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Eigenvalues of parametric rank one perturbations of matrix pencils

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Abstract

The behavior of eigenvalues of regular matrix pencils under rank one perturbations which depend on a scalar parameter is studied. In particular we address the change of the algebraic multiplicities, the change of the eigenvalues for small parameter variations as well as the asymptotic eigenvalue behavior as the parameter tends to infinity. Besides that, an interlacing result for rank one perturbations of matrix pencils is obtained. Finally, we apply the result to a redesign problem for electrical circuits.

Keywords: rank one, matrix pencils, eigenvalue perturbation, circuit redesign, interlacing
2010 MSC: 15A18, 15A22, 47A55

1. Introduction

In this note, we study the behavior of the eigenvalues of regular matrix pencils $sE - A$, $E, A \in \mathbb{C}^{n\times n}$ under parameter-dependent perturbations of the form

$$sF_\tau - G_\tau := \tau(as - b)uv^*, \quad \tau \in \mathbb{C}, \tag{1}$$

where $a, b \in \mathbb{C}$ and $u, v \in \mathbb{C}^n$ are fixed and only τ is allowed to vary. This is expressed by the subscript τ when we write $sF_\tau - G_\tau$.

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Such variations of a single parameter appear e.g. in the redesign of complex networks consisting of certain dynamic elements when only one of these elements is changed. For example in electrical circuits, the parameter τ might be the capacity of a capacitor or the resistance of a resistor and in the context of modified nodal analysis [1] both, u and v, are the difference of two unit vectors $e_i - e_j$ indicating that the new capacitor is placed between the i-th and j-th node of the electrical network, cf. Section 7. Moreover, such perturbations appear also in mechanical mass-spring-damper-systems, see [2, 3, 4] and the references therein.

In each of the following sections, we focus on a certain property of the eigenvalues subject to perturbations of the form (1):

(i) *algebraic multiplicities* of the perturbed pencil in dependence of the parameter τ (see Theorem 3.2);

(ii) *asymptotic behavior* for large values of τ via Rouché's theorem (see Proposition 4.1);

(iii) *local behavior* for small values of τ via Taylor approximation (see Proposition 5.1);

(iv) *interlacing* of real eigenvalues (see Proposition 6.3).

For matrices, i.e. $E = I_n$, $a = 0$ and $b = 1$, a detailed study of (i) and (ii) was conducted in [5, 6, 7]. We extend (some) of the results to matrix pencils. Similar methods for matrix pencils were applied in [8, 9] to study the rank-one nearest singular matrix pencils for a given regular matrix pencil.

There are several results on the local behavior as $\tau \to 0$ for analytic matrix-valued functions, see e.g. [10, 11, 12, 13], which generalize the classical results for matrices obtained e.g. in [14]. Our contribution regarding (iii) given in Section 5 is a first order expansion at semi-simple eigenvalues.

Closely related is [15] where a similar expansion for singular pencils and simple eigenvalues and [16] where an even more general expansion for analytic matrix-valued functions at semi-simple eigenvalues subject to perturbations of arbitrary rank was obtained. However since we restrict to rank-one perturbations we are able to obtain an simpler explicit expression for the first-order coefficient.

Regarding (iv), we consider in Section 6 pencils with $E, A \in \mathbb{R}^{n\times n}$ subject to real-valued perturbations, i.e. $\tau, a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$. Then it follows from the local behavior (iii) that for each eigenvalue of the unperturbed pencil there is either a real-valued eigenvalue curve emerging from this point, or the eigenvalue remains stationary for all parameter values.

In the first case, we can give a sign $+1$ or -1 to the eigenvalues depending on the direction where the eigenvalue curve moves. Under the additional assumption that two consecutive signs of eigenvalues are equal, we are able to show that at least one eigenvalue of the perturbed pencil is contained in the interval between the two eigenvalues of the unperturbed pencil. In the case where all eigenvalues have the same sign this allows us to conclude the interlacing of real eigenvalues without further structural assumptions on the pencil coefficients.

This extends classical interlacing results for Hermitian matrices under rank-one perturbations [17, 18], see also [19] for related results on perturbations of singular values of matrices and [20, 21, 22] for related results on matrix pencils.

Finally, in Section 7 we demonstrate how the results for (i)-(iv) might be applied to the redesign of electrical circuits. Here we focus on two particular examples: a low pass filter and a two-stage amplifier. For each circuit, the task is to insert new capacitors or resistors, in such a way that the eigenvalues of the modified circuit are located in a prescribed region of the complex plane. In this setting, a single capacitor or resistor can be modeled as a rank-one perturbation (1) and one has to determine appropriate values for $\tau > 0$ to achieve this goal.

2. Preliminaries

We consider square matrix pencils $sE - A$ with $E, A \in \mathbb{C}^{n\times n}$ and denote it briefly by $sE - A \in \mathbb{C}^{n\times n}[s]$, where $\mathbb{C}^{n\times n}[s]$ is the ring of polynomials with coefficients in $\mathbb{C}^{n\times n}$. For obvious reasons we always assume that the natural number n is larger or equal to one.

We focus on the class of regular pencils and their spectrum [23, 24]. In the following, the natural numbers $\mathbb{N}$ will always include 0. The *spectrum* or the *set of eigenvalues* of the matrix pencil $sE - A \in \mathbb{C}[s]^{n\times n}$ is the set

$$\sigma(E, A) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - A) = 0\}, \quad \text{if } E \text{ is invertible},$$

and

$$\sigma(E, A) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - A) = 0\} \cup \{\infty\}, \quad \text{if } E \text{ is singular}.$$

The pencil $sE-A$ is called *regular* if $\det(sE-A)$ is not the zero polynomial. Otherwise the pencil is called *singular*.

If $sE-A$ is regular then the *algebraic multiplicity* of $\lambda \in \mathbb{C} \cup \{\infty\}$ is given by

$$\mathrm{am}_\lambda(E,A) := \begin{cases} \text{multiplicity of } \lambda \text{ as a root of } \det(sE-A), & \text{if } \lambda \neq \infty, \\ n - \deg\det(sE-A), & \text{if } \lambda = \infty. \end{cases}$$

The *geometric multiplicity* of $\lambda \in \mathbb{C} \cup \{\infty\}$ is defined as

$$\mathrm{gm}_\lambda(E,A) := \begin{cases} \dim\ker(\lambda E - A), & \text{if } \lambda \neq \infty, \\ \dim\ker E, & \text{if } \lambda = \infty. \end{cases}$$

We say that an eigenvalue $\lambda \in \sigma(E,A)$ is *simple* if $\mathrm{am}_\lambda(E,A) = 1$ and *semi-simple* if $\mathrm{am}_\lambda(E,A) = \mathrm{gm}_\lambda(E,A)$.

Obviously, the spectrum of a regular matrix pencil is a finite subset of the extended complex plane $\mathbb{C} \cup \{\infty\}$. If $E = I_n$ then $\sigma(I_n, A)$ is the set of eigenvalues of the matrix A and is denoted by $\sigma(A)$.

Let $k \in \mathbb{N}$ with $k \geq 1$ and denote by I_k the $k \times k$ identity matrix and by $J_k(\lambda)$ the $k \times k$ Jordan block corresponding to an eigenvalue $\lambda \in \mathbb{C}$. The Weierstraß form is a canonical form for regular matrix pencils. It consists of blocks of the following types

$$sI_k - J_k(\lambda) := \begin{bmatrix} s-\lambda & -1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & -1 \\ & & & & s-\lambda \end{bmatrix} \in \mathbb{C}[s]^{k\times k},$$

$$sN_k - I_k := \begin{bmatrix} -1 & s & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & s \\ & & & & -1 \end{bmatrix} \in \mathbb{C}[s]^{k\times k}.$$

More precisely, let $\ell_\alpha \in \mathbb{N} \setminus \{0\}$ and $\alpha = (\alpha_1, \ldots, \alpha_{\ell_\alpha}) \in (\mathbb{N} \setminus \{0\})^{\ell_\alpha}$ be a multi-index and assume that the entries are non-increasing, i.e. $\alpha_1 \geq \alpha_2 \ldots \geq$

$\alpha_{\ell_\alpha} \geq 1$. Set

$$N_\alpha := \begin{bmatrix} N_{\alpha_1} & & & \\ & N_{\alpha_2} & & \\ & & \ddots & \\ & & & N_{\alpha_{\ell_\alpha}} \end{bmatrix}$$

and $|\alpha| := \sum_{i=1}^{\ell_\alpha} \alpha_i$. For $\lambda \in \mathbb{C}$, $\ell_\lambda \in \mathbb{N} \setminus \{0\}$, and a multi-index $s(\lambda) = (s_1(\lambda), \ldots, s_{\ell_\lambda}(\lambda)) \in (\mathbb{N} \setminus \{0\})^{\ell_\lambda}$ with $s_1(\lambda) \geq \ldots \geq s_{\ell_\lambda}(\lambda) \geq 1$, we define $J_{s(\lambda)}$ as a block diagonal matrix of Jordan blocks of sizes $s_1(\lambda), \ldots, s_{\ell_\lambda}(\lambda)$

$$J_{s(\lambda)} := \begin{bmatrix} J_{s_1(\lambda)}(\lambda) & & \\ & \ddots & \\ & & J_{s_{\ell_\lambda(\lambda)}}(\lambda) \end{bmatrix}.$$

It is a well-known result, see e.g. [25, Theorem XII.3], that for regular $sE - A \in \mathbb{C}[s]^{n \times n}$ there exist some invertible $S, T \in \mathbb{C}^{n \times n}$ and $r \in \mathbb{N}$ such that

$$S(sE - A)T = \begin{bmatrix} sI_r - \operatorname{diag}(J_{s(\lambda_1)}, \ldots, J_{s(\lambda_l)}) & 0 \\ 0 & sN_\alpha - I_{n-r} \end{bmatrix} \tag{2}$$

for some $l \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$, and a multi-index α. Here $l = 0$ or $\alpha = 0$ indicates that the corresponding entry is non-existing. The block diagonal form (2) is called *Weierstraß form*. In this situation we have

$$\operatorname{gm}_\lambda(E, A) = l_\lambda \quad \text{and} \quad \operatorname{am}_\lambda(E, A) = \sum_{j=1}^{l_\lambda} s_j(\lambda). \tag{3}$$

and, hence, the rational function

$$s \mapsto (sE - A)^{-1} \quad s \in \mathbb{C} \setminus \sigma(E, A) \tag{4}$$

has at $s = \lambda_j$ a pole of order $s_1(\lambda_j)$ for all $j = 1, \ldots, l$. Note that the size of the largest entry of the multi-index α in the Weierstraß form (2) is called the *index* of the matrix pencil.

3. Algebraic multiplicities of the perturbed pencil in dependence of the parameter τ

In this section we always assume $a, b \in \mathbb{C}$ such that $(a, b) \neq (0, 0)$.

Definition 3.1. *Let $sE - A \in \mathbb{C}[s]^{n\times n}$ be a regular matrix pencil, let $u, v \in \mathbb{C}^n$ and consider the rational function*

$$s \mapsto (as - b)v^*(sE - A)^{-1}u, \quad s \in \mathbb{C} \setminus \sigma(E, A). \tag{5}$$

Denote by o_λ the order of λ as a pole of (5). Then for all $\lambda \in \sigma(E, A) \setminus \{\infty\}$ we consider

$$s_{u,v}^{a,b}(\lambda) := \begin{cases} o_\lambda, & \text{if } \lambda \text{ is a pole of (5)}, \\ 0, & \text{if } \lambda \text{ is not a pole of (5)}. \end{cases}$$

