Manifestations of Symmetries and Their Breaking in Hydrodynamics and Holography

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Abstract

Symmetries are fundamental to physical theories: they are used to classify, constrain and simplify. In reality, however, symmetries tend to be broken; the study of symmetry breaking thus constitutes an important endeavour. This thesis presents work pertaining to symmetry and symmetry breaking – in particular to their manifestations in relativistic hydrodynamics and holography.

Pseudo-spontaneous breaking of $U(1)$ symmetry and phase relaxation are studied by extending the hydrodynamic analysis of superfluids. For sufficiently small explicit breaking parameters a hydrodynamic regime may be approximated. The hydrodynamic modes display the effects of pseudo-spontaneous symmetry breaking and phase relaxation. It is furthermore shown that, in the probe limit and in the absence of charge relaxation, the DC conductivity becomes finite but nevertheless the AC conductivity does not present a Drude-peak.

A viscoelastic hydrodynamic framework for systems with spontaneously broken translational invariance is also presented. Strain pressure – an effect which appears when such systems do not minimise the free energy – is discussed. It is shown that the temperature derivative of the strain pressure need not vanish even when strain pressure is absent. The hydrodynamic modes are found, for which the repercussions of strain pressure and its temperature derivative are discussed for different scenarios.

The hydrodynamic frameworks are tested numerically against the dynamics of the lowest quasi-normal modes of holographic models. The presence of strain pressure and its temperature derivative is confirmed by comparison to the dynamics of a massive gravity model. The hydrodynamic framework for pseudo-spontaneous $U(1)$ symmetry breaking is tested against two modifications of the standard holographic superfluid; their dynamics are considered in the probe limit. The hydrodynamic frameworks match the quasi-normal modes for both models. It is shown that phase relaxation may appear due to the interplay of explicit and spontaneous symmetry breaking, in which case it behaves in accordance with a proposed universal relation. The finite nature of the AC and DC conductivity, in the relevant regime, is also confirmed.

Finally, motivated by its relevance for a holographic duality involving flat spacetimes, two-dimensional BMS symmetry is considered. The highest-weight representation of the $\mathfrak{bms}_3$ algebra is formulated in terms of the oscillator formalism. The tools inherent to this construction are used to prove that $\mathfrak{bms}_3$-blocks exponentiate in the semiclassical limit. In the semiclassical context also two examples of vacuum $\mathfrak{bms}_3$-blocks – the perturbatively heavy, and heavy-light vacuum $\mathfrak{bms}_3$-blocks – are calculated.

Zusammenfassung

Symmetrien sind grundlegend für physikalische Theorien: Sie werden zu deren Klassifikation, Einschränkung und Vereinfachung verwendet. In der Realität sind Symmetrien aber meist gebrochen, wodurch das Studium der Symmetriebrechung wichtig wird. Diese Dissertation befasst sich mit Symmetrien und deren Brechung – insbesondere mit deren Manifestationen in der relativistischen Hydrodynamik und Holographie.

Durch eine Erweiterung der hydrodynamischen Analyse von Superfluiden werden die pseudospontane Brechung einer $U(1)$-Symmetrie und die Phasenrelaxation untersucht. Für genügend kleine explizit symmetriebrechende Parameter kann ein hydrodynamisches Regime approximiert werden. Die hydrodynamischen Moden zeigen die Effekte pseudospontaner Symmetriebrechung und Phasenrelaxation.Weiterhin wird gezeigt, dass, im Grenzfall eines verschwindenden Energie-Impulstensors und ohne Ladungsrelaxation, die Gleichstromleitfähigkeit endlich wird, die Wechselstromleitfähigkeit aber kein Drude-Maximum zeigt. Weiterhin wird ein viskoelastisch-hydrodynamischer Zugang für Systeme mit spontan gebrochener Translationssymmetrie präsentiert. Der Spannungsdruck – ein Effekt, der auch auftritt, wenn solche Systeme die freie Energie nicht minimieren – wird diskutiert. Es wird gezeigt, dass die Ableitung des Spannungsdruckes nach der Temperatur auch bei verschwindendem Spannungsdruck endlich sein kann. Es werden die hydrodynamischen Moden und die Auswirkungen von Spannungsdruck und dessen Temperaturableitung darauf in verschiedenen Szenarien diskutiert.

Die hydrodynamischen Zugänge werden numerisch gegen die Dynamik der niedrigsten Quasinormalmoden von holographischen Modellen getestet. Ein Vergleich mit einem massiven Modell der Gravitation zeigt die Existenz von Spannungsdruck und dessen Temperaturableitung. Der hydrodynamische Zugang für pseudospontane Symmetriebrechung wird gegen zwei Modifikationen des üblichen holographischen Superfluids getestet; deren Dynamik wird im Grenzfall ohne Rückreaktion betrachtet. Es wird gezeigt, dass Phasenrelaxation aufgrund des Zusammenspiels von expliziter und spontaner Symmetriebrechung auftreten kann. In diesem Fall verhält sie sich gemäß einer vorgeschlagenen universellen Relation. Weiterhin wird die endliche Natur der Gleich- und Wechselstromleitfähigkeit in dem relevanten Regime bestätigt.

Zu guter Letzt wird zweidimensionale BMS-Symmetrie aufgrund ihrer Relevanz für eine holographische Dualität mit flachen Raumzeiten betrachtet. Die Darstellung der $\mathfrak{bms}_3$-Algebra mit dem höchsten Gewicht wird mittels der Oszillatorkonstruktion formuliert. Mittels dieser Konstruktion wird gezeigt, dass $\mathfrak{bms}_3$-Blöcke im semiklassischen Grenzfall exponenzieren. Zudem werden im semiklassischen Kontext auch zwei Beispiele von Vakuum-$\mathfrak{bms}_3$-Blöcken – die perturbativ schweren und die schwer-leichten $\mathfrak{bms}_3$-Blöcke – berechnet.

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Contents

1 Introduction

Humans display a propensity toward symmetry; it appears in our art, architecture and design. Moreover, many of the physical frameworks from which we have derived much of our modern understanding of Nature are fundamentally reliant on the mathematical formalisation of symmetry. For instance, according to Noether's theorem – which states that each continuous symmetry is associated to a conserved quantity – energy and momentum conservation arises as a consequence of translational invariance in time and space, respectively.

Of particular relevance for fundamental physics is the manifestation of symmetry in quantum field theory, which finds applications in domains including high energy and condensed matter physics [5–8]. The role of symmetry in quantum field theory is aptly illustrated by the Standard Model of particle physics [9–12], wherein three out of the four fundamental forces of Nature – the strong and weak nuclear forces and the electromagnetic force – as well as all observed elementary particles are classified by their properties under symmetry transformations. The fourth fundamental force, gravity, has yet to realise a complete quantum description.

Beyond microscopic quantum field theories (such as the Standard Model of particle physics) symmetry is also essential to the construction of effective field theories, for which the main goal is to describe specific properties of a system without laying claim to the exact mechanisms behind them [13]. One such effective theory is hydrodynamics [14, 15], which captures the late-time, small-momentum dynamics of massless degrees of freedom in many-body systems at finite temperature. A hydrodynamic description of a theory is fundamentally dependent on its conserved quantities and thus – invoking Noether's theorem – its symmetries. The nomenclature recalls the liquid state of matter but, in fact, hydrodynamics is a universal framework and its applicability ranges from the quark-gluon plasma produced in heavy-ion collisions [16–18] to the behaviour of electrons in graphene [19, 20]. Hydrodynamics will be a focal point of this thesis.

A physical phenomenon need not submit itself to one unique description; if two distinct theories have the same number of degrees of freedom and yield equivalent physical outcomes they are referred to as dual. Dualities appear in many contexts but one common trait is that they are symmetry preserving. A duality which has been the recipient of much attention over the past two decades is the holographic duality, a conjecture which equates a gravitational theory to a quantum field theory with one spatial dimension lower. An intuitive interpretation of the holographic duality is that the quantum field theory inhabits the boundary of the gravitational theory. The most developed incarnation of the holographic duality is the AdS/CFT correspondence [21–23], where the gravitational theory is a string theory [24–27] – a candidate theory for quantum gravity – in asymptotically anti-de Sitter (AdS) spacetime, and the quantum field theory is a conformal field theory (CFT) [28]. Anti-de Sitter spacetime has a negative cosmological constant; its bulk has a constant negative curvature while its boundary is flat, i.e. without a cosmological constant. The symmetry at the boundary of anti-de Sitter spacetime is that of conformal (angle-preserving) transformations [29], which in turn is inherited by the quantum field theory.

The AdS/CFT correspondence may in principle be utilised in two directions: to study string theory – and hence quantum gravity – via quantum field theory, or vice versa. In a certain regime the AdS/CFT duality relates a strongly interacting quantum field theory to a weakly interacting gravitational theory. Understanding the dynamics of strongly interacting systems is a persistent frustration in physics in general; however, the AdS/CFT correspondence promises to make such analyses more manageable by allowing insights to originate from a weakly interacting theory [30–34].

Hydrodynamics has a formal realisation in AdS/CFT via the so-called fluid/gravity correspondence [35]. In this setting the properties of a black hole in the gravitational theory are imposed on the boundary quantum field theory, and the Einstein equations are equivalent to the hydrodynamic conservation equations. Fluctuations of the black hole give rise to dynamics in the quantum field theory, which under certain circumstances may be captured by hydrodynamics. For example, this paradigm has unveiled a universal lower bound for the ratio of shear viscosity over entropy in a fluid [36], as well as found novel hydrodynamic phenomena [37–40] which have since been experimentally verified [41].

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Regardless of their mathematical utility and prevalence in theoretical frameworks, Nature tends to break symmetries in reality. Occasionally a symmetry is broken in a way which may be systematically avoided in a first approximation; often, though, the symmetry breaking must be incorporated into the relevant analysis in order for the properties of the system displaying symmetry breaking to be fully understood. Two symmetry breaking mechanisms will be of particular interest in this thesis: spontaneous symmetry breaking and explicit symmetry breaking.

Spontaneous symmetry breaking arises when the ground state of a theory no longer obeys the symmetry, or symmetries, of its defining equations [42]. The conserved quantity associated to a symmetry is not affected if the symmetry is spontaneously broken. If the spontaneous breaking is of a continuous symmetry a new massless degree of freedom – the Goldstone boson – must be accounted for in the dynamics of the system [43–45]. Spontaneous symmetry breaking emerges in many areas of physics; for example, it is the effects of spontaneous symmetry breaking which generate the masses of the massive elementary particles in the Standard Model of particle physics [46–48]. Spontaneous symmetry breaking may also be used as an indicator for phase transitions [49]. The study of spontaneous symmetry breaking has a long history and is still ongoing, see for instance [50–56].

Explicit symmetry breaking occurs at the level of the equations which define a theory. When a symmetry is explicitly broken it is no longer possible to associate it to a conserved quantity. A straightforward example: Defects in the structure of a crystal explicitly break spatial translational invariance; as a result momentum is no longer conserved.

Spontaneous symmetry breaking may coincide with an explicit breaking of the same symmetry – if the explicit breaking is slight compared to the spontaneous breaking the combined breaking is called pseudo-spontaneous [57]. In the pseudo-spontaneous regime the Goldstone bosons associated to the spontaneous breaking gain a mass (which increases with the amount of explicit breaking) and are referred to as pseudo-Goldstone bosons. Pions – the lightest composite particles – may be described as pseudo-Goldstone bosons [58].

In addition to explicit symmetry breaking, Goldstone bosons may fall victim to a dampening effect called phase relaxation [59, 60]. Phase relaxation can be viewed as a breaking of the internal symmetries of the Goldstone boson [61–64] but may arise independently of other explicit symmetry breaking and does not affect the conserved quantities.

The gentle touch with which spontaneous symmetry breaking treats the conserved quantities of a theory, and the massless nature of the Goldstone bosons, implies that investigations of its effects, in an appropriate regime, fall within the scope of hydrodynamics. A superfluid, such as liquid Helium, may be modelled as a two-component fluid where the superfluid component is due to a spontaneously broken phase symmetry [65–70]. Moreover, elasticity may be described as an effect of spontaneous breaking of translational invariance and may also be integrated into a hydrodynamic description [60, 71–73].

The destructive effects of explicit symmetry breaking, however, make its presence difficult to reconcile with hydrodynamics – a priori. Nevertheless, if the explicit breaking is sufficiently small one may approximate a regime where the conserved quantities are sufficiently intact and long-lived for the application of hydrodynamics; in the case of pseudo-spontaneous breaking this also results in a small mass for the pseudo-Goldstone. Such an approach is called generalised hydrodynamics or quasi-hydrodynamics [74]. Even an infinitesimal explicit breaking plays a role in the hydrodynamic theory.

Symmetry breaking may also be included in holographic models [75]. This is mainly achieved by constructing a gravity theory whose boundary behaviour breaks the desired symmetry in the dual quantum field theory. In such scenarios the significance is placed on the dynamics of the quantum field theory. Different gravitational theories may give rise to the same symmetry breaking.

* *

When the AdS/CFT duality is employed to study quantum field theories via gravitational theories the results are generically valid for flat spacetimes, which is standard, albeit with the addition of conformal symmetry. The inverse implementation gives insight into quantum gravity in spacetimes with a negative cosmological constant. At large, intergalactic length scales our Universe displays a very small but positive cosmological constant [76, 77]; however, at astrophysical scales, for instance for black holes, the influence of the cosmological constant is negligible. Thus, although conceptually useful, the knowledge gained about quantum gravity from AdS/CFT is at odds with reality. It is therefore imperative for the holographic approach to quantum gravity that there exists a duality where the spacetime of the gravitational theory is something other than anti-de Sitter. Let us turn to flatspace holography [78–81].

An example of a flatspace holographic duality is not yet known. The foundations of such a duality may however be motivated from symmetry considerations, assuming the principles which hold for AdS/CFT are transferable.

The boundary symmetries of flat space are enhanced from those in the bulk of the spacetime, which consist of boost, translational and rotational invariance: Translations may be angle dependent while rotations become generalised – this is known as Bondi-Meltzer-Sachs (BMS) symmetry [82,83]. It thus seems plausible that information about quantum gravity in flat spacetimes may be extracted from BMS-invariant quantum field theories.

BMS symmetry may be cast in two forms: a version which applies to non-relativistic systems (where the speeds are small compared to the speed of light) and an ultra-relativistic version (where the speeds approach the speed of light); the symmetry is the same but the mathematical properties differ between the two [84,85]. In the context of flatspace holography the differences between the implementations of BMS symmetry would also be reflected in the gravitational dual.

* * *

The work in this thesis aims to contribute to the stories told above.

The second chapter lays the foundations for the AdS/CFT correspondence and introduces features which will be taken advantage of further along in the thesis. The dynamic equivalence between string theory in anti-de Sitter spacetime and conformal field theory is discussed, as well as simplifying circumstances which make the duality more tractable. The holographic dictionary between fields in the gravity theory and objects in the conformal field theory is derived for the simplest case and presented for some other quantities of interest. Also, finite temperature quantum field theory and linear response theory is presented from a holographic perspective.

Chapter 3 begins with a review of hydrodynamics, where key concepts – such as the constitutive relations and transport properties – are presented. Certain features of linear response theory are also shown from the hydrodynamic perspective. Subsequently, symmetry breaking in hydrodynamics is considered. A review of superfluidity is followed by novel results due to the addition of explicit symmetry breaking and phase relaxation [3]. Furthermore, spontaneous translational symmetry breaking is considered in the context of hydrodynamics, and the appearance of a new effect caused by thermodynamic instability – called the strain pressure – is discussed [2, 73].

Hydrodynamics and holography are combined in chapter 4. Three classes of holographic models which give rise to the symmetry breakings of interest are presented: one for translational symmetry breaking [86–89] and two which are dual to a superfluid with explicit symmetry breaking [90–92]. Numerical analysis is utilised to validate the hydrodynamic frameworks of chapter three by comparing the relevant, holographically dual quantities [1–3].

Chapter 5 takes a more formal approach than the earlier chapters; it considers aspects of BMS symmetry motivated by a putative flatspace holographic duality. A non-relativistic representation of BMS symmetry is found in the so-called oscillator construction [4]. The validity and utility of this representation is argued by calculating BMS field theory quantities using methods intrinsic to the oscillator construction.

Lastly, the thesis concludes with a discussion and outlook in chapter 6.

2 Holography

In physics two theories are said to be dual when they describe the same physical properties from two different perspectives. The holographic principle [93–96] states that all information of a spacetime volume may be contained in its spatial boundary. An example of a *holographic duality* is the equivalence between superstring theory or M-theory on $\text{AdS}_{d+1} \times C$ – where C is a compact space such that the total dimension adds up to ten or eleven – and a d-dimensional (super)conformal field theory located on the flat conformal boundary of the spacetime; this is known as the AdS/CFT correspondence [21]. The AdS/CFT correspondence may be approached in two ways: the so-called top-down and bottom-up approaches. In the top-down approach the full theories are known on both sides of the duality but must be truncated in order to be tractable. The bottom-up approach is used to model specific properties of a boundary conformal field theory by the means of a gravity theory in the bulk. In this chapter the foundations for future considerations involving the AdS/CFT duality will be presented mainly from the bottom-up perspective. The effects of supersymmetry, which in principle enter via superstring theory, will not be considered since they will not appear in the applications of later chapters – the reason for this is mentioned below equation (2.27). Moreover, the compact space C is irrelevant in the following discussion since Kaluza-Klein reduction makes the fields of the theory dependent only on the asymptotically AdS spacetime. Much of the material covered in this chapter is standard to the topic and may be sourced from a range of literature; see for instance [30, 97–102].

2.1 Duality of Partition Functions

The strong form of the AdS/CFT duality states that the partition function of a superstring theory in an asymptotically $(d+1)$-dimensional anti-de Sitter spacetime is equivalent to the partition function of a d-dimensional conformal field theory living on the flat conformal boundary of the AdS spacetime [22, 23]; it may schematically be expressed as

$$Z_{\text{CFT}} = Z_{\text{String}}\big|_{\text{conformal boundary}}\,, \tag{2.1}$$

where Z denotes the partition function. The implication of the evaluation at the conformal boundary will become clear in section 2.2.1. The two partition functions may advantageously be considered separately.

The CFT partition function is given by

$$Z_{\text{CFT}}\left[\gamma_{\mu\nu},\{A_\mu^N\},\{\phi_s^M\}\right]=e^{-W_{\text{CFT}}\left[\gamma_{\mu\nu},\{A_\mu^N\},\{\phi_s^M\}\right]}, \tag{2.2}$$

with W_{CFT} being the generating functional of connected correlation functions. W_{CFT} may be taken to be

$$\begin{aligned}
&W_{\text{CFT}}\left[\gamma_{\mu\nu},\{A_\mu^N\},\{\phi_s^M\}\right]\\
&\quad=-\ln\left\langle\exp\left[\int \mathrm{d}^d x\sqrt{-\gamma}\left(\frac{1}{2}T^{\mu\nu}(x)\gamma_{\mu\nu}(x)+J_N^\mu(x)A_\mu^N(x)+\mathcal{O}_M(x)\phi_s^M(x)\right)\right]\right\rangle,
\end{aligned} \tag{2.3}$$

where $\mathcal{O}_M$ are composite scalar operators; J_N^μ are conserved symmetry currents; $T^{\mu\nu}$ is the boundary energy-momentum tensor; and the respective sources are the scalar sources ϕ_s^M; gauge fields A_μ^N; and metric $\gamma_{\mu\nu}$ with determinant γ. The vacuum expectation values of the operators follow from the variation of the logarithm of the partition function with respect to the sources, i.e.

$$\langle\mathcal{O}_M(x)\rangle_{\text{CFT}}=-\left.\frac{\delta W_{\text{CFT}}}{\delta\phi_s^M(x)}\right|_{\substack{\gamma_{\mu\nu}(x)=\eta_{\mu\nu}\\ A_\mu^N(x)=0\\ \phi_s^M(x)=0}}, \tag{2.4a}$$

$$\langle J_N^\mu(x)\rangle_{\text{CFT}}=-\left.\frac{\delta W_{\text{CFT}}}{\delta A_\mu^N(x)}\right|_{\substack{\gamma_{\mu\nu}(x)=\eta_{\mu\nu}\\ A_\mu^N(x)=0\\ \phi_s^M(x)=0}}, \tag{2.4b}$$

$$\langle T^{\mu\nu}(x)\rangle_{\text{CFT}}=-\frac{2}{\sqrt{-\gamma}}\left.\frac{\delta W_{\text{CFT}}}{\delta\gamma_{\mu\nu}(x)}\right|_{\substack{\gamma_{\mu\nu}(x)=\eta_{\mu\nu}\\ A_\mu^N(x)=0\\ \phi_s^M(x)=0}}, \tag{2.4c}$$

where $\eta_{\mu\nu}=\operatorname{diag}(-1,1,\ldots,1)$ is the Minkowski metric. By extension a connected n-point correlation function of scalar operators is given by

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_{\text{CFT}}=-\left.\frac{\delta^n W_{\text{CFT}}}{\delta\phi_s^1(x_1)\delta\phi_s^2(x_2)\cdots\delta\phi_s^n(x_n)}\right|_{\substack{\gamma_{\mu\nu}(x)=\eta_{\mu\nu}\\ A_\mu^N(x)=0\\ \phi_s^M(x)=0}}, \tag{2.5}$$

and similarly for correlators involving also symmetry currents (including the energy-momentum tensor). In the above expression the operators, and hence the sources, may be identical.

The treatment of the string theory partition function is more challenging; the main obstacle which needs to be overcome is the fact that the full partition function Z_{String}

is not explicitly known nor is there a way to compute it.[1] Fortunately, our ignorance may be circumvented by weakening the duality by a sequence of two limits – first the semiclassical limit of string theory (ignoring loop corrections) followed by the point-particle limit – after which the theory takes the form of semiclassical supergravity. In this regime the string partition function may be given by a saddle point approximation, i.e.

$$Z_{\text{String}} \approx e^{-\mathcal{S}^{\text{o.s.}}_{\text{Sugra}}}, \tag{2.6}$$

where $\mathcal{S}^{\text{o.s.}}_{\text{Sugra}}$ is the Euclidean on-shell action of the supergravity theory in the bulk spacetime.

Recalling the correspondence as given in equation (2.1) it may – given the above discussion – be expressed in the weaker form

$$Z_{\text{CFT}} = e^{-\mathcal{S}^{\text{o.s.}}_{\text{Sugra}}}\Big|_{\text{conformal boundary}}. \tag{2.7}$$

The semi-classical limit of the bulk string theory results in a weak string coupling; consequently, if the boundary conformal field theory is a gauge theory the rank of its gauge group must be large. Moreover, the point-particle limit in the bulk theory enforces a large coupling in the boundary field theory. The relation (2.7) thus equates a weakly coupled semiclassical theory of gravity to a strongly coupled quantum conformal field theory without gravity.

2.2 Field-operator Maps

The AdS/CFT duality relies on the matching of degrees of freedom of asymptotically $(d+1)$-dimensional anti-de Sitter spacetime to those of a d-dimensional conformal field theory. A guiding principle may be found in the asymptotic symmetries of the bulk spacetime and their relation to the symmetries of the dual field theory. In particular the mappings between fields of the gravitational theory and operators of the conformal field theory must respect the representations under which the objects transform. Scalar fields of the gravitational theory map to scalar operators in the field theory; similarly gauge fields in AdS map to conserved currents of the CFT, and so on. To make these principles clear it is helpful to consider the simplest example for which the duality is applicable: the massive scalar sector of supergravity in asymptotically $(d+1)$-dimensional AdS spacetime. Backreaction onto the metric due to the scalar field will be neglected.

[1]This topic has seen recent progress [103].

2.2.1 Scalar Field

Considering the full bulk supergravity action for a scalar field is beyond the scope of this thesis; however a simplified toy model – which is sufficiently enlightening – is given by

$$\mathcal{S}_{\text{Sugra}} = -\frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}u \, \sqrt{-g} \left(g^{ab} \partial_a \phi \, \partial_b \phi + m^2 \phi^2 \right) , \tag{2.8}$$

where $\phi(u, x)$ is a scalar field with mass m; u is the radial coordinate; and $g_{\mu\nu}$ is the metric with determinant g.

The geometry of the bulk anti-de Sitter spacetime is determined by the line element, which in Poincaré coordinates reads

$$\mathrm{d}s^2 = g_{ab} \, \mathrm{d}x^a \, \mathrm{d}x^b = \frac{\ell^2}{u^2} \left(\mathrm{d}u^2 + \eta_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu \right) , \tag{2.9}$$

where the radial coordinate is excluded from the coordinates with greek indices, and ℓ is the radius of curvature of AdS. The conformal boundary of AdS is located at $u = 0$.

The equation of motion for the scalar field is the Klein-Gordon equation in AdS,

$$\frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} g^{ab} \partial_b \phi \right) - m^2 \phi = 0. \tag{2.10}$$

Plugging in all relevant quantities, the above equation takes the form

$$u^2 \partial_u^2 \phi - (d - 1) u \partial_u \phi + (u^2 \Box_\eta - m^2 \ell^2) \phi = 0, \tag{2.11}$$

where $\Box_\eta$ is the d'Alembert operator with Minkowski metric.

For the current purpose it is sufficient to solve equation (2.11) at the conformal boundary. To this end, an ansatz which is well-behaved near the boundary is

$$\phi(u, x) = u^\alpha \phi(x) + \dots , \tag{2.12}$$

where the ellipsis signify sub-leading terms for $u \to 0$ and α is a constant to be determined. Plugging the above ansatz into equation (2.11) and keeping only leading terms when $u \to 0$ results in a quadratic equation for α, with roots

$$\alpha_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 \ell^2}. \tag{2.13}$$

The solutions $\alpha_\pm$ are real if $m^2 \ell^2 \geq -d^2/4$, which means that the scalar fields in anti-de Sitter spacetime may have a negative mass while remaining well-behaved – this is the Breitenlohner-Freedman bound [104, 105]. Assuming that α_+ and α_- are not related by an integer,[2] the near boundary solution takes the form of an expansion with two independent coefficients

$$\phi(u,x) = \phi_-(x)u^{d-\alpha_+} + \ldots + \phi_+(x)u^{\alpha_+} + \ldots, \tag{2.14}$$

where $d - \alpha_+ = \alpha_-$ and the ellipsis denote subleading terms for each $\alpha_\pm$ whose coefficients are determined from the respective $\phi_\pm$.[3] The leading term of the expansion (2.14) is referred to as non-normalisable while the sub-leading term is called normalisable; this jargon stems from the behaviour of the on-shell action when using these solutions.

The discussion so far has taken place in the bulk side of the correspondence, however the relationship to the boundary conformal field theory is of equal interest. Consider thus an unspecified d-dimensional conformal field theory action with an additional source term, i.e.

$$\mathcal{S}_{\text{boundary}} = \mathcal{S}_{\text{CFT}} - \int \mathrm{d}^d x\, \mathcal{O}(x)\phi_s(x), \tag{2.15}$$

where $\mathcal{O}(x)$ is a scalar operator with scaling dimension Δ and $\phi_s(x)$ is its source. In a following step, consider a constant rescaling of the boundary coordinates of the form

$$x \mapsto \tilde{x} = \lambda x, \tag{2.16}$$

for which the transformation behaviour of the scalar operator is fixed by conformal invariance, while the transformation of the source may be determined by invariance of the action; the fields thus transform as

$$\mathcal{O}(x) \mapsto \tilde{\mathcal{O}}(\tilde{x}) = \lambda^{-\Delta}\mathcal{O}(x), \qquad \phi_s(x) \mapsto \tilde{\phi}_s(\tilde{x}) = \lambda^{\Delta-d}\phi_s(x). \tag{2.17}$$

The bulk gravitational theory should be invariant under isometries, which includes rescalings. Invariance of the line element (2.9) under the transformation of the boundary variables (2.16) requires the radial coordinate to behave as

$$u \mapsto \tilde{u} = \lambda u. \tag{2.18}$$

Furthermore, also the scalar fields in the gravity theory should be invariant under isometries and hence, as a consequence of the above transformation of the radial coor-

[2] If the two solutions α_+ and α_- are related by an integer the near boundary expansion may contain additional logarithmic terms.

[3] The coefficients ϕ_+ and ϕ_- are only independent at the conformal boundary; regularity of the scalar field in the bulk imposes non-trivial relations between the two coefficients.

dinate, the coefficients of the expansion (2.14) must transform as

$$\phi_+(x) \mapsto \tilde{\phi}_+(\tilde{x}) = \lambda^{-\alpha_+}\phi_+(x), \qquad \phi_-(x) \mapsto \tilde{\phi}_-(\tilde{x}) = \lambda^{\alpha_+ - d}\phi_-(x). \tag{2.19}$$

By comparing the above transformation properties to the transformations of the CFT fields (2.17) an intriguing picture appears. By identifying $\Delta \equiv \alpha_+$ the transformation properties of the two sets of fields suggest the equivalences

$$\phi_-(x) \sim \phi_s(x), \qquad \phi_+(x) \sim \langle \mathcal{O}(x) \rangle_{\text{CFT}}\,, \tag{2.20}$$

i.e. that the mode $\phi_+(x)$ is dual to the vacuum expectation value $\langle \mathcal{O}(x) \rangle$, while $\phi_-(x)$ is dual to the source $\phi_s(x)$.[4]

Although the association (2.20) appears in a somewhat ad-hoc manner the map between operators and fields may be derived in more stringent terms via the process of holographic renormalisation. A full treatment of this procedure is beyond the scope of this thesis but it may be summarised as follows: The weak form of the duality formulated in terms of partition functions, as in equation (2.7), in essence equates the CFT generating functional with the on-shell Euclidean action of supergravity; divergent terms in the bulk theory must thus be taken care of in order for the CFT quantities which arise from the generating functional to be free of divergences associated with the near-boundary behaviour of the supergravity fields – this requires holographic renormalisation. In the present context one then finds the precise relationships

$$\langle \mathcal{O}(x) \rangle_{\text{CFT}} = \ell^{d-1}(2\Delta - d)\phi_+(x), \qquad \phi_s(x) = \phi_-(x), \tag{2.21}$$

at the conformal boundary. The second of the above identifications fixes a value for the leading contribution to the scalar field and hence imposes (generalised) Dirichlet boundary conditions at the conformal boundary. See for instance [106, 107] for more involved treatments of holographic renormalisation.

