we explicitly describe the conjugacy classes of certain elements in D'. Their particular structure allows us to choose a set of generators for H with exceptionally nice properties (see Section 4.1.3).

We use the following notation: Let $\pi\colon G \to D$ denote the canonical projection map onto D and write $\bar{g} := \pi(g)$ for the image of $g \in G$ in D (similarly for subgroups of G). Note that the group D is of the form $\bar{P} \rtimes \bar{H}$. Since the Sylow p-subgroup $\bar{P}$ of D is abelian, we can apply the structural results derived in Section 2.6. Moreover, the lengths of all conjugacy classes $C \in \mathrm{Cl}(D)$ with $C \subseteq \bar{P}$ are coprime to p by (2.11). Note that the group $\bar{H}$ is isomorphic to H since we have $H \cap P' = 1$. For this reason, we usually identify $\bar{H}$ and H.

4.1.1 Decomposition of D'

In this section, we derive a decomposition of the group D' into a direct product of certain subgroups, which will be fundamental for the entire chapter. Before doing so, we collect several elementary properties of G and D, which will be used frequently.

Remark 4.2. Instead of Hypothesis 4.1, we only assume that G is a finite group of the form $P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H. Theorem 2.4 yields $P = C_P(H)[P,H]$ and $\bar{P} = C_{\bar{P}}(H) \times [\bar{P},H]$. Note that the subgroup $[P,H]$ is normal in $G = PH$ (see [30, Theorem 1.5.5]) and similarly, we obtain $[\bar{P},H] \trianglelefteq D$. Since H is abelian and P' is a normal subgroup of G, this yields

$$G' = [PH, PH] = [P, PH] = P'[P,H].$$

In particular, we obtain $D' = [\bar{P}, H]$. ◁

We now study the structure of the center $Z(D)$ and related subgroups.

Remark 4.3. Again, let G be a finite group of the form $P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H such that $O_{p'}(G) = 1$ holds.

(i) By the preceding remark, we obtain

$$\bar{P} = C_{\bar{P}}(H) \times [\bar{P},H] = C_{\bar{P}}(H) \times D'. \tag{4.1}$$

In particular, we obtain $C_{D'}(H) = C_{\bar{P}}(H) \cap D' = 1$.

(ii) Consider an element $h \in C_H(D')$. By (4.1), we have $h \in C_H(\bar{P}) = C_H(P/P')$. Since P' is contained in $\Phi(P)$, the element h acts trivially on $P/\Phi(P)$ and hence also on P by Theorem 2.3. Since H is abelian, this implies $h \in Z(G) \cap H \subseteq O_{p'}(G) = 1$. This yields $O_{p'}(D) \subseteq C_H(\bar{P}) \subseteq C_H(D') = 1$ by Remark 2.20.

(iii) Theorem 2.1 yields $C_D(\bar{P}) = \bar{P}$. In particular, the center of D is given by

$$Z(D) = C_D(\bar{P}) \cap C_D(H) = \bar{P} \cap C_D(H) = C_{\bar{P}}(H). \quad ◁$$

We now move to the decomposition of D' announced at the beginning of this part.

Theorem 4.4. *Let G be a finite group with $O_{p'}(G) = 1$ satisfying $\mathrm{soc}(ZFG) \trianglelefteq FG$ and write $G = P \rtimes H$ as before. We set $D = G/P'$. Then there exists a normal subgroup $L \trianglelefteq G$ with $P' \subseteq L$ satisfying the following:*

(i) $D' = (L/P') \times Z_D$ *with* $Z_D := C_{G'}(P)P'/P'$*, and* $T := L/P'$ *is elementary abelian.*

(ii) There exists a decomposition

$$L/P' = L_1/P' \times \ldots \times L_n/P'$$

for some $n \in \mathbb{N}_0$ *such that* $L_1, \ldots, L_n \trianglelefteq G$ *are normal subgroups which satisfy* $P' \subsetneq L_i \subseteq G'$ *and* $T_i := L_i/P'$ *is a simple* $\mathbb{F}_pH$*-module for* $i = 1, \ldots, n$.

Proof. Consider the normal subgroup $N := \{x \in G' : x^p \in P'\}$ of G. Since $G' = [P, G]$ holds by Remark 4.2, applying Lemma 2.12 to P yields $N^+ \cdot C^+ = 0$ for all $C \in \mathrm{Cl}(G)$ with $C \not\subseteq C_G(P)$. As $C_G(P) = Z(P)$ holds by Theorem 2.1, the element $(NZ(P))^+$ is contained in $\mathrm{soc}(ZFG)$. By Remark 2.9, this yields $G' \subseteq NZ(P)$ and hence

$$G' = NZ(P) \cap G' = N(Z(P) \cap G') = NC_{G'}(P)$$

by Dedekind's identity. Since the quotient group N/P' is elementary abelian, its H-invariant subgroup $N \cap C_{G'}(P)P'/P'$ has an H-invariant complement in N/P' by Maschke's Theorem (see [20, Theorem 3.3.2]). This means that there exists a normal subgroup $L \trianglelefteq G$ containing P' such that $N/P' = L/P' \times (N \cap C_{G'}(P)P')/P'$ holds. Note that L/P' is elementary abelian as well. With this, we obtain

$$G'/P' = (N/P')(C_{G'}(P)P'/P') = L/P' \times C_{G'}(P)P'/P'.$$

By Maschke's theorem, the $\mathbb{F}_pH$-module L/P' decomposes into a direct sum $T_1 \times \ldots \times T_n$ of simple $\mathbb{F}_pH$-modules $T_1, \ldots, T_n$ for some $n \in \mathbb{N}_0$. In particular, T_i is a minimal normal subgroup of D for $i = 1, \ldots, n$ and the correspondence theorem yields $T_i = L_i/P'$ for some normal subgroup $L_i \trianglelefteq G$ with $P' \subseteq L_i \subseteq G'$. Since T_i is a simple module, P' is a proper subgroup of L_i. □

Example 4.5.

(i) Let F be an algebraically closed field of characteristic $p = 2$ and consider the special linear group $G = \mathrm{SL}_2(\mathbb{F}_3)$. Note that it is of the form $G = G' \rtimes H$ with $G' \cong Q_8$ and a cyclic subgroup H of order three. We have $G'' = Z(G) = Z(G')$ and this group has order two. Moreover, conjugation with elements of H permutes the nontrivial elements in G'/G'' transitively.

We claim that $\mathrm{soc}(ZFG)$ is an ideal in FG. Analogously to Example 2.49, we need to show that $\dim S_h = 1$ holds for all $h \in H$, where $S_h := \mathrm{soc}(ZFG) \cap FhG'$ is the homogeneous component of $\mathrm{soc}(ZFG)$ with respect to the H-grading described in Remark 2.27. In G', we have the G-conjugacy classes $\{1\}$, $Z(G)\backslash\{1\}$ and $G'\backslash Z(G)$. Each of the other two cosets of G' splits into two G-conjugacy classes, which arise from each other by multiplication with the nontrivial element in $Z(G)$. In particular, we have $Z(G)^+ \cdot [h]^+ = (hG')^+$ for any $1 \neq h \in H$, that is, $Z(G)^+$ and $[h]^+$ are not contained in $\mathrm{soc}(ZFG)$. As in Example 2.49, this yields $\dim S_h = \dim S_1 = 1$, which implies $\mathrm{soc}(ZFG) \subseteq (G')^+ \cdot FG$. By Lemma 2.8, $\mathrm{soc}(ZFG)$ is an ideal of FG.

Since conjugation with H permutes the nontrivial elements in $D' = G'/G'' \cong C_2 \times C_2$ transitively, D' is a simple $\mathbb{F}_2H$-module. The decomposition given in Theorem 4.4 therefore consists of a single module, that is, we have $n = 1$, $T = D'$ and $Z_D = 1$.

(ii) Now let F be an algebraically closed field of characteristic $p = 3$ and consider the group $G = \texttt{SmallGroup}(216, 86)$ in GAP. In Example 2.49, we showed that $\operatorname{soc}(ZFG)$ is an ideal of FG. The group G is of the form $G' \rtimes H$ with $G' \in \operatorname{Syl}_3(G)$ and a cyclic group H of order eight, which acts transitively on the nontrivial elements of $D' = G'/G'' \cong C_3 \times C_3$. As in the first example, the decomposition of D' given in Theorem 4.4 therefore consists of a single module. ◁

In order to find a group G satisfying $\operatorname{soc}(ZFG) \trianglelefteq FG$ for which the decomposition of D' given in Theorem 4.4 consists of n simple modules $T_1, \ldots, T_n$ for some $n \in \mathbb{N}$, one can simply take the direct (or, more generally, central) product of n copies of one of the groups in Example 4.5 (see Lemma 2.43).

4.1.2 A special set of conjugacy classes

We now introduce a subset of the elements in H whose conjugacy classes have a particularly simple structure. Their existence will form the basis for the derivation of the following sections. In the first two results, we only assume that $D' = G'/P'$ has the structure described in Theorem 4.4 instead of requiring $\operatorname{soc}(ZFG) \trianglelefteq FG$.

Remark 4.6. Let G be a finite group of the form $P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H such that $O_{p'}(G) = 1$ holds. Moreover, assume that $D' \cong T_1 \times \ldots \times T_n \times Z_D$ holds for $Z_D := C_{G'}(P)P'/P'$ and minimal normal subgroups $T_1, \ldots, T_n$ ($n \in \mathbb{N}_0$) of D. For $i = 1, \ldots, n$, we consider the subgroup

$$N_i := \bigoplus_{j \neq i} T_j \times Z_D \trianglelefteq D. \tag{4.2}$$

Note that we have $D' = T_i \times N_i$. Furthermore, we consider the preimage $M_i := \pi^{-1}(N_i)$ of N_i under the canonical projection $\pi\colon G \to D$. Note that M_i is a normal subgroup of G. Explicitly, it is given by

$$M_i = \prod_{j \neq i} L_j \cdot C_{G'}(P)P', \tag{4.3}$$

where we set $L_j = \pi^{-1}(T_j)$ for $j = 1, \ldots, n$ as before. ◁

We are interested in the conjugacy classes of the nontrivial elements of the centralizer $C_H(M_i)$. This is motivated by the following result:

Lemma 4.7. *Let G be a finite group of the form $P \rtimes H$ with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H such that $O_{p'}(G) = 1$ holds. Moreover, assume that $D' \cong T_1 \times \ldots \times T_n \times Z_D$ holds for $Z_D := C_{G'}(P)P'/P'$ and minimal normal subgroups $T_1, \ldots, T_n$ of D. For $i = 1, \ldots, n$, we consider the subgroup M_i defined in* (4.3)*. Then the following hold:*

(i) The centralizer $C_H(M_i)$ is a cyclic subgroup of H.

(ii) Let $1 \neq h \in H$ and set $C := [h] \in \operatorname{Cl}(G)$. Then we have $h \in C_H(M_i)$ if and only if $C \in \operatorname{Cl}_{p',P'}(G)$ holds and its image $\bar{C} \in \operatorname{Cl}(D)$ is of the form $\bar{C} = T_i \cdot \bar{h}$.

Proof.

(i) We consider the restriction to $C_H(M_i)$ of the canonical projection onto $H/C_H(T_i)$ (identifying H and $\bar{H}$ as usual):

$$\varphi\colon C_H(M_i) \to H/C_H(T_i),\ h \mapsto hC_H(T_i). \tag{4.4}$$

We have $\mathrm{Ker}(\varphi) \subseteq C_H(T_i) \cap C_H(N_i) = C_H(D') = 1$ by Remark 4.3, so φ is injective. Therefore, the group $C_H(M_i)$ is isomorphic to a subgroup of the group $H/C_H(T_i)$. Since T_i is a simple $\mathbb{F}_pH$-module, the latter group is cyclic by [20, Theorem 3.2.3], which implies that $C_H(M_i)$ is cyclic as well.

(ii) First let $h \in C_H(M_i)$. By Lemma 2.45, $\bar{C}$ can be expressed in the form $\bar{C} = U \cdot \bar{h}$ with a normal subgroup $U \trianglelefteq D$. Since $\bar{P}$ is abelian, the map

$$\alpha\colon T_i \to T_i,\ t \mapsto [t, \bar{h}]$$

is a group homomorphism. The element $\bar{h}$ centralizes N_i and since $\bar{h} \notin C_H(D') = 1$ (see Remark 4.3) and $D' = T_i \times N_i$ hold, the map α is nontrivial. Since T_i is a simple $\mathbb{F}_pH$-module, Schur's Lemma (see [48, Theorem 2.1.1]) implies that α is an isomorphism. In particular, T_i is contained in U and since $U \cap N_i = 1$ follows by Remark 2.17, we obtain $T_i = U$, so $\bar{C}$ is of the form $\bar{C} = T_i \cdot \bar{h}$. Furthermore, $h \in C_H(M_i) \subseteq C_H(P')$ implies $[h] \subseteq C_G(P')$ since this is a normal subgroup of G. Applying Lemma 2.42 to the group P' then yields $C \in \mathrm{Cl}_{p',P'}(G)$.

Conversely, assume that $C \in \mathrm{Cl}_{p',P'}(G)$ and $\bar{C} = T_i \cdot \bar{h}$ hold. Then $\nu_{P'}(C^+)$ is a nonzero scalar multiple of T_i^+. Here, $\nu_{P'}\colon FG \to FD$ denotes the canonical projection (see (2.6)). On the other hand, the image of T_i^+ in $F[G/M_i] \cong F[D/N_i]$ is given by $(D'/N_i)^+$, which shows that the image of C^+ in $F[G/M_i]$ is nontrivial. By Corollary 2.13, this implies $C \subseteq C_G(M_i)$. □

In general, the map φ given in (4.4) induces an isomorphism between $C_H(M_i)$ and a proper subgroup of $H/C_H(T_i)$. In particular, we will see that the group H acts transitively on the nontrivial elements of T_i, whereas the action of $C_H(M_i)$ is not necessarily transitive.

By Lemma 4.7, the images of the conjugacy classes of the elements in $C_H(M_i)\backslash\{1\}$ have a particularly simple shape. From now on until the end of Section 4.1, we additionally assume that $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds, so we return to Hypothesis 4.1. This ensures that the centralizer $C_H(M_i)$ is nontrivial, that is, it guarantees the existence of such a conjugacy class. In order to prove this result, we need the following observation:

Remark 4.8. Let $Z := \pi(Z(P))$ denote the image of $Z(P)$ in D. Then $Z \cap D' = Z_D$ holds: Using Dedekind's identity, we obtain

$$\pi^{-1}(Z \cap D') = \pi^{-1}(Z) \cap \pi^{-1}(D') = Z(P)P' \cap G' = (Z(P) \cap G')P' = C_{G'}(P)P'.$$

Since the map π is surjective, this yields $Z \cap D' = \pi\big(C_{G'}(P)P'\big) = Z_D$. ◁

Lemma 4.9. *Let G be a finite group which satisfies Hypothesis 4.1. Then the centralizer $C_H(M_i)$ is nontrivial for $i = 1, \ldots, n$.*

Proof. Let $i \in \{1, \ldots, n\}$. By Lemma 4.7, it suffices to show that there exists an element $h \in H$ with $[h] \in \mathrm{Cl}_{p',P'}(G)$ such that $[\bar{h}] = T_i \cdot \bar{h}$ holds. Set $Z \coloneqq \pi(Z(P))$ as before. Dedekind's identity together with Remark 4.8 shows that the subgroup

$$ZN_i \cap D' = (Z \cap D')N_i = Z_D N_i = N_i \tag{4.5}$$

is properly contained in D'. In particular, we have $(ZN_i)^+ \notin \mathrm{Ann}_{ZFD}(\overline{\mathrm{Cl}}^+_{p',P'}(G))$ since the latter set is contained in $(D')^+ \cdot FD$ by Theorem 2.39. Thus there exists a conjugacy class $C \in \mathrm{Cl}_{p',P'}(G)$ with $(ZN_i)^+ \cdot b_{\bar{C}} \neq 0$, where $b_{\bar{C}}$ denotes the basis element of $J(ZFD)$ corresponding to $\bar{C}$ (see Definition 2.22). If C is contained in $Z(P)$, we have $\bar{C} \subseteq Z \subseteq Z(\bar{P})$ and hence

$$(ZN_i)^+ \cdot b_{\bar{C}} = (ZN_i)^+ \cdot (\bar{C}^+ - |\bar{C}| \cdot 1) = |\bar{C}| \cdot (ZN_i)^+ - |\bar{C}| \cdot (ZN_i)^+ = 0,$$

which is a contradiction. It follows that $C \nsubseteq Z(P)$ holds, so $|C|$ is divisible by p. Since $C \in \mathrm{Cl}_{p',P'}(G)$ holds, also $|\bar{C}|$ is divisible by p, which yields $\bar{C} \nsubseteq \bar{P}$ and hence $C \nsubseteq P$. Now let $g \in C$ and decompose $g = g_p \cdot g_{p'}$ into its p-part and p'-part. By Corollary 2.13, we have $g \in C_G(P')$. By conjugating with a suitable element, we may assume that $h \coloneqq g_{p'}$ is contained in H. Since h is a power of g, it centralizes P' as well. This yields $[h] \in \mathrm{Cl}_{p',P'}(G)$ by Lemma 2.42. By Corollary 2.46, $(ZN_i)^+ \cdot \bar{C}^+ \neq 0$ implies $(ZN_i)^+ \cdot [\bar{h}]^+ \neq 0$. This yields $\bar{h} \in C_H(ZN_i)$ (see Corollary 2.13). Since we have $\bar{h} \notin C_H(D') = 1$, the element must act nontrivially on T_i. As in the proof of Lemma 4.7, taking commutators with $\bar{h}$ therefore induces an automorphism on T_i. This implies $[\bar{h}] = T_i \cdot \bar{h}$. □

Example 4.10. We determine the centralizers $C_H(M_i)$ for the groups in Example 4.5. First consider the special linear group $G = \mathrm{SL}_2(\mathbb{F}_3)$. Recall that we have $n = 1$ in the decomposition given in Theorem 4.4. We obtain $M_1 = G'' = Z(G)$, which yields $C_H(M_1) = H$ for all Hall $2'$-subgroups H of G. Also for $G = \texttt{SmallGroup}(216, 86)$, we have $n = 1$ and $M_1 = G''$. Consider a Hall $3'$-subgroup H of G. Then $H = \langle h \rangle$ is cyclic of order eight and acts on G'' by inversion, so $C_H(M_1) = \langle h^2 \rangle$ is a group of order four in this case. ◁

In the next section, we will make use of the existence of these particular elements in order to determine the structure of the D-conjugacy classes of the elements in T.

4.1.3 Conjugacy classes in T and the structure of H

Throughout, we assume that G satisfies Hypothesis 4.1. In the following, we describe the D-conjugacy classes of the elements in T. Their special structure enables us to choose a particularly nice set of generators of the Hall p'-subgroup H afterwards. We begin with the following preliminary result:

Proposition 4.11. *Every normal subgroup $N \subseteq T$ of D is of the form $N = T_{i_1} \times \ldots \times T_{i_\ell}$ for some $\ell \in \mathbb{N}_0$ and distinct indices $i_1, \ldots, i_\ell \in \{1, \ldots, n\}$.*

Proof. We may assume $N \neq 1$. Note that the $\mathbb{F}_p H$-modules $T_1, \ldots, T_n$ are pairwise non-isomorphic since they have distinct kernels by Lemma 4.7 and Lemma 4.9. Since N is a semisimple $\mathbb{F}_p H$-submodule of T, it decomposes in the form $N = T_{i_1} \times \ldots \times T_{i_\ell}$ for some $\ell \in \mathbb{N}$ and some $i_1, \ldots, i_\ell \in \{1, \ldots, n\}$ by [13, page 46]. □

Lemma 4.12. *For $t, t' \in T$, we write $t = t_1 \cdots t_n$ and $t' = t'_1 \cdots t'_n$ with $t_i, t'_i \in T_i$ for $i = 1, \ldots, n$. Then the elements t and t' are conjugate in D if and only if $t_i = 1$ is equivalent to $t'_i = 1$ for $i = 1, \ldots, n$.*

Proof. Assume that t and t' are conjugate. For $d \in D$, we have

$$dtd^{-1} = \prod_{i=1}^{n} dt_i d^{-1}$$

with $dt_i d^{-1} \in T_i$ for $i = 1, \ldots, n$. In particular, $t_i = 1$ implies $dt_i d^{-1} = 1$ and hence $t'_i = 1$. By symmetry, we obtain the equivalence $t_i = 1 \Leftrightarrow t'_i = 1$ for $i = 1, \ldots, n$.

Now assume that t and t' satisfy the equivalence $t_i = 1 \Leftrightarrow t'_i = 1$ for $i = 1, \ldots, n$. We show that they are conjugate in D by using induction on $m \coloneqq |\{i \in \{1, \ldots, n\} : t_i \neq 1\}|$. Without loss of generality, we assume $t_i \neq 1$ for $i = 1, \ldots, m$ and $t_{m+1} = \ldots = t_n = 1$. For $m = 0$, there is nothing to show. For $m \in \{1, \ldots, n\}$, we consider the group

$$W \coloneqq T_{m+1} \times \ldots \times T_n.$$

Note that this is a normal subgroup of D. Moreover, set $\ell \coloneqq |[t]|$ and $\ell' \coloneqq |[t']|$, and recall that both numbers are coprime to p since $\bar{P}$ is abelian. We now show

$$y \coloneqq \left(\ell'[t]^+ - \ell[t']^+\right) \cdot (ZW)^+ \in \mathrm{Ann}_{ZFD}(\overline{\mathrm{Cl}}^+_{p',P'}(G)),$$

where $Z \coloneqq \pi(Z(P))$ denotes the image of $Z(P)$ in D as before. Note that $ZW \cap D' = (Z \cap D')W = Z_D \times W$ holds by Dedekind's identity combined with Remark 4.8 and hence we have $ZW \cap (T_1 \times \ldots \times T_m) = 1$. By construction, it follows that $y \in ZFD$.

Consider a conjugacy class $C \in \mathrm{Cl}_{p',P'}(G)$. If C is contained in $Z(P)$, we obtain $y \cdot b_{\bar{C}} = 0$ as in the proof of Lemma 4.9. Otherwise both $|C|$ and $|\bar{C}|$ are divisible by p (see (2.11)) and we have $b_{\bar{C}} = \bar{C}^+$. In particular, $\bar{C}$ is not contained in $\bar{P}$. By Corollary 2.46, we have $y \cdot \bar{C}^+ = 0$ if $y \cdot [\bar{h}]^+ = 0$ holds for the element $h \in H$ with $C \subseteq hP$. By Lemma 2.45, $[\bar{h}]$ is of the form $U\bar{h}$ for some nontrivial normal subgroup $U \subseteq D'$ of D. It therefore suffices to show that $y \cdot U^+ = 0$ holds in order to prove that y annihilates $b_{\bar{C}}$. To this end, we first assume $U = T_i$ for some $i \in \{1, \ldots, m\}$. By induction, we obtain $C_i \coloneqq [t_1 \cdots t_{i-1} \cdot t_{i+1} \cdots t_m] = [t'_1 \cdots t'_{i-1} \cdot t'_{i+1} \cdots t'_m]$. By Remark 2.10, we have

$$[t]^+ \cdot T_i^+ = \frac{\ell}{|C_i|} \cdot C_i^+ \cdot T_i^+$$

and we obtain a similar formula for $[t']^+ \cdot T_i^+$. This yields

$$\left(\ell' \cdot [t]^+ - \ell \cdot [t']^+\right) \cdot T_i^+ = \ell' \cdot \frac{\ell}{|C_i|} \cdot C_i^+ \cdot T_i^+ - \ell \cdot \frac{\ell'}{|C_i|} \cdot C_i^+ \cdot T_i^+ = 0$$

and hence also $y \cdot T_i^+ = 0$.

Now we show that $y \cdot U^+ = 0$ holds for an arbitrary subgroup $1 \neq U \subseteq D'$ which is normal in D. If $\nu_{ZW}(U^+) = 0$ holds, then we obtain $U^+ \cdot (ZW)^+ = 0$ and hence $y \cdot U^+ = 0$. In the case $\nu_{ZW}(U^+) \neq 0$, we have $U \cap ZW = U \cap Z_D W = 1$ since U is a union of cosets of $U \cap ZW$, which is a p-group. Hence $U^+ \cdot (ZW)^+ = U_T^+ \cdot (ZW)^+$ follows, where

$$U_T := \{t \in T_1 \times \ldots \times T_m : tc \in U \text{ for some } c \in Z_D W\}$$

denotes the projection of U onto $T_1 \times \ldots \times T_m$. Since y is a multiple of $(ZW)^+$ in FD, we also obtain $y \cdot U^+ = y \cdot U_T^+$. Note that U_T is a normal subgroup of D with $|U_T| = |U|$, so by Proposition 4.11, it is of the form $U_T = T_{i_1} \times \ldots \times T_{i_r}$ for some $r \in \mathbb{N}$ and $i_1, \ldots, i_r \in \{1, \ldots, m\}$. We obtain $y \cdot U_T^+ = 0$ since U_T is a union of cosets of T_{i_1} and y annihilates $T_{i_1}^+$ by the above.

This shows $y \in \mathrm{Ann}_{ZFD}(\overline{\mathrm{Cl}}_{p',P'}^+(G))$. If $[t] \neq [t']$ holds, then the element y is nonzero. Furthermore, it has non-constant coefficients on T since the coefficient of $1 \in T$ in y is zero. This contradicts the assumption $\mathrm{soc}(ZFG) \trianglelefteq FG$ by Theorem 2.39. Hence t and t' must be conjugate in D. □

We now examine the influence of this result on the structure of H. Since T_i is a simple $\mathbb{F}_p H$-module for $i = 1, \ldots, n$, it follows that $H/C_H(T_i)$ is cyclic (see [20, Theorem 3.2.3]). We fix an element $x_i \in H$ such that $x_i C_H(T_i)$ generates $H/C_H(T_i)$. The D-conjugacy class structure of the elements in T now enables us to choose the generators $x_1, \ldots, x_n \in H$ in such a way that $\langle x_i \rangle$ centralizes T_j for $j \neq i$ and acts transitively on $T_i \backslash \{1\}$ by conjugation.

Lemma 4.13. *The canonical map*

$$\rho \colon H/C_H(T) \to H/C_H(T_1) \times \ldots \times H/C_H(T_n),\ hC_H(T) \mapsto \big(hC_H(T_1), \ldots, hC_H(T_n)\big)$$

is an isomorphism. In particular, there exist $e_1, \ldots, e_n \in H$ with $H = \langle e_1, \ldots, e_n, C_H(T)\rangle$ and $\langle e_i C_H(T_i)\rangle = H/C_H(T_i)$ such that $\langle e_i \rangle$ acts transitively on $T_i \backslash \{1\}$ and centralizes T_j for $i, j \in \{1, \ldots, n\}$ with $j \neq i$.

Proof. We consider the map

$$\rho' \colon H \to H/C_H(T_1) \times \ldots \times H/C_H(T_n),\ h \mapsto \big(hC_H(T_1), \ldots, hC_H(T_n)\big).$$

Clearly, this is a group homomorphism with kernel $C_H(T_1) \cap \ldots \cap C_H(T_n) = C_H(T)$. We claim that ρ' is surjective. To see that this holds, we fix an element $t_i \in T_i \backslash \{1\}$ for $i = 1, \ldots, n$. By Lemma 4.12, the elements $t_1 \cdots t_n$ and $x_1 t_1 x_1^{-1} \cdot t_2 \cdots t_n$ are conjugate in D. Since $\bar{P}$ is abelian, there even exists an element $h \in H$ with

$$h(t_1 \cdots t_n)h^{-1} = x_1 t_1 x_1^{-1} \cdot t_2 \cdots t_n.$$

Since T_i is a normal subgroup of D, $ht_ih^{-1} \in T_i$ follows for $i = 1, \ldots, n$ and hence we have $ht_1h^{-1} = x_1 t_1 x_1^{-1}$ as well as $ht_ih^{-1} = t_i$ for $i = 2, \ldots, n$. This yields

$$\rho'(h) = \big(x_1 C_H(T_1), C_H(T_2), \ldots, C_H(T_n)\big).$$

Since this can be carried out similarly for all indices, ρ' is surjective and hence the map ρ is an isomorphism. For $i = 1, \ldots, n$, we fix an element $e_i \in H$ with

$$e_i C_H(T) = \rho^{-1}\big(C_H(T_1), \ldots, C_H(T_{i-1}), x_i C_H(T_i), C_H(T_{i+1}), \ldots, C_H(T_n)\big).$$

Then e_i acts trivially on T_j for $j \neq i$ and $e_i C_H(T_i)$ generates $H/C_H(T_i)$. Since all nontrivial elements in T_i are conjugate by elements in H (see Lemma 4.12), $\langle e_i \rangle$ acts transitively on $T_i \backslash \{1\}$. Furthermore, $H/C_H(T)$ is generated by the images of $e_1, \ldots, e_n$, which yields $H = \langle e_1, \ldots, e_n, C_H(T) \rangle$. □

From now on, we always choose generators $x_i := e_i$ $(i = 1, \ldots, n)$ having the properties described in the preceding lemma. The following theorem summarizes the results on the structure of D obtained in this section:

Theorem 4.14. *Let G be a finite group with $O_{p'}(G) = 1$ such that $\operatorname{soc}(ZFG) \trianglelefteq FG$ holds. Let $P \in \operatorname{Syl}_p(G)$ and fix a Hall p'-subgroup H of G. As before, we set $D := G/P'$. Then this group has the following properties:*

(D1) There exists some $n \in \mathbb{N}_0$ such that the derived subgroup D' decomposes as

$$D' = T \times Z_D = T_1 \times \ldots \times T_n \times Z_D \tag{4.6}$$

with elementary abelian normal subgroups $T_1, \ldots, T_n \trianglelefteq D$ that are simple $\mathbb{F}_p H$-modules and $Z_D := C_{G'}(P)P'/P'$.

(D2) There exist $e_1, \ldots, e_n \in H$ with $H = \langle e_1, \ldots, e_n, C_H(T) \rangle$ and $\langle e_i C_H(T_i) \rangle = H/C_H(T_i)$ such that $\langle e_i \rangle$ acts transitively on $T_i \backslash \{1\}$ and centralizes T_j for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

(D3) For $i = 1, \ldots, n$, the centralizer $C_H(M_i)$ is nontrivial. Here, M_i denotes the preimage of $\prod_{j \neq i} T_j \times Z_D$ under the projection onto D.

Later, we provide a reduction to the case $P = G'$. There, we examine the special situation $Z(G') \subseteq G''$ (see Section 4.3), which translates to $Z_D = 1$ in the decomposition given in (4.6). We show that D decomposes as the direct product of the subgroups $\langle T_i, e_i \rangle$ for $i = 1, \ldots, n$, which are then isomorphic to affine linear groups (see Lemma 4.31). In this case, the properties (D1)–(D3) are therefore already sufficient to determine the structure of D.

At the end of this section, we apply our results to the case that G is a Frobenius group.

Example 4.15 (Frobenius groups)**.** Let G be a finite Frobenius group with Frobenius kernel K and Frobenius complement A. Recall that K is the Fitting subgroup of G and that A is nontrivial by definition. We claim that $\operatorname{soc}(ZFG)$ is an ideal of FG if and only if $K = G'$ is an abelian Sylow p-subgroup of G. In this case, the Hall p'-subgroups of G are cyclic.

First suppose that $\operatorname{soc}(ZFG)$ is an ideal of FG. As before, $G = P \rtimes H$ is a semidirect product of $P \in \operatorname{Syl}_p(G)$ and an abelian Hall p'-subgroup H of G (see Theorem 2.15). The Sylow subgroup P is a normal nilpotent subgroup of G and hence contained in the Fitting subgroup K. Because of $K \cap A = 1$, it follows that A is a p'-group, so it is contained in a Hall p'-subgroup of G. Without loss of generality, we may assume $A \subseteq H$. Now for every element $h \in H$, we have $hAh^{-1} = A$ since H is abelian and hence $A = H$ follows from the fact that $A \cap gAg^{-1} = 1$ holds for every $g \in G \backslash A$. With this, we obtain $K = P$.

By Theorem 2.4, the group P decomposes as $P = C_P(H)[P, H]$. Since G is a Frobenius group, we obtain $C_P(H) = 1$, which yields $P = G'$. Moreover, note that $O_{p'}(G) = 1$ follows by Remark 2.20. Again, we consider the group $D = G/G''$ and write $D = T_1 \times \ldots \times T_n \times Z_D$ with $Z_D = Z(G')/G''$ as in Theorem 4.14. If G' is not abelian, that is, if $n \geq 1$ holds, then the subgroup $C_H(M_1)$ is nontrivial by Theorem 4.14. But any element $1 \neq h \in C_H(M_1)$ centralizes the nontrivial subgroup G'', which is a contradiction. This yields $G' = Z(G')$, that is, G' is abelian. Note that if p is an odd prime number, then this also follows directly from [24, Satz V.8.18]. Moreover, the group H is cyclic by [24, Hauptsatz V.8.7].

Conversely, if G is a Frobenius group with an abelian Frobenius kernel $G' \in \mathrm{Syl}_p(G)$ and Frobenius complement A, then G is a semidirect product of G' by A. In particular, A is an abelian p'-group and hence $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds by Theorem 2.47. $\triangleleft$

4.2 Structure of G

In the last section, we investigated the structure of $D = G/P'$. The information obtained therein is now exploited in order to understand the structure of the group G itself. We have encountered a set of particularly relevant conjugacy classes in Section 4.1.2 whose structure will be further investigated in Section 4.2.1. These results will form the basis for the second part of this section, which mainly deals with the structure of P. Finally, we focus on the decomposition of G' into a central product in Section 4.2.3.

4.2.1 Conjugacy classes of elements in $C_H(M_i)$

Throughout this section, we make the following assumption on G:

Hypothesis 4.16. Let $G = P \rtimes H$ be a finite group with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H such that $O_{p'}(G) = 1$ holds. Moreover, we assume that G satisfies the conditions (D1)–(D3) described in Theorem 4.14.

Similarly to Section 4.1.2, we replace the requirement $\mathrm{soc}(ZFG) \trianglelefteq FG$ by the (weaker) necessary conditions on the structure of D collected in Theorem 4.14. This is essential since we will eventually use the results of this section to show that these conditions, combined with some other assumptions, are actually sufficient for $\mathrm{soc}(ZFG) \trianglelefteq FG$ to hold.