For $\lambda = \infty$ we define $s_{u,v}^{a,b}(\infty)$ as the order of the pole of

$$s \mapsto (a - bs)v^*(-sA + E)^{-1}u$$

at $s = 0$ and we set $s_{u,v}^{a,b}(\infty) := 0$ if there is no such pole. Further, we introduce

$$M_{u,v}^{a,b}(E, A) := \sum_{\mu\in\sigma(E,A)} s_{u,v}^{a,b}(\mu)$$

and we consider the polynomials

$$m_{u,v}^{a,b}(s) := \prod_{\lambda\in\sigma(E,A)\setminus\{\infty\}} (s - \lambda)^{s_{u,v}^{a,b}(\lambda)}, \tag{6}$$

$$p(s) := m_{u,v}^{a,b}(s)(as - b)v^*(sE - A)^{-1}u. \tag{7}$$

By the definition of $s_{u,v}^{a,b}$ and by (4) it follows for all $\lambda \in \mathbb{C} \cup \{\infty\}$ that

$$s_{u,v}^{a,b}(\lambda) \leq s_1(\lambda). \tag{8}$$

We have in view of (2)

$$\begin{aligned} v^*(sE - A)^{-1}u &= v^*T\left(S(sE - A)T\right)^{-1} Su \\ &= v^*T \begin{bmatrix} sI - \operatorname{diag}(J_{s(\lambda_1)}, \ldots, J_{s(\lambda_l)}) & 0 \\ 0 & sN_\alpha - I \end{bmatrix}^{-1} Su, \end{aligned}$$

where $l, \ell_\alpha \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$ are pairwise distinct, the multi-indices $s(\lambda_1), \ldots, s(\lambda_l)$ and $\alpha = (\alpha_1, \ldots, \alpha_{\ell_\alpha})$ are as in (2). In what follows, we decompose v^*T and Su according to the block structure of the matrix in (2). In order to simplify notation we set

$$w := (v^*T)^* \quad \text{and} \quad z := Su \tag{9}$$

and write

$$w = \begin{bmatrix} w^1 \\ \vdots \\ w^l \\ w^{\infty,1} \\ \vdots \\ w^{\infty,\ell_\alpha} \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} z^1 \\ \vdots \\ z^l \\ z^{\infty,1} \\ \vdots \\ z^{\infty,\ell_\alpha} \end{bmatrix}, \tag{10}$$

where w^j, z^j, for $1 \le j \le l$, are vectors of size $|s(\lambda_j)|$ which correspond to the block in (2) given by $J_{s(\lambda_j)}$ and $w^{\infty,m}, z^{\infty,m}$, for $1 \le m \le \ell_\alpha$, are vectors of size α_m which correspond to the block in (2) given by the m-th entry of α. For $w^{\infty,m}, z^{\infty,m}$ we write

$$w^{\infty,m} = \begin{bmatrix} w_1^{\infty,m} \\ \vdots \\ w_{\alpha_m}^{\infty,m} \end{bmatrix} \quad \text{and} \quad z^{\infty,m} = \begin{bmatrix} z_1^{\infty,m} \\ \vdots \\ z_{\alpha_m}^{\infty,m} \end{bmatrix}.$$

In addition, we also have to label all entries of w^j, z^j, $1 \le j \le l$, in (10). Note that $s(\lambda_j)$ is itself a multi-index, $s(\lambda_j) = (s_1(\lambda_j), \ldots, s_{\ell_{\lambda_j}}(\lambda_j))$, and we write for

$$w^j = \begin{bmatrix} w^{j,1} \\ \vdots \\ w^{j,\ell_{\lambda_j}} \end{bmatrix} \quad \text{and} \quad z^j = \begin{bmatrix} z^{j,1} \\ \vdots \\ z^{j,\ell_{\lambda_j}} \end{bmatrix},$$

where $w^{j,k}$ and $z^{j,k}$ for $1 \le k \le \ell_{\lambda_j}$ are vectors of size $s_k(\lambda_j)$,

$$w^{j,k} = \begin{bmatrix} w_1^{j,k} \\ \vdots \\ w_{s_k(\lambda_j)}^{j,k} \end{bmatrix} \quad \text{and} \quad z^{j,k} = \begin{bmatrix} z_1^{j,k} \\ \vdots \\ z_{s_k(\lambda_j)}^{j,k} \end{bmatrix}. \tag{11}$$

In the next theorem we study the algebraic multiplicities in dependence of the parameter τ. A similar result for generic perturbations of matrices was obtained in [5, Proposition 2.2].

Theorem 3.2. *Let $sE - A \in \mathbb{C}[s]^{n\times n}$ be a regular matrix pencil and let $sF_\tau - G_\tau$ as in* (1)*. Then for all $\tau \in \mathbb{C} \setminus \{0\}$ and all $\lambda \in \mathbb{C} \cup \{\infty\}$*

$$\mathrm{am}_\lambda(E + F_\tau, A + G_\tau) = \mathrm{am}_\lambda(E, A) - s_{u,v}^{a,b}(\lambda), \qquad \text{if } s_{u,v}^{a,b}(\lambda) > 0, \tag{12}$$

$$\mathrm{am}_\lambda(E + F_\tau, A + G_\tau) \ge \mathrm{am}_\lambda(E, A), \qquad \text{if } s_{u,v}^{a,b}(\lambda) = 0 \tag{13}$$

and we have

$$\sum_{\mu\in\sigma(E+F_\tau,A+G_\tau)\setminus\sigma(E,A)} \mathrm{am}_\mu(E+F_\tau, A+G_\tau) \le M_{u,v}^{a,b}(E,A). \tag{14}$$

In the case that the function $s_{u,v}^{a,b}$ *is identically equal to zero we have for all* $\lambda \in \mathbb{C}\cup\{\infty\}$

$$\mathrm{am}_\lambda(E+F_\tau, A+G_\tau) = \mathrm{am}_\lambda(E,A) \quad \textit{for all } \tau\in\mathbb{C}. \tag{15}$$

In the case that the function $s_{u,v}^{a,b}$ *is not identically equal to zero, we obtain the following characterizations.*

(a) *The pencil* $s(E+F_\tau)-(A+G_\tau)$ *is regular for all* $\tau\in\mathbb{C}$.

(b) *Let* $\mu\in\mathbb{C}\setminus\sigma(E,A)$ *then*

$$p(\mu)\neq 0 \quad \textit{and} \quad \tau = -\tfrac{m_{u,v}^{a,b}(\mu)}{p(\mu)} \quad \iff \quad \mu\in\sigma(E+F_\tau, A+G_\tau).$$

(c) *Let* $\lambda\in\sigma(E,A)$ *be simple and assume* $s_{u,v}^{a,b}(\lambda)=1$. *Then*

$$\lambda\notin\sigma(E+F_\tau, A+G_\tau) \quad \textit{for all } \tau\in\mathbb{C}\setminus\{0\}.$$

(d) *Let* $\lambda\in\sigma(E,A)$ *and* $s_{u,v}^{a,b}(\lambda)=0$. *Then the rational function* $\varphi(s) = \tau(as-b)v^*(sE-A)^{-1}u$ *has no pole in* λ *and we have*

$$\mathrm{am}_\lambda(E+F_\tau, A+G_\tau) = \mathrm{am}_\lambda(E,A) + \varphi_\lambda, \tag{16}$$

where φ_λ *denotes the order of the zero of the function* $1+\tau\varphi$ *in* λ.

Moreover, there is a full characterization for the function $s_{u,v}^{a,b}$ *to be identically equal to zero. It depends on whether* a *is zero, and whether* ∞ *or* b/a *is in the spectrum of the pencil. We collect the different possibilities below.*

I. *If* $a=0$, $\infty\notin\sigma(E,A)$ *or* $a\neq 0$, $b/a\notin\sigma(E,A)$, *then the function* $s_{u,v}^{a,b}$ *is identically equal to zero if and only if (in the notation introduced in*

(10)–(11)) *for each* $1 \le j \le l$, $1 \le k \le \ell_{\lambda_j}$, *and each* $1 \le m \le \ell_\alpha$ *the following relations are fulfilled*

$$w^j \perp z^j,\ w^{\infty,m} \perp z^{\infty,m} \text{ and } \tag{17}$$

$$\begin{bmatrix} w_1^{j,k} \\ \vdots \\ w_{s_k(\lambda_j)-1}^{j,k} \end{bmatrix} \perp \begin{bmatrix} z_2^{j,k} \\ \vdots \\ z_{s_k(\lambda_j)}^{j,k} \end{bmatrix}, \begin{bmatrix} w_1^{j,k} \\ \vdots \\ w_{s_k(\lambda_j)-2}^{j,k} \end{bmatrix} \perp \begin{bmatrix} z_3^{j,k} \\ \vdots \\ z_{s_k(\lambda_j)}^{j,k} \end{bmatrix}, \ldots, \overline{w_1^{j,k}} z_{s_k(\lambda_j)}^{j,k} = 0, \tag{18}$$

$$\begin{bmatrix} w_1^{\infty,m} \\ \vdots \\ w_{\alpha_m-1}^{\infty,m} \end{bmatrix} \perp \begin{bmatrix} z_2^{\infty,m} \\ \vdots \\ z_{\alpha_m}^{j,k} \end{bmatrix}, \begin{bmatrix} w_1^{\infty,m} \\ \vdots \\ w_{\alpha_m-2}^{\infty,m} \end{bmatrix} \perp \begin{bmatrix} z_3^{\infty,m} \\ \vdots \\ z_{\alpha_m}^{\infty,m} \end{bmatrix}, \ldots, \overline{w_1^{\infty,m}} z_{\alpha_m}^{\infty,m} = 0. \tag{19}$$

Moreover, the polynomial p *in* (7) *is identically equal to zero if and only if the function* $s_{u,v}^{a,b}$ *is identically equal to zero.*

II. *If* $a \ne 0$ *and* $b/a \in \sigma(E, A)$. *Assume for simplicity that the first eigenvalue* λ_1 *equals* b/a. *Then the function* $s_{u,v}^{a,b}$ *is identically equal to zero if and only if for each* $2 \le j \le l$, $1 \le k \le \ell_{\lambda_j}$, $1 \le m \le \ell_\alpha$ *we have* (17)–(19) *and for* $j = 1$ *we have* (18)–(19) *and* $w^{\infty,m} \perp z^{\infty,m}$.

III. *If* $a = 0$, $\infty \in \sigma(E, A)$, *then the function* $s_{u,v}^{a,b}$ *is identically equal to zero if and only if for each* $1 \le j \le l$, $1 \le k \le \ell_{\lambda_j}$, $1 \le m \le \ell_\alpha$ (18)–(19) *holds and for* $1 \le j \le l$ *we have* $w^j \perp z^j$.

Proof. The Sylvester's determinant identity yields for $\lambda \in \mathbb{C}$

$$\begin{aligned} \det(s(E+F_\tau)-(A+G_\tau)) &= \det(sE-A)\det(I_n + (sE-A)^{-1}\tau(as-b)uv^*) \\ &= \det(sE-A)(1+\tau(as-b)v^*(sE-A)^{-1}u) \qquad (20) \\ &= \frac{\det(sE-A)}{m_{u,v}^{a,b}(s)}(m_{u,v}^{a,b}(s) + \tau p(s)). \qquad (21) \end{aligned}$$

Then for $\lambda \ne \infty$, (13) and (16) follow from (20). By construction, the polynomial $m_{u,v}^{a,b}$ is non-zero. The same applies to $\det(sE-A)/m_{u,v}^{a,b}(s)$, see (8). Let $\lambda \in \mathbb{C}$ and assume $s_{u,v}^{a,b}(\lambda) > 0$. Then $p(\lambda) \ne 0$ and

$$m_{u,v}^{a,b}(\lambda) + \tau p(\lambda) = \tau p(\lambda) \ne 0,$$

follows. This shows (12) and statement (a) for $\lambda \ne \infty$.

It is no restriction to assume that the matrix pencil $sE - A$ in (2) is already in Weierstraß form, i.e.

$$sE - A = \begin{bmatrix} sI_{n-|\alpha|} - \operatorname{diag}(J_{s(\lambda_1)}, \ldots, J_{s(\lambda_l)}) & 0 \\ 0 & sN_\alpha - I_{|\alpha|} \end{bmatrix}, \tag{22}$$

where $l, \ell_\alpha \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$, the multi-indices $s(\lambda_1), \ldots, s(\lambda_l)$ and α are as in (2). Hence for the dual pencil we have

$$-sA + E = \begin{bmatrix} I_{n-|\alpha|} - s\ \operatorname{diag}(J_{s(\lambda_1)}, \ldots, J_{s(\lambda_l)}) & 0 \\ 0 & N_\alpha - sI_{|\alpha|} \end{bmatrix}.$$

For $1 \leq j \leq l$ and the corresponding multi-index $s(\lambda_j) = (s_1(\lambda_j), \ldots, s_{\ell_{\lambda_j}}(\lambda_j))$ we have for $1 \leq k \leq \ell_{\lambda_j}$

$$(I - sJ_{s_k(\lambda_j)})^{-1} = \\ \begin{bmatrix} (1-s\lambda_j)^{-1} & s(1-s\lambda_j)^{-2} & \cdots & s^{s_k(\lambda_j)-1}(1-s\lambda_j)^{-s_k(\lambda_j)} \\ & (1-s\lambda_j)^{-1} & \cdots & s^{s_k(\lambda_j)-2}(1-s\lambda_j)^{-s_k(\lambda_j)+1} \\ & & \ddots & \vdots \\ & & & (1-s\lambda_j)^{-1} \end{bmatrix}$$

and for $1 \leq m \leq \ell_\alpha$ and $\alpha = (\alpha_1, \ldots, \alpha_{\ell_\alpha}) \in (\mathbb{N} \setminus \{0\})^{\ell_\alpha}$ we have

$$(N_{\alpha_m} - sI_{\alpha_m})^{-1} = - \begin{bmatrix} s^{-1} & s^{-2} & \cdots & s^{-\alpha_m} \\ & s^{-1} & \cdots & s^{-\alpha_m+1} \\ & & \ddots & \vdots \\ & & & s^{-1} \end{bmatrix}. \tag{23}$$