With the prescription (2.21) at hand the evaluation at the boundary written in (2.1) and (2.7) can be made more transparent by expressing the AdS/CFT duality for a scalar field as

$$Z_{\text{CFT}}\left[\phi_s(x)\right] = e^{-\mathcal{S}^{\text{o.s.}}_{\text{Sugra}}[\phi(x,u)]}\Big|_{\lim\limits_{u\to 0}\phi(u,x)u^{\Delta-d}=\phi_s(x)}\,, \tag{2.22}$$

where $\mathcal{S}^{\text{o.s.}}_{\text{Sugra}}$ is the renormalised Euclidean on-shell bulk action [106], and the evaluation

[4] The identification $\Delta \equiv \alpha_+$ assumes 'standard quantisation' and (generalised) Dirichlet boundary conditions for the bulk scalar field. If the scaling dimension of the scalar operator lies within the range $d/2 - 1 < \Delta \leq d/2$ it is instead possible to impose 'alternate quantisation' with the identifications $\phi_+(x) \sim \phi_s(x)$ and $\phi_-(x) \sim \langle \mathcal{O}(x) \rangle_{\text{CFT}}$, which leads to (generalised) Neumann boundary conditions [75].

at the conformal boundary extracts $\phi_-(x) = \lim_{u\to 0} \phi(u,x)u^{\Delta-d}$ and equates it to the scalar source $\phi_s(x)$ in the boundary conformal field theory. The expression for a connected n-point correlation function of identical scalar operators in the boundary CFT then follows straightforwardly from (2.5), i.e.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_n)\rangle_{\text{CFT}} = -\left.\frac{\delta^n \mathcal{S}^{\text{o.s.}}_{\text{Sugra}}\left[\phi(x,u)\right]}{\delta\phi_s(x_1)\delta\phi_s(x_2)\cdots\delta\phi_s(x_n)}\right|_{\substack{\lim_{u\to 0}\phi(x,u)u^{\Delta-d}=\phi_s(x)\\ \phi_s(x)=0}}. \tag{2.23}$$

The above result can be generalised to non-identical scalars by including further source terms, as well as interaction terms, in the bulk action.

2.2.2 Energy-momentum Tensor and Current

The analysis carried out in the previous section becomes substantially more technically involved when applying it to other quantities of interest, such as the boundary energy-momentum tensor $\langle T^{\mu\nu}\rangle$ or conserved charge current $\langle J^\mu\rangle$; nevertheless, the principles are similar. By matching transformation properties, and performing holographic renormalisation, the desired operator dualities may be derived.

The energy-momentum tensor of the conformal field theory is dual to the boundary components of the fluctuating bulk metric $g_{\mu\nu}(u,x)$ via

$$\langle T^{\mu\nu}\rangle_{\text{CFT}} = \lim_{u\to 0}\left[\frac{\ell^{d-1}}{u^{d/2-1}}\frac{1}{8\pi\text{G}}\left(K_{\mu\nu} - K\gamma_{\mu\nu} - c_1\frac{d-1}{\ell}\gamma_{\mu\nu} + c_2\frac{\ell}{d-2}G_{\mu\nu}\right)\right], \tag{2.24}$$

where G is the $(d+1)$-dimensional gravitational constant; $K_{\mu\nu}$ is the extrinsic curvature tensor projected on the conformal boundary and K is its trace; $G_{\mu\nu}$ is the Einstein tensor for the induced boundary metric $\gamma_{\mu\nu}$; and $c_1 = 1$ for $d \geq 2$, $c_2 = 1$ for $d \geq 3$ and zero otherwise [106, 108]. Moreover, the CFT energy-momentum tensor is sourced by the leading contribution to the near-boundary expansion of the full AdS metric g_{mn}.

Finally, fluctuations of gauge fields $A^\mu_N(u,x)$ in the bulk geometry are dual to conserved symmetry currents J^μ_N of the boundary conformal field theory in accordance with

$$\langle J^\mu_N\rangle_{\text{CFT}} = \lim_{u\to 0}\frac{1}{u^{d-1}}\gamma^{\mu\alpha}\partial_u A^N_\alpha, \tag{2.25}$$

where J^μ_N is sourced by the leading term of the near-boundary expansion of A^k_N [109, 110].[5]

At first glance the right-hand sides of (2.24) and (2.25) appear divergent as $u \to 0$; however, one finds that the terms proportional to inverse powers of u cancel with

[5]Note that N acts as a label and hence its placement is of no concern.

contributions from the other quantities in the equation.

2.3 Beyond the Vacuum

When establishing the AdS/CFT duality – as has been done in the previous sections – it is natural to present it in its simplest form: pure anti-de Sitter gravity in the bulk being dual to a vacuum conformal field theory on the conformal boundary. However, the applications which will be presented in this thesis require a boundary field theory at finite temperature; the realisation of this property and the subsequent implementation of linear response theory – within the context of holography – will be covered in this section.

2.3.1 Finite Temperature and Black Brane Thermodynamics

Finite Temperature in QFT

At finite temperature T a quantum field theory experiences thermal fluctuations – in addition to quantum fluctuations – which affect the macroscopic state of the system. The thermal partition function is given by

$$Z(\beta) \equiv \mathrm{tr}_{\mathcal{H}} e^{-\beta H} = \int \mathrm{d}\Theta \, \langle \Theta | e^{-\beta H} | \Theta \rangle \,, \tag{2.26}$$

where the trace runs over the Hilbert space $\mathcal{H}$ with basis $|\Theta\rangle$, H is the Hamiltonian, and $\beta = 1/T$. Wick-rotating the time-direction $\tau \equiv it$ allows the integrand in (2.26) to be interpreted as a transition amplitude between an initial state at imaginary time τ and final state at $\tau + \beta$; this may in turn be evaluated via the path integral of the Euclidean action. Evaluating the standard integration with respect to Θ the result is

$$Z(\beta) = \int \mathcal{D}\Theta \, e^{-\mathcal{S}_{\mathrm{E}}} \Big|_{\Theta(\tau) = \pm\Theta(\tau+\beta)} \,, \tag{2.27}$$

where Θ is a fundamental field and anti-periodic boundary condition only apply to fermionic fields – periodic boundary conditions otherwise. The distinction between fermions and bosons in terms of the boundary conditions results in spontaneous breaking of any supersymmetry.

In equation (2.27) an important property of finite temperature in QFT becomes evident, namely that it can be captured by Euclidean QFT with a compactified dimension with periodicity[6]

$$\tau \sim \tau + \beta. \tag{2.28}$$

With the introduction of a temperature ones gains access to thermodynamics. For now it is sufficient to introduce the free energy

$$F = -T \ln Z(\beta). \tag{2.29}$$

Black Brane in AdS/CFT

By the principles of the holographic duality the introduction of a temperature for the boundary theory necessitates a dual property in the bulk. In semiclassical gravity the simplest thermodynamic object is a black brane. Consider thus the $(d+1)$-dimensional Euclidean geometry of a black brane in asymptotically anti-de Sitter spacetime – it is given by the line element

$$\mathrm{d}s^2 = \frac{\ell^2}{u^2}\left(f(u)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}u^2}{f(u)} + \mathrm{d}\mathbf{x}^2 \right), \tag{2.30}$$

where $\mathbf{x}$ denotes the vector of spatial boundary coordinates; the emblackening factor $f(u)$ defines the horizon located at u_h via $f(u_h) = 0$ while $f'(u_h) \neq 0$; and $f(0) = 1$ such that AdS spacetime is recovered at the conformal boundary. Taylor expanding the emblackening factor near the horizon, i.e.

$$f(u) = f(u_h) + (u - u_h) f'(u_h) + \ldots, \tag{2.31}$$

and defining the new coordinates

$$\rho = 4\frac{u - u_h}{\left|f'(u_h)\right|}, \qquad \varphi = \frac{\tau}{2}\left|f'(u_h)\right|, \tag{2.32}$$

the line element takes the approximate form

$$\mathrm{d}s^2 \approx \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\varphi^2 + \mathrm{d}\mathbf{x}^2. \tag{2.33}$$

The line element (2.33) defines a plane in polar coordinates (ρ, φ) if the conical singularity at $\rho = 0$ is avoided – this requires that the angular coordinate is periodic, $\varphi \sim \varphi + 2\pi$, which in terms of τ gives

$$\tau \sim \tau + \frac{4\pi}{\left|f'(u_h)\right|}. \tag{2.34}$$

[6] Another perspective is that the circumference β of the compactified dimension is infinite at vanishing temperature.

The above identification is a generic feature of black branes in Euclidean space.

Invoking the AdS/CFT correspondence the non-radial coordinates naturally map from the bulk to the boundary; the periodicity (2.34) may thus be compared to the thermal QFT expression (2.28) and thus concluding that the bulk/boundary-system has the temperature

$$T = \frac{\left|f'(u_h)\right|}{4\pi}, \tag{2.35}$$

which agrees with the temperature of black hole radiation first derived by Hawking [111].

In addition to temperature the black brane provides a useful notion of entropy given by the Bekenstein-Hawking entropy [112, 113]

$$S_{\mathrm{BH}} = \frac{A}{4\mathrm{G}}, \tag{2.36}$$

where A is the area of the black brane horizon

$$A = \int \mathrm{d}^{d-1}x \sqrt{g_{d-1}|_{u=u_h}}. \tag{2.37}$$

Since black brane horizons are non-compact the above integral is proportional to the infinite volume of $\mathbb{R}^{d-1}$; it is hence advantageous to work with densities within the context of holography – in this case entropy density is

$$s = \frac{1}{4\mathrm{G}} \frac{A}{\mathrm{Vol}(\mathbb{R}^{d-1})}. \tag{2.38}$$

Although the above analysis has been done in Euclidean signature the properties carry over to Lorentzian signature.

Finite charge

In chapter 4 the holographic duality is applied to study boundary systems with finite temperature and $U(1)$ charge which is carried by the symmetry current. As mentioned in section 2.2.2 the symmetry currents of the boundary CFT are dual to gauge fields in the bulk. Introducing gauge fields to the bulk theory will in principle affect the Einstein equations and hence, at finite temperature, also the black brane. The analysis required in such a situation is beyond the scope of this thesis; in fact, the upcoming considerations of holographic theories with finite charge will be done in the probe limit, meaning that the metric and hence the black brane are unaffected.

2.3.2 Linear Response Theory

The presence of a black brane places the holographic system at finite temperature while also establishing a notion of thermodynamic equilibrium. Future applications in this thesis concern non-equilibrium dynamics of thermal systems. Far-from equilibrium dynamics of thermal systems is a highly non-trivial subject; however, the the near-equilibrium regime is more tractable and lies within the domain of linear response theory. In this section the framework of linear response will be presented for a scalar operator but it generalises to arbitrary fields.[7] In order to discuss time-dependent dynamics the analysis must be done in Lorentzian spacetime.

In Quantum Field Theory

Linear response arises from a deformation of a time-independent background Hamiltonian H by a term of the form

$$\delta H = -\int \mathrm{d}^d x\, \delta\phi_s(x)\mathcal{O}(x), \tag{2.39}$$

where $\delta\phi_s$ is a infinitesimal fluctuation of the source of the scalar operator $\mathcal{O}$. The above deformation of the Hamiltonian in turn shifts the expectation value of the scalar operator $\langle\mathcal{O}(x)\rangle$ by

$$\delta\left\langle\mathcal{O}(x)\right\rangle = -\int \mathrm{d}^d x'\, G^{\mathrm{R}}_{\mathcal{O}\mathcal{O}}(x-x')\delta\phi_s(x') + \mathcal{O}(\delta\phi_s^2), \tag{2.40}$$

where the retarded Green's function for bosonic operators defined as

$$G^{\mathrm{R}}_{ab}(x-x') = -i\theta(t-t')\left\langle[a(x), b(x')]\right\rangle, \tag{2.41}$$

where θ is the Heaviside function and $a(x)$, $b(x)$ represent generic bosonic operators.[8] The retarded Green's function is non-zero only if $t' < t$; this introduces causality since only sources in the past may influence the expectation value $\delta\left\langle\mathcal{O}(x)\right\rangle$.

Using the Fourier transform of the Green's function (2.41) at $x' = 0$,

$$G^{\mathrm{R}}_{ab}(\omega, \mathbf{k}) = \int \mathrm{d}^{d-1}\mathbf{x}\, \mathrm{d}t\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\, G^{\mathrm{R}}_{ab}(t, \mathbf{x}), \tag{2.42}$$

where ω is the frequency and $\mathbf{k}$ the spatial momentum vector, the integral in (2.40)

[7]Linear response does in general not require finite temperature, however it is particularly relevant for systems at finite temperature.

[8]The same expression but with the commutator replaced by an anti-commutator defines the retarded Green's function for fermionic opeators.

may be evaluated – this results in the linear order expression

$$\delta \langle \mathcal{O}(\omega, \mathbf{k}) \rangle = G^{\mathrm{R}}_{\mathcal{O}\mathcal{O}}(\omega, \mathbf{k}) \delta\phi_s(\omega, \mathbf{k}) + \mathcal{O}(\delta\phi_s^2). \tag{2.43}$$

Hence, in linear response the Fourier-space retarded Green's function of scalar operators can be calculated using the formula

$$G^{\mathrm{R}}_{\mathcal{O}\mathcal{O}}(\omega, \mathbf{k}) = \frac{\delta \langle \mathcal{O}(\omega, \mathbf{k}) \rangle}{\delta\phi_s(\omega, \mathbf{k})} + \mathcal{O}(\delta\phi_s). \tag{2.44}$$

More generally the linear retarded Green's function between two non-specific operators is given by

$$G^{\mathrm{R}}_{ab}(\omega, \mathbf{k}) = \frac{\delta \langle a(\omega, \mathbf{k}) \rangle}{\delta j_b(\omega, \mathbf{k})} + \mathcal{O}(j_b), \tag{2.45}$$

where δj_b denotes the fluctuation of the source of some operator $b(\omega, \mathbf{k})$.

In AdS/CFT

The concepts of linear response theory may be applied within the holographic context; this will be done via the means of a scalar field fluctuation $\delta\phi(u, x)$ in an asymptotically AdS black brane spacetime and its dual CFT. Using the Lorentzian analogue of the metric (2.30) in the formula (2.10) the Fourier transformed (with respect to the boundary coordinates) equation of motion for the scalar fluctuation is

$$u^{d+1}\partial_u \left[u^{1-d} f(u) \partial_u \delta\phi(u, k) \right] + \left(\frac{u^2}{f(u)}\omega^2 - u^2\mathbf{k}^2 \right) \delta\phi(u, k) - m^2\ell^2 \delta\phi(u, k) = 0, \tag{2.46}$$

where $k = (\omega, \mathbf{k})$ is the d-dimensional momentum vector and the derivatives with respect to boundary coordinates have been evaluated.

The spacetime remains asymptotically anti-de Sitter and hence the result of the analysis made for the scalar field in section 2.2.1 may be applied identically to $\delta\phi$; the identification with the CFT quantities is thus

$$\delta \langle \mathcal{O}(\omega, \mathbf{k}) \rangle_{\mathrm{CFT}} = \ell^{d-1}(2\Delta - d)\delta\phi_+(\omega, \mathbf{k}), \qquad \delta\phi_s(\omega, \mathbf{k}) = \delta\phi_-(\omega, \mathbf{k}), \tag{2.47}$$

where the subscripts are the same as in section 2.2.1.

The asymptotic analysis does not account for the causal structure which is important when calculating the retarded Green's functions; getting a handle on this requires a venture into the bulk to solve the equations of motion near the black brane horizon.[9] Assuming homogeneity for simplicity, a suitable ansatz which is well-behaved near the

[9] A near-horizon analysis must also be done for the scalar field background. However, for the background the horizon conditions are chosen to ensure regularity.

horizon at u_h is

$$\delta\phi(u,k) = (u_h - u)^\kappa w(k) + \dots, \tag{2.48}$$

where the ellipsis denotes subleading terms. Using the above ansatz together with the Taylor expansion (2.31) the leading terms of (2.46) as $u \to u_h$ are

$$\kappa^2 f'(u_h) + \frac{\omega^2}{f'(u_h)} = 0, \tag{2.49}$$

which has the roots

$$\kappa = \pm\frac{i\omega}{4\pi T}, \tag{2.50}$$

where the temperature enters via (2.35). The two roots above correspond to different behaviours of the fluctuation – this can be seen by restoring the time-dependency; schematically it reads

$$\delta\phi(u,x) \sim e^{-i\omega\left[t \mp \frac{1}{4\pi T}\ln(1-u)\right]} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.51}$$

where u_h has been set to one for clarity. Keeping the expression in brackets constant as t increases means that u must increase when the relative sign is plus, which is the minus solution of (2.50) – since $u = 0$ corresponds to the conformal boundary this means that this fluctuation is approaching the black brane and hence this solution is referred to as infalling. The opposite is true for the plus solution of (2.50), which is referred to as the outgoing solution. Due to the presence of the black brane horizon the infalling solution can only be affected by sources in past, and similarly the outgoing solutions are only affected by sources in the future. Thus the choice of boundary condition at the horizon has causal implications.

The discussion above drives one to the conclusion that – using the identifications (2.47) – the boundary retarded Green's function (2.44) can be calculated in terms of bulk quantities as

$$G^{\mathrm{R}}_{\mathcal{O}\mathcal{O}}(\omega,\mathbf{k}) = \ell^{d-1}(2\Delta - d)\frac{\delta\phi_+(\omega,\mathbf{k})}{\delta\phi_-(\omega,\mathbf{k})}\bigg|_{\text{infalling}}, \tag{2.52}$$

where the fluctuations satisfy infalling boundary conditions at the horizon. Similar expressions hold for other operators.

In later chapters it will become evident that in addition to the retarded Green's functions themselves also their poles are of interest. These poles lie in the lower-half complex frequency plane and correspond to the quasi-normal modes of the bulk spacetime. The quasi-normal modes are probed by imposing homogeneous Dirichlet boundary conditions at the conformal boundary, i.e. setting the sources of the fluctuations (but not necessarily the background) to zero. Calculating the Green's functions requires inhomogenous Dirichlet boundary conditions.

3 Hydrodynamics

Hydrodynamics is a universal framework which captures the long-wavelength, late-time dynamics of massless degrees of freedom in many-body system at finite temperature. Long-wavelength qualifies as distance scales much greater than the characteristic length scale of the system – usually the mean free path – while late-time means that sufficient time should have passed such that the system is perturbatively near thermodynamic equilibrium.

Hydrodynamics has been fruitfully applied to several physical phenomena, for instance in neutron stars [114], quark-gluon plasma [16–18], and graphene [19, 20]. The manifestation of the framework in relativistic quantum field theory – which is relevant for the examples mentioned in the previous breath – is referred to as relativistic hydrodynamics; here the adjective signifies the inclusion of Lorentz invariance rather than the speed at which the particles travel. The contents of this chapter treat various aspects of relativistic hydrodynamics in two spatial dimensions. Applications to holographic systems will guide some discussions throughout the chapter.

3.1 Foundations

This first section reviews some fundamental notions of a parity preserving normal fluid which will be of importance further along this thesis, following [15]. For more exhaustive treatments of the topic see for instance [14, 15].

3.1.1 Thermodynamics

Given the requirement that the system must be at finite temperature makes thermodynamics a natural starting point. The thermodynamic incarnation of energy conservation is captured by the first law of thermodynamics, which states that the change in total energy of a system is given by the energy supplied by heat and charge flux, minus the work done by the system. In differential form the first law may be expressed as

$$\mathrm{d}U = T\,\mathrm{d}S + \mu\,\mathrm{d}N - p\,\mathrm{d}V, \tag{3.1}$$

where U is the total energy; T is the temperature and S the entropy; μ, N are the chemical potential and particle number respectively; and p is the pressure while V is the volume. From the above expression one may deduce the functional dependence of the thermodynamic potential $U = U(S, V, N)$ and thermodynamic relations

$$T = \left(\frac{\partial U}{\partial S}\right)_{V,N}, \quad \mu = \left(\frac{\partial U}{\partial N}\right)_{V,S}, \quad -p = \left(\frac{\partial U}{\partial V}\right)_{S,N}, \tag{3.2}$$

where the subscripts below the parenthesis denote quantities which are kept fixed. Thermodynamic quantities related through derivatives, such as those in equation (3.2), are said to be conjugate to each other. Homogeneity imposes that the thermodynamic potential (3.1) must behave as $U(\lambda S, \lambda V, \lambda N) = \lambda U(S, V, N)$ under rescalings of extensive parameters – this means that (3.1) may be integrated to obtain

$$U = TS + \mu N - pV. \tag{3.3}$$

As noted in section 2.3.1 it is more natural to work with densities rather than total quantities in the context of holography; for the quantities presented so far it follows that

$$s = \frac{S}{V}, \quad \varepsilon = \frac{U}{V}, \quad \rho = \frac{N}{V}, \tag{3.4}$$

for entropy density, energy density, and particle density, respectively. Acting with a differential on the energy density and simplifying leads to the relation

$$\mathrm{d}\varepsilon = T\,\mathrm{d}s + \mu\,\mathrm{d}\rho, \tag{3.5}$$

and thus $\varepsilon = \varepsilon(s, \rho)$. The above expression can be compared to

$$\varepsilon = sT + \mu\rho - p, \tag{3.6}$$

which follows from $\varepsilon = U/V$; this result defines the enthaply density

$$h \equiv \varepsilon + p = Ts + \mu\rho. \tag{3.7}$$

From the expression for the energy density (3.6) the free energy density is defined as

$$f_\varepsilon \equiv \varepsilon - sT, \tag{3.8}$$

which is equal to $f_\varepsilon = \mu\rho - p$. Taking the differential of the definition of the free energy density results in

$$\mathrm{d}f_\varepsilon = \mu\,\mathrm{d}\rho - s\,\mathrm{d}T, \tag{3.9}$$

hence the free energy has the functional dependence $f_\varepsilon = f_\varepsilon(T,\rho)$.

In relativistic quantum field theory particle number, and by extension particle density, is not conserved; it is hence convenient to define the grand canonical potential (density)

$$\Omega = f_\varepsilon - \mu\rho, \tag{3.10}$$

In differential form equation (3.10) reads

$$\mathrm{d}\Omega = -s\,\mathrm{d}T - \rho\,\mathrm{d}\mu, \tag{3.11}$$

and thus $\Omega = \Omega(T,\mu)$. The definition (3.10) gives $\Omega = -p$, which means $p = p(T,\mu)$; this leads to the equation of state

$$\mathrm{d}p = s\,\mathrm{d}T + \rho\,\mathrm{d}\mu, \tag{3.12}$$

which uniquely fixes the quantities $p(T,\mu)$, $s(T,\mu)$, $\rho(T,\mu)$ and $\varepsilon = sT + \mu\rho - p$.

3.1.2 Near Equilibrium Dynamics

In this section the analysis moves beyond thermodynamics and into the realm of hydrodynamics. In the process of doing so a three-dimensional fluid velocity vector-field $u^\mu(t,x)$, which is normalised such that $u^\mu u_\mu = -1$, is introduced. Moreover, the temperature and chemical potential are promoted to scalar fields $T(t,x)$ and $\mu(t,x)$. In order to satisfy near-equilibrium requirement of hydrodynamics the newly defined fields are taken to be slowly varying, meaning that their spacetime derivatives may be used as perturbative parameters.

The next step is to consider the hydrodynamic equations – these are merely the conservation equations of the system. A generic quantum field theory demands Poincaré invariance; hence, as a consequence of Noether's theorem, the hydrodynamic regime of a quantum field theory should in principle obey energy-momentum conservation[1]

$$\partial_\mu \langle T^{\mu\nu}\rangle = 0, \tag{3.13a}$$

where the symmetric energy-momentum tensor $\langle T^{\mu\nu}\rangle$ is the conserved current for spacetime translational invariance.[2] Moreover, if one allows for the presence of an internal $U(1)$-symmetry the corresponding equation of charge conservation appears,

$$\partial_\mu \langle J^\mu\rangle = 0, \tag{3.13b}$$

[1] As will become clear in the following there exists hydrodynamic systems where energy and momentum conservation may be neglected.

[2] Invariance under rotations and boosts follows from a symmetric $T^{\mu\nu}$.

with $\langle J^\mu \rangle$ being the $U(1)$-current.

3.1.3 Constitutive relations

The slowly varying nature of the hydrodynamic regime allows the conserved currents to be expressed as gradient expansions in terms of the hydrodynamic variables $u^\mu(t,x)$, $T(t,x)$ and $\mu(t,x)$ since each additional derivative of a term decreases its contribution. The derivative expansions of the conserved currents are referred to as the constitutive relations.

Zeroth-order

The first terms of the constitutive relations – at zeroth-order in derivatives – is found by acting with a Lorentz transformation on the static equilibrium energy-momentum tensor and charge current, given by $\langle T^{\mu\nu} \rangle = \mathrm{diag}(\varepsilon, p, p)$ and $\langle J^\mu \rangle = (\rho, 0, 0)$ respectively; the resulting expressions are

$$\langle T^{\mu\nu} \rangle = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu}, \tag{3.14a}$$

$$\langle J^\mu \rangle = \rho u^\mu, \tag{3.14b}$$

where the projector is given by

$$\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu, \tag{3.15}$$

with $\eta^{\mu\nu}$ being the Minkowski metric.

At zeroth-order hydrodynamics describes an ideal fluid, meaning a fluid on which any perturbation propagates indefinitely. Ideal fluids are not necessarily realistic, the more realistic description is captured at first-order in derivatives.

First-order

Subtleties arise when considering hydrodynamics at first-order in derivatives. Although the fields $u^\mu(t,x)$, $T(t,x)$ and $\mu(t,x)$ are well-defined at equilibrium their out-of-equilibrium definitions may differ by derivative corrections which vanish as their gradients go to zero – this is referred to as the freedom to choose a fluid frame. This freedom has the consequence that the form of the first-order constitutive relations is only fixed up to artefacts which depend on the frame choice. The physical observables, however, are not affected by the frame; thus, the choice of frame is a matter of convenience.

The full procedure of finding the constitutive relations is beyond the present scope.

In short, one first writes down all possible terms which satisfy the tensor structure of the current, after which one restricts the allowed terms based on fluid-specific symmetry considerations and frame choice. In this thesis the frame of choice is the Landau frame [14], which is defined by $\langle T^{\mu\nu}\rangle u_\nu = -\varepsilon u^\mu$ and $\langle J^\mu\rangle u_\mu = -\rho$; in this frame the constitutive relations take the form

$$\langle T^{\mu\nu}\rangle = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta_1 \Delta^{\mu\nu}\partial_\lambda u^\lambda + \mathcal{O}(\partial^2), \tag{3.16a}$$

$$\langle J^\mu\rangle = \rho u^\mu - \sigma_0 T \Delta^{\mu\nu}\partial_\nu \left(\frac{\mu}{T}\right) + \mathcal{O}(\partial^2), \tag{3.16b}$$

with the shear tensor defined as

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \eta_{\alpha\beta}\partial_\mu u^\mu\right). \tag{3.17}$$

The coefficients arising at first order in the constitutive relations are called transport coefficients; by the second law of thermodynamics they must be non-negative, and they are functions of the temperature and chemical potential. $\sigma_0(T,\mu)$ is the charge conductivity; $\eta(T,\mu)$ and $\zeta_1(T,\mu)$ are the shear and bulk viscosities, respectively, and they determine the fluids response to shear and bulk strain. Formulae for the transport coefficients are discussed in section 3.1.5.

3.1.4 Hydrodynamic Modes

The next step in the hydrodynamic process is studying the behaviour of small fluctuations of the hydrodynamic fields. The dynamics of the system is constrained by the conservation equations (3.13). The constitutive relations (3.16) are expressed in terms of fluctuations of the hydrodynamic quantities around thermodynamic equilibrium and terms up to linear order in fluctuations are considered. The equilibrium state is characterised by

$$u^\mu = (1,0,0), \qquad T = \text{constant} \equiv \bar{T}, \qquad \mu = \text{constant} \equiv \bar{\mu}, \tag{3.18}$$

where the bar denotes constant equilibrium quantities. Including the fluctuations is straightforward – the fields read

$$u^\mu(t,x) = (1, \delta u^i(t,x)), \qquad T(t,x) = \bar{T} + \delta T(t,x), \qquad \mu(t,x) = \bar{\mu} + \delta\mu(t,x), \tag{3.19}$$

where the local fluctuations are taken to be much smaller than the constant values.

Although the analysis so far has been expressed in terms of the fluid velocity u^μ, temperature T and chemical potential μ, the remaining analysis will mainly be done in their conjugate variables; in terms of the components of the conserved currents they are

$$\pi^i(t,x) = T^{ti}(t,x), \qquad \varepsilon(t,x) = T^{tt}(t,x), \qquad \rho(t,x) = J^t(t,x), \tag{3.20}$$

i.e. from left to right: the momentum density, energy density, and charge density, respectively. In this context the old variables (3.18) are referred to as sources. In terms of the variables (3.20) the conservation equations (3.13) linearise and decompose to take the form

$$\partial_t \delta\varepsilon(t,x) + \partial_i \delta\pi^i(t,x) = 0, \tag{3.21a}$$

$$\partial_t \delta\pi^i(t,x) + \partial_j \delta T^{ij}(t,x) = 0, \tag{3.21b}$$

$$\partial_t \delta\rho(t,x) + \partial_i \delta J^i(t,x) = 0. \tag{3.21c}$$

The linearised constitutive relations for the energy-momentum tensor and charge current are still expressed in terms of the velocity, temperature and chemical potential; using (3.19) the conserved currents take the form

$$\partial_j \delta T^{ij} = \partial^i \delta p - \eta \left(\partial_j \partial^i \delta u^j + \partial_j \partial^j \delta u^i - \partial^i \partial_k \delta u^k \right) - \zeta_1 \partial^i \partial_k \delta u^k, \tag{3.22a}$$

$$\partial_i \delta J^i = \rho \partial_i \delta u^i - \sigma_0 \left(\partial_i \partial^i \delta\mu - \frac{\mu}{T} \partial_i \partial^i \delta T \right), \tag{3.22b}$$

where the pressure has been linearised, and derivatives of equilibrium quantities vanish. The bars on equilibrium quantities have been dropped. The above expressions may be written in terms of the energy density, momentum density and charge density by using

$$\delta p = \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \delta\varepsilon + \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \delta\rho, \tag{3.23a}$$

$$\delta \mu = \left(\frac{\partial \mu}{\partial \varepsilon}\right)_\rho \delta\varepsilon + \left(\frac{\partial \mu}{\partial \rho}\right)_\varepsilon \delta\rho, \tag{3.23b}$$

$$\delta T = \left(\frac{\partial T}{\partial \varepsilon}\right)_\rho \delta\varepsilon + \left(\frac{\partial T}{\partial \rho}\right)_\varepsilon \delta\rho, \tag{3.23c}$$

$$\delta u^i = \frac{\delta \pi^i}{h}, \tag{3.23d}$$

with the enthalpy h as defined in equation (3.7). Note that the opposite transformation – expressing the variables in terms of the sources – is facilitated by the susceptibility matrix χ_{ab},

$$\begin{pmatrix} \delta\varepsilon \\ \delta\pi^i \\ \delta\rho \end{pmatrix} = \begin{pmatrix} T\left(\frac{\partial\varepsilon}{\partial T}\right)_{\mu/T} & 0 & \left(\frac{\partial\varepsilon}{\partial\mu}\right)_T \\ 0 & h & 0 \\ T\left(\frac{\partial\rho}{\partial T}\right)_{\mu/T} & 0 & \left(\frac{\partial\rho}{\partial\mu}\right)_T \end{pmatrix} \begin{pmatrix} \delta T/T \\ \delta u^i \\ \delta\mu - \frac{\mu}{T}\delta T \end{pmatrix} \tag{3.24}$$

Using the transformations (3.23) the currents (3.22) now become

$$\partial_j \delta T^{ij} = \left(\frac{\partial p}{\partial\varepsilon}\right)_\rho \partial^i\delta\varepsilon + \left(\frac{\partial p}{\partial\rho}\right)_\varepsilon \partial^i\delta\rho - \frac{\eta}{h}\left(\partial_j\partial^i\delta\pi^j + \partial_j\partial^j\delta\pi^i - \partial^i\partial_k\delta\pi^k\right) - \frac{\zeta_1}{h}\partial^i\partial_k\delta\pi^k, \tag{3.25a}$$

$$\partial_i\delta J^i = \frac{\rho}{h}\partial_i\delta\pi^i - \sigma_0\alpha_1\partial_i\partial^i\delta\varepsilon - \sigma\alpha_2\partial_i\partial^i\delta\rho, \tag{3.25b}$$

with

$$\alpha_1 = \left(\frac{\partial\mu}{\partial\varepsilon}\right)_\rho - \frac{\mu}{T}\left(\frac{\partial T}{\partial\varepsilon}\right)_\rho, \qquad \alpha_2 = \left(\frac{\partial\mu}{\partial\rho}\right)_\varepsilon - \frac{\mu}{T}\left(\frac{\partial T}{\partial\rho}\right)_\varepsilon \tag{3.26}$$

The derivatives in the constitutive relations may be evaluated by Fourier transforming the fluctuations while also, without loss of generality, choosing the momentum to be in the x-direction; the Fourier transform takes the form

$$\delta\vartheta(t,x) = \frac{1}{2\pi}\int \mathrm{d}\omega\mathrm{d}k\, e^{-i\omega t + ikx}\delta\vartheta(\omega,k), \tag{3.27}$$

where $\vartheta(t,x)$ is some unspecified function representing the hydrodynamic variables and k is the momentum in the x-direction. Although the momentum has been chosen to flow in the x-direction the momentum density may still fluctuate in both spatial directions.