In this section, we use the notation from Theorem 4.14. Additionally, we set $L_i \coloneqq \pi^{-1}(T_i)$ for $i = 1, \ldots, n$. Recall the definition of the normal subgroups

$$M_i \coloneqq \pi^{-1}\left(\prod_{j \neq i} T_j \times Z_D\right).$$

By assumption, the centralizer $C_H(M_i)$ is nontrivial for $i = 1, \ldots, n$. Since its elements play an important role in our derivation, our aim now is to describe the structure of their conjugacy classes in G. Recall that the images of these classes in D are of a particularly simple shape (see Lemma 4.7). For $g \in G$, we write $[g] = U_g g$ with $U_g \subseteq G'$ as in (2.1). For an element $1 \neq h \in C_H(M_i)$, we will then show that all nontrivial elements in U_h are conjugate and that for any other element $1 \neq h' \in C_H(M_i)$, the nontrivial elements of U_h

and $U_{h'}$ are conjugate as well. Requiring $y \cdot [h]^+ = 0$ for some $y \in FG$ therefore often yields $y \cdot [h']^+ = 0$ for all $h' \in C_H(M_i)\backslash\{1\}$ automatically.

We make the following observation on the subgroups $T_1, \ldots, T_n$:

Remark 4.17. For $i = 1, \ldots, n$, the normal subgroup T_i consists of at least three elements: By assumption, we have $T_i \neq 1$. Assume $|T_i| = 2$, so we have $T_i = \{1, t\}$ for some $t \in D'$ with $t^2 = 1$. By Remark 4.3, we obtain $t \in Z(D) \cap D' \subseteq C_{\bar{P}}(H) \cap D' = 1$, which is a contradiction. ◁

Moreover, we make use of the following representation of the elements in T_i:

Remark 4.18. For $i = 1, \ldots, n$, we set $s_i := |T_i| - 1$ and fix an element $f \in G'$ with $\bar{f} \in T_i\backslash\{1\}$. Since $\langle e_i \rangle$ acts transitively on $T_i\backslash\{1\}$, we have

$$T_i = \{1\} \cup \left\{ e_i^k \bar{f} e_i^{-k} \colon 0 \leq k \leq s_i - 1 \right\}. \qquad ◁$$

Now we collect several properties of the conjugacy classes of the elements in $C_H(M_i)\backslash\{1\}$, which will be crucial for the derivation in the following sections.

Lemma 4.19. *Let $G = P \rtimes H$ be a finite group with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H such that $O_{p'}(G) = 1$ holds. Moreover, assume that G has the properties (D1)–(D3) described in Theorem 4.14. Let $i \in \{1, \ldots, n\}$ and set $s_i := |T_i| - 1$. For any element $1 \neq h_i \in C_H(M_i)$, the following hold:*

(i) Its conjugacy class $C := [h_i] \in \mathrm{Cl}(G)$ is of the form $C = U_{h_i} \cdot h_i$ with

$$U_{h_i} = \{1\} \cup \left\{ e_i^k g_{h_i} e_i^{-k} \colon 0 \leq k \leq s_i - 1 \right\} \tag{4.7}$$

for some $g_i := g_{h_i} \in L_i\backslash P'$.

(ii) For any $a \in C_G(H)$ with $\bar{a} \in C_D(T_i)$, we have $[a, g_i] = 1$. In particular, g_i centralizes $C_H(T)$ as well as $C_P(H)$ and commutes with e_j for $j \in \{1, \ldots, n\}$ with $j \neq i$.

(iii) The element g_i is conjugate to $[e_i^k, g_i]$ for $k = 1, \ldots, s_i - 1$, and $[g_i]$ is a real conjugacy class.

(iv) Every element $a \in G'$ can be decomposed in the form

$$a = a_1 \cdots a_n \cdot z \tag{4.8}$$

with $a_i \in \{1\} \cup \{e_i^k g_i e_i^{-k} \colon k \in \mathbb{N}\}$ and $z \in C_{G'}(P)P'$.

(v) For $h_i, h_i' \in C_H(M_i)\backslash\{1\}$, the elements g_{h_i} and $g_{h_i'}$ defined in (i) are conjugate in G.

Proof. For the entire proof, we fix an element $f \in L_i\backslash P'$ as in Remark 4.18.

(i) Write $C = U_{h_i} \cdot h_i$ with $U_{h_i} = \{[g, h_i] \colon g \in G\} \subseteq G'$ (see (2.1)). Since H is abelian, we have $H \subseteq C_G(h_i)$ and hence $|C|$ is a power of p. Since $C \in \mathrm{Cl}_{p',P'}(G)$ holds and by using Lemma 4.7, we have $|C| = |\bar{C}| = |T_i|$. Now consider the set

$$R := \{1\} \cup \left\{ e_i^k f e_i^{-k} \colon 0 \leq k \leq s_i - 1 \right\}.$$

By Remark 4.18, we have $\pi(R) = T_i$. Since $\bar{h}_i$ acts on $T_i \backslash \{1\}$ without fixed points and centralizes $N_i = \pi(M_i)$ as well as H, the elements in $\pi(R)$ form a system of representatives for the cosets of $C_D(\bar{h}_i)$ in D. In particular, the elements in R lie in pairwise different cosets of $C_G(h_i)$ and since $|C| = |\bar{C}|$ holds, they form a system of representatives for the cosets of $C_G(h_i)$ in G. This yields $U_{h_i} = \{[g, h_i] \colon g \in R\}$ and hence we obtain

$$U_{h_i} = \left\{[e_i^k f e_i^{-k}, h_i] \colon 0 \leq k \leq s_i - 1\right\} \cup \{1\} = \left\{e_i^k [f, h_i] e_i^{-k} \colon 0 \leq k \leq s_i - 1\right\} \cup \{1\}.$$

The last equality follows from the fact that e_i^k and h_i commute since H is abelian. We set $g_{h_i} := [f, h_i] \in L_i$. Since $\bar{h}_i$ acts on $T_i \backslash \{1\}$ without fixed points (see Remark 4.18), we have $[\bar{f}, \bar{h}_i] \in T_i \backslash \{1\}$, which yields $g_{h_i} \notin P'$. Hence U_{h_i} has the form given in (4.7).

(ii) Let $a \in C_G(H)$ with $\bar{a} \in C_D(T_i)$. In particular, we have $\bar{a}\bar{g}_i\bar{a}^{-1} = \bar{g}_i$ and hence $a g_i a^{-1} \in g_i P'$. Since h_i and a commute and we have $g_i h_i \in [h_i]$, we obtain

$$a g_i h_i a^{-1} = a g_i a^{-1} h_i \in [h_i] \cap g_i h_i P' = \{g_i h_i\}.$$

Here, we use that $|C| = |\bar{C}|$ holds (see (i)), so every coset of P' in G contains at most one element of C. This yields $a g_i a^{-1} = g_i$. For the second statement, note that e_j is contained in $C_H(T_i)$ for $j \neq i$. Moreover, the image of $C_P(H)$ in D centralizes T_i since $\bar{P}$ is abelian. The other prerequisites are clearly satisfied, so the first part of the statement can be applied in all stated cases.

(iii) Let $a \in G$. Since C is invariant under conjugation, we obtain

$$U_{h_i} \cdot h_i = C = aCa^{-1} = aU_{h_i}a^{-1} \cdot a h_i a^{-1}.$$

Since all nontrivial elements in U_{h_i} are conjugate by (i), this also follows for the nontrivial elements in $aU_{h_i}a^{-1} = U_{h_i}[a, h_i]^{-1}$. Varying $a \in G$, the commutator $[a, h_i]$ runs over all elements of U_{h_i}. This shows that for every $u \in U_{h_i}$, the nontrivial elements of $U_{h_i}u^{-1}$ are conjugate.

Now we choose two distinct nontrivial elements $u, u' \in U_{h_i}$. This is possible since $|U_{h_i}| = |T_i| \geq 3$ holds by Remark 4.17. By (i), u and u' are conjugate in G. By the above, we then obtain

$$u'u^{-1} \sim u^{-1} \sim u'^{-1} \sim uu'^{-1} = (u'u^{-1})^{-1}$$

and hence the conjugacy class of u^{-1} is a real conjugacy class, which implies that also $[u]$ is real. In particular, setting $u := g_i$ and $u' := e_i^k g_i e_i^{-k}$ for some $k \in \{1, \ldots, s_i - 1\}$, this yields $g_i \sim g_i^{-1} \sim [e_i^k, g_i]$.

(iv) This follows by condition (D1) in Theorem 4.14 together with Remark 4.18.

(v) Let $h_i, h_i' \in C_H(M_i) \backslash \{1\}$ and set $g_i := g_{h_i}$ and $g_i' := g_{h_i'}$ as in (i). Since we can replace g_i' by $e_i^k g_i' e_i^{-k}$ for some $k \in \{0, \ldots, s_i - 1\}$, we may assume $g_i' \in g_i P'$, that is, $g_i' = g_i c$ for some $c \in P'$. Now write $h_i C_H(T_i) = e_i^k C_H(T_i)$ for some $k \in \{1, \ldots, s_i - 1\}$.

Since h_i acts trivially on $N_i = \pi(M_i)$ and e_i^k centralizes T_m for $m \neq i$, we have $h_i e_i^{-k} \in C_H(T)$, that is, there exists an element $j \in C_H(T)$ with $h_i = e_i^k j$. Then

$$[h_i, g_i] = [e_i^k j, g_i] = e_i^k [j, g_i] e_i^{-k} \cdot [e_i^k, g_i] = [e_i^k, g_i]$$

follows by (ii). Similarly, we obtain $[h_i, g_i'] = [e_i^k, g_i']$. Since h_i centralizes P', using (iii) yields

$$g_i' \sim [e_i^k, g_i'] = [h_i, g_i'] = [h_i, g_i c] = [h_i, g_i] = [e_i^k, g_i] \sim g_i. \qquad \square$$

Note that for $1 \neq h_i \in C_H(M_i)$, an element g_{h_i} which satisfies (4.7) is unique up to conjugation with powers of e_i. In the preceding proof, we have chosen $g_{h_i} := [f, h_i]$.

4.2.2 Structure of P

From now on until the end of Section 4.2, we again assume that G satisfies Hypothesis 4.1. In this section, we focus on the structure of the Sylow p-subgroup P. Our main result is a decomposition of G into a central product of the centralizer $C_P(H)$ and a subgroup G_H of the form $G_H' \rtimes H$ with $G_H' \in \mathrm{Syl}_p(G_H)$. Since the structure of the p-group $C_P(H)$ is determined in Theorem 3.1, this allows us to focus on groups G with $G' \in \mathrm{Syl}_p(G)$ in the following parts. At the end of this section, we use the decomposition of G in order to describe $\mathrm{soc}(ZFG)$ in case that this space is an ideal of FG.

Again, we use the notation of the previous parts and, in particular, of Theorem 4.14. For $i = 1, \ldots, n$, we fix a nontrivial element $h_i \in C_H(M_i)$ (see (D3) in Theorem 4.14) and set $g_i := g_{h_i}$, where g_{h_i} is defined as in Lemma 4.19 (i).

We begin our investigation by proving the fundamental result that the derived subgroup G' has nilpotency class at most two if $\mathrm{soc}(ZFG)$ is an ideal of FG.

Lemma 4.20. *We have*

$$\langle [P, G'], \Phi(G') \rangle \subseteq Z(G').$$

Furthermore, we obtain $G'' \subseteq Z(P)$ and $G''' = 1$. In particular, G' is of nilpotency class at most two.

Proof. Let C be a critical subgroup of P (see Theorem 2.2). In particular, C is characteristic in P, so also in G, and we have $C_P(C) = Z(C)$. Applying Lemma 2.28 with $N := C$ yields $C^+ \in \mathrm{soc}(ZFG)$ and hence we obtain $G' \subseteq C$ by Remark 2.9. By Theorem 2.2 (ii), we have

$$[P, G'] \subseteq [P, C] \subseteq Z(C) \subseteq C_G(G').$$

Since $[P, G'] \subseteq G'$ holds, this implies $[P, G'] \subseteq Z(G')$, which yields $[[P, G'], G'] = 1$. With the three subgroups lemma (see [30, Satz 1.5.6]), we obtain $[G'', P] = [[G', G'], P] = 1$. Hence G'' is contained in $Z(P)$, which implies $G''' = 1$. Finally, for every $x \in G' \subseteq C$, we have $x^p \in \Phi(C) \cap G' \subseteq Z(C) \cap G' \subseteq Z(G')$ by Theorem 2.2 (i). Together with $G'' \subseteq Z(G')$, this yields $\Phi(G') = (G')^p \cdot G'' \subseteq Z(G')$. $\square$

As announced at the beginning of this section, we now derive a decomposition of G into a central product of $C_P(H)$ and a subgroup G_H with $G_H' \in \mathrm{Syl}_p(G_H)$. This is achieved by decomposing P as the central product $P = C_P(H) * [G', H]$. The main step will be to show that in the given situation, the centralizer $C_P(H)$ is normal in G. Recall that

$$P = C_P(H)[P, H] = C_P(H)G' \tag{4.9}$$

holds by Theorem 2.4 and since we have $[P, H] \subseteq G' \subseteq P$. This yields $G = HC_P(H)G'$. Finally, we observe that since all elements of the form $e_i^{\ell} g_i e_i^{-\ell}$ for $i = 1, \ldots, n$ and $\ell \in \mathbb{Z}$ centralize $C_P(H)$ by Lemma 4.19 (ii), the decomposition of the elements in G' given in (4.8) yields

$$[C_P(H), G'] = [C_P(H), C_{G'}(P)P'] = [C_P(H), P']. \tag{4.10}$$

The following observation concerning the derived subgroup P' is a key step towards the desired decomposition of G.

Proposition 4.21. *The subgroup P' decomposes as $P' = C_{P'}(H) \cdot G''$.*

Proof. Note that the group $C_{P'}(H) \cdot G''$ is normal in G since for any element $x = ug$ with $u \in HC_P(H)$ and $g \in G'$, we have

$$xC_{P'}(H)x^{-1} = (ug)C_{P'}(H)(ug)^{-1} \subseteq uC_{P'}(H) \cdot G''u^{-1} \subseteq C_{P'}(H) \cdot G''.$$

We therefore consider the quotient group $\tilde{G} \coloneqq G/C_{P'}(H)G''$ and set $\tilde{P}$ to be the image of P in $\tilde{G}$. Then $C_{\tilde{P}}(H)$ is abelian as it is the image of $C_P(H)$ under the quotient map (see Theorem 2.5) and we have $C_P(H)' \subseteq C_{P'}(H)$. Analogously to (4.9), $\tilde{P}$ decomposes in the form $\tilde{P} = C_{\tilde{P}}(H) \cdot \tilde{G}'$ and we obtain

$$\tilde{P}' = [\tilde{P}, \tilde{P}] = [C_{\tilde{P}}(H) \cdot \tilde{G}', C_{\tilde{P}}(H) \cdot \tilde{G}'] = [C_{\tilde{P}}(H), \tilde{G}'],$$

since we have $C_{\tilde{P}}(H)' = \tilde{G}'' = 1$ and $[C_{\tilde{P}}(H), \tilde{G}']$ is normal in $\tilde{P} = C_{\tilde{P}}(H) \cdot \tilde{G}'$ (see [30, Theorem 1.5.5]). By (4.10), this yields

$$\tilde{P}' = [C_{\tilde{P}}(H), \tilde{G}'] = [C_{\tilde{P}}(H), \tilde{P}'] = [C_{\tilde{P}}(H) \cdot \tilde{G}', \tilde{P}'] = [\tilde{P}, \tilde{P}'].$$

In the third step, we used that $[\tilde{G}', \tilde{P}'] = 1$ holds since $\tilde{G}'$ is abelian. Thus, starting from $\tilde{P}'$, the lower central series of $\tilde{P}$ is stationary. Since $\tilde{P}$ is a nilpotent group, this yields $\tilde{P}' = 1$ and hence $P' \subseteq C_{P'}(H) \cdot G''$. The other inclusion is clear. □

With this, we can show that $C_P(H)$ is a normal subgroup of G, which will then allow us to decompose G.

Lemma 4.22. *The centralizer $C_P(H)$ is a normal subgroup of G.*

Proof. We have

$$[C_P(H), G'] = [C_P(H), P'] = [C_P(H), C_{P'}(H) \cdot G''] \subseteq C_P(H).$$

The first equality is due to (4.10), the second follows by Proposition 4.21 and for the last inclusion, we use that G'' centralizes $C_P(H) \subseteq P$ (see Lemma 4.20). Since $C_P(H)$ is normalized by $HC_P(H)$, we have

$$[C_P(H), G] = [C_P(H), HC_P(H)G'] \subseteq C_P(H),$$

which proves that $C_P(H)$ is a normal subgroup of G. $\square$

In order to show that P is the central product of $C_P(H)$ and $[G', H]$, we need to prove that these two groups commute element-wise.

Proposition 4.23. *The elements of $C_P(H)$ and $[G', H]$ commute.*

Proof. We first show that the intersection $C_P(H) \cap [G', H] = C_{G'}(H) \cap [G', H]$ is contained in $Z(G)$. Since $\tilde{G}' := G'/G''$ is abelian, it decomposes as $\tilde{G}' = C_{\tilde{G}'}(H) \times [\tilde{G}', H]$ by Theorem 2.4. For any $x \in C_{G'}(H) \cap [G', H]$, we obtain $xG'' \in C_{\tilde{G}'}(H) \cap [\tilde{G}', H] = 1$ and hence $x \in G'' \subseteq Z(P)$ follows, which implies $x \in Z(P) \cap C_P(H) \subseteq Z(G)$.

Now we prove that the subgroups $C_P(H)$ and $[G', H]$ commute element-wise. To this end, let $a \in [G', H]$ and write $a = a_1 \cdots a_n \cdot z$ with $a_i \in \{1\} \cup \{e_i^k g_i e_i^{-k} : k \in \mathbb{Z}\}$ and $z \in C_{G'}(P)P'$ as in (4.8). Since the elements of $C_P(H)$ commute with H-conjugates of $g_1, \ldots, g_n$ by Lemma 4.19, it remains to show that z centralizes $C_P(H)$. Note that $a_1, \ldots, a_n$ are contained in $[G', H]$ by Lemma 4.19 (i), so we have $z \in C_{G'}(P)P' \cap [G', H]$. Write $z = cu$ for some $c \in C_{C_{G'}(P)P'}(H)$ and some $u \in [C_{G'}(P)P', H]$ (see Theorem 2.4). Since z and u are contained in $[G', H]$, we also have $c \in [G', H]$ and hence the first part of this proof yields $c \in C_P(H) \cap [G', H] \subseteq Z(G)$. Note that we have

$$[P', H] = [C_{P'}(H) \cdot G'', H] \subseteq G'' \subseteq C_{G'}(P)$$

by Proposition 4.21 together with Lemma 4.20 and hence $u \in C_{G'}(P)$ follows. In particular, $z = cu$ centralizes $C_P(H)$. $\square$

We can now prove the principal result of this section.

Theorem 4.24. *Let $G = P \rtimes H$ be a finite group with $P \in \operatorname{Syl}_p(G)$ and an abelian p'-group H such that $O_{p'}(G) = 1$ holds. Then $\operatorname{soc}(ZFG)$ is an ideal of FG if and only if the following conditions hold:*

(i) $G = C_P(H) * G_H$ is a central product of the centralizer $C_P(H)$ and $G_H := H[G', H]$.

(ii) $\operatorname{soc}(ZFC_P(H))$ and $\operatorname{soc}(ZFG_H)$ are ideals in $FC_P(H)$ and FG_H, respectively.

Proof. If the conditions (i) and (ii) hold, then Lemma 2.43 yields $\operatorname{soc}(ZFG) \trianglelefteq FG$. From now on, we therefore assume that $\operatorname{soc}(ZFG)$ is an ideal of FG. We first prove that G is the central product of $C_P(H)$ and G_H. Note that G_H is a normal subgroup of G since we have $[G', H] = [HC_{P'}(H)G', G] = [G, H]$ and the latter group is normal in G. Moreover, this yields $[P, H] = [G', H]$ and hence we obtain $P = C_P(H)[G', H]$ by (4.9). Since the subgroups $C_P(H)$ and $[G', H]$ commute element-wise by Proposition 4.23, we obtain $P = C_P(H) * [G', H]$. This implies $G = C_P(H) * G_H$. Lemma 2.43 then yields $\operatorname{soc}(ZFC_P(H)) \trianglelefteq FC_P(H)$ and $\operatorname{soc}(ZFG_H) \trianglelefteq FG_H$. $\square$

In the following, we may therefore treat the group algebras of $C_P(H)$ and G_H separately.

Remark 4.25. Assume that $\operatorname{soc}(ZFG) \trianglelefteq FG$ holds. Since $C_P(H)$ is a p-group, its structure is determined by Theorem 3.1. In particular, it follows that $C_P(H)$, and hence P, are metabelian in this situation. In case that p is odd, we even obtain the stronger condition $c(C_P(H)) \leq 2$. By Lemma 4.20, we have $c(G') \leq 2$, so also $P = C_P(H) * G'$ is of nilpotency class at most two in this case. $\triangleleft$

It therefore remains to consider the subgroup G_H. We show that it has the structure given in Hypothesis 4.1, albeit with the additional restriction that the derived subgroup G'_H is the (unique) Sylow p-subgroup of G_H.

Lemma 4.26. *The group $G_H = H[G', H]$ decomposes as $G_H = G'_H \rtimes H$ and $G'_H = [G', H]$ is a Sylow p-subgroup of G_H.*

Proof. We have $[G', H] \cap H = 1$ and hence $G_H = [G', H] \rtimes H$ with $[G', H] \in \mathrm{Syl}_p(G_H)$. It remains to show that $[G', H]$ is the derived subgroup of G_H. By Theorem 4.24, we have $[G', H] = [C_P(H)' \cdot G'_H, H] = [G'_H, H] \subseteq G'_H$. The other inclusion is clear since $G'_H \subseteq G'$ is a p-group. □

Since G_H is a normal subgroup of G, we obtain $O_{p'}(G_H) \subseteq O_{p'}(G) = 1$. In the following, we can therefore restrict our investigation to the group G_H while using all results derived in the previous sections.

At the end of this part, we state a first application of Theorem 4.24 by determining $\mathrm{soc}(ZFG)$ in case that this is an ideal of FG.

Lemma 4.27. *Let G be a finite group with $O_{p'}(G) = 1$ which satisfies $\mathrm{soc}(ZFG) \trianglelefteq FG$. Then $\mathrm{soc}(ZFG)$ is given by*

$$\mathrm{soc}(ZFG) = (Z(P)G')^+ \cdot FG.$$

Proof. Note that $Z(C_P(H)) \subseteq Z(G)$ holds since we have $G = C_P(H) * G_H$ by Theorem 4.24. Remark 2.14 therefore yields $\mathrm{soc}(ZFG) \subseteq Z(C_P(H))^+ \cdot FG$. Together with Lemma 2.8, this implies

$$\mathrm{soc}(ZFG) \subseteq (Z(C_P(H))G')^+ \cdot FG \subseteq (Z(P)G')^+ \cdot FG.$$

In the last step, we use the identity $Z(P) = Z(C_P(H))Z(G'_H) \subseteq Z(C_P(H))G'$. In order to prove the converse inclusion, set $N := Z(P)G' \trianglelefteq G$ and consider the element $N^+ \in ZFG$. We show that N^+ annihilates the basis elements of $J(ZFG)$ given in Theorem 2.23. For any $z \in Z(P)\backslash\{1\}$, we have $[z] \subseteq N$ and hence

$$N^+ \cdot b_{[z]} = N^+ \cdot ([z]^+ - |[z]| \cdot 1) = 0.$$

Now consider a conjugacy class $C \in \mathrm{Cl}(G)$ such that p divides $|C|$. Since $C \subseteq gG'$ holds for some $g \in G$, we have $(G')^+ \cdot C^+ = 0$ and hence $N^+ \cdot b_C = N^+ \cdot C^+ = 0$. This shows $N^+ \in \mathrm{soc}(ZFG)$ and hence we have $N^+ \cdot FG \subseteq \mathrm{soc}(ZFG)$, which proves the equality. □

Note that this generalizes the corresponding result for p-groups stated in Theorem 3.1.

4.2.3 Structure of G'

Again, we assume that G is a finite group satisfying Hypothesis 4.1. In this section, we refine our results on the structure of G' by applying Theorem 4.24. Our aim is to decompose G' as a central product of $C_{G'}(P)P'$ and the subgroups $L_i = \pi^{-1}(T_i)$ defined in Theorem 4.4 ($i = 1, \dots, n$). We use the terminology from the previous parts. As before, we fix a nontrivial element $h_i \in C_H(M_i)$ for $i = 1, \dots, n$ (see Theorem 4.14) and denote by g_i a corresponding element of $L_i \backslash P'$ as defined in Lemma 4.19. Our first result is a decomposition of $Z(G')$:

Lemma 4.28. *We have* $Z(G') = C_{G'}(P)P'$.

Proof. By Theorem 4.24, we have $G' = C_P(H)' \cdot G'_H$ and hence

$$[C_P(H)', G'] = [C_P(H)', C_P(H)' \cdot G'_H] = [C_P(H)', C_P(H)'] = 1, \tag{4.11}$$

since $C_P(H)$ is metabelian by Remark 4.25 and centralizes G_H. Moreover, we obtain $P' = C_P(H)' \cdot G'' \subseteq Z(G')$ by (4.11) and since $G'' \subseteq Z(P)$ holds by Lemma 4.20. This proves the inclusion $C_{G'}(P)P' \subseteq Z(G')$. Now suppose that $C_{G'}(P)P'$ is a proper subgroup of $Z(G')$. This means that $Z := \pi(Z(G'))$ is a normal subgroup of D which properly contains $Z_D = C_{G'}(P)P'/P'$. By Proposition 4.11, it therefore has the form

$$Z = T_{i_1} \times \ldots \times T_{i_k} \times Z_D$$

for some $k \in \mathbb{N}$ and distinct indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Without loss of generality, we assume $i_1 = 1$. In particular, we have $\pi(g_1) \in Z$, which yields $g_1 d \in Z(G')$ for some $d \in P' \subseteq Z(G')$. Hence g_1 is contained in $Z(G')$ as well. Since g_1 centralizes $C_P(H)$ by Lemma 4.19 (ii), we have $g_1 \in C_{G'}(P)$, which yields $\pi(g_1) \in Z_D \cap T_1 = 1$. This is a contradiction to $g_1 \notin P'$, which proves the equality $C_{G'}(P)P' = Z(G')$. □

Now we proceed with the decomposition of G' into a central product which we announced at the beginning of this part:

Theorem 4.29. *Let G be finite group for which* $\operatorname{soc}(ZFG) \trianglelefteq FG$ *holds and consider the subgroups $L_1, \ldots, L_n$ introduced in Theorem 4.4. Then the group G' decomposes as the central product*

$$G' = L_1 * \ldots * L_n * Z(G').$$

Proof. Recall that the subgroups $L_1, \ldots, L_n$ together with $Z(G') = C_{G'}(P)P'$ generate G' (see Theorem 4.4). It therefore remains to show that the elements of L_i and L_j commute for $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Since $C_{G'}(L_i)$ is a normal subgroup of G containing $Z(G') = C_{G'}(P)P'$, its image in D is of the form

$$\pi\big(C_{G'}(L_i)\big) = T_{i_1} \times \ldots \times T_{i_k} \times Z_D$$

for some $k \in \mathbb{N}_0$ and distinct indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$ by Proposition 4.11. By Corollary 2.29 and Remark 2.9, we have $L_i \cdot C_{G'}(L_i) = G'$. Since $\pi(L_i) = T_i$ holds, we obtain $N_i \subseteq \pi(C_{G'}(L_i))$ for the normal subgroup N_i of D defined in (4.2). Since $P' \subseteq Z(G') \subseteq C_{G'}(L_i)$ holds by the previous lemma, we obtain

$$\prod_{j \neq i} L_j \cdot Z(G') = \pi^{-1}(N_i) \subseteq C_{G'}(L_i)P' = C_{G'}(L_i).$$

In particular, this shows $[L_i, L_j] = 1$ for $j \neq i$, which finishes the proof. □

In the next section, we study the problem in the special case $C_{G'}(P) \subseteq P'$. In this case, the structural information derived in this section will enable us to classify the groups G which satisfy $\operatorname{soc}(ZFG) \trianglelefteq FG$.

4.3 Case $C_{G'}(P) \subseteq P'$

Let $G = P \rtimes H$ be a finite group which satisfies the conditions (D1)–(D3) in Theorem 4.14. In this chapter, we additionally assume $C_{G'}(P) \subseteq P'$, that is, the term Z_D in the decomposition of D' given in (D1) vanishes. In this situation, we determine conditions under which $\operatorname{soc}(ZFG) \trianglelefteq FG$ holds.

Throughout, we use the notation from the preceding parts. This section is organized as follows: First, we decompose the quotient group D as a direct product of affine linear groups (see Section 4.3.1) and derive a corresponding decomposition of G into a central product (see Section 4.3.2). By Lemma 2.43, we may then restrict our investigation to the case that D' is a simple $\mathbb{F}_pH$-module. For this simplified situation, we determine necessary and sufficient conditions for $\operatorname{soc}(ZFG) \trianglelefteq FG$ in Section 4.3.3. In Section 4.3.4, we collect our preceding results in order to classify the finite groups G of the form $P \rtimes H$ with $C_{G'}(P) \subseteq P'$ which satisfy $\operatorname{soc}(ZFG) \trianglelefteq FG$.

4.3.1 Structure of D

As before, we begin by investigating the quotient group $D = G/P'$ and use the additional assumption $C_{G'}(P) \subseteq P'$ in order to simplify the statements given in Theorem 4.14. By Theorem 4.24, we may restrict ourselves to the case $P = G'$, for which the condition $C_{G'}(P) \subseteq P'$ translates to $Z(G') \subseteq G''$. Note that the converse inclusion is given by Lemma 4.20 if $\operatorname{soc}(ZFG)$ is an ideal of FG, so we usually assume $G'' = Z(G')$.

Remark 4.30. Let G be a finite group of the form $G' \rtimes H$ with $G' \in \operatorname{Syl}_p(G)$ and an abelian p'-group H. Moreover, we assume that $Z(G') = G''$ and $O_{p'}(G) = 1$ hold and that G satisfies the conditions (D1) and (D2) in Theorem 4.14. In particular, D' splits as a direct product $T_1 \times \ldots \times T_n$ of minimal normal subgroups of D for some $n \in \mathbb{N}_0$ since $Z(G')/G'' = 1$ holds by assumption. Observe that D' is elementary abelian in this case. In this decomposition, $n = 0$ implies $G' = 1$, which yields $G = O_{p'}(G) = 1$. We therefore focus on the case $n \geq 1$. There, the assumption $Z(G') = G''$ implies that G' has nilpotency class exactly two. Since $C_H(D') = 1$ holds by Remark 4.3, the group H is of the form $\langle e_1, \ldots, e_n \rangle$ by condition (D2), where $\langle e_i \rangle$ centralizes T_j for $j \neq i$ and acts transitively on $T_i \backslash \{1\}$ for $i = 1, \ldots, n$. Note that we have

$$\langle e_1 \rangle \cap \langle e_2, \ldots, e_n \rangle \subseteq C_H(T_2 \times \ldots \times T_n) \cap C_H(T_1) = C_H(D') = 1.$$

Inductively, this yields $H \cong \langle e_1 \rangle \times \ldots \times \langle e_n \rangle$. In particular, we have $\operatorname{ord}(e_i) = |T_i| - 1 =: s_i$ and the group $\langle e_i \rangle$ acts on $T_i \backslash \{1\}$ without fixed points. ◁

In the given situation, we obtain a natural decomposition of D into a direct product of affine linear groups:

Lemma 4.31. *Let G be a finite group of the form $G' \rtimes H$ with $G' \in \operatorname{Syl}_p(G)$ and an abelian p'-group H. Moreover, we assume that $Z(G') = G''$ and $O_{p'}(G) = 1$ hold. Then G satisfies the conditions (D1) and (D2) in Theorem 4.14 if and only if there exist $n \in \mathbb{N}_0$ and $d_1, \ldots, d_n \in \mathbb{N}$ with*

$$D \cong \operatorname{AGL}(1, p^{d_1}) \times \ldots \times \operatorname{AGL}(1, p^{d_n}). \tag{4.12}$$

Here, $\mathrm{AGL}(1,p^\ell) \cong \mathbb{F}_{p^\ell} \rtimes \mathbb{F}_{p^\ell}^\times$ *is the one-dimensional affine linear group over* $\mathbb{F}_{p^\ell}$ *for* $\ell \in \mathbb{N}$.

Proof. First assume that G satisfies the conditions (D1) and (D2) in Theorem 4.14. For $i = 1, \dots, n$, we set $A_i := \langle e_i, T_i \rangle$. Note that D is generated by $A_1, \dots, A_n$ and that the elements of A_i and A_j commute for $i \neq j$ (see Remark 4.30). In particular, the subgroups $A_1, \dots, A_n$ are normal in D. Note that for any $i, j \in \{1, \dots, n\}$ with $i \neq j$, we have $A_i \cap A_j \subseteq C_{D'}(H) = 1$ (see Remark 4.3). Since every element in D can be expressed uniquely in the form $a_1 \cdots a_n$ with $a_i \in A_i$ for $i = 1, \dots, n$, we obtain $D = A_1 \times \ldots \times A_n$. Moreover, the group $\langle e_i \rangle$ acts transitively and without fixed points on $T_i \backslash \{1\}$, so $A_i \cong \mathrm{AGL}(1, |T_i|)$ follows analogously to the proof of [37, Theorem 4.2]. Conversely, if D is a direct product of one-dimensional affine linear groups as in (4.12), then the conditions (D1) and (D2) in Theorem 4.14 are naturally satisfied. □

4.3.2 Decomposition of G into a central product

In the preceding section, we showed that the group D is isomorphic to a direct product $\mathrm{AGL}(1,p^{d_1}) \times \ldots \times \mathrm{AGL}(1,p^{d_n})$ of affine linear groups if G satisfies the conditions described in Theorem 4.14. The aim of this section is to derive a corresponding decomposition of G into a central product $Q_1 * \ldots * Q_n$ with $Q_i/Q_i'' \cong \mathrm{AGL}(1,p^{d_i})$ for $i = 1, \dots, n$ if $\mathrm{soc}(ZFG)$ is an ideal of FG. This allows us to restrict ourselves to the case $n = 1$ in the subsequent section. Throughout, we make the following assumption:

Hypothesis 4.32. Let G be a finite group with $O_{p'}(G) = 1$ for which $\mathrm{soc}(ZFG) \trianglelefteq FG$ holds. Moreover, we assume that G' is a Sylow p-subgroup of G with $Z(G') = G''$. As usual, we fix a Hall p'-subgroup H of G.