By the definition of the spectrum, the dual pencil $-sA + E$ has spectrum $\sigma(-A, -E)$ which coincides with $\sigma(A, E)$. Hence, in what follows, we denote this spectrum by $\sigma(A, E)$. Moreover, by $\mathrm{am}_\lambda(A, E)$ the algebraic multiplicity of $\lambda \in \mathbb{C} \cup \{\infty\}$ of the dual pencil, see Section 2. The above calculations imply for 0, ∞, and $\lambda \neq 0$

$$\begin{aligned} \mathrm{am}_\infty(A, E) &= \mathrm{am}_0(E, A), & \infty \in \sigma(A, E) &\iff & 0 \in \sigma(E, A), \\ \mathrm{am}_0(A, E) &= \mathrm{am}_\infty(E, A), & 0 \in \sigma(A, E) &\iff & \infty \in \sigma(E, A), \\ \mathrm{am}_\lambda(A, E) &= \mathrm{am}_{1/\lambda}(E, A), & \lambda \in \sigma(A, E) &\iff & \tfrac{1}{\lambda} \in \sigma(E, A). \end{aligned} \tag{24}$$

Replace in (5) the expression $(sE-A)^{-1}$ by the inverse $(-sA+E)^{-1}$ of the dual pencil $-sA+E$ and consider also the dual perturbation $-sG_\tau+F_\tau$. The corresponding pole order function defined as in (5) is denoted by $(s_{u,v}^{a,b})^\sharp(\mu)$ for $\mu \in \mathbb{C}$. It is the order of μ as a pole of

$$s \mapsto (-bs+a)v^*(-As+E)^{-1}u. \tag{25}$$

Clearly, this gives

$$(s_{u,v}^{a,b})^\sharp(0) := s_{u,v}^{a,b}(\infty). \tag{26}$$

As (12), (13) and statement (a) are proved for $\lambda = 0$, we can apply this to the dual pencil (for $\lambda = 0$) and the dual perturbation $-sG_\tau + F_\tau$. With the above identities and (26) we obtain

$$\begin{aligned}
\mathrm{am}_\infty(E+F_\tau, A+G_\tau) &= \mathrm{am}_0(A+G_\tau, E+F_\tau)\\
&= \mathrm{am}_0(A,E) - (s_{u,v}^{a,b})^\sharp(0)\\
&= \mathrm{am}_\infty(E,A) - s_{u,v}^{a,b}(\infty).
\end{aligned}$$

This shows (12), (16) for $\lambda = \infty$. For $\lambda = \infty$ (13) follows in the same way. Now (15) follows from (13) and the fact that

$$\sum_{\lambda\in\sigma(E,A)} \mathrm{am}_\lambda(E,A) = \sum_{\lambda\in\sigma(E+F_\tau,A+G_\tau)} \mathrm{am}_\lambda(E+F_\tau, A+G_\tau) = n.$$

We show (a) under the assumption $s_{u,v}^{a,b}(\infty) > 0$. Hence, by (26), $(s_{u,v}^{a,b})^\sharp(0) > 0$. Again, as (a) is proved already for $\lambda = 0$, we apply it to the dual pencil and the dual perturbation and obtain that

$$s(-A-G_\tau) + E + F_\tau$$

is regular for all $\tau \in \mathbb{C}$. By the above relationship (24) between the spectra of a pencil and it's dual, we see that (a) also holds in the case $s_{u,v}^{a,b}(\infty) > 0$.

We show (14). From (12) and (13) we conclude

$$\sum_{\lambda\in\sigma(E,A)} \mathrm{am}_\lambda(E+F_\tau, A+G_\tau) \geq \sum_{\lambda\in\sigma(E,A)} \mathrm{am}_\lambda(E,A) - s_{u,v}^{a,b}(\lambda) = n - M_{u,v}^{a,b}(E,A)$$

and therefore

$$\sum_{\lambda\in\sigma(E+F_\tau,A+G_\tau)\setminus\sigma(E,A)} \mathrm{am}_\lambda(E+F_\tau,A+G_\tau) = n - \sum_{\lambda\in\sigma(E,A)} \mathrm{am}_\lambda(E+F_\tau,A+G_\tau)$$
$$\leq M_{u,v}^{a,b}(E,A).$$

To prove (b), let $\mu\in\mathbb{C}\setminus\sigma(E,A)$ with $p(\mu)\neq 0$, then $\tau=-\frac{m_{u,v}^{a,b}(\mu)}{p(\mu)}$ and (21) yield that $\mu\in\sigma(E+F_\tau,A+G_\tau)$. Conversely, $\mu\in\sigma(E+F_\tau,A+G_\tau)$, together with $\det(\mu E-A)\neq 0$ and (21) imply that $m_{u,v}^{a,b}(\mu)=-\tau p(\mu)$. Since $\mu\notin\sigma(E,A)$, $m_{u,v}^{a,b}(\mu)\neq 0$ and therefore $p(\mu)\neq 0$ and $\tau=-\frac{m_{u,v}^{a,b}(\lambda)}{p(\mu)}$. Note, that (c) follows from (12) and the definition of $\mathrm{am}_\lambda(E,A)$.

It remains to show I–III. Assume that $s_{u,v}^{a,b}$ vanishes everywhere. By definition of the quantity $s_{u,v}^{a,b}$, the function

$$s\mapsto (as-b)v^*(sE-A)^{-1}u \text{ is holomorphic in } \mathbb{C} \text{ and} \tag{27}$$
$$s\mapsto (a-bs)v^*(-sA+E)^{-1}u \text{ is holomorphic in a neighborhood of } 0. \tag{28}$$

The assumptions in statement I on a and b imply that

$$s\mapsto v^*(sE-A)^{-1}u \text{ is holomorphic in } \mathbb{C} \text{ and} \tag{29}$$
$$s\mapsto v^*(-sA+E)^{-1}u \text{ is holomorphic in a neighborhood of } 0. \tag{30}$$

Again, it is no restriction to assume that the matrix pencil $sE-A$ in (2) is already in Weierstraß form (22), i.e. the matrices S and T in (2) equal identity. This implies for (9)

$$w=v \quad\text{and}\quad z=u.$$

In what follows, we use the notation introduced for w and z in (10)–(11) in the same way for v and u. We consider the left upper corner of the block matrix in (22) and choose one block associated to λ_j, $1\leq j\leq l$. For this we consider the multi-index $s(\lambda_j)=(s_1(\lambda_j),\ldots,s_{\ell_{\lambda_j}}(\lambda_j))$ and, in view of (11), we obtain for some k with $1\leq k\leq \ell_{\lambda_j}$

$$\begin{bmatrix} v_1^{j,k} \\ \vdots \\ v_{s_k(\lambda_j)}^{j,k}\end{bmatrix}^* \left(sI-J_{s_k(\lambda_j)}(\lambda_j)\right)^{-1} \begin{bmatrix} u_1^{j,k} \\ \vdots \\ u_{s_k(\lambda_j)}^{j,k}\end{bmatrix}. \tag{31}$$

We have

$$\left(sI - J_{s_k(\lambda_j)}(\lambda_j)\right)^{-1} = \begin{bmatrix} (s-\lambda_j)^{-1} & (s-\lambda_j)^{-2} & \cdots & (s-\lambda_j)^{-s_k(\lambda_j)} \\ & (s-\lambda_j)^{-1} & \cdots & (s-\lambda_j)^{-s_k(\lambda_j)+1} \\ & & \ddots & \vdots \\ & & & (s-\lambda_j)^{-1} \end{bmatrix},$$

which is a meromorphic function in $\mathbb{C}$. In order to have a holomorphic function in $\mathbb{C}$, see (29), all coefficients of the poles in (31) has to vanish. Therefore, for all $s \in \mathbb{C}$

$$\begin{aligned} &\overline{v}_1^{j,k}\left(u_1^{j,k}(s-\lambda_j)^{-1} + u_2^{j,k}(s-\lambda_j)^{-2} + \cdots + u_{s_k(\lambda_j)}^{j,k}(s-\lambda_j)^{-s_k(\lambda_j)}\right) \\ &+\overline{v}_2^{j,k}\left(u_2^{j,k}(s-\lambda_j)^{-1} + u_3^{j,k}(s-\lambda_j)^{-2} + \cdots + u_{s_k(\lambda_j)}^{j,k}(s-\lambda_j)^{-s_k(\lambda_j)+1}\right) \\ &+ \cdots + \overline{v}_{s_k(\lambda_j)}^{j,k} u_{s_k(\lambda_j)}^{j,k}(s-\lambda_j)^{-1} = 0 \end{aligned}$$

and all coefficients in front of the expressions $(s-\lambda_j)^{-1}, \ldots, (s-\lambda_j)^{s_k(\lambda_j)}$ vanish. This show $w^j \perp z^j$ in (17) (recall that $w = v$ and $z = u$ and, hence, $w^j = v^j$ and $z^j = u^j$) and (18).

In order to show the second half of (17) and (19) we use that the function in (30) is holomorphic in a neighborhood of zero. Then, by (23) and a similar argument as above, we see that the coefficients of different powers of s^{-1} have to vanish and (17)–(19) is proved.

Conversely, if (17)–(19) hold then the above calculations show that $s_{u,v}^{a,b}$ is identically equal to zero.

The statement on the polynomial p is a direct consequence of what we have proved above and the definition (7) of p.

In order to show statement II we proceed as in the proof of statement I and therefore (27) and (28) hold. From this follows (30) and instead of (29) we have

$$s \mapsto v^*(sE - A)^{-1}u \text{ is holomorphic in } \mathbb{C} \setminus \{b/a\} \\ \text{and has a pole of order} \leq 1 \text{ in } b/a.$$

Now statement II follows in the same way as in the proof of statement I.

In order to show statement III we proceed as in the proof of statement I. Therefore (29) holds and instead of (30) we have

$$s \mapsto v^*(-sA + E)^{-1}u \text{ is meromorphic and has a pole of order} \leq 1 \text{ in } 0$$

and, again, statement III follows in the same way. □

4. Asymptotic behavior of the eigenvalues of the perturbed pencil

In the next proposition, we study the asymptotic behavior of the eigenvalues as $\tau \to \infty$. An analogous result for generic perturbations of matrices was obtained previously in [5, Proposition 4.1]. The proof presented here is mainly based on this result.

Proposition 4.1. *Let $sE-A \in \mathbb{C}[s]^{n\times n}$ be a regular matrix pencil, let $sF_\tau - G_\tau = \tau(as-b)uv^*$ as in* (1) *and $\frac{b}{a} \notin \sigma(E,A)$, $a \neq 0$. Assume that one of the relations* (17)–(19) *is not satisfied.*

Then $\deg p$ *eigenvalues of $s(E+\tau auv^*) - (A+\tau buv^*)$ converge to the roots of p and one eigenvalue converges to $\frac{b}{a}$ as $\tau \to \infty$.*

Proof. By Theorem 3.2 I. the polynomial p is non-zero. We rewrite the right factor in (21) and consider

$$\tau^{-1} m_{u,v}^{a,b} + p \quad \text{for } \tau \in \mathbb{C} \setminus \{0\} \tag{32}$$

in a neighborhood $D_j(\varepsilon)$ of the pairwise distinct roots $\{\mu_j\}_{j=1}^d$ of p with $d \leq \deg p$ given by $D_j(\varepsilon) := \{\lambda \in \mathbb{C} \mid |\lambda - \mu_j| < \varepsilon\}$ with $j = 1,\dots,d$. Here $\varepsilon > 0$ is chosen such that the discs $D_j(\varepsilon)$ are pairwise disjoint. As $\tau \to \infty$ the polynomial in (32) converges on $\cup_{j=1}^d D_j(\varepsilon)$ uniformly to p. Hence there exists $\tau(\varepsilon) > 0$ such that for all $|\tau| \geq \tau(\varepsilon)$ we have

$$|p(z) - (\tau^{-1} m_{u,v}^{a,b}(z) + p(z))| = |\tau^{-1} m_{u,v}^{a,b}(z)| < 1$$

for all $z \in \partial D_j(\varepsilon)$ and all $j = 1,\dots,d$. Rouché's theorem yields that the number of zeros of p and $\tau^{-1} m_{u,v}^{a,b} + p$ inside of $D_j(\varepsilon)$ coincide for $|\tau| \geq \tau(\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, the roots of $\tau^{-1} m_{u,v}^{a,b} + p$ converges to the roots of p. Further, by (21), the roots of $\tau^{-1} m_{u,v}^{a,b} + p$ are eigenvalues of $s(E+F_\tau) - (A+G_\tau)$. Finally, one eigenvalue converges to $\frac{b}{a}$ which happens to be a root of p. $\square$

A result similar to Proposition 4.1 holds also in the case where $a = 0$. Here we need to assume that $\infty \notin \sigma(E,A)$, i.e. E is invertible, and one can show, e.g. by using the dual pencil, that one eigenvalue converges to ∞ as $\tau \to \infty$.