When choosing the momentum in the x-direction the conservation equations simply decompose into two sectors – one set parallel to and one set transverse to the momentum flow. After evaluating the derivatives the set of longitudinal equations becomes

$$0 = -i\omega\delta\varepsilon + ik\delta\pi^x, \tag{3.28a}$$

$$0 = \left(-i\omega + \frac{\eta+\zeta_1}{h}k^2\right)\delta\pi^x + ik\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho \delta\varepsilon + ik\left(\frac{\partial p}{\partial\rho}\right)_\varepsilon \delta\rho, \tag{3.28b}$$

$$0 = \left(-i\omega + \sigma\alpha_2 k^2\right)\delta\rho + \sigma_0\alpha_1 k^2\delta\varepsilon + ik\frac{\rho}{h}\delta\pi^x, \tag{3.28c}$$

while the transverse sector only contains the equation of momentum conservation

$$\left(-i\omega + \frac{\eta}{h}k^2\right)\delta\pi^y = 0. \tag{3.29}$$

Solving the conservation equations for ω results in the hydrodynamic modes. Equation (3.29) is solved to find a diffusive mode with dispersion relation

$$\omega(k) = -iDk^2, \qquad D = \frac{\eta}{h}. \tag{3.30}$$

The longitudinal equations are somewhat trickier – to minimise the clutter the presence of charge will be ignored for now. The system of equations with (3.28a) and (3.28b) can be cast into the form of a matrix equation

$$\begin{pmatrix} -i\omega & ik \\ ik\frac{\partial p}{\partial \varepsilon} & -i\omega + \frac{\eta+\zeta_1}{h}k^2 \end{pmatrix} \begin{pmatrix} \delta\varepsilon \\ \delta\pi^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.31}$$

The above equation has non-trivial solutions only if the coefficient matrix is non-invertible, i.e. if its determinant is zero – the modes are thus found by taking the determinant, setting it to zero and solving the resulting equation for ω pertubatively in k; this results in a pair of damped propagating sound modes with dispersion relation

$$\omega(k) = \pm v_s k - \frac{i}{2}\Gamma_s k^2 + \dots, \tag{3.32}$$

where the ellipsis signify higher order terms in k. The speed of sound v_s and attenuation constant Γ_s are given by

$$v_s^2 = \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho, \qquad \text{and} \qquad \Gamma_s = \frac{\eta+\zeta_1}{h}. \tag{3.33}$$

3.1.5 Retarded Green's functions and Kubo Formulae

Clearly the transport coefficients play a large part in the behaviour of the hydrodynamic modes in addition to being properties of the fluid themselves. Formulae for the transport coefficients may be found in terms of retarded Green's functions of the system, as defined in section 2.3.2 of chapter 2.

In hydrodynamics the matrix of all retarded Green's functions is calculated from the linearised constitutive relations using the equation [15, 115]

$$G^{\mathrm{R}}(\omega, \mathbf{k}) = -\left(\mathbb{1} + i\omega K^{-1}\right)\chi, \tag{3.34}$$

where $\mathbf{k}$ is the spatial momentum vector, K is the coefficient matrix of the hydrodynamic equations – as in equation (3.31) – and χ is the susceptibility matrix. Since K^{-1} is proportional to $1/\det K$ some basic manipulation of the formula (3.34) shows that the hydrodynamic modes correspond to the poles of the retarded Green's functions.

Note also the limiting behaviour

$$G^{\mathrm{R}}_{ab}(0,\mathbf{k}) = -\chi_{ab}. \tag{3.35}$$

Setting the momentum in the x-direction and using the linearised equations (3.28) together with the susceptibility matrix (3.24) a set of particularly relevant Green's functions whose expansions in momentum read

$$G^{\mathrm{R}}_{\pi^x\pi^x}(\omega,k) = \frac{k^2}{\omega^2}\left[h\left(\frac{\partial p}{\partial \varepsilon}\right)_\rho + \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \rho - i\omega(\eta+\zeta)\right] + \dots, \tag{3.36a}$$

$$G^{\mathrm{R}}_{\rho\rho}(\omega,k) = \frac{k^2}{\omega^2}\left[\frac{\rho^2}{h} - i\omega\sigma_0\right] + \dots, \tag{3.36b}$$

and from the transverse momentum conservation equation (3.29) one finds

$$G^{\mathrm{R}}_{\pi^y\pi^y}(\omega,k) = -i\frac{\eta k^2}{\omega} + \dots. \tag{3.36c}$$

To arrive at the above equations the thermodynamic relations

$$1 = \alpha_1\chi_{13} + \alpha_2\chi_{33}, \tag{3.37a}$$

$$\rho = \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \chi_{13} + \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \chi_{33}, \tag{3.37b}$$

have been used. From the retarded Green's functions (3.36) the so-called Kubo formulae for the transport coefficients may be deduced, yielding

$$\eta = -\lim_{\omega\to 0}\omega\lim_{k\to 0}\frac{1}{k^2}\,\mathrm{Im}\left[G^{\mathrm{R}}_{\pi^y\pi^y}(\omega,k)\right], \tag{3.38a}$$

$$\sigma_0 = -\lim_{\omega\to 0}\omega\lim_{k\to 0}\frac{1}{k^2}\,\mathrm{Im}\left[G^{\mathrm{R}}_{\rho\rho}(\omega,k)\right], \tag{3.38b}$$

$$\eta+\zeta = -\lim_{\omega\to 0}\omega\lim_{k\to 0}\frac{1}{k^2}\,\mathrm{Im}\left[G^{\mathrm{R}}_{\pi^x\pi^x}(\omega,k)\right]. \tag{3.38c}$$

The Green's functions of the hydrodynamic variables may be expressed in terms of the currents by using the conservation equations and the definition (2.41).

Hydrodynamics is not equipped with the means of determining the values of Green's functions or transport coefficients for specific phenomena. Nevertheless, in holography the definition of the retarded Green's function (2.45) allows for the calculations of numerical values of transport coefficients from bulk data – this also gives access to the hydrodynamic modes via the poles of the retarded Green's functions [116]. Hydrodynamics in the context of holography will be returned to in chapter 4.

3.2 Symmetry Breaking in Hydrodynamics

The hydrodynamic discussion above relies heavily on symmetries and their conserved currents. In reality, however, very few systems satisfy exact symmetries – hydrodynamics is nevertheless not a lost cause. This section will be devoted to the application of hydrodynamics to systems with broken symmetries; the focus will be set on systems which display two forms of symmetry breaking: spontaneous as well as explicit symmetry breaking.

Spontaneous symmetry breaking occurs when the ground state of a system no longer satisfies the symmetry of its theory; the symmetry currents, however, are still conserved. Once a continuous symmetry is spontaneously broken the system gains a new massless degree of freedom – a Goldstone boson – for each broken symmetry [43–45]. Spontaneous symmetry breaking appears in a vast amount of physical systems, ranging from high energy physics to condensed matter, and underlies several hydrodynamic phenomena [117]. Two fluids with properties due to spontaneous symmetry breaking will be considered in this section. Namely, superfluidity can be described by a hydrodynamic theory with a spontaneously broken global $U(1)$ symmetry [67–70], while spontaneously broken translational symmetry gives rise to elastic properties in a fluid [60, 71–73].

A symmetry is explicitly broken when it occurs at the level of the equations which define a theory; in turn the conservation of the current corresponding to the broken symmetry is rendered void. If a small explicit breaking occurs in a background with a spontaneous breaking of the same symmetry the associated Goldstone bosons become massive – they are then referred to as pseudo-Goldstone bosons and the breaking is called pseudo-spontaneous [57]. Pseudo-spontaneous breaking of $U(1)$ symmetry will be considered in the context of a superfluid [3].

3.2.1 Review of Relativistic Superfluid

A relativistic superfluid is taken to consist of two components, a normal fluid component and a superfluid component [65, 66]. The superfluid component arises due to a spontaneously broken $U(1)$ symmetry; the corresponding Goldstone boson $\varphi(t, x)$ is identified with the phase of the scalar condensate and must be accounted for in the hydrodynamic analysis [118]. Much of the review below follows [69, 70].

Constitutive Relations

The Goldstone is shifted $\varphi(x) \mapsto \varphi(x) + \lambda_g(x)$ under local $U(1)$ transformations [69]; hence it is necessary to work with the gauge invariant quantity

$$\xi_\mu(x) \equiv \partial_\mu \varphi(x) - A^{\text{ext}}_\mu(x), \tag{3.39}$$

with external gauge field A^{ext}_μ which couples to the $U(1)$-current and transforms as $A^{\text{ext}}_\mu(x) \mapsto A^{\text{ext}}_\mu(x) + \partial_\mu \lambda_g(x)$. In the following the gauge will be fixed such that $A^{\text{ext}}_\mu = 0$. The dynamics of φ are governed by the Josephson relation, which results from the fact that the Goldstone boson is canonically conjugate to the conserved charge of the spontaneously broken symmetry [67]; for superfluids it reads

$$u^\mu \xi_\mu = -\mu. \tag{3.40}$$

The chemical potential μ is zeroth-order in derivatives and hence $\xi_\mu = \partial_\mu \varphi$ is to be considered zeroth-order in derivatives also.

The Josephson relation may be expanded in a gradient expansion in the same way as the energy-momentum tensor and charge current. Moreover, superfluid-effects will make their presence known in the constitutive relations of the energy-momentum tensor as well as the charge current. The constitutive relations and the expansion of the Josephson relation read [70]

$$\begin{aligned}\langle T^{\mu\nu} \rangle = {} & \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu} + 2\rho_s \mu n^{(\mu} u^{\nu)} + \rho_s \mu n^\mu n^\nu \\ & - \eta \sigma^{\mu\nu} - \eta_s \sigma^{\mu\nu}_s - \zeta_1 \Delta^{\mu\nu} \partial_\lambda u^\lambda - \zeta_2 \Delta^{\mu\nu} \partial_\sigma \left(\rho_s n^\sigma\right) + \mathcal{O}(\partial^2),\end{aligned} \tag{3.41a}$$

$$\langle J^\mu \rangle = \rho_t u^\mu + \rho_s n^\mu - \sigma_0 T \Delta^{\mu\nu} \partial_\nu \left(\frac{\mu}{T}\right) + \mathcal{O}(\partial^2), \tag{3.41b}$$

$$u^\mu \partial_\mu \left(\Delta_\nu{}^\rho \xi_\rho\right) = \Delta_\nu{}^\rho \left[-\partial_\rho \mu + \zeta_3 \partial_\rho \partial_\mu \left(\rho_s n^\mu\right)\right] + \mathcal{O}(\partial^2), \tag{3.41c}$$

where ρ_s is the superfluid charge density; the total charge density is $\rho_t = \rho_n + \rho_s$, with ρ_n the charge density of the normal fluid component; η_s is the superfluid viscosity; ζ_2 and ζ_3 are bulk viscosities of the superfluid component; and n^μ is the relative superfluid velocity,[3] which is related to the $U(1)$ Goldstone boson through

$$\mu n^\mu \equiv \Delta^{\mu\nu} \xi_\nu. \tag{3.42}$$

In accordance with (3.42) the hydrodynamic variable is taken to be the spatial components of ξ_μ. The superfluid shear tensor $\sigma^{\mu\nu}_s$ takes the form of equation (3.17) with the

[3] The relative superfluid velocity is defined in the frame where the normal fluid velocity vanishes. The relative superfluid velocity is taken to be small, $n^\mu \ll 1$.

normal fluid velocity replaced by the superfluid velocity. Note that equation (3.41c) is the spatial derivative of the Josephson relation, such that the equation may be expressed in terms the hydrodynamic variable ξ_i.

For applications to holography and for compactness of the expressions conformal invariance will be assumed in the following – tracelessness of the energy-momentum tensor then requires that ζ_1 and ζ_2 must vanish. The superfluid viscosity is set to zero, $\eta_s = 0$. In addition to ξ_i the hydrodynamic variables are taken to be the temperature T; the momentum density π^i; and the chemical potential μ. The momentum is taken to flow along the x-direction and hence the superfluid dynamics contribute only in the longitudinal sector.

Linear Dynamics

The derivative expansions (3.41) may now be linearised in fluctuations and the hydrodynamic dispersion relations calculated. The Goldstone is linearised as $\varphi \to \bar{\varphi} + \delta\varphi$ but since it contributes to the constitutive relations with a derivative it appears as $\xi_i \equiv \delta\xi_i$.

In the chosen variables the linearised longitudinal conservation equations take the form

$$0 = -i\omega \left[T \left(\frac{\partial s}{\partial T} \right)_\mu + \mu \left(\frac{\partial \rho_t}{\partial T} \right)_\mu \right] \delta T - i\omega \left[T \left(\frac{\partial s}{\partial \mu} \right)_T + \mu \left(\frac{\partial \rho_t}{\partial \mu} \right)_T \right] \delta\mu + ik\delta\pi^x, \tag{3.43a}$$

$$0 = \left(-i\omega + \frac{\eta}{h} \right) \delta\pi^x + isk\,\delta T + i\rho_t k\,\delta\mu - \rho_s \frac{\eta}{h} k^2 \delta\xi^x, \tag{3.43b}$$

$$0 = \left[-i\omega \left(\frac{\partial \rho_t}{\partial T} \right)_\mu - \sigma_0 \frac{\mu}{T} k^2 \right] \delta T + \left[-i\omega \left(\frac{\partial \rho_t}{\partial \mu} \right)_T + \sigma_0 k^2 \right] \delta\mu + i\frac{\rho_n}{h} k\delta\pi^x + i\frac{\rho_s sT}{\mu h} k\delta\xi^x, \tag{3.43c}$$

and the Josephson relation to first order in fluctuations reads

$$\left(-i\omega + \frac{\rho_s \zeta_3 w}{\mu h} k^2 \right) \delta\xi^x + ik\delta\mu - \frac{\zeta_3 \rho_s}{h} k^2 \delta\pi^x = 0, \tag{3.43d}$$

where the total enthalpy density $w = h + \mu\rho_s = \varepsilon + p$ acts as the superfluid momentum susceptibility [69].

A unique property of a superfluid is that the superfluid component alone can carry hydrodynamic modes without the presence of the energy-momentum tensor, in addition to the temperature and normal fluid velocity being held fixed; this regime will be refered to as the probe-limit. The relevant linearised equations are then

$$0 = \left[-i\omega\left(\frac{\partial\rho_t}{\partial\mu}\right)_T + \sigma_0 k^2\right]\delta\mu + i\frac{\rho_s s T}{\mu h}k\delta\xi^x, \tag{3.44a}$$

$$0 = \left(-i\omega + \frac{\rho_s\zeta_3 w}{\mu h}k^2\right)\delta\xi^x + ik\delta\mu. \tag{3.44b}$$

Conformal invariance allows the pressure to be expressed in the scaling form

$$p = T^3\varrho(\mu/T) \tag{3.45}$$

where $\varrho(\mu/T)$ is an arbitrary dimensionless function. Using the above form of the pressure it follows from the equation of state (3.12) that the total charge density ρ_t and entropy s can be expressed as

$$\rho_t = \left(\frac{\partial p}{\partial\mu}\right)_T = T^2\varrho'(\mu/T), \tag{3.46}$$

$$s = \left(\frac{\partial p}{\partial T}\right)_\mu = 3T^2\varrho(\mu/T) - T\mu\varrho'(\mu/T). \tag{3.47}$$

In a further step the entropy per particle,

$$\sigma = s/\rho_t, \tag{3.48}$$

may be introduced [70]. The thermodynamic derivatives $(\partial\rho_t/\partial T)_\mu$, $(\partial\rho_t/\partial\mu)_T$, $(\partial s/\partial T)_\mu$, $(\partial s/\partial\mu)_T$ are then expressed in terms of $(\partial\sigma/\partial T)_\mu$; they read

$$\left(\frac{\partial\rho_t}{\partial T}\right)_\mu = \frac{\partial^2 p}{\partial T\partial\mu} = \left(\frac{\partial s}{\partial\mu}\right)_T = \frac{\rho_t}{w}\left[2s - \rho_t T\left(\frac{\partial\sigma}{\partial T}\right)_\mu\right], \tag{3.49a}$$

$$\left(\frac{\partial\rho_t}{\partial\mu}\right)_T = \frac{\rho_t}{w}\left[2s + \rho_t T^2\left(\frac{\partial\sigma}{\partial T}\right)_\mu\right], \tag{3.49b}$$

$$\left(\frac{\partial s}{\partial T}\right)_\mu = \frac{2(sT - \mu\rho_t)}{T^2} + \frac{\mu^2}{T^2}\frac{\rho_t}{w}\left[2\rho_t + \frac{\rho_t T^2}{\mu}\left(\frac{\partial\sigma}{\partial T}\right)_\mu\right]. \tag{3.49c}$$

Hydrodynamic Modes

Following the procedure outlined in section 3.1.4 the four equations (3.43) yield four hydrodynamic modes: two pairs of propagating modes, respectively called first and second sound, with dispersion relations

$$\omega(k) = \pm v_j k - \frac{i}{2}\Gamma_j\, k^2, \qquad j = 1, 2. \tag{3.50}$$

The speed and attenuation of first sound are

$$v_1^2 = \frac{1}{2}, \qquad \Gamma_1 = \frac{\eta}{w}, \tag{3.51}$$

where the speed of sound is constrained by conformality. For second sound the same coefficients are

$$v_2^2 = \frac{\rho_s \sigma^2}{(\partial\sigma/\partial T)_\mu w}, \qquad \Gamma_2 = \sigma_0 \frac{\mu w}{(\partial\sigma/\partial T)_\mu T^2 \rho_t^2} + \eta \frac{\mu\rho_s}{wh} + \zeta_3 \frac{\rho_s w}{\mu h}. \tag{3.52}$$

The reduced conservation equations (3.44) give rise to a pair of propagating sound modes, called fourth sound, with speed and attenuation of sound given by

$$v_4^2 = \frac{\rho_s}{\mu(\partial\rho_t/\partial\mu)}, \qquad \Gamma_4 = \frac{\sigma_0}{(\partial\rho_t/\partial\mu)} + \zeta_3 \frac{\rho_s}{\mu}. \tag{3.53}$$

Kubo Formulae and AC Conductivity

Kubo formulae are required for the new transport coefficients. In order to implement the canonical approach as in equation (3.34) it is more convenient to work with energy density ε and charge density ρ_t as variables, instead of the temperature T and chemical potential μ. The susceptibility matrix is then given by [119]

$$\begin{pmatrix} \delta\varepsilon \\ \delta\pi_x \\ \delta\rho_t \\ \delta\xi_x \end{pmatrix} = \begin{pmatrix} T\left(\frac{\partial\epsilon}{\partial T}\right)_{\mu/T} & 0 & \left(\frac{\partial\epsilon}{\partial\mu}\right)_T & 0 \\ 0 & w & 0 & \mu \\ T\left(\frac{\partial\rho_t}{\partial T}\right)_{\mu/T} & 0 & \left(\frac{\partial\rho_t}{\partial\mu}\right)_T & 0 \\ 0 & \mu & 0 & \frac{\mu}{\rho_s} \end{pmatrix} \begin{pmatrix} \delta T/T \\ \delta u_x \\ \delta\mu - \frac{\mu}{T}\delta T \\ \rho_s \delta n_x \end{pmatrix}. \tag{3.54}$$

In these variables the linearised conservation equations and Josephson relation read

$$0 = -i\omega\delta\varepsilon + ik\delta\pi^x, \tag{3.55}$$

$$0 = -i\omega\delta\pi^x + ik\beta_1\delta\varepsilon + ik\beta_2\delta\rho_t + \frac{\eta}{h}k^2\delta\pi^x - \frac{\eta}{h}\rho_s k^2\delta\xi^x, \tag{3.56}$$

$$0 = -i\omega\delta\rho_t + ik\frac{\rho_n}{h}\delta\pi^x + ik\frac{\rho_s sT}{\mu h}\delta\xi^x + k^2\frac{\kappa}{T}\left(\alpha_1\delta\varepsilon + \alpha_2\delta\rho_t\right), \tag{3.57}$$

$$0 = -i\omega\delta\xi^x + ik\gamma_1\delta\varepsilon + ik\gamma_2\delta\rho_t + k^2\frac{\zeta_3\rho_s}{h}\delta\pi^x - k^2\frac{h+\mu\rho_s}{h\mu}\zeta_3\rho_s\delta\xi^x, \tag{3.58}$$

where α_1, α_2 are given by (3.26) with the substitution $\rho \to \rho_t$ and

$$\beta_1 = \left(\frac{\partial p}{\partial \varepsilon}\right)_{\rho_t}, \qquad \beta_2 = \left(\frac{\partial p}{\partial \rho_t}\right)_{\varepsilon}, \tag{3.59}$$

$$\gamma_1 = \left(\frac{\partial \mu}{\partial \varepsilon}\right)_{\rho_t}, \qquad \gamma_2 = \left(\frac{\partial \mu}{\partial \rho_t}\right)_{\varepsilon}. \tag{3.60}$$

Implementing the canonical Green's function computation (3.34) results in the new Kubo formula

$$\zeta_3 = -\lim_{\omega\to 0}\omega \lim_{k\to 0}\frac{1}{k^2}\,\mathrm{Im}\left[G^{\mathrm{R}}_{\xi_x\xi_x}(\omega,k)\right] = \lim_{\omega\to 0}\omega\lim_{k\to 0}\,\mathrm{Im}\left[G^{\mathrm{R}}_{\varphi\varphi}(\omega,k)\right], \tag{3.61}$$

where the second equality follows from the Fourier-space relation $\delta\xi_x = ik\delta\varphi$ together with the definition of the retarded Green's function (2.41). The other transport coefficients are given by (3.38).

Finally, the retarded Green's functions allow for the definition of the frequency dependent AC conductivity $\sigma(\omega)$ as [15]

$$\sigma(\omega) \equiv \lim_{k\to 0}\frac{i}{\omega}\left[G^{\mathrm{R}}_{J^xJ^x}(\omega,k) - G^{\mathrm{R}}_{J^xJ^x}(0,k)\right]. \tag{3.62}$$

For superfluids as presented in this sections $G^{\mathrm{R}}_{J^xJ^x}(0,k) = 0$; thus

$$\sigma(\omega) = \sigma_0 + \frac{i}{\omega}\left[\frac{\rho_n^2}{h} + \frac{\rho_s}{\mu}\right]. \tag{3.63}$$

The imaginary part of the above AC conductivity has a $1/\omega$-pole at $\omega = 0$; according to the Kramer-Kronig relation the real part of the AC conductivity must therefore also contain a delta-distribution valued term – which is implicit in hydrodynamics – representing an infinite DC conductivity [71, 120]. This discussion also holds for normal fluids, i.e. when $\rho_s = 0$; there the infinite DC conductivity is a consequence of translational invariance (momentum conservation) and may be relaxed by breaking this symmetry explicitly [60, 60, 121, 122] – this does not hold for superfluids.

3.2.2 Broken Superfluid

In this section the well-known relativistic superfluid of the previous section will be presented in a novel setting, namely by considering the effects of an additional explicit breaking of $U(1)$ symmetry [3]. The main way in which an explicitly broken symmetry affects a hydrodynamic setup is by turning the conservation equation of the affected symmetry current into a non-conservation equation. In accordance with the discussion at the beginning of this section – and using the intuition gained from similar considerations for translational symmetry [60, 122] – upon the addition of explicit symmetry breaking the non-conservation equation for the $U(1)$-current reads

$$\partial_\mu \langle J^\mu \rangle = \Gamma\, u_\mu \langle J^\mu \rangle + m\varphi, \tag{3.64}$$

where Γ is the charge relaxation rate and m is related to the mass of the pseudo-Goldstone, which is still denoted as φ [3]. The term proportional to the charge relaxation contributes at equilibrium; this subtlety may be addressed by either assuming that the relaxation time is so long that such contributions may be ignored, or by only considering this term for fluctuations of the charge current.[4] In order for the fluctuations of the charge current to be long-lived Γ and m are assumed to be sufficiently small. When introducing an explicit breaking all the hydrodynamic fields become dependent on the breaking parameter, in addition to T and μ.

When explicitly breaking the $U(1)$-symmetry the traceless nature of the energy-momentum tensor is violated unless the explicit breaking is done by a marginal deformation. For simplicity and compactness of the expressions this scenario will be assumed.

In addition to the non-conservation equation (3.64) the Josephson relation may be altered to take the form

$$\left(-i\omega + \Omega + \frac{\rho_s \zeta_3 w}{\mu h} k^2\right) \delta\xi^x + ik\delta\mu - \frac{\zeta_3 \rho_s}{h} k^2 \delta\pi^x = 0, \tag{3.65}$$

where Ω is the phase relaxation [59, 60, 122]. Solving the relaxed Josephson relation (without the spatial derivative) for φ yields an exponential damping due to Ω. In principle phase relaxation may appear independently of explicit symmetry breaking.

Dispersion Relations

The equations (3.64) and (3.65) may be used to calculate the dispersion relations in complete analogy to the previous section – considering the charge relaxation only for fluctuations allows the terms on the right-hand side of (3.64) to be linearised simply. The analysis results in a pair of propagating sound modes with dispersion relation

$$\omega_0(k) = \pm v_0 k - \frac{i}{2}\Gamma_0 k^2, \tag{3.66}$$

where

$$v_0^2 = \frac{1}{2}, \qquad \Gamma_0 = \frac{\eta}{h}, \tag{3.67}$$

[4] The second option may be realised by considering a driven system.

which is remarkably similar to the normal fluid results (3.33) with conformal speed of sound $(\partial p/\partial\varepsilon)_\rho = 1/2$ and $\zeta_1 = 0$. Moreover, the spectrum contains a pair of gapped modes with dispersion relation

$$\omega(k) = \alpha_\pm - iD_\pm k^2, \tag{3.68}$$

where the gap takes the form

$$\alpha_\pm = -\frac{i}{2}(\Gamma + \Omega) \pm \sqrt{\frac{ms\mu}{(\partial\sigma/\partial T)_\mu T\rho_t^2} - \frac{(\Gamma-\Omega)^2}{4}}. \tag{3.69}$$

The expressions for the constants $D_\pm$ are too unwieldy to display in full here; for the purpose of illustration, setting $\Gamma = \Omega = 0$ gives

$$D_\pm = \sigma_0 \frac{\mu w}{2(\partial\sigma/\partial T)_\mu T^2\rho_t^2} + \zeta_3 \frac{\rho_s w}{2\mu h} \pm \frac{i}{2}\frac{s^{3/2}\sqrt{T}\rho_s}{h\rho_t\sqrt{(\partial\sigma/\partial T)_\mu m\mu}}, \tag{3.70}$$

where the imaginary term will contribute a k^2-term to the real part of the dispersion relation. The limit of zero explicit breaking and phase relaxation must be taken at the level of the conservation equations and Josephson relation, respectively, before calculating the dispersion relations.

The results (3.67) and (3.70) lend themselves to an interesting observation, namely that the sum of the coefficients of the k^2-terms of the dispersion relations is the same in the broken paradigm as in the non-broken state, i.e.[5]

$$D_+ + D_- + \Gamma_0 = \Gamma_1 + \Gamma_2. \tag{3.71}$$

The above sum rule also holds when the charge and phase relaxation are non-zero.

The broken superfluid can still support a mode in the probe limit. When the $U(1)$-symmetry is pseudo-spontaneously broken this mode has the dispersion relation

$$\omega(k) = \alpha_\pm^{\mathrm{p}} - iD_\pm^{\mathrm{p}} k^2, \tag{3.72}$$

where

$$\alpha_\pm^{\mathrm{p}} = -\frac{i}{2}(\Gamma + \Omega) \pm \sqrt{\frac{m}{(\partial\rho_t/\partial\mu)} - \frac{(\Gamma-\Omega)^2}{4}}, \tag{3.73}$$

[5]There is a slight artefact of convention here. Propagating modes exclude a factor 1/2 in the definition of Γ_i. More strictly. the sum involves the full k^2-coefficients for each of the pair of propagating modes, which is equivalent to summing the attenuation constants as they have been defined previously. With this in mind, the speeds of sound sum to zero in both the non-broken and broken case.

and

$$D_\pm^{\rm p} = \frac{1}{2}\left(\frac{\sigma_0}{(\partial\rho_t/\partial\mu)} + \zeta_3\frac{\rho_s}{\mu}\right) \pm \frac{i}{2}\frac{\zeta_3\rho_s(\partial\rho_t/\partial\mu)(\Gamma-\Omega) + 2\rho_s - \sigma_0\mu(\Gamma-\Omega)}{\mu\sqrt{4m(\partial\rho_t/\partial\mu) - (\partial\rho_t/\partial\mu)^2(\Gamma-\Omega)^2}}. \qquad (3.74)$$

The sum of the k^2-coefficients still agree in both the broken and unbroken case,

$$D_+^{\rm p} + D_-^{\rm p} = \Gamma_4. \qquad (3.75)$$

The superscript p will be dropped in section 4.2.