Recall that H is of the form $\langle e_1 \rangle \times \ldots \times \langle e_n \rangle$ by Remark 4.30. As before, we fix a nontrivial element $h_i \in C_H(M_i)$ for $i = 1, \dots, n$ (see (D3) in Theorem 4.14) and consider a corresponding element $g_i \in L_i \backslash G''$ as defined in Lemma 4.19. Here, L_i denotes the preimage of T_i under the projection onto D as usual. For $i = 1, \dots, n$, we consider the subgroup

$$Q_i := \langle g_i, e_i \rangle. \tag{4.13}$$

This leads to the desired decomposition of G into a central product:

Lemma 4.33. *The group G decomposes as $G = Q_1 * \ldots * Q_n$.*

Proof. We set $Q := \langle Q_1, \dots, Q_n \rangle$ and show that this subgroup coincides with G. Since $H = \langle e_1, \dots, e_n \rangle$ is contained in Q, it suffices to show $G' \subseteq Q$. By (4.8), any element $u \in G'$ can be written in the form $u = u_1 \cdots u_n \cdot g$ with $u_i \in \{1\} \cup \{e_i^k g_i e_i^{-k} : k \in \mathbb{Z}\} \subseteq Q_i$ and $g \in G''$. Theorem 4.29 yields $G'' = L_1' \cdots L_n'$. Since the elements in L_i are of the form $e_i^k g_i e_i^{-k} d$ for some $k \in \mathbb{Z}$ and $d \in G'' = Z(G')$ by (4.8), the derived subgroup L_i' is generated by commutators of the form $[e_i^k g_i e_i^{-k}, e_i^m g_i e_i^{-m}]$ for $k, m \in \mathbb{Z}$, so it is contained in Q_i. This yields $G'' \subseteq Q$, which shows $Q = G$. Note that $[Q_i, Q_j] = 1$ holds for $i \neq j$ since the respective generators commute by Lemma 4.19 (ii) and Theorem 4.29. Hence $Q_1, \dots, Q_n$ are normal subgroups of G and we obtain $G = Q_1 * \ldots * Q_n$ as claimed. □

The aim is to reduce our investigation to the study of the individual factors $Q_1, \dots, Q_n$ of the central product. To this end, we need the following observations:

Lemma 4.34. *Let $i \in \{1, \ldots, n\}$.*

(i) The group Q_i satisfies Hypothesis 4.32.

(ii) Q_i'/Q_i'' is a minimal normal subgroup of Q_i/Q_i''.

Proof.

(i) Since Q_j centralizes Q_i for $j \neq i$ by Lemma 4.33, every normal p'-subgroup of Q_i is normal in G and hence we have $O_{p'}(Q_i) \subseteq O_{p'}(G) = 1$. Moreover, $\operatorname{soc}(ZFQ_i) \trianglelefteq FQ_i$ follows by Lemma 2.43. This implies $Q_i'' \subseteq Z(Q_i')$ by Lemma 4.20. Therefore, it remains to show that Q_i' is a Sylow p-subgroup of Q_i and that $Z(Q_i')$ is contained in Q_i''.

We first show $Q_i' \in \operatorname{Syl}_p(Q_i)$. Clearly, we have $[e_i, g_i] \in Q_i'$. The elements $[e_i, g_i]$ and g_i are conjugate in G by Lemma 4.19 (iii), so also in Q_i since Q_j centralizes Q_i for $j \neq i$ by Lemma 4.33. Since Q_i' is a normal subgroup of Q_i, this yields $g_i \in Q_i'$. Hence $e_i Q_i'$ generates Q_i/Q_i', which shows that this quotient is a p'-group. This implies that Q_i', which is a p-group, is a Sylow p-subgroup of Q_i.

Now we show that $Z(Q_i')$ is contained in Q_i''. Applying Remark 4.3 (i) to the group Q_i yields $C_{Q_i'/Q_i''}(e_i) = 1$ and hence $C_{Q_i'}(e_i) \subseteq Q_i''$. Note that we have

$$Z(Q_i') \subseteq Z(G') \cap Q_i \subseteq G'' \cap Q_i.$$

By Lemma 4.33, any element $g \in G''$ can be expressed in the form $q_i \cdot a$ for some $q_i \in Q_i''$ and $a \in \prod_{j \neq i} Q_j'' \subseteq C_{G''}(e_i)$. If g is additionally contained in Q_i, then $a = q_i^{-1} g \in Q_i'$ follows, which yields $a \in C_{Q_i'}(e_i)$. With this, we obtain

$$Z(Q_i') \subseteq G'' \cap Q_i \subseteq C_{Q_i'}(e_i) Q_i'' = Q_i''. \tag{4.14}$$

(ii) As before, we set $\bar{Q}_i$ to be the image of Q_i in D. Clearly, we have $Q_i'' \subseteq Q_i \cap G''$, and by (4.14), we obtain equality. In particular, this yields

$$Q_i/Q_i'' = Q_i/Q_i \cap G'' \cong Q_i G''/G'' = \bar{Q}_i,$$

so it suffices to show that $\bar{Q}_i'$ is a minimal normal subgroup of $\bar{Q}_i$. Note that $\bar{Q}_i'$ is centralized by e_j for $j \neq i$, so we obtain $\bar{Q}_i' \subseteq T_i$. Since $g_i \in Q_i \backslash G''$ holds, the group $\bar{Q}_i'$ is a nontrivial normal subgroup of D, which implies $\bar{Q}_i' = T_i$ since T_i is a minimal normal subgroup of D. In particular, $\bar{Q}_i'$ is a minimal normal subgroup of $\bar{Q}_i$. □

Combining Lemma 4.33 and Lemma 4.34, we obtain the main result of this section:

Theorem 4.35. *Let G be a finite group with $G' \in \operatorname{Syl}_p(G)$ such that* $\operatorname{soc}(ZFG) \trianglelefteq FG$ *holds. Moreover, we assume $Z(G') = G''$ and $O_{p'}(G) = 1$. Then there exist $n \in \mathbb{N}_0$ and normal subgroups $Q_1, \ldots, Q_n$ of G with*

$$G = Q_1 * \ldots * Q_n$$

such that the following hold for $i = 1, \ldots, n$:

(i) The group Q_i satisfies Hypothesis 4.32.

(ii) Q_i'/Q_i'' is a minimal normal subgroup of Q_i/Q_i''.

Remark 4.36. Note that if G is a group which is a central product of subgroups $Q_1, \ldots, Q_n$ with $\mathrm{soc}(ZFQ_i) \trianglelefteq FQ_i$ for $i = 1, \ldots, n$, then $\mathrm{soc}(ZFG) \trianglelefteq FG$ follows by Lemma 2.43. In this sense, the converse of Theorem 4.35 also holds. ◁

The above decomposition is related to those introduced earlier in the following way:

Remark 4.37. By Lemma 4.34, we have $\bar{Q}_i' = T_i$ for $i = 1, \ldots, n$. Since L_i is the preimage of T_i in G, this yields $Q_i'G'' = L_i$. In this sense, the decomposition $G' = Q_1' * \ldots * Q_n'$ can be viewed as a refinement of the one given in Theorem 4.29. Moreover, note that

$$Q_i/Q_i'' \cong T_i \rtimes \langle e_i \rangle \cong \mathrm{AGL}(1, |T_i|)$$

follows by the proof of Lemma 4.34 as well as Lemma 4.31. This relates the decompositions of G and D given in Lemmas 4.33 and 4.31, respectively. ◁

In the following, we treat the groups $Q_1, \ldots, Q_n$ individually. The crucial observation is that if we apply Theorem 4.14 to the group Q_i, then the decomposition of Q_i'/Q_i'' stated therein consists of a single term since Q_i'/Q_i'' is a minimal normal subgroup of Q_i/Q_i''. This allows us to restrict to the case $n = 1$ in our further investigation.

4.3.3 Special case: D' simple $\mathbb{F}_pH$-module

In this section, we consider the special case that D' is a simple $\mathbb{F}_pH$-module, that is, a minimal normal subgroup of D. In contrast to the preceding parts, we do not only draw consequences concerning the structure of G from the assumption $\mathrm{soc}(ZFG) \trianglelefteq FG$, but we also state sufficient conditions for this property. For this reason, we limit ourselves to the following assumption:

Hypothesis 4.38. Let G be a finite group of the form $G = G' \rtimes H$ with $G' \in \mathrm{Syl}_p(G)$ and an abelian p'-group H such that $Z(G') = G''$ and $O_{p'}(G) = 1$ hold. Moreover, we assume that G has the properties (D1)–(D3) in Theorem 4.14, and that G'/G'' is a minimal normal subgroup of $D := G/G''$.

Recall that in this context, we have $D \cong \mathrm{AGL}(1, |D'|)$ by Lemma 4.31. In particular, $H = \langle e_1 \rangle$ is cyclic of order $s := |D'| - 1$. The condition (D3) in Theorem 4.14 reads $C_H(G'') \neq 1$. As before, we fix an element $1 \neq h_1 \in C_H(G'')$ and a corresponding element $g_1 := g_{h_1} \in G' \backslash G''$ as defined in Lemma 4.19.

In the first part of this section, we investigate the structure of the conjugacy classes of certain elements in G. These results will form the basis for the characterization of the groups G satisfying Hypothesis 4.38 which have the property $\mathrm{soc}(ZFG) \trianglelefteq FG$. It is given in Theorem 4.42, which we prove in the second part of this section.

4.3.3.1 Conjugacy classes in G

We now collect some preliminary results on the conjugacy class structure of the elements in G, with a particular focus on the elements in G'.

By (4.8), every element in $G'\backslash G''$ can be written in the form $e_1^k g_1 e_1^{-k} d$ for some $d \in G''$ and $k \in \mathbb{Z}$. For $g \in G'$, we consider the set of commutators

$$C_g \coloneqq \left\{[a, g] \colon a \in G'\right\}.$$

Note that we have $[g]_{G'} = C_g \cdot g$. By Lemma 2.45, C_g is a subgroup of G'' since G' is of nilpotency class at most two. For the fixed element $g_1 \in G'\backslash G''$, we abbreviate C_{g_1} by C. With this, the conjugacy classes of the elements in G' can be characterized in the following way:

Lemma 4.39.

(i) We have $C = \langle [e_1^m g_1 e_1^{-m}, g_1] \colon m \in \mathbb{Z}\rangle$.

(ii) Consider an arbitrary element $x \in G'\backslash G''$ and write $x = e_1^k g_1 e_1^{-k} d$ for some $k \in \mathbb{Z}$ and $d \in G''$. Then we have $C_x = e_1^k C e_1^{-k}$.

(iii) For any $x \in G'\backslash G''$, we have $[x]_{G'} = [x] \cap xG''$, which yields

$$[x] = \bigcup_{m=0}^{s-1} e_1^m \cdot [x]_{G'} \cdot e_1^{-m}. \tag{4.15}$$

Proof.

(i) Since every element in G' can be written in the form $e_1^m g_1 e_1^{-m} d$ with $m \in \mathbb{Z}$ and $d \in G'' = Z(G')$, we have

$$C = \left\{[a, g_1] \colon a \in G'\right\} = \langle [e_1^m g_1 e_1^{-m}, g_1] \colon m \in \mathbb{Z}\rangle.$$

(ii) Since G'' centralizes G', we have $[a, x] = [a, e_1^k g_1 e_1^{-k} d] = [a, e_1^k g_1 e_1^{-k}]$ for all $a \in G'$ and hence

$$C_x = \left\{\left[a, e_1^k g_1 e_1^{-k}\right] \colon a \in G'\right\} = \left\{e_1^k [a, g_1] e_1^{-k} \colon a \in G'\right\} = e_1^k C e_1^{-k}.$$

In the second step, we use that conjugation with e_1^{-k} permutes the elements of G'.

(iii) We first prove $[x]_{G'} = [x] \cap xG''$. Clearly, we have $[x]_{G'} \subseteq [x] \cap xG''$. On the other hand, assume $kxk^{-1} \in xG''$ for some $k \in G$ and write $k = e_1^\ell u$ with $\ell \in \mathbb{Z}$ and $u \in G'$. Since $uxu^{-1} \in xG''$ holds, conjugation with e_1^ℓ fixes xG''. By Remark 4.30, this implies $e_1^\ell = 1$ and hence we have $k \in G'$, which shows the equality.

Now we prove the second part of the statement. Clearly, the set on the right hand side of (4.15) is contained in $[x]$. On the other hand, the conjugates of x are contained in $G'\backslash G'' = \bigcup_{m=0}^{s-1} e_1^m x e_1^{-m} G''$ and for any $m \in \mathbb{Z}$, conjugating with e_1^m induces a bijection between $[x]_{G'} = [x] \cap xG''$ and $[x] \cap e_1^m x e_1^{-m} G''$. □

We observe that the G'-conjugacy classes of the elements in $G' \backslash G''$ are all of size $|C|$. Furthermore, the preceding lemma yields

$$G'' = \langle C_g \colon g \in G' \rangle = \left\langle \bigcup_{k=0}^{s-1} e_1^k C e_1^{-k} \right\rangle .$$

For the conjugacy classes of G which are not contained in G', we make use of the following observation:

Lemma 4.40. *Every conjugacy class $K \in \mathrm{Cl}(G)$ with $K \nsubseteq G'$ contains an element $e_1^k d$ with $k \in \mathbb{Z}$ and $d \in G''$, and we have $e_1^k[d] \subseteq K$.*

Proof. As before, we denote the image of an element $g \in G$ in D by $\bar{g}$ (similarly for subsets of G). Let $k \in \mathbb{Z}$ with $K \subseteq e_1^k G'$ and set $B := [e_1^k] \in \mathrm{Cl}(G)$. Since $\bar{e}_1^k$ is not contained $C_H(D') = 1$ (see Remark 4.3), we obtain $|\bar{B}| > 1$. By Lemma 2.45, $\bar{B}$ is of the form $\bar{e}_1^k N$ for a subgroup $N \subseteq D'$ which is normal in D. Since D' is a minimal normal subgroup of D, we obtain $N = D'$ and hence $\bar{B} = \bar{K}$. In particular, K contains an element $e_1^k d$ with $d \in G''$. Since G' centralizes G'', we have $[d] = \{e_1^\ell d e_1^{-\ell} \colon \ell \in \mathbb{Z}\}$, which yields $e_1^k[d] \subseteq [e_1^k d] = K$. □

We conclude this part with the following observation on the conjugacy classes of elements in $H \backslash C_H(G'')$:

Remark 4.41. Consider the conjugacy class $K := [h]$ of an element $h \in H \backslash C_H(G'')$ and write $K = U \cdot h$ with $U \subseteq G'$ as in (2.1). As h centralizes H, we have

$$U = \{[g,h] \colon g \in G\} = \left\{[g,h] \colon g \in G'\right\}.$$

Since h acts on $D' \backslash \{1\}$ without fixed points (see Remark 4.30), we deduce that $[g,h] \in G''$ for some $g \in G'$ is equivalent to $g \in G''$. Moreover, we have $[a_1 a_2, h] = [a_1, h] \cdot [a_2, h]$ for all $a_1, a_2 \in G''$ since G'' centralizes G'. This shows that $N := U \cap G''$ is a normal subgroup of G. Since h acts nontrivially on G'', we have $N \neq 1$.

Now consider an orbit B of the conjugation action of G'' on $[h]$ and let $b \in B$. We claim that $B = Nb$ holds. To this end, write $b = ghg^{-1}$ for some $g \in G'$. For any $n \in N$, there exists some $d \in G''$ with $[d,h] = n$ by the above. Since $[d,b] = [d, ghg^{-1}] = [d,h] = n$ follows from the fact that g centralizes G'', we have $nb = dbd^{-1} \in B$. On the other hand, $[d,h] \in N$ holds for all $d \in G''$ and hence $B \subseteq Nb$ follows, which proves the equality. In particular, $[h]$, and hence also U, is a disjoint union of cosets of N. ◁

4.3.3.2 Main result

In this section, we study groups G which satisfy Hypothesis 4.38 and provide necessary as well as sufficient conditions under which $\mathrm{soc}(ZFG)$ is an ideal in FG. The aim of this section is the proof of the following statement:

Theorem 4.42. *Let G be a finite group of the form $G' \rtimes H$ with $G' \in \mathrm{Syl}_p(G)$ and an abelian p'-group H such that $O_{p'}(G) = 1$ and $Z(G') = G''$ hold. Moreover, we assume that G has the properties (D1)–(D3) in Theorem 4.14 and that G'/G'' is a minimal normal subgroup of $D := G/G''$. Then the following are equivalent:*

(i) $\mathrm{soc}(ZFG) \trianglelefteq FG$.

(ii) $Z(G)C_g = G''$ *holds for all* $g \in G' \backslash G''$*, that is,* $G'/Z(G)$ *is a Camina group (see Section 2.1).*

(iii) $Z(G)C_g = G''$ *holds for some* $g \in G' \backslash G''$*.*

The proof will be split into several parts. Note that the conditions (ii) and (iii) are equivalent by Lemma 4.39. We show that condition (ii) implies (i) before proving that condition (iii) follows from (i).

Proof of the implication (ii) ⇒ (i) in Theorem 4.42. Let G be a finite group which satisfies Hypothesis 4.38 and assume that $Z(G)C_g = G''$ holds for all $g \in G' \backslash G''$. Consider an element $y = \sum_{g \in G} a_g g \in \mathrm{soc}(ZFG)$. By Remark 2.9, we need to show that the coefficients of y are constant on the cosets of G' in G.

We first consider the coefficients of elements in G'. For $g \in G''$, we obtain $a_g = a_1$ by Remark 2.25 since $G'' = Z(G')$ holds by assumption. Now let $g \in G' \backslash G''$. Then g has a G-conjugate of the form $g_1 d$ with $d \in G''$. By assumption, we can write $d = cz$ with $z \in Z(G)$ and $c \in C$. Then we obtain

$$a_g = a_{g_1 cz} = a_{g_1 c} = a_{g_1} = a_1.$$

In the second step, we used $z \in Z(G)$, which yields $a_{g_1 cz} = a_{g_1 c}$ by Remark 2.14. The third equality follows from the fact that $g_1 c = c g_1$ is conjugate to g_1 by definition of C. In the last step, we used that $[h_1] = \{1\} \cup \{e_1^k g_1 e_1^{-k} \colon 0 \leq k \leq s-1\}$ holds for the fixed element $h_1 \in C_H(G'')$ (see Lemma 4.19). By Remark 2.25, $y \cdot [h_1]^+ = 0$ implies

$$a_1 + s \cdot a_{g_1} = a_1 + \sum_{i=0}^{s-1} a_{e_1^k g_1 e_1^{-k}} = 0$$

and hence $a_{g_1} = a_1$ follows. This shows that the coefficients of y on G' are constant.

Now consider the coset $e_1^k G'$ for some $1 \leq k \leq s-1$ and let $g \in G'$. By Lemma 4.40, we have $[e_1^k g] = [e_1^k d]$ for some $d \in G''$ and hence $a_{e_1^k g} = a_{e_1^k d}$ follows. Either $d = 1$ holds or we obtain $y \cdot b_{[d^{-1}]} = 0$ for the basis element $b_{[d^{-1}]}$ of $J(ZFG)$ corresponding to $[d^{-1}]$ (see Definition 2.22). Setting ℓ to be the length of $[d]$, Remark 2.25 then yields

$$\ell \cdot a_{e_1^k} - \sum_{d' \in [d]} a_{e_1^k d'} = 0.$$

All coefficients occurring in the sum are equal since the elements $e_1^k d'$ with $d' \in [d]$ are conjugate by Lemma 4.40. The above condition therefore reads $\ell \cdot a_{e_1^k} - \ell \cdot a_{e_1^k d} = 0$, which implies $a_{e_1^k d} = a_{e_1^k}$ since ℓ is coprime to p. Hence the coefficients of y are constant on $e_1^k G'$. □

In the remaining part of this section, we prove that the first condition stated in Theorem 4.42 implies the third one. Additionally to Hypothesis 4.38, we therefore require that $\mathrm{soc}(ZFG)$ is an ideal of FG. In order to derive a contradiction, we assume in the following that $Z(G)C$ is a proper subgroup of G'' and construct an element $y \in \mathrm{soc}(ZFG)$ which is not contained in $(G')^+ \cdot FG$:

Construction 4.43. Since the group D' is elementary abelian (see Remark 4.30), we have $g^p \in G'' = Z(G')$ for all $g \in G'$. By [24, Hilfssatz III.1.3], the group G'' is elementary abelian, so it can be viewed as an $\mathbb{F}_p$-vector space. In particular, there exists a nontrivial group homomorphism $\alpha\colon G'' \to \mathbb{F}_p$ with constant values on the cosets of $Z(G)C$ in G''. For instance, one could extend an $\mathbb{F}_p$-basis of $Z(G)C$ to a basis of G'' and map each element of G'' to the coefficient of a fixed basis vector $v \in G''\backslash Z(G)C$. In the following, we interpret the field $\mathbb{F}_p$ as a subset of F. For $g \in G$, we define

$$a_g := \begin{cases} \alpha(u) & \text{if } g \sim g_1 u \text{ holds for some } u \in G'' \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 4.44. *The map $G \to G,\ g \mapsto a_g$ introduced in Construction 4.43 is well-defined.*

Proof. Let $g \in G$ and assume that g is conjugate to g_1u_1 and g_1u_2 with $u_1, u_2 \in G''$. Then g_1u_2 is contained in

$$[g_1u_1] \cap g_1u_1G'' = [g_1u_1]_{G'} = Cg_1u_1 = g_1u_1C$$

(see Lemma 4.39), which yields $u_2 \in u_1C$ and hence $\alpha(u_1) = \alpha(u_2)$. In particular, the image of g is well-defined. □

Remark 4.45. Since the map α is a group homomorphism, we have

$$a_{g_1u_1u_2} = a_{g_1u_1} + a_{g_1u_2} \tag{4.16}$$

for all $u_1, u_2 \in G''$. Note that $a_g = a_h$ holds for all conjugate elements $g, h \in G$. For $k = 1, \ldots, s-1$, conjugation with e_1^k therefore yields

$$a_{e_1^k g_1 e_1^{-k} u_1 u_2} = a_{e_1^k g_1 e_1^{-k} u_1} + a_{e_1^k g_1 e_1^{-k} u_2}$$

for all $u_1, u_2 \in G''$. ◁

In the following, we consider the element $y := \sum_{g \in G} a_g g \in FG$ with the coefficients described in Construction 4.43. Clearly, we have $y \in ZFG$ and the coefficients of y are not constant on G' since α is nontrivial. Our aim for the remainder of the section is to show that $y \in \mathrm{soc}(ZFG)$ holds. To this end, we need two auxiliary results.

Remark 4.46. For any $g \in G''$, the element $t := \prod_{g' \in [g]} g' \in G'' = Z(G')$ is invariant under conjugation with e_1, which implies $t \in Z(G)$. By (4.16), we obtain

$$\sum_{g' \in [g]} a_{g_1 g'} = a_{g_1 \prod_{g' \in [g]} g'} = a_{g_1 t} = \alpha(t) = \alpha(1) = 0, \tag{4.17}$$

since α has constant values on the cosets of $Z(G)C$ in G''. ◁

The following observation will later ensure that y annihilates $[h]^+$ for all elements $h \in H$ which do not centralize G''.

Proposition 4.47. *Let $U \subseteq G''$ be a subgroup with $|U| > 2$. Then we have $y \cdot U^+ = 0$.*

Proof. By Remark 2.25, we need to show that $\sum_{u\in U} a_{wu^{-1}} = 0$ holds for all $w \in G$. Note that all summands are zero for $w \notin G'$ or $w \in G''$, so it suffices to consider an element $w \in G'\backslash G''$. We write $w = e_1^k g_1 e_1^{-k} d$ for some $k \in \mathbb{Z}$ and some $d \in G''$. Then we have

$$\sum_{u\in U} a_{wu^{-1}} = \sum_{u\in U} a_{e_1^k g_1 e_1^{-k} du^{-1}} = |U| \cdot a_{e_1^k g_1 e_1^{-k} d} + \sum_{u\in U} a_{e_1^k g_1 e_1^{-k} u^{-1}} = a_{e_1^k g_1 e_1^{-k} \prod_{u\in U} u^{-1}}$$
$$= a_{e_1^k g_1 e_1^{-k}} = 0.$$

In the second and third step, we apply the homomorphism properties described in Remark 4.45. Moreover, we use that p divides $|U|$ and that $\prod_{u\in U} u^{-1} = 1$ holds since U is elementary abelian of order at least three (see Lemma 3.11). The last step follows from the fact that $e_1^k g_1 e_1^{-k}$ is conjugate to g_1 and we have $\alpha(1) = 0$. □

With these preliminaries, we now show that y annihilates the basis element b_C of $J(ZFG)$ for any conjugacy class $1 \neq C \in \mathrm{Cl}(G)$ (see Definition 2.22). We begin with the basis elements corresponding to conjugacy classes which are contained in G'.

Lemma 4.48. *For $1 \neq g \in G'$, we obtain $y \cdot b_{[g]} = 0$.*

Proof. First, we assume $g \in G''$, that is, we have $b_{[g]} = [g]^+ - \ell \cdot 1$ with $\ell := |[g]|$. By (2.13), we need to show that

$$\sum_{g'\in[g]} a_{tg'^{-1}} = \ell \cdot a_t$$

holds for all $t \in G$. Again, all summands are zero for $t \in G''$ or $t \notin G'$, so let $t \in G'\backslash G''$ and write $t = e_1^k g_1 e_1^{-k} d'$ for some $d' \in G''$. Setting $d := e_1^{-k} d' e_1^k$, we have $t = e_1^k g_1 d e_1^{-k}$. Then

$$\sum_{g'\in[g]} a_{tg'^{-1}} = \sum_{g'\in[g]} a_{e_1^k g_1 d e_1^{-k} g'^{-1}} = \sum_{g'\in[g]} a_{g_1 d e_1^{-k} g'^{-1} e_1^k} = \sum_{g'\in[g]} a_{g_1 d g'^{-1}}$$
$$= \ell \cdot a_{g_1 d} + \sum_{g'\in[g^{-1}]} a_{g_1 g'} = \ell \cdot a_{g_1 d} = \ell \cdot a_t.$$

In the third step, we use that conjugation with e_1^k permutes the elements of $[g]$. The fourth equality is due to (4.16) and in the fifth step, we make use of (4.17).

Now assume $g \in G'\backslash G''$. Then g is conjugate to an element of the form $g_1 d$ for some $d \in G''$ and without loss of generality, we may assume $g = g_1 d$. Note that the group C is nontrivial since $g_1 \notin Z(G')$ holds. Suppose that $|C| = 2$ holds. By Lemma 4.39, it follows that all conjugacy classes in G' are of length at most two. Since $G'' = Z(G')$ holds, G' is an extraspecial group by [26, Proposition 3.1], which implies $|G''| = 2$. Hence C coincides with G'', which is a contradiction to the assumption $Z(G)C < G''$. This shows that $|C| > 2$ holds. By Lemma 4.39, $[g]$ is a union of cosets of subgroups of the form $e_1^\ell C e_1^{-\ell}$ with $\ell \in \mathbb{Z}$. Since we have $|C| > 2$, Proposition 4.47 yields $y \cdot (e_1^\ell C e_1^{-\ell})^+ = 0$ and hence y annihilates $b_{[g]} = [g]^+$. □

Now we show that y annihilates the basis elements b_C corresponding to conjugacy classes $C \in \mathrm{Cl}(G)$ which are not contained in G'. This problem is solved in two steps. The first one concerns the conjugacy classes of the elements in $C_H(G'')\backslash\{1\}$ and is far more complicated than the second, in which we summarize the above results (see Lemma 4.50).

Proposition 4.49. *The element y annihilates every class sum $[h]^+$ for $1 \neq h \in C_H(G'')$.*

Proof. Recall that the class $[h]$ is of the form $U \cdot h$ with

$$U := \{1\} \cup \left\{e_1^\ell g_1' e_1^{-\ell} \colon 0 \leq \ell \leq s-1\right\}$$

for some $g_1' \in G'\backslash G''$ which is conjugate to g_1 (see Lemma 4.19). By conjugating, we can choose $g_1' \in g_1 G''$. By Lemma 4.39, we then have $g_1' \in g_1 G'' \cap [g_1] = [g_1]_{G'} = g_1 C$, so we write $g_1' = g_1 c$ for some $c \in C$. Moreover, note that the condition $y \cdot [h]^+ = 0$ can be replaced by $y \cdot U^+ = 0$. By (2.12), we need to show that

$$\sum_{u \in U} a_{tu^{-1}} = 0 \tag{4.18}$$

holds for all $t \in G$. We may assume $t \in G'$ since all summands are zero otherwise.

First consider an element $t \in G'\backslash G''$ and write $t = e_1^k g_1 d' e_1^{-k}$ for some $k \in \{0, \ldots, s-1\}$ and some $d' \in G''$. Setting $d := c^{-1}d'$, this yields $t = e_1^k g_1' d e_1^{-k}$. We have

$$\sum_{u \in U} a_{tu^{-1}} = \sum_{u \in U} a_{e_1^{-k} t u^{-1} e_1^k} = \sum_{u \in U} a_{g_1' d e_1^{-k} u^{-1} e_1^k} = \sum_{u \in U} a_{g_1' u^{-1} d},$$

since conjugation with e_1^{-k} permutes the elements of U and we have $d \in Z(G')$. Furthermore, we obtain

$$\begin{aligned}\left\{g_1' u^{-1} d \colon u \in U\right\} &= \left\{g_1'(e_1^\ell g_1'^{-1} e_1^{-\ell}) d \colon 0 \leq \ell \leq s-1\right\} \cup \{g_1' d\} \\ &= \left\{[g_1', e_1^\ell] d \colon 0 \leq \ell \leq s-1\right\} \cup \{g_1' d\} =: X.\end{aligned}$$

For $\ell \in \{1, \ldots, s-1\}$, Lemma 4.19 yields $[g_1', e_1^\ell] = [e_1^\ell, g_1']^{-1} \sim g_1'^{-1} \sim g_1'$. By Lemma 4.39, this implies that for any $d \neq x \in X$, there exist $d_x \in C_{g_1'}$ and $m_x \in \{0, \ldots, s-1\}$ with

$$x = e_1^{m_x} g_1' d_x e_1^{-m_x} d.$$

Note that $C_{g_1'} = C$ holds because g_1' is contained in $g_1 G''$ (see Lemma 4.39). Since the elements in U form a system of representatives for the cosets of G'' in G', the same holds for the elements in X since they arise from the elements in U by multiplication with $g_1' d$. This implies that the correspondence $x \leftrightarrow m_x$ is one-to-one. Setting $v_{m_x} := d_x$ for $m_x \in \{0, \ldots, s-1\}$, this yields $x = e_1^{m_x} g_1' v_{m_x} e_1^{-m_x} d$ for all $x \in X\backslash\{d\}$ and hence

$$X = \left\{e_1^m g_1' v_m e_1^{-m} d \colon 0 \leq m \leq s-1\right\} \cup \{d\}.$$

Since we have $a_d = 0$, this implies

$$\sum_{u \in U} a_{tu^{-1}} = \sum_{x \in X} a_x = \sum_{m=0}^{s-1} a_{e_1^m g_1' v_m e_1^{-m} d} = \sum_{m=0}^{s-1} a_{g_1' v_m e_1^{-m} d e_1^m} = \sum_{m=0}^{s-1} a_{g_1 c v_m e_1^{-m} d e_1^m},$$

since the coefficients of y are constant under conjugation with e_1^{-m}. Recall that $g_1' = g_1c$ holds for some $c \in C$. We obtain

$$\sum_{m=0}^{s-1} a_{g_1 c v_m e_1^{-m} d e_1^m} = \sum_{m=0}^{s-1} a_{g_1 e_1^{-m} d e_1^m} = \frac{s}{|[d]|} \sum_{d' \in [d]} a_{g_1 d'} = 0. \tag{4.19}$$

In the first step, we use that $cv_m \in C$ holds for $m = 0, \ldots, s-1$, which yields

$$g_1 c v_m e_1^{-m} d e_1^m \in C g_1 e_1^{-m} d e_1^m = [g_1 e_1^{-m} d e_1^m]_{G'}$$

by Lemma 4.39. The second equality in (4.19) follows since the element $e_1^{-m} d e_1^m$ traverses the conjugacy class $[d] = \{e_1^\ell d e_1^{-\ell} \colon 0 \leq \ell \leq |[d]| - 1\}$ exactly $s/|[d]|$ times. In the last step, we apply (4.17). This shows that the identity given in (4.18) holds for $t \in G' \backslash G''$.

Now let $t \in G''$. By Lemma 4.19, there exists an element $g \in G$ with $g_1'^{-1} = g g_1 g^{-1}$. With this, we obtain

$$\sum_{u \in U} a_{tu^{-1}} = a_t + \sum_{i=0}^{s-1} a_{t e_1^i g_1'^{-1} e_1^{-i}} = \sum_{i=0}^{s-1} a_{t(e_1^i g) g_1 (e_1^i g)^{-1}} = \sum_{i=0}^{s-1} a_{g_1 (e_1^i g)^{-1} t (e_1^i g)}.$$

Since t centralizes G', we may assume $g \in H$, so e_1 and g commute. With $t' = g^{-1}tg$, we therefore have

$$\sum_{i=0}^{s-1} a_{g_1 (e_1^i g)^{-1} t (e_1^i g)} = \sum_{i=0}^{s-1} a_{g_1 e_1^{-i} t' e_1^i} = a_{g_1 \prod_{i=0}^{s-1} e_1^{-i} t' e_1^i} = 0,$$

where we use Remark 4.45 and the last equality follows by (4.19). Hence the equality given in (4.18) holds for every $t \in G$, which completes the proof. □

This settles the case that $C = [h]$ is a conjugacy class of an element in $1 \neq h \in C_H(G'')$. Now we gather our results in order to prove the following statement:

Lemma 4.50. *For any $g \in G \backslash G'$, we have $y \cdot [g]^+ = 0$.*

Proof. By Lemma 4.40, it suffices to show that y annihilates all conjugacy class sums of the form $[e_1^k d]^+$ with $k \in \{1, \ldots, s-1\}$ and $d \in G''$. Note that a system of representatives for the cosets of $C_G(e_1^k)$ in G can be chosen in G' since e_1^k commutes with all elements of H. Similarly, a system of representatives for the cosets of $C_G(d)$ in G can be found in $H = \langle e_1 \rangle$ since d centralizes G'. Since d commutes with the elements of G' and e_1^k centralizes $\langle e_1 \rangle$, we have $[e_1^k d] = [e_1^k] \cdot [d]$. Moreover, the group G' acts on $[e_1^k d]$ by conjugation with orbits of the form $[e_1^k] d'$ with $d' \in [d]$. In particular, $[e_1^k d]$ is a disjoint union of sets of this form and hence $[e_1^k d]^+$ is a multiple of $[e_1^k]^+$ in FG. It therefore suffices to prove $y \cdot [e_1^k]^+ = 0$ in order to show that y annihilates $[e_1^k d]^+$.