5. The local behavior of the eigenvalues for small changes of the parameter τ

In the following, we consider the local behavior of semi-simple eigenvalues for small changes of the parameter τ. Recall that for a regular matrix pencil $sE - A$ we always have for $\lambda \in \mathbb{C} \cup \{\infty\}$

$$|\mathrm{gm}_\lambda(E, A) - \mathrm{gm}_\lambda(E + F_\tau, A + G_\tau)| \leq 1, \tag{33}$$

see, e.g., [26, 27]. In particular, this implies that if λ is not in the spectrum of the pencil $sE - A$, the corresponding kernel of the perturbed pencil is at most one-dimensional.

This leads to the following situation which is described in Proposition 5.1 below: Given a semi-simple eigenvalue λ_0 of the pencil $sE - A$. Then

$$\mathrm{gm}_{\lambda_0}(E, A) = \mathrm{am}_{\lambda_0}(E, A) = l_{\lambda_0}$$

and, hence, $s_j(\lambda_0) = 1$ for all $1 \leq j \leq l_{\lambda_0}$, cf. (3). By (8) the quantity $s_{u,v}^{a,b}(\lambda_0)$ equals zero or one. If $s_{u,v}^{a,b}(\lambda_0) = 1$ then there is a τ-dependent smooth eigenvalue curve c defined in a neighborhood of zero starting in λ, that is, $c(0) = \lambda_0$, and each $c(\tau)$ is an eigenvalue of $s(E + F_\tau) - (A + G_\tau)$. On the other hand, if $s_{u,v}^{a,b}(\lambda) = 0$ then the eigenvalue λ_0 remains stationary.

In the case of a semi-simple eigenvalue $\lambda_0 \in \sigma(E, A) \backslash \{\infty\}$, the Weierstraß form (2) can be written as

$$S(sE - A)T = \begin{bmatrix} (s - \lambda_0)I_{l_{\lambda_0}} & 0 \\ 0 & s\hat{E} - \hat{A} \end{bmatrix}, \quad S = \begin{bmatrix} S_{\lambda_0} \\ \hat{S} \end{bmatrix}, \quad T = [T_{\lambda_0}, \hat{T}] \tag{34}$$

for some invertible $S, T \in \mathbb{C}^{n \times n}$ and $S_{\lambda_0}, T_{\lambda_0}^* \in \mathbb{C}^{l_{\lambda_0} \times n}$ such that the pencil $s\hat{E} - \hat{A}$ is in Weierstraß form with $\sigma(\hat{E}, \hat{A}) = \sigma(E, A) \setminus \{\lambda_0\}$. For a corresponding result in the chordal distance, we refer to [28, Theorem VI.2.2]. A related result for analytic matrix-valued functions and perturbations of higher rank is given in [16, Theorem 6].

Proposition 5.1. *Let $sE - A \in \mathbb{C}[s]^{n \times n}$ be a regular matrix pencil and $sF_\tau - G_\tau = \tau(as - b)uv^*$ and let $\lambda_0 \in \sigma(E, A) \backslash \{\infty\}$ be semi-simple. Let S, T invertible and $S_{\lambda_0}, T_{\lambda_0}$ such that* (34) *and $\sigma(\hat{E}, \hat{A}) = \sigma(E, A) \setminus \{\lambda_0\}$ hold.*

Then $s_{u,v}^{a,b}(\lambda_0) \leq 1$ and we have $(a\lambda_0 - b)v^* T_{\lambda_0} S_{\lambda_0} u \neq 0$ if and only if $s_{u,v}^{a,b}(\lambda_0) = 1$.

(i) *If* $s_{u,v}^{a,b}(\lambda_0) = 1$ *then there exists* $\varepsilon > 0$ *and a smooth (that is, an infinitely often differentiable) function* $c : (-\varepsilon, \varepsilon) \to \mathbb{R}$ *with* $c(0) = \lambda_0$, $c(\tau) \in \sigma(E + F_\tau, A + G_\tau)$ *and for* $\tau \in (-\varepsilon, \varepsilon)$

$$c(\tau) = \lambda_0 + \tau(b - a\lambda_0)v^* T_{\lambda_0} S_{\lambda_0} u + o(\tau) \tag{35}$$

such that $c(\tau) \notin \sigma(E, A)$ *for* $\tau \neq 0$.

(ii) *If* $s_{u,v}^{a,b}(\lambda_0) = 0$ *then the eigenvalue* λ_0 *is stationary in the following sense: There exists a neighborhood* U *around* λ_0 *and* $\varepsilon > 0$ *such that*

$$U \cap \sigma(E + F_\tau, A + G_\tau) = \{\lambda_0\}$$

for $\tau \in (-\varepsilon, \varepsilon)$.

Proof. In what follows, we set to improve readability $k := l_{\lambda_0}$. We consider the polynomial Φ in the variables τ and λ given by

$$\Phi(\lambda, \tau) := \frac{\det(\lambda E - A + \tau(a\lambda - b)uv^*)}{(\lambda - \lambda_0)^{k-1}}, \quad \Phi(\lambda_0, 0) = 0.$$

The function Φ is a polynomial. Indeed, for $\tau = 0$ the function $\lambda \mapsto \det(\lambda E - A)$ has, by assumption, a zero of order k at λ_0. Moreover, for $\tau \neq 0$, it follows from (33) that the function $\lambda \mapsto \det(\lambda E - A + \tau(a\lambda - b)uv^*)$ has a zero of order at least $k - 1$ at λ_0.

The determinant can be rewritten with Sylvester's identity

$$\det(\lambda E - A + \tau(a\lambda - b)uv^*) = \det(\lambda E - A)(1 + \tau(a\lambda - b)v^*(\lambda E - A)^{-1}u) \tag{36}$$

Furthermore, we rewrite

$$\begin{aligned} v^*(\lambda E - A)^{-1}u &= v^*T(S(\lambda E - A)T)^{-1}Su \\ &= v^*T_{\lambda_0}(\lambda - \lambda_0)^{-1}S_{\lambda_0}u + v^*\hat{T}(\lambda\hat{E} - \hat{A})^{-1}\hat{S}u. \end{aligned} \tag{37}$$

By assumption $\lambda \mapsto (\lambda\hat{E} - \hat{A})^{-1}$ has no pole in λ_0. Consequently, $\lambda \mapsto (a\lambda - b)v^*(\lambda E - A)^{-1}u$ has either no pole in λ_0, or a pole of order one, i.e. $s_{u,v}^{a,b}(\lambda_0) \leq 1$. We inspect the first summand in (37) which leads to $(a\lambda_0 - b)v^*T_{\lambda_0}S_{\lambda_0}u = 0$ if and only if $s_{u,v}^{a,b}(\lambda_0) = 0$.

Assume $s_{u,v}^{a,b}(\lambda_0) = 1$. Invoking

$$\det(\lambda E - A) = \det(ST)^{-1}(\lambda - \lambda_0)^k \det(\lambda \hat{E} - \hat{A}), \tag{38}$$

we obtain for the partial derivatives together with (36)

$$\tfrac{\partial}{\partial \lambda}\Phi(\lambda_0, 0) = \tfrac{\partial}{\partial \lambda}\left(\tfrac{\det(\lambda E - A)}{(\lambda-\lambda_0)^{k-1}}(1 + \tau(a\lambda - b)v^*(\lambda E - A)^{-1}u)\right)\Big|_{\lambda=\lambda_0,\tau=0}$$

$$= \tfrac{\partial}{\partial \lambda}\left(\det(ST)^{-1}(\lambda - \lambda_0)\det(\lambda\hat{E} - \hat{A})(1 + \tau(a\lambda - b)v^*(\lambda E - A)^{-1}u\right)\Big|_{\lambda=\lambda_0,\tau=0}.$$

In order to shorten the expressions, we introduce

$$\begin{aligned} f(\lambda) &:= \det(\lambda\hat{E} - \hat{A}),\\ g(\lambda, \tau) &:= (\lambda - \lambda_0)(1 + \tau(a\lambda - b)v^*(\lambda E - A)^{-1}u),\\ h(\lambda) &:= (a\lambda - b)v^*(\lambda - \lambda_0)(\lambda E - A)^{-1}u. \end{aligned}$$

Note that by $s_{u,v}^{a,b}(\lambda_0) = 1$, the functions f, g and h are polynomials (here g in the two variables λ and τ). Therefore, we continue with $g(\lambda_0, 0) = 0$ and

$$\begin{aligned} \tfrac{\partial}{\partial \lambda}\Phi(\lambda_0, 0) &= \det(ST)^{-1}\left(f'(\lambda)g(\lambda,\tau) + f(\lambda)\tfrac{\partial}{\partial\lambda}g(\lambda,\tau)\right)\Big|_{\lambda=\lambda_0,\tau=0}\\ &= \det(ST)^{-1}\left(f'(\lambda)g(\lambda,\tau) + f(\lambda)(1 + \tau h'(\lambda))\right)\Big|_{\lambda=\lambda_0,\tau=0}\\ &= \det(ST)^{-1}\det(\lambda_0\hat{E} - \hat{A}) \neq 0 \end{aligned}$$

and with (37) and (38)

$$\begin{aligned} \tfrac{\partial}{\partial \tau}\Phi(\lambda_0, 0) &= \left[(a\lambda - b)v^*(\lambda E - A)^{-1}u\frac{\det(\lambda E - A)}{(\lambda - \lambda_0)^{k-1}}\right]\Bigg|_{\lambda=\lambda_0}\\ &= \left[(a\lambda - b)v^*T_{\lambda_0}(\lambda - \lambda_0)^{-1}S_{\lambda_0}u\frac{\det(\lambda E - A)}{(\lambda - \lambda_0)^{k-1}}\right]\Bigg|_{\lambda=\lambda_0}\\ &\quad + \left[(a\lambda - b)v^*\hat{T}(\lambda\hat{E} - \hat{A})^{-1}\hat{S}u\frac{\det(\lambda E - A)}{(\lambda - \lambda_0)^{k-1}}\right]\Bigg|_{\lambda=\lambda_0}\\ &= \left[(a\lambda - b)v^*T_{\lambda_0}S_{\lambda_0}u\frac{\det(\lambda E - A)}{(\lambda - \lambda_0)^{k}}\right]\Bigg|_{\lambda=\lambda_0}\\ &= (a\lambda_0 - b)v^*T_{\lambda_0}S_{\lambda_0}u\det(ST)^{-1}\det(\lambda_0\hat{E} - \hat{A}). \end{aligned}$$

Thus, by the implicit function theorem there exists $\hat{\varepsilon} > 0$ and $c : (-\hat{\varepsilon}, \hat{\varepsilon}) \to \mathbb{R}$ such that $\Phi(c(\tau), \tau) = 0$ for all $\tau \in (-\hat{\varepsilon}, \hat{\varepsilon})$ meaning that $c(\tau)$ is an eigenvalue of $sE - A + \tau(as - b)uv^*$ and the derivative is given by

$$\tfrac{d}{d\tau}c(0) = -\frac{\frac{\partial}{\partial\tau}\Phi(\lambda_0, 0)}{\frac{\partial}{\partial\lambda}\Phi(\lambda_0, 0)} = (b - a\lambda_0)v^* T_{\lambda_0} S_{\lambda_0} u$$

Since $s^{a,b}_{u,v}(\lambda_0) = 1$ we have $\frac{d}{d\tau}c(0) = (b - a\lambda_0)v^* T_{\lambda_0} S_{\lambda_0} u \neq 0$ and, hence, (35) holds for some $\varepsilon \leq \hat{\varepsilon}$. Since Φ is smooth, also c is smooth which proves (i).

In order to prove (ii) observe that by $s^{a,b}_{u,v}(\lambda_0) = 0$ the rational function $\lambda \mapsto \tau(a\lambda - b)v^*(\lambda E - A)^{-1}u$ has no pole in λ_0. Hence there exists $\varepsilon > 0$ with $|\tau(a\lambda - b)v^*(\lambda E - A)^{-1}u| < 1$ for $\tau \in (-\varepsilon, \varepsilon)$ and (ii) follows from (36). $\square$

Remark 5.2. *The local perturbation behavior at ∞ can be described by the dual pencil at $\lambda_0 = 0$. For this, let $\infty \in \sigma(E, A)$ be semi-simple. Then* (34) *holds with $(s - \lambda_0)I_{l_{\lambda_0}}$ replaced by $-I_{l_\infty}$,*

$$S(sE - A)T = \begin{bmatrix} -I_{l_\infty} & 0 \\ 0 & s\hat{E} - \hat{A} \end{bmatrix}, \quad S = \begin{bmatrix} S_\infty \\ \hat{S} \end{bmatrix}, \quad T = [T_\infty, \hat{T}]$$

for some invertible $S, T \in \mathbb{C}^{n \times n}$ and $S_\infty, T_\infty^* \in \mathbb{C}^{l_\infty \times n}$ such that $\sigma(\hat{E}, \hat{A}) = \sigma(E, A) \setminus \{\infty\}$.