Probe-limit Green's Functions

The attention will now be directed towards the retarded Green's functions, which may be computed using the canonical approach (3.34). For simplicity only the probe limit dynamics will be considered, and the charge relaxation rate Γ will be set to zero.[6] Using the same set of variables as in section 3.2.1 the susceptibility matrix is given by

$$\chi = \begin{pmatrix} \chi_{\rho\rho} & 0 \\ 0 & \tilde{\chi}_{\xi\xi} \end{pmatrix}, \qquad (3.76)$$

where $\chi_{\rho\rho} = \partial\rho_t/\partial\mu$ and $\tilde{\chi}_{\xi\xi}$ is the susceptibility of the pseudo-Goldstone boson. Due to the explicit symmetry breaking, a general k^2-dependent parameterisation $\tilde{\chi}_{\xi\xi} = \chi_{\xi\xi} f(k)$ will be allowed, where $\chi_{\xi\xi}$ is a positive constant. The form of $\tilde{\chi}_{\xi\xi}$ is determined by considering the retarded Green's functions[7]

$$G^{\rm R}_{\varphi J^t}(\omega,k) = \frac{i\mu\chi_{\rho\rho}\omega}{\mathfrak{p}(\omega,k)}, \qquad (3.77a)$$

$$G^{\rm R}_{J^t\varphi}(\omega,k) = \frac{i\omega(m\mu + k^2\rho_s)\chi_{\xi\xi}\chi_{\rho\rho} f(k)}{\mathfrak{p}(\omega,k)}, \qquad (3.77b)$$

where $\mathfrak{p}(\omega,k)$ denotes the polynomial

$$\mathfrak{p}(\omega,k) = m\mu - \mu\chi_{\rho\rho}\omega(\omega + i\Omega) + k^2\left(\rho_s - i\zeta_3\rho_s\chi_{\rho\rho}\omega + \mu\sigma_0(\Omega - i\omega)\right) + k^4\zeta_3\rho_s\sigma_0. \qquad (3.78)$$

The symmetry requirement $G^{\rm R}_{\varphi J^t}(\omega,k) = G^{\rm R}_{J^t\varphi}(\omega,k)$ fixes $f(k)$ such that the pseudo-Goldstone susceptibility is given by

$$\tilde{\chi}_{\xi\xi} = \frac{k^2\chi_{\xi\xi}}{k^2 + \mathfrak{m}^2}, \qquad (3.79)$$

[6] This limit is suitable for the holographic model in section 4.2.1.
[7] The subscripts of the retarded Green's functions will use $J^t = \rho_t$ for notational clarity.

which constitutes a Gell-Mann-Oakes-Renner relation [58,123] with $\mathfrak{m}^2$ being the mass of the pseudo-Golstone boson given by

$$\mathfrak{m}^2 = \chi_{\xi\xi}\, m\,, \qquad \chi_{\xi\xi} = \frac{\mu}{\rho_s}. \tag{3.80}$$

The zero-frequency Green's function may be deduced from the limiting behaviour (3.35), i.e.

$$G_{\varphi\varphi}(0,k) = \frac{\chi_{\xi\xi}}{k^2 + \mathfrak{m}^2}. \tag{3.81}$$

Proceeding, the retarded Green's functions at $k = 0$ are given by

$$G^{\mathrm{R}}_{J^tJ^t}(\omega,0) = \frac{m\mu\chi_{\rho\rho}}{m\mu - \mu\chi_{\rho\rho}\omega(\omega + i\Omega)}\,, \tag{3.82a}$$

$$G^{\mathrm{R}}_{\varphi J^t}(\omega,0) = \frac{i\mu\chi_{\rho\rho}\omega}{m\mu - \mu\chi_{\rho\rho}\omega(\omega + i\Omega)}\,, \tag{3.82b}$$

and also

$$\lim_{k\to 0}\frac{1}{k}G^{\mathrm{R}}_{J^xJ^t}(\omega,k) = -\frac{\chi_{\rho\rho}\omega\left(\rho_s + \mu\sigma_0(\Omega - i\omega)\right)}{m\mu - \mu\chi_{\rho\rho}\omega(\omega + i\Omega)}, \tag{3.82c}$$

$$\lim_{k\to 0}\frac{1}{k}G^{\mathrm{R}}_{J^x\varphi}(\omega,k) = \frac{i\rho_s + \mu\sigma_0\omega + \rho_s\chi_{\rho\rho}\omega\Omega/m}{m\mu - \mu\chi_{\rho\rho}\omega(\omega + i\Omega)}. \tag{3.82d}$$

Furthermore, the spatial current-current correlator yields

$$G^{\mathrm{R}}_{J^xJ^x}(\omega,0) = \frac{\rho_s}{\mu} - i\,\sigma_0\,\omega\,, \tag{3.83a}$$

$$G^{\mathrm{R}}_{J^xJ^x}(0,k) = \frac{m\rho_s}{m\mu + k^2\rho_s}, \tag{3.83b}$$

from which the low-frequency AC conductivity follows,

$$\sigma(\omega) \equiv \lim_{k\to 0}\frac{i}{\omega}\left[G^{\mathrm{R}}_{J^xJ^x}(\omega,k) - G^{\mathrm{R}}_{J^xJ^x}(0,k)\right] = \sigma_0. \tag{3.84}$$

Neither the phase relaxation Ω or the parameter m contribute to the above AC conductivity; nevertheless, this result differs from the AC conductivity of a superfluid (3.63) – there is no longer a pole at $\omega = 0$ in the imaginary part, hence the DC conductivity is finite. The way the DC conductivity is rendered finite in (3.84) differs from that of [59], where phase relaxation appears without explicit symmetry breaking.

Expanding some of the retarded Green's functions (3.82) in ω yields

$$G^{\mathrm{R}}_{J^tJ^t}(\omega,0) = -\chi_{\rho\rho} - \frac{i\omega}{m}\chi^2_{\rho\rho}\Omega + \mathcal{O}(\omega^2), \tag{3.85a}$$

$$G^{\mathrm{R}}_{\varphi J^t}(\omega,0) = \frac{i\omega}{m}\chi_{\rho\rho} - \frac{\omega^2}{m^2}\chi^2_{\rho\rho}\Omega + \mathcal{O}(\omega^3); \tag{3.85b}$$

from which the following Kubo formula may be extracted

$$\Omega = \lim_{\omega\to 0}\frac{m}{\omega\chi^2_{\rho\rho}}\mathrm{Im}\left[G^{\mathrm{R}}_{J^tJ^t}(\omega,0)\right], \tag{3.86}$$

or similarly using $G^{\mathrm{R}}_{\varphi J^t}(\omega,0)$. The result (3.83a) provides the Kubo formula for σ_0 as

$$\sigma_0 = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\left[G^{\mathrm{R}}_{J^xJ^x}(\omega,0)\right]. \tag{3.87}$$

Finally, using the transport coefficients above, the superfluid bulk viscosity ζ_3 may be determined from

$$\lim_{\omega\to 0}\frac{1}{\omega}\lim_{k\to 0}\frac{\partial^2}{\partial k^2}\mathrm{Im}\left[G^{\mathrm{R}}_{J^tJ^t}(\omega,k)\right] = -2\frac{\chi^2_{\rho\rho}}{m^2\mu}\left(\zeta_3 m\rho_s - \rho_s\Omega - \mu\sigma_0\Omega^2\right). \tag{3.88}$$

The formula for ζ_3 above is not the same as the simple superfluid Kubo formula (3.61), which no longer holds.

3.2.3 Viscoelasticity

Spontaneous symmetry breaking of translational invariance gives rise to elastic behaviour; in hydrodynamics, paired with viscosity, such a system is called viscoelastic. This section will be concerned with charge-neutral viscoelastic hydrodynamics. The general principles of spontaneous symmetry breaking in hydrodynamics carry over from the superfluid example in section 3.2.1 and will hence not be repeated in detail.

The spontaneous symmetry breaking is accompanied by one Goldstone boson $\Phi^i(t,x)$ for each direction in which the translations are broken. Of interest here is a theory where the translations are spontaneously broken in both spatial directions.[8] The Goldstone boson of broken translations is referred to as the phonon – in the current analysis the two phonons have the equilibrium value $\Phi^i = \alpha x^i$, where $\alpha(T)$ can be viewed as a parameter controlling the symmetry breaking [2, 73].

A distinct aspect of the viscoelastic hydrodynamics discussed in this section is the

[8] Goldstone bosons associated to the spontaneous breaking of rotations may be gapped out of the hydrodynamic spectrum when translational invariance is spontaneously broken in at least one direction.

presence of the so-called strain pressure $\mathcal{P}(T,\alpha)$, which is defined by

$$\mathcal{P} = \langle T^{xx} \rangle - p, \tag{3.89}$$

and is hence a measure of the difference between the mechanical pressure $\langle T^{xx} \rangle$ and thermodynamic pressure p. A non-zero strain pressure signifies that a system does not minimise the free energy, i.e. that it is an excited state which is being sustained by an external strain. The hydrodynamic formalisation of strain pressure and its effects on the constitutive relations were presented in [73]. Further insights for configurations with vanishing strain pressure at equilibrium were discussed in [2].

Constitutive Relations

The first point to consider is the constitutive relations; for detailed derivations of the quantities involved the reader is referred to [73]. The main building block is the spacetime derivative of the phonons,

$$e^i_\mu = \partial_\mu \Phi^i, \qquad i = 1,2 \tag{3.90}$$

which is to be considered as zeroth-order in the derivative expansions. From the above quantity one may define the metric-like object

$$h^{ij} = e^i_\mu e^{j\mu}, \tag{3.91}$$

which acts as $e_{i\mu} = h^{-1}_{ij} e^j_\mu$ and $h_{\mu\nu} = h^{-1}_{ij} e^i_\mu e^j_\nu$. The strain tensor is defined in terms of the metric (3.91) as

$$u_{\mu\nu} = \frac{1}{2}\left(h^{-1}_{ij} - \delta_{ij}/\alpha^2\right) e^i_\mu e^j_\nu. \tag{3.92}$$

The constitutive relation of the energy-momentum tensor reads

$$\begin{aligned}\langle T^{\mu\nu} \rangle = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu} + \left(\mathcal{P} + T\mathcal{P}'\right) u^\lambda{}_\lambda \Delta^{\mu\nu} + \mathcal{P} h^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta_1 \Delta^{\mu\nu}\partial_\rho u^\rho \\ - 2G u^{\mu\nu} - (B-G) u^\lambda{}_\lambda h^{\mu\nu} + \mathcal{O}(\partial^2),\end{aligned} \tag{3.93}$$

where B and G are the bulk and shear elastic moduli respectively, and $\chi_{\pi\pi} = \varepsilon + p + \mathcal{P}$ is the momentum susceptibility. Prime denotes partial derivative with respect to the temperature T for fixed α; this means that there is a point to be made regarding the role of $\mathcal{P}'$: If the equilibrium configuration for some $\alpha_*(T)$ is such that $\mathcal{P}(T,\alpha_*) = 0$ then the total derivative should vanish,

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}T} = \left(\frac{\partial \mathcal{P}}{\partial T}\right)_\alpha + \frac{\mathrm{d}\alpha}{\mathrm{d}T}\frac{\partial \mathcal{P}}{\partial \alpha}\bigg|_{\alpha=\alpha_*} = 0, \tag{3.94}$$

which means that $\mathcal{P}' = (\partial\mathcal{P}/\partial T)_\alpha$ may be non-zero [2]. Hence $\mathcal{P}'$ should in principle be considered even for systems which are unstrained at equilibrium – in fact $\mathcal{P}'$ may be accounted for as a thermodynamic susceptibility related to thermal expansion [2, 124].

Similarly to the superfluid, the dynamics of the Goldstones are constrained by a Josephson relation. For the two phonons the Josephson relations read

$$u^\mu e^i_\mu = \frac{h^{ij}}{\gamma}\partial_\mu\left(\mathcal{P}e^\mu_j - (B-G)u^\lambda{}_\lambda e^\mu_j - 2Gu^{\mu\nu}e_{j\nu}\right) + \mathcal{O}(\partial^2), \tag{3.95}$$

where γ is a dissipative coefficient which is characteristic of spontaneously broken translations.

Linear components

With the energy-momentum tensor and Josephson relation at hand the dispersion relations may be calculated upon linearisation of the constitutive relations. The hydrodynamic variables are the temperature T, spatial fluid velocity u^i, and the spatial derivative of the phonon fluctuations. The relevant object containing the phonons up to linear order in fluctuations is

$$e^i_\mu = \partial_\mu\Phi^i = \alpha\delta^i_\mu + \partial_\mu\delta\Phi^i; \tag{3.96}$$

for simplicity α will be set to one. Using the above expression, together with the fluctuating forms of T and u^i in (3.19), the components of the energy-momentum tensor (3.93) up to linear order in fluctuations read

$$\delta T^{tt} = \varepsilon'\delta T + (T\mathcal{P}' - \mathcal{P})\partial_j\delta\Phi^j, \tag{3.97a}$$

$$\delta T^{ti} = \chi_{\pi\pi}u^i - \frac{s\mathcal{P}(T\mathcal{P}'-\mathcal{P})}{\gamma\chi_{\pi\pi}}\partial^i\delta T + \frac{sT\mathcal{P}}{\gamma\chi_{\pi\pi}}\left[2G\partial_j\partial^{(i}\Phi^{j)} - (\mathcal{P}-B+G)\,\partial^i\partial_j\delta\Phi^j\right], \tag{3.97b}$$

$$\delta T^{ij} = \left(p' + \mathcal{P}'\right)\delta^{ij}\delta T - \eta\sigma^{ij} - \zeta_1\delta^{ij}\partial_k u^k + (\mathcal{P}-B+G)\,\delta^{ij}\partial_j\delta\Phi^j - 2G\partial^{(i}\delta\Phi^{j)}. \tag{3.97c}$$

Moreover, the Josephson relation reads

$$\partial_t\delta\Phi^i = u^i - \frac{s\,(T\mathcal{P}'-\mathcal{P})}{\gamma\chi_{\pi\pi}}\partial^i\delta T + \frac{sT}{\gamma\chi_{\pi\pi}}\left[2G\partial_j\partial^{(i}\Phi^{j)} - (\mathcal{P}-B+G)\,\partial^i\partial_j\delta\Phi^j\right]. \tag{3.98}$$

With the information above the hydrodynamic modes may be found by following the procedure in section 3.1.4; the components (3.97) are readily plugged into the energy-momentum conservation equations, while a spatial derivative must be applied to the Josephson relation (3.98) in order to express it in terms of the spatial derivative of the phonon fluctuation.

Hydrodynamic Modes and Kubo Formulae

The equations again split into sets transverse and longitudinal to the momentum with the two phonons contributing to their respective sector. The bulk elastic modulus allows for a propagating mode in the transverse sector with dispersion relation

$$\omega(k) = \pm v_\perp k - \frac{i}{2}\Gamma_\perp k^2, \tag{3.99}$$

where

$$v_\perp^2 = \frac{G}{\chi_{\pi\pi}}, \qquad \Gamma_\perp = \frac{\eta}{\chi_{\pi\pi}} + \frac{G}{\gamma}\frac{s^2T^2}{\chi_{\pi\pi}^2}. \tag{3.100}$$

The longitudinal sector contains one propagating and one diffusive mode with dispersion relations

$$\omega(k) = \pm v_\parallel k - \frac{i}{2}\Gamma_\parallel k^2, \qquad \text{and} \qquad \omega(k) = -iD_\parallel k^2, \tag{3.101}$$

where the coefficients are given by

$$v_\parallel^2 = \frac{(s+\mathcal{P}')^2}{s'\chi_{\pi\pi}} + \frac{B+G-\mathcal{P}}{\chi_{\pi\pi}}, \tag{3.102a}$$

$$\Gamma_\parallel = \frac{\eta+\zeta_1}{\chi_{\pi\pi}} + \frac{T^2s^2v_\parallel^2}{\sigma\chi_{\pi\pi}}\left(1 - \frac{s+\mathcal{P}'}{Ts'v_\parallel^2}\right)^2, \tag{3.102b}$$

$$D_\parallel = \frac{s^2}{\gamma s'}\frac{B+G-\mathcal{P}}{\chi_{\pi\pi}v_\parallel^2}. \tag{3.102c}$$

For applications to holography conformal invariance enforces a traceless energy-momentum tensor; this leads to the constraints

$$\varepsilon = 2(p+\mathcal{P}), \qquad T\mathcal{P}' = 3\mathcal{P} - 2B, \qquad \zeta_1 = 0. \tag{3.103}$$

Moreover, in the conformal setting the speeds of sound of the transverse and longitudinal sectors are related by

$$v_\parallel^2 = 1/2 + v_\perp^2. \tag{3.104}$$

The scale invariance also affects the attenuation and diffusion constants in the longitudinal sector, which become

$$\Gamma_\parallel = \frac{\eta}{\chi_{\pi\pi}} + \frac{s^2T^2G^2}{\gamma\chi_{\pi\pi}^3 v_\parallel^2}, \qquad D_\parallel = \frac{s^2T(B+G-\mathcal{P})}{\gamma(s+\mathcal{P}')(\chi_{\pi\pi}+2G)}, \tag{3.105}$$

where it is evident that strain pressure only contributes to the diffusion constant of the diffusive mode.

Expressions for the modes in configurations where the strain pressure vanishes at equilibrium follow directly from the expressions (3.102) and (3.105) by setting $\mathcal{P}$ – but not $\mathcal{P}'$ – to zero [2], and using $\chi_{\pi\pi} = \varepsilon + p$. Furthermore, a non-zero value of $\mathcal{P}'$ also allows for a non-zero bulk modulus in scale invariant theories [2], which follows from the second equation in (3.103).

Finally, some relevant Kubo formulae are

$$G = \lim_{\omega\to 0} \omega^2 \lim_{k\to 0} \frac{1}{k^2} \mathrm{Re}\, G^{\mathrm{R}}_{\pi^y\pi^y}, \tag{3.106a}$$

$$\frac{(\epsilon+p)^2}{\gamma\chi^2_{\pi\pi}} = \lim_{\omega\to 0} \lim_{k\to 0} \omega\, \mathrm{Im}\, G^{\mathrm{R}}_{\Phi^x\Phi^x}, \tag{3.106b}$$

$$\chi_{\pi\pi} v_\parallel^2 = \lim_{\omega\to 0} \omega^2 \lim_{k\to 0} \frac{1}{k^2} \mathrm{Re}\, G^{\mathrm{R}}_{\pi^x\pi^x}, \tag{3.106c}$$

where (3.106c) can be used to indirectly obtain the bulk modulus B.

4 Symmetry Breaking and Hydrodynamics in Holography

Testing theoretical understandings of systems and phenomena requires the engineering of models which posses the properties that one wishes to probe. In the context of applied holography this is commonly achieved by the means of bottom-up model building; in this approach a gravitational theory is constructed with the purpose of generating some desired properties of the assumed dual conformal field theory, which is then studied via the holographic dictionary.

Within the scope of this thesis a major motivation for bottom-up holography lies in confirming the hydrodynamic analysis of chapter 3. In particular, the circumstances where strain pressure appears, and the independent nature of its temperature derivative, cannot be confirmed by the hydrodynamic analysis itself. Moreover, the validity of – or the regime of applicability for – hydrodynamics with explicit symmetry breaking is not self-evident. A holographic perspective is also advantageous when exploring universal properties. Furthermore, successful bottom-up holography provides practical evidence in favour of the AdS/CFT duality, which is ultimately still a conjecture.

In this chapter the two settings introduced in chapter 3 will be reproduced by holographic models: one model which gives rise to spontaneous breaking of translational invariance in the dual conformal field theory, and two models where the boundary theory displays pseudo-spontaneously broken $U(1)$ symmetry. The former is referred to as a massive gravity model [86–88], while the two latter models are modifications of a holographic superfluid [90].[1] The numerical investigation of the hydrodynamic behaviour and other aspects of the respective dual conformal field theories is done using pseudo-spectral methods [130, 131].

Note that the analysis which takes place in this chapter does not constitute a fluid/gravity approach [35]. Simply put, the fluid/gravity correspondence is used to find the constitutive relations through a gradient expansion of dual quantities, which are constrained by the Einstein equations.

[1]Translational symmetry breaking has also been implemented holographically using other models, for example [125–128]. Pseudo-spontaneous breaking of $U(1)$ symmetry was considered in another holographic setting in [129].

4.1 Holographic Massive Gravity

Translations may be broken by considering a spatially modulated scalar field; if this is done in a gravitational theory the metric fluctuations gain an effective mass [132]. In the holographic context an effective theory for this scenario is given by the $(3+1)$-dimensional asymptotically anti-de Sitter massive gravity model defined by the bulk action [86–88]

$$\mathcal{S}_{\mathrm{MG}} = m_{\mathrm{P}}^2 \int \mathrm{d}^3x\,\mathrm{d}u\,\sqrt{-g}\left[\frac{R}{2} + \frac{3}{\ell^2} - m^2 V(\mathcal{I}^{IJ})\right], \tag{4.1}$$

where u is the radial coordinate; R is the Ricci scalar; m_{P} is the Planck mass; ℓ is the AdS-radius which will, without loss of generality, be set to one in the following; m is a parameter related to the graviton mass; and the scalar fields ϕ^I with $I = 1, 2$ enter via the kinetic matrix

$$\mathcal{I}^{IJ} = g^{mn}\partial_m\phi^I\partial_n\phi^J. \tag{4.2}$$

Lowercase Latin indices run over the full set of spacetime coordinates. Moreover, $8\pi\mathrm{G_N}$ has been set to one. For isotropic breaking of translational invariance the potential in (4.1) may be chosen as $V(\mathcal{I}^{IJ}) = V(X, Z)$, with

$$X = \frac{1}{2}\mathrm{tr}\left[\mathcal{I}^{IJ}\right] \quad \text{and} \quad Z = \det\left[\mathcal{I}^{IJ}\right]; \tag{4.3}$$

see for example [133]. Most emphasis in this section will be placed on models with potentials depending on X.

The equations of motion for the scalar field resulting from (4.1) admit a radially constant, time-independent solution of the form

$$\phi^I(\mathbf{x}) = \alpha x^I, \tag{4.4}$$

where $\mathbf{x}$ denotes the spatial boundary coordinates and α is a constant which is related to m through field redefinitions. The scalar fields ϕ^I break the translational invariance of the boundary theory; the nature of the breaking depends on the boundary conditions as well as the form of the potential $V(X, Z)$ – these points will be returned to in more detail below.

The black brane geometry satisfying the Einstein equations of the bulk spacetime is captured by the line element in Eddington-Finkelstein coordinates,

$$\mathrm{d}s^2 = \frac{1}{u^2}\left[-f(u)\,\mathrm{d}t^2 - 2\,\mathrm{d}t\,\mathrm{d}u + \mathrm{d}x^2 + \mathrm{d}y^2\right], \tag{4.5}$$

where the radial coordinate spans $u \in [0, u_h]$. The choice of coordinate system is motivated by the numerical methods which will be used in the following. The conformal boundary is located at $u = 0$ while the black brane horizon is defined by $f(u_h) = 0$, where the emblackening factor takes the form

$$f(u) = 1 - \frac{u^3}{u_h^3} - u^3 \int_u^{u_h} \frac{m^2}{\mathfrak{h}^4} V(\alpha^2\mathfrak{h}^2, \alpha^4\mathfrak{h}^4)\, \mathrm{d}\mathfrak{h}, \tag{4.6}$$

with integration variable $\mathfrak{h}$.

As discussed in section 2.3.1 of chapter 2, the presence of a black brane gives rise to a finite temperature in the bulk and boundary theories and hence thermodynamic quantities may be defined. The ensamble of choice is the grand canonical ensamble. The temperature T of the boundary conformal field theory is identified with the Hawking temperature (2.35) of the bulk theory, while the entropy density s is proportional the black brane horizon area as in (2.38). For the model (4.1) these quantites are given by

$$T = \frac{3 - m^2 V_h}{4\pi u_h}, \qquad s = \frac{2\pi}{u_h}, \tag{4.7}$$

with the definition $V_h = V(\alpha^2 u_h^2, \alpha^4 u_h^4)$. In accordance with (2.29) and (2.22) the free energy density f_ε of the dual conformal field theory follows from the renormalised Euclidean on-shell bulk action in the bulk; at zero charge the discussion below equation (3.10) yields $f_\varepsilon = -p$ and hence the thermodynamic pressure for models with potentials (4.3) is given by

$$p = \frac{1}{2u_h^3} - \frac{m^2}{u_h^3}\left(\frac{1}{2}V_h - U_h\right), \tag{4.8}$$

with the definition

$$U_h = -u^3 \int_0^{u_h} \mathfrak{h}^{-4} V(\alpha^2\mathfrak{h}^2, \alpha^4\mathfrak{h}^4)\, \mathrm{d}\mathfrak{h}, \tag{4.9}$$

where $\mathfrak{h}$ is an integration variable. The above integral is finite for potentials $V(X, Z)$ which fall off faster than u^3 and the conformal boundary [2].[2]

The expectation value of the energy-momentum tensor is given by (2.24), which yields the energy density formula

$$\varepsilon = \langle T^{tt} \rangle = \frac{1}{u_h^3} - \frac{m^2}{u_h^3} U_h. \tag{4.10}$$

Finally, a particularly notable bulk formula is that for the strain pressure (3.89) which takes the form [2]

$$\mathcal{P} = \frac{m^2}{u_h^3}\left(\frac{1}{2}V_h - \frac{3}{2}U_h\right). \tag{4.11}$$

It is evident from the above equation that strain pressure may be present in the models

[2] Cases where this does not hold will be returned to below.

defined by the action (4.1) – such models will hence be referred to as strained. It is however possible to find an $\alpha = \alpha_*$ as a solution to $V_h = 3U_h$, resulting in unstrained models with $\mathcal{P} = 0$. Both strained as well as unstrained scenarios will be considered below.

Beyond thermodynamic quantities, holographic formulae may also be constructed for some of the transport coefficients of the viscoelastic hydrodynamics [2]. Using (3.103) and (4.11) the expression for the conformal bulk elastic modulus reads

$$B = \frac{m^2}{4u_h^3}\left[3V_h - 9U_h + \frac{u_h(\partial V_h/\partial u_h)(m^2 V_h - 3)}{m^2\left(V_h - u_h(\partial V_h/\partial u_h)\right) - 3}\right]. \tag{4.12}$$

Moreover, the dissipative coefficient for the Goldstone is given by

$$\gamma = \frac{m^2}{2\alpha^2 u_h^3}\frac{\partial V_h}{\partial u_h}. \tag{4.13}$$

The shear elastic modulus G and shear viscosity η must be calculated using the Kubo formulae (3.106a) and (3.38a) respectively. For the purpose of holographic calculations it is convenient to display the Kubo formulae in terms of quantities which are easily accessible in the bulk; using momentum conservation and the definition (2.44) the transport coefficients of interest are given by

$$\eta = -\lim_{\omega\to 0}\lim_{k\to 0}\frac{1}{\omega}\text{Im}\left[G^{\text{R}}_{T^{xy}T^{xy}}(\omega,k)\right], \tag{4.14a}$$

$$G = -\lim_{\omega\to 0}\lim_{k\to 0}\text{Re}\left[G^{\text{R}}_{T^{xy}T^{xy}}(\omega,k)\right] \tag{4.14b}$$

Holographically the retarded Green's functions above are found using expressions akin to formula (2.52), with the appropriate substitutions for the boundary energy-momentum tensor components.

4.1.1 Strained Models

The first models to be considered are the strained massive gravity models with monomial potentials of the form [89, 134]

$$V(X,Z) = X^N,\ Z^M. \tag{4.15}$$

The strain in these models stems from the equilibrium state $\phi^I = \alpha x^I$ not being a minimum of the free energy, which is discussed below equation (4.20).

Symmetry Breaking Mechanism

Whether potentials of the form (4.15) result in spontaneous or explicit breaking of translational invariance in the dual conformal field theory depends on the value of the exponents N and M, as well as the quantisation scheme and associated boundary conditions [89]. The breaking mechanism becomes evident from the the near-boundary expansion of the Stückelberg fields ϕ^I, which is given by [89, 134]

$$\phi^I(u,x) = \phi^I_{(0)}(x) + \ldots + \phi^I_{(1)}(x)u^a + \ldots, \tag{4.16}$$

where the ellipsis denote subleading terms. The solution for the scalar field (4.4) is radially constant and may hence be identified as

$$\phi^I_{(0)}(x) = \alpha x^I. \tag{4.17}$$

The value of the exponent a in the expansion (4.16) depends on the potential and is given by

$$a = \begin{cases} 5 - 2N & \text{if} \quad V(X,Z) = X^N, \\ 5 - 4M & \text{if} \quad V(X,Z) = Z^M. \end{cases} \tag{4.18}$$

Assuming standard quantisation the reasoning in section 2.2.1 of chapter 2 may be applied here: The leading term in the asymptotic expansion (4.16) is the source of a dual operator in the bulk; the subleading term is dual to the vacuum expectation value of said bulk operator.

The fall-off behaviour of (4.16) is determined by the exponents (4.18). For monomial potentials with $N < 5/2$ or $M < 5/4$ the exponent a is positive and hence $\phi^I_{(0)}$ is leading as $u \to 0$; the dual theory thus contains a source given by (4.17), which breaks translational invariance explicitly. If $N > 5/2$ or $M > 5/4$ then $\phi^I_{(0)}$ is subleading and is hence dual to an expectation value, while the source term with coefficient $\phi^I_{(1)}$ must vanish in order to preserve the radially constant profile of the scalar field; in such a situation the ground-state vacuum expectation value breaks translational invariance without a corresponding source – this is spontaneous symmetry breaking of translational invariance.

The scalar field enters with the term m and the constant α and hence their interplay influences the amount of symmetry breaking. In the numerical calculations the definitions will be fixed such that the breaking is controlled by m; however, in the boundary conformal theory only dimensionless quantities should be considered – the only other dimensionful equilibrium parameter is the temperature T and hence the dimensionless breaking scale is chosen to be m/T. Other quantities will also be made dimensionless by using factors of T.