If $e_1^k \in C_G(G'')$ holds, then $y \cdot [e_1^k]^+ = 0$ follows by Proposition 4.49. Now let $e_1^k \notin C_G(G'')$ and write $[e_1^k] = U e_1^k$ with $U \subseteq G'$ as in (2.1). By Remark 4.41, U is a union of cosets of the normal subgroup $N := U \cap G''$, which is nontrivial, so it suffices to show $y \cdot N^+ = 0$. If $|N| = 2$ holds, we have $N \subseteq Z(G)$, which is a contradiction since e_1^k does not commute with its nontrivial commutators by Remark 2.17. Hence we have $|N| > 2$ and Proposition 4.47 yields $y \cdot N^+ = 0$. Summarizing, we obtain $y \cdot [e_1^k d]^+ = 0$ in all cases. □

With these preliminary results, we can complete the proof of Theorem 4.42:

Proof of the implication (i) ⇒ (iii) in Theorem 4.42. Let G be a finite group satisfying Hypothesis 4.38 and assume that $\mathrm{soc}(ZFG)$ is an ideal of FG. Suppose that $Z(G)C$ is a proper subgroup of G'' and consider the element $y = \sum_{g \in G} a_g g$ with the coefficients described in Construction 4.43. Clearly, we have $y \in ZFG$. By Lemmas 4.48 and 4.50, y annihilates the basis elements of $J(ZFG)$ given in Theorem 2.23. This shows that $y \in \mathrm{soc}(ZFG)$ holds. By Remark 2.9, this is a contradiction to $\mathrm{soc}(ZFG) \trianglelefteq FG$ since y has non-constant coefficients on G', so $Z(G)C = G''$ follows. □

4.3.4 Characterization of the groups G with $\mathrm{soc}(ZFG) \trianglelefteq FG$

We finally collect the results obtained in this section in order to classify the finite groups G of the form $P \rtimes H$ with $C_{G'}(P) \subseteq P'$ for which $\mathrm{soc}(ZFG)$ is an ideal in FG. Recall that we may restrict to the case $O_{p'}(G) = 1$ by Lemma 2.19.

Theorem 4.51. *Let $G = P \rtimes H$ with $P \in \mathrm{Syl}_p(G)$ and an abelian p'-group H such that $O_{p'}(G) = 1$ and $C_{G'}(P) \subseteq P'$ hold. Then $\mathrm{soc}(ZFG)$ is an ideal of FG if and only if there exist normal subgroups $K, Q_1, \ldots, Q_n$ of G for some $n \in \mathbb{N}_0$ such that*

$$G = K * Q_1 * \ldots * Q_n$$

is a central product and the following hold:

(i) K is a p-group with $\mathrm{soc}(ZFK) \trianglelefteq FK$,

(ii) $Q_i = Q_i' \rtimes H_i$, where Q_i' is a p-group of nilpotency class exactly two and H_i is a cyclic group of order $|Q_i'/Q_i''| - 1$ such that $Q_i/Q_i'' \cong \mathrm{AGL}(1, |Q_i'/Q_i''|)$ holds,

(iii) $C_{H_i}(Q_i'') \neq 1$,

(iv) $Q_i'/Z(Q_i)$ is a Camina group.

In this case, we have $\mathrm{soc}(ZFG) = (Z(P)G')^+ \cdot FG$.

Proof. First assume that $\mathrm{soc}(ZFG)$ is an ideal of FG. By Lemma 4.27, we then have $\mathrm{soc}(ZFG) = (Z(P)G')^+ \cdot FG$. By Theorem 4.24, G decomposes in the form $K * G_H$ with $K := C_P(H)$ and $G_H := H[G', H]$. Moreover, $\mathrm{soc}(ZFK)$ and $\mathrm{soc}(ZFG_H)$ are ideals in FK and FG_H, respectively.

In the following, we focus on the group G_H. Lemma 4.26 shows $G_H' \in \mathrm{Syl}_p(G_H)$ and we have $O_{p'}(G_H) \subseteq O_{p'}(G) = 1$. Furthermore, we have $Z(G_H') \subseteq Z(G') \subseteq G'' = G_H''$ since $C_P(H)$ is metabelian by Remark 4.25. This shows that G_H satisfies Hypothesis 4.32. By Lemma 4.33, there exists a decomposition $G_H = Q_1 * \ldots * Q_n$ into a central product such that Q_i satisfies Hypothesis 4.32 and Q_i'/Q_i'' is a minimal normal subgroup of Q_i/Q_i'' for $i = 1, \ldots, n$. In particular, we have $\mathrm{soc}(ZFQ_i) \trianglelefteq FQ_i$. By Lemma 4.31, we have $Q_i/Q_i'' \cong \mathrm{AGL}(1, |Q_i'/Q_i''|)$. In particular, Q_i/Q_i' is cyclic of order $|Q_i'/Q_i''| - 1$. The condition $C_H(Q_i'') \neq 1$ follows from Theorem 4.14, applied to the group Q_i. Since Q_i satisfies the prerequisites of Theorem 4.42, $Q_i'/Z(Q_i)$ is a Camina group.

Conversely, assume that $G = K * Q_1 * \ldots * Q_n$ is a central product of normal subgroups $K, Q_1, \ldots, Q_n$ which satisfy the conditions (i) – (iv). Note that $O_{p'}(G) = 1$ implies $O_{p'}(Q_i) = 1$ for $i = 1, \ldots, n$. By Lemma 4.31 and property (iii), Q_i satisfies the conditions (D1)–(D3) given in Theorem 4.14. Moreover, the condition (ii) implies that Q_i'/Q_i'' is a simple $\mathbb{F}_p H_i$-module. Since $Z(Q_i')$ is a proper subgroup of Q_i', this yields $Z(Q_i)/Q_i'' = 1$ and hence $Z(Q_i') = Q_i''$. By Theorem 4.42, we obtain $\operatorname{soc}(ZFQ_i) \trianglelefteq FQ_i$. Hence $\operatorname{soc}(ZFG) \trianglelefteq FG$ follows by Lemma 2.43. □

4.4 Case $C_{G'}(P) \nsubseteq P'$

In this section, we investigate the question under which conditions $\operatorname{soc}(ZFG)$ is an ideal of FG in general, that is, we drop the assumption $C_{G'}(P) \subseteq P'$ from the previous section. By Theorem 4.24 and Lemma 2.19, we may again assume $P = G'$ and $O_{p'}(G) = 1$. More precisely, we require the following throughout this section:

Hypothesis 4.52. Let G be a finite group in which $G' = O_p(G)$ holds and G' has nilpotency class at most two, and fix a Hall p'-subgroup H of G (see Theorem 2.15). Moreover, we assume $O_{p'}(G) = 1$ and that G has the properties (D1)–(D3) given in Theorem 4.14.

Recall that H is abelian in this situation. As before, we first consider the quotient group $D \coloneqq G/G''$. By the property (D1) in Theorem 4.14, we have $D' \cong T \times Z_D$, where $T = T_1 \times \ldots \times T_n$ is an elementary abelian p-group and $Z_D = Z(G')/G''$ can have a larger exponent. In order to apply methods from linear algebra, it is convenient to pass to the quotient group $G_\Phi \coloneqq G/\Phi(G')$. Its derived subgroup G_Φ', which is elementary abelian, decomposes in the form $T \times Z$, where Z denotes the image of $Z(G')$ under the projection onto G_Φ. We collect some preliminary results on the structure of G_Φ in Section 4.4.1.

Since the conjugation action of H on T was already analyzed in the preceding sections, we focus on the action of H and, in particular, of its subgroup $C_T \coloneqq C_H(T)$ on Z. Setting $C_Z \coloneqq C_H(Z)$, this leads to the following conjecture:

Conjecture 4.53. *Let G be a group which satisfies Hypothesis 4.52. We claim that then the following implication holds:*

$$\operatorname{soc}(ZFG) \trianglelefteq FG \Rightarrow H = C_T \times C_Z. \tag{4.20}$$

In Section 4.4.2, we provide evidence for Conjecture 4.53. In particular, we prove it in the special case that Z is a simple $\mathbb{F}_p H$-module. In Section 4.4.3, we conversely assume that $H = C_T \times C_Z$ holds. This will lead to a decomposition of G into a direct product. To its factors, one can apply the results from Section 4.3 to decide whether $\operatorname{soc}(ZFG)$ is an ideal of FG. In other words, combined with certain conditions from the preceding sections, the condition $H = C_T \times C_Z$ ensures that $\operatorname{soc}(ZFG)$ is an ideal of FG.

4.4.1 Structure of $G/\Phi(G')$

As explained at the beginning, we mainly consider the group $G_\Phi := G/\Phi(G')$ in this section. Note that G_Φ is of the form $G'_\Phi \rtimes H$ if we identify H and its image in G_Φ as usual. Studying G_Φ instead of D is convenient since the elementary abelian group G'_Φ can be interpreted as an $\mathbb{F}_p$-vector space, which often allows us to identify H with a subgroup of the general linear group $\mathrm{GL}(G'_\Phi)$. On the other hand, any element in H which acts trivially on G'_Φ also acts trivially on G' by Theorem 2.3.

We first relate the structure of G_Φ to that of D.

Remark 4.54. We have

$$\Phi(G')/G'' = \Phi(G'/G'') = \Phi(D') = \Phi(T \times Z_D) = \Phi(T) \times \Phi(Z_D) = \Phi(Z_D),$$

since T is elementary abelian. This yields

$$G'_\Phi = G'/\Phi(G') \cong (G'/G'')/(\Phi(G')/G'') = D'/\Phi(Z_D) \cong T \times (Z_D/\Phi(Z_D)).$$

Note that this is even an isomorphism of $\mathbb{F}_pH$-modules. Identifying T_i with its image in G_Φ for $i = 1, \ldots, n$ and setting $Z := Z_D/\Phi(Z_D)$, this yields

$$G'_\Phi \cong T_1 \times \ldots \times T_n \times Z. \tag{4.21}$$

Note that we have

$$Z = Z_D/\Phi(Z_D) \cong (Z(G')/G'')/(\Phi(G')/G'') \cong Z(G')/\Phi(G').$$

Since Z is a semisimple $\mathbb{F}_pH$-module, it can be decomposed into a direct sum $Z_1 \times \ldots \times Z_k$ of simple $\mathbb{F}_pH$-modules $Z_1, \ldots, Z_k$ for some $k \in \mathbb{N}_0$. ◁

We collect some first results regarding the conjugation action of the group H on G'_Φ.

Remark 4.55.

(i) If $h \in H$ acts trivially on the subgroup Z of G_Φ, then it also acts trivially on Z_D by Theorem 2.3. This way, we can identify the centralizers C_Z and $C_H(Z_D)$. Similarly, since T_i and $(T_i \times \Phi(Z_D))/\Phi(Z_D) \subseteq G_\Phi$ are isomorphic $\mathbb{F}_pH$-modules, we simply write $C_H(T_i)$ for the centralizer of either group in H.

(ii) If an element $j \in C_T$ centralizes Z, then j acts trivially on G'_Φ. By Theorem 2.3 and Theorem 2.1, this yields $j \in C_H(G') = C_G(G') \cap H = 1$. In other words, we have

$$C_T \cap C_Z = 1. \tag{4.22}$$

(iii) For $i = 1, \ldots, n$ and every $h \in H$, the centralizer $C_{Z_i}(h)$ is a normal subgroup of G_Φ since it is H-invariant by [20, Lemma 2.6.2] and G'_Φ is abelian. As Z_i is a simple $\mathbb{F}_pH$-module, we either have $C_{Z_i}(h) = 1$, that is, h acts on $Z_i \backslash \{1\}$ without fixed points, or $C_{Z_i}(h) = Z_i$, that is, the operation of h on Z_i is trivial. ◁

In the following, we further analyze the action of H on Z. For $i = 1, \dots, k$, the group $H_i := H/C_H(Z_i)$ is cyclic since Z_i is a simple $\mathbb{F}_pH$-module (see [20, Theorem 3.2.3]), and it can be viewed as a subgroup of $\mathrm{Aut}(Z_i)$. More precisely, we now show that H_i can be embedded into a cyclic group of automorphisms of Z_i that acts transitively on $Z_i \backslash \{1\}$. In the following, we set $p^{z_i} := |Z_i|$ for $i = 1, \dots, k$.

Lemma 4.56. *For $i = 1, \dots, k$, there exists an element $A_i \in \mathrm{Aut}(Z_i)$ of order $p^{z_i} - 1$ with $H_i \subseteq \langle A_i \rangle \subseteq \mathrm{Aut}(Z_i)$.*

Proof. The group algebra $\mathbb{F}_pH_i$ is semisimple. By Wedderburn's theorem (see [47, Theorem I.6.3]), there exists an isomorphism of $\mathbb{F}_p$-algebras

$$\mathbb{F}_pH_i \cong \mathrm{Mat}_{n_1}(F_1) \oplus \dots \oplus \mathrm{Mat}_{n_r}(F_r) \tag{4.23}$$

for some $n_1, \dots, n_r \in \mathbb{N}$ and skew fields $F_1, \dots, F_r$. Since $\mathbb{F}_pH_i$ has a finite number of elements, $F_1, \dots, F_r$ are finite as well, which implies that they are even fields. Moreover, since $\mathbb{F}_pH_i$ is commutative, we have $n_1 = \dots = n_r = 1$. There exists an index $j \in \{1, \dots, r\}$ such that Z_i and F_j are isomorphic $\mathbb{F}_pH_i$-modules. In particular, we have $|F_j| = p^{z_i}$. We obtain $\mathrm{End}_{\mathbb{F}_pH_i}(F_j) = \mathrm{End}_{F_j}(F_j)$ since all direct summands in (4.23), except for F_j, annihilate the $\mathbb{F}_pH_i$-module F_j. By [47, Lemma I.6.1], the $\mathbb{F}_p$-algebra $\mathrm{End}_{F_j}(F_j)$ is isomorphic to F_j. Since the group H_i acts faithfully on F_j by left multiplication, it can be identified with a subgroup of $\mathrm{End}_{F_j}(F_j)^\times \cong F_j^\times$. Taking A_i to be a generator of this cyclic group, the claim follows. □

The proof also shows that the group $\langle A_i \rangle$ acts transitively on $Z_i \backslash \{1\}$. Moreover, note that every nonzero $\mathbb{F}_pH_i$-endomorphism of Z_i is of the form A_i^k for some $k \in \mathbb{Z}$.

4.4.2 Conjecture

In this section, we discuss Conjecture 4.53 in detail. After some preliminary results, which are given in Section 4.4.2.1, we prove the conjecture for the special case that Z is a simple $\mathbb{F}_pH$-module in Section 4.4.2.2. Subsequently, we describe a possible generalization (see Section 4.4.2.3). Additionally to Hypothesis 4.52, we assume throughout this section that $\mathrm{soc}(ZFG)$ is an ideal of FG.

4.4.2.1 Preliminary results

Here, we collect several auxiliary results which will be needed later. As before, we fix a nontrivial element $h_i \in C_H(M_i)$ (see condition (D3) in Theorem 4.14) and set $g_i := g_{h_i}$ to be a corresponding element in G' as defined in Lemma 4.19.

In the following, we use the abbreviations $A_\Phi := \overline{\mathrm{Cl}}_{p',\Phi(G')}(G)$ and $B_\Phi := \overline{\mathrm{Cl}}^+_{p',\Phi(G')}(G)$ (see Definition 2.34). The crucial fact that we use in our derivation is that

$$\mathrm{Ann}_{ZFG_\Phi}(B_\Phi) \subseteq (G'_\Phi)^+ \cdot FG_\Phi$$

is a necessary condition for $\mathrm{soc}(ZFG) \trianglelefteq FG$ (see Theorem 2.39). To simplify the calculations, we show that B_Φ may be replaced by a smaller set in the above annihilator.

Lemma 4.57. *The annihilator* $\mathrm{Ann}_{ZFG_\Phi}(B_\Phi)$ *is given by*

$$S := \bigcap_{i=1}^{n} \mathrm{Ann}_{ZFG_\Phi}(T_i^+) \cap \bigcap_{z \in Z} \mathrm{Ann}_{ZFG_\Phi}\left([z]^+ - |[z]| \cdot 1\right).$$

Proof. First let $y \in S \subseteq ZFG_\Phi$. We show that $y \cdot b_{\tilde{C}} = 0$ holds for all $\tilde{C} \in A_\Phi$. To this end, let $C \in \mathrm{Cl}_{p',\Phi(G')}(G)$ be a preimage of $\tilde{C}$. If p does not divide $|C|$, then C is contained in $Z(G')$ and hence we have $\tilde{C} \subseteq Z$. This yields $y \cdot b_{\tilde{C}} = y \cdot (\tilde{C}^+ - |\tilde{C}| \cdot 1) = 0$ since y is contained in S. Now assume that p divides $|C|$ and hence also $|\tilde{C}|$. In particular, we have $b_{\tilde{C}} = \tilde{C}^+$. Since G'_Φ is abelian, (2.11) yields $\tilde{C} \nsubseteq G'_\Phi$. Let $x \in \tilde{C}$ and write $x = hg$ for some $1 \neq h \in H$ and some $g \in G'_\Phi$. If y annihilates $[h]^+$, then $y \cdot \tilde{C}^+ = 0$ follows by Corollary 2.46. It therefore suffices to show that $y \cdot [h]^+ = 0$ holds for all $1 \neq h \in H$. We write $[h] = U_h \cdot h$ with $U_h \subseteq G'_\Phi$ as in (2.1). By Remark 4.55 (ii), we have $C_H(G'_\Phi) = 1$ and hence h acts nontrivially on G'_Φ.

If $[h, T_i] \neq 1$ holds for some $i \in \{1, \ldots, n\}$, we obtain $T_i \subseteq U_h$ since all nontrivial elements in T_i are conjugate (see condition (D2) in Theorem 4.14) and U_h is a normal subgroup of G_Φ by Lemma 2.45. In this case, U_h is a union of cosets of T_i, which yields $y \cdot [h]^+ = y \cdot U_h^+ \cdot h = 0$ since y annihilates T_i^+ by assumption.

If $[h, T_i] = 1$ holds for $i = 1, \ldots, n$, then there exists an index $i \in \{1, \ldots, k\}$ with $[h, Z_i] \neq 1$. By Remark 4.55, h acts on $Z_i \backslash \{1\}$ without fixed points, so U_h is a union of cosets of Z_i. We obtain

$$y \cdot Z_i^+ = y \cdot \left(\sum_{[z] \subseteq Z_i} [z]^+ \right) = y \cdot \left(\sum_{[z] \subseteq Z_i} [z]^+ - |z| \cdot 1 \right) = 0,$$

since $|Z_i|$ is divisible by p and y annihilates elements of the form $[z]^+ - |[z]| \cdot 1$ with $z \in Z$ by assumption. This yields $y \cdot [h]^+ = 0$ and hence S is contained in $\mathrm{Ann}_{ZFG_\Phi}(B_\Phi)$.

For the other inclusion, consider the conjugacy class $C := [h_i]$ of the nontrivial element $h_i \in C_H(M_i)$ fixed at the beginning of this section ($i \in \{1, \ldots, n\}$). By Lemma 4.7, its image $\tilde{C}$ in G_Φ is of the form $T_i \cdot h_i$ and we have $\tilde{C} \in A_\Phi$. This yields

$$\mathrm{Ann}_{ZFG_\Phi}(B_\Phi) \subseteq \mathrm{Ann}_{ZFG_\Phi}(T_i^+).$$

Now let $z \in Z \backslash \{1\}$ and consider a preimage $z' \in Z(G')$. Since $|[z']|$ is not divisible by p, we have $[z] \in A_\Phi$, which yields $\mathrm{Ann}_{ZFG_\Phi}(B_\Phi) \subseteq \mathrm{Ann}_{ZFG_\Phi}([z]^+ - |[z]| \cdot 1)$. This shows that $\mathrm{Ann}_{ZFG_\Phi}(B_\Phi)$ is contained in S, which proves the equality. □

For $i = 1, \ldots, n$, we write t_i for the image of the fixed element g_i in D' or G'_Φ (with the identification of T_i and its image in G'_Φ given in Remark 4.54) and set $t := t_1 \cdots t_n$. In the next sections, we will be mainly concerned with the question under which conditions an element $y \in \mathrm{Ann}_{ZFG_\Phi}(B_\Phi)$ has constant coefficients on tZ. For this reason, it is essential to know which elements in tZ are conjugate to a given element tz with $z \in Z$:

Lemma 4.58. *For* $z \in Z$, *we have* $[tz] \cap tZ = t[z]_{C_T}$, *where* $[z]_{C_T}$ *denotes the set of elements that arise from* z *by conjugating with elements of* C_T.

Proof. Clearly, $t[z]_{C_T}$ is contained in $[tz] \cap tZ$ since conjugation with elements of C_T fixes t. Now consider an element $z' \in Z$ with $tz' \in [tz] \cap tZ$. Since G'_Φ is abelian, there exists an element $h \in H$ such that $tz' = htzh^{-1}$ holds. Writing $h = e_1^{r_1} \cdots e_n^{r_n} \cdot j$ for some $r_1, \dots, r_n \in \mathbb{Z}$ and $j \in C_T$, this reads

$$tz' = ht_1 \cdots t_n \cdot zh^{-1} = e_1^{r_1} t_1 e_1^{-r_1} \cdots e_n^{r_n} t_n e_n^{-r_n} \cdot hzh^{-1},$$

since j commutes with $t_1, \dots, t_n$ and e_i centralizes T_ℓ for $\ell \neq i$. Therefore, we have $e_i^{r_i} t_i e_i^{-r_i} = t_i$ and hence $e_i^{r_i} \in C_T$ since $\langle e_i \rangle$ acts transitively on $T_i \backslash \{1\}$ for $i = 1, \dots, n$ (see condition (D2) in Theorem 4.14). This implies $h \in C_T$ and hence $tz' = thzh^{-1} \in t[z]_{C_T}$ follows as claimed. □

In the next section, we prove that H is of the form $C_T \times C_Z$ if Z is a simple $\mathbb{F}_p H$-module. By the preceding lemma, this implies $t[z] = t[z]_{C_T} \subseteq [tz]$ for all $z \in Z$.

4.4.2.2 Special case: Z simple $\mathbb{F}_p H$-module

In this section, we assume that Z is a simple $\mathbb{F}_p H$-module, which corresponds to the case $k = 1$ in Remark 4.54. The aim of this section is the proof of the following result:

Lemma 4.59. *Let G be a group which satisfies Hypothesis 4.52 and suppose that* $\operatorname{soc}(ZFG)$ *is an ideal of FG. Write $G'_\Phi = T \times Z$ as before and assume that Z is a simple $\mathbb{F}_p H$-module. Then we obtain $H \cong C_T \times C_Z$.*

In the following, we consider the action of the quotient group $H_1 := H/C_Z$ on Z. Recall that H_1 is a cyclic group and that the action of H_1 on Z is free (see Remark 4.55 (iii)). By (4.22), the intersection $C_T \cap C_Z$ is trivial and hence from now on, $C_T \cong C_T \times C_Z / C_Z$ will be identified with a subgroup of H_1.

In the following, we suppose that $C_T \times C_Z$ is a proper subgroup of H or, equivalently, that C_T is a proper subgroup of H_1. Under this assumption, we construct an element in $\operatorname{Ann}_{ZFG_\Phi}(B_\Phi)$ with non-constant coefficients on G'_Φ. Its existence is a contradiction to $\operatorname{soc}(ZFG) \trianglelefteq FG$ by Theorem 2.39. We proceed in two steps: First, we derive a contradiction in the special case that $e_i \notin C_T \times C_Z$ holds for $i = 1, \dots, n$. This is then used to prove the general result.

Case 1: In the following, we assume that e_i is not contained in $C_T \times C_Z$ for $i = 1, \dots, n$.

In order to apply methods from linear algebra, we interpret the elementary abelian group Z as an $\mathbb{F}_p$-vector space in the following.

Remark 4.60. Let $p^z := |Z|$. Since Z is elementary abelian, there exists an isomorphism $\varphi\colon Z \to V_Z$ onto the additive group V_Z of an $\mathbb{F}_p$-vector space of dimension z. Moreover, there is an isomorphism $\Phi\colon \operatorname{Aut}(Z) \to \operatorname{GL}(z, p)$ (see [24, Bemerkung I.13.13]). By identifying H_1 with its image in $\operatorname{GL}(z, p)$, V_Z becomes an $\mathbb{F}_p H_1$-module, which is isomorphic to Z.

By Lemma 4.56, there exists a Singer cycle $A \in \operatorname{GL}(z, p)$ with $\Phi(H_1) \subseteq \langle A \rangle$. Write $\Phi(H_1) = \langle A^m \rangle$ with $m := (p^z - 1)/|H_1|$. Setting $d := |C_T|$ and $k := (p^z - 1)/d$, we have $\Phi(C_T) = \langle A^k \rangle$. Here, we view C_T as a subgroup of H_1 as explained before. Fixing a nonzero element $f \in V_Z$, the elements of V_Z can be labeled in the form $0, Af, A^2 f, \ldots, A^{p^z-1} f$.

In the following, we usually identify H_1 and $\Phi(H_1)$. Note that the action of H_1 on Z is given by conjugation, whereas $H_1 \cong \Phi(H_1)$ acts on V_Z by matrix-vector multiplication. ◁

Since $d = |C_T| < |H_1| \leq p^z - 1$ holds by assumption, A^d is not the identity matrix and hence there exist indices $\lambda_1, \lambda_2 \in \{1, \ldots, z\}$ with $[A^d]_{\lambda_1\lambda_2} \neq [\mathbb{1}]_{\lambda_1\lambda_2}$. Here, $[A]_{\lambda_1\lambda_2}$ denotes the entry of the matrix A in the position (λ_1, λ_2). We now define a map

$$\alpha\colon V_Z \to \mathbb{F}_p,\ \alpha(x) = \begin{cases} 0 & x = 0 \\ \left[A^{id}\right]_{\lambda_1\lambda_2} & x = A^i f \text{ for some } i \in \mathbb{Z} \end{cases}. \tag{4.24}$$

Note that α is well-defined as $A^{i_1} f = A^{i_2} f$ for $i_1, i_2 \in \mathbb{Z}$ implies $A^{i_1} = A^{i_2}$ since A acts on the nonzero elements of V_Z without fixed points. Furthermore, α is non-constant since

$$\alpha(Af) = [A^d]_{\lambda_1\lambda_2} \neq [\mathbb{1}]_{\lambda_1\lambda_2} = [A^{kd}]_{\lambda_1\lambda_2} = \alpha(A^k f)$$

holds by our choice of λ_1 and λ_2.

Remark 4.61. The map α is invariant under the action of C_T in the sense that for any $j \in C_T$ and $x \in V_Z$, we have $\alpha(jx) = \alpha(x)$. This is clear for $x = 0$. Now write $j = A^{\ell k}$ and $x = A^i f$ for some $\ell, i \in \mathbb{Z}$. Then we have

$$\alpha(jx) = \alpha\big(A^{\ell k + i} f\big) = [A^{(\ell k + i)d}]_{\lambda_1\lambda_2} = [A^{id}]_{\lambda_1\lambda_2} = \alpha(x),$$

since $A^{kd} = A^{p^z-1} = \mathbb{1}$ holds. ◁

The following property of α will be the key ingredient to ensure that the element $y \in FG_\Phi$ which we later construct from α annihilates the basis elements of $J(ZFG_\Phi)$ given in Theorem 2.23.

Proposition 4.62. *Consider a subgroup $L \subseteq H_1$ with $L \nsubseteq C_T$. For $x \in V_Z$, we have*

$$\sum_{\ell \in L} \alpha(\ell x) = 0.$$

If $|L| > d$ holds, then for any $x, u \in V_Z$, we obtain

$$\frac{1}{|L|} \sum_{\ell \in L} \alpha(u + \ell x) = \alpha(u).$$

Proof. Note that $L = \langle A^{cm} \rangle$ holds for $c := |H_1|/|L|$. We now prove the first equality stated above. It is obviously true for $x = 0$, so let $x = A^w f$ for some $w \in \mathbb{Z}$. We obtain

$$\sum_{\ell \in L} \alpha(\ell x) = \sum_{i=1}^{|L|} \alpha\big(A^{cmi+w} f\big) = \sum_{i=1}^{|L|} \left[A^{(cmi+w)d}\right]_{\lambda_1\lambda_2} = \left[\sum_{i=1}^{|L|} A^{(cmi+w)d}\right]_{\lambda_1\lambda_2}.$$

Set $B := \sum_{i=1}^{|L|} A^{(cmi+w)d}$. Multiplication with B defines an $\mathbb{F}_pH_1$-endomorphism of V_Z. By the proof of Lemma 4.56, using that V_Z and Z are isomorphic $\mathbb{F}_pH_1$-modules, we either have $B = 0$ or B is of the form A^b for some $b \in \mathbb{Z}$. Assume that the latter case applies, so B is invertible. We observe that

$$A^{cmd} \cdot B = \sum_{i=1}^{|L|} A^{(cm(i+1)+w)d} = \sum_{i=1}^{|L|} A^{(cmi+w)d} = B$$

holds since we have $A^{(cm(|L|+1))d} = A^{cmd}$. Since B is invertible, this yields $A^{cmd} = \mathbb{1}$, which is a contradiction to $A^{cm} \notin C_T$. Therefore, we have $B = 0$ and $\sum_{\ell\in L} \alpha(\ell x) = [B]_{\lambda_1\lambda_2} = 0$ follows as claimed.

Now we prove the second part of the statement. To this end, we assume $|L| > d$. For $u = 0$, the claim follows from the first part of this proof, so let $u = A^r f$ for some $r \in \mathbb{Z}$. This yields

$$\sum_{\ell\in L} \alpha(u + \ell x) = \sum_{i=1}^{|L|} \alpha\big(A^r f + A^{cmi+w} f\big) = \sum_{i=1}^{|L|} \alpha\big(A^r(\mathbb{1} + A^{cmi+w-r})f\big).$$

For $i = 1, \ldots, |L|$, multiplication with $\tilde{A} := A^r(\mathbb{1} + A^{cmi+w-r})$ defines an $\mathbb{F}_pH_1$-endomorphism of V_Z. As before, either we have $\tilde{A} = 0$ and hence $\alpha(\tilde{A}f) = 0$, or we have $\tilde{A} = A^b$ for some $b \in \mathbb{Z}$, which yields $\alpha(\tilde{A}f) = [A^{bd}]_{\lambda_1\lambda_2}$. In both cases, we obtain $\alpha(\tilde{A}f) = [\tilde{A}^d]_{\lambda_1\lambda_2}$. This implies

$$\begin{aligned}\sum_{i=1}^{|L|} \alpha\big(A^r(\mathbb{1} + A^{cmi+w-r})f\big) &= \sum_{i=1}^{|L|} \left[\left(A^r\left(\mathbb{1} + A^{(cmi+w-r)}\right)\right)^d\right]_{\lambda_1\lambda_2} \\ &= \left[A^{rd} \sum_{i=1}^{|L|} \left(\mathbb{1} + A^{(cmi+w-r)}\right)^d\right]_{\lambda_1\lambda_2}.\end{aligned}$$

The last equality follows from the fact that A commutes with $\mathbb{1}+A^b$ for all $b \in \mathbb{Z}$. Applying the binomial theorem yields

$$\left[\sum_{i=1}^{|L|} \left(\mathbb{1} + A^{(cmi+w-r)}\right)^d\right]_{\lambda_1\lambda_2} = \left[\sum_{i=1}^{|L|}\sum_{v=0}^{d} \binom{d}{v} A^{(cmi+w-r)v}\right]_{\lambda_1\lambda_2} = \left[\sum_{v=0}^{d} \binom{d}{v} \cdot B_v\right]_{\lambda_1\lambda_2}$$

with $B_v := \sum_{i=1}^{|L|} A^{(cmi+w-r)v}$ for $v = 0, \ldots, d$. Note that the matrix B_v is invariant under multiplication with A^{cmv}. For $v \neq 0$, the latter is not the identity matrix since we have $\operatorname{ord}(A^{cm}) = |L| > d$ by assumption, so we obtain $B_v = 0$ as before. This yields

$$\sum_{\ell\in L} \alpha(u+\ell x) = \left[A^{rd} \sum_{v=0}^{d} \binom{d}{v} \cdot B_v\right]_{\lambda_1\lambda_2} = \left[A^{rd}\left(|L| \cdot \mathbb{1}\right)\right]_{\lambda_1\lambda_2} = |L|\cdot\alpha(A^r f) = |L|\cdot\alpha(u). \quad \square$$

With the map α defined in (4.24), we can construct an element $y \in \mathrm{Ann}_{ZFG_\Phi}(B_\Phi)$ which is not contained in $(G'_\Phi)^+ \cdot FG_\Phi$:

Construction 4.63. As before, we set t to be the image of $g_1 \cdots g_n$ in G_Φ and denote by $\varphi\colon Z \to V_Z$ the isomorphism between Z and V_Z. For $g \in G_\Phi$, we set

$$a_g = \begin{cases} \alpha\big(\varphi(z_g)\big) & \text{if } g \text{ is conjugate to } tz_g \text{ for some } z_g \in Z \\ 0 & \text{otherwise} \end{cases}.$$

Note that this map is well-defined: If $g \in G'_\Phi$ is conjugate to elements tz_1 and tz_2 with $z_1, z_2 \in Z$, then z_1 and z_2 are conjugate by some element in C_T (see Lemma 4.58), which yields $\alpha\big(\varphi(z_1)\big) = \alpha\big(\varphi(z_2)\big)$ by Remark 4.61. In the following, we consider the element $y \coloneqq \sum_{g \in G_\Phi} a_g g \in FG_\Phi$.

By construction, we have $y \in ZFG_\Phi$. Moreover, y is not contained in $(G'_\Phi)^+ \cdot FG_\Phi$ since the map α is non-constant. It remains to show that the element y annihilates $J(ZFG_\Phi)$. By Lemma 4.57, this reduces to verifying that y annihilates all basis elements $b_{[u]}$ with $u \in Z\backslash\{1\}$ as well as the sums T_i^+ for $i = 1, \dots, n$.