The local eigenvalue curve at ∞ is by (24) *given by the local eigenvalue curve of the dual pencil $-sA + E$ at $\lambda_0 = 0$. From above, the dual pencil fulfills*

$$S(sA - E)T = \begin{bmatrix} sI_{l_\infty} & 0 \\ 0 & s\hat{A} - \hat{E} \end{bmatrix}.$$

As consequence of Proposition 5.1 applied to the dual pencil and in view of (25) *and* (26) *one has $av^* T_\infty S_\infty u \neq 0$ if and only if $s^{a,b}_{u,v}(\infty) = 1$. In this case, there exists $\varepsilon > 0$ and $c : (-\varepsilon, 0) \cup (0, \varepsilon) \to \mathbb{R}$ which is for $\tau \neq 0$ given by*

$$c(\tau)^{-1} = \tau a v^* T_\infty S_\infty u + o(\tau)$$

and fulfills $c(\tau) \in \sigma(E + F_\tau, A + G_\tau)$.

6. Interlacing of real eigenvalues

From Proposition 5.1 we see that the numbers $(a\lambda_0 - b)v^*T_{\lambda_0}S_{\lambda_0}u$ can be used to characterize the invariance of eigenvalues under perturbations. In particular, in the case of real matrices and real (semi-simple) eigenvalues, the sign of this quantity describes whether eigenvalues increase or decrease after perturbations. In this section we collect various interlacing properties.

To guarantee that $(a\lambda_0-b)v^*T_{\lambda_0}S_{\lambda_0}u$ is a real number, we assume throughout the section that $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$. Furthermore, one can show that for $\lambda_0 \in \mathbb{R}$ we already have $T_{\lambda_0}^T, S_{\lambda_0} \in \mathbb{R}^{\ell_{\lambda_0}\times n}$. This follows after a transformation to quasi-Weierstraß form over $\mathbb{R}$, see [29, Theorem 2.6], and a subsequent use of the real Jordan form [30, Section 3.4.1].

Definition 6.1. *Let $sE - A \in \mathbb{R}[s]^{n\times n}$ be a regular matrix pencil, $a, b \in \mathbb{R}$, $u, v \in \mathbb{R}^n$, let $\lambda \in \sigma(E, A) \cap (\mathbb{R} \cup \{\infty\})$ be a semi-simple eigenvalue and let $S, T \in \mathbb{C}^{n\times n}$ be invertible such that $S(sE - A)T$ is in Weierstraß form* (34). *Define*

$$\varepsilon_{u,v}^{a,b}(\lambda) := \operatorname{sgn}\left((b - a\lambda)v^T T_\lambda S_\lambda u\right), \quad \lambda \neq \infty,$$

with $\operatorname{sgn} 0 := 0$ *and* $\varepsilon_{u,v}^{a,b}(\pm\infty) := \operatorname{sgn}(-av^T T_\infty S_\infty u)$*, where S_∞ and T_∞ were introduced in Remark 5.2.*

Note that for pencils $sE - A$ with $E = E^*$ and $A = A^*$ and perturbations satisfying $u = v$ the above signs are given by the *characteristic signs* of the Hermitian pencil, see e.g. [31, 32].

In the case of semi-simple eigenvalues there is a close connection between $\varepsilon_{u,v}^{a,b}$ and $s_{u,v}^{a,b}$.

Lemma 6.2. *Let $sE - A \in \mathbb{R}^{n\times n}[s]$ be a regular matrix pencil. Let all eigenvalues in $\sigma(E, A) \cap (\mathbb{R} \cup \{\infty\})$ be semi-simple. Then for $\mu_0 \in \sigma(E, A) \cap (\mathbb{R} \cup \{\infty\})$*

$$\begin{aligned} \varepsilon_{u,v}^{a,b}(\mu_0) = 0 \quad &\Longleftrightarrow \quad s_{u,v}^{a,b}(\mu_0) = 0, \\ \varepsilon_{u,v}^{a,b}(\mu_0) \neq 0 \quad &\Longleftrightarrow \quad s_{u,v}^{a,b}(\mu_0) = 1. \end{aligned}$$

Proof. We show that

$$v^T(\lambda E - A)^{-1}u = \sum_{i=1}^{m} \frac{v^T T_{\lambda_i} S_{\lambda_i} u}{\lambda - \lambda_i} - v^T T_\infty S_\infty u + \phi(\lambda), \tag{39}$$

where ϕ is a rational function with poles in a finite subset of $\sigma(E, A) \setminus \mathbb{R}$, $\lim_{|\lambda|\to\infty} \phi(\lambda) = 0$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_m$ denote all real eigenvalues of $sE - A$. By assumption, all real eigenvalues $\lambda_1, \ldots, \lambda_m$ are semi-simple, i.e., $s(\lambda_j) = 1$ and $\mathrm{am}_{\lambda_j}(E, A) = \mathrm{gm}_{\lambda_j}(E, A) = l_{\lambda_j}$ for $j = 1, \ldots, m$, see (3). Using the Weierstraß form over $\mathbb{R}$, there exists $S, T \in \mathbb{R}^{n\times n}$ such that

$$S(\lambda E - A)T = \mathrm{diag}((\lambda - \lambda_1)I_{l_{\lambda_1}}, \ldots, (\lambda - \lambda_m)I_{l_{\lambda_m}}, -I_{n-r}, \lambda I_{r'} - J_{\mathbb{C}\setminus\mathbb{R}}).$$

where $J_{\mathbb{C}\setminus\mathbb{R}} \in \mathbb{R}^{r'\times r'}$ for some $r' \geq 1$ contains all blocks of the real Jordan canonical form which correspond to complex conjugate eigenvalues. Hence

$$v^T(\lambda E - A)^{-1}u = \sum_{i=1}^{m} \frac{v^T T_{\lambda_i} S_{\lambda_i} u}{\lambda - \lambda_i} - v^T T_\infty S_\infty u + v^T T_{\mathbb{C}\setminus\mathbb{R}}(\lambda - J_{\mathbb{C}\setminus\mathbb{R}})^{-1} S_{\mathbb{C}\setminus\mathbb{R}} u$$

where $S_{\mathbb{C}\setminus\mathbb{R}} \in \mathbb{R}^{r'\times n}$ are the last r' rows of S and $T_{\mathbb{C}\setminus\mathbb{R}} \in \mathbb{R}^{n\times r'}$ are the last r' columns of T. It remains to define ϕ as the last summand. By definition the inverse exists for all $\lambda \in \mathbb{R}$, it is real-valued and has the desired growth property at ∞. The statement of Lemma 6.2 follows now from (39) and the fact that μ_0 is semi-simple. □

If all matrices E, A, F_τ, G_τ are real and if the sign $\varepsilon^{a,b}_{u,v}(\lambda_0)$ is positive for a real semi-simple eigenvalue λ_0, then Proposition 5.1 together with Lemma 6.2 show that the perturbed pencil has an eigenvalue to the right of λ_0. This observation will be made more precise in Proposition 6.3 below: Between two real eigenvalues of the unperturbed pencil with the same sign 1 there is an eigenvalue of the perturbed pencil for all $\tau \in \mathbb{R}$.

In the formulation of the theorem, we consider the *total algebraic multiplicity* of all eigenvalues of a regular matrix pencil $sE - A \in \mathbb{R}[s]^{n\times n}$ which are contained in a subset S of the extended complex plane $\mathbb{C} \cup \{\infty\}$ and denote it by

$$\mathrm{am}_S(E, A) := \sum_{\lambda\in S} \mathrm{am}_\lambda(E, A).$$

By definition, the above sum contains only finitely many non-zero summands.

Proposition 6.3. *Let $sE - A \in \mathbb{R}^{n\times n}[s]$ be a regular matrix pencil and the pencil $sF_\tau - G_\tau$ is given by (1) with $a, b \in \mathbb{R}$, $u, v \in \mathbb{R}^n$. Furthermore, let all eigenvalues in $\sigma(E, A) \cap (\mathbb{R} \cup \{\infty\})$ be semi-simple. Let $\mu_1, \mu_2 \in \sigma(E, A) \cap (\mathbb{R} \cup \{\infty\})$ with*

$$-\infty \leq \mu_1 < \mu_2 \leq \infty \quad \textit{and} \quad \varepsilon^{a,b}_{u,v}(\mu_1) = \varepsilon^{a,b}_{u,v}(\mu_2) = 1,$$

i.e. $(b - a\mu_1)v^T T_{\mu_1} S_{\mu_1} u > 0$ *and* $(b - a\mu_2)v^T T_{\mu_2} S_{\mu_2} u > 0$, *if* $\mu_1 \neq -\infty$ *or* $\mu_2 \neq \infty$ *or* $-av^T T_\infty S_\infty u > 0$ *otherwise. Then the following holds.*

(a) *Let* μ_1 *and* μ_2 *be finite. If* $\varepsilon^{a,b}_{u,v}(\mu) = 0$ *for all* $\mu \in (\mu_1, \mu_2) \cap \sigma(E, A)$, *then for* $\tau \in \mathbb{R} \setminus \{0\}$

$$\mathrm{am}_{(\mu_1,\mu_2)}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{(\mu_1,\mu_2)}(E, A) + 1. \tag{40}$$

If $\varepsilon^{a,b}_{u,v}(\mu) = 0$ *for all* $\mu \in (\{\infty\} \cup \mathbb{R} \setminus (\mu_1, \mu_2)) \cap \sigma(E, A)$, *then for* $\tau \in \mathbb{R} \setminus \{0\}$

$$\mathrm{am}_{\{\infty\}\cup\mathbb{R}\setminus(\mu_1,\mu_2)}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{\{\infty\}\cup\mathbb{R}\setminus(\mu_1,\mu_2)}(E, A) + 1.$$

(b) *Let* $\mu_1 = -\infty$ *and* μ_2 *be finite. Then* $a \neq 0$. *If* $\varepsilon^{a,b}_{u,v}(\mu) = 0$ *for all* $\mu \in (-\infty, \mu_2) \cap \sigma(E, A)$, *then for* $\tau \in \mathbb{R} \setminus \{0\}$

$$\mathrm{am}_{(-\infty,\mu_2)}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{(-\infty,\mu_2)}(E, A) + 1.$$

If $\varepsilon^{a,b}_{u,v}(\mu) = 0$ *for all* $\mu \in (\mu_2, \infty) \cap \sigma(E, A)$, *then for* $\tau \in \mathbb{R} \setminus \{0\}$

$$\mathrm{am}_{(\mu_2,\infty)}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{(\mu_2,\infty)}(E, A) + 1.$$

(c) *Let* μ_1 *be finite and* $\mu_2 = \infty$. *Then* $a \neq 0$. *If* $\varepsilon^{a,b}_{u,v}(\mu) = 0$ *for all* $\mu \in (-\infty, \mu_1) \cap \sigma(E, A)$, *then for* $\tau \in \mathbb{R} \setminus \{0\}$

$$\mathrm{am}_{(-\infty,\mu_1)}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{(-\infty,\mu_1)}(E, A) + 1.$$

If $\varepsilon^{a,b}_{u,v}(\mu) = 0$ *for all* $\mu \in (\mu_1, \infty) \cap \sigma(E, A)$, *then for* $\tau \in \mathbb{R} \setminus \{0\}$

$$\mathrm{am}_{(\mu_1,\infty)}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{(\mu_1,\infty)}(E, A) + 1.$$

(d) *Let* $\mu_1 = -\infty$ *and* $\mu_2 = \infty$. *Then* $a \neq 0$. *If* $\varepsilon^{a,b}_{u,v}(\mu) = 0$ *for all* $\mu \in \mathbb{R} \cap \sigma(E, A)$, *then for* $\tau \in \mathbb{R} \setminus \{0\}$

$$\mathrm{am}_{\mathbb{R}}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{\mathbb{R}}(E, A) + 1.$$

In the case $a = 0$ *then, by definition,* $\varepsilon^{a,b}_{u,v}(\pm\infty) = 0$ *and* μ_1, μ_2 *are finite.*

Proof. <u>*Step 1:*</u> By Lemma 6.2, $\varepsilon_{u,v}^{a,b}(\mu) = 1$ implies $s_{u,v}^{a,b}(\mu) = 1$ and $\varepsilon_{u,v}^{a,b}(\mu) = 0$ implies $s_{u,v}^{a,b}(\mu) = 0$. As $\varepsilon_{u,v}^{a,b}(\mu_1) = 1$, the function $s_{u,v}^{a,b}$ is not identically to zero. Then p is not identically to zero, see Theorem 3.2, where $m_{u,v}^{a,b}$ and p are given by (6) and (7). We define a rational function ψ by

$$\begin{aligned}\psi(\lambda) &:= -\frac{p(\lambda)}{m_{u,v}^{a,b}(\lambda)} = (b - a\lambda)v^T(\lambda E - A)^{-1}u \\ &= (b - a\lambda)\left(\sum_{i=1}^{m} \frac{v^T T_{\lambda_i} S_{\lambda_i} u}{\lambda - \lambda_i} - v^T T_\infty S_\infty u + \phi(\lambda)\right),\end{aligned}$$

where we used the representation from (39). By assumption the function ψ has on the real lines only poles in those λ_i with $\varepsilon_{u,v}^{a,b}(\lambda_i) = 1$, more precisely, the behaviour of ψ close to the simple poles μ_1, μ_2 and at $\pm\infty$ is given by the following. For finite numbers μ_1, μ_2 we have

$$\begin{array}{lcl} \lim\limits_{\lambda \downarrow \mu_1} \psi(\lambda) = \infty, & & \lim\limits_{\lambda \downarrow \mu_2} \psi(\lambda) = \infty, \\ & \text{and} & \\ \lim\limits_{\lambda \uparrow \mu_1} \psi(\lambda) = -\infty, & & \lim\limits_{\lambda \uparrow \mu_2} \psi(\lambda) = -\infty. \end{array} \tag{41}$$