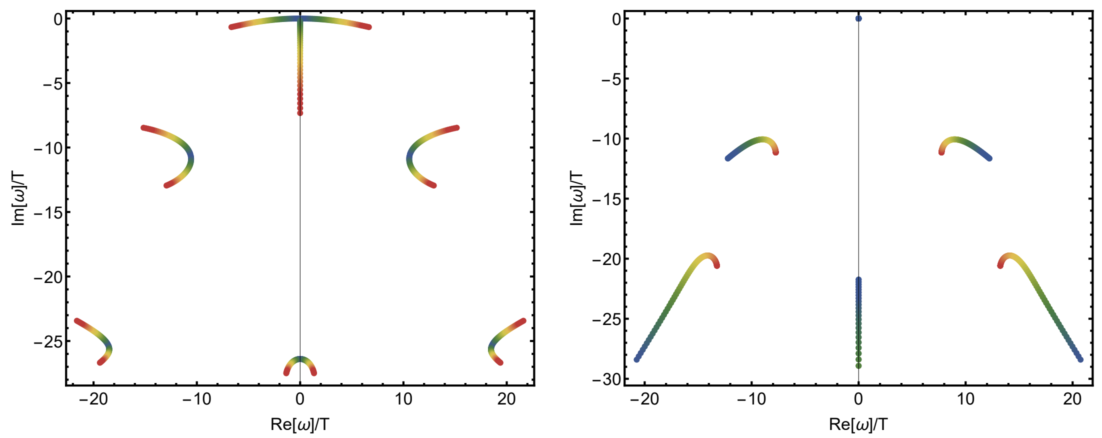

Figure 4.1: Spectrum of quasi-normal modes for the massive gravity model (4.1) with potential $V(X,Z) = X^3$. Blue dots signify low values of the functional dependence while red dots show high values. **Left:** The quasi-normal mode spectrum for fixed $m/T = 0.179$ as a function of the dimensionless momentum $k/T \in [0.186, 7.45]$. **Right:** The quasi-normal mode spectrum for vanishing momentum as a function of the dimensionless parameter $m/T \in [0, 6.5]$.

Quasi-normal Modes

Employing pseudo-spectral methods it is possible to numerically calculate the retarded Green's functions using bulk degrees of freedom, in accordance with equation (2.52). Results will be presented for the longitudinal sector of the system [1,2] – the transverse sector of the present model was studied and matched to hydrodynamics in [89, 134].

Imposing homogeneous Dirichlet boundary conditions sets the sources to zero at the conformal boundary, which together with infalling boundary conditions at the horizon yields the quasi-normal modes of the gravitational theory. The quasi-normal modes are independent of the the retarded Green's function used to calculate them. The spectrum of some quasi-normal modes for $V(X,Z) = X^N$ with $N = 3$ are displayed in figure 4.1.

As discussed in section 3.1.5 of chapter 3, hydrodynamic modes are poles of the retarded Green's functions and are hence dual to specific quasi-normal modes. In figure 4.1 the hydrodynamic modes are the quasi-normal modes which are located at the origin for $k/T = 0$. The hydrodynamic sector of the spectrum thus contains a pair of propagating modes with real and imaginary parts, and a purely imaginary diffusive mode. The speed and attenuation of sound, as well as diffusion constants, are extracted by fitting the lowest quasi-normal modes with polynomials quadratic in the momentum – this process is repeated for several values of the breaking parameter m/T, after which the results are presented as data points in plots, such as the right panel in figure 4.2.

Hydrodynamic Behaviour

The behaviour of the hydrodynamic modes extracted from the quasi-normal modes may be compared to the theoretical predictions of the viscoelastic hydrodynamics presented in section 3.2.3. It was shown in [1] that the viscoelastic hydrodynamic theories available the time – which did not contain strain pressure – could not accurately describe the hydrodynamic diffusive mode which appears in the quasi-normal mode spectrum of a massive gravity theory with spontaneously broken translational invariance.

The horizon formulae for the thermodynamic quantities and transport coefficients require the evaluation of V_h and U_h, which results in

$$V_h = \alpha^{2N} u_h^{2N}, \qquad U_h = \frac{\alpha^{2N} u_h^{2N}}{3 - 2N}. \tag{4.19}$$

The horizon formulae thus read

$$\begin{gathered} T = \frac{3 - m^2 V_h}{4\pi\, u_h}, \qquad p = \frac{1}{2u_h^3}\left(1 - \frac{2N-1}{2N-3} m^2 V_h\right), \\ \varepsilon = \frac{1}{u_h^3}\left(1 + \frac{m^2 V_h}{2N-3}\right), \quad \mathcal{P} = \frac{N}{2N-3}\frac{m^2 V_h}{u_h^3}, \qquad \mathcal{P}' = -\frac{4\pi}{u_h^2}\frac{N m^2 V_h}{3 + (2N-1) m^2 V_h}, \\ B = \frac{N m^2 V_h}{2u_h^3}\left(\frac{3}{2N-3} + \frac{3 - m^2 V_h}{3 + (2N-1) m^2 V_h}\right), \quad \gamma = \frac{N m^2 V_h}{\alpha^2 u_h^4}. \end{gathered} \tag{4.20}$$

The shear elastic modulus G and shear viscosity η are computed using the Kubo formulae (4.14). Furthermore, from the viscoelastic hydrodynamic theory of section 3.2.3 the longitudinal attenuation of sound $\Gamma_\parallel$ and diffusion constant $D_\parallel$ are given by

$$\Gamma_\parallel = \frac{\eta}{\chi_{\pi\pi}} + \frac{s^2 T^2 G^2}{\gamma \chi_{\pi\pi}^3 v_\parallel^2}, \qquad D_\parallel = \frac{s^2 T (B + G - \mathcal{P})}{\gamma (s + \mathcal{P}')(\chi_{\pi\pi} + 2G)}. \tag{4.21}$$

Using the relation $f_\varepsilon = -p$ it is evident form the expression for the pressure in the first line of (4.20) that the free energy receives positive contributions due to the scalar field for all $N \notin [1/2, 3/2]$. This illustrates that the background solution $\phi^I = \alpha x^I$ raises the free energy within the regime of spontaneous breaking of translational invariance – it acts as strain in the dual field theory. The would-be minimal solution with $\alpha = 0$ does not provide a well-defined vacuum due to the gravitational theory being strongly coupled at this point [89].

The information needed to compare the quasi-normal mode dynamics to the predictions provided by viscoelastic hydrodynamics is now at hand. For simplicity, and without loss of generality, the numerical data has been produced with $\alpha = u_h = 1$.

The speed of longitudinal sound is fixed by conformal invariance – the comparison

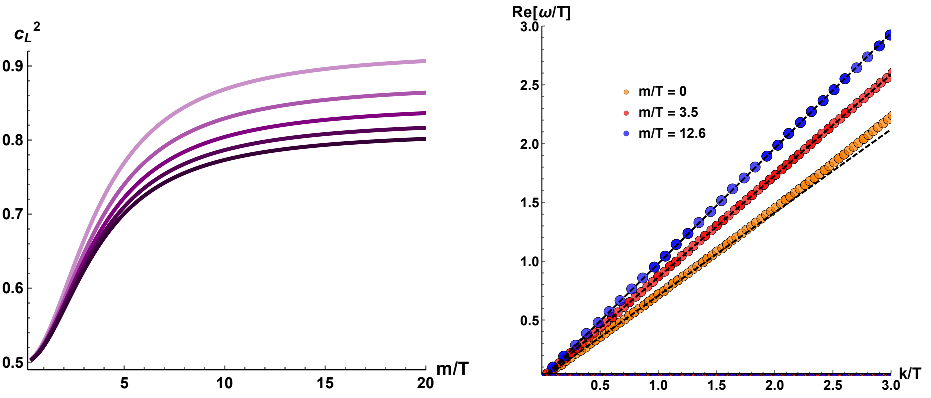

Figure 4.2: **Left:** Speed of longitunidal sound for models with potentials $V(X,Z) = X^N$ with $N = 3,4,5,6,7,8$ (from lighter to darker colour) as a function of the dimensionless breaking parameter m/T. **Right:** Real part of the dispersion relation of the propagating quasi-normal modes for $N = 3$ at various $m/T \in \{0, 3.5, 12.6\}$. The dashed black lines show the speed of sound which is fixed by conformal invariance.

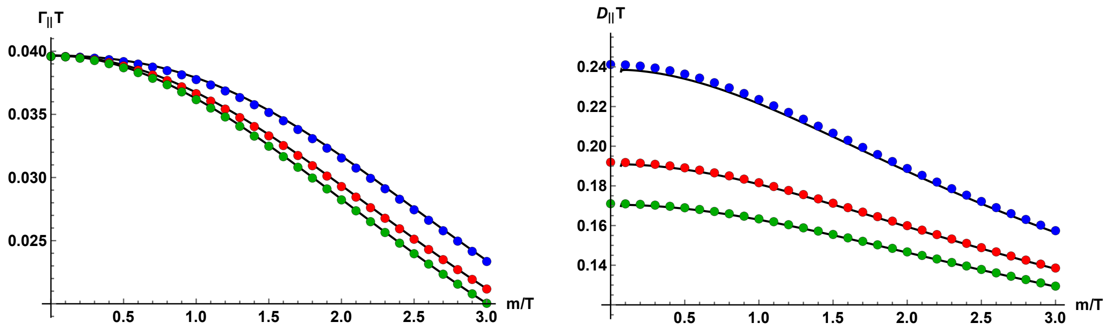

Figure 4.3: Quasi normal mode data for the longitudinal sound attenuation and diffusion constants, for models with $V(X,Z) = X^N$ and $N = 3,4,5$ (dots, from top to bottom) as functions of the dimensionless parameter m/T, compared to their hydrodynamic predictions $\Gamma_{||}$ and $D_{||}$ from (4.21) (solid lines).

between the (conformal) hydrodynamic relation $v_{\|}^2 = 1/2 + v_{\perp}^2$ and quasi-normal modes is displayed in the right-hand panel of figure 4.2 (the speed of sound is given by the slope of the graphs). The agreement is good.

Quasi-normal mode data – as a function of m/T and for several values of the exponent N – is compared to the hydrodynamic expressions for the longitudinal sound attenuation constant $\Gamma_{\|}$ and diffusion constant $D_{\|}$, given by equation (4.21), in figure 4.3. The match is excellent, and in particular the diffusion constant illustrates the importance of strain pressure in the hydrodynamic theory.

The analysis continues to models with potentials of the form $V(X,Z) = Z^M$, which yields $V_h = \alpha^4 u_h^4$ and $U_h = \alpha^4 u_h^4/(3-4M)$. The relevant horizon formulae are obtained by substituting $N \to 2M$ in equation (4.20). For models with $V(X,Z) = Z^M$ the shear elastic modulus G is zero. The speed of sound is again constrained by conformal

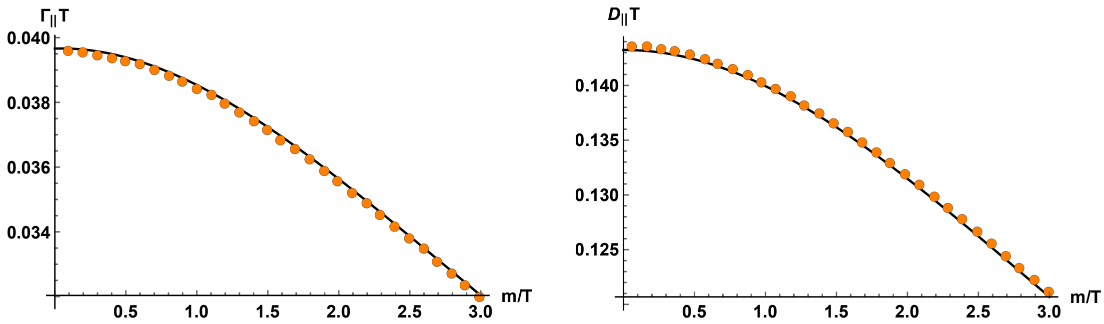

Figure 4.4: Quasi normal mode data for the longitudinal sound attenuation and diffusion constants for the model $V(X,Z) = Z^2$ (dots) as a function of the dimensionless parameter m/T, compared to their hydrodynamic predictions $\Gamma_{\|}$ and $D_{\|}$ from (4.21) (solid lines).

invariance and is trivially satisfied. The comparison between quasi-normal modes and the hydrodynamic expressions (4.21) for the sound attenuation and diffusion constant is presented in figure 4.4. The match is again very good.

4.1.2 Unstrained Models

Although they are theoretically and principally interesting, the strained models with monomial potentials (4.15) do not have clear real-world applications or interpretations due to their thermodynamic quirks. However, by implementing models with polynomial potentials it is possible to find an $\alpha = \alpha_*$ such that the free energy is minimised. The model which will be considered here has the potential [2, 135]

$$V(X,Z) = X + \lambda X^2, \tag{4.22}$$

for some constant λ. This model is plagued by its own unphysical behaviour, as will become clear; it nevertheless serves a purpose as a toy model for illustrating some key aspects of the hydrodynamic theory.

Symmetry Breaking

The asymptotic behaviour of polynomial potentials such as (4.22) is controlled by the leading monomial. For the case at hand this means that the near-boundary expansion (4.16) becomes

$$\phi^I(u,x) = \phi^I_{(0)}(x) + \ldots + \phi^I_{(1)}(x)u^3 + \ldots, \tag{4.23}$$

which according to standard quantisation would mean that the translational symmetry is explicitly broken by a non-vanishing source term. However, employing alternate quantisation the role of the leading and subleading terms are switched [75] – hence if one assumes alternate quantisation the symmetry breaking becomes spontaneous in the same way as for the monomial models (4.15). Alternate quantisation in turn imposes Neumann boundary conditions at the conformal boundary.

Hydrodynamic Behaviour

The potential (4.22) behaves as u^2 at the conformal boundary, meaning that the definition of U_h must be changed in this instance [2]; it is now instead defined as

$$U_h = u_h^3 \int_{u_h}^{\infty} \mathfrak{h}^{-4} V(\alpha^2 \mathfrak{h}^2, \alpha^4 \mathfrak{h}^4)\, \mathrm{d}\mathfrak{h}. \tag{4.24}$$

Evaluating the the integral in V_h, and U_h as defined above, yields

$$V_h = \alpha^2 + u_h^2 + \lambda\alpha^4 u_h^4, \qquad U_h = \alpha^2 + u_h^2 - \lambda\alpha^4 u_h^4. \tag{4.25}$$

Using these values in the formula (4.11) the strain pressure $\mathcal{P}$ is set to zero for an equilibrium value α_* by solving the constraint with $\alpha \neq 0$ – this results in

$$\alpha_* = \frac{1}{2\lambda u_h^2}. \tag{4.26}$$

The hydrodynamic parameters then become

$$T = \frac{3}{4\pi u_h}\left(1 - \frac{m^2}{4\lambda}\right), \qquad s = \frac{2\pi}{u_h^2}, \quad p = \frac{1}{2u_h^3}\left(1 - \frac{m^2}{4\lambda}\right),$$
$$\epsilon = \frac{1}{u_h^3}\left(1 - \frac{m^2}{4\lambda}\right), \quad \mathcal{P}' = -\frac{4\pi}{3u_h^2}\frac{m^2}{\lambda + 5m^2/12}, \qquad B = \frac{m^2}{2\lambda u_h^3}\frac{\lambda - m^2/4}{\lambda + 5m^2/12}, \qquad \gamma = \frac{2m^2}{u_h^2}, \tag{4.27}$$

In the above equations it is manifest that the temperature derivative of the strain pressure, for fixed α, is non-zero; this in turn allows for a non-zero bulk modulus B as a consequence of the scale invariance constraints in equation (3.103). In this regime the longitudinal diffusion constant of the hydrodynamic theory takes the form

$$D_\parallel = \frac{s^2 T(B+G)}{\gamma(s+\mathcal{P}')(\chi_{\pi\pi} + 2G)}, \tag{4.28}$$

with $\chi_{\pi\pi} = \varepsilon + p$.

Computing the values for G and η numerically the trasverse speed of sound $v_\perp^2 = G/\chi_{\pi\pi}$ and diffusion constant (4.28) may be compared to the quasi-normal mode data of the holographic model; the results are shown in figure 4.5, as functions of m/T. The speed of transverse sound is imaginary due to a negative shear elastic modulus G but nevertheless the hydrodynamic predictions are valid. Most importantly, matching the

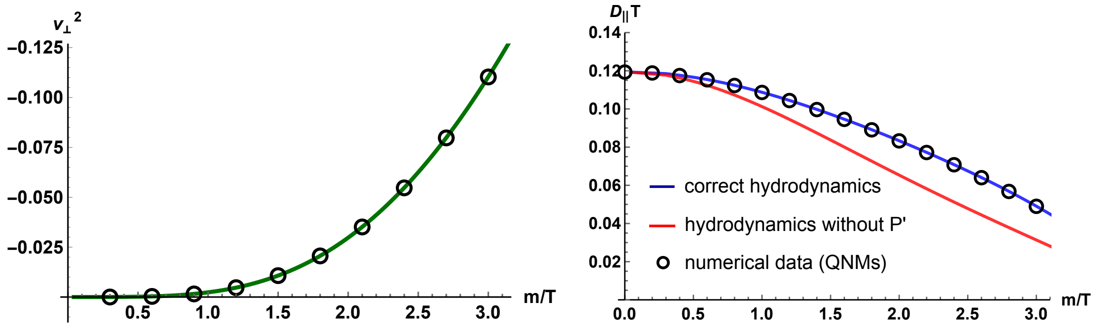

Figure 4.5: Quantities of interest for $V(X,Z) = X + X^2/2$ and $\mathcal{P} = 0$; $u_h = 1$ and thus $\alpha_* = 1$. **Left:** Quasi-normal modes for transverse speed of sound $v_\perp^2$ (circles) alongside the hydrodynamic predictions (solid lines). **Right:** Quasi-normal mode data for $D_\parallel$ (circles) alongside the hydrodynamic prediction (4.21) (solid lines).

diffusive constant $D_\parallel$ evidently relies on the presence of $\mathcal{P}'$.[3]

4.2 Holographic Broken Superfluid

The pseudo-spontaneous breaking of $U(1)$ symmetry will be studied holographically in this section [3]. The theories are formulated in the probe limit, meaning that the backreaction of the matter fields onto the metric is ignored. The principles behind the numerical computations are the same as in the previous section and hence their details will be spared.

4.2.1 Holographic Superfluid

The basis of the holographic analysis is the model proposed in [90], which is given by the $U(1)$-invariant bulk action

$$\mathcal{S}_{\text{SF}} = -\int \mathrm{d}^3x\,\mathrm{d}u\,\sqrt{-g}\left[\frac{1}{4}F^{mn}F_{mn} + D_m\psi(D^m\psi)^* + M^2\psi\psi^*\right], \tag{4.29}$$

where u is the radial coordinate and $\psi(u)$ is a radially dependent complex scalar field with mass M; $F \equiv \mathrm{d}A$ is the field strength tensor of the $U(1)$ gauge field A; and the covariant derivative is defined as $D_m \equiv \partial_m - iqA_m$, where q is the charge of the scalar operator dual to ψ. The lowercase Latin indices span all bulk spacetime coordinates while Greek indices do not include the radial direction. The AdS radius of curvature ℓ has been set to one.

The bulk geometry is that of a AdS_4 Schwarzschild black hole in Eddington-Finkelstein

[3] Signatures of an equivalent phenomena were also seen in [136].

coordinates and is given by the line element

$$\mathrm{d}s^2 = \frac{1}{u^2}\left[-f(u)\,\mathrm{d}t^2 - 2\,\mathrm{d}t\,\mathrm{d}u + \mathrm{d}x^2 + \mathrm{d}y^2\right], \qquad \text{with} \qquad f(u) = 1 - u^3, \tag{4.30}$$

where $u \in [0,1]$. The conformal boundary is located at $u = 0$ and the black hole horizon is set at $u_h = 1$. The probe limit is manifest from the simple form of the emblackening factor $f(u)$ which, from equation (2.35), gives rise to a constant temperature

$$T = \frac{3}{4\pi}. \tag{4.31}$$

The equilibrium configuration of the theory defined by (4.29) may be determined by making the radially dependent ansätze

$$A = A_t(u)\,\mathrm{d}t, \qquad \psi(u) = \psi_1(u) - i\psi_2(u), \tag{4.32}$$

which ensure rotational invariance. The asymptotic expansions for the scalar fields $\psi_{1,2}$ and gauge field component A_t are given by

$$\psi(u) = \psi_I^{(l)} u^{3-\Delta} + \ldots + \psi_I^{(s)} u^{\Delta} + \ldots, \qquad (I = 1,2) \tag{4.33a}$$

$$A_t(u) = A_t^{(l)} + A_t^{(s)} u + \ldots, \tag{4.33b}$$

where Δ is the scaling dimension of the complex scalar operator $\mathcal{O}$ dual to ψ; the superscripts (l), (s) denote the leading and subleading term respectively; and the ellipsis denotes terms at higher order in u. Standard quantisation is assumed. The chemical potential μ of the boundary theory is defined as the source of $\langle J^t \rangle$ and is hence identified as

$$\mu = A_t^{(l)}. \tag{4.34}$$

In order to model a superfluid the theory must exhibit spontaneous symmetry breaking, where the charged operator $\mathcal{O}$ condenses below a critical temperature T_c. Choosing the boundary condition $\psi_I^{(l)} = 0$ removes the source for $\mathcal{O}$. Moreover, the standard value for the scalar field mass $M^2 = -2$ results in $\Delta = 2$ and hence $\mathcal{O}$ is relevant in the infrared. In the following sections the consequences of adding a small explicit breaking of $U(1)$ symmetry on top of the spontaneous breaking will be considered.

4.2.2 Holographic Superfluid with Sourced Charged Scalar

The first means by which an explicit breaking of $U(1)$ symmetry is introduced is by sourcing the scalar operator $\mathcal{O}$ of the boundary theory [3]. The source λ is taken to be real, which in terms of the boundary expansion (4.33) corresponds to choosing $\lambda = \psi_1^{(l)}$

with $\psi_2^{(l)} = 0$. Holographic renormalisation results in the expectation value

$$\langle \mathcal{O} \rangle = 2\psi_1^{(s)} + 2i\psi_2^{(s)} + 2iq\mu\psi_1^{(l)} = 2\psi_1^{(s)}, \tag{4.35}$$

where the second equality follows from the constraint $\psi_2^{(s)} = -q\mu\psi_1^{(l)}$, which is specific to Eddington-Finkelstein coordinates.

The source λ controls the explicit breaking but must be presented in a dimensionless form; one possibility is λ/T. Increasing the value of λ/T corresponds to more symmetry breaking. In the following λ/T will be chosen to be infinitesimal such that the breaking may be treated as pseudo-spontaneous.

The non-conservation equation for the $U(1)$ current of the boundary theory is given by

$$\partial_\mu \langle J^\mu \rangle = \frac{iq}{2} \left[\psi^{(l)} \langle \mathcal{O}^* \rangle - \left(\psi^{(l)} \right)^* \langle \mathcal{O} \rangle \right]. \tag{4.36}$$

For real λ the expectation value $\langle \mathcal{O} \rangle$ is real at equilibrium, hence the right-hand side of the above equation vanishes; away from equilibrium it instead reads

$$\partial_\mu \langle J^\mu \rangle = \lambda q \mathrm{Im} \langle \mathcal{O} \rangle , \tag{4.37}$$

see also [123,129,137]. Comparing the non-conservation equation (4.37) to the one used in the hydrodynamic calculations, equation (3.64), it is evident that this mechanism for explicit breaking of $U(1)$ symmetry does not result in charge relaxation, i.e. $\Gamma = 0$. Since φ in the hydrodynamic theory is the phase of the condensate an appropriate identification is $\varphi \equiv \mathrm{Im} \langle \mathcal{O} \rangle / \langle \mathcal{O} \rangle_{\mathrm{eq}}$, from which the relationship $m = \lambda q \langle \mathcal{O} \rangle_{\mathrm{eq}}$ follows, where $\langle \mathcal{O} \rangle_{\mathrm{eq}}$ is the expectation value at equilibrium.[4]

The behaviour of this setup lends itself to be studied using numerical techniques. Without loss of generality $q = 1$ will be fixed in the remainder of this chapter.

Scalar Condensate and Goldstone Correlation Function

Without the presence of explicit symmetry breaking the scalar condensate $\langle \mathcal{O} \rangle$ acts as the order parameter for the superfluid phase transition; it becomes non-zero at a critical temperature T_{c}, indicating that the superfluid has formed. The behaviour of $\langle \mathcal{O} \rangle$ in the presence of explicit breaking is thus of interest – the numerical results for the dimensionless condensate $\langle \mathcal{O} \rangle / T$ are shown in the left panel of figure 4.6. As the dimensionless breaking parameter λ/T is increased the value of the dimensionless condensate increases and the phase transition goes from sharp to continuous, making is difficult to define a critical temperature T_{c}.

[4]The subscript denoting equilibrium will be dropped in the following.

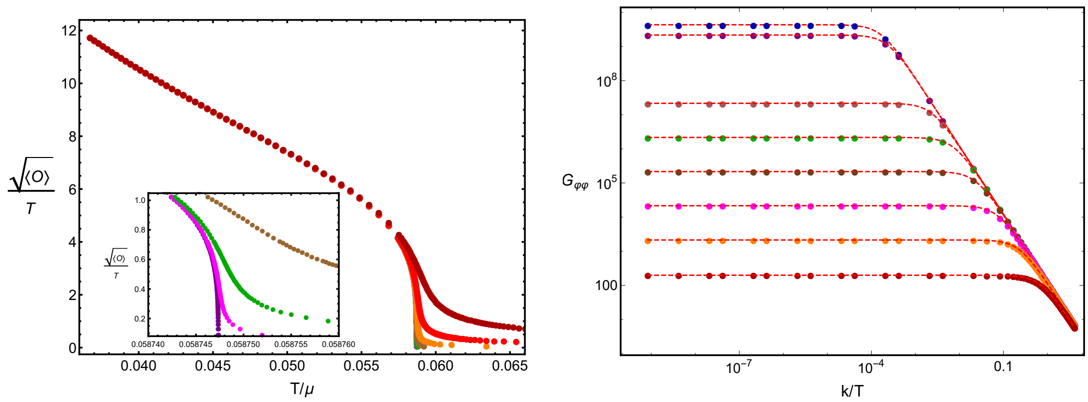

Figure 4.6: Colours indicate different values of the dimensionless source λ/T – blue is 5×10^{-10}, purple is 10^{-9}, light brown is 10^{-7}, green is 10^{-6}, dark brown is 10^{-5}, magenta is 10^{-4}, orange is 10^{-3}, and red is 10^{-2}. **Left:** numerical results for the dimensionless scalar condensate, near the superfluid phase transition, as a function of temperature and for different values of the dimensionless source λ/T. **Right:** the pseudo-Goldstone correlator at zero frequency and finite momentum with fixed $T/\mu = 0.0575$, for different values of the source λ/T. The dashed lines indicate the hydrodynamic result presented in equation (4.38)

Moreover, in the right panel of figure 4.6 the zero-frequency Green's function $G_{\varphi\varphi}(0,k)$ obtained from holography is compared to the hydrodynamic formula of section 3.2.2,

$$G_{\varphi\varphi}(0,k) = \frac{\chi_{\xi\xi}}{k^2 + \mathfrak{m}^2}, \tag{4.38}$$

for small amounts of explicit breaking. The agreement is excellent for a large range of values of $G_{\varphi\varphi}(0,k)$ and confirms the pseudo-Goldstone nature of φ.

Zero-momentum Excitations

The quasi-normal modes at zero momentum, for various values of the breaking parameter, are shown in figure 4.7. At zero explicit breaking the spectrum contains a pair of sound modes as well as a non-hydrodynamic diffusive mode with imaginary gap. As the explicit breaking increases the gapless modes acquire a complex gap while the non-hydrodynamic mode moves away from the origin along the imaginary axis. For large explicit breakings the mode with a purely imaginary gap is closer to the origin than the modes with complex gaps; in this regime the hydrodynamic description must fail [74].

Since it has been established that the scenario considered here does not display charge relaxation the behaviour of the modes are in accordance with the dispersion relation

$$\omega = -i\frac{\Omega}{2} \pm \sqrt{\omega_0^2 - \frac{\Omega^2}{4}}, \qquad \omega_0^2 = \frac{m}{(\partial\rho_t/\partial\mu)}, \tag{4.39}$$

with a positive square root. The above dispersion relation stems from the hydrodynamic expression (3.73), and ω_0 denotes the pinning frequency. The superscript used to denote probe-limit quantities in section 3.2.2 will be dropped.

The behaviour of the phase relaxation Ω and pinning frequency ω_0, as functions of the explicit breaking, may be extracted directly from the zero-momentum quasi-normal modes – the results are shown in figure 4.8. The pinning frequency, displayed in the left panel, decreases linearly with the explicit breaking scale and vanishes at zero explicit breaking; this is a realisation of the Gell-Mann-Oaks-Renner relation [58]. Similarly, in the right panel the phase relaxation Ω is linearly dependent on the explicit breaking scale and vanishes at zero explicit breaking; the second property illustrates that in this model Ω arises due to the interplay of explicit and spontaneous symmetry breaking.

The phase relaxation Ω and pinning frequency ω_0 are displayed as functions of the dimensionless temperature T/μ in figure 4.9; the dots denote quasi-normal mode data while the dashed lines indicate the Kubo formula for Ω, given in equation (3.86), and the definitions of the pinning frequency in the dispersion relation (4.39). The hydrodynamic results are in good agreement with the quasi-normal modes for small values of λ/T; for larger values of the explicit breaking the agreement is good until the mass of the pseudo-Goldstone is of the same order as the square root of the scalar condensate.

Finite-momentum Excitations

If the hydrodynamic framework constructed in section 3.2.2 is valid, the finite momentum quasi-normal modes of the current treatment of the model (4.29) should obey the dispersion relations (3.72) with $\Gamma = 0$ and

$$D_\pm = \frac{1}{2}\left(\frac{\sigma_0}{(\partial\rho_t/\partial\mu)} + \zeta_3\frac{\rho_s}{\mu}\right) \pm \frac{i}{2}\frac{2\rho_s - \zeta_3\rho_s(\partial\rho_t/\partial\mu)\Omega + \sigma_0\mu\Omega}{\mu\sqrt{4m(\partial\rho_t/\partial\mu) - (\partial\rho_t/\partial\mu)^2\Omega^2}}. \tag{4.40a}$$

The real and imaginary parts of (4.40) are plotted alongside quasi-normal mode data in the top panels of figure 4.10 – the agreement is good for several values of λ/T. In the bottom panel of figure 4.10 the absolute difference between the hydrodynamic expression (4.40) and the quasi-normal modes increases with the breaking λ/T but also decreases with the dimensionless temperature T/μ. For smaller values of T/μ the value of the scalar condensate is larger, as is evident from figure 4.6, meaning the effects of the explicit breaking become relatively smaller and hence the pseudo-spontaneous approximation holds stronger.

