# Full-analytic frequency domain gravitational wave forms from eccentric compact binaries to second post-Newtonian accuracy

**M Tessmer and G Schäfer**

Friedrich-Schiller-Universität, Max-Wien-Platz 1, 07743 Jena, Germany

E-mail: m.tessmer@uni-jena.de

**Abstract.** Full-analytic gravitational wave forms for inspiralling eccentric compact binaries of arbitary mass ratio have been provided in the frequency domain for the case of vanishing spins. Tail terms are not considered. In the given prescription, the semi-analytical property of recent descriptions, which demand inverting the higher order Kepler equation numerically, but keeping all other computations analytic is avoided.

## 1. Introduction

The dynamics of inspiralling compact binaries, consisting of neutron stars, black holes or a mixture of both, can be modelled accurately in the post-Newtonian (PN) approximation to general relativity. Roughly speaking, the PN approximation provides the equations of motion of a compact binary as corrections to the Newtonian equations of motion in powers of $(v/c)^2 \sim G\mathcal{M}/(c^2R)$, where $v$, $\mathcal{M}$, and $R$ are the characteristic orbital velocity, the total mass, and the typical orbital separation of the binary respectively. The PN accurate compact binary dynamics is required to construct temporally evolving gravitational wave forms associated with inspiralling compact binaries. Those wave froms are essential to make functioning gravitational wave astronomy with ground, and space based laser interferometric detectors, for which only the frequency domain of gravitational waves is crucial. We like to avoid numerical computations, because of limited computer power resources, so we have aimed to generate analytical frequency domain expressions (see also [1] for a parameter estimation concerning the second PN accurate case of decaying orbits, assuming small eccentricities, and [2] for eccentric orbits using the Newtonian equations of motion and a post-circular expansion). For the sake of simplicity, we have employed the tool of the spin-2 tensor spherical harmonics, as done in an older publication [3]. These harmonics are decomposed appropriately, where after several re-casts, the final expressions can be Fourier transformed analytically.

## 2. Essential ingredients

### *2.1. Multipole expansion of the radiation field far away*

The far-zone radiation field for a source at the coordinate origin, and an observer at point $\mathbf{R}$ can be expressed in terms of symmetric trace-free (STF) Cartesian tensors with basis:

$$N_{\mathcal{A}_{l-2}} := N_{a_1} \cdots N_{a_{l-2}}, \tag{1}$$

where $\mathbf{N}:= \mathbf{R}/\mathrm{R}$ is the direction of the line of sight (source – observer), given in [4, 5] as:

$$h_{km}^{TT}(\mathbf{R},t) = \frac{G}{c^4 R}\mathcal{P}_{kmij}(\mathbf{N})\sum_{l=2}^{\infty}\left[\left(\frac{1}{c}\right)^{l-2}\left(\frac{4}{l!}\right)\overset{(l)}{\mathcal{I}}_{ij\mathcal{A}_{l-2}}\left(t-\frac{R}{c}\right)N_{\mathcal{A}_{l-2}} + \left(\frac{1}{c}\right)^{l-1}\left(\frac{8l}{(l+1)!}\right)\epsilon_{pq(i}\overset{(l)}{\mathcal{J}}_{j)p\mathcal{A}_{l-2}}\left(t-\frac{R}{c}\right)N_{q\mathcal{A}_{l-2}}\right]. \tag{2}$$

The projector $\mathcal{P}_{kmij}$ extracts the STF parts from the explicit sum of time derivatives of mass-type ($\mathcal{I}$), and current-type ($\mathcal{J}$) multipole moments. We have truncated this series beyond 2 $PN$ order and take care of the *instantaneous point mass parts only.* Because the wave form depends on the direction of $\mathbf{N}$, the frequency domain will also do so. Therefore, to deal with irreducible components, we have recast $h_{ij}^{TT}$ into the basis of the spin-2 tensor spherical harmonics ($T_{ij}^{E2}$ and $T_{ij}^{B2}$, and the scalar expansion coefficients $I$ and $S$ in the expression below have been processed further) as:

$$\begin{aligned} h_{ij}^{TT} &= \frac{\eta}{c^4 R}\Bigg\{\sum_{m=-2}^{2}\overset{(2)}{I}{}^{2m}T_{ij}^{E2,2m} \\ &+ c^{-1}\left[\sum_{m=-2}^{2}\overset{(2)}{S}{}^{2m}T_{ij}^{B2,2m} + \sum_{m=-3}^{3}\overset{(3)}{I}{}^{3m}T_{ij}^{E2,3m}\right] \\ &+ c^{-2}\left[\sum_{m=-3}^{3}\overset{(3)}{S}{}^{3m}T_{ij}^{B2,3m} + \sum_{m=-4}^{4}\overset{(4)}{I}{}^{4m}T_{ij}^{E2,4m}\right] \\ &+ c^{-3}\left[\sum_{m=-4}^{4}\overset{(4)}{S}{}^{4m}T_{ij}^{B2,4m} + \sum_{m=-5}^{5}\overset{(5)}{I}{}^{5m}T_{ij}^{E2,5m}\right] \\ &+ c^{-4}\left[\sum_{m=-5}^{5}\overset{(5)}{S}{}^{5m}T_{ij}^{B2,5m} + \sum_{m=-6}^{6}\overset{(6)}{I}{}^{6m}T_{ij}^{E2,6m}\right]\Bigg\}. \end{aligned} \tag{3}$$