At infinity we have for the case $\infty \notin \sigma(E, A)$ that $T_\infty = S_\infty = 0$ and, hence,

$$\begin{aligned}\lim_{\lambda \to \infty} \psi(\lambda) &= \lim_{\lambda \to \infty} (b - a\lambda)\left(\sum_{i=1}^{m} \frac{v^T T_{\lambda_i} S_{\lambda_i} u}{\lambda - \lambda_i} + \phi(\lambda)\right) \\ &= -a \sum_{i=1}^{m} v^T T_{\lambda_i} S_{\lambda_i} u + \lim_{\lambda \to \infty} (b - a\lambda)\phi(\lambda).\end{aligned}$$

As ϕ has finitely many poles in $\mathbb{C} \setminus \mathbb{R}$, the limit $\lim_{\lambda \to \infty}(b - a\lambda)\phi(\lambda)$ is a finite complex number ϕ_∞ and we set

$$\lim_{\lambda \to \infty} \psi(\lambda) = a \sum_{i=1}^{m} v^T T_{\lambda_i} S_{\lambda_i} u + \phi_\infty =: \psi_\infty.$$

Note that in the case $a = 0$ one has $\psi_\infty = 0$. One obtains the same value for $\lim_{\lambda \to -\infty} \psi(\lambda)$. The behaviour of ψ at ∞ and at $-\infty$ is as follows

$$\lim_{\lambda \to -\infty} \psi(\lambda) = \begin{cases} \infty & \text{if } \varepsilon_{u,v}^{a,b}(-\infty) = 1, \\ 0 & \text{if } \varepsilon_{u,v}^{a,b}(-\infty) = 0 \text{ or if } a = 0, \\ \psi_\infty & \text{if } \infty \notin \sigma(E, A), \end{cases} \tag{42}$$

$$\lim_{\lambda\to\infty}\psi(\lambda)=\begin{cases}-\infty & \text{if } \varepsilon_{u,v}^{a,b}(\infty)=1,\\ 0 & \text{if } \varepsilon_{u,v}^{a,b}(\infty)=0 \text{ or if } a=0,\\ \psi_\infty & \text{if } \infty\notin\sigma(E,A).\end{cases} \tag{43}$$

Step 2: We show (a). Then, by assumption, μ_1 and μ_2 are finite. If $\varepsilon_{u,v}^{a,b}(\mu)=0$ for all $\mu\in(\mu_1,\mu_2)\cap\sigma(E,A)$, then ψ is locally holomorphic in (μ_1,μ_2) and (41) holds. Thus, by the intermediate value theorem for $\tau\in\mathbb{R}\setminus\{0\}$, we find $\lambda_\tau\in(\mu_1,\mu_2)$ such that

$$\tau^{-1}=\psi(\lambda). \tag{44}$$

If $\lambda_\tau\in\sigma(E,A)$ then we have with the function φ from (d) in Theorem 3.2

$$\tau^{-1}=\psi(\lambda_\tau)=\frac{-p(\lambda_\tau)}{m_{u,v}^{a,b}(\lambda_\tau)}=-\varphi(\lambda_\tau),$$

hence $p(\lambda_\tau)\neq 0$. Now (40) follows from (d) and (13) in Theorem 3.2.

If $\lambda_\tau\notin\sigma(E,A)$ then (b) in Theorem 3.2 implies $\lambda_\tau\in\sigma(E+F_\tau,A+G_\tau)$ and, with (13), (40) follows.

In order to prove the second assertion in (a) assume that $\varepsilon_{u,v}^{a,b}(\mu)=0$ for all $\mu\in(\{\infty\}\cup\mathbb{R}\setminus(\mu_1,\mu_2))\cap\sigma(E,A)$. Then either $\infty\in\sigma(E,A)$ with $\varepsilon_{u,v}^{a,b}(\infty)=\varepsilon_{u,v}^{a,b}(-\infty)=0$ or $\infty\notin\sigma(E,A)$. Then the limit behaviour of the function ψ at μ_1, μ_2 and $\pm\infty$ together with the intermediate value theorem yields the existence of at least one solution (at least two solutions in the case $\varepsilon_{u,v}^{a,b}(\infty)=1$) of (44) in $\{\infty\}\cup\mathbb{R}\setminus(\mu_1,\mu_2)$. Using the same arguments as above shows (a).

Step 3: Note that by definition $\varepsilon_{u,v}^{a,b}(\infty)=\varepsilon_{u,v}^{a,b}(-\infty)$, hence $\varepsilon_{u,v}^{a,b}(\infty)=1$ implies $\varepsilon_{u,v}^{a,b}(-\infty)=1$ and vice versa. Therefore, (b) and (c) are equivalent. Assume $\mu_1=-\infty$ and μ_2 is finite. Then by (41)–(43)

$$\lim_{\lambda\downarrow\mu_2}\psi(\lambda)=\infty,\quad \lim_{\lambda\uparrow\mu_2}\psi(\lambda)=-\infty,\quad \lim_{\lambda\to-\infty}\psi(\lambda)=\infty,\quad \lim_{\lambda\to\infty}\psi(\lambda)=-\infty,$$

and (b) follows from the intermediate value theorem in a similar way as above. The same applies to (d). □

In Proposition 6.3 we always have two different real numbers or infinity with signs equal to one. But it may happen that there is only one real number with this property.

Corollary 6.4. *Let $sE - A \in \mathbb{R}^{n\times n}[s]$ be a regular matrix pencil and the pencil $sF_\tau - G_\tau$ is given by (1) with $a, b \in \mathbb{R}$, $u, v \in \mathbb{R}^n$. Let all eigenvalues in $\sigma(E,A) \cap (\mathbb{R} \cup \{\infty\})$ be semi-simple. Let $\mu_1 \in \sigma(E,A) \cap (\mathbb{R} \cup \{\infty\})$ with $\varepsilon^{a,b}_{u,v}(\mu_1) = 1$ and let $\varepsilon^{a,b}_{u,v}(\mu) = 0$ for all $\mu \in ((\{\infty\} \cup \mathbb{R}) \setminus \{\mu_1\}) \cap \sigma(E,A)$. Then for $\tau \in \mathbb{R} \setminus \{0\}$*

$$\mathrm{am}_{(\{\infty\}\cup\mathbb{R})\setminus\{\mu_1\}}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{(\{\infty\}\cup\mathbb{R})\setminus\{\mu_1\}}(E,A) + 1.$$

Proof. If μ_1 is finite we have $\varepsilon^{a,b}_{u,v}(\infty) = \varepsilon^{a,b}_{u,v}(-\infty) = 0$. Then by (41)–(43) the limit of ψ at $\pm\infty$ exists and is finite and

$$\lim_{\lambda \downarrow \mu_1} \psi(\lambda) = \infty, \quad \lim_{\lambda \uparrow \mu_1} \psi(\lambda) = -\infty.$$

The assertion follows from the intermediate value theorem in the same way as in Proposition 6.3.

If μ_1 is not finite we have $\varepsilon^{a,b}_{u,v}(\infty) = \varepsilon^{a,b}_{u,v}(-\infty) = 1$. Then the assertion follows from (42), (43) and the intermediate value theorem. $\square$

If in Proposition 6.3 we have $a \neq 0$, then the function ψ has no pole in b/a. Hence we have $\psi(b/a) = 0$ for all $\lambda \in \sigma(E,A)$ with the property $\varepsilon^{a,b}_{u,v}(\lambda) = 0$ or with the property $\lambda \notin \sigma(E,A)$. The fact that b/a is a zero of the function ψ can be used to find a smaller interval for an eigenvalue of the perturbed pencil. As the argument is based on the intermediate value theorem in the same way as in Proposition 6.3, we obtain the following corollary.

Corollary 6.5. *Let μ_1 be as in Corollary 6.4 and let $a \neq 0$.*

(1) *If $\varepsilon^{a,b}_{u,v}(\mu) = 0$ for all $\mu \in (\mu_1, b/a] \cap \sigma(E,A)$. Then for all $\tau > 0$*

$$\mathrm{am}_{(\mu_1, b/a)}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{(\mu_1, b/a)}(E,A) + 1.$$

(2) *If $\varepsilon^{a,b}_{u,v}(\mu) = 0$ for all $\mu \in [b/a, \mu_1) \cap \sigma(E,A)$. Then for all $\tau < 0$*

$$\mathrm{am}_{(b/a, \mu_1)}(E + F_\tau, A + G_\tau) \geq \mathrm{am}_{(b/a, \mu_1)}(E,A) + 1.$$

In the following, we study the case where all signs $\varepsilon^{a,b}_{u,v}(\lambda)$ are non-positive (analogously for non-negative) and show in the next corollary that the eigenvalues interlace, i.e. roughly speaking that for all parameter values $\tau \in \mathbb{R}$ there is only one eigenvalue of $s(E + F_\tau) - (A + G_\tau)$ between two consecutive eigenvalues of $sE - A$. We will distinguish two cases, depending on the value of $\varepsilon^{a,b}_{u,v}(-\infty) = \varepsilon_{u,b}(\infty)$.

Corollary 6.6. *Under the assumptions of Proposition 6.3, assume* $\sigma(E,A) \subseteq \mathbb{R}\cup\{\infty\}$ *and* $\varepsilon^{a,b}_{u,v}(\lambda) \geq 0$ *for all* $\lambda \in \sigma(E,A)$. *Assume that* $\varepsilon^{a,b}_{u,v}(-\infty) = 0$ *or* $\infty \notin \sigma(E,A)$.

Denote by μ_j, $j = 1,\ldots,m$, $m \leq n$, *those real eigenvalues with positive sign and assume that there exists at least one eigenvalue with this property, i.e.* $m \geq 1$, *ordered in the following way*

$$-\infty < \mu_1 < \mu_2 < \ldots < \mu_m < \infty \quad \textit{and} \quad \varepsilon^{a,b}_{u,v}(\mu_j) = 1,\ j = 1,\ldots,m.$$

Then $\sigma(E+F_\tau, A+G_\tau) \subseteq \mathbb{R}\cup\{\infty\}$ *for all* $\tau \in \mathbb{R}$. *For* $\lambda \in \sigma(E,A) \setminus \{\mu_1,\ldots,\mu_m\}$

$$\mathrm{am}_\lambda(E,A) + 1 \geq \mathrm{am}_\lambda(E+F_\tau, A+G_\tau) \geq \mathrm{am}_\lambda(E,A), \tag{45}$$

and for $\lambda \in \{\mu_1,\ldots,\mu_m\}$

$$\mathrm{am}_\lambda(E+F_\tau, A+G_\tau) = \mathrm{am}_\lambda(E,A) - 1. \tag{46}$$

For $j = 1,\ldots,m-1$

$$\mathrm{am}_{(\mu_j,\mu_{j+1})}(E+F_\tau, A+G_\tau) = \mathrm{am}_{(\mu_j,\mu_{j+1})}(E,A) + 1, \tag{47}$$

$$\mathrm{am}_{(-\infty,\mu_1)\cup(\mu_m,\infty)}(E+F_\tau, A+G_\tau) = \mathrm{am}_{(-\infty,\mu_1)\cup(\mu_m,\infty)}(E,A) + 1, \tag{48}$$

Proof. As $m \geq 1$ and $\varepsilon^{a,b}_{u,v}(\mu_1) \geq 1$, Lemma 6.2 implies that $s^{a,b}_{u,v}$ is not identically zero. Therefore by Theorem 3.2 the pencil $s(E+F_\tau) - (A+G_\tau)$ is regular. Thus, the algebraic multiplicities of all eigenvalues add up to n. Statement (46) follows from Theorem 3.2. Hence, together with (13),

$$\mathrm{am}_{\sigma(E,A)}(E+F_\tau, A+G_\tau) \geq n - m. \tag{49}$$

If $m = 1$ then by Corollary 6.4 the algebraic multiplicity of $s(E+F_\tau) - (A+G_\tau)$ in $\mathbb{R}\setminus\{\mu_1\}$ increases by one which shows together with (49) that (48) holds and that all eigenvalues of $s(E+F_\tau) - (A+G_\tau)$ are real. If $m \geq 2$ then item (a) in Proposition 6.3 applied to the intervals (μ_j,μ_{j+1}), $j = 1,\ldots,m-1$, and to the interval (μ_1,μ_m) shows (47) and (48). Then (46)–(48) and (13) imply (45). Together with (49) it follows that all eigenvalues of $s(E+F_\tau) - (A+G_\tau)$ are real. □

There is an analogue statement for the case $\varepsilon^{a,b}_{u,v}(-\infty) = \varepsilon^{a,b}_{u,v}(\infty) = 1$. Its proof is similar to Corollary 6.6 and uses items (a), (b) and (d) from Proposition 6.3.