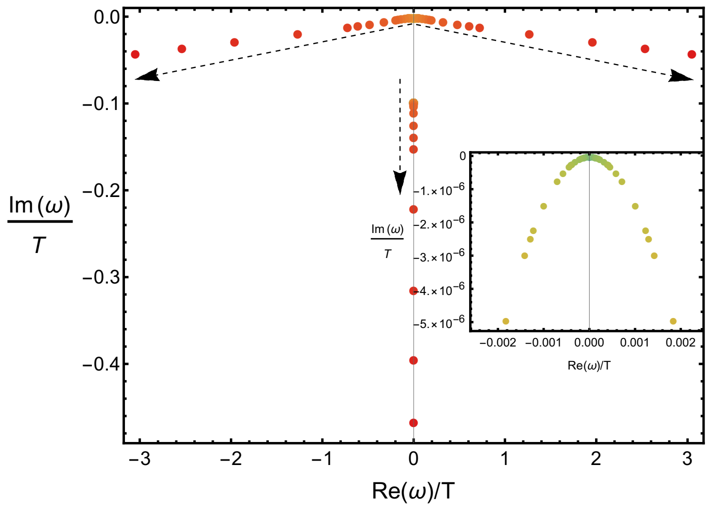

Figure 4.7: The lowest modes in the quasi-normal mode spectrum, at zero momentum, for values of the source λ/T in the range $[10^{-16}, 0.1]$ at fixed $T/\mu = 0.0582$. The dashed arrows show the direction of movement towards the limit of strong explicit breaking, $\lambda/T \gg 1$. The inset displays the modes in the limit of infinitesimal explicit breaking, $\lambda/T \ll 1$; a pair of sound modes may be identified – which gain a complex gap once explicit breaking is added – as well as a pseudo-diffusive mode which moves away from the origin as the breaking increases.

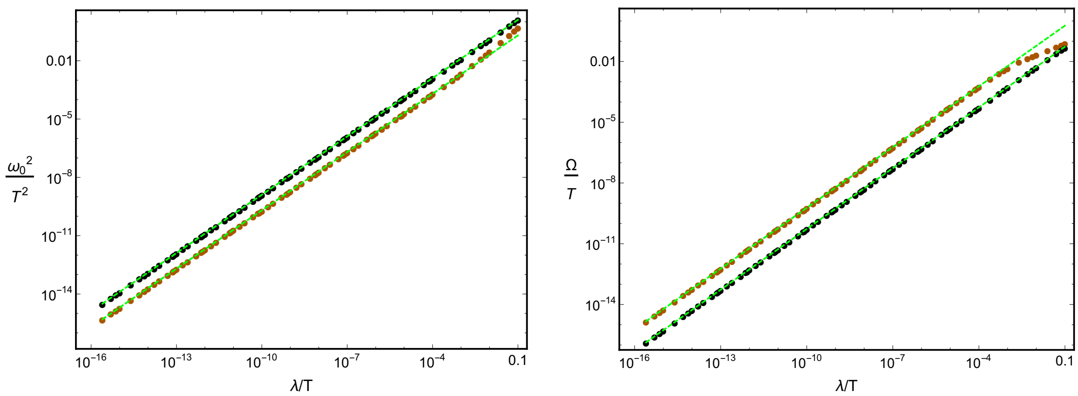

Figure 4.8: The dimensionless pinning frequency ω_0^2/T^2 (**left**) and the dimensionless phase relaxation rate Ω/T (**right**) as a function of the dimensionless source λ/T. The dashed green lines are a linear fit to the data, emphasising that at small explicit breaking both quantities are linear in the explicit breaking parameter. Colours indicate fixed T/μ – light brown is 0.0582 and black is 0.0434.

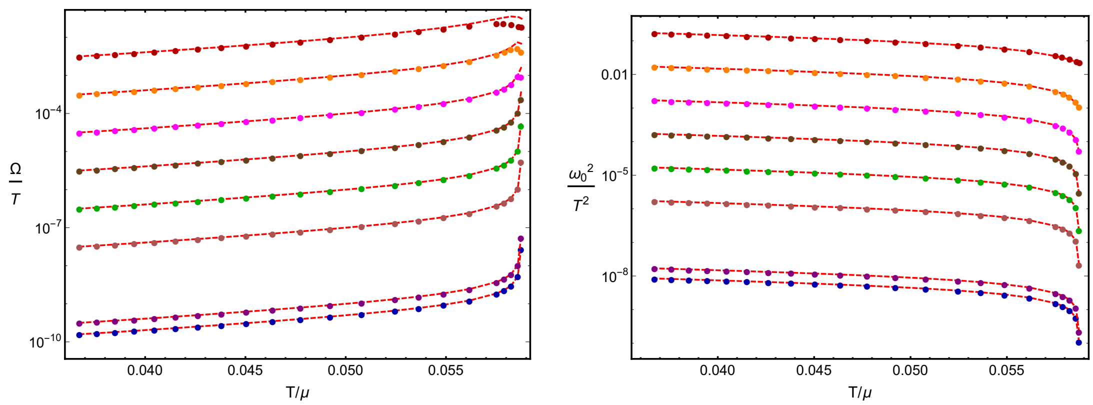

Figure 4.9: The dimensionless phase relaxation (**left**) and the dimensionless pinning frequency (**right**) as functions of the dimensionless temperature T/μ for different values of the explicit breaking parameter λ/T. Quasi-normal mode data is displayed by dots while the dashed lines in the left panel are values of the Kubo formula in (3.86) and in the right panel they correspond to the definition (4.39). The colours indicate different values of the dimensionless source λ/T – blue is 5×10^{-10}, purple is 10^{-9}, light brown is 10^{-7}, green is 10^{-6}, dark brown is 10^{-5}, magenta is 10^{-4}, orange is 10^{-3}, and red is 10^{-2}.

Universal Behaviour of Phase Relaxation

It has been observed in several works [134, 136, 138–148], in various contexts, that phase relaxation Ω which arises due to the interplay between explicit and spontaneous symmetry breaking obeys the relation

$$\Omega = \mathfrak{m}^2 D_\xi, \tag{4.41}$$

where $\mathfrak{m}$ is the mass of the pseudo-Goldstone and D_ξ denotes the Goldstone diffusivity which can be calculated from the decoupled Josephson relation in the purely spontaneous state. Using the superfluid quantities of section 3.2.2 the phase relaxation Ω for pseudo-spontaneous breaking of $U(1)$ symmetry reads

$$\Omega = \omega_0^2 \zeta_3 \chi_{\rho\rho}, \tag{4.42}$$

where $\chi_{\rho\rho} = \partial\rho_t/\partial\mu$ is the charge susceptibility.

Numerical data for the dimensionless quotient $\omega_0^2\zeta_3\chi_{\rho\rho}/\Omega$, for the model currently under consideration, is plotted in figure 4.11. All quantities on the right-hand side of (4.42) have been calculated from the state without explicit breaking. The quotient tends to one for small explicit breakings, validating the formula (4.42) in the pseudo-spontaneous regime. Analogously to figure 4.10, the agreement is better for smaller values of T/μ. Since $\Omega \leq \omega_0^2\zeta_3\chi_{\rho\rho}$ for all breakings the relation (4.42) may also be

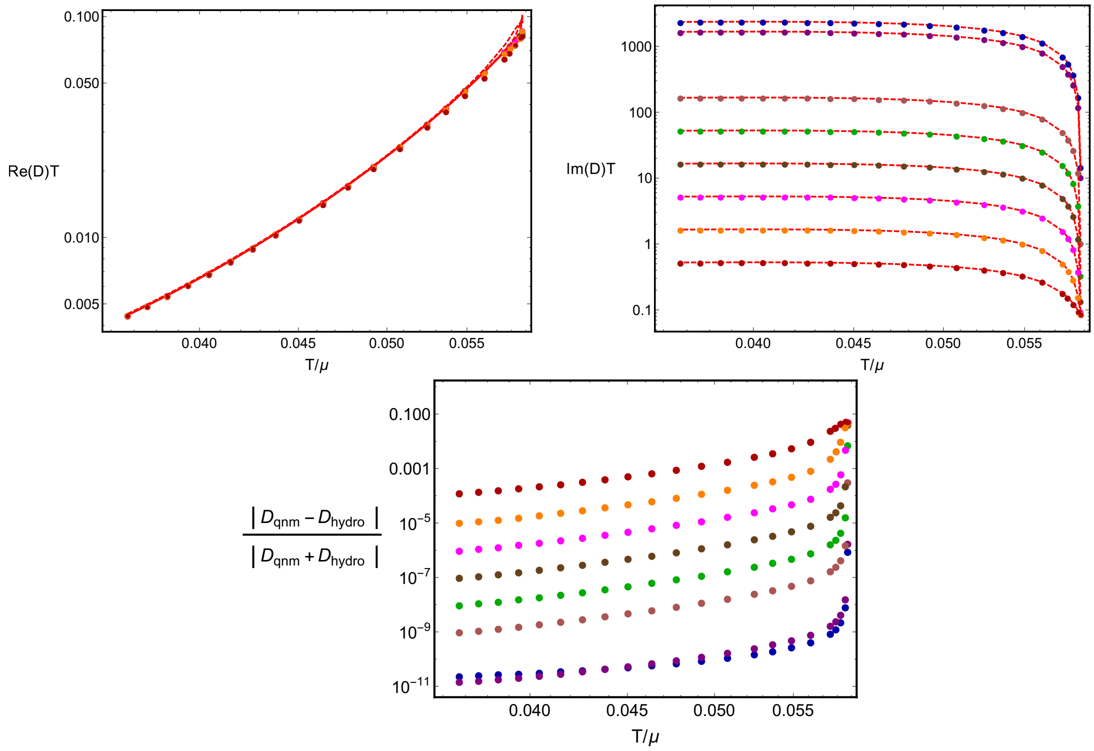

Figure 4.10: Top to panels display a comparison between the k^2-coefficients of the lowest quasi-normal modes (dots) versus the real and imaginary parts of the hydrodynamic formula (4.40) (lines), as a function of the dimensionless temperature T/μ. The bottom panel plots the agreement between the quasi-normal mode data and the hydrodynamic formula; it shows increasing agreement by lowering the temperature, i.e. by making the spontaneous symmetry breaking order parameter larger compared to the explicit breaking scale λ/T. The colours indicate different values of the dimensionless source λ/T – blue is 5×10^{-10}, purple is 10^{-9}, light brown is 10^{-7}, green is 10^{-6}, dark brown is 10^{-5}, magenta is 10^{-4}, orange is 10^{-3}, and red is 10^{-2}.

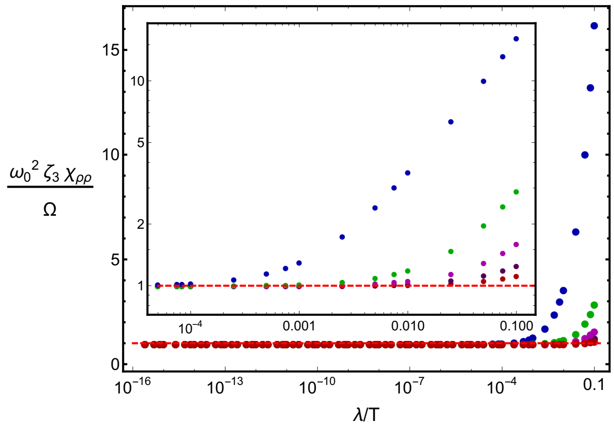

Figure 4.11: The dimensionless ratio $\omega_0^2\zeta_3\chi_{\rho\rho}/\Omega$ as a function of the dimensionless source λ/T. At small explicit breaking the ratio tends to one and thus validates the relation in equation (4.42). Different colours indicate different values of the dimensionless scale $T/\mu = \{0.0582, 0.0555, 0.0519, 0.0477, 0.0434\}$, starting from blue. The inset zooms into the range of larger values of the explicit breaking. The agreement is better for larger chemical potentials (or lower temperatures) which correspond to a larger spontaneous symmetry breaking scale and therefore a better approximation for the pseudo-spontaneous approximation.

viewed as an upper bound on the phase relaxation.

Electrical Conductivity

The pseudo-spontaneous hydrodynamic analysis of section 3.2.2 concluded that the AC conductivity in this regime does not display any influence from the phase relaxation Ω or the parameter m, nor a pole at zero frequency; as a consequence the DC conductivity is finite.

The frequency dependent conductivity is plotted for different values of the dimensionless source λ/T in figure 4.12. The hydrodynamic prediction (3.84) is obeyed at lowest order in the frequency, i.e.

$$\sigma(\omega) = \sigma_0 + \mathcal{O}(\omega^2) \ . \tag{4.43}$$

The real part of the conductivity grows with a ω^2 scaling which is beyond the present scope. Introducing the breaking parameter decreases the value of the real part of $\sigma(\omega)$; the expectation would be that as the explicit breaking increases a gap appears where the conductivity is zero at low frequencies. The imaginary part is zero at $\omega/T = 0$.

The hydrodynamic results and the holographic results presented in figure 4.12 may be compared to those of [59], where phase relaxation in a superfluid appears due to vortices and hence without explicit symmetry breaking. The AC conductivity of such

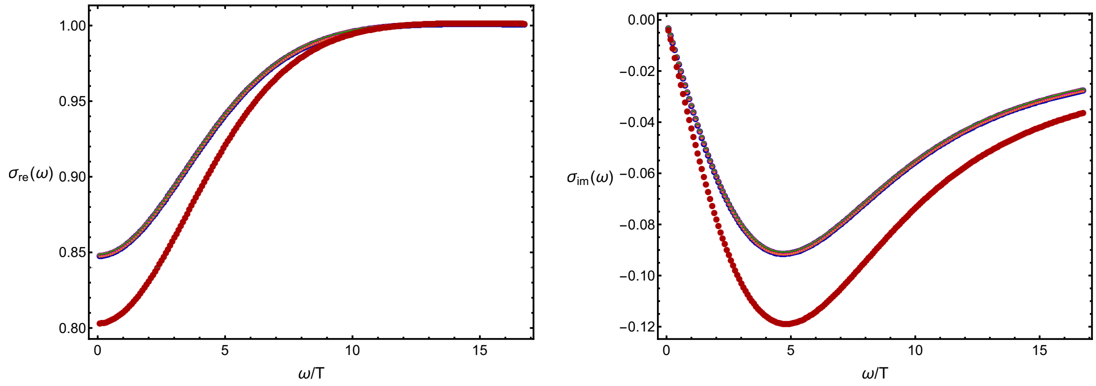

Figure 4.12: Real (**left**) and imaginary (**right**) parts of the AC conductivity as a function of the dimensionless frequency ω/T, for different values of the dimensionless source λ/T. The dimensionless temperature is fixed at $T/\mu = 0.0579$. The colours indicate different values of the dimensionless source λ/T – blue is 5×0^{-10}, purple is 10^{-9}, light brown is 10^{-7}, green is 10^{-6}, dark brown is 10^{-5}, magenta is 10^{-4}, orange is 10^{-3}, and red is 10^{-2}. The data from most cases are on top of each other.

systems receive contributions due to the phase relaxation in a way which produces a Drude-peak as well as a finite DC conductivity.

4.2.3 Holographic Superfluid with Massive Gauge Field

Another method of implementing pseudo-spontaneous breaking of $U(1)$ symmetry in holography is by modifying the action (4.29) to be of the form [91, 92, 149–151]

$$\mathcal{S} = \int \mathrm{d}^3x\, \mathrm{d}u\, \sqrt{-g} \left[-\frac{1}{4} F_{mn} F^{mm} - \frac{M_A^2}{2} \left(A_m - \partial_m \theta\right)\left(A^m - \partial^m \theta\right) - |D\psi|^2 - M^2 |\psi|^2 \right], \tag{4.44}$$

where M_A is the mass of the bulk gauge field A_m. The appearance of a mass-term for the gauge field breaks the local $U(1)$ symmetry which transforms $A_m \mapsto A_m + \partial_m \lambda_g$; this symmetry has in turn been restored by introducing the Stückelberg field θ which transforms as $\theta \mapsto \theta - \lambda_g$ under local $U(1)$ transformations. The configuration is taken to be static, with $\partial_t \theta = 0$. The geometry of the spacetime is unchanged and still given by (4.30).

The ansätze (4.32) are taken also for the model (4.44), however the boundary expansion of the gauge field (4.33b) is in this case modified to

$$A_t(u) \sim A_t^{(l)}\, u^{-\Delta_A}(1+\dots) + A_t^{(s)}\, u^{1+\Delta_A}(1+\dots) \qquad \text{with} \qquad \Delta_A = \frac{1}{2}\left(-1 + \sqrt{1 + 4\, M_A^2}\right). \tag{4.45}$$

The expectation value of the boundary $U(1)$ current, $\langle J^\mu \rangle$, is related to $A_\mu^{(s)} - \partial_\mu \theta^{(s)}$

and its scaling dimension is $[J^\mu] = 2 + \Delta_A$. Δ_A is thus an anomalous scaling dimension which grows with M_A and indicates that the symmetry is broken and the current no longer conserved. One may again identify the a boundary quantity $\mu \equiv A_t^{(l)}$ – however, this should not be interpreted as the chemical potential but rather as a source (with mass-dimension $1 - \Delta_A$) in the boundary field theory.

The dynamics of the complex scalar field ψ are unchanged from section 4.2.1. In contrast to the approach of section 4.2.2 the leading term of the boundary expansion (4.33a) can be made to vanish by imposing the boundary condition $\psi_I^{(l)} = 0$; when a solution for the subleading term exists the scalar sector exhibits spontaneous breaking of $U(1)$ symmetry, which for small values of M_A results in a combined pseudo-spontaneous breaking regime [3].

The form of the non-conservation equation for the boundary $U(1)$ current $\langle J^\mu \rangle$ may be argued for when the dual field theory is four-dimensional [3, 91, 92]; an analogous expression may on phenomenological grounds be taken also for a three-dimensional field theory, i.e.

$$\partial_\mu \langle J^\mu \rangle = -\Gamma \langle J^t \rangle , \tag{4.46}$$

where Γ is the charge relaxation rate [3]. Comparing the above expression to the hydrodynamic relation (3.64) it is evident that the parameter m, and hence the pinning frequency ω_0, vanishes in the model (4.44). It will also become clear that phase relaxation Ω is not present in this model.

Phase Diagram

The first observable of interest for the model (4.44) is the dimensionless scalar condensate – the numerical data are displayed as a function of the dimensionless temperature, for several values of the breaking parameter M_A, in figure 4.13. When a finite explicit breaking is introduced the value of the dimensionless condensate increases but the phase transition remains sharp. The clearly defined phase transition in this model may be compared to that of the model in the previous section, where it was smeared; this difference may be understood intuitively by remembering that in the previous model the pseudo-Goldstone bosons relax and thus no clear definition of order exists.

Quasi-normal Modes and Hydrodynamics

The quasi-normal modes of the model (4.44) are plotted in figure 4.14. Comparing the structure of the modes to the hydrodynamic result for the gap, equation (3.73), the inevitable conclusion is that the phase relaxation Ω and the pinning frequency ω_0 vanish. The charge relaxation rate obtained from the quasi-normal modes is plotted for various breakings in figure 4.16 – they show remarkable agreement to the leading

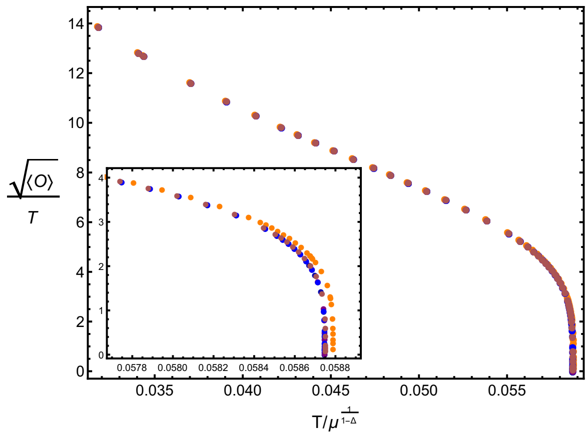

Figure 4.13: Plot of the dimensionless scalar condensate as a function of the dimensionless temperature $T/\mu^{1/(1-\Delta_A)}$, for different amounts of explicit breaking. Colours indicate different values of the breaking parameter M_A^2 – dark blue is 10^{-8}, blue is 10^{-6}, purple is 10^{-5}, light brown is 10^{-4}, and orange is 10^{-3}. The condensate and the critical temperature grow with the breaking but the phase transition remains sharp.

order expression

$$\Gamma = \frac{M_A^2}{\chi_{\rho\rho}} + \dots , \tag{4.47}$$

where the ellipsis indicate terms of higher order in M_A and the susceptibility $\chi_{\rho\rho}$ is calculated in the state without explicit symmetry breaking. Similar behaviour has been observed in [86, 88, 91, 92, 152, 153].

When $\Omega = \omega_0 = 0$ the hydrodynamic predictions for the modes (3.72) reduce to

$$\omega(k) = -iD_+k^2 + \dots , \qquad \omega(k) = -i\,\Gamma - i\,D_-\,k^2 + \dots , \tag{4.48}$$

where

$$D_+ = \zeta_3\frac{\rho_s}{\mu} + \frac{1}{\Gamma}\frac{\rho_s}{\mu\chi_{\rho\rho}} , \qquad D_- = \frac{\sigma_0}{\chi_{\rho\rho}} - \frac{1}{\Gamma}\frac{\rho_s}{\mu\chi_{\rho\rho}} . \tag{4.49}$$

In figure (4.16) the k^2-coefficients (4.49) are compared to the quasi-normal mode data for different amounts of breaking; the quantities entering $D_\pm$, except for Γ, have been extracted from the state without explicit $U(1)$ symmetry breaking. Moreover, also in the right-hand panel of figure (4.14) the coefficients (4.49) are compared to the quasi-normal modes as a function of the dimensionless momentum k/T – the agreement is good for small k/T and small values of the breaking parameter.

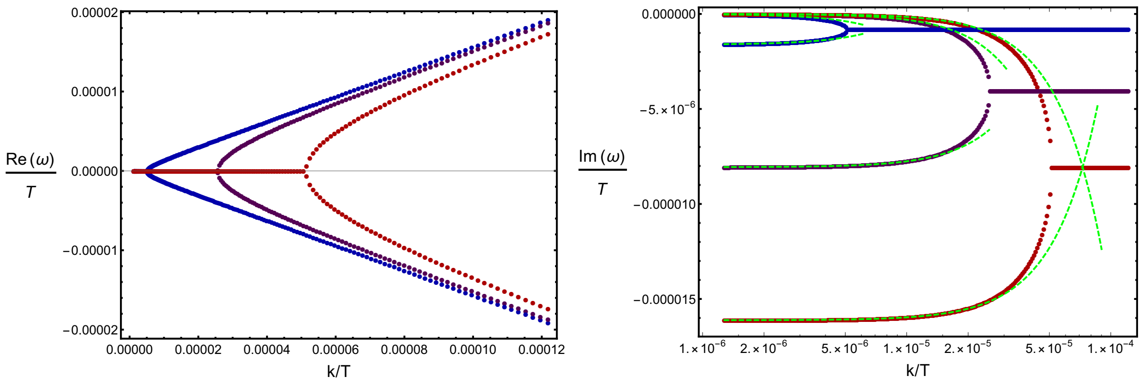

Figure 4.14: Low-lying quasi-normal modes as a function of the dimensionless momentum k/T (dots). In the right panel the dashed line represent the hydrodynamic predictions (4.48) and (4.49), with transport coefficient and susceptibility data extracted from the purely spontaneous phase. Colours indicate different values of the breaking parameter M_A^2 – blue is 10^{-6}, purple is 10^{-5}, and red is 5×10^{-5}. Values are shown for fixed dimensionless temperature $T/\mu^{1/(1-\Delta_A)} = 0.0582$.

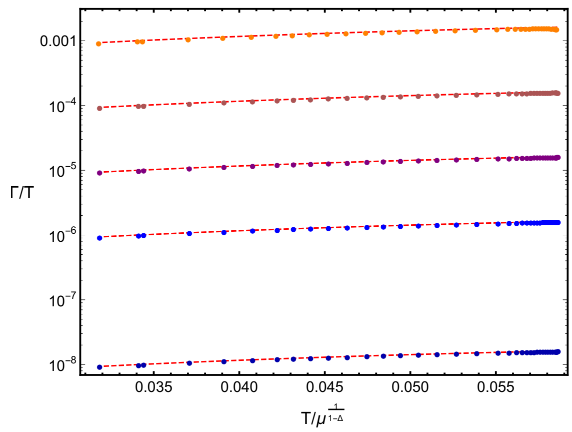

Figure 4.15: The dimensionless charge relaxation rate Γ/T extracted from quasi-normal modes (dots) compared to the formula (4.47)(dashed lines), as functions of the dimensionless temperature $T/\mu^{1/(1-\Delta_A)}$. Colours indicate different values of the breaking parameter M_A^2 – dark blue is 10^{-8}, blue is 10^{-6}, purple is 10^{-5}, light brown is 10^{-4}, and orange is 10^{-3}. The agreement is remarkable.

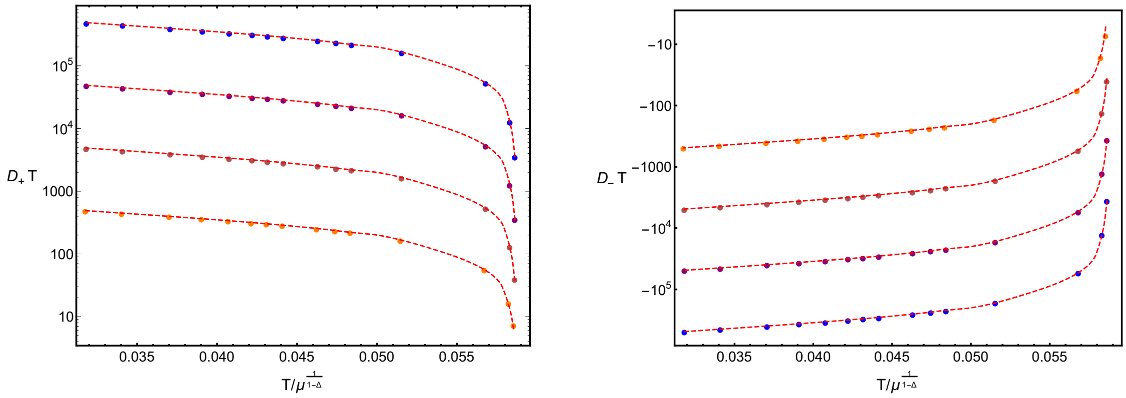

Figure 4.16: Dots denote the dimensionless k^2-coefficients entering the dispersion relations of the two lowest quasi-normal modes, extracted for different amounts of explicit breaking. Dashed lines are hydrodynamic predictions (4.49) where transport coefficients and susceptibilities are computed in the purely spontaneous background. Both quantities are computed for the same dimensionless temperature $T/\mu^{1/(1-\Delta_A)}$. Colours indicate different values of the breaking parameter M_A^2 – blue is 10^{-6}, purple is 10^{-5}, light brown is 10^{-4}, and orange is 10^{-3}. **Left:** Diffusion constant of gapless mode. **Right:** The quadratic coefficient of the damped mode.

5 BMS Symmetry and the Oscillator Construction

The main goal across the past three chapters has been to understand aspects of symmetries in dynamic systems and within this context applying holographic techniques in order to probe quantities of interest. The contents of this chapter will focus on aspects of symmetry motivated by the holographic principle itself.

By virtue of the bottom-up approach to the AdS/CFT correspondence (which was employed in chapters 2 and 4) a bulk gravitational theory has, in this thesis, mainly been used to study the dynamics of the dual boundary conformal field theory; nevertheless, the correspondence may be used in reverse in order to gain insight into formal aspects of the gravitational theory governing the bulk, which in the strong form of the correspondence has the potential to reveal secrets to the nature of quantum gravity. However, our Universe is not asymptotically anti-de Sitter and hence there exists a non-trivial gap between reality and any understanding gained by studying quantum gravity in the setting of the AdS/CFT correspondence. This raises the question whether the holographic principle may be applied to a gravitational theories in more realistic spacetimes, for example one which is asymptotically flat – such a holographic duality is called flatspace holography.

There are currently no concrete examples of flatspace holographic models;[1] there are however symmetry criteria which must be fulfilled by any candidate theory. In the AdS/CFT correspondence the conformal invariance of the boundary field theory is aligned with the asymptotic symmetry group of $(d+1)$-dimensional anti-de Sitter spacetime, namely $SO(d,2)$, which is isomorphic to the conformal group. For asymptotically flat spacetimes the asymptotic symmetry is not Poincaré symmetry but rather Bondi-Metzner-Sachs (BMS) symmetry [82, 83], where translational and rotational invariance is promoted to invariance under supertranslations and superrotations [183]. Assuming the generality of the holographic principle, and following the example of the AdS/CFT correspondence, it is reasonable to conclude that a quantum field theory dual to an asymptotically flat gravitational theory must obey BMS symmetry [79, 85].

[1] There is nevertheless progress being made in the field of flatspace holography, see for example [85, 154–182] for an incomplete list.

The properties of the BMS group, and its algebra $\mathfrak{bms}$, depend on the spacetime dimensions. Much like the conformal Virasoro group [28] the incarnation of the BMS group in two spacetime dimensions is particularly constraining due to the infinite number of generators in the corresponding $\mathfrak{bms}_3$ algebra. There are two representation theories of $\mathfrak{bms}_3$ and they are both related to two copies of the conformal Virasoro algebra via separate İnönü-Wigner contractions: the highest-weight representation – also called the Galilean conformal algebra $\mathfrak{gca}_2$ – is a result of non-relativistic contraction while the induced representation is ultra-relativistic [84, 85]. Each representation may be suitable for quantum field theories dual to the appropriate limits of gravitational theories in three spacetime dimensions, for instance Newton-Cartan gravity [184].

In this chapter the highest-weight representation of $\mathfrak{bms}_3$ will be considered; its so-called oscillator construction will be derived and used to calculate $\mathfrak{bms}_3$-blocks [4], i.e. objects equivalent to conformal blocks for BMS-invariant theories. The oscillator formalism has also been applied to the Virasoro algebra in [185–188].

5.1 The $\mathfrak{bms}_3$ Module

The first task is to formally introduce some important aspects of the highest-weight representation of $\mathfrak{bms}_3$, which will also be referred to as the $\mathfrak{bms}_3$ module [84].