Lemma 4.64. *Let $u \in Z\backslash\{1\}$ and consider the element $b_{[u]} = [u]^+ - |[u]| \cdot 1 \in J(ZFG_\Phi)$. Then $y \cdot b_{[u]} = 0$ holds.*

Proof. By Remark 2.25, we need to show that for every $x \in G_\Phi$, we have

$$\sum_{v \in [u]} a_{xv^{-1}} = |[u]| \cdot a_x. \tag{4.25}$$

All summands in (4.25) are zero unless x is of the form $x = t'z_x$ with $z_x \in Z$ and a conjugate t' of t. Since conjugation permutes the elements of $[u]$ and the coefficients of y are constant on conjugacy classes, we may assume $t' = t$. This yields

$$\sum_{v \in [u]} a_{xv^{-1}} = \sum_{v \in [u]} a_{tz_xv^{-1}} = \sum_{v \in [u]} \alpha\big(\varphi(z_xv^{-1})\big) = \sum_{v \in [u]} \alpha\big(\varphi(z_x) - \varphi(v)\big).$$

Note that $[u]$ is the orbit of u under the action of H_1. Since every nontrivial element of H_1 acts on $Z\backslash\{1\}$ without fixed points, we obtain $|[u]| = |H_1|$. Since φ is an isomorphism of $\mathbb{F}_pH_1$-modules, we furthermore obtain $\varphi(v) = h\varphi(u)$ if $v = huh^{-1}$ holds for some $h \in H_1$. With this, Proposition 4.62 yields

$$\sum_{v \in [u]} \alpha\big(\varphi(z_x) - \varphi(v)\big) = \sum_{h \in H_1} \alpha\big(\varphi(z_x) - h\varphi(u)\big) = |H_1| \cdot \alpha\big(\varphi(z_x)\big) = |[u]| \cdot a_x. \qquad \square$$

Lemma 4.65. *For $i = 1, \dots, n$, we have $y \cdot T_i^+ = 0$.*

Proof. It suffices to show the claim for $i = 1$. Recall that the group T_1 is given by

$$T_1 = \{1\} \cup \{e_1^v t_1 e_1^{-v} \colon v = 0, \dots, s_1 - 1\}$$

with $s_1 = |T_1| - 1$. By Remark 2.25, we need to show that

$$a_x + \sum_{v=0}^{s_1-1} a_{xe_1^v t_1 e_1^{-v}} = 0 \tag{4.26}$$

holds for all $x \in G_\Phi$. As in the preceding proof, it suffices to consider $x \in G'_\Phi$. Write $x = x_1 \cdots x_n \cdot u'$ with $x_i \in T_i$ for $i = 1, \ldots, n$ and $u' \in Z$ (see (4.21)) and observe that all summands in (4.26) are zero unless $x_2, \ldots, x_n$ are nontrivial. By conjugating with a suitable element $e \in \langle e_2, \ldots, e_n \rangle$ and using that G'_Φ is abelian, we then find

$$a_x + \sum_{v=0}^{s_1-1} a_{xe_1^v t_1 e_1^{-v}} = a_{x_1 t_2 \cdots t_n u} + \sum_{v=0}^{s_1-1} a_{e_1^v t_1 e_1^{-v} \cdot x_1 t_2 \cdots t_n u} \tag{4.27}$$

with $u = eu'e^{-1}$. Note that $T_1 = \{x_1\} \cup \{e_1^v t_1 e_1^{-v} x_1 \colon 0 \le v \le s_1 - 1\}$ holds since multiplying with x_1 permutes the elements of T_1. Hence by reordering the summands in (4.27), we may assume $x_1 = 1$. Then the first term in the above expression is zero. Note that $e_1^{s_1}$ acts trivially on T_1 and that $s_1 = \operatorname{ord}(e_1 C_H(T_1))$ divides $\operatorname{ord}(e_1)$. In particular, we have

$$\sum_{v=0}^{s_1-1} a_{e_1^v t_1 e_1^{-v} \cdot t_2 \cdots t_n u} = \frac{s_1}{\operatorname{ord}(e_1)} \sum_{v=0}^{\operatorname{ord}(e_1)-1} a_{e_1^v t_1 e_1^{-v} \cdot t_2 \cdots t_n u} = \frac{s_1}{\operatorname{ord}(e_1)} \sum_{v=0}^{\operatorname{ord}(e_1)-1} a_{te_1^{-v} u e_1^v}.$$

The last equality follows from the fact that the coefficients of y are constant under conjugation with e_1^{-v}. Now consider the subgroup $L \coloneqq \langle e_1 C_Z \rangle$ of H_1. By assumption, we have $e_1 \notin C_T \times C_Z$ and hence $L \not\subseteq C_T$. Note that $|L|$ divides $\operatorname{ord}(e_1)$. We then obtain

$$\frac{s_1}{\operatorname{ord}(e_1)} \sum_{v=0}^{\operatorname{ord}(e_1)-1} a_{te_1^{-v} u e_1^v} = \frac{s_1}{\operatorname{ord}(e_1)} \cdot \frac{\operatorname{ord}(e_1)}{|L|} \sum_{v=0}^{|L|-1} a_{te_1^{-v} u e_1^v} = \frac{s_1}{|L|} \sum_{\ell \in L} \alpha\big(\ell \varphi(u)\big) = 0.$$

Here, we use that $e_1^{|L|}$ acts trivially on Z. The last equality follows by Proposition 4.62. □

Lemmas 4.64 and 4.65, together with the reduction given in Lemma 4.57, show that the element y annihilates B_Φ. As $y \in ZFG_\Phi$ is not contained in $(G'_\Phi)^+ \cdot FG_\Phi$, this is a contradiction to $\operatorname{soc}(ZFG) \trianglelefteq FG$ by Theorem 2.39.

Case 2: Again, we suppose that $C_T \times C_Z$ is a proper subgroup of H, but we drop the assumption that $e_i \notin C_T \times C_Z$ holds for $i = 1, \ldots, n$. Without loss of generality, we may assume $e_1, \ldots, e_\ell \notin C_T \times C_Z$ and $e_{\ell+1}, \ldots, e_n \in C_T \times C_Z$ for some $\ell \in \{1, \ldots, n\}$. For $i = \ell+1, \ldots, n$, there exists an element $j_i \in C_T$ with $e_i j_i^{-1} \in C_Z$. Note that the action of e_i and $e_i j_i^{-1}$ on T coincides and that we have

$$H = \langle e_1, \ldots, e_\ell, e_{\ell+1} j_{\ell+1}^{-1}, \ldots, e_n j_n^{-1}, C_T \rangle.$$

By replacing e_i by $e_i j_i^{-1}$ for $i = \ell+1, \ldots, n$, we may therefore assume that $e_{\ell+1}, \ldots, e_n$ act trivially on Z. Since the elements in H do not commute with their nontrivial commutators by Remark 2.17, we obtain $[e] \subseteq e(T_{\ell+1} \times \ldots \times T_n)$ for any $e \in \langle e_{\ell+1}, \ldots, e_n \rangle$ and hence the subgroup $K \coloneqq \langle e_{\ell+1}, \ldots, e_n, T_{\ell+1}, \ldots, T_n \rangle$ is normal in G_Φ. We now consider the quotient group $\hat{G} := G_\Phi/K$. Observe that $H = \langle e_1, \ldots, e_\ell, C_T \rangle \times \langle e_{\ell+1}, \ldots, e_n \rangle$ holds since any element in the intersection of the two factors lies in $C_T \cap C_Z = 1$ (see (4.22)). For the image $\hat{H}$ of H in $\hat{G}$, we therefore obtain

$$\hat{H} \cong H/\langle e_{\ell+1}, \ldots, e_n \rangle \cong \langle e_1, \ldots, e_\ell, C_T \rangle.$$

Note that we have $K \cap G'_\Phi = T_{\ell+1} \times \ldots \times T_n$. Hence $\hat{G}'$ decomposes as $\hat{T}_1 \times \ldots \times \hat{T}_\ell \times \hat{Z}$ with $\hat{T}_i \cong T_i$ for $i = 1, \ldots, n$ and $\hat{Z} \cong Z$. Identifying $\hat{H}$ and $\langle e_1, \ldots, e_\ell, C_T \rangle$, this is even an isomorphism of $\mathbb{F}_p\hat{H}$-modules. This yields

$$C_{\hat{H}}(\hat{T}_1 \times \ldots \times \hat{T}_\ell) \cong C_H(T_1 \times \ldots \times T_\ell) \cap \langle e_1, \ldots, e_\ell, C_T \rangle = C_T$$

and

$$C_{\hat{H}}(\hat{Z}) \cong C_Z \cap \langle e_1, \ldots, e_\ell, C_T \rangle.$$

In particular, $e_i \notin C_{\hat{H}}(\hat{T}) \cdot C_{\hat{H}}(\hat{Z})$ follows for $i \in \{1, \ldots, \ell\}$. Since $\hat{G}$ satisfies the prerequisites of Lemma 4.59, the derivation of the first case, applied to the group $\hat{G}$, leads to a contradiction. Hence we obtain $H = C_T \times C_Z$, which finishes the proof of Lemma 4.59.

Remark 4.66. A similar, but more tedious computation shows that $H = C_T \times C_Z$ follows from $\operatorname{soc}(ZFG) \trianglelefteq FG$ also in the case where Z is of the form $Z_1 \times Z_2$ provided that certain restrictions on the action of C_T and H are given. However, it is not clear how to extend this construction to the general setting. $\triangleleft$

4.4.2.3 Outlook: Generalization

We conclude this section on Conjecture 4.53 with a few remarks concerning the general case. To this end, we assume that Z decomposes in the form $Z_1 \times \ldots \times Z_k$ with minimal normal subgroups $Z_1, \ldots, Z_k$ of G_Φ (see Remark 4.54). By applying Lemma 4.59 to suitable quotient groups of G_Φ, we obtain $H = C_T \cdot C_H(Z_i)$ for $i = 1, \ldots, k$. Nevertheless, $C_T \times C_Z$ might be a proper subgroup of H. In this case, the results from the previous section can be generalized as follows:

Remark 4.67. We consider a map $\alpha\colon Z \to F$ with the following properties:

(i) The map α is invariant under the action of C_T in the sense that $\alpha(u) = \alpha(juj^{-1})$ holds for all $u \in Z$ and $j \in C_T$.

(ii) We have $\frac{1}{|[u]|} \sum_{u' \in [u]} \alpha(xu') = \alpha(x)$ for all $x, u \in Z$.

(iii) We have $\frac{1}{s_i} \sum_{k=0}^{s_i-1} \alpha(xe_i^k u e_i^{-k}) = \alpha(x)$ for all $x, u \in Z$ and $i = 1, \ldots, n$.

With this map, one can construct an element $y := \sum_{g \in G_\Phi} a_g g$ in $\operatorname{Ann}_{ZFG_\Phi}(B_\Phi)$ by setting

$$a_g = \begin{cases} \alpha(z_g) & \text{if } g \text{ is conjugate to } tz_g \text{ for some } z_g \in Z \\ \alpha(1) & \text{otherwise} \end{cases}.$$

The proof goes along the lines of that in the preceding subsection. In particular, the first condition ensures that $a_{tz_1} = a_{tz_2}$ holds if tz_1 and tz_2 are conjugate by Lemma 4.58. The second property is used to show that y annihilates all basis elements of the form

$[u]^+ - |[u]| \cdot 1$ for $1 \neq u \in Z$, whereas the third ensures that $y \cdot T_i^+ = 0$ holds for $i = 1, \ldots, n$. Note that for the special case that Z is a simple $\mathbb{F}_pH$-module, the map α introduced in Construction 4.63 satisfies the above conditions: The first property is given by Remark 4.61 and the last two conditions are covered by the stronger properties given in Proposition 4.62.

If $C_T \times C_Z = H$ holds, then setting $x = 1$ in (ii) yields

$$\alpha(u) = \frac{1}{|[u]|} \sum_{u \in [u]} \alpha(u') = \alpha(1)$$

for all $u \in Z$ since we have $[u] = [u]_{C_T}$ in this case and hence all summands are equal by (i). Hence the map α is constant and we have $y = \alpha(1)G_\Phi^+ \in (G'_\Phi)^+ \cdot FG_\Phi$. Conversely, computer-aided experiments suggest that whenever $C_T \cdot C_Z$ is a proper subgroup of H, that is, whenever one finds an element $z \in Z$ with $[z]_{C_T} \subsetneq [z]$, then a non-constant map α satisfying the properties (i) – (iii) exists. The above construction then yields an element $y \in \mathrm{Ann}_{ZFG_\Phi}(B_\Phi)$ which is not contained in $(G'_\Phi)^+ \cdot FG_\Phi$. This is a contradiction to $\mathrm{soc}(ZFG) \trianglelefteq FG$ by Theorem 2.39.

Therefore, we conjecture that $H = C_T \times C_Z$ holds if $\mathrm{soc}(ZFG)$ is an ideal of FG. ◁

4.4.3 Converse statement

Here, we consider the case that $H = C_T \times C_Z$ holds, that is, we require the following:

Hypothesis 4.68. Let G be a finite group in which $G' = O_p(G)$ has nilpotency class at most two and let H be a Hall p'-subgroup of G. Moreover, we assume $O_{p'}(G) = 1$ and suppose that G satisfies the conditions (D1)–(D3) given in Theorem 4.14. Setting $G_\Phi = G/\Phi(G')$ and writing $G'_\Phi = T \times Z$ as in Remark 4.54, we additionally assume that $H = C_T \times C_Z$ holds for $C_T = C_H(T)$ and $C_Z = C_H(Z)$.

Under this assumption, we show that G can be decomposed into a direct product $G_T \times G_Z$, where G_T is a group which satisfies the prerequisites of Theorem 4.51 and G_Z is a group with an abelian Sylow p-subgroup. In particular, $\mathrm{soc}(ZFG)$ is an ideal in FG if and only if $\mathrm{soc}(ZFG_T)$ is an ideal in FG_T, and the latter can be verified by using the criterion stated in Theorem 4.51.

Again, we write $H = \langle e_1, \ldots e_n, C_T \rangle$ as in condition (D2) in Theorem 4.14. By replacing e_i by $e_i j_i^{-1}$ for a suitable element $j_i \in C_T$ as in the second case of the proof of Lemma 4.59 ($i = 1, \ldots, n$), we may assume that $e_1, \ldots, e_n$ act trivially on Z. As before, we set L to be the preimage of T under the projection onto D.

Remark 4.69. Since $D = T \times Z_D$ holds, we have $G' = LZ(G')$ and $L \cap Z(G') = G''$. Note that we have $L' = G'' \subseteq \Phi(L)$. On the other hand, $T \cong L/G''$ is elementary abelian, which yields $\Phi(L) = G''$. By Theorem 2.3, C_T acts trivially on L. ◁

We begin with the following preliminary result on the center of G':

Proposition 4.70. *We have $Z(G') = G'' \times [Z(G'), C_T]$. Moreover, C_T centralizes G'' and C_Z acts trivially on $[Z(G'), C_T]$.*

Proof. Theorem 2.4 yields the decomposition

$$Z(G') = C_{Z(G')}(C_T) \times [Z(G'), C_T], \tag{4.28}$$

so it remains to show that $G'' = C_{Z(G')}(C_T)$ holds. By Remark 4.69, C_T centralizes $G'' \subseteq L$. For the other inclusion, we consider the quotient group $Z_D = Z(G')/G''$. Since the elements of C_Z act trivially on Z_D by Remark 4.55, we have

$$1 = C_{Z_D}(H) = C_{Z_D}(C_T \times C_Z) = C_{Z_D}(C_T) \cap C_{Z_D}(C_Z) = C_{Z_D}(C_T),$$

where the first equality follows by Remark 4.3. This implies $C_{Z(G')}(C_T) \subseteq G''$. The fact that C_Z acts trivially on Z_D additionally yields $[C_Z, Z(G')] \subseteq G''$. On the other hand, $[Z(G'), C_T]$ is a normal subgroup of G since it is centralized by G' and conjugation with elements of H fixes both $Z(G')$ and C_T. Hence we obtain

$$\big[C_Z, \big[Z(G'), C_T\big]\big] \subseteq G'' \cap \big[Z(G'), C_T\big] = C_{Z(G')}(C_T) \cap [Z(G'), C_T] = 1$$

by (4.28) and hence C_Z acts trivially on $[Z(G'), C_T]$. □

We conclude this part by showing that G decomposes as a direct product $G = G_T \times G_Z$ of a group G_T satisfying the prerequisites of Theorem 4.51 and a group G_Z whose Sylow p-subgroup is abelian.

Theorem 4.71. *Let G be a finite group in which $G' = O_p(G)$ is of nilpotency class at most two and $O_{p'}(G) = 1$ holds. Moreover, assume that G satisfies the conditions (D1)–(D3) given in Theorem 4.14. Writing $G'/\Phi(G') = T \times Z$ as in Remark 4.54, we assume that a Hall p'-subgroup H of G decomposes as $C_T \times C_Z$ with $C_T = C_H(T)$ and $C_Z = C_H(Z)$. Then we have*

$$G = G_T \times G_Z,$$

where $G_T := L \rtimes C_Z$ satisfies the prerequisites of Theorem 4.51 and the Sylow p-subgroup of $G_Z := [Z(G'), C_T] \rtimes C_T$ is abelian. In particular, $\operatorname{soc}(ZFG) \trianglelefteq FG$ *is equivalent to* $\operatorname{soc}(ZFG_T) \trianglelefteq FG_T$.

Proof. By Remark 4.69 and Proposition 4.70, we obtain $G' = L \times [Z(G'), C_T]$. Since $H = C_T \times C_Z$ holds, G is generated by G_T and G_Z. Note that C_T centralizes L by Remark 4.69 and C_Z centralizes $[Z(G'), C_T]$ by Proposition 4.70. Hence the groups G_T and G_Z commute element-wise. Since their intersection is trivial, we have $G = G_T \times G_Z$. Note that the group G_T is of the form $G_T' \rtimes C_Z$ and that we have

$$Z(G_T') = Z(G') \cap L = G'' = G_T'',$$

where the last equality follows from the decomposition of G as a direct product together with the fact that $G_Z' = [Z(G'), C_T]$ is abelian. Moreover, we have $O_{p'}(G_T) \subseteq O_{p'}(G) = 1$. This shows that G_T satisfies the prerequisites of Theorem 4.51. Since G_Z' is abelian, $\operatorname{soc}(ZFG_Z)$ is an ideal of FG_Z by Theorem 2.47. The condition $\operatorname{soc}(ZFG) \trianglelefteq FG$ is therefore equivalent to $\operatorname{soc}(ZFG_T) \trianglelefteq FG_T$ by Lemma 2.43. □

If $\operatorname{soc}(ZFG)$ is an ideal of FG, the structure of G is therefore altered by the additional direct summand G_Z in comparison to the groups described in Theorem 4.51.

Chapter 5

Symmetric local algebras

In this chapter, we mainly consider symmetric local algebras. Our aim is to find examples A of minimal dimension such that $J(Z(A)) \ntrianglelefteq A$ or $\operatorname{soc}(Z(A)) \ntrianglelefteq A$ hold, respectively. An important tool will be the investigation of the Loewy structure of A, for which the necessary theoretical background is developed in Section 5.1. After that, we consider two special classes of symmetric local algebras. First, we examine quantum complete intersection algebras, which arise as basic algebras of certain non-nilpotent blocks (see Section 5.2). Afterwards, we consider a construction which extends an arbitrary finite-dimensional local algebra A to a symmetric local algebra $T(A)$ of dimension $2 \cdot \dim A$ (see Section 5.3). With the help of these examples, we determine the minimal dimension of a symmetric local algebra A with $J(Z(A)) \ntrianglelefteq A$ and we find both upper and lower bounds for the minimal dimension of a symmetric local algebra A with $\operatorname{soc}(Z(A)) \ntrianglelefteq A$ (see Section 5.4).

5.1 Loewy structure

Let F be an algebraically closed field of arbitrary characteristic and consider a finite-dimensional (unitary) F-algebra A with Jacobson radical $J := J(A)$. Note that we do not require A to be local at this point. In this section, we collect some results on the Loewy structure of A.

Since the ideal J is nilpotent (see Lemma 1.3), there exists a minimal natural number $\ell \in \mathbb{N}$ such that $J^\ell = 0$ holds, where J^ℓ denotes the ℓ-th power of J. We consider the decreasing chain of ideals

$$A \supseteq J \supseteq J^2 \supseteq \ldots \supseteq J^\ell = 0.$$

This series is called the *Loewy series* of A and its length $\ell\ell(A) := \ell$ is called the *Loewy length* of A. For $i = 0, \ldots, \ell - 1$, the i-th *Loewy layer* of A is the quotient space J^i/J^{i+1}.

The following statement will be used frequently to determine a set of generators for a Loewy layer J^{i+m}/J^{i+m+1}, provided that a generating set for J^i/J^{i+1} is already known.

Lemma 5.1 ([34, Lemma E]). *Let I be an ideal of A and let $n, m \in \mathbb{N}$ be natural numbers with $m \leq n$. Suppose that*

$$I^n = F\{x_{i1} \cdots x_{in} : i = 1, \ldots, d\} + I^{n+1}$$

holds for some $d \in \mathbb{N}$ and elements $x_{ij} \in I$. Then we have

$$I^{n+m} = F\{x_{j1} \cdots x_{jm} x_{i1} \cdots x_{in} : i, j = 1, \ldots, d\} + I^{n+m+1}.$$

Moreover, we frequently need the following result on local algebras:

Lemma 5.2 ([10, Lemma 0.3])**.** *Suppose that A is a local algebra. If* $\dim J^i/J^{i+1} = 1$ *holds for some $i \in \mathbb{N}$, then we have $J^i \subseteq Z(A)$.*

Note that a local algebra A can be expressed in the form $A = F \cdot 1 \oplus J$ and hence we obtain

$$K(A) = [A, A] = [J, J] \subseteq J^2. \tag{5.1}$$

We now move to the investigation of symmetric local algebras. Throughout, we widely use the following properties:

Lemma 5.3 ([27, Lemma 2.1])**.** *Let A be a symmetric local algebra. Then:*

(i) $\dim \operatorname{soc}(A) = 1$.

(ii) $\operatorname{soc}(A) \subseteq \operatorname{soc}(Z(A))$.

(iii) $K(A) \cap \operatorname{soc}(A) = 0$.

(iv) $Z(A)$ is local.

(v) We have $J^{\ell-1} = \operatorname{soc}(A)$, where $\ell := \ell\ell(A)$ denotes the Loewy length of A.

In the following, we gather some results which demonstrate that prescribing the structure of the center of a symmetric local algebra has a strong influence on the structure of the algebra itself. We begin with a refined version of Lemma 5.2 for symmetric local algebras:

Lemma 5.4 ([34, Lemma G])**.** *Let A be a symmetric local algebra and suppose that* $\dim J^i/J^{i+1} = 1$ *holds for some $i \in \mathbb{N}$. Then we have $J^{i-1} \subseteq Z(A)$.*

The following statements show that there are few possibilities for the structure of a symmetric local algebra A if its center is of small dimension.

Theorem 5.5 ([34, Theorem B])**.** *If A is a symmetric local algebra with* $\dim Z(A) \leq 4$, *then A is commutative.*

In the following theorem, we summarize several results obtained in [9] and we use the ideas of the proofs presented therein.

Theorem 5.6. *Let A be a symmetric local algebra with* $\dim Z(A) = 5$. *Then one of the following cases occurs:*

(i) $\dim A = 5$ *and A is commutative.*

(ii) $\dim A = 8$ *and there are two possibilities for the Loewy structure of A:*

- *(a)* $\dim J/J^2 = \dim J^2/J^3 = 3$, $\dim J^3/J^4 = 1$ *and* $J^4 = 0$, *or*
- *(b)* $\dim J/J^2 = \dim J^2/J^3 = \dim J^3/J^4 = 2$, $\dim J^4/J^5 = 1$ *and* $J^5 = 0$.

Proof. The fact that the algebra A is of dimension five or eight is the main result of [9] and [11]. If A is of dimension five, then $A = Z(A)$ holds and hence A is commutative. Now we consider the case $\dim A = 8$. If $J^3 = 0$ holds, then Lemma 5.3 yields $\dim J^2/J^3 \leq 1$ and by Lemma 5.4, we obtain $J \subseteq Z(A)$. This implies that A is commutative, which is a contradiction. Hence we have $J^3 \neq 0$, which yields $\dim J^3/J^4 \geq 1$ by Nakayama's lemma. If $\dim J^3/J^4 \geq 2$ holds, then Lemma 5.1 yields $\dim J/J^2 \geq 2$ and $\dim J^2/J^3 \geq 2$. Furthermore, we have $\dim J^4/J^5 \geq 1$ by Lemma 5.3. As $\dim A = 8$ and $\dim A/J = 1$ hold, this implies that A has the Loewy structure given in (b). It remains to consider the case $\dim J^3/J^4 = 1$. Here, we obtain $J^2 \subseteq J(A) \cap Z(A) = J(Z(A))$ by Lemma 5.4 and hence $\dim J^2 \leq 4$ follows. On the other hand, we have $K(A) \subseteq J^2$ and hence

$$(J^2)^\perp \subseteq K(A)^\perp \cap J = Z(A) \cap J = J(Z(A)),$$

which yields $\dim (J^2)^\perp \leq \dim J(Z(A)) = 4$. Lemma 1.19 (i) then implies $\dim J^2 = \dim (J^2)^\perp = 4$ and hence $J^2 = J(Z(A)) = (J^2)^\perp$. By Lemma 1.19 (v), this implies $J^4 = J^2 \cdot J^2 = J^2 \cdot (J^2)^\perp = 0$ and hence we obtain $\dim J^3 = 1$, so A has the Loewy structure given in (a). □

Both Loewy structures given in the theorem occur as the following example demonstrates.

Example 5.7.

(i) For an algebraically closed field K with $\mathrm{char}(K) = 3$, we consider the algebra

$$A = K\langle X, Y, Z\rangle / \langle X^4, Y^2, Z^2, YX + XY, ZX + XZ, YZ - X^2, ZY + X^2\rangle.$$

Here, $K\langle X, Y, Z\rangle$ denotes the free algebra in variables X, Y, Z. One can show that $\dim A = 8$ and $\dim Z(A) = 5$ hold, that A is a symmetric local algebra and that the Loewy structure of A is of type (a) in Theorem 5.6 (ii).

(ii) Let K be a field of characteristic $p = 2$. Then the group algebra KD_8 of the dihedral group of order eight over K has dimension eight, a five-dimensional center and a Loewy structure of the second type described in Theorem 5.6 (ii). ◁

We conclude this collection of results on the Loewy structure of symmetric local algebras with the following lemma, which is stated in the proof of [39, Theorem 3.2]. It even holds without the assumption of locality. The proof uses the same arguments as the one of [11, Lemma 1.1].

Lemma 5.8. *Let A be a symmetric algebra. Then either A is commutative or we have* $\dim A \geq \dim Z(A) + 3$.

Proof. Suppose that A is not commutative. Clearly, this implies $\dim A > \dim Z(A)$. Assume that $\dim A = \dim Z(A) + 1$ holds. Then A is of the form $A = Fx \oplus Z(A)$ for some $x \in A$. But then we have $K(A) = [Fx \oplus Z(A), Fx \oplus Z(A)] = 0$, a contradiction to A being non-commutative. If $\dim A = \dim Z(A) + 2$ holds, then A can be expressed in the form $A = Fx \oplus Fy \oplus Z(A)$ for some elements $x, y \in A$. Then $K(A) \subseteq F[x, y]$ follows and hence we have $\dim K(A) \leq 1$. Since A is symmetric, this yields $\dim Z(A) = \dim A - \dim K(A) \geq \dim A - 1 > \dim A - 2 = \dim Z(A)$, which is a contradiction. Hence we obtain $\dim A \geq \dim Z(A) + 3$. □

Example 5.7 demonstrates that this bound is tight.

5.2 Quantum complete intersection algebras

Throughout, let F be an algebraically closed field of characteristic $p > 0$. In this section, we study our main problem for quantum complete intersection algebras, which form a special class of symmetric local F-algebras. They arise, for example, as the basic algebras of certain non-nilpotent blocks containing a single simple module (see [2] and [23]).

We begin our investigation by defining quantum complete intersection algebras and excluding some trivial cases (see Section 5.2.1). Moreover, we fix a canonical basis of such an algebra A. In Section 5.2.2, we then study the structure of the basis elements y which are contained in $Z(A)$ and state conditions which ensure that $Ay \subseteq Z(A)$ holds. Using these, we find criteria for $J(Z(A))$ or $\operatorname{soc}(Z(A))$ to be ideals of A in Sections 5.2.3 and 5.2.4, respectively. We conclude this part by providing counterexamples to some open questions from the preceding chapters in Section 5.2.5.

5.2.1 Definition and conventions

Before introducing the concept of quantum complete intersection algebras, we point out that there exist various definitions of these algebras in the literature, including some for which the resulting algebras are not necessarily symmetric. In our treatment, we always refer to the following definition, which is taken from [2]:

Definition 5.9. Let $c \in \mathbb{N}$. A *quantum complete intersection algebra* is an F-algebra A of the form

$$A := F\langle X_1, \ldots, X_c\rangle / \langle X_1^p, \ldots, X_c^p, X_iX_j - q_{ij}X_jX_i\rangle_{i>j}. \tag{5.2}$$

Here, $F\langle X_1, \ldots, X_c\rangle$ denotes the free algebra in variables $X_1, \ldots, X_c$ over F and the *quantum parameters* $q_{ij} \in F$ are required to satisfy the relations $q_{ij}^{p-1} = q_{ii} = q_{ij}q_{ji} = 1$ for all indices $i, j \in \{1, \ldots, c\}$.

By [3, Lemma 3.1], the algebra A defined in (5.2) is symmetric. For $i = 1, \ldots, c$, we denote the image of X_i in A by x_i. Then the set

$$B_A := \{x_1^{r_1} \cdots x_c^{r_c} \colon 0 \leq r_1, \ldots, r_c \leq p-1\}$$

forms an F-basis of A. In particular, A is of dimension p^c. Since all nontrivial elements in this basis are nilpotent, it follows that A is a local F-algebra.

Remark 5.10. As mentioned at the beginning, the definition of quantum complete intersection algebras is not consistent in the literature. For example, Holloway and Kessar [23] replace the condition $x_1^p = \ldots = x_c^p = 0$ by

$$x_1^{p^{\ell_1}} = \ldots = x_c^{p^{\ell_c}} = 0 \text{ for some } \ell_1, \ldots, \ell_c \in \mathbb{N}.$$

Bergh [3] even considers relations of the form $x_1^{a_1} = \ldots = x_c^{a_c} = 0$ for arbitrary exponents $a_1, \ldots, a_c \geq 2$ together with relations $x_ix_j - q_{ij}x_jx_i = 0$ for $i > j$, thereby omitting the restriction $q_{ij}^{p-1} = 1$ on the quantum parameters. It should be noted that the algebras arising in the latter way are in general not symmetric. ◁

We first exclude the case $p = 2$ and establish some conventions:

Remark 5.11.

(i) For $p = 2$, the assumption $q_{ij}^{p-1} = 1$ reads $q_{ij} = 1$ for all $i, j \in \{1, \ldots, c\}$ and hence the resulting algebra A is commutative, so $J(Z(A)) \trianglelefteq A$ and $\mathrm{soc}(Z(A)) \trianglelefteq A$ hold by Example 1.5. In the following, we therefore assume $p \geq 3$.

(ii) Let p be an odd prime number. Since $q_{ij}^{p-1} = 1$ holds for all $i, j \in \{1, \ldots, c\}$, we may write $q_{ij} = q^{t_{ij}}$ for some

$$t_{ij} \in \left\{-\frac{p-1}{2}, \ldots, \frac{p-1}{2}\right\}$$

and a fixed element $q \in F^\times$ of order $p - 1$. Moreover, since we have $q_{ij} = q_{ji}^{-1}$, we may require $t_{ij} = -t_{ji}$ for $i, j \in \{1, \ldots, c\}$. Then the matrix $T := (t_{ij})_{ij}$ consisting of the exponents t_{ij} is skew-symmetric. ◁

Summarizing, we make the following assumption throughout our investigation of quantum complete intersection algebras:

Hypothesis 5.12. Let F be an algebraically closed field of odd characteristic p and let

$$A := F\langle X_1, \ldots, X_c\rangle / \langle X_1^p, \ldots, X_c^p, X_iX_j - q_{ij}X_jX_i\rangle_{i>j}.$$

denote a quantum complete intersection algebra with quantum parameters $q_{ij} = q^{t_{ij}}$ for a fixed element $q \in F^\times$ of order $p - 1$ and a skew-symmetric matrix $T = (t_{ij})_{ij}$ with entries

$$t_{ij} \in \left\{-\frac{p-1}{2}, \ldots, \frac{p-1}{2}\right\}.$$

For the sake of brevity, we introduce the following notation: We set $R := \{0, \ldots, p - 1\}$ and $S := \{0, \ldots, p-2\}$. If not stated otherwise, a bold variable, such as $\mathbf{r}$, denotes a vector of length c with entries $r_1, \ldots, r_c \in R$. In this situation, we set $x^{\mathbf{r}} := x_1^{r_1} \cdots x_c^{r_c}$. Moreover, for $i = 1, \ldots, c$, we define an auxiliary vector $\mathbf{r}_{i+} := (r_1, \ldots, r_{i-1}, r_i+1, r_{i+1}, \ldots, r_c)$ arising from $\mathbf{r}$ by increasing the i-th component by one.

5.2.2 Ideals in the center

In this section, we study conditions under which certain subsets of $Z(A)$ are ideals in A. First, we note that the structure of the basis elements in B_A gives rise to an $\mathbb{N}_0^c$-grading of A in a natural way, which allows us to restrict our investigation to the elements that are homogeneous with respect to this grading.

Remark 5.13.

(i) Let $\mathbf{n} = (n_1, \ldots, n_c) \in \mathbb{N}_0^c$ and set $A_{\mathbf{n}} := Fx^{\mathbf{n}}$. If $n_i \geq p$ holds for some $i \in \{1, \ldots, c\}$, we have $A_{\mathbf{n}} = 0$. The algebra A decomposes as

$$A = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^c} A_{\mathbf{n}}$$

and since $A_{\mathbf{n}} \cdot A_{\mathbf{m}} \subseteq A_{\mathbf{n}+\mathbf{m}}$ holds for any $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^c$, this defines an $\mathbb{N}_0^c$-grading on A (see Remark 2.26).