Corollary 6.7. *Under the assumptions of Proposition 6.3, assume* $\sigma(E,A) \subseteq \mathbb{R} \cup \{\infty\}$ *and* $\varepsilon_{u,v}^{a,b}(\lambda) \geq 0$ *for all* $\lambda \in \sigma(E,A)$. *Assume that* $\varepsilon_{u,v}^{a,b}(-\infty) = 1$.

Denote by μ_j, $j = 1, \dots, m+1$, $m+1 \leq n$, *those real eigenvalues with positive sign ordered in the following way*

$$-\infty = \mu_1 < \mu_2 < \dots < \mu_{m+1} = \infty \quad \text{and} \quad \varepsilon_{u,v}^{a,b}(\mu_j) = 1,\ j = 1, \dots, m+1.$$

Then $\sigma(E + F_\tau, A + G_\tau) \subseteq \mathbb{R} \cup \{\infty\}$ *for all* $\tau \in \mathbb{R}$. *For* $\lambda \in \sigma(E,A) \setminus \{\mu_1, \dots, \mu_m\}$

$$\mathrm{am}_\lambda(E,A) + 1 \geq \mathrm{am}_\lambda(E + F_\tau, A + G_\tau) \geq \mathrm{am}_\lambda(E,A), \tag{50}$$

and for $\lambda \in \{\mu_1, \dots, \mu_m\}$

$$\mathrm{am}_\lambda(E + F_\tau, A + G_\tau) = \mathrm{am}_\lambda(E,A) - 1. \tag{51}$$

For $j = 1, \dots, m$

$$\mathrm{am}_{(\mu_j,\mu_{j+1})}(E + F_\tau, A + G_\tau) = \mathrm{am}_{(\mu_j,\mu_{j+1})}(E,A) + 1, \tag{52}$$

Corollary 6.8. *Under the assumptions of Proposition 6.3 assume* $\sigma(E,A) \subseteq \mathbb{R} \cup \{\infty\}$ *and* $\varepsilon_{u,v}^{a,b}(\lambda) = 0$ *for all* $\lambda \in \sigma(E,A)$. *Then* $\sigma(E + F_\tau, A + G_\tau) \subseteq \mathbb{R} \cup \{\infty\}$ *for all* $\tau \in \mathbb{R}$ *and the algebraic multiplicities coincides*

$$\mathrm{am}_\lambda(E + F_\tau, A + G_\tau) = \mathrm{am}_\lambda(E,A) \qquad \text{for } \lambda \in \mathbb{R} \cup \{\infty\}. \tag{53}$$

Proof. It follows from Lemma 6.2 that the function $s_{u,v}^{a,b}$ is the zero function. Then the statement follows from (15). □

7. Application to electrical networks

7.1. Low-pass filter

As a first example, we consider an RC-low-pass filter shown in Figure 1. This circuit consists of a capacitor with value $C > 0$ interconnected in row to a resistor with value $R > 0$ and there are voltage in- and outputs u_{in} and u_{out}. Altough this is a very basic example, the conclusions from this example can be generalized to arbitrary circuits consisting of resistors, capacitors and current or voltage sources.

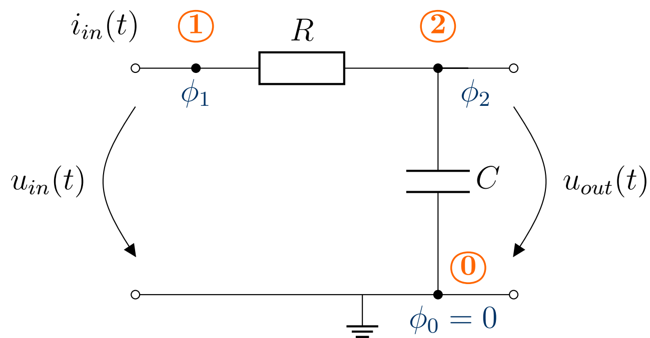

Figure 1: The circuit graph of a low-pass filter with nodes 0, 1 and 2 with corresponding node potentials ϕ_0, ϕ_1, and ϕ_2. Here the node 0 is grounded.

Using the modified nodal analysis [1], the underlying differential-algebraic equation of a low-pass filter with $u_{in} = 0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=:E} \begin{pmatrix} \phi_1 \\ \phi_2 \\ i_{in} \end{pmatrix} = \underbrace{\begin{bmatrix} -R^{-1} & R^{-1} & -1 \\ R^{-1} & -R^{-1} & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=:A} \begin{pmatrix} \phi_1 \\ \phi_2 \\ i_{in} \end{pmatrix}$$

where ϕ_1 and ϕ_2 are the node potentials at the nodes 1 and 2, respectively.

The Weierstraß form is given by

$$S(sE - A)T = \begin{bmatrix} s+(CR)^{-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & C^{-1/2} & -C^{-1/2}R^{-1} \\ 1 & 0 & (2R)^{-1} \\ 0 & 0 & 1 \end{bmatrix},$$

$$T = \begin{bmatrix} 0 & 0 & 1 \\ C^{-1/2} & 0 & 0 \\ R^{-1}C^{-1/2} & -1 & -(2R)^{-1} \end{bmatrix}.$$

The eigenvalues of the low-pass filter $\sigma(E, A) = \{-(RC)^{-1}, \infty\}$ are semi-simple with

$$\mathrm{am}_{-(RC)^{-1}}(E, A) = \mathrm{gm}_{-(RC)^{-1}}(E, A) = 1 \text{ and } \mathrm{am}_{\infty}(E, A) = \mathrm{gm}_{\infty}(E, A) = 2.$$

The aim of network redesign is to change the network topology in such a way that the location of the eigenvalues after this change has a positive influence on the stability of the circuit. Possible modifications of the low pass filter in Figure 1 can be the insertion of additional resistors or capacitors or the change of the parameter values R and C.

All of the above changes can be modeled as rank-one perturbations of the form $sF_\tau - G_\tau$ as described in (1) and $\tau > 0$ is either the capacity or the resistivity of the circuit elements. Hence we can describe the change of eigenvalues in dependence on τ using the results from the previous sections.

We investigate the numbers $s_{u,v}^{a,b}(-(RC)^{-1})$ and $s_{u,v}^{a,b}(\infty)$ from Section 3 as well as their signs $\varepsilon_{u,v}^{a,b}(-(RC)^{-1})$ and $\varepsilon_{u,v}^{a,b}(\infty)$ from Section 6. Here the following matrices are essential

$$S_{-(RC)^{-1}} = \begin{bmatrix} 0 & C^{-1/2} & C^{-1/2}R^{-1} \end{bmatrix}, \quad T_{-(RC)^{-1}} = \begin{bmatrix} 0 \\ C^{-1/2} \\ R^{-1}C^{-1/2} \end{bmatrix} = S_{-(RC)^{-1}}^T$$

and

$$S_\infty = \begin{bmatrix} 1 & 0 & (2R)^{-1} \\ 0 & 0 & 1 \end{bmatrix}, \quad T_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & -(2R)^{-1} \end{bmatrix}.$$

We study now the influence of a new capacitor or a new resistor that is introduced in the low pass filter. A new capacitor between the nodes (1) and (0) with capacity τ can be described by adding to $sE - A$ the pencil

$$sF_\tau - G_\tau = s\tau \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

that is, we have in the representation (1)

$$a = 1, \; b = 0, \; \text{and } u = v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =: e_1.$$

We conclude

$$\begin{aligned} \varepsilon_{e_1,e_1}^{1,0}(-(RC)^{-1}) &= \operatorname{sgn}\left((b + a(RC)^{-1})e_1^T T_{-(RC)^{-1}} S_{-(RC)^{-1}} e_1\right) \\ &= \operatorname{sgn}\left((RC)^{-1} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ C^{-1/2} \\ R^{-1}C^{-1/2} \end{bmatrix} \begin{bmatrix} 0 & C^{-1/2} & C^{-1/2}R^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} \varepsilon_{e_1,e_1}^{1,0}(\infty) &= \operatorname{sgn}\left(-av^T T_\infty S_\infty u\right) \\ &= \left(- \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & -(2R)^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & (2R)^{-1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \\ &= 0. \end{aligned}$$

Hence, we obtain by Lemma 6.2

$$s_{e_1,e_1}^{1,0}(-(RC)^{-1}) = 0 \text{ and } s_{e_1,e_1}^{1,0}(\infty) = 0,$$

hence $s^{1,0}_{e_1,e_1}$ is identically equal to zero. As a consequence of (15) the eigenvalues $-(RC)^{-1}$ and ∞ together with their algebraic multiplicites remain stationary for all $\tau \geq 0$. Concerning the geometric multiplicities at ∞ one observes $\operatorname{rank}(E) = 1$ and $\operatorname{rank}(E + F_\tau) = 2$ for $\tau > 0$, hence

$$\operatorname{am}_\infty(E, A) = 2, \quad \operatorname{gm}_\infty(E, A) = 2, \tag{54}$$

$$\operatorname{am}_\infty(E + F_\tau, A + G_\tau) = 2, \quad \operatorname{gm}_\infty(E + F_\tau, A + G_\tau) = 1. \tag{55}$$

As the eigenvalues remain stationary the choice of a new capacity between the nodes ① and ⓪ is not favorable for redesign.

On the other hand a new resistor with resistance $\tau > 0$ can be described by

$$sF_\tau - G_\tau = -\tau^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

that is, we have in the representation (1)

$$a = 0, \ b = 1, \ \text{and } u = v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =: e_1.$$

The same calculations as above show $s^{0,1}_{e_1,e_1}$ is identically equal to zero and the eigenvalues remain stationary for all $\tau > 0$.

A new capacitor between the nodes ① and ② with capacity $\tau > 0$ can be modeled by

$$sF_\tau - G_\tau = s\tau \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix},$$

that is, we have in the representation (1)

$$a = 1, \ b = 0, \ \text{and } u = v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Here we obtain

$$\begin{aligned} \varepsilon^{1,0}_{u,u}(-(RC)^{-1}) &= \operatorname{sgn}\left((b + a(RC)^{-1})v^T T_{-(RC)^{-1}} S_{-(RC)^{-1}} u\right) \\ &= \operatorname{sgn}\left((RC)^{-1} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ C^{-1/2} \\ R^{-1}C^{-1/2} \end{bmatrix} \begin{bmatrix} 0 & C^{-1/2} & C^{-1/2}R^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) \\ &= \operatorname{sgn}\left((RC)^{-1}C^{-1}\right) = 1, \end{aligned}$$

hence, by Lemma 6.2, $s^{1,0}_{u,u}(-(RC)^{-1}) = 1$. This means that the eigenvalue curve starting at $-(RC)^{-1}$ moves towards 0 (see Proposition 5.1) and as $b/a = 0$ there is by Corollary 6.5 one eigenvalue in the interval $(-(RC)^{-1}, 0)$.

Furthermore, we have

$$\begin{aligned}\varepsilon_{u,u}^{1,0}(\infty) &= \operatorname{sgn}\left(-av^T T_\infty S_\infty u = -\left[\begin{smallmatrix}1 & -1 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & 1\\ 0 & 0\\ -1 & -(2R)^{-1}\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0 & (2R)^{-1}\\ 0 & 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\ -1\\ 0\end{smallmatrix}\right]\right)\\ &= 0.\end{aligned}$$

As $\operatorname{rank}(E+F_\tau) = 2$ for $\tau > 0$, again (54) and (55) hold. Hence the insertion of a capacitor between the nodes ① and ② does not improve the stability nor the robustness of the circuit.