In the context of asymptotically flat gravity the generators of $\mathfrak{bms}_3$ are those of superrotations L_m and supertranslations M_n; for $m, n \in \mathbb{Z}$ they satisfy the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_{\text{L}}}{12}m(m^2-1)\delta_{m,-n}, \tag{5.1a}$$

$$[L_m, M_n] = (m-n)M_{m+n} + \frac{c_{\text{M}}}{12}m(m^2-1)\delta_{m,-n}, \tag{5.1b}$$

$$[M_m, M_n] = 0\,, \tag{5.1c}$$

where c_{L} and c_{M} are (non-negative and real) central charges. Superrotations generate diffeomorphisms of S^1 at asymptotic null infinity and supertranslations are angle dependent translations.

A primary state $|\Delta, \xi\rangle$ is defined by the eigenvalue equations

$$L_0|\Delta,\xi\rangle = \Delta|\Delta,\xi\rangle \qquad \text{and} \qquad M_0|\Delta,\xi\rangle = \xi|\Delta,\xi\rangle\,, \tag{5.2}$$

where Δ is the scaling dimension and ξ is the rapidity. A primary state must also satisfy

$$L_n|\Delta,\xi\rangle = 0 \qquad \text{and} \qquad M_n|\Delta,\xi\rangle = 0\,, \tag{5.3}$$

for $n > 0$. A $\mathfrak{bms}_3$ primary operator $\mathcal{O}_{\Delta,\xi}(t,x)$ is defined from the primary state via the operator-state correspondence

$$|\Delta,\xi\rangle \equiv \lim_{t,x\to 0} \mathcal{O}_{\Delta,\xi}(t,x)\,|0\rangle\,, \tag{5.4}$$

where t and x are coordinates on the plane. The vacuum state $|0\rangle$ is a primary state with $\Delta = \xi = 0$ which is annihilated by L_n and M_n with $n \geq -1$. The $\mathfrak{bms}_3$ generators act on $\mathcal{O}_{\Delta,\xi}(t,x)$ as

$$[L_n, \mathcal{O}_{\Delta,\xi}(t,x)] = -\mathcal{L}_n\, \mathcal{O}_{\Delta,\xi}(t,x), \tag{5.5a}$$

$$[M_n, \mathcal{O}_{\Delta,\xi}(t,x)] = -\mathcal{M}_n\, \mathcal{O}_{\Delta,\xi}(t,x), \tag{5.5b}$$

where the differential operators are given by [189]

$$\mathcal{L}_n = -t^{n+1}\partial_t - (n+1)t^n x\partial_x - (n+1)(t^n\Delta + nt^{n-1}x\xi)\,, \tag{5.6a}$$

$$\mathcal{M}_n = -t^{n+1}\partial_x - (n+1)\xi t^n. \tag{5.6b}$$

The operators $\mathcal{L}_n$ and $\mathcal{M}_n$ satisfy the commutation relations (5.1) with vanishing central charges.

The $\mathfrak{bms}_3$ module $\mathcal{B}^{c_L,c_M}_{\Delta,\xi}$ is constructed by acting on the primary state $|\Delta,\xi\rangle$ with an ordered string of the operators L_{-n} and M_{-n} with $n > 0$. The vector space of the highest-weight representation of $\mathfrak{bms}_3$ is thus spanned by the basis vectors

$$|(m_1,\ldots,m_s),(n_1,\ldots,n_l);\Delta,\xi\rangle = L_{-m_1}\cdots L_{-m_s}M_{-n_1}\cdots M_{-n_l}|\Delta,\xi\rangle, \tag{5.7}$$

where $m_1 \geq \ldots \geq m_s \geq 1$ and $n_1 \geq \ldots \geq n_l \geq 1$. Imposing the adjoint relations $L_n^\dagger = L_{-n}$ and $M_n^\dagger = M_{-n}$ uniquely defines the Hermitian product $\langle q|p\rangle$ for states $|p\rangle\,, |q\rangle \in \mathcal{B}^{c_L,c_M}_{\Delta,\xi}$. The Hermitian product of two basis vectors defines an element of the Gram matrix; the general formula is given by

$$\begin{aligned}&\Big\langle (m_i')_{i=1}^{s'},(n_j')_{j=1}^{l'};\Delta,\xi\,\Big|(m_i)_{i=1}^{s},(n_j)_{j=1}^{l};\Delta,\xi\Big\rangle \\ &\qquad = \Big\langle \Delta,\xi\Big|\prod_{j=l'}^{1} M_{n_j'}\prod_{i=s'}^{1} L_{m_i'}\prod_{i=1}^{s} L_{-m_i}\prod_{j=1}^{l} M_{-n_j}\Big|\Delta,\xi\Big\rangle,\end{aligned} \tag{5.8}$$

with the compact notation $(m_i)_{i=1}^s \equiv (m_1,\ldots,m_s)$. The Hermitian product of the highest-weight representation of $\mathfrak{bms}_3$ is in general not positive semi-definite and hence the corresponding representation will generically be non-unitary.

5.2 The Oscillator Construction

The oscillator formalism – or construction – gets its name by associating differential operators to oscillator modes of a quantum field theory, consequently also expressing the generators of the algebra in terms of said differential operators. The oscillator construction of the Virasoro algebra may be found through a linear-dilaton theory [187]. As mentioned at the beginning of this chapter the highest-weight representation of $\mathfrak{bms}_3$ is related to two copies of the Virasoro algebra via a non-relativistic contraction; its oscillator construction will thus be derived by performing a non-relativistic contraction of a generalised, two-dimensional linear-dilaton-like theory [4].

5.2.1 Unravelling the Module

The generalised linear-dilaton-like theory is defined by expressing the components of the two-dimensional conformal stress-tensor $T \equiv T_{zz}$ and $\bar{T} \equiv T_{\bar{z}\bar{z}}$, where the bar denotes quantities belonging to the anti-holomorphic sector of the two-dimensional conformal algebra, as

$$T = \sum_{m=-\infty}^{\infty} L_m^{\mathfrak{vir}} z^{-m+2}, \qquad \text{with} \quad L_m^{\mathfrak{vir}} = \frac{1}{2} \sum_{n=-\infty}^{\infty} {:}\alpha_{m-n}\alpha_n{:} + i(m+1)V\alpha_m \tag{5.9a}$$

and

$$\bar{T} = \sum_{m=-\infty}^{\infty} \bar{L}_m^{\mathfrak{vir}} \bar{z}^{-m+2}, \qquad \text{with} \quad \bar{L}_m^{\mathfrak{vir}} = \frac{1}{2} \sum_{n=-\infty}^{\infty} {:}\bar{\alpha}_{m-n}\bar{\alpha}_n{:} + i(m+1)\bar{V}\bar{\alpha}_m, \tag{5.9b}$$

where $L_m^{\mathfrak{vir}}$, $\bar{L}_m^{\mathfrak{vir}}$ are generators of the Virasoro algebra and satisfy a commutation relation reminiscent of (5.1a); V, $\bar{V}$ are complex constants which do not relate by complex conjugation; and the colons denote normal-ordering. There is no action principle associated to the kind of theory considered here – which is why it is referred to is as 'dilaton-like'. The generators obey the conjugation $\left(L_m^{\mathfrak{vir}}\right)^\dagger = L_{-m}^{\mathfrak{vir}}$ and $\left(\bar{L}_m^{\mathfrak{vir}}\right)^\dagger = \bar{L}_{-m}^{\mathfrak{vir}}$ and the oscillators α_m, $\bar{\alpha}_m$ satisfy the canonical commutation relations

$$[\alpha_m, \alpha_n] = [\bar{\alpha}_m, \bar{\alpha}_n] = m\,\delta_{m+n,0}, \qquad [\alpha_m, \bar{\alpha}_n] = 0. \tag{5.10}$$

Non-relativistic Contraction

The $\mathfrak{bms}_3$ generators will be found by taking the non-relativistic limit on the level of the generators $L_m^{\mathfrak{vir}}$ and $\bar{L}_m^{\mathfrak{vir}}$. The contraction is equivalent to taking the linear combinations [84, 190]

$$L_m \equiv \lim_{\epsilon\to 0}\left(L_m^{\mathfrak{vir}} + \bar{L}_m^{\mathfrak{vir}}\right), \tag{5.11a}$$

$$M_m \equiv \lim_{\epsilon\to 0}\epsilon\left(L_m^{\mathfrak{vir}} - \bar{L}_m^{\mathfrak{vir}}\right). \tag{5.11b}$$

The contractions (5.11) stem from the coordinate rescalings

$$t \mapsto t, \qquad x \mapsto \epsilon x, \tag{5.12}$$

which means that the velocities of the theory tend to zero in the limit $\epsilon \to 0$, in units where $c = 1$ – thus the regime is non-relativistic.

For the case at hand the limiting parameter ϵ is introduced into the generators (5.9) by defining a new pair of oscillator modes as

$$\beta_m = \frac{1}{\sqrt{\epsilon}}\left(\alpha_m - i\bar{\alpha}_m\right), \quad \gamma_m = \sqrt{\epsilon}\left(\alpha_m + i\bar{\alpha}_m\right), \tag{5.13}$$

together with the linear combinations

$$W_{\text{L}} = \frac{1}{2\sqrt{\epsilon}}\left(V - i\bar{V}\right), \quad W_{\text{M}} = \frac{\sqrt{\epsilon}}{2}\left(V + i\bar{V}\right). \tag{5.14}$$

The above oscillators satisfy the commutation relations

$$[\beta_m, \gamma_n] = 2m\delta_{m+n,0}, \qquad [\beta_m, \beta_n] = [\gamma_m, \gamma_n] = 0, \tag{5.15}$$

which follow the commutation relations (5.10). Imposing the adjoint transformation properties $\beta_m^\dagger = \beta_{-m}$ and $\gamma_m^\dagger = \gamma_{-m}$ the Virasoro oscillators must transform like $\alpha_m^\dagger = \alpha_{-m}$ and $\bar{\alpha}_m^\dagger = -\bar{\alpha}_{-m}$, for $m \neq 0$. Preserving the adjoint property of the Virasoro generators then requires $V^* = V$ and $\bar{V}^* = -\bar{V}$, together with $\alpha_0^\dagger = \alpha_0 + 2iV$ and $\bar{\alpha}_0^\dagger = -\bar{\alpha}_0 - 2i\bar{V}$.

Expressing the Virasoro generators $L_m^{\mathfrak{vir}}$ and $\bar{L}_m^{\mathfrak{vir}}$ in terms of the oscillators (5.13) and redefinitions (5.14), and taking the non-relativistic limits (5.11), results in

$$L_m = \frac{1}{4}\sum_{n=-\infty}^{\infty} :\beta_{m-n}\gamma_n + \gamma_{m-n}\beta_n: + i(m+1)\left(W_{\text{L}}\gamma_m + W_{\text{M}}\beta_m\right), \tag{5.16a}$$

$$M_m = \frac{1}{4}\sum_{n=-\infty}^{\infty} :\gamma_{m-n}\gamma_n: + i(m+1)W_{\text{M}}\gamma_m. \tag{5.16b}$$

The generators above satisfy the $\mathfrak{bms}_3$ algebra (5.1) with the central charges $c_{\text{L}} = 2 + 48W_{\text{L}}W_{\text{M}}$ and $c_{\text{M}} = 24W_{\text{M}}^2$. The adjoints $L_m^\dagger = L_{-m}$ and $M_m^\dagger = M_{-m}$ are enforced;

the transformation properties of all quantities follow from the above redefinitions.

The task of assigning differential operators to the oscillator modes is made simpler by expressing the generators L_n and M_n in terms of oscillators which satisfy commutation relations akin to the canonical ones (5.10). To this end another set of oscillators are introduced as

$$a_m = \frac{1}{2}\left(\beta_m + \gamma_m\right), \qquad \hat{a}_m = \frac{1}{2}\left(\beta_m - \gamma_m\right), \tag{5.17}$$

which satisfy the commutation relations

$$[a_m, a_n] = m\delta_{m+n,0}\,, \qquad [\hat{a}_m, \hat{a}_n] = -m\delta_{m+n,0}, \qquad [a_m, \hat{a}_n] = 0\,. \tag{5.18}$$

In terms of the oscillators (5.17) the $\mathfrak{bms}_3$ generators become

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} {:}a_{m-n}a_n - \hat{a}_{m-n}\hat{a}_n{:} + i(m+1)\left[\left(W_{\text{M}} + W_{\text{L}}\right)a_m + \left(W_{\text{M}} - W_{\text{L}}\right)\hat{a}_m\right], \tag{5.19a}$$

$$M_m = \frac{1}{4}\sum_{n=-\infty}^{\infty} {:}a_{m-n}a_n - (a_{m-n}\hat{a}_n + \hat{a}_{m-n}a_n) + \hat{a}_{m-n}\hat{a}_n{:} + i(m+1)W_{\text{M}}(a_m - \hat{a}_m). \tag{5.19b}$$

The adjoint properties $L_m^\dagger = L_{-m}$ and $M_m^\dagger = M_{-m}$ are satisfied if $a_m^\dagger = a_{-m}$ and $a_m^\dagger = \hat{a}_{-m}$ for $m \neq 0$, and

$$a_0^\dagger = a_0 + 2i(W_{\text{M}} + W_{\text{L}}), \qquad \hat{a}_0^\dagger = \hat{a}_0 - 2i(W_{\text{M}} - W_{\text{L}}). \tag{5.20}$$

The oscillators a_m and $\hat{a}_m$ in general have complex eigenvalues – the real parts of their eigenvalues are respectively parametrised by λ_1 and λ_2 while the imaginary parts are fixed by the adjoints (5.20). Defining $\mu_1 = -(W_{\text{M}} + W_{\text{L}})/\sqrt{2}$ and $\mu_2 = (W_{\text{M}} - W_{\text{L}})/\sqrt{2}$ the eigenvalues take the form

$$a_0 \equiv \sqrt{2}\lambda_1 + i\sqrt{2}\mu_1\,, \qquad \hat{a}_0 \equiv \sqrt{2}\lambda_2 + i\sqrt{2}\mu_2, \tag{5.21}$$

where factors of $\sqrt{2}$ are a normalisation choice.

Differential Operators and Generators

Differential operators may now be assigned to the oscillator modes (5.17) and thus completing the oscillator formalism expressions for the $\mathfrak{bms}_3$-generators. The following assignments are employed

$$a_m = \frac{i}{2\sqrt{2}}\left(\partial_{v_m^{(1)}} + \partial_{v_m^{(2)}}\right), \qquad a_{-m} = -im\sqrt{2}\left(v_m^{(1)} + v_m^{(2)}\right), \tag{5.22a}$$

$$\hat{a}_m = \frac{i}{2\sqrt{2}}\left(\partial_{v_m^{(1)}} - \partial_{v_m^{(2)}}\right), \qquad \hat{a}_{-m} = im\sqrt{2}\left(v_m^{(1)} - v_m^{(2)}\right), \tag{5.22b}$$

where $v_m^{(1)}$ and $v_m^{(2)}$, with $m \in \mathbb{N}$, are complex variables which will be referred to as oscillator variables [4, 185–188].

Plugging the definitions (5.22) into the generators (5.19), accounting for the normal ordering and using (5.21) for the terms with $n = 0$, the oscillator construction expressions for generators (which are denoted in lowercase) read

$$l_0 = \Delta + \sum_{n=1}^{\infty} n\left(v_n^{(1)}\partial_{v_n^{(1)}} + v_n^{(2)}\partial_{v_n^{(2)}}\right), \tag{5.23a}$$

$$l_k = \sum_{n=1}^{\infty} n\left(v_n^{(1)}\partial_{v_{k+n}^{(1)}} + v_n^{(2)}\partial_{v_{k+n}^{(2)}}\right) - \frac{1}{4}\sum_{n=1}^{k-1}\partial_{v_n^{(1)}}\partial_{v_{k-n}^{(2)}} + A_k\partial_{v_k^{(1)}} + B_k\partial_{v_k^{(2)}}, \tag{5.23b}$$

$$l_{-k} = \sum_{n=1}^{\infty}(k+n)\left(v_{k+n}^{(1)}\partial_{v_n^{(1)}} + v_{k+n}^{(2)}\partial_{v_n^{(2)}}\right) - 4\sum_{n=1}^{k-1} n(k-n)v_n^{(1)}v_{k-n}^{(2)} + 4k\hat{B}_k v_k^{(1)} + 4k\hat{A}_k v_k^{(2)}, \tag{5.23c}$$

as well as,

$$m_0 = \xi + \sum_{n=1}^{\infty} n v_n^{(1)}\partial_{v_n^{(2)}}, \tag{5.24a}$$

$$m_k = \sum_{n=1}^{\infty} n v_n^{(1)}\partial_{v_{k+n}^{(2)}} - \frac{1}{8}\sum_{n=1}^{k-1}\partial_{v_{k-n}^{(2)}}\partial_{v_n^{(2)}} + A_k\partial_{v_k^{(2)}}, \tag{5.24b}$$

$$m_{-k} = \sum_{n=1}^{\infty}(k+n)v_{k+n}^{(1)}\partial_{v_n^{(2)}} - 2\sum_{n=1}^{k-1} n(k-n)v_{k-n}^{(1)}v_n^{(1)} + 4k\hat{A}_k v_k^{(1)}, \tag{5.24c}$$

with $k > 0$. In (5.23a) and (5.24a) the scaling dimension Δ and rapidity ξ have been identified as

$$\Delta \equiv \lambda_1^2 - \lambda_2^2 + \mu_1^2 - \mu_2^2, \tag{5.25a}$$

$$\xi \equiv \frac{1}{2}\left[(\lambda_1 - \lambda_2)^2 + (\mu_1 - \mu_2)^2\right], \tag{5.25b}$$

which have been used together with the values of the central charges $c_{\text{L}} = 2 + 24\left(\mu_1^2 - \mu_2^2\right)$

and $c_{\mathrm{M}} = 12\left(\mu_1 - \mu_2\right)^2$ to express the coefficients

$$A_k = -\frac{i}{2}\sqrt{2\xi - \frac{c_{\mathrm{M}}}{12}} - k\sqrt{\frac{c_{\mathrm{M}}}{48}}, \qquad B_k = i\frac{c_{\mathrm{L}} - 2 - 24\Delta}{48\sqrt{2\xi - \frac{c_{\mathrm{M}}}{12}}} - k\frac{c_{\mathrm{L}} - 2}{48\sqrt{\frac{c_{\mathrm{M}}}{12}}}, \tag{5.26a}$$

$$\hat{A}_k = \frac{i}{2}\sqrt{2\xi - \frac{c_{\mathrm{M}}}{12}} - k\sqrt{\frac{c_{\mathrm{M}}}{48}}, \qquad \hat{B}_k = -i\frac{c_{\mathrm{L}} - 2 - 24\Delta}{48\sqrt{2\xi - \frac{c_{\mathrm{M}}}{12}}} - k\frac{c_{\mathrm{L}} - 2}{48\sqrt{\frac{c_{\mathrm{M}}}{12}}}. \tag{5.26b}$$

The above coefficients are related by complex conjugation, $\hat{A}_k = A_k^*$ and $\hat{B}_k = B_k^*$, if $\xi \geq c_{\mathrm{M}}/24$ and for which case $l_n^\dagger = l_{-n}$ and $m_n^\dagger = m_{-n}$. However, if $\xi < c_{\mathrm{M}}/24$ the coefficients (5.26) are real and hence independent; preserving the desired adjoint property requires an analytic continuation which will be discussed in section 5.3.2. It is assumed that $\xi \geq c_{\mathrm{M}}/24$ unless otherwise stated.

States, Hermitian Product and Gram Matrix

In the oscillator formalism the role of a state is played by functions depending on the full, infinite set of oscillator variables. Equivalence between the $\mathfrak{bms}_3$ module $\mathcal{B}_{\Delta,\xi}^{c_{\mathrm{L}},c_{\mathrm{M}}}$, presented in section 5.1, and the oscillator construction of the highest-weight representation of $\mathfrak{bms}_3$ follows from the definition

$$f_p(v) \equiv \langle v|p\rangle\ , \tag{5.27a}$$

where $|p\rangle \in \mathcal{B}_{\Delta,\xi}^{c_{\mathrm{L}},c_{\mathrm{M}}}$ and $\langle v| \equiv |\bar{v}\rangle^\dagger$ is a generalised coherent state of the $\mathfrak{bms}_3$ module [4], where v denotes the infinite set of all oscillator variables $v_n^{(1)}$ and $v_n^{(2)}$, i.e. $|v\rangle \equiv |v^{(1)}, v^{(2)}\rangle$ and the dropped indices indicate a full set. Formally the oscillator generators act as $l_n f_p(v) = \langle v|L_n|p\rangle$, and similarly for m_n. There is also a dual function defined by

$$\overline{f_q(v)} = \overline{\langle v|q\rangle} \equiv \langle q|\bar{v}\rangle \tag{5.27b}$$

with $|q\rangle \in \mathcal{B}_{\Delta,\xi}^{c_{\mathrm{L}},c_{\mathrm{M}}}$. The overline operation maps $v_n^{(i)} \mapsto \bar{v}_n^{(i)}$ and acts as complex conjugation. The dual functions are acted on by barred generators $\bar{l}_{-n}$ and $\bar{m}_{-n}$ which are found by acting with an overline operation on the generators l_{-n} and m_{-n}. In terms of states $\bar{l}_{-n}\overline{f_q(v)} = \langle q|L_n|\bar{v}\rangle$ and similarly for $\bar{m}_{-n}$.

The non-constant terms of l_0 and m_0 depend on derivatives with respect to the oscillator variables, hence the requirements of the primary state $|\Delta, \xi\rangle$ are satisfied by a constant function which is chosen as

$$f_{\Delta,\xi}(v) = \langle v|\Delta, \xi\rangle \equiv \mathbf{1}. \tag{5.28}$$

In the oscillator formalism the primary state properties (5.2) and (5.3) thus read

$$l_0 \cdot \mathbf{1} = \Delta\,, \qquad m_0 \cdot \mathbf{1} = \xi\,, \tag{5.29a}$$

$$l_k \cdot \mathbf{1} = 0\,, \qquad m_k \cdot \mathbf{1} = 0\,, \tag{5.29b}$$

for $k > 0$. The basis vectors defined in equation (5.7) are thus polynomials given by $\prod_{i=1}^{s} l_{-m_i} \prod_{j=1}^{l} m_{-n_j} \cdot \mathbf{1}$.

The Hermitian product of the oscillator construction is uniquely defined by the adjoints $l_n^\dagger = l_{-n}$ and $m_n^\dagger = m_{-n}$. The appropriate expression, in terms of functions defined in equations (5.27), is found by inserting the completeness relation of the form

$$\int_{\mathbb{C}^\infty} [\mathrm{d}^2 v]_{\Delta,\xi}\, |\bar{v}\rangle \langle v| = \mathbb{1} \tag{5.30}$$

into the Hermitian product $\langle q|p\rangle$ discussed below equation (5.7). The resulting Hermitian product for the oscillator construction of the highest-weight representation of $\mathfrak{bms}_3$ thus reads

$$(f_q, g_p) = \int_{\mathbb{C}^\infty} [\mathrm{d}^2 v]_{\Delta,\xi}\, \overline{f_q(v)} g_p(v), \tag{5.31}$$

where the measure is given by

$$[\mathrm{d}^2 v]_{\Delta,\xi} = \prod_{n=1}^{\infty} 16n^2 \exp\left[-4n\left(v_n^{(1)}\bar{v}_n^{(2)} + v_n^{(2)}\bar{v}_n^{(1)}\right)\right] \mathrm{d}^2 v_n^{(1)}\, \mathrm{d}^2 v_n^{(2)}, \tag{5.32}$$

with $\mathrm{d}^2 v_n^{(i)} = \mathrm{d}v_n^{(i)} \mathrm{d}\bar{v}_n^{(i)}$. The form of the measure $[\mathrm{d}^2 v]_{\Delta,\xi}$ is argued for in appendix A of [4].

The Hermitian product of oscillator monomials satisfies the orthogonality-like relation

$$\left(\left(v_m^{(1)}\right)^a \left(v_m^{(2)}\right)^b, \left(v_m^{(1)}\right)^c \left(v_m^{(2)}\right)^d\right) = \frac{a!\,b!}{(4m)^{a+b}} \delta_{a,d}\delta_{b,c}. \tag{5.33}$$

In the oscillator formalism the general Gram-matrix element (5.8) is given by

$$\left(l_{-m'_1} \cdots l_{-m'_{s'}} m_{-n'_1} \cdots m_{-n'_{l'}} \cdot \mathbf{1}\,,\, l_{-m_1} \cdots l_{-m_s} m_{-n_1} \cdots m_{-n_l} \cdot \mathbf{1}\right). \tag{5.34}$$

Since the basis vectors are polynomials in $v_n^{(1)}$ and $v_n^{(2)}$ the orthogonality relation (5.33) may be applied to their constituent monomials. The entries of of the lowest-level $\mathfrak{bms}_3$ Gram matrix read

$$\begin{aligned} (l_{-1} \cdot \mathbf{1}, l_{-1} \cdot \mathbf{1}) &= 2\Delta\,, & (l_{-1} \cdot \mathbf{1}, m_{-1} \cdot \mathbf{1}) &= 2\xi\,, \\ (m_{-1} \cdot \mathbf{1}, l_{-1} \cdot \mathbf{1}) &= 2\xi\,, & (m_{-1} \cdot \mathbf{1}, m_{-1} \cdot \mathbf{1}) &= 0\,. \end{aligned} \tag{5.35}$$

The above matrix components match the results of [191].

The non-unitarity of the highest-weight representation of $\mathfrak{bms}_3$ is manifest from the Gram matrix given by the entries in (5.35); if $\xi \neq 0$ it has one positive and one negative eigenvalue, resulting in an indefinite Hermitian product.[2]

5.2.2 Correlation Functions and $\mathfrak{bms}_3$-blocks

One of the main features of the oscillator formalism is that is provides a unique way of determining correlation functions and, by extension, $\mathfrak{bms}_3$-blocks.

Two- and Three-point Correlation Functions

The oscillator formalism expressions for a two-point correlation function of primary operators is found by inserting the completeness relation (5.30) into its definition, which gives

$$\langle 0|\mathcal{O}_{\Delta_1,\xi_1}(t_1,x_1)\mathcal{O}_{\Delta_2,\xi_2}(t_2,x_2)|0\rangle = \int_{\mathbb{C}^\infty} [\mathrm{d}^2 v]_{\Delta,\xi}\, \chi_{\Delta_1,\xi_1;\Delta,\xi}(t_1,x_1;\bar{v})\psi_{\Delta_2,\xi_2;\Delta,\xi}(t_2,x_2;v); \tag{5.36}$$

and similarly a three-point correlation function takes the form

$$\begin{aligned}\langle 0|&\mathcal{O}_{\Delta_1,\xi_1}(t_1,x_1)\mathcal{O}_{\Delta_2,\xi_2}(t_2,x_2)\mathcal{O}_{\Delta_3,\xi_3}(t_3,x_3)|0\rangle \\ &= \int_{\mathbb{C}^\infty} [\mathrm{d}^2 v]_{\Delta,\xi}\, \chi_{\Delta_{1,2},\xi_{1,2};\Delta,\xi}(t_{1,2},x_{1,2};\bar{v})\psi_{\Delta_3,\xi_3;\Delta,\xi}(t_3,x_3;v), && (5.37a)\\ &= \int_{\mathbb{C}^\infty} [\mathrm{d}^2 v]_{\Delta,\xi}\, \chi_{\Delta_1,\xi_1;\Delta,\xi}(t_1,x_1;\bar{v})\psi_{\Delta_{2,3},\xi_{2,3};\Delta,\xi}(t_{2,3},x_{2,3};v). && (5.37b)\end{aligned}$$

The right-hand sides of the above equations define the wave functions of the oscillator construction; the level-one wave functions are given by

$$\psi_{\Delta_2,\xi_2;\Delta,\xi}(t_2,x_2;v) = \langle v|\mathcal{O}_{\Delta_2,\xi_2}(t_2,x_2)|0\rangle, \tag{5.38a}$$

$$\chi_{\Delta_1,\xi_1;\Delta,\xi}(t_1,x_1;\bar{v}) = \langle 0|\mathcal{O}_{\Delta_1,\xi_1}(t_1,x_1)|\bar{v}\rangle, \tag{5.38b}$$

and the level-two wave functions are

$$\psi_{\Delta_{2,3},\xi_{2,3};\Delta,\xi}(t_2,x_2,t_3,x_3;v) = \langle v|\mathcal{O}_{\Delta_2,\xi_2}(t_2,x_2)\mathcal{O}_{\Delta_3,\xi_3}(t_3,x_3)|0\rangle\,, \tag{5.39a}$$

$$\chi_{\Delta_{1,2},\xi_{1,2};\Delta,\xi}(t_1,x_1,t_2,x_2;\bar{v}) = \langle 0|\mathcal{O}_{\Delta_2,\xi_2}(t_2,x_2)\mathcal{O}_{\Delta_1,\xi_1}(t_1,x_1)|\bar{v}\rangle\,. \tag{5.39b}$$

[2] An exception arises for the case $c_{\mathrm{M}} = 0$ and $\xi = 0$; then the $\mathfrak{bms}_3$ representation considered here reduces to a Virasoro highest-weight representation with central charge c_{L} and conformal dimension $h = \Delta$, provided that one takes a quotient with respect to the null states $M_{-n}|\Delta,0\rangle$ with $n \in \mathbb{N}$ [84].

$\psi_{\Delta_{i,j},\xi_{i,j};\Delta,\xi}$ will be referred to as the wave function and $\chi_{\Delta_{i,j},\xi_{i,j};\Delta,\xi}$ as the dual wave function. The subscripts Δ_i and ξ_i label the scaling dimension and rapidity of the external operators, while Δ and ξ label the $\mathfrak{bms}_3$ module $\mathcal{B}^{c_\mathrm{L},c_\mathrm{M}}_{\Delta,\xi}$.

Using that $L_n\,|0\rangle = M_n\,|0\rangle = 0$ for $n \geq -1$ it follows that $\langle v|\mathcal{O}_{\Delta_2,\xi_2}L_n|0\rangle = \langle v|\mathcal{O}_{\Delta_2,\xi_2}M_n|0\rangle = 0$; expressing the product inside the braket in terms of a commutator and using the definitions of the differential operators (5.5) two sets of differential equations for the wave function $\psi_{\Delta_2,\xi_2;\Delta,\xi}(t_2,x_2;v)$ appear,

$$\left(l_n^{(\Delta,\xi)} + \mathcal{L}_n^{(\Delta_2,\xi_2)}\right)\psi_{\Delta_2,\xi_2;\Delta,\xi}(t_2,x_2;v) = 0, \tag{5.40a}$$

$$\left(m_n^{(\Delta,\xi)} + \mathcal{M}_n^{(\Delta_2,\xi_2)}\right)\psi_{\Delta_2,\xi_2;\Delta,\xi}(t_2,x_2;v) = 0, \tag{5.40b}$$

for $n \geq -1$. By similar arguments the dual wave function $\chi_{\Delta_1,\xi_1;\Delta,\xi}(t_1,x_1;\bar{v})$ must satisfy the following set of differential equations,

$$\left(\bar{l}_n^{(\Delta,\xi)} - \mathcal{L}_{-n}^{(\Delta_1,\xi_1)}\right)\chi_{\Delta_1,\xi_1;\Delta,\xi}(t_1,x_1;\bar{v}) = 0, \tag{5.41a}$$

$$\left(\bar{m}_n^{(\Delta,\xi)} - \mathcal{M}_{-n}^{(\Delta_1,\xi_1)}\right)\chi_{\Delta_1,\xi_1;\Delta,\xi}(t_1,x_1;\bar{v}) = 0, \tag{5.41b}$$

for $n \geq -1$. The wave functions connect oscillator variables and coordinates of the complex plane and hence the superscripts emphasise the domains of action of the generators. Solving both sets of differential equations for $n \in \{-1,0,1,2\}$ fixes the wave functions also for all $n > 2$; this is related to the fact that two-point functions are completely determined by the globally well-defined generators L_n and M_n with $n \in \{-1,0,1\}$, while satisfying the $n = 2$ differential equations guarantees a solution for all n.