(ii) The center $Z(A)$ is a graded subalgebra of A. To see that this is true, we consider an element $0 \neq y \in Z(A)$ and write $y = \sum_{k=1}^{n} \alpha_k x^{\mathbf{r}^k}$ with nonzero coefficients $\alpha_1, \ldots, \alpha_n \in F$ and pairwise distinct vectors $\mathbf{r}^1, \ldots, \mathbf{r}^n \in R^c$. For all $i \in \{1, \ldots, c\}$, we have $x_i \cdot y = y \cdot x_i$. Note that for each occurring basis element $x^{\mathbf{r}^k}$, we either have $x_i \cdot x^{\mathbf{r}^k} = 0 = x^{\mathbf{r}^k} \cdot x_i$ or $x_i \cdot x^{\mathbf{r}^k}$ is a nonzero multiple of the basis element $x^{\mathbf{r}^k_{i+}}$. Comparing the coefficients then yields

$$x_i \cdot x^{\mathbf{r}^k} = x^{\mathbf{r}^k} \cdot x_i.$$

This shows that the element $x^{\mathbf{r}^k}$ is contained in $Z(A)$ for $k = 1, \ldots, n$. In particular, the set $Z(A) \cap B_A$, which consists of homogeneous elements with respect to the grading given in (i), forms an F-basis for $Z(A)$. Hence $Z(A)$ is a graded subalgebra of A. Note that $J(Z(A))$ and $\operatorname{soc}(Z(A))$ are homogeneous subspaces of A as well. ◁

In the following lemma, we determine which elements of the basis B_A are contained in $Z(A)$. It will form the basis for all further results of this section.

Lemma 5.14. *Let $y = x^{\mathbf{r}} \in B_A$ with $\mathbf{r} \in R^c$ be a basis element. Then $y \in Z(A)$ holds if and only if for all $i \in \{1, \ldots, c\}$, we have*

$$r_i = p - 1 \text{ or } \sum_{j=1}^{c} t_{ij} r_j \equiv 0 \pmod{p-1}. \tag{5.3}$$

Proof. Note that the condition $y \in Z(A)$ is equivalent to $x_i \cdot y = y \cdot x_i$ for $i = 1, \ldots, c$. We fix an index $i \in \{1, \ldots, c\}$. If $r_i = p - 1$ holds, then $x_i \cdot y = 0 = y \cdot x_i$ follows in any case. Now assume $r_i < p - 1$, which is equivalent to $x^{\mathbf{r}_{i+}} \neq 0$. In this case, the condition $x_i \cdot y = y \cdot x_i$ can be restated in the following way:

$$\prod_{j=1}^{i-1} q^{t_{ij} r_j} \cdot x^{\mathbf{r}_{i+}} = \prod_{j=1}^{i-1} q_{ij}^{r_j} \cdot x^{\mathbf{r}_{i+}} = x_i \cdot y = y \cdot x_i = \prod_{j=i+1}^{c} q_{ji}^{r_j} \cdot x^{\mathbf{r}_{i+}} = \prod_{j=i+1}^{c} q^{t_{ji} r_j} \cdot x^{\mathbf{r}_{i+}}.$$

Since we have $\operatorname{ord}(q) = p - 1$, comparing the coefficient of the two expressions yields

$$\sum_{j=1}^{i-1} t_{ij} r_j \equiv \sum_{j=i+1}^{c} t_{ji} r_j \pmod{p-1}.$$

This is equivalent to

$$\sum_{j=1}^{c} t_{ij} r_j \equiv 0 \pmod{p-1},$$

since $t_{ji} = -t_{ij}$ holds for $i, j \in \{1, \ldots, c\}$ by assumption. Summarizing, we have $y \in Z(A)$ if and only if $x_i \cdot y = y \cdot x_i$ holds for $i = 1, \ldots, c$, which in turn is the case if and only if one of the conditions given in (5.3) is satisfied. □

Remark 5.15. A basis element $y = x^{\mathbf{r}} \in B_A$ with $\mathbf{r} \in R^c$ is uniquely determined by the choice of the set

$$I_y := \{i \in \{1, \ldots, c\} \colon r_i < p - 1\}$$

together with the truncation $\mathbf{r}_{I_y} \in S^{|I_y|}$ of the vector $\mathbf{r}$ to the indices in I_y. In this formalism, the statement of the preceding lemma reads

$$y \in Z(A) \Leftrightarrow I_y = \emptyset \text{ or } T_{I_y} \cdot \mathbf{r}_{I_y} \equiv 0 \pmod{p-1}.$$

Here, $T_{I_y} := (t_{ij})_{i,j \in I_y}$ denotes the restriction of the matrix T to rows and columns with indices in I_y and the above congruence relation is understood entry-wise. In this way, we obtain a correspondence between the elements in $Z(A) \cap B_A$ and the set

$$\Big\{(I, \mathbf{v})\colon \emptyset \neq I \subseteq \{1, \dots, c\},\ \mathbf{v} \in S^{|I|},\ T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}\Big\} \cup \{\emptyset\}. \tag{5.4}$$

For this reason, we may write $\mathbf{r}_{I,\mathbf{v}}$ for the vector $\mathbf{r} \in R^c$ corresponding to the tuple $(I, \mathbf{v})$. Note that the empty set corresponds to the element $x_1^{p-1} \cdots x_c^{p-1} \in Z(A)$, which spans the one-dimensional subspace $\mathrm{soc}(A)$. In order to determine the set given in (5.4), we use that the elements in S^ℓ are in bijective correspondence with the elements in $(\mathbb{Z}/(p-1)\mathbb{Z})^\ell$ for any $\ell \in \mathbb{N}$. ◁

Now we describe the basis elements $y \in Z(A) \cap B_A$ for which the principal ideal Ay is contained in $Z(A)$.

Lemma 5.16. *Consider a basis element $y = x^{\mathbf{r}} \in Z(A)$ with $\mathbf{r} \in R^c$. Then we have $Ay \subseteq Z(A)$ if and only if $I_y = \emptyset$ or $T_{I_y} = 0$ hold.*

Proof. For $I_y = \emptyset$, we have $y = x_1^{p-1} \cdots x_c^{p-1} \in \mathrm{soc}(A)$ and hence $Ay \subseteq \mathrm{soc}(A) \subseteq Z(A)$ follows. Now we assume $I_y \neq \emptyset$. First suppose $Ay \subseteq Z(A)$. We fix an index $m \in I_y$ and write $x_m \cdot y = \lambda \cdot x^{\mathbf{r}_{m+}}$ for some $\lambda \in F^\times$. Note that $x_m \cdot y \in Z(A)$ implies $x^{\mathbf{r}_{m+}} \in Z(A)$. For any $i \in I_y$, we now need to show that $t_{im} = 0$ holds. Since we have $t_{mm} = 0$ by assumption, we may suppose $i \neq m$. This implies $i \in I_{x^{\mathbf{r}_{m+}}}$ and hence (5.3), applied to $x^{\mathbf{r}_{m+}}$, yields

$$0 \equiv t_{im} \cdot (r_m + 1) + \sum_{j \neq m} t_{ij} r_j = t_{im} + \sum_{j=1}^{c} t_{ij} r_j \equiv t_{im} \pmod{p-1},$$

which implies $t_{im} = 0$ by our assumption on T. In the last congruence, we used the condition $y \in Z(A)$ together with (5.3).

On the other hand, assume $T_{I_y} = 0$ and consider an element $x^{\mathbf{s}} \in B_A$ with $\mathbf{s} \in R^c$ such that $x^{\mathbf{s}} \cdot y$ is nonzero. Note that $x^{\mathbf{s}} \cdot y$ is a scalar multiple of $y' := x^{\mathbf{v}}$ with $\mathbf{v} := \mathbf{r} + \mathbf{s}$ and that we have $I_{y'} \subseteq I_y$. This yields $T_{I_{y'}} = 0$, which implies

$$T_{I_{y'}} \cdot \mathbf{v}_{I_{y'}} \equiv 0 \pmod{p-1}.$$

By Remark 5.15, this yields $y' \in Z(A)$ and hence $x^{\mathbf{s}} \cdot y \in Z(A)$ follows. Summarizing, we obtain $Ay \subseteq Z(A)$ in this case. □

Remark 5.17. In other words, for a nonempty subset $I \subseteq \{1, \dots, c\}$ with $T_I \neq 0$, any solution $\mathbf{v} \in S^{|I|}$ of the congruence system $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ gives rise to an element $y = x^{\mathbf{r}_{I,\mathbf{v}}} \in Z(A) \cap B_A$ with $Ay \not\subseteq Z(A)$ by the correspondence given in Remark 5.15. Note that the tuple $(I, \mathbf{v})$ with $I = \{1, \dots, c\}$ and $\mathbf{v} = 0$ corresponds to the element $y = 1$.

In particular, we find an element $y \in J(Z(A)) \cap B_A$ with $Ay \not\subseteq Z(A)$ if and only if there exists a nonempty proper subset $I \subseteq \{1, \ldots, c\}$ with $T_I \neq 0$ or the congruence relation $T \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ has a nonzero solution $\mathbf{v} \in S^c$. ◁

5.2.3 Jacobson radical

The following result demonstrates that apart from a single exceptional case, $J(Z(A))$ is only an ideal of A if the algebra is commutative.

Lemma 5.18. *We have $J(Z(A)) \trianglelefteq A$ if and only if one of the following holds:*

(i) A is commutative.

(ii) $c = 2$ and $\operatorname{ord}(q_{21}) = p - 1$.

In the first case, we have $J(Z(A)) = J(A)$, and in the second, we obtain

$$J(Z(A)) = F\left\{x_1^{r_1} x_2^{r_2} \colon r_1, r_2 \in R,\ r_1 = p-1 \text{ or } r_2 = p-1\right\}. \tag{5.5}$$

Proof. If A is commutative, we have $J(Z(A)) = J(A) \trianglelefteq A$. In the following, we therefore assume that A is not commutative, which translates to $T \neq 0$ by our assumption on T. In particular, this implies $c > 1$. We begin with the case $c = 2$, thereby using the criterion from Remark 5.17. Since the diagonal elements of the matrix T are zero, it suffices to determine whether there exists a nontrivial solution of $T \cdot \mathbf{v} \equiv 0 \pmod{p-1}$. Since T is of the form

$$T = \begin{pmatrix} 0 & -t_{21} \\ t_{21} & 0 \end{pmatrix},$$

this is the case if and only if t_{21} is not invertible modulo $p - 1$, which is equivalent to $\operatorname{ord}(q_{21}) < p - 1$. Otherwise, $J(Z(A))$ is an ideal of A and Remark 5.15 yields

$$\begin{aligned} J(Z(A)) \cap B_A &= \left\{x_1^{p-1} x_2^{p-1}\right\} \cup \left\{x^{\mathbf{r}_{I,v}} \colon I \subseteq \{1,2\},\ |I| = 1,\ v \in S\right\} \\ &= \left\{x_1^{p-1} x_2^{v} \colon v \in R\right\} \cup \left\{x_1^{v} x_2^{p-1} \colon v \in R\right\}. \end{aligned}$$

Finally, let $c \geq 3$. Since T is nonzero, there exists a subset $I \subseteq \{1, \ldots, c\}$ with $T_I \neq 0$ such that $|I| > 1$ is odd. Since T_I is skew-symmetric, this yields $\det(T_I) = 0$ and hence $\operatorname{Ker}(T_I) \neq 0$. Clearly, then also the congruence system $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ has a nontrivial solution. By Remark 5.17, this implies $J(Z(A)) \ntrianglelefteq A$. □

5.2.4 Socle

In this section, we examine the socle of $Z(A)$. We focus on the case that the quantum parameters q_{ij} are of order $p - 1$ for $i \neq j$ and derive a characterization of the quantum complete intersection algebras A which satisfy $\operatorname{soc}(Z(A)) \trianglelefteq A$ in terms of the corresponding matrix T. However, if not stated otherwise, we assume that A is an arbitrary quantum complete intersection algebra subject only to the restrictions stated in Hypothesis 5.12.

First we determine under which conditions a basis element $x \in Z(A) \cap B_A$ is contained in $\operatorname{soc}(Z(A))$, thereby using the correspondence from Remark 5.15. To this end, we need the following definition:

Definition 5.19. Let $I \subseteq \{1, \ldots, c\}$ be a nonempty subset and consider a solution $\mathbf{v} \in S^{|I|}$ of $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$. We define a vector $\tilde{\mathbf{v}} = (\tilde{v}_1, \ldots, \tilde{v}_c) \in S^c$ by setting $\tilde{v}_i := v_i$ for $i \in I$ and $\tilde{v}_i = 0$ otherwise. If $T \cdot \tilde{\mathbf{v}} \equiv 0 \pmod{p-1}$ holds, we say that the solution $\mathbf{v}$ *can be extended* to T.

With this, we characterize the elements in $\operatorname{soc}(Z(A)) \cap B_A$:

Lemma 5.20. *Let $I \subseteq \{1, \ldots, c\}$ be a nonempty subset and consider a vector $\mathbf{u} \in S^{|I|}$. Then $y := x^{\mathbf{r}_{I,\mathbf{u}}}$ is contained in $\operatorname{soc}(Z(A))$ if and only if the following three conditions hold:*

(i) $T_I \cdot \mathbf{u} \equiv 0 \pmod{p-1}$.

(ii) $u_i \geq 1$ *for* $i = 1, \ldots, c$.

(iii) For every solution $0 \neq \mathbf{v} \in S^{|I|}$ of $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ which can be extended to T, there exists an index $i \in I$ with $u_i > p - 1 - v_i$.

Proof. First assume $y \in \operatorname{soc}(Z(A))$. Then the property (i) follows by Lemma 5.18. Since y annihilates $x_i^{p-1} \in J(Z(A))$, we obtain $u_i \geq 1$ for $i = 1, \ldots, c$. Now let $0 \neq \mathbf{v} \in S^{|I|}$ be a solution of the congruence relation $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ which can be extended to T. We consider the basis element $z := x^{\tilde{\mathbf{v}}}$, where $\tilde{\mathbf{v}}$ denotes the extension of $\mathbf{v}$. Then $z \in J(Z(A))$ follows since we have $T \cdot \tilde{\mathbf{v}} \equiv 0 \pmod{p-1}$ by assumption and $\tilde{\mathbf{v}}$ is nonzero. This implies $yz = 0$, so there exists an index $i \in \{1, \ldots, c\}$ with $u_i > p - 1 - v_i$. Since $\tilde{v}_\ell = 0$ holds for every index $\ell \notin I$, we have $i \in I$.

Conversely, assume that $\mathbf{u} \in S^{|I|}$ satisfies the conditions (i) – (iii) and consider a basis element $z := x^{\mathbf{s}} \in J(Z(A))$ for some $\mathbf{s} \in R^c$. Note that $\mathbf{s}$ is nonzero. We claim that $yz = 0$ holds. To show this, we may assume $s_\ell = 0$ for $\ell \notin I$ and $s_i \in S$ for $i \in I$ since we directly obtain $yz = 0$ otherwise. In particular, we have $I_z = \{1, \ldots, c\}$, so the condition $z \in J(Z(A))$ implies $T \cdot \mathbf{s} \equiv 0 \pmod{p-1}$ (see Lemma 5.18). Note that $\mathbf{s}$ is the extension of $\mathbf{s}_I$ by zeros, so we have $T_I \cdot \mathbf{s}_I \equiv 0 \pmod{p-1}$. By (iii), there exists an index $i \in I$ with $u_i > p - 1 - s_i$ and hence y annihilates $z = x^{\mathbf{s}}$. This yields $y \in \operatorname{soc}(Z(A))$. □

In the situation of Lemma 5.20, the question arises whether the vector $\mathbf{u} \in S^{|I|}$ itself can be extended to T if $x^{\mathbf{r}_{I,\mathbf{u}}}$ is contained in $\operatorname{soc}(Z(A))$. This turns out to be impossible:

Remark 5.21. Let $I \subseteq \{1, \ldots, c\}$ be a nonempty subset and consider a vector $\mathbf{u} \in S^{|I|}$ such that $y = x^{\mathbf{r}_{I,\mathbf{u}}}$ is contained in $\operatorname{soc}(Z(A))$. Suppose that the solution $\mathbf{u}$ can be extended to T. Since $u_i \geq 1$ holds for all $i \in I$, we have $v_i := p - 1 - u_i \in S$. For the vector $\mathbf{v} := (v_i)_{i \in I}$, we have

$$T_I \cdot \mathbf{v} \equiv T_I \cdot (-\mathbf{u}) \equiv 0 \pmod{p-1}.$$

Moreover, also $\mathbf{v}$ can be extended to T, which yields $x^{\tilde{\mathbf{v}}} \in J(Z(A))$. Note that $y \cdot x^{\tilde{\mathbf{v}}}$ is a nonzero multiple of $x_1^{p-1} \cdots x_c^{p-1}$, which contradicts $y \in \operatorname{soc}(Z(A))$. Hence $\mathbf{u}$ cannot be extended to T. In particular, I is a proper subset of $\{1, \ldots, c\}$ in this situation. ◁

This leads to the following observation:

Example 5.22. For $c \in \{1, 2\}$, $\operatorname{soc}(Z(A))$ is an ideal of A: If $c = 1$ holds, then the algebra A is commutative and hence $\operatorname{soc}(Z(A)) = \operatorname{soc}(A)$ is an ideal of A. Now let $c = 2$ and consider an element $y \in \operatorname{soc}(Z(A)) \cap B_A$. First assume that y is of the form $x^{\mathbf{r}_{I,\mathbf{u}}}$ for some nonempty subset $I \subseteq \{1, 2\}$ and some $\mathbf{u} \in S^{|I|}$. By Remark 5.21, we have $|I| = 1$ and hence $Ay \subseteq Z(A)$ follows by Lemma 5.16. Otherwise, we have $y \in \mathrm{soc}(A)$ (see Remark 5.15) and hence $Ay \subseteq \mathrm{soc}(Z(A))$ follows. This shows that $\mathrm{soc}(Z(A))$ is an ideal of A in this case. ◁

We now move to the characterization of the property $\mathrm{soc}(Z(A)) \trianglelefteq A$ in the case where all quantum parameters q_{ij} are of order $p-1$ for $i \neq j$. We begin with some preliminary results on the structure of the solution set of the congruence relation $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$.

Remark 5.23. Consider indices $i, j, k \in \{1, \ldots, c\}$ with $i < j < k$ such that t_{ji}, t_{ki} and t_{kj} are invertible modulo $p-1$ and set $I := \{i, j, k\}$. Then T_I has the form

$$T_I = \begin{pmatrix} 0 & -t_{ji} & -t_{ki} \\ t_{ji} & 0 & -t_{kj} \\ t_{ki} & t_{kj} & 0 \end{pmatrix}.$$

In particular, the set of solutions of $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ is given by $\mathbb{Z}(t_{kj}, -t_{ki}, t_{ji})^T$. ◁

For $i, j, k \in \{1, \ldots, c\}$ with $i < j < k$, we define $\mathbf{v}_{ijk}$ to be the unique vector in S^3 with $\mathbf{v}_{ijk} \equiv (t_{kj}, -t_{ki}, t_{ji})^T \pmod{p-1}$. Moreover, let $\mathbf{r}_{ijk} := \tilde{\mathbf{v}}_{ijk}$ be the extension of $\mathbf{v}_{ijk}$ by zeros. Later, we will encounter the situation that $T \cdot \mathbf{r}_{ijk} \equiv 0 \pmod{p-1}$ holds for all $i < j < k$. Under this condition, we obtain the following description of the set of solutions of the congruence relation $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$:

Proposition 5.24. *For $i, j, k \in \{1, \ldots, c\}$, let $\mathrm{ord}(q_{ij}) = p-1$ for $i \neq j$ and assume that $T \cdot \mathbf{r}_{ijk} \equiv 0 \pmod{p-1}$ holds for $i < j < k$. For a subset $I \subseteq \{1, \ldots, c\}$ with $|I| \geq 3$, the set of solutions of the congruence relation $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ is given by*

$$\mathbb{Z}\,\{(\mathbf{r}_{ijk})_I : i, j, k \in I,\ i < j < k\}.$$

Proof. Since $T \cdot \mathbf{r}_{ijk} \equiv 0 \pmod{p-1}$ holds for all $i, j, k \in I$ with $i < j < k$, we also obtain $T_I \cdot (\mathbf{r}_{ijk})_I \equiv 0 \pmod{p-1}$ since the remaining entries of the vector $\mathbf{r}_{ijk}$ are zero. For the converse inclusion, we may assume $I = \{1, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$. Then T_I is of the form

$$T_I = \begin{pmatrix} 0 & -t_{21} & -t_{31} & -t_{41} & \ldots & -t_{\ell 1} \\ t_{21} & 0 & -t_{32} & -t_{42} & \ldots & -t_{\ell 2} \\ t_{31} & t_{32} & 0 & -t_{43} & \ldots & -t_{\ell 3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{\ell 1} & t_{\ell 2} & t_{\ell 3} & t_{\ell 4} & \ldots & 0 \end{pmatrix}.$$

Since we have $\mathrm{ord}(q_{ij}) = p-1$, the entries t_{ij} are invertible modulo $p-1$ for $i \neq j$ and hence Gaussian elimination leads to the matrix

$$\begin{pmatrix} t_{21} & 0 & -t_{32} & -t_{42} & \ldots & -t_{\ell 2} \\ 0 & -t_{21} & -t_{31} & -t_{41} & \ldots & -t_{\ell 1} \\ 0 & 0 & a_{33} & a_{34} & \ldots & a_{3\ell} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{\ell 3} & a_{\ell 4} & \ldots & a_{\ell\ell} \end{pmatrix}$$

with $a_{ij} = t_{i1}t_{j2} - t_{i2}t_{j1} + t_{ij}t_{21}$ for $i, j \in \{3, \ldots, \ell\}$. Note that this term is equal to the product of the i-th row of T and the vector $\mathbf{r}_{12j}$. Since $T \cdot \mathbf{r}_{12j} \equiv 0 \pmod{p-1}$ holds by assumption, we obtain $a_{ij} \equiv 0 \pmod{p-1}$. For $m \in \{3, \ldots, \ell\}$, the m-th basis solution of the above congruence system is given by $(\mathbf{r}_{12m})_I$ and an arbitrary solution is an integral linear combination of vectors of this form. □

We now describe the quantum complete intersection algebras A with $\operatorname{ord}(q_{ij}) = p-1$ for $i \neq j$ in which $\operatorname{soc}(Z(A))$ is an ideal.

Lemma 5.25. *Assume that* $\operatorname{ord}(q_{ij}) = p - 1$ *holds for all* $i, j \in \{1, \ldots, c\}$ *with* $i \neq j$. *Then the following are equivalent:*

(i) $\operatorname{soc}(Z(A)) \trianglelefteq A$.

(ii) $T \cdot \mathbf{r}_{ijk} \equiv 0 \pmod{p-1}$ *holds for all* $i, j, k \in \{1, \ldots, c\}$ *with* $i < j < k$.

In this case, we have

$$\operatorname{soc}(Z(A)) = F\left\{ x_i^{r_i} \cdot \prod_{j \neq i} x_j^{p-1} : i \in \{1, \ldots, c\} \text{ and } r_i \in \{1, \ldots, p-1\} \right\}. \tag{5.6}$$

Proof. For $c \leq 2$, we have $\operatorname{soc}(Z(A)) \trianglelefteq A$ by Example 5.22 and the second property is trivially satisfied. With the explicit description of $J(Z(A))$ given in Lemma 5.18, one easily verifies that $\operatorname{soc}(Z(A))$ is of the form given in (5.6). In the following, we therefore consider the case $c \geq 3$.

First assume $\operatorname{soc}(Z(A)) \trianglelefteq A$ and consider a subset $I = \{i, j, k\} \subseteq \{1, \ldots, c\}$ with $i < j < k$. Suppose that $T \cdot \mathbf{r}_{ijk} \not\equiv 0 \pmod{p-1}$ holds. By Remark 5.23, $\mathbf{v} = 0$ is then the only solution of $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ in S^3 which can be extended to T. Then the vector $\mathbf{v}_{ijk} \in S^3$ satisfies the criteria given in Lemma 5.20 and we obtain $y := x^{\mathbf{r}_{I,\mathbf{v}_{ijk}}} \in \operatorname{soc}(Z(A))$. Since T_I is nonzero, Lemma 5.16 yields $Ay \not\subseteq Z(A)$, which is a contradiction to the assumption $\operatorname{soc}(Z(A)) \trianglelefteq A$.

Now assume conversely that for any choice of $i, j, k \in \{1, \ldots, c\}$ with $i < j < k$, we have $T \cdot \mathbf{r}_{ijk} \equiv 0 \pmod{p-1}$. We show that $Ay \subseteq Z(A)$ holds for all $y \in \operatorname{soc}(Z(A)) \cap B_A$. Note that this is clear for $y \in \operatorname{soc}(A)$, so by Remark 5.15, we may write $y = x^{\mathbf{r}_{I,\mathbf{u}}}$ for some nonempty set $I \subseteq \{1, \ldots, c\}$ and a vector $\mathbf{u} \in S^{|I|}$ satisfying $T_I \cdot \mathbf{u} \equiv 0 \pmod{p-1}$. If $|I| \geq 3$ holds, then $\mathbf{u}$ can be extended to T by Proposition 5.24, which is a contradiction to Remark 5.21. Now suppose that $I = \{i, j\}$ consists of two elements. Since t_{ij} is invertible modulo $p - 1$, the condition $T_I \cdot \mathbf{u} \equiv 0 \pmod{p-1}$ implies $\mathbf{u} \equiv 0 \pmod{p-1}$. Since we have $1 \leq u_i, u_j < p - 1$ by Lemma 5.20, this is a contradiction. This yields $|I| = 1$ and hence $Ay \subseteq Z(A)$ follows by Lemma 5.16. Hence we have

$$y = x_i^{r_i} \cdot \prod_{j \neq i} x_j^{p-1}$$

for some $i \in \{1, \ldots, c\}$ and some $r_i \in \{1, \ldots, p-1\}$. Conversely, every element of this form is contained in $\operatorname{soc}(Z(A))$, so this set has the structure given in (5.6). □

In this situation, $\operatorname{soc}(Z(A))$ is an ideal of A for $c \leq 3$. This extends Example 5.22.

Remark 5.26. The condition $T \cdot \mathbf{r}_{ijk} \equiv 0 \pmod{p-1}$ is equivalent to

$$t_{\ell i}t_{kj} - t_{\ell j}t_{ki} + t_{\ell k}t_{ji} \equiv 0 \pmod{p-1} \text{ for all } \ell \in \{1, \dots, c\}.$$

If i, j, k, ℓ are pairwise distinct, then the square of the left expression, up to sign, equals the determinant of the matrix T_J with $J = \{i, j, k, \ell\}$. Hence $\det(T_J) \equiv 0 \pmod{p-1}$ for all subsets $J \subseteq \{1, \dots, c\}$ with $|J| = 4$ is a necessary condition for $\operatorname{soc}(Z(A)) \trianglelefteq A$. ◁

In the special case that $\operatorname{ord}(q_{ij}) = p-1$ holds for all $i, j \in \{1, \dots, c\}$ with $i \neq j$, Lemma 5.25 characterizes the quantum complete intersection algebras A which satisfy $\operatorname{soc}(Z(A)) \trianglelefteq A$ in terms of the matrix T. The key insight we used is the fact that the solution set of the congruence relation $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ can be described easily. In the general case, one can proceed similarly, but there might be more solutions of the congruence system, which complicates the problem. At the end of this section, we examine this question for the special case where all quantum parameters q_{ij} for $i > j$ are equal.

Lemma 5.27. *Assume that there exists some $t \in \{1, \dots, p-2\}$ such that $q_{ij} = q^t$ holds for all indices $i, j \in \{1, \dots, c\}$ with $i > j$. Then $\operatorname{soc}(Z(A)) \trianglelefteq A$ is equivalent to $c \in \{1, 2, 3\}$.*

Proof. Note that $u := \operatorname{ord}(q^t)$ is a divisor of $p-1$ and that we have $u \geq 2$ by assumption. By Example 5.22, $\operatorname{soc}(Z(A))$ is an ideal of A for $c \leq 2$. Now let $c = 3$ and consider an element $y := x^{\mathbf{w}} \in \operatorname{soc}(Z(A))$ with $\mathbf{w} \in R^3$. By Remark 5.21, I_y is a proper subset of $\{1, 2, 3\}$. Suppose that $|I_y| = 2$ holds. Without loss of generality, we assume $I_y = \{1, 2\}$. Lemma 5.18 yields

$$0 \equiv T_{I_y} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \pmod{p-1},$$

so w_1 and w_2 are divisible by u. On the other hand, we have $w_1, w_2 > p-1-u$ since x_1^u and x_2^u are contained in $J(Z(A))$ by Lemma 5.16. Since $w_1, w_2 < p-1$ holds by assumption, this is a contradiction. Hence $|I_y| \leq 1$ follows, which implies $Ay \subseteq Z(A)$ by Lemma 5.16.

Finally let $c \geq 4$ and set $I := \{1, 2, 3\}$. A vector $\mathbf{v} \in \mathbb{Z}^3$ satisfies $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ if and only if $\mathbf{v} \equiv \lambda \cdot (1, -1, 1)^T \pmod{u}$ holds for some $\lambda \in \mathbb{Z}$. In particular, $\mathbf{w} := (p-u, p-2, p-u)^T$ is a solution to the above congruence system since u divides $p-1$. We show $y := x^{\mathbf{r}_{I,\mathbf{w}}} \in \operatorname{soc}(Z(A))$ by using Lemma 5.20. The first two conditions stated therein are clearly satisfied. In order to verify the third one, let $0 \neq \mathbf{v} \in S^3$ be a solution to the congruence system $T_I \cdot \mathbf{v} \equiv 0 \pmod{p-1}$ which can be extended to T. We write $\mathbf{v} = (v_1, v_2, v_3)^T = (\lambda + k_1 u, -\lambda + k_2 u, \lambda + k_3 u)^T$ for some $k_1, k_2, k_3, \lambda \in \mathbb{Z}$. Denoting the extension of $\mathbf{v}$ by $\tilde{\mathbf{v}}$ as usual, the condition $T \cdot \tilde{\mathbf{v}} \equiv 0 \pmod{p-1}$ then yields

$$0 \equiv t \cdot ((\lambda + k_1 u) + (-\lambda + k_2 u) + (\lambda + k_3 u)) \equiv t\lambda \pmod{p-1}$$

since $c \geq 4$ holds, so λ is a multiple of u and hence u divides all entries of $\mathbf{v}$. Since $\mathbf{v}$ is nonzero, this implies $v_i \geq u \geq 2$ for some $i \in \{1, 2, 3\}$, which yields $w_i > p-1-v_i$. This shows $y \in \operatorname{soc}(Z(A))$ and since the matrix T_I is nonzero, we obtain $Ay \not\subseteq Z(A)$ by Lemma 5.16, which implies $\operatorname{soc}(Z(A)) \ntrianglelefteq A$. □

5.2.5 Applications

To conclude Section 5.2, we use our results on quantum complete intersection algebras to provide counterexamples to some open questions arising in the preceding chapters. The first example demonstrates that for arbitrary symmetric local algebras, $\operatorname{soc}(Z(A))$ is not necessarily a principal ideal if it is an ideal of A.

Remark 5.28. For a finite p-group G, the group algebra FG is local. If G satisfies $\operatorname{soc}(ZFG) \trianglelefteq FG$, then $\operatorname{soc}(ZFG)$ is of the form $\operatorname{soc}(ZFG) = (Z(G)G')^+ \cdot FG$ by Theorem 3.1. In other words, $\operatorname{soc}(ZFG)$ is the principal ideal of FG generated by the central element $(Z(G)G')^+$. A similar statement holds for the group algebras of arbitrary finite groups (see Lemma 4.27).

In contrast, for an arbitrary symmetric local algebra A, the condition $\operatorname{soc}(Z(A)) \trianglelefteq A$ does in general not imply that $\operatorname{soc}(Z(A))$ is of the form As for some $s \in Z(A)$. To see that this is true, let F be an algebraically closed field of odd characteristic p and consider the quantum complete intersection algebra

$$A = F\langle X_1, X_2\rangle / \langle X_1^p, X_2^p, X_2X_1 - qX_1X_2\rangle$$

for some $q \in F^\times$ with $\operatorname{ord}(q) = p-1$. Again, we write x_1 and x_2 for the images of X_1 and X_2 in A, respectively. By Example 5.22, $\operatorname{soc}(Z(A))$ is an ideal of A. Assume that there exists an element $s \in Z(A)$ with $\operatorname{soc}(Z(A)) = As$. By (5.6), we may write

$$s = \sum_{i=1}^{p-1} \alpha_i x_1^{p-1} x_2^i + \sum_{j=1}^{p-2} \beta_j x_1^j x_2^{p-1}$$

for some coefficients $\alpha_1, \ldots, \alpha_{p-1}, \beta_1, \ldots, \beta_{p-2} \in F$. Since $x_1^{p-1}x_2$ and $x_1x_2^{p-1}$ are contained in $\operatorname{soc}(Z(A))$ by Lemma 5.25, we find elements $a_1, a_2 \in A$ with $a_1 s = x_1^{p-1}x_2$ and $a_2 s = x_1 x_2^{p-1}$. Now write $a_i = f_i + j_i$ with $f_i \in F$ and $j_i \in J(A)$ for $i = 1, 2$. This yields

$$x_1^{p-1}x_2 = a_1 s = f_1\alpha_1 x_1^{p-1}x_2 + f_1\beta_1 x_1 x_2^{p-1} + r,$$

where r is a sum of basis elements which contain a factor $x_1^{p-1}x_2^2$ or $x_1^2x_2^{p-1}$. Comparing the coefficients yields $f_1\alpha_1 = 1$ and $f_1\beta_1 = 0$, which implies $\beta_1 = 0$. The analogous argument for $x_1x_2^{p-1}$ yields $f_2\beta_1 = 1$, which is a contradiction. This shows that $\operatorname{soc}(Z(A))$ is not a principal ideal. ◁

Next, we see an example of a symmetric algebra Q which satisfies $\operatorname{soc}(ZFQ) \trianglelefteq Q$ and a (non-symmetric) quotient algebra A of Q in which $\operatorname{soc}(Z(A))$ is not an ideal (compare this to Lemma 1.25).