The numbers in round brackets denote time derivatives, which are computed using the quasi-Keplerian parameterization, which we have introduced below. The conservative dynamics of the binary can be reduced to a one-body dynamics in a non-precessing plane. We can connect the radial separation $r$ and the orbital phase $\phi$ to the elapsed time $(t - t_0)$ via the 2 $PN$ Kepler equation:

$$r = a_r(1 - e_r\cos\mathcal{E}) , \tag{4}$$

$$\frac{(\phi-\phi_0)}{\Phi/2\pi} = v + c^{-4}f_{4\phi}\sin 2v + c^{-4}g_{4\phi}\sin 3v , \tag{5}$$

$$\mathcal{M} := \mathcal{N}(t-t_0) = \mathcal{E} - e_t\sin\mathcal{E} + c^{-4}\Big[F_{v-\mathcal{E}}(v-\mathcal{E}) + F_v\sin v\Big] , \tag{6}$$

$$defining \qquad v := 2\arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{\mathcal{E}}{2}\right] , \tag{7}$$

with $\mathcal{N} := 2\pi/P$ as the mean motion ($\mathcal{P}$ being the radial period), the functions $a_r$ (semi-major axis), $e_{\{r,\ t,\ \phi\}}$ (radial, time and phase eccentricity), $f_{4\phi}$ and $g_{4\phi}$ are the PN accurate functions of the binding energy $|E|$, and the orbital angular momentum $L$ (see e.g., [6]). Using the above quasi-Keplerian parameterization, and PN accurate expressions for the acceleration in terms of the separation and velocity, we have been able to express the higher order time derivatives as functions of $r$ and $\phi$.

## 3. Expansion procedure

Our way to full-analytical Fourier-domain wave forms can be compressed symbolically as in the schedule below:

(i) Write down time derivative of $I^{lm}$ and $S^{lm}$ as functions of the eccentric anomaly $\mathcal{E}$ and collect for *even* and *odd* functions separately.

(ii) Convert *even terms in* $\mathcal{E}$ to $\sum_{m\,\geq\,0} c_m \cos m\mathcal{E}$, and *odd terms* to $\sum_{m\,>\,0} s_m \sin m\mathcal{E}$.

(iii) Compute coefficients of $\cos m\mathcal{E} = \sum_{j\,\geq\,0} \gamma_j \cos j\mathcal{M}$, and $\sin m\mathcal{E} = \sum_{j\,>\,0} \sigma_j \sin j\mathcal{M}$.

(iv) Perform the Fourier transformation of sines and cosines, which is straightforward.

(v) For the inclusion of radiation reaction, the method of the stationary phase will be applied for the appearing integrals.

## 4. Technicalities

The aim is to reduce the treatment of the scalar coefficients $I$ and $S$ to that of their constituents. We shall first extract the very slow periastron precession from the phase, and convert it to an overall factor. Defining $A(\mathcal{E}) := 1 - e_t \cos(\mathcal{E})$, the structure of the components reads:

$$I^{lm} = e^{-im\phi} f_{lm}(A(\mathcal{E}),\ \sin(\mathcal{E})), \tag{8}$$

where $f_{lm}$ being PN accurate functions, and similarly for $S$. Extraction of the periastron shift yields:

$$I^{lm} = e^{-im\mathcal{K}\mathcal{M}} \left( f^{*}_{lm}(A(\mathcal{E})) + \sin(\mathcal{E}) \tilde{f}^{*}_{lm}(A(\varepsilon)) \right) , \tag{9}$$

where $f^*$ and $\tilde{f}^*$ are some polynomials in $A(\mathcal{E})$. An infinite Taylor series in $e_t$, and a successive sorting for trigonometric functions to positive multiples of $\mathcal{E}$ gives:

$$A^n(\mathcal{E}) = \sum_{j=1}^{\infty} A_j^{(n)} \cos(j\mathcal{E}), \quad \textit{and} \quad \frac{\sin(\mathcal{E})}{A^n(\mathcal{E})} = \sum_{j=0}^{\infty} S_j^{(n)} \sin(j\mathcal{E}) \ . \tag{10}$$

These trigonometric functions of $j\mathcal{E}$ can be further expanded as:

$$\sin(n\mathcal{E}) = \sum_{j=1}^{\infty} \mathcal{S}_j^{(n)} \sin(j\mathcal{M}) \ , \quad \textit{and} \quad \cos(n\mathcal{E}) = \sum_{j=0}^{\infty} \mathcal{C}_j^{(n)} \sin(j\mathcal{M}) \ . \tag{11}$$

The coefficients $\mathcal{S}_j^{(n)}$ and $\mathcal{C}_j^{(n)}$ can be obtained by computing Fourier-Bessel integrals, like $\mathcal{S}_j^{(n)} = \int \sin(n\mathcal{E}(\mathcal{M})) sin(j\mathcal{M}) d\mathcal{M}$, inserting the 2 $PN$ Kepler equation and expanding each term up to $c^{-4}$.