7.2. Two-stage CMOS operational amplifier

As a second example we consider a two-stage complementary metal-oxide-semiconductor (CMOS amplifier) used for certain BluRay applications in [33]. The coefficients of the underlying differential-algebraic equation are given by

$$E = 10^{-14}\begin{bmatrix} -48.8 & 0 & 0 & 4 & 22.4 & 22.4 & 0 & 0 & 0 & 0 & 0\\ 0 & -27.88 & 2 & 0 & 0 & 4 & 0 & 21.88 & 0 & 0 & 0\\ 0 & 2 & -31.9 & 0 & 4 & 4 & 0 & 21.9 & 0 & 0 & 0\\ 4 & 0 & 0 & -75.2 & 0 & 4 & 67.2 & 0 & 0 & 0 & 0\\ 22.4 & 0 & 4 & 0 & -26.4 & 0 & 0 & 0 & 0 & 0 & 0\\ 22.4 & 4 & 4 & 4 & 0 & -134.4 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 67.2 & 0 & 0 & -67.2 & 0 & 0 & 0 & 0\\ 0 & 21.88 & 21.9 & 0 & 0 & 0 & 0 & -43.78 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = 10^{-5}\begin{bmatrix} 18.4 & -0.080 & -0.080 & 13 & -9.03 & -9.04 & -13.13 & 0 & 0 & 0 & 0\\ -9.12 & 5.47 & 0 & 0 & 0 & 9.04 & 0 & -5.39 & 0 & 0 & 0\\ -9.11 & 5.31 & 0.16 & 0 & 9.03 & 0 & 0 & -5.39 & 0 & 0 & 0\\ 0 & 0 & 0 & 10.79 & 0 & 0 & -10.79 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 11.3 & 13.7 & 0 & 0.22 & -13.82 & -11.4 & 0 & 0 & 0\\ -0.13 & 0 & 0 & -37.49 & 0 & -0.12 & 37.74 & 0 & 1 & 0 & 0\\ 0 & -10.70 & -11.38 & 0 & 0 & -0.097 & 0 & 22.17 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The model contains eight transistors which are labeled in Figure 2 by the letter M.

The matrix pencil $sE - A \in \mathbb{R}[s]^{11\times 11}$ associated to the two stage CMOS amplifier which is shown in Figure 2 has the eigenvalues

$$\lambda_1 \approx -4.00\cdot 10^8, \quad \lambda_2 \approx -2.81\cdot 10^8, \quad \lambda_3 \approx -1.29\cdot 10^8$$

and one pair of complex eigenvalues λ_4, λ_5 given by

$$\lambda_{4,5} \approx (0.15 \pm 1.37i)\cdot 10^8,$$

and an eigenvalue at ∞ with $\operatorname{am}_\infty(E,A) = \operatorname{gm}_\infty(E,A) = 6$.

From the eigenvalue locations we see that the underlying differential-algebraic equation is unstable and hence a typical aim of network redesign,

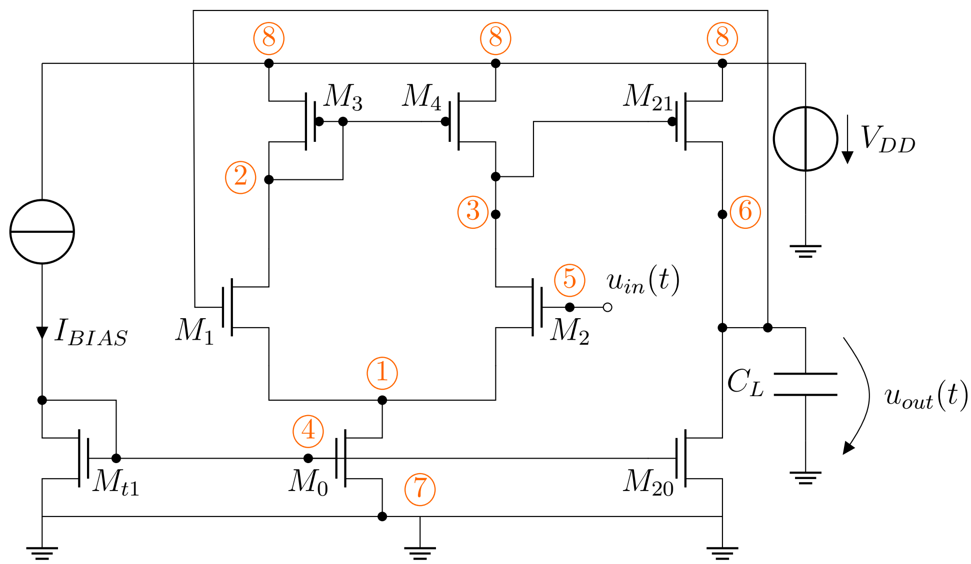

Figure 2: The two-stage CMOS OpAmp without precompensation and capacity $C_L = 1\text{pF}$ at the voltage output. There is also a closed feedback loop from M_1 to C_L.

also called *compensation*, would be to insert new capacities in such a way that the underlying circuit is stable meaning that all eigenvalues of the modified pencil have negative real part. In industrial circuit design one typically demands the newly inserted capacities to be as small as possible to have a low production cost. This leads to a high-dimensional non-convex optimization problem for which solution algorithms were proposed in [33] and more recently in [34].

One of the main approaches for stabilization of the circuit is based on the *Miller effect*, where one introduces a new capacitor between the nodes ③ and ⑥. If we choose

$$sF_\tau - G_\tau = s\tau(e_3 - e_6)(e_3 - e_6)^T$$

where $e_3, e_6 \in \mathbb{R}^{11}$ are two canonical unit vectors, that is, we have in the representation (1)

$$a = 1,\ b = 0,\ \text{and}\ u = v = e_3 - e_6.$$

Then for $\tau = 10^{-12}$ the eigenvalue curve of the complex poles moves towards

$$\lambda_{4,5}(\tau) \approx (-0.22 \pm 6.41i) \cdot 10^7$$

which stabilizes the circuit. Furthermore, we can compute the signs

$$\varepsilon_{u,v}^{a,b}(\lambda_1) = \varepsilon_{u,v}^{a,b}(\lambda_2) = 1, \quad \varepsilon_{u,v}^{a,b}(\lambda_3) = -1, \quad \varepsilon_{u,v}^{a,b}(\infty) = 0.$$

Hence by Proposition 6.3 for all $\tau > 0$ there is one eigenvalue in the interval (λ_1, λ_2). Since the eigenvalues λ_2 and λ_3 have a different sign, by Proposition 5.1 there are two eigenvalues in the interval (λ_2, λ_3) for small values of $\tau > 0$. For larger values of τ the eigenvalue curves possibly leave the real axis. The eigenvalue locations for large values of τ are by Proposition 4.1 close to the roots of the polynomial p defined in (7) which are for our example given by

$$z_1 \approx -3.98 \cdot 10^8, \quad z_2 \approx -2.66 \cdot 10^8, \quad z_3 \approx -1.30 \cdot 10^8, \quad z_4 \approx -0.15 \cdot 10^8. \tag{56}$$

and one eigenvalue converges to $\frac{b}{a} = 0$. Moreover, since $\varepsilon_{u,v}^{a,b}(\infty) = 0$ holds, Lemma 6.2 implies $s_{u,v}^{a,b}(\infty) = 0$ and using Theorem 3.2 we conclude that $\mathrm{am}_\infty(E+F_\tau, A+G_\tau) \geq \mathrm{am}_\infty(E, A) = 6$ holds for all $\tau > 0$. Hence, $\mathrm{am}_\infty(E+F_\tau, A+G_\tau) = 6$ holds for $\tau > 0$ sufficiently large.

In particular, for τ large enough all eigenvalues are located on the real axis. We can also exclude a bifurcation of the eigenvalue curves in the interval (λ_2, λ_3). If there would exist a point where the eigenvalue curves starting at λ_2 and λ_3 meet then by Theorem 3.2 for each value the eigenvalue curve attains in the interval, the corresponding parameter value τ is unique. Hence the eigenvalue curves will not enter the interval (λ_2, λ_3) again. However because of the convergence to the values $z_1, \ldots, z_4$ and 0 it can also not enter each of the intervals $(-\infty, \lambda_1)$, (λ_1, λ_2), $(\lambda_3, 0)$ and $(0, \infty)$. Furthermore, it can also not remain complex. This is a contradiction and hence there is no eigenvalue bifurcation in the interval (λ_2, λ_3).

An advantage of the Miller compensation between 3 and 6 is that it stabilizes the circuit already for small capacities. The smallest $\tau > 0$ which stabilizes the circuit can be obtained from Theorem 3.2 (b). Let $\mu = ir$ for some $r > 0$, then $\mu \notin \sigma(E, A)$ which implies $p(\mu) \neq 0$ and

$$\tau = -\frac{m_{u,v}^{a,b}(\mu)}{p(\mu)} = -\frac{1}{\mu(e_3-e_6)^T(\mu E-A)^{-1}(e_3-e_6)}.$$

One can verify numerically that in the interval $[0, 10^{10}]$ the only value for which $\operatorname{Im}\tau = 0$ holds is given approximately by $r = 7.68 \cdot 10^7$ which results in $\tau = 6.11 \cdot 10^{-13}$.

Another approach to compensation is to use a combination of a resistor and a capacitor which are connected in row between the nodes ③ and ⑥. However this results in a rank-two perturbation of the underlying matrix pencil. In the matrix case this was studied in [35].

A position for a capacitor which is more suitable is between the nodes ① and ③, that is in (1)

$$a = 1,\ b = 0,\ \text{and}\ u = v = e_1 - e_3.$$

The value τ for which the modified circuit becomes stable is given as a solution of

$$\tau = -\frac{m_{u,v}^{a,b}(\mu)}{p(\mu)} = -\frac{1}{\mu(e_1-e_3)^T(\mu E-A)^{-1}(e_1-e_3)}$$

for some $\mu = ir$, $r > 0$. The only real solution is given for $r = 1.16 \cdot 10^8$ by $\tau = 2.54 \cdot 10^{-13}$. The eigenvalue curves are shown in Figure 3. We observe a pole bifurcation in (λ_1, λ_2) and the eigenvalue curves enter the interval $(\lambda_3, 0)$.

In this sense, numerical computations show that the most sensitive capacitor is between the nodes ② and ③, that is

$$a = 1,\ b = 0 \text{ and } u = v = e_2 - e_3.$$

Here the coefficient of the first order term in Proposition 5.1 is given by

$$-a\lambda_4 v^T T_{\lambda_4} S_{\lambda_4} u = \lambda_4(-2.7173 + 0.0384i) \cdot 10^{12}.$$

Because of this large value one would expect that already small value of τ are sufficient for stabilization. We can make use of Theorem 3.2 (b) to find the smallest value τ which stabilizes the circuit. Here we consider

$$\tau = -\frac{m_{u,v}^{a,b}(\mu)}{p(\mu)} = -\frac{1}{(\mu(e_2-e_3)^T(\mu E-A)^{-1}(e_2-e_3)}$$

for $\mu = ir$ and $r \in [0, 10^{10}]$. In the specified parameter range, the only real solution τ is given by for $r = 1.93 \cdot 10^7$ by $\tau = 1.01 \cdot 10^{-11}$, which is one order of magnitude larger then the capacities that were previously used for stabilization.

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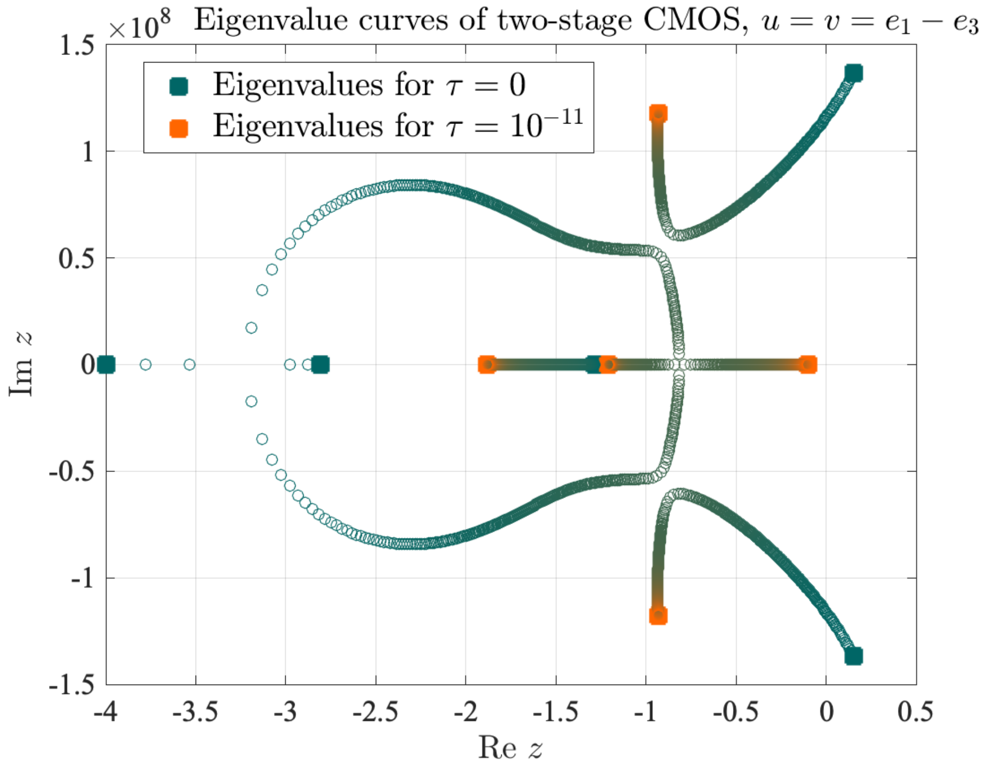

Figure 3: The figure shows the eigenvalue curves for the two-stage CMOS after perturbation with $sF_\tau - G_\tau = s\tau(e_1 - e_3)^T(e_1 - e_3)$ and τ varies from 0 to 10^{-11}].

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