Example 5.29. Let F be an algebraically closed field of characteristic $p = 5$ and consider the (non-symmetric) local algebra

$$A \coloneqq F\langle X_1, X_2\rangle / \langle X_1^2, X_2^4, X_1X_2 + X_2X_1\rangle.$$

Again, the images of X_1 and X_2 in A are denoted by x_1 and x_2, respectively. Note that an F-basis of A is given by

$$\left\{1, x_2, x_2^2, x_2^3, x_1, x_1x_2, x_1x_2^2, x_1x_2^3\right\}.$$

A short computation shows that $J(Z(A)) = \mathrm{soc}(Z(A)) = F\{x_2^2, x_1x_2^3\}$ is two-dimensional. By multiplying with the basis elements of A, we see that

$$A \cdot J(Z(A)) = A \cdot \mathrm{soc}(Z(A)) = F\left\{x_2^2, x_2^3, x_1x_2^2, x_1x_2^3\right\}$$

is of dimension four, so $J(Z(A)) = \mathrm{soc}(Z(A))$ is not an ideal in A. Now consider the quantum complete intersection algebra

$$Q := F\langle X_1, X_2\rangle / \langle X_1^5, X_2^5, X_1X_2 + X_2X_1\rangle.$$

This algebra is symmetric and satisfies $\mathrm{soc}(Z(Q)) \trianglelefteq Q$ by Lemma 5.27. Note that A can be viewed as a quotient algebra of Q. This demonstrates that for a symmetric algebra Q with $\mathrm{soc}(Z(Q)) \trianglelefteq Q$ and an ideal I of Q, the quotient algebra Q/I does not necessarily have the property $\mathrm{soc}(Z(Q/I)) \trianglelefteq Q/I$ if it is not symmetric. This proves that the prerequisites in Lemma 1.25 are necessary. ◁

5.3 Trivial extension algebras

Let F be an algebraically closed field of arbitrary characteristic and consider a finite-dimensional F-algebra A. In this section, we introduce a certain symmetric algebra $T(A)$, the trivial extension algebra, associated with A. These algebras arise in various contexts in the representation theory of finite-dimensional algebras (for instance, see [16]). We will use them in the next section in order to find symmetric local algebras in which the socle of the center is not an ideal. Here, we exhibit conditions on A which ensure that the Jacobson radical or the socle of the center of $T(A)$ are ideals in $T(A)$, respectively.

We consider the dual space $\mathrm{Hom}_F(A, F)$ of A, that is, the vector space of F-linear forms on A. It becomes an A-A-bimodule by setting

$$(af)(x) := f(xa) \text{ and } (fa)(x) := f(ax)$$

for all $x, a \in A$ and $f \in \mathrm{Hom}_F(A, F)$. This allows us to make the following definition:

Definition 5.30. For a finite-dimensional F-algebra A, we set $T(A) := A \oplus \mathrm{Hom}_F(A, F)$. Endowed with the multiplication law

$$(a, f) \cdot (b, g) := (ab, ag + fb)$$

for all $a, b \in A$ and $f, g \in \mathrm{Hom}_F(A, F)$, the space $T(A)$ becomes an F-algebra, the *trivial extension algebra* of A.

By [47, Example IV.2.7], the trivial extension algebra $T := T(A)$ is symmetric and a corresponding non-degenerate associative symmetric bilinear form is given by

$$\beta\colon T \times T \to F,\ ((a, f), (b, g)) \mapsto f(b) + g(a).$$

It should be noted that as in the case of quantum complete intersection algebras, there exist generalized versions of the above definition in the literature.

From now on until the end of this section, we additionally assume that A is a local algebra. In this case, the corresponding trivial extension algebra T is also local (see [10, Lemma 4.1]). In the following, we view $\mathrm{Hom}_F(A/K(A), F)$ as a subset of $\mathrm{Hom}_F(A, F)$ by identifying the map $f \in \mathrm{Hom}_F(A/K(A), F)$ with $f^* \in \mathrm{Hom}_F(A, F)$, where f^* is defined by $f^*(x) := f(x + K(A))$ for all $x \in A$. By [10, Lemma 4.3], the center of T is given by

$$Z(T) = Z(A) \oplus \mathrm{Hom}_F(A/K(A), F).$$

For its Jacobson radical, we claim that the following holds:

$$J(Z(T)) = J(Z(A)) \oplus \mathrm{Hom}_F(A/K(A), F). \tag{5.7}$$

To verify this identity, we consider elements $j \in J(Z(A))$ and $f \in \mathrm{Hom}_F(A/K(A), F)$. Inductively, one can show that

$$(j, f)^k = (j^k, kj^{k-1}f)$$

holds for any $k \in \mathbb{N}$. Since j is nilpotent, there exists an exponent $n \in \mathbb{N}$ with $j^n = 0$. By the above identity, we then have $(j, f)^{n+1} = 0$ and hence (j, f) is nilpotent, which implies $(j, f) \in J(T) \cap Z(T) = J(Z(T))$. Since $Z(T)$ is local, we have

$$\dim J(Z(T)) = \dim Z(T) - 1 = \dim(J(Z(A)) \oplus \mathrm{Hom}_F(A/K(A), F))$$

and we obtain the equality in (5.7).

Now we answer the question under which conditions the radical $J(Z(T))$ is an ideal of T.

Theorem 5.31. *Let A be a local F-algebra with trivial extension algebra T. Then $J(Z(T))$ is an ideal in T if and only if $J(Z(A))$ and $K(A)$ are ideals in A.*

Proof. First assume that $J(Z(T))$ is an ideal in T. We begin by showing that $J(Z(A))$ is an ideal of A in this case. To this end, let $b \in J(Z(A))$ and consider an arbitrary element $a \in A$. Since $(b, 0) \in J(Z(T))$ holds, we obtain $(ab, 0) = (a, 0) \cdot (b, 0) \in J(Z(T))$, which implies $ab \in J(Z(A))$ by (5.7). This shows that $J(Z(A))$ is an ideal of A. Now we prove that $K(A)$ is an ideal of A. To this end, let $g \in \mathrm{Hom}_F(A/K(A), F)$. The element $(0, g)$ is contained in $J(Z(T))$, so for any $a \in A$, we have $(0, ag) = (a, 0) \cdot (0, g) \in J(Z(T))$. This implies $ag \in \mathrm{Hom}_F(A/K(A), F)$, that is, $g(xa) = 0$ holds for all $x \in K(A)$. Since $g \in \mathrm{Hom}_F(A/K(A), F)$ was an arbitrary element, we obtain

$$xA \subseteq \bigcap_{g \in \mathrm{Hom}_F(A/K(A), F)} \mathrm{Ker}(g) = K(A)$$

and hence $K(A) \cdot A = A \cdot K(A) \subseteq K(A)$ follows (see Lemma 1.7). This shows that $K(A)$ is an ideal of A.

Conversely, assume that $J(Z(A))$ and $K(A)$ are ideals of A and let $(b, g) \in J(Z(T))$. We show that for any $(a, f) \in A$, the product $(a, f) \cdot (b, g) = (ab, ag + bf)$ is contained in $J(Z(T))$. Note that $ab \in J(Z(A))$ holds since $J(Z(A))$ is an ideal of A. For all $x \in K(A)$, we obtain $(ag + fb)(x) = g(xa) + f(bx) = 0$ since we have $xa \in K(A) \subseteq \mathrm{Ker}(g)$ and $bx \in bK(A) = 0$ (see Lemma 1.9 (ii)). Hence $ag + fb$ can be viewed as an element of $\mathrm{Hom}_F(A/K(A), F)$ and we obtain $(ab, ag + bf) \in J(Z(T))$. This implies that $J(Z(T))$ is an ideal of T. □

In the special case that the algebra A is symmetric, $J(Z(T))$ is an ideal of T if and only if A is commutative (see Remark 1.21). Therefore, we mostly focus on non-symmetric algebras as bases for the trivial extension.

In the second part of this section, we investigate the corresponding problem for the socle. As a first step, we determine the structure of $\mathrm{soc}(Z(T))$.

Lemma 5.32. *Set $I := K(A) + A \cdot J(Z(A))$. Then the socle of $Z(T)$ is given by*

$$\mathrm{soc}(Z(T)) = \{(b, g) \in \mathrm{soc}(Z(A)) \times \mathrm{Hom}_F(A, F) \colon Ab \subseteq K(A),\ g(I) = 0\}. \tag{5.8}$$

Proof. Since T is a local algebra of dimension at least two, we have $\mathrm{soc}(Z(T)) \subseteq J(Z(T))$ (see Remark 1.14). First consider an element $t := (b, g) \in \mathrm{soc}(Z(T))$. For any $a \in J(Z(A))$, one has $(a, 0) \in J(Z(T))$ and hence $0 = (a, 0) \cdot t = (ab, ag)$. In particular, this yields $J(Z(A)) \cdot b = 0$, which implies $b \in \mathrm{soc}(Z(A))$. The condition $ag = 0$ is equivalent to $g(xa) = 0$ for all $x \in A$. This yields $g(y) = 0$ for all $y \in A \cdot J(Z(A))$ and hence $g(I) = 0$ follows from the fact that we have $g \in \mathrm{Hom}_F(A/K(A), F)$.

Now consider an arbitrary element $(a, f) \in J(Z(T))$. The condition $0 = (a, f) \cdot t = (ab, ag + fb)$ yields $g(xa) + f(bx) = 0$ for all $x \in A$. Since $xa \in A \cdot J(Z(A)) \subseteq \mathrm{Ker}(g)$ holds by the above, we obtain $f(bx) = 0$. This yields

$$Ab = bA \subseteq \bigcap_{f \in \mathrm{Hom}_F(A/K(A),F)} \mathrm{Ker}(f) = K(A).$$

Now we show the other inclusion in (5.8). To this end, let $b \in \mathrm{soc}(Z(A))$ with $Ab \subseteq K(A)$ and $g \in \mathrm{Hom}_F(A, F)$ with $g(I) = 0$. Note that the element (b, g) is contained in $Z(T)$. Now consider an arbitrary element $(a, f) \in J(Z(T))$. Because of $b \in \mathrm{soc}(Z(A))$, we have $ab = 0$. Moreover, for any $x \in A$, we obtain $(ag + bf)(x) = g(xa) + f(bx) = 0$ since we have $bx = xb \in Ab \subseteq K(A) \subseteq \mathrm{Ker}(f)$ and $xa \in A \cdot J(Z(A)) \subseteq \mathrm{Ker}(g)$. This shows $(a, f) \cdot (b, g) = 0$ and hence $(b, g) \in \mathrm{soc}(Z(T))$. □

We now characterize the trivial extension algebras T in which $\mathrm{soc}(Z(T))$ is an ideal.

Theorem 5.33. *Let A be a local F-algebra with trivial extension algebra T. Then the socle $\mathrm{soc}(Z(T))$ is an ideal of T if and only if the following two conditions hold:*

(i) $I := K(A) + A \cdot J(Z(A))$ is an ideal of A.

(ii) For all $b \in \mathrm{soc}(Z(A))$ with $Ab \subseteq K(A)$, we have $Ab \subseteq Z(A)$.

Proof. Note that the first condition is equivalent to $A \cdot K(A) \subseteq K(A) + A \cdot J(Z(A))$ as $A \cdot J(Z(A))$ is an ideal of A and $A \cdot K(A) = K(A) \cdot A$ holds by Lemma 1.7. First assume that $\mathrm{soc}(Z(T))$ is an ideal in T and consider an element $t := (b, g) \in \mathrm{soc}(Z(T))$. The previous lemma yields $b \in \operatorname{soc}(Z(A))$ and $Ab \subseteq K(A)$. For an arbitrary element $(a, f) \in T$, we have

$$(ab, ag + fb) = (a, f) \cdot (b, g) \in \operatorname{soc}(Z(T)).$$

In particular, this implies $ab \in \operatorname{soc}(Z(A))$, so we obtain $Ab \subseteq Z(A)$. By Remark 1.10, this yields $K(A) \cdot b = 0$ and hence we have $I \cdot b = b \cdot I = 0$. Moreover, the preceding result yields $(ag + fb)(I) = 0$, that is, we obtain $g(xa) = g(xa) + f(bx) = 0$ for all $x \in I$ since $bx \in bI = 0$ holds. This yields

$$A \cdot I = I \cdot A \subseteq \bigcap_{\substack{g \in \operatorname{Hom}_F(A,F),\\ I \subseteq \operatorname{Ker}(g)}} \operatorname{Ker}(g) = I.$$

Conversely, assume that A satisfies the conditions (i) and (ii). Consider $(b, g) \in \operatorname{soc}(Z(T))$ and let $(a, f) \in T$ be an arbitrary element. We show that $(ab, ag + fb) = (a, f) \cdot (b, g)$ is contained in $\operatorname{soc}(Z(T))$ by using the characterization given in (5.8). By (ii), we have $ab \in Z(A)$, which implies $ab \in \operatorname{soc}(Z(A))$. Moreover, it follows that $A(ab) \subseteq Ab \subseteq K(A)$. It remains to show that I is contained in $\operatorname{Ker}(ag + fb)$. Note that for any $x \in I$, we have $(ag + fb)(x) = g(xa) + f(bx) = f(bx)$ since $xa \in I \subseteq \operatorname{Ker}(g)$ holds by (i) and (5.8). First let $x \in K(A)$. By (5.8), we have $Ab \subseteq K(A)$, which implies $Ab \subseteq Z(A)$ by (ii). Remark 1.10 then yields $bx \in b \cdot K(A) = 0$ and hence $f(bx) = 0$. For $x \in A \cdot J(Z(A))$, we have $bx = 0$ since $b \in \operatorname{soc}(Z(A))$ holds. Hence $f(bx) = 0$ follows as well. This shows $(ag + fb)(I) = 0$, so $(a, f) \cdot (b, g)$ is contained in $\operatorname{soc}(Z(T))$ by (5.8) and hence $\operatorname{soc}(Z(T))$ is an ideal of T as claimed. □

In this section, we related the properties $J(Z(T)) \trianglelefteq T$ and $\operatorname{soc}(Z(T)) \trianglelefteq T$ to certain conditions on the basis algebra A of the trivial extension. In the next section, we use these criteria to construct a symmetric local algebra T of small dimension in which $\operatorname{soc}(Z(T))$ is not an ideal.

5.4 Small counterexamples

Throughout this section, F is assumed to be an algebraically closed field of arbitrary characteristic and A is a finite-dimensional symmetric local F-algebra. We are interested in finding examples of such algebras A of minimal dimension which satisfy $J(Z(A)) \ntrianglelefteq A$ or $\operatorname{soc}(Z(A)) \ntrianglelefteq A$, respectively.

5.4.1 Jacobson radical

In this section, we investigate the problem for the Jacobson radical $J(Z(A))$. In [39, Theorem 3.2], it is shown that $J(Z(A)) \trianglelefteq A$ holds for every symmetric local F-algebra A of dimension at most ten. Here, we refine this bound by showing that this statement remains valid for $\dim A = 11$ and providing an example of a twelve-dimensional symmetric local algebra A with $J(Z(A)) \ntrianglelefteq A$.

Throughout, we abbreviate the powers $J^i(A) := J(A)^i$ of the Jacobson radical by J^i. The main result in this section is the following theorem:

Theorem 5.34. *Let A be a symmetric local F-algebra. If* $\dim(A) \leq 11$ *holds, then* $J(Z(A))$ *is an ideal in A.*

Proof. By [39, Theorem 3.2], the statement holds for $\dim A \leq 10$, so we restrict our investigation to the case $\dim A = 11$. Suppose that $J(Z(A))$ is not an ideal of A. In particular, A is not commutative. Theorems 5.5 and 5.6 yield $\dim Z(A) \geq 6$, and since $\dim A \geq \dim Z(A)+3$ holds by Lemma 5.8, we have $\dim Z(A) \in \{6,7,8\}$. By Remark 1.10, there exists an element $z \in J(Z(A))$ with $z \cdot K(A) \neq 0$, that is, $K(A) \not\subseteq (Az)^\perp$. Setting $A' := A/(Az)^\perp$, this algebra is symmetric and local (see Lemma 1.22). Moreover, we have $K(A') = K(A) + (Az)^\perp/(Az)^\perp \neq 0$ (see Lemma 1.8), so A' is not commutative. Again, Theorems 5.5 and 5.6 yield $\dim A' = 8$ or $\dim Z(A') \geq 6$. In the latter case, Lemma 5.8 implies $\dim A' \geq 9$. In any case, we obtain $8 \leq \dim A' = \dim A - \dim (Az)^\perp = \dim Az$. Since $z \in J$ holds, we have $Az \subseteq J \subsetneq A$ and hence $\dim Az \in \{8,9,10\}$.

If $\dim Az = 10$ holds, then we have $J = Az$. Writing $A = F\cdot 1 \oplus J$ yields $J = Az = Fz + Jz$, so $J = Fz + J^2$. This shows that the algebra A is generated by $\{1, z\}$ and hence it is commutative, which is a contradiction. Now suppose $\dim Az = 9$. Since $z \in J$ holds, we have $Jz \subseteq J^2$ and hence $\dim J^2 \geq 8$. On the other hand, Lemma 5.4 yields $\dim J/J^2 \geq 2$, which implies $\dim J^2 = 8$. In particular, z is not contained in J^2. Hence there exists some element $a \in J$ with $J = F\{a, z\} + J^2$. This means that A is generated, as an algebra, by a subset of $\{1, a, z\}$, so A is commutative. This is a contradiction.

It follows that $\dim Az = \dim A' = 8$ holds, which yields $\dim (Az)^\perp = 3$. In particular, Az is not contained in $(Az)^\perp$, which implies $z \notin (Az)^\perp$. Since A' is not commutative, we obtain $5 \leq \dim Z(A') \leq \dim A' - 3 = 5$ (see Theorem 5.5 and Lemma 5.8). This yields $\dim Z(A') = 5$ and hence we have $\dim K(A') = \dim Z(A')^\perp = 3$. Moreover, note that the restriction $\dim J/J^2 \geq 2$ (see Lemma 5.4) implies $\dim J^2 \leq 8$. On the other hand, we have $Jz \subseteq J^2$. Since $Az = Fz \oplus Jz$ holds, we obtain $\dim Jz = 7$ and hence $\dim J^2 \in \{7, 8\}$. Note that in the case $\dim J^2 = 7 = \dim Jz$, the element z is not contained in J^2.

The two possibilities for the dimensions of the Loewy layers of A' are given in Theorem 5.6 (ii). Assume that the Loewy structure of A' is of type (a), so $\dim J^2(A') = 4$ holds. Since we have

$$J^2(A') = J^2 + (Az)^\perp/(Az)^\perp \cong J^2/J^2 \cap (Az)^\perp, \tag{5.9}$$

we obtain

$$\dim J^2 = \dim J^2(A') + \dim J^2 \cap (Az)^\perp \leq \dim J^2(A') + \dim (Az)^\perp = 7.$$

By the above, this implies $J^2 = Jz$ and $\dim J^2 = 7$. As we already observed that $z \notin J^2$ holds in this case, there exist elements $a, b \in J$ with $J = F\{z, a, b\} + J^2$. We obtain $K(A) \subseteq F[a, b] + J^3$ and hence

$$\dim K(A) \cap J^3 \geq \dim K(A) - 1. \tag{5.10}$$

Let $s \in J\backslash(Az)^\perp$ with $s + (Az)^\perp \in \mathrm{soc}(A')$. In particular, we have $s \notin \mathrm{soc}(A) \subseteq (Az)^\perp$ and since $K(A') \cap \mathrm{soc}(A') = 0$ holds by Lemma 5.3 (iii), we obtain $s \notin K(A)$ by Lemma 1.8. By

Theorem 5.6 (ii), $J^3(A')$ is one-dimensional, which yields $J^3(A') = \operatorname{soc}(A')$ by Lemma 5.3. Therefore, we can choose $s \in J^3$. This implies

$$\dim J^3 \geq \dim(Fs \oplus \operatorname{soc}(A) \oplus K(A) \cap J^3) = \dim K(A) \cap J^3 + 2.$$

Together with (5.10), we obtain

$$\dim K(A) \leq \dim K(A) \cap J^3 + 1 \leq \dim J^3 - 1. \tag{5.11}$$

Recall that $\dim Z(A) \in \{6, 7, 8\}$ holds. If we have $\dim Z(A) \in \{6, 7\}$, then $\dim K(A) = 11 - \dim Z(A) \geq 4$ follows and hence $\dim J^3 \geq 5$ holds. This yields $\dim J^2/J^3 \leq 2$, which is a contradiction to $\dim J^2(A')/J^3(A') = 3$. Hence we have $\dim Z(A) = 8$ and $\dim K(A) = 3 = \dim K(A')$, which implies $K(A) \cap (Az)^\perp = 0$. By (5.10), we have $\dim K(A) \cap J^3 \geq 2$, which yields $\dim J^3(A') \geq 2$ since we have $(K(A) \cap J^3) \cap (Az)^\perp = 0$. This is a contradiction to $J^3(A')$ being one-dimensional, so this situation cannot arise.

Now assume that A' has a Loewy structure of type (b) in Theorem 5.6 (ii). In particular, we have $\dim J^2(A') = 5$ and $\dim J^3(A') = 3$. If $\dim J^2 = 7$, so $J^2 = Jz$, holds, then $\dim J^2 \cap (Az)^\perp = 2$ follows by (5.9). Since we have $z \notin J^2 \cup (Az)^\perp$, we find elements $\ell \in (Az)^\perp$ and $a \in J$ with $J = F\{z, \ell, a\} + J^2$. This yields $K(A) \subseteq F[\ell, a] + J^3$ and hence $K(A') \subseteq J^3(A')$. Since $\dim K(A') = 3 = \dim J^3(A')$ holds, we obtain equality, which is a contradiction to Remark 1.21 since A' is a non-commutative symmetric algebra. Hence we have $\dim J^2 = 8$. If $z \in J^2$ holds, then we have $Jz \subseteq J^3$, which implies $\dim J^3 \geq 7$ and hence $\dim J^2/J^3 \leq 1$. This is a contradiction to $J \not\subseteq Z(A)$ (see Lemma 5.4). Otherwise, there exists an element $a \in J$ with $J = \{z, a\} + J^2$. This implies that A is generated by $\{1, z, a\}$ as an algebra and hence A is commutative, which is a contradiction. Thus A cannot have a Loewy structure of type (b) in Theorem 5.6, either. We conclude that $J(Z(A))$ is an ideal of A if $\dim A = 11$ holds. □

Now we complete this derivation by presenting an example of a symmetric local algebra A of dimension twelve in which $J(Z(A))$ is not an ideal.

Example 5.35. Let F be an algebraically closed field of odd characteristic p. We consider the unitary subalgebra A of $\operatorname{Mat}_{12}(F)$ generated by the matrices

$$M = \begin{pmatrix}
. & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & .
\end{pmatrix}$$

and

$$N = \begin{pmatrix} . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . & . & . \\ . & -1 & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & -1 & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & -1 & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & 1 & . \end{pmatrix}.$$

For the sake of readability, zero entries are represented by dots. A short computation shows $M^7 = M^5N = 0$. Furthermore, we have $NM = -MN$ and $N^2 = M^2$. Hence the set

$$B := \{\mathbb{1}, M, N, M^2, MN, M^3, M^2N, M^4, M^3N, M^5, M^4N, M^6\}$$

generates A as an F-vector space and we can verify directly that these elements are linearly independent, so B forms an F-basis of A. In particular, we obtain $\dim A = 12$. Since all nontrivial basis elements are nilpotent, the algebra A is local.

Let $s \in \operatorname{soc}(A)$ and write $s = \sum_{v \in B} c_v v$ with coefficients $c_v \in F$. The condition $sM^6 = 0$ then translates to $c_{\mathbb{1}} = 0$ since all other products vanish. Due to $M^5N = M^7 = 0$, we then obtain

$$0 = s \cdot M^5 = c_M \cdot M^6,$$

so $c_M = 0$, and

$$0 = s \cdot M^4N = c_N \cdot M^6,$$

which yields $c_N = 0$. Continuing this way, we obtain $s = c_{M^6} \cdot M^6$. Conversely, M^6 annihilates all basis elements in B apart from the identity matrix and hence we have $M^6 \in \operatorname{soc}(A)$. This shows that $\operatorname{soc}(A) = FM^6$ is one-dimensional.

Next, we consider the commutator space of A. Recall that the matrices M and N satisfy the identities $NM = -MN$ and $N^2 = M^2$. For $i, j \in \{0, \dots, 5\}$, we obtain

$$[M^i, M^jN] = M^{i+j}N - M^jN \cdot M^i = M^{i+j}N - (-1)^i M^{i+j}N = (1 - (-1)^i) \cdot M^{i+j}N.$$

Moreover, we have

$$[M^iN, M^jN] = M^iN \cdot M^jN - M^jN \cdot M^iN = ((-1)^j - (-1)^i) \cdot M^{i+j+2}.$$

Note that the last term is zero if i and j have the same parity, that is, if $i + j + 2$ is even. This shows that $K(A)$ is spanned by the set $\{MN, M^2N, M^3N, M^4N, M^3, M^5\}$ and hence we obtain $\dim K(A) = 6$. In particular, note that $K(A) \cap \operatorname{soc}(A) = 0$ holds.

Thus we can define a linear form $\lambda\colon A \to F$ by setting $\lambda(M^6) = 1$, $\lambda(v) = 0$ for $v \in K(A)$, choosing arbitrary values of λ for the remaining elements in B and extending this linearly to A. Then $\lambda(ab) - \lambda(ba) = \lambda(ab - ba) = \lambda([a, b]) = 0$ holds and hence λ is symmetric. Moreover, note that for any $0 \neq c \in A$, there exists an element $d \in A$ with $dc = fM^6$ for some $f \in F^\times$ and hence $\mathrm{soc}(A)$ is contained in every nontrivial left ideal I of A. In particular, this implies $I \not\subseteq \mathrm{Ker}(\lambda)$. By [47, Theorem IV.2.2], this shows that (A, λ) is a symmetric local algebra.

By direct computation, we show that the set $\{\mathbb{1}, M^2, M^4, M^5, M^4N, M^6\}$ is contained in $Z(A)$. These elements even form an F-basis of the center since we have $\dim Z(A) = \dim A - \dim K(A) = 6$. As $Z(A)$ is a local algebra, we have $M^2 \in J(Z(A))$. However, note that $M \cdot M^2 = M^3$ is not contained in $Z(A)$, which implies that $J(Z(A))$ is not an ideal of A. ◁

The example shows that the bound given in Theorem 5.34 is tight.

5.4.2 Socle

In this section, we investigate the analogous problem for the socle. Again, all occurring F-algebras are assumed to be finite-dimensional. First we show that $\mathrm{soc}(Z(A)) \trianglelefteq A$ holds for all symmetric local algebras A of dimension at most 16. In the second part of this section, we prove that there exists a trivial extension algebra T of dimension 20 with $\mathrm{soc}(Z(T)) \ntrianglelefteq T$. Additionally, we show that this is the optimal bound achievable in this way since $\mathrm{soc}(Z(T)) \trianglelefteq T$ holds for all trivial extension algebras T of dimension at most 18. Summarizing, this yields $\dim A \in \{17, 18, 19, 20\}$ for a symmetric local algebra A of minimal dimension in which $\mathrm{soc}(Z(A))$ is not an ideal.

We first collect some properties of these algebras:

Remark 5.36. Let A be a symmetric local algebra in which $\mathrm{soc}(Z(A))$ is not an ideal. By Remark 1.10, there exists an element $z \in \mathrm{soc}(Z(A))$ with $z \cdot K(A) \neq 0$ and we set $A' := A/(Az)^\perp$ as before. By Lemma 1.22, A' is a symmetric local algebra. As in the proof of Theorem 5.34, it follows that A' is not commutative, which yields $\dim Z(A') \geq 5$ (see Theorems 5.5 and 5.6) as well as

$$\dim Az = \dim A' \geq \dim Z(A') + 3 \geq 8 \tag{5.12}$$

by Lemma 5.8. Since $z \in \mathrm{soc}(Z(A)) \subseteq J(Z(A))$ holds (see Remark (1.14)), we obtain $z^2 = 0$. In particular, we have $Az \subseteq (Az)^\perp$ and hence Lemma 1.19 yields

$$\dim A = \dim Az + \dim (Az)^\perp \geq 2 \cdot \dim Az \geq 16. \tag{5.13}$$

Moreover, A not being commutative implies $\dim Z(A) \geq 6$ by Theorems 5.5 and 5.6. ◁

Our next aim is to exclude the possibility that A has dimension 16.

Theorem 5.37. *Let A be a symmetric local F-algebra. If $\dim(A) \leq 16$ holds, then $\mathrm{soc}(Z(A))$ is an ideal in A.*

Proof. Assume that A is a symmetric local algebra of dimension at most 16 in which $\mathrm{soc}(Z(A))$ is not an ideal. Remark 5.36 then yields $\dim A = 16$. Again, let $z \in A$ with

$z \cdot K(A) \neq 0$ and set $A' \coloneqq A/(Az)^{\perp}$. In this case, we obtain equality in (5.13) and hence

$$\dim A' = \dim Az = \dim (Az)^{\perp} = 8,$$

which implies $Az = (Az)^{\perp}$. Furthermore, we obtain $\dim Z(A') = 5$ by Theorem 5.5 and Lemma 5.8, which yields $3 = \dim Z(A')^{\perp} = \dim K(A') = \dim K(A) + (Az)^{\perp}/(Az)^{\perp}$. This implies $\dim K(A) + (Az)^{\perp} = 11$ and hence

$$5 = \dim (K(A) + (Az)^{\perp})^{\perp} = \dim Z(A) \cap Az = \dim Z(A) \cap (Az)^{\perp}.$$

Since $J(Z(A))$ is contained in $Z(A) \cap (Az)^{\perp}$, we find $\dim Z(A) - 1 = \dim J(Z(A)) \leq 5$ and hence $\dim Z(A) \leq 6$. Remark 5.36 yields $\dim Z(A) = 6$ and hence $\dim K(A) = 10$.

We first show $\dim J^2 = 12$. The two possibilities for the Loewy structure of A' are given in Theorem 5.6 (ii). In both cases, we have $J^3(A') \neq 0$ and hence $\operatorname{soc}(A') \subseteq J^3(A')$. Let $s \in A$ with $0 \neq s + (Az)^{\perp} \in \operatorname{soc}(A')$. In particular, we may choose $s \in J^3$ and we have $s \notin K(A) + (Az)^{\perp}$ since $K(A') \cap \operatorname{soc}(A') = 0$ holds by Lemma 5.3. This shows that the sum of subspaces $K(A) + \operatorname{soc}(A) + Fs \subseteq J^2$ is in fact direct. It follows that

$$\dim J^2 \geq \dim K(A) + \dim \operatorname{soc}(A) + \dim Fs = \dim K(A) + 2 = 12,$$

which yields $\dim J/J^2 \leq 3$. By Lemma 5.4, we have $\dim J/J^2 \geq 2$ and by the main theorem in [15], $\dim J/J^2 = 2$ implies $\dim A \leq 12$. This shows that $\dim J/J^2 = 3$ and hence $\dim J^2 = 12$ holds.

Now we prove that $\dim J^3 = 9$ and $\dim J^4 = 7$ hold. We have $\dim K(A) \cap (Az)^{\perp} = \dim K(A) - \dim K(A') = 7$. Since $K(A)$ and $\operatorname{soc}(A)$ intersect trivially (see Lemma 5.3), comparing the dimensions yields $(Az)^{\perp} = K(A) \cap (Az)^{\perp} \oplus \operatorname{soc}(A)$. In particular, we obtain $(Az)^{\perp} \subseteq J^2$, which yields $\dim J^2(A') = \dim J^2 - \dim (Az)^{\perp} = 4$. This shows that the Loewy structure of A' is of type (a) in Theorem 5.6 (ii). We then obtain $\dim J^3(A') = 1$ and hence $\dim J^2/J^3 \geq \dim J^2(A')/J^3(A') = 3$. On the other hand, writing $J = F\{a, b, c\} + J^2$ for elements $a, b, c \in J$, we obtain $K(A) \subseteq F\{[a, b], [b, c], [a, c]\} + J^3$ and hence

$$\dim K(A) \leq \dim K(A) \cap J^3 + 3. \tag{5.14}$$

Since $\operatorname{soc}(A) \subseteq J^3$ and $s \in J^3$ hold, we obtain

$$\dim J^3 \geq \dim K(A) \cap J^3 \oplus \operatorname{soc}(A) \oplus Fs = \dim K(A) \cap J^3 + 2. \tag{5.15}$$

Combining (5.14) and (5.15) yields $\dim J^3 \geq \dim K(A) - 1 = 9$, which implies $\dim J^3 = 9$ by the above. Because of $\dim J^3 \cap (Az)^{\perp} = \dim J^3 - \dim J^3(A') = 8$, we obtain $(Az)^{\perp} \subseteq J^3$. This yields $J^3 = (Az)^{\perp} \oplus Fs$ as s is not contained in $(Az)^{\perp}$. In particular, this implies $z \in J^3$ and hence $Jz \subseteq J^4$, which shows that $\dim J^4 \geq 7$ holds. By Lemma 5.4, we have $\dim J^3/J^4 \geq 2$ and hence $J^4 = Jz$ follows.

In particular, we obtain $J^3 \cdot J^4 = ((Az)^{\perp} \oplus Fs) \cdot Jz = 0$ since we have $sJ \subseteq (Az)^{\perp}$. Lemma 5.3 (iii) now implies $\dim J^6 \leq 1$, which yields $\dim J^4/J^5 \geq 3$ or $\dim J^5/J^6 \geq 3$. By the derivation in [19, Chapter 2.2], which is summarized in Lemma A.1, this is impossible. Hence every symmetric local algebra A of dimension 16 satisfies $\operatorname{soc}(Z(A)) \trianglelefteq A$. □

In the remainder of this section, we find a local F-algebra A of minimal dimension such that $\operatorname{soc}(Z(T)) \ntrianglelefteq T$ holds for the corresponding trivial intersection algebra $T := T(A)$ (see Section 5.3). First, we show that $\operatorname{soc}(Z(T))$ is an ideal of T if $\dim A \leq 9$ holds. To this end, we check the criteria stated in Theorem 5.33. By Theorem 5.37, it only remains to consider the case $\dim T = 18$, that is, $\dim A = 9$. As before, we set $J := J(A)$.

Proposition 5.38. *Let A be a local F-algebra with $\dim A = 9$. In this case, the subspace $I := K(A) + A \cdot J(Z(A))$ is an ideal of A.*

Proof. Suppose that I is not an ideal of A, that is, $A \cdot K(A) = K(A) \cdot A$ is not contained in I. Note that we have $A = F \cdot 1 \oplus J$ as well as $K(A) \subseteq J^2$ (see (5.1)). If $J^3 \subseteq Z(A)$ holds, then this implies

$$A \cdot K(A) \subseteq K(A) + J \cdot K(A) \subseteq K(A) + J^3 \subseteq K(A) + A \cdot J(Z(A)) = I,$$

which contradicts the assumption. Hence J^3 is not contained in $Z(A)$. In particular, the quotient spaces J/J^2, J^2/J^3 and J^3/J^4 are of dimension at least two by Lemma 5.2. Moreover, this yields $0 \neq [A, J^3] = [J, J^3] \subseteq J^4$ and hence $J^4 \neq 0$. In particular, we obtain $\dim J/J^2 \leq 3$.