## 5. Results for the conservative case

Collecting all the intermediate results, we have obtained a time-domain representation for the non-tail part of the gravitational wave, having sines and cosines of multiples of the mean anomaly. The frequency domain in the conservative case, from here onwards, is trivial to obtain. It is to be emphasized that everything is given in the positive domain of the spectrum [7]:

$$\overset{(a)}{I}{}^{am} = \mathcal{I}_{am} e^{-mi\mathcal{K}\mathcal{M}} \sum_{j=0}^{\infty} \left\{ \sin(j\mathcal{M}) \ \overset{(a)}{I}{}^{am}_{Sj} + \cos(j\mathcal{M}) \ \overset{(a)}{I}{}^{am}_{Cj} \right\} , \quad \textit{and} \tag{12}$$

$$\overset{(b)}{S}{}^{bm} = \mathcal{S}_{bm} e^{-mi\mathcal{K}\mathcal{M}} \sum_{j=0}^{\infty} \left\{ \sin(j\mathcal{M}) \ \overset{(b)}{S}{}^{bm}_{Sj} + \cos(j\mathcal{M}) \ \overset{(b)}{S}{}^{bm}_{Cj} \right\} . \tag{13}$$

## 6. Including radiation reaction

Binary systems in the early part of their evolution are characterized by $|E|$ and $L$ as slowly varying functions of time. In a stage where the variation is estimated to be small enough, the stationary phase approximation can be applied to compute the frequency domain. We have used the Peters-Matthews equations for the orbital decay of $|E|$ and $e_t$ [8]. Under these conditions, it turns out that the Fourier domain of any term appearing in Eqs. (12) and (13), shorthand notated by $\tilde{h}(f)$, omitting the multipole indices, takes the prototype form:

$$\tilde{h}(f) = \int \mathcal{B}(t) e^{2\pi i f t - \phi(t)} dt. \tag{14}$$

Here, $\mathcal{B}(t)$ is an amplitude, slowly varying in time. Having found the stationary point in $\dot{\Phi} = 0$, where $\Phi(t) := 2\pi f t - \phi(t)$, the quantity $\tilde{h}$, consisting of complex sine and cosine contributions $S$ and $C$, which are to be read off from Eq. (14), and finally written as:

$$\overset{(n)}{\tilde{h}}(f)_{n,m} = \frac{1}{c^4 R} \sum_{j=0}^{\infty} \frac{1}{2} \left( i S^*_{m(j>0)} + \Big(1 + \theta(m)\delta_{0j}\Big) C^*_{mj} \right) \times \left[ \frac{e^{i\left(\Phi_{mj}(t^*_{mj}) - \pi/4\right)}}{\sqrt{\ddot{\phi}_{mj}(t^*_{mj})}} \right] e^{-im\phi_0} \,, \tag{15}$$

$$\ddot{\phi}_{mj}(t^*_{mj}) \equiv \mathcal{N} \,.\, (j + m\mathcal{K})|_{t=t^*_{mj}} \,. \tag{16}$$

The function $\theta(m)$ is "zero" if $m = 0$ and "one" if $m > 0$. A quantity with a star has to be evaluated at the stationary point, which in turn, depends on the number $m$, and the position $j$ in the summation.

## 7. Outlook

For a complete picture of the frequency domain wave forms, tail effects in the far-zone amplitude are necessary to be included. The inclusion of spin interactions and higher-order radiation reaction effects are demanded too.

## Acknowledgements

This project is funded by the DFG (Deutsche Forschungsgemeinschaft) through SFB TR - 7, and by the DLR (Deutsches Zentrum für Luft - und Raumfahrt). MT wishes to thank Johannes Hartung and Steven Hergt, for many useful discussions.

## References

[1] Królak A, Kokkotas K D and Schäfer G 1995 *Phys. Rev. D* **52** 2089
[2] Yunes N, Arun K G, Berti E and Will C M 2009 *Phys. Rev. D* **80** 084001
[3] Junker W and Schäfer G 1992 *MNRAS* **254** 146
[4] Thorne K S 1980 *Rev. Mod. Phys.* **52** 299
[5] Blanchet L, Damour T and Iyer B R 1995 *Phys. Rev. D* **51** 5360
[6] Memmesheimer R M, Gopakumar A and Schäfer G 2004 *Phys. Rev. D* **70** 104011
[7] Tessmer M and Schäfer G 2011 *Ann. Phys. (Berlin)* **523** 813
[8] Peters P C and Matthews J 1963 *Phys. Rev.* **131** 435