First assume $\dim J/J^2 = 2$, so J can be written in the form $F\{a, b\} + J^2$ for some elements $a, b \in J$. The proof of [10, Lemma 3.5] can be adapted to this situation (see Lemma A.2), showing that a and b can be chosen in such a way that the elements $a^2 + J^3$ and $ab + J^3$ form an F-basis of J^2/J^3 (by possibly replacing A by its opposite algebra). Lemma 5.1 then yields $\dim J^i/J^{i+1} \leq 2$ for $i \geq 3$. By the above, the only possibility for the Loewy structure is $\dim J = 8$, $\dim J^2 = 6$, $\dim J^3 = 4$ and $\dim J^4 = 2$. We either have $J^5 = 0$, which implies $J^4 \subseteq Z(A)$ as before, or $\dim J^5 = 1$, which implies $\dim J^4/J^5 = 1$ and hence $J^4 \subseteq Z(A)$ by Lemma 5.2. In any case, we therefore obtain $J^4 \subseteq Z(A)$. Note that $K(A) \subseteq F[a, b] + J^3$ holds. Moreover, we have

$$a \cdot [a, b] = a \cdot (ab - ba) = a(ab) - (ab)a = [a, ab] \in I \tag{5.16}$$

and similarly we obtain $b \cdot [a, b] = [ba, b] \in I$. This implies

$$J \cdot K(A) \subseteq F\{a[a, b], b[a, b]\} + J^4 \subseteq I$$

and hence $A \cdot K(A) \subseteq K(A) + J \cdot K(A) \subseteq I$, which contradicts the assumption.

Now we assume $\dim J/J^2 = 3$, that is, we have $J = F\{a, b, c\} + J^2$ for some elements $a, b, c \in J$. Since J^2/J^3 and J^3/J^4 are of dimension at least two and J^4 is nonzero, we obtain $\dim J^3 = 3$ and $\dim J^4 = 1$. By assumption, there exists an element $k \in K(A)$ with $Ak \nsubseteq I$. Since $K(A) \subseteq F\{[a, b], [a, c], [b, c]\} + J^3$ holds, k can be expressed in the form

$$k = f_1[a, b] + f_2[a, c] + f_3[b, c] + j$$

with $f_1, f_2, f_3 \in F$ and $j \in J^3 \cap K(A)$. Note that $Aj \subseteq Fj + J^4 \subseteq I$ holds. Without loss of generality, we may therefore assume $A[a, b] \nsubseteq I$. Since $a[a, b]$ and $b[a, b]$ are contained in $K(A)$ (see (5.16)) and $J^2[a, b] \subseteq J^4 \subseteq I$ holds, we obtain $c[a, b] \notin I$. Since $\dim J^2/J^3 = 2$ holds, the elements $[a, b] + J^3$, $[a, c] + J^3$ and $[b, c] + J^3$ are linearly dependent in J^2/J^3, so there exists a nonzero tuple $(\lambda_0, \lambda_1, \lambda_2) \in F^3$ with

$$\lambda_0[a,b] + \lambda_1[a,c] + \lambda_2[b,c] \in J^3.$$

First assume $\lambda_0 \neq 0$. By scaling, we may assume $[a,b] \in \lambda_1[a,c] + \lambda_2[b,c] + J^3$. In this case, we obtain

$$c[a,b] \in \lambda_1 c[a,c] + \lambda_2 c[b,c] + J^4 \subseteq I,$$

since $c[a,c]$ and $c[b,c]$ are contained in $K(A)$ (see (5.16)). This is a contradiction, so $\lambda_0 = 0$ follows. This yields $[\lambda_1 a + \lambda_2 b, c] = \lambda_1[a,c] + \lambda_2[b,c] \in J^3$. By scaling and possibly exchanging a and b, we may assume $\lambda_2 = 1$. This yields

$$\begin{aligned} c[a, \lambda_1 a + b] &= ca(\lambda_1 a + b) - c(\lambda_1 a + b)a \\ &\equiv ca(\lambda_1 a + b) - (\lambda_1 a + b)ca \\ &= [ca, \lambda_1 a + b] \pmod{J^4}. \end{aligned}$$

Now we have

$$c[a,b] = c[a, \lambda_1 a + b] \in [ca, \lambda_1 a + b] + J^4 \subseteq K(A) + J^4 \subseteq I,$$

which is a contradiction. This shows that $A \cdot I \subseteq I$ holds and hence I is an ideal of A. □

Proposition 5.39. *Let A be a local F-algebra with* $\dim A = 9$. *For every $z \in \operatorname{soc}(Z(A))$ with $Az \subseteq K(A)$, we have $Az \subseteq Z(A)$.*

Proof. Assume that there exists an element $z \in \operatorname{soc}(Z(A))$ with $Az \subseteq K(A)$ such that $Az \nsubseteq Z(A)$ holds. The latter property is equivalent to $K(A) \cdot z \neq 0$ by Remark 1.10. Since we have $z \in K(A) \subseteq J^2$ (see (5.1)), we obtain $J^3 \nsubseteq Z(A)$ and $J^4 \neq 0$. As in the proof of Proposition 5.38, it follows that J/J^2, J^2/J^3 and J^3/J^4 are at least two-dimensional. In particular, assuming $\dim J/J^2 \geq 4$ yields $\dim J^2 \leq 4$, $\dim J^3 \leq 2$ and $J^4 = 0$, which is a contradiction. In the following, we therefore distinguish the cases $\dim J/J^2 = 2$ and $\dim J/J^2 = 3$.

First assume $\dim J/J^2 = 2$, that is, we have $\dim J^2 = 6$. We write $J = F\{a,b\} + J^2$ for some elements $a, b \in J$. By the above, we have $\dim J^3 \leq 4$ and $\dim J^4 \leq 2$. As in the proof of Proposition 5.38, we obtain $J^4 \subseteq Z(A)$. Since Az is not contained in $Z(A)$, this implies $z \notin J^3$. Since we have $K(A) \subseteq F[a,b] + J^3$, the element z can be expressed in the form $z = f[a,b] + j$ for some $0 \neq f \in F$ and some $j \in J^3$. By scaling, we may assume $z = [a,b] + j$. Since we have $Az \nsubseteq Z(A)$ and $J^2 z \subseteq J^4 \subseteq Z(A)$, it follows that $az \notin Z(A)$ or $bz \notin Z(A)$ holds. Without loss of generality, we assume $az \notin Z(A)$. As $[az, a] = 0$ holds, we must have $z[a,b] = [az, b] \neq 0$ because A is generated, as an algebra, by a subset of $\{1, a, b\}$. We have $0 = z^2 = z[a,b] + zj$, which implies $zj \neq 0$ and hence $J^5 \neq 0$. This yields $\dim J^3 = 4$, $\dim J^4 = 2$ and $\dim J^5 = 1$. Note that $bz \notin Z(A)$ follows because of $[a, bz] = z[a,b] \neq 0$.

In particular, az and bz are not contained in J^4. Assume $J^3 = F\{az, bz\} + J^4$ and write $j = f_1 az + f_2 bz + j'$ with $f_1, f_2 \in F$ and $j' \in J^4$. But then we obtain $zj = 0$ since $z^2 = 0$ as well as $zj' \in zJ^4 \subseteq J^6 = 0$ hold, which is a contradiction to $zj \neq 0$. This shows that $az + J^4$ and $bz + J^4$ are linearly dependent in J^3/J^4. Without loss of generality, we may assume that $az = \lambda_1 bz + j'$ holds for some $\lambda_1 \in F$ and some $j' \in J^4$. This implies $z(a - \lambda_1 b) = j'$ and hence $z[a,b] = z[a - \lambda_1 b, b] = [j', b] = 0$ due to $j' \in J^4 \subseteq Z(A)$. This is a contradiction to $z[a,b] \neq 0$, so the case $\dim J/J^2 = 2$ cannot arise.

Now assume $\dim J/J^2 = 3$, so $\dim J^2 = 5$, and write $J = F\{a,b,c\} + J^2$ for some $a,b,c \in J$. Similarly to the above derivation, we obtain $\dim J^3 = 3$ and $\dim J^4 = 1$. Again, we have $J^4 \subseteq Z(A)$, which yields $z \in J^2 \backslash J^3$ and hence $K(A) \not\subseteq J^3$. Suppose that $\dim F\{[a,b],[a,c],[b,c]\} + J^3 = \dim J^3 + 1$ holds. Without loss of generality, we may assume $[a,b] \notin J^3$ and write $[a,c] + J^3 = \lambda_1[a,b] + J^3$ and $[b,c] + J^3 = -\lambda_2[a,b] + J^3$ for some $\lambda_1, \lambda_2 \in F$. Replacing c by $\bar{c} := c - \lambda_1 b - \lambda_2 a$ yields $[a,\bar{c}] = [b,\bar{c}] \in J^3$ and $J = F\{a,b,\bar{c}\} + J^2$. We then obtain $K(A) \subseteq F[a,b] + J^3$, so by scaling, we may write $z = [a,b] + j$ for some $j \in J^3$. But then $0 = z^2 = z[a,b] + zj = z[a,b]$ follows since $zj \in J^5 = 0$ holds. This yields $z \cdot K(A) \subseteq Fz[a,b] + J^5 = 0$, which is a contradiction.

Hence we have $\dim F\{[a,b],[b,c],[a,c]\} + J^3/J^3 = 2 = \dim J^2/J^3$. As $z \notin J^3$ holds, we may assume without loss of generality that $[a,b] + J^3$ and $z + J^3$ form a basis of J^2/J^3. If $z[a,b] = 0$ holds, we obtain $K(A)z \subseteq J^2 z \subseteq F\{[a,b]z, z^2\} + J^5 = 0$, which contradicts our assumption. Hence we have $z[a,b] \neq 0$, which implies $az, bz \notin Z(A)$. Suppose that $az + J^4$ and $bz + J^4$ are linearly dependent in J^3/J^4. Without loss of generality, we may assume $bz = \lambda_1 az + j$ for some $\lambda_1 \in F$ and some $j \in J^4$. Then we obtain $z[a,b] = [a,bz] = [a, \lambda_1 az + j] = 0$ since we have $j \in J^4 \subseteq Z(A)$. Again, this is a contradiction, so we have $J^3 = F\{az, bz\} + J^4$. By assumption, we obtain $J^2 = F\{[a,b], z\} + J^3 \subseteq F[a,b] + Az \subseteq K(A)$. Together with (5.1), this yields $K(A) = J^2$. The derivation in [10, page 15] can also be used in our present context to show that this is impossible (see Lemma A.3). □

Combining the two preceding results, we obtain the following:

Theorem 5.40. *Let A be a local F-algebra with trivial extension algebra $T := T(A)$. If $\dim A \leq 9$, that is, $\dim T \leq 18$ holds, then $\mathrm{soc}(Z(T))$ is an ideal of T.*

Proof. For $\dim A \leq 8$, we have $\dim T \leq 16$ and hence $\mathrm{soc}(Z(T)) \trianglelefteq T$ holds by Theorem 5.37. Now let $\dim A = 9$. By the preceding results, $K(A) + A \cdot J(Z(A))$ is an ideal of A and $Az \subseteq Z(A)$ follows for all $z \in \mathrm{soc}(Z(A))$ with $Az \subseteq K(A)$. Therefore, the conditions (i) and (ii) stated in Theorem 5.33 are fulfilled and hence $\mathrm{soc}(Z(T))$ is an ideal of T. □

In other words, any local trivial extension algebra $T := T(A)$ with $\mathrm{soc}(Z(T)) \ntrianglelefteq T$ is of dimension at least 20. The following example shows that this bound is tight.

Example 5.41. Let F be an algebraically closed field of characteristic $p = 2$. We consider the unitary subalgebra A of $\mathrm{Mat}_{10}(F)$ generated by the matrices

$$M = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot \end{pmatrix}.$$

Again, zero entries are represented by dots. One can check that these matrices satisfy the relations $M^6 = N^2 = 0$ as well as

$$NM = M^2 + MN + M^3 + M^2N.$$

As an F-vector space, the algebra A is therefore spanned by elements of the form $M^{\ell_1}N^{\ell_2}$ with $\ell_1 \in \{0, \dots, 5\}$ and $\ell_2 \in \{0, 1\}$. Moreover, we can check directly that $M^4N = M^5$ and hence $M^5N = M^6 = 0$ holds, which yields $\dim A \leq 10$. On the other hand, we can verify that the elements in

$$B := \left\{ M^{\ell_1}N^{\ell_2} \colon \ell_1 \in \{0, \dots, 4\},\ \ell_2 \in \{0, 1\} \right\}$$

are linearly independent and hence form a basis for A. This shows that A is of dimension ten. Note that A is local since all nontrivial elements in B are nilpotent.

As before, we set $J := J(A)$. We have $M^3N \in J^4$ and hence $J^4 \not\subseteq Z(A)$. By Lemma 5.2, this yields $\dim J^i/J^{i+1} \geq 2$ for $i = 1, \dots, 4$. On the other hand, we have $M^5 \in J^5$ and hence $\dim J^5 \geq 1$. This is only possible if $\dim J^i/J^{i+1} = 2$ holds for $i = 1, \dots, 4$. In particular, it follows that the elements $M^i + J^{i+1}$ and $M^{i-1}N + J^{i+1}$ form a basis of J^i/J^{i+1}. By direct computation, we show that $K(A)$ is of the form

$$K(A) = F\left\{M^2, M^3 + M^3N, M^2N + M^3N, M^4, M^5\right\}.$$

Furthermore, we have

$$J(Z(A)) = F\{M^4, M^5\} \subseteq K(A)$$

and $A \cdot J(Z(A)) = J(Z(A))$. Since $M^2 \in K(A)$ holds, but M^3 is not contained in $K(A)$, we obtain $A \cdot K(A) \not\subseteq K(A) = K(A) + A \cdot J(Z(A))$ and hence $K(A) + A \cdot J(Z(A))$ is not an ideal of A. Setting $T := T(A)$ to be the trivial extension algebra of A, we have $\dim T = 20$, and Theorem 5.33 implies that $\operatorname{soc}(Z(T))$ is not an ideal in T. $\triangleleft$

Note that this result, combined with Theorem 5.37, yields $\dim A \in \{17, 18, 19, 20\}$ for a symmetric local algebra A of minimal dimension in which $\operatorname{soc}(Z(A))$ is not an ideal.

Appendix

In Chapter 5, we used several results of [10] and [19], adapted to our given setting. These modified versions are presented in the following. The corresponding proofs largely follow the lines of the original sources [10] and [19].

Throughout, let F be an algebraically closed field of arbitrary characteristic and let A be a finite-dimensional F-algebra. As before, $J := J(A)$ denotes the Jacobson radical of A. We begin with the following statement on symmetric local algebras with a six-dimensional center, which is used in Theorem 5.37. The main ideas of the proof can be found in [19, Chapter 2.2].

Lemma A.1. *Let A be a symmetric local F-algebra with $\dim Z(A) = 6$ which satisfies $\dim J/J^2 = \dim J^2/J^3 = 3$ and $\dim J^3/J^4 = 2$ as well as $\dim K(A) = \dim K(A) \cap J^3 + 3$. Then $\dim J^i/J^{i+1} \leq 2$ holds for all $i \geq 4$.*

Proof. In order to derive a contradiction, we assume that $\dim J^i/J^{i+1} \geq 3$ holds for some $i \geq 4$. Write $J = F\{a, b, c\} + J^2$ with elements $a, b, c \in J$. By [15, Lemma 2.6], we may assume that $J^2 = F\{a^2, ab, ac\} + J^3$ or $J^2 = F\{a^2, ab, ba\} + J^3$ holds. In the first case, we obtain $J^3 = F\{a^3, a^2b, a^2c\} + J^4$ by Lemma 5.1. If we have $J^3 = F\{a^3, a^2b\} + J^4$, then Lemma 5.1 yields $J^k = F\{a^k, a^{k-1}b\} + J^{k+1}$ for $k \geq 4$, so $\dim J^k/J^{k+1} \leq 2$, which is a contradiction. We can argue similarly for all other choices of the two basis elements of J^3/J^4, so this case cannot arise. We therefore assume $J^2 = F\{a^2, ab, ba\} + J^3$. Then there exist coefficients $\alpha_i, \beta_i, \gamma_i \in F$ $(i = 1, \ldots, 6)$ with

$$\begin{aligned}
ac &\equiv \alpha_1 a^2 + \beta_1 ab + \gamma_1 ba \pmod{J^3} \\
b^2 &\equiv \alpha_2 a^2 + \beta_2 ab + \gamma_2 ba \pmod{J^3} \\
bc &\equiv \alpha_3 a^2 + \beta_3 ab + \gamma_3 ba \pmod{J^3} \\
ca &\equiv \alpha_4 a^2 + \beta_4 ab + \gamma_4 ba \pmod{J^3} \\
cb &\equiv \alpha_5 a^2 + \beta_5 ab + \gamma_5 ba \pmod{J^3} \\
c^2 &\equiv \alpha_6 a^2 + \beta_6 ab + \gamma_6 ba \pmod{J^3}.
\end{aligned} \tag{A.1}$$

We may assume $ac \in Fa^2 + Fab + J^3$, which yields $\gamma_1 = 0$, since we are in the case $J^2 = F\{a^2, ab, ac\} + J^3$ otherwise. By going over to the opposite algebra of A, we similarly obtain $\beta_4 = 0$. By replacing c by $c - \alpha_1 a - \beta_1 b$, we may assume $\alpha_1 = \beta_1 = 0$. Furthermore, by replacing b by $b - \gamma_2 a$, we suppose that $\gamma_2 = 0$ holds. Now consider elements $\lambda, \mu, \nu \in F$ and set $v := \lambda a + \mu b + \nu c$. If $J^2 = F\{va, vb, vc\} + J^3$ holds, then replacing a by v brings us back to the case $J^2 = F\{a^2, ab, ac\} + J^3$. Hence we suppose that $va + J^3$, $vb + J^3$ and

$vc + J^3$ are linearly dependent in J^2/J^3. Inserting the relations given in (A.1), we obtain

$$\begin{aligned} va &\equiv (\lambda + \alpha_4\nu)a^2 + (\mu + \gamma_4\nu)ba && \pmod{J^3} \\ vb &\equiv (\alpha_2\mu + \alpha_5\nu)a^2 + (\lambda + \beta_2\mu + \beta_5\nu)ab + \gamma_5\nu ba && \pmod{J^3} \\ vc &\equiv (\alpha_3\mu + \alpha_6\nu)a^2 + (\beta_3\mu + \beta_6\nu)ab + (\gamma_3\mu + \gamma_6\nu)ba && \pmod{J^3}. \end{aligned}$$

Since $a^2 + J^3$, $ab + J^3$ and $ba + J^3$ form a basis of J^2/J^3, the linear dependency of $va + J^3$, $vb + J^3$ and $vc + J^3$ translates to

$$\begin{aligned} 0 &= \det \begin{pmatrix} \lambda + \alpha_4\nu & 0 & \mu + \gamma_4\nu \\ \alpha_2\mu + \alpha_5\nu & \lambda + \beta_2\mu + \beta_5\nu & \gamma_5\nu \\ \alpha_3\mu + \alpha_6\nu & \beta_3\mu + \beta_6\nu & \gamma_3\mu + \gamma_6\nu \end{pmatrix} \\ &= \gamma_3\lambda^2\mu + \gamma_6\lambda^2\nu + (\beta_2\gamma_3 - \alpha_3)\lambda\mu^2 + (\alpha_4\gamma_3 - \alpha_3\gamma_4 - \alpha_6 + \beta_2\gamma_6 + \beta_5\gamma_3 - \beta_3\gamma_5)\lambda\mu\nu \\ &\quad + (\alpha_4\gamma_6 - \alpha_6\gamma_4 + \beta_5\gamma_6 - \beta_6\gamma_5)\lambda\nu^2 + (\alpha_2\beta_3 - \alpha_3\beta_2)\mu^3 \\ &\quad + (\alpha_2(\beta_3\gamma_4 + \beta_6) - \alpha_3(\beta_2\gamma_4 + \beta_5) + \alpha_4\beta_2\gamma_3 + \alpha_5\beta_3 + \alpha_5\beta_3 - \alpha_6\beta_2)\mu^2\nu \\ &\quad + (\alpha_2\beta_6\gamma_4 - \alpha_3\beta_5\gamma_4 + \alpha_4(\beta_2\gamma_6 - \beta_3\gamma_5 + \beta_5\gamma_3) + \alpha_5(\beta_3\gamma_4 + \beta_6) - \alpha_6(\beta_2\gamma_4 + \beta_5))\mu\nu^2 \\ &\quad + (\alpha_4(\beta_5\gamma_6 - \beta_6\gamma_5) + (\alpha_5\beta_6 - \alpha_6\beta_5)\gamma_4)\nu^3. \end{aligned}$$

Since this holds for all choices of $\lambda, \mu, \nu \in F$, this implies $\gamma_3 = \gamma_6 = \alpha_3 = \alpha_2\beta_3 = 0$ as well as $\alpha_6 = -\beta_3\gamma_5$. Similarly, the vectors $av + J^3$, $bv + J^3$ and $cv + J^3$ are linearly dependent in J^2/J^3 and by inserting the corresponding expressions, we obtain

$$\begin{aligned} 0 &= \det \begin{pmatrix} \lambda & \mu & 0 \\ \alpha_2\mu & \beta_2\mu + \beta_3\nu & \lambda \\ \alpha_4\lambda + \alpha_5\mu - \beta_3\gamma_5\nu & \beta_5\mu + \beta_6\nu & \gamma_4\lambda + \gamma_5\mu \end{pmatrix} \\ &= (\alpha_4 + \beta_2\gamma_4 - \beta_5)\lambda^2\mu + (\beta_3\gamma_4 - \beta_6)\lambda^2\nu + (\beta_2\gamma_5 - \alpha_2\gamma_4 + \alpha_5)\lambda\mu^2 - \alpha_2\gamma_5\mu^3. \end{aligned}$$

This implies $\alpha_5 = \alpha_2\gamma_4 - \beta_2\gamma_5$, $\beta_5 = \alpha_4 + \beta_2\gamma_4$ and $\alpha_2\gamma_5 = 0$. Lemma 5.1 yields

$$J^3 = F\{a^3, a^2b, aba, ba^2, bab, b^2a\} + J^4 = F\{a^3, a^2b, aba, ba^2, bab\} + J^4. \tag{A.2}$$

Here, we use that $b^2a \in F\{a^3, aba\} + J^3$ holds by (A.1) together with $\gamma_2 = 0$.

Assume $\alpha_4 \neq 0$. Then we can replace a by $\alpha_4 a$ and assume $\alpha_4 = 1$. The condition $0 \equiv (ac)a - a(ca) \pmod{J^4}$ yields

$$a^3 \equiv -\gamma_4 aba \pmod{J^4} \tag{A.3}$$

and $0 \equiv (bc)a - b(ca) \pmod{J^4}$ yields

$$ba^2 \equiv (\alpha_2\gamma_4^2 + \beta_3 - \beta_2\gamma_4)aba \pmod{J^4}. \tag{A.4}$$

If $\beta_3 \neq 0$ holds, then we obtain

$$0 \equiv (b^2)c - b(bc) \equiv (\alpha_2 a^2 + \beta_2 ab)c - \beta_3 bab \equiv \beta_2\beta_3 a^2b - \beta_3 bab \pmod{J^4}$$

and hence $bab \equiv \beta_2 a^2 b \pmod{J^4}$. Together with (A.3) and (A.4), this congruence shows that $J^2 = F\{a^2b, aba\} + J^3$ holds (see (A.2)), which implies $\dim J^i/J^{i+1} \leq 2$ for all $i \geq 3$ by Lemma 5.1. This is a contradiction and hence $\beta_3 = 0$ follows. If β_2 is nonzero, we may assume $\beta_2 = 1$ by replacing b by $\beta_2^{-1}b$. Then $0 \equiv (b^2)b - b(b^2) \pmod{J^4}$ yields

$$bab \equiv (\alpha_2 + 1)a^2b - \alpha_2^2\gamma_4^2 aba \pmod{J^4},$$

which leads to a contradiction as before. Hence we have $\beta_2 = 0$, which yields $\alpha_5 = \alpha_2\gamma_4$ and $\beta_5 = 1$. The condition $0 \equiv (bc)b - b(cb) \pmod{J^4}$ translates to $bab \equiv (\alpha_2\gamma_4\gamma_5 - \alpha_2^2\gamma_4^3)aba \pmod{J^4}$ and we obtain a contradiction in the same way as above.

We therefore have $\alpha_4 = 0$. Since $\dim K(A) = \dim K(A) \cap J^3 + 3$ holds, the elements $[a,b] + J^3$, $[a,c] + J^3$ and $[b,c] + J^3$ are linearly independent in J^2/J^3. In particular, $[a,c]$ is nonzero. This implies $\gamma_4 \neq 0$ and, by replacing b by $\gamma_4 b$, we may assume $\gamma_4 = 1$. Note that this yields

$$0 \equiv a(ca) - (ac)a \equiv aba \pmod{J^4}. \tag{A.5}$$

Moreover, we have

$$\begin{aligned}
[a,b] &\equiv ab - ba && \pmod{J^3}\\
[a,c] &\equiv -ba && \pmod{J^3}\\
[b,c] &\equiv -\alpha_5 a^2 + (\beta_3 - \beta_5)ab - \gamma_5 ba && \pmod{J^3}.
\end{aligned}$$

Since $[a,b] + J^3$, $[a,c] + J^3$ and $[b,c] + J^3$ are linearly independent in J^2/J^3, this implies $\alpha_5 \neq 0$. Since $\alpha_5 = \alpha_2 - \beta_2\gamma_5$ holds, either α_2 or β_2 is nonzero. If $\alpha_2 = 0$ holds, then we may replace a by $\beta_2 a$ and assume $\beta_2 = 1$. This yields $\beta_5 = \alpha_2 + \beta_2\gamma_4 = 1$ and hence

$$\begin{aligned}
0 = (b^2)b - b(b^2) &\equiv a^2b - bab && \pmod{J^4}\\
0 = (ac)b - a(cb) &\equiv -\alpha_5 a^3 - a^2b && \pmod{J^4}.
\end{aligned}$$

Together with (A.5), this yields $J^3 = F\{a^3, ba^2\} + J^4$ (see (A.2)), which leads to a contradiction as in the case $\alpha_4 \neq 0$. Now if $\alpha_2 \neq 0$ holds, then $\alpha_2\beta_3 = \alpha_2\gamma_5 = 0$ implies $\beta_3 = \gamma_5 = 0$ and hence we obtain

$$\begin{aligned}
0 \equiv (ac)b - a(cb) &\equiv -\alpha_5 a^3 - \beta_5 a^2 b && \pmod{J^4}\\
0 \equiv (bc)b - b(cb) &\equiv -\alpha_5 ba^2 - \beta_5 bab && \pmod{J^4}.
\end{aligned}$$

The first congruence relation yields $a^3 \in Fa^2b + J^4$ since $\alpha_5 = \alpha_2$ is nonzero. The second identity implies $ba^2 \in Fbab + J^4$. Together with (A.5), we obtain $J^3 = F\{a^2b, bab\} + J^4$ (see (A.2)). By [40, Lemma 3.2], this yields $\dim J^i/J^{i+1} \leq 2$ for all $i \geq 3$, which is a contradiction. □

Now we move to the investigation of certain (not necessarily symmetric) local algebras. The following statement is the content of [10, Lemmas 2.5, 2.7, 3.3 and 3.5], adapted to our situation.

Lemma A.2. *Let A be a local F-algebra with* $\dim J/J^2 = 2$ *and* $\dim J^2/J^3 \geq 2$ *or* $\dim J/J^2 = 3$ *and* $\dim J^2/J^3 = \dim J^3/J^4 = 2$. *Then the following hold:*

(i) *There exists an element $x \in J$ with $x^2 \notin J^3$.*

(ii) There exist elements $a, b \in J$ such that $a^2 + J^3$ and $ab + J^3$ or $a^2 + J^3$ and $ba + J^3$ are linearly independent in J^2/J^3.

Proof.

(i) Assume that $x^2 \in J^3$ holds for all $x \in J$. First let $\dim J/J^2 = 2$ and $\dim J^2/J^3 \geq 2$ and write $J = F\{a, b\} + J^2$ for some $a, b \in J$. Then we obtain

$$0 \equiv (a+b)^2 \equiv a^2 + ab + ba + b^2 \equiv ab + ba \pmod{J^3},$$

which yields $ba \equiv -ab \pmod{J^3}$. By Lemma 5.1, we have $J^2 = F\{a^2, ab, ba, b^2\} + J^3 \subseteq Fab + J^3$, which is a contradiction to $\dim J^2/J^3 \geq 2$.

Now let $\dim J/J^2 = 3$ and $\dim J^2/J^3 = \dim J^3/J^4 = 2$ and write $J = F\{a, b, c\} + J^2$ for some $a, b, c \in J$. By assumption, we have $a^2, b^2, c^2, (a+b)^2, (a+c)^2, (b+c)^2 \in J^3$, which implies $ba \equiv -ab \pmod{J^3}$, $ca \equiv -ac \pmod{J^3}$ and $cb \equiv -bc \pmod{J^3}$ as before. With this, Lemma 5.1 yields $J^2 = F\{ab, ac, bc\} + J^3$ and hence

$$J^3 = F\{a^2b, a^2c, abc, bab, bac, b^2c\} + J^4 = Fabc + J^4,$$

which is a contradiction to $\dim J^3/J^4 = 2$.

(ii) First assume $\dim J/J^2 = 2$ and $\dim J^2/J^3 \geq 2$. By (i), there exists an element $a \in J$ with $a^2 \notin J^3$. We choose an element $b \in J$ such that $J = F\{a, b\} + J^2$ holds. Again, we obtain $J^2 = F\{a^2, ab, ba, b^2\} + J^3$ and we may assume $ab, ba \in a^2 + J^3$ since the claim follows otherwise. By exchanging the roles of a and b, we additionally may assume $ab, ba \in b^2 + J^3$. In particular, there exist coefficients $\alpha, \beta \in F$ with $ab \equiv \alpha a^2 \equiv \beta b^2 \pmod{J^3}$. Since $a^2 + J^3$ and $b^2 + J^3$ form a basis of J^2/J^3, this yields $\alpha = \beta = 0$ and hence $ab \in J^3$. Analogously, one shows $ba \in J^3$. Then

$$\begin{aligned}(a+b)^2 &= a^2 + ab + ba + b^2 \equiv a^2 + b^2 \pmod{J^3}\\ (a+b)b &= ab + b^2 \equiv b^2 \pmod{J^3},\end{aligned}$$

so $(a+b)^2 + J^3$ and $(a+b)b + J^3$ are linearly independent in J^2/J^3, which proves the claim.

Now assume $\dim J/J^2 = 3$ and $\dim J^2/J^3 = \dim J^3/J^4 = 2$. By (i), we may write $J = F\{a, b, c\} + J^2$ for some $a, b, c \in J$ with $a^2 \notin J^3$. We may assume $ab, ac, ba, ca \in Fa^2 + J^3$ since the claim is proven otherwise. By Lemma 5.1, this implies

$$J^2 = F\{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3 = F\{a^2, b^2, bc, cb, c^2\} + J^3.$$

Assume $b^2 \notin Fa^2 + J^3$. By exchanging the roles of a and b, we may assume $ab, ba, bc, cb \in Fb^2 + J^3$. Analogously to the above, one can then show that $(a+b)^2 + J^3$ and $(a+b)b + J^3$ are linearly independent in J^2/J^3. Hence let $b^2 \in Fa^2 + J^3$ and $c^2 \in Fa^2 + J^3$. Without loss of generality, we may assume $J^2 = F\{a^2, bc\} + J^3$. But then $J^3 = F\{a^3, abc, ba^2, b^2c\} + J^4 = Fa^3 + J^4$ follows by Lemma 5.1, which is a contradiction to $\dim J^3/J^4 = 2$. □

The preceding lemma is applied to prove the following modified version of the argument in [10, page 15], which is used in Proposition 5.39.

Lemma A.3. *Let A be a local F-algebra which satisfies* $\dim J/J^2 = 3$ *and* $\dim J^2/J^3 = \dim J^3/J^4 = 2$. *Then $K(A)$ is a proper subset of J^2.*

Proof. Note that $K(A)$ is contained in J^2 by (5.1). In order to derive a contradiction, we assume $K(A) = J^2$. By the preceding lemma and possibly replacing A by its opposite algebra, we find elements $a, b, c \in J$ such that $J = F\{a, b, c\} + J^2$ and $J^2 = F\{a^2, ab\} + J^3$ hold. By Lemma 5.1, $a^3 + J^4$ and $a^2b + J^4$ form an F-basis of J^3/J^4. Write $ac \equiv \alpha a^2 + \beta ab \pmod{J^3}$ for some $\alpha, \beta \in F$. By replacing c by $\bar{c} := c - \alpha a - \beta b$, we may assume $ac \in J^3$. Furthermore, there exist coefficients $\alpha_i, \beta_i \in F$ $(i = 1, \ldots, 4)$ with

$$\begin{aligned} bc &\equiv \alpha_1 a^2 + \beta_1 ab \pmod{J^3} \\ ca &\equiv \alpha_2 a^2 + \beta_2 ab \pmod{J^3} \\ cb &\equiv \alpha_3 a^2 + \beta_3 ab \pmod{J^3} \\ c^2 &\equiv \alpha_4 a^2 + \beta_4 ab \pmod{J^3}. \end{aligned}$$

With this, we obtain

$$\begin{aligned} 0 &\equiv (ac)a \equiv a(ca) \equiv \alpha_2 a^3 + \beta_2 a^2 b \pmod{J^4} \\ 0 &\equiv (ac)b \equiv a(cb) \equiv \alpha_3 a^3 + \beta_3 a^2 b \pmod{J^4} \\ 0 &\equiv (ac)c \equiv ac^2 \quad \equiv \alpha_4 a^3 + \beta_4 a^2 b \pmod{J^4}. \end{aligned}$$

Comparing the coefficients yields $\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = \alpha_4 = \beta_4 = 0$. This implies

$$0 \equiv bc^2 \equiv (bc)c \equiv \alpha_1 a^2 c + \beta_1 abc \equiv \beta_1 abc = \beta_1(\alpha_1 a^3 + \beta_1 a^2 b) \pmod{J^4},$$

which yields $\beta_1 = 0$. Furthermore, we obtain

$$0 \equiv b(cb) \equiv (bc)b \equiv \alpha_1 a^2 b \pmod{J^4}$$

and hence $\alpha_1 = 0$. This yields $[a, c], [b, c] \in J^3$ and hence

$$J^2 = K(A) \subseteq F\{[a, b], [a, c], [b, c]\} + J^3 \subseteq F\{[a, b]\} + J^3,$$

which is a contradiction to $\dim J^2/J^3 = 2$. □

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Hiermit erkläre ich,

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Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts haben mich folgende Personen unterstützt: Prof. Dr. Burkhard Külshammer.

Ich habe die gleiche, eine in wesentlichen Teilen ähnliche bzw. eine andere Abhandlung bereits bei einer anderen Hochschule als Dissertation eingereicht: Nein.

Jena, den 15. September 2022

Sofia Bettina Brenner