The differential equations (5.40) and (5.41) have non-trivial solutions only if $\Delta_2 = \Delta$, $\xi_2 = \xi$, and $\Delta_1 = \Delta$, $\xi_1 = \xi$. To keep the notation compact degenerate subscripts of the wave functions will be dropped. The level-one $\mathfrak{bms}_3$ wave functions read [4]

$$\psi_{\Delta,\xi}(t_2,x_2;v) = \exp\left[4\hat{A}_1\sum_{n=1}^{\infty}\left(t_2^n v_n^{(2)} + n\,x_2\,t_2^{n-1}v_n^{(1)}\right) + 4\hat{B}_1\sum_{n=1}^{\infty}t_2^n v_n^{(1)}\right], \tag{5.42a}$$

$$\chi_{\Delta,\xi}(t_1,x_1;\bar{v}) = t_1^{-2\Delta}\,\mathrm{e}^{-2\xi\frac{x_1}{t_1}}\exp\left[4A_1\sum_{n=1}^{\infty}\left(t_1^{-n}\bar{v}_n^{(2)} - n\,x_1\,t_1^{-n-1}\bar{v}_n^{(1)}\right) + 4B_1\sum_{n=1}^{\infty}t_1^{-n}\bar{v}_n^{(1)}\right], \tag{5.42b}$$

where the coefficients A_1, B_1 and $\hat{A}_1$, $\hat{B}_1$ are given by equations (5.26a) and (5.26b) for $k = 1$. From the solutions (5.42) the relationship between the wave functions can be determined to be

$$\chi_{\Delta,\xi}(t_1,x_1;\bar{v}) = t_1^{-2\Delta}\,\mathrm{e}^{-2\xi\frac{x_1}{t_1}}\,\overline{\psi_{\Delta,\xi}\left(t_1^{-1},-x_1t_1^{-2};v\right)}, \tag{5.43}$$

which also may be motivated by the operator-state correspondence for bra-states

$$\langle\Delta,\xi| = \lim_{t\to\infty} t^{2\Delta}\,\mathrm{e}^{2\xi\frac{x}{t}}\,\langle 0|\,\mathcal{O}_{\Delta,\xi}(t,x)\,. \tag{5.44}$$

The $\mathfrak{bms}_3$ two-point correlation function is found by plugging the wave functions (5.42) into (5.36), power expanding, and using the orthogonality relation (5.33) on the resulting monomials; if $\Delta_1=\Delta_2=\Delta$ and $\xi_1=\xi_2=\xi$ the expression reads

$$\langle 0|\,\mathcal{O}_{\Delta_1,\xi_1}(t_1,x_1)\mathcal{O}_{\Delta_2,\xi_2}(t_2,x_2)\,|0\rangle = (t_1-t_2)^{-2\Delta}e^{-\frac{2\xi(x_1-x_2)}{(t_1-t_2)}}\,, \tag{5.45}$$

and zero otherwise. The above result is in agreement with [189].

The reasoning that lead to the differential equations for the level-one wave functions may be analogously applied to the second-level wave functions. The level-two wave function is thus constrained by the equations

$$\left(l_n^{(\Delta,\xi)} + \mathcal{L}_n^{(\Delta_1,\xi_1)} + \mathcal{L}_n^{(\Delta_2,\xi_2)}\right)\psi_{\Delta_{1,2},\xi_{1,2};\Delta,\xi}(t_1,x_1,t_2,x_2;v) = 0, \tag{5.46a}$$

$$\left(m_n^{(\Delta,\xi)} + \mathcal{M}_n^{(\Delta_1,\xi_1)} + \mathcal{M}_n^{(\Delta_2,\xi_2)}\right)\psi_{\Delta_{1,2},\xi_{1,2};\Delta,\xi}(t_1,x_1,t_2,x_2;v) = 0, \tag{5.46b}$$

and similarly its dual wave function must obey

$$\left(\bar{l}_n^{(\Delta,\xi)} - \mathcal{L}_{-n}^{(\Delta_3,\xi_3)} - \mathcal{L}_{-n}^{(\Delta_4,\xi_4)}\right)\chi_{\Delta_{3,4},\xi_{3,4};\Delta,\xi}(t_3,x_3,t_4,x_4;\bar{v}) = 0, \tag{5.47a}$$

$$\left(\bar{m}_n^{(\Delta,\xi)} - \mathcal{M}_{-n}^{(\Delta_3,\xi_3)} - \mathcal{M}_{-n}^{(\Delta_4,\xi_4)}\right)\chi_{\Delta_{3,4},\xi_{3,4};\Delta,\xi}(t_3,x_3,t_4,x_4;\bar{v}) = 0, \tag{5.47b}$$

for $n\geq -1$. As a consequence of the transformation between bra and ket-states (5.44) the level-two wave functions are related by

$$\begin{aligned}\chi_{\Delta_{3,4},\xi_{3,4};\Delta,\xi}&(t_3,x_3,t_4,x_4;\bar{v})\\ &= t_3^{-2\Delta_3}\,\mathrm{e}^{-2\xi_3\frac{x_3}{t_3}}\,t_4^{-2\Delta_4}\,\mathrm{e}^{-2\xi_4\frac{x_4}{t_4}}\,\overline{\psi_{\Delta_{3,4},\xi_{3,4};\Delta,\xi}(t_3^{-1},-x_3t_3^{-2},t_4^{-1},-x_4t_4^{-2};v)}.\end{aligned} \tag{5.48}$$

No closed-form solutions for the level-two wave functions have been found. However, it is possible to employ the semiclassical limit in order to find approximate solutions for the second-level wave functions; this point is returned to in section 5.3.

$\mathfrak{bms}_3$-blocks

Four-point correlation functions allow for the definition of a $\mathfrak{bms}_3$-block $\mathcal{B}_{\Delta_{\text{tot}},\xi_{\text{tot}};\Delta,\xi}(t,x)$ by

$$\mathcal{B}_{\Delta_{\text{tot}},\xi_{\text{tot}};\Delta,\xi}(t,x) = \langle 0|\mathcal{O}_{\Delta_4,\xi_4}(t_4,x_4)\mathcal{O}_{\Delta_3,\xi_3}(t_3,x_3)\mathcal{P}_{\Delta,\xi}\mathcal{O}_{\Delta_1,\xi_1}(t_1,x_1)\mathcal{O}_{\Delta_2,\xi_2}(t_2,x_2)|0\rangle, \tag{5.49}$$

where $\mathcal{P}_{\Delta,\xi}$ is the projector onto the $\mathfrak{bms}_3$ module $\mathfrak{B}^{c_L,c_M}_{\Delta,\xi}$, which acts as the unit operator when restricted to the module; and $\Delta_{\text{tot}} = \{\Delta_1,\Delta_2,\Delta_3,\Delta_4\}$ and $\xi_{\text{tot}} = \{\xi_1,\xi_2,\xi_3,\xi_4\}$.

Inserting a completeness relation allows the $\mathfrak{bms}_3$-block to be expressed in terms of second-level wave functions; without loss of generality the point configuration is restricted to

$$\{(t_i,x_i)\} = \{(t,x),(0,0),(1,0),(\infty,0)\}, \tag{5.50}$$

such that the $\mathfrak{bms}_3$-block $\mathcal{B}_{\Delta_{\text{tot}},\xi_{\text{tot}};\Delta,\xi}(t,x)$ is given by

$$\begin{aligned}\mathcal{B}&_{\Delta_{\text{tot}},\xi_{\text{tot}};\Delta,\xi}(t,x)\\ &= \lim_{\substack{t_4\to\infty\\ x_4\to 0}} t_4^{2\Delta_4} e^{2\xi_4\frac{x_4}{t_4}} \int_{\mathbb{C}^\infty} [\mathrm{d}^2 v]_{\Delta,\xi}\, \chi_{\Delta_{3,4},\xi_{3,4};\Delta,\xi}(1,0,t_4,x_4;\bar{v})\psi_{\Delta_{1,2},\xi_{1,2};\Delta,\xi}(t,x,0,0;v)\,,\end{aligned} \tag{5.51}$$

where the prefactor appears due to (5.44). Furthermore, the point configuration (5.50) simplifies the relationship between the second-level wave function and its dual; it takes the form

$$\chi_{\Delta_{3,4},\xi_{3,4};\Delta,\xi}(1,0,t_4,x_4;\bar{v}) = t_4^{-2\Delta_4}\,\mathrm{e}^{-2\xi_4\frac{x_4}{t_4}}\,\overline{\psi_{\Delta_{3,4},\xi_{3,4};\Delta,\xi}(1,0,t_4^{-1},-x_4t_4^{-2};v)}. \tag{5.52}$$

Plugging in the above relation into equation (5.51) and implementing the limits results in the compact formula

$$\mathcal{B}_{\Delta_{\text{tot}},\xi_{\text{tot}};\Delta,\xi}(t,x) = \int_{\mathbb{C}^\infty} [\mathrm{d}^2 v]_{\Delta,\xi}\,\overline{\psi_{\Delta_{3,4},\xi_{3,4};\Delta,\xi}(1,0,0,0;v)}\,\psi_{\Delta_{1,2},\xi_{1,2};\Delta,\xi}(t,x,0,0;v). \tag{5.53}$$

5.3 Semiclassical $\mathfrak{bms}_3$-blocks

The remainder of this chapter will be dedicated to $\mathfrak{bms}_3$-blocks. From the field theory perspective $\mathfrak{bms}$-blocks are of relevance in the bootstrap approach; while in the flatspace holography context the so called heavy-light $\mathfrak{bms}_3$-blocks contain information about probe fields in dual, non-trivial asymptotically flat spacetimes.

Calculating $\mathfrak{bms}_3$-blocks in full generality is a difficult task; hence some simplifying conditions must be implemented. One of the simplifications is the limit given by $c_{\rm M} \to \infty$, with $\Delta/c_{\rm M}$, $\Delta_i/c_{\rm M}$; $\xi/c_{\rm M}$, $\xi_i/c_{\rm M}$, and $c_{\rm L}/c_{\rm M}$ kept fixed; where $\Delta, \xi \in \mathcal{B}^{c_{\rm L},c_{\rm M}}_{\Delta,\xi}$ are the scaling dimension and rapidity of the internal primary operator of each block, while Δ_i, ξ_i with $i = 1, .., 4$ denote the scaling dimensions and rapidities of the external operators. In the context of flatspace holography the central charge $c_{\rm M}$ is dual to the inverse of Newton's constant [192]; hence the limit considered here corresponds to a semiclassical limit of an infinitesimal Newton constant. The semiclassical limit will be implemented by introducing the auxiliary parameter μ as follows,

$$c_{\rm M} = \mu^2 \tilde{c}_{\rm M}\,, \quad c_{\rm L} = \mu^2 \tilde{c}_{\rm L}\,, \quad \Delta = \mu^2 \tilde{\Delta}\,, \quad \Delta_i = \mu^2 \tilde{\Delta}_i\,, \quad \xi = \mu^2 \tilde{\xi}\,, \quad \xi_i = \mu^2 \tilde{\xi}_i\,; \tag{5.54}$$

hence the the semiclassical limit corresponds to $\mu \to \infty$ while keeping the tilde-quantities fixed. Due to its relation to Newton's constant $c_{\rm M}$ is dimensionful; when appropriate the dimensionality will be assigned to the tilde-quantities while μ is dimensionless. Dimensionful quantities will always appear in dimensionless products in physical objects.

5.3.1 Wave Functions and Exponentiation in semiclassical Limit

Determining the $\mathfrak{bms}_3$-blocks requires information about the wave functions; however, as mentioned below equation (5.48) no closed form solutions have been found for them. Nevertheless, in the point configuration (5.50) the set of equations (5.46) with $n = 0$ fixes the wave function to be of the form

$$\psi_{\Delta_{1,2},\xi_{1,2};\Delta,\xi}(t, x, 0, 0; v) = t^{\Delta-\Delta_1-\Delta_2}\, \mathrm{e}^{\frac{x}{t}(\xi-\xi_1-\xi_2)}\, F(\eta, \nu), \tag{5.55}$$

where $F(\eta, \nu)$ is an unknown functionw which depend on the combination variables

$$\eta_n = t^n v^{(1)}_n, \qquad \nu_n = n t^{n-1} x v^{(1)}_n + t^n v^{(2)}_n, \tag{5.56}$$

where $n \in \mathbb{N}$.

As part of the semiclassical limit a saddle-point approximation may be implemented to evaluate the integral in (5.53). To this end the exponential function in the measure (5.32) motivates the re-scaling of the oscillator variables

$$v^{(i)}_n \mapsto \mu v^{(i)}_n, \qquad \bar{v}^{(i)}_n \mapsto \mu \bar{v}^{(i)}_n, \tag{5.57}$$

and as a consequence of this new variables σ_n and κ_n are defined by

$$\eta_n = \mu \sigma_n, \qquad \nu_n = \mu \kappa_n. \tag{5.58}$$

Also, an exponential ansatz for the undetermined function in the wave function (5.55) may be made as

$$F(\sigma,\kappa) = \exp\left[\mu^2 S(\sigma,\kappa)\right] . \tag{5.59}$$

Differential Equations

The function $S(\sigma,\kappa)$ is uniquely[3] determined by the differential equations (5.46) with $n \geq 1$.[4] Using the expressions for l_k given in (5.23b), m_k given in (5.24b), and $\mathcal{L}_k$ and $\mathcal{M}_k$ given in (5.6), the leading terms of the differential equations in the limit $\mu \to \infty$ read

$$\begin{aligned} 0 = &\sum_{n=1}^{\infty} n\left(\sigma_n\left(\partial_{\sigma_{k+n}} S - \partial_{\sigma_n} S\right) + \kappa_n\left(\partial_{\kappa_{k+n}} S - \partial_{\kappa_n} S\right)\right) \\ &- \frac{1}{4}\sum_{n=1}^{k-1} \partial_{\sigma_n} S\, \partial_{\kappa_{k-n}} S + \tilde{A}_k \partial_{\sigma_k} S + \tilde{B}_k \partial_{\kappa_k} S - \left(\tilde{\Delta} + k\tilde{\Delta}_1 - \tilde{\Delta}_2\right) , \end{aligned} \tag{5.60a}$$

$$0 = \sum_{n=1}^{\infty} n\sigma_n\left(\partial_{\kappa_{k+n}} S - \partial_{\kappa_n} S\right) - \frac{1}{8}\sum_{n=1}^{k-1} \partial_{\kappa_{k-n}} S\, \partial_{\kappa_n} S + \tilde{A}_k \partial_{\kappa_k} S - \left(\tilde{\xi} + k\tilde{\xi}_1 - \tilde{\xi}_2\right) , \tag{5.60b}$$

with the definitions

$$A_k = \mu\left(-\frac{i}{2}\sqrt{2\tilde{\xi} - \frac{\tilde{c}_{\mathrm{M}}}{12}} - k\sqrt{\frac{\tilde{c}_{\mathrm{M}}}{48}}\right) \equiv \mu \cdot \tilde{A}_k, \tag{5.61a}$$

$$B_k = \mu\left(i\frac{\tilde{c}_{\mathrm{L}} - 24\tilde{\Delta}}{48\sqrt{2\xi - \frac{\tilde{c}_{\mathrm{M}}}{12}}} - k\frac{\tilde{c}_{\mathrm{L}}}{48\sqrt{\frac{\tilde{c}_{\mathrm{M}}}{12}}}\right) \equiv \mu \cdot \tilde{B}_k. \tag{5.61b}$$

for $k \geq 1$.

Proof of Exponentiation

Plugging the ansatz (5.59) into the $n = 0$ solution of the wave function (5.55), and using relation (5.48), results in exponential wave functions. Expressing the variables in terms t, x and $v_n^{(i)}$, $\bar{v}_n^{(i)}$ the expression for the $\mathfrak{bms}_3$-block given in (5.53) reads

$$\begin{aligned} &\mathcal{B}_{\Delta_{\mathrm{tot}},\xi_{\mathrm{tot}};\Delta,\xi}(t,x) \\ &\sim t^{\Delta-\Delta_1-\Delta_2} e^{\frac{x}{t}(\xi-\xi_1-\xi_2)} \int_{\mathbb{C}^\infty} \left(\prod_{n=1}^{\infty} 16n^2\mu^4\, \mathrm{d}^2 v_n^{(1)} \mathrm{d}^2 v_n^{(2)}\right) \exp\left[\mu^2 \mathcal{I}(t,x;v,\bar{v})\right] , \end{aligned} \tag{5.62}$$

[3]See appendix B in [4] for the proof of uniqueness for the functions appearing in the ansatz (5.59).
[4]The $n = -1$ equation may be used to reinstate the dependence on the second set of variables.

with the definitions

$$\mathcal{I}(t,x;v,\bar{v}) = -4\sum_{n=1}^{\infty} n\left(v_n^{(1)}\bar{v}_n^{(2)} + v_n^{(2)}\bar{v}_n^{(1)}\right) + S(t,x;v) + \overline{S(v)}\,. \tag{5.63}$$

That the overall factor of μ^2 in the exponential function appears as a consequence of the rescaling (5.57).

The semiclassical limit is implemented by taking $\mu \to \infty$; in this regime the exponent of the exponential function in (5.62) is large and the integral is dominated by the stationary points of $\mathcal{I}(t,x;v,\bar{v})$ – it is then possible to use the saddle-point approximation in order to evaluate the integral. The stationary points $(w,\bar{w})$ are found by extremising $\mathcal{I}(t,x;v,\bar{v})$, they satisfy[5]

$$w_m^{(1)} = \frac{1}{4m}\frac{\partial \bar{S}}{\partial \bar{v}_m^{(2)}}\bigg|_{\bar{v}^{(i)}=\bar{w}^{(i)}}, \qquad w_m^{(2)} = \frac{1}{4m}\frac{\partial \bar{S}}{\partial \bar{v}_m^{(1)}}\bigg|_{\bar{v}^{(i)}=\bar{w}^{(i)}}, \tag{5.64a}$$

$$\bar{w}_m^{(1)} = \frac{1}{4m}\frac{\partial S}{\partial v_m^{(2)}}\bigg|_{v^{(i)}=w^{(i)}}, \qquad \bar{w}_m^{(2)} = \frac{1}{4m}\frac{\partial S}{\partial v_m^{(1)}}\bigg|_{v^{(i)}=w^{(i)}}. \tag{5.64b}$$

Plugging the stationary points (5.64) into the integrand of equation (5.62) results in the saddle-point approximation of a $\mathfrak{bms}_3$-block

$$\mathcal{B}_{\Delta_{\text{tot}},\xi_{\text{tot}};\Delta,\xi}(t,x) \approx t^{\Delta-\Delta_1-\Delta_2}\exp\left[\frac{x}{t}\left(\xi-\xi_1-\xi_2\right) + \mu^2\mathcal{I}(t,x;w,\bar{w})\right], \tag{5.65}$$

where the overall factor $16n^2\mu^4$ in (5.62) cancels with the determinant of the Hessian evaluated at the stationary point. The factor $t^{\Delta-\Delta_1-\Delta_2}$ may be expressed as an exponential of a logarithm and thus the $\mathfrak{bms}_3$-block takes a unique exponential form in the semiclassical limit [4].

5.3.2 Perturbatively Heavy Vacuum $\mathfrak{bms}_3$-block

Two examples of $\mathfrak{bms}_3$-blocks belonging to the vacuum module $\mathcal{B}_{0,0}^{c_L,c_M}$ will be computed in the following, using the oscillator construction presented in this chapter and the saddle-point approximation outlined in section 5.3.1. The perturbatively heavy vacuum $\mathfrak{bms}_3$-block will be the first example; here $\Delta = \xi = 0$ and the external parameters Δ_i and ξ_i are infinitesimal and of the same order ϵ. For simplicity the external operators are pairwise identified such that $\Delta_1 = \Delta_2$, $\xi_1 = \xi_2$ and $\Delta_3 = \Delta_4$, $\xi_3 = \xi_4$. The scaling dimension and rapidity are set to zero by choosing $\tilde{\Delta} = \tilde{\xi} = 0$ before taking $\mu \to \infty$.

[5]It is for simplicity assumed that there exists only a single stationary point $(w,\bar{w})$.

Analytic Behaviour

When $\xi = \Delta = 0$ the coefficients defined in equations (5.26) become purely real due to negative expressions under the square root; choosing the branch $\sqrt{-1} = +i$ the resulting coefficients read

$$A_k = -\sqrt{\frac{c_{\mathrm{M}}}{48}}(k-1) \equiv A_k^+\,, \qquad B_k = -\frac{c_{\mathrm{L}}}{2\sqrt{48 c_{\mathrm{M}}}}(k-1) \equiv B_k^+\,, \tag{5.66a}$$

$$\hat{A}_k = -\sqrt{\frac{c_{\mathrm{M}}}{48}}(k+1) \equiv A_k^-\,, \qquad \hat{B}_k = -\frac{c_{\mathrm{L}}}{2\sqrt{48 c_{\mathrm{M}}}}(k+1) \equiv B_k^-\,. \tag{5.66b}$$

The absence of imaginary terms in the above expressions renders them immune to the complex conjugation defined with the overline operation; consequently the $\mathfrak{bms}_3$ generators l_n and m_n, and by extension the differential equations (5.60), become independent of their barred counterparts.[6] This motivates a new notation for the $\mathfrak{bms}_3$ generators, namely $l_n \equiv l_n^+$, $m_n \equiv m_n^+$ and $\bar{l}_n \equiv l_n^-$, $\bar{m}_n \equiv m_n^-$, where the superscripts correspond to the superscripts of the coefficients (5.66) which appear in their respective expressions.

The ineffectiveness of the overline operation means that the relation between the second-level wave functions, given in equation (5.48), is invalidated – there are instead two independent wave functions which are defined with respect to the notation in equation (5.53) as

$$\psi_{\Delta_1,\xi_1;0,0}(t,x,0,0;v) \equiv \psi^+_{\Delta_1,\xi_1;0,0}(t,x,0,0;v), \tag{5.67a}$$

$$\overline{\psi_{\Delta_3,\xi_3;0,0}(1,0,0,0;v)} \equiv \psi^-_{\Delta_3,\xi_3;0,0}(1,0,0,0;\bar{v}), \tag{5.67b}$$

where the superscripts of the wave functions are the same as the superscripts of the generators which appear in their differential equations. Subsequently, the $n = 0$ equations are independent of the coefficients (5.66) which means that the partial solution (5.55) may be used to immediately deduce the forms of the wave functions

$$\psi^+_{\Delta_1,\xi_1;0,0}(t,x,0,0;v) = t^{-2\Delta_1}\,\mathrm{e}^{-2\xi_1\frac{x}{t}}\,F^+(\eta,\nu), \tag{5.68a}$$

$$\psi^-_{\Delta_3,\xi_3;0,0}(1,0,0,0;\bar{v}) = F^-(\bar{\eta},\bar{\nu}). \tag{5.68b}$$

Following the reasoning of section 5.3.1 exponential ansätze for the wave functions are given as

$$F^+(\sigma,\kappa) = \exp\left[\mu^2 S^+(\sigma,\kappa)\right], \qquad F^-(\bar{\sigma},\bar{\kappa}) = \exp\left[\mu^2 S^-(\bar{\sigma},\bar{\kappa})\right]. \tag{5.69}$$

[6]This situation was alluded to below equations (5.26).

The two functions S^+ and S^- must be determined separately.

Solving the Differential Equations

The differential equations (5.60) are general enough that their expressions may serve as equations for the wave functions (5.67) given the appropriate substitutions. For $\psi^+_{\Delta_1,\xi_1;0,0}(t,x,0,0;v)$ they read

$$\begin{aligned}0 = \sum_{n=1}^{\infty} n\Big(\sigma_n\left(\partial_{\sigma_{k+n}}S^+ - \partial_{\sigma_n}S^+\right) + \kappa_n\left(\partial_{\kappa_{k+n}}S^+ - \partial_{\kappa_n}S^+\right)\Big)\\ - \frac{1}{4}\sum_{n=1}^{k-1}\partial_{\sigma_n}S^+\,\partial_{\kappa_{k-n}}S^+ + \tilde{A}^+_k\partial_{\sigma_k}S^+ + \tilde{B}^+_k\partial_{\kappa_k}S^+ - \tilde{\Delta}_1(k-1),\end{aligned} \tag{5.70a}$$

$$0 = \sum_{n=1}^{\infty} n\sigma_n\left(\partial_{\kappa_{k+n}}S^+ - \partial_{\kappa_n}S^+\right) - \frac{1}{8}\sum_{n=1}^{k-1}\partial_{\kappa_{k-n}}S^+\,\partial_{\kappa_n}S^+ + \tilde{A}^+_k\partial_{\kappa_k}S^+ - \tilde{\xi}_1(k-1); \tag{5.70b}$$

and for $\psi^-_{\Delta_3,\xi_3;\Delta,\xi}(1,0,0,0;\bar{v})$ they read

$$\begin{aligned}0 = \sum_{n=1}^{\infty} n\Big(\bar{\sigma}_n\left(\partial_{\bar{\sigma}_{k+n}}S^- - \partial_{\bar{\sigma}_n}S^-\right) + \bar{\kappa}_n\left(\partial_{\bar{\kappa}_{k+n}}S^- - \partial_{\bar{\kappa}_n}S^-\right)\Big)\\ - \frac{1}{4}\sum_{n=1}^{k-1}\partial_{\bar{\sigma}_n}S^-\,\partial_{\bar{\kappa}_{k-n}}S^- + \tilde{A}^-_k\partial_{\bar{\sigma}_k}S^- + \tilde{B}^-_k\partial_{\bar{\kappa}_k}S^- - \tilde{\Delta}_3(k-1),\end{aligned} \tag{5.71a}$$

$$0 = \sum_{n=1}^{\infty} n\bar{\sigma}_n\left(\partial_{\bar{\kappa}_{k+n}}S^- - \partial_{\bar{\kappa}_n}S^-\right) - \frac{1}{8}\sum_{n=1}^{k-1}\partial_{\bar{\kappa}_{k-n}}S^-\,\partial_{\bar{\kappa}_n}S^- + \tilde{A}^-_k\partial_{\bar{\kappa}_k}S^- - \tilde{\xi}_3(k-1), \tag{5.71b}$$

where the tilde denotes quantities which are kept fixed in the limit $\mu \to \infty$.

The infinitesimal nature of the external scaling dimensions and rapidities allows the equations (5.70) and (5.71) to be simplified. S^+ and S^- may be treaded as expansions in the infinitesimal parameters $\tilde{\Delta}_1$, $\tilde{\xi}_1$ and $\tilde{\Delta}_3$, $\tilde{\xi}_3$, respectively, with leading order ϵ.[7] The leading order of equations (5.70) and (5.71) is ϵ, hence terms quadratic in derivatives of $S^\pm$ are of sub-leading order ϵ^2 and may be dropped. Moreover, the saddle point coordinates (5.64) – using the expansions of $S^\pm$ – have the leading order ϵ; this means that products of oscillator variables with $S^\pm$ are sub-leading when evaluated at the saddle point – for the current purposes such terms may be dropped, too. The

[7] The leading terms of $S^+(\sigma,\kappa)$ and $S^-(\bar{\sigma},\bar{\kappa})$ may be constant functions of order one. Such constant functions only contribute to the $\mathfrak{bms}_3$-block as inconsequential multiplicative factors and may thus be dropped.

equations (5.70) thus reduce to the linear differential equations

$$0 = \tilde{A}_k^+ \partial_{\sigma_k} S^+ + \tilde{B}_k^+ \partial_{\kappa_k} S^+ - \tilde{\Delta}_1(k-1) + \mathcal{O}\big(\epsilon^2\big), \tag{5.72a}$$

$$0 = \tilde{A}_k^+ \partial_{\kappa_k} S^+ - \tilde{\xi}_1(k-1) + \mathcal{O}\big(\epsilon^2\big), \tag{5.72b}$$

while (5.71) become

$$0 = \tilde{A}_k^- \partial_{\bar{\sigma}_k} S^- + \tilde{B}_k^- \partial_{\bar{\kappa}_k} S^- - \tilde{\Delta}_3(k-1) + \mathcal{O}\big(\epsilon^2\big), \tag{5.73a}$$

$$0 = \tilde{A}_k^- \partial_{\bar{\kappa}_k} S^- - \tilde{\xi}_3(k-1) + \mathcal{O}\big(\epsilon^2\big), \tag{5.73b}$$

where the omitted terms are indicated by their order at the stationary point. Plugging the necessary quantities into the differential equations (5.72b) and (5.73b), and integrating, results in

$$S^+(\sigma,\kappa) = -\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\tilde{\xi}_1 \sum_{n=1}^{\infty} \kappa_n + f(\sigma) + \mathcal{O}\big(\epsilon^3\big), \tag{5.74a}$$

$$S^-(\bar{\sigma},\bar{\kappa}) = -\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\tilde{\xi}_3 \sum_{n=1}^{\infty} \frac{n-1}{n+1}\bar{\kappa}_n + g(\bar{\sigma}) + \mathcal{O}\big(\epsilon^3\big), \tag{5.74b}$$

with constants of integration $f(\sigma)$ and $g(\bar{\sigma})$. The functions $f(\sigma)$ and $g(\bar{\sigma})$ are determined by integrating the equations (5.72a) and (5.73a) after the insertion of the above results; this yields

$$f(\sigma) = -\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\left(\tilde{\Delta}_1 - \tilde{\xi}_1 \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}}\right) \sum_{n=1}^{\infty} \sigma_n + \mathcal{O}\big(\epsilon^3\big), \tag{5.75a}$$

$$g(\bar{\sigma}) = -\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\left(\tilde{\Delta}_3 - \tilde{\xi}_3 \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}}\right) \sum_{n=1}^{\infty} \frac{n-1}{n+1}\bar{\sigma}_n + \mathcal{O}\big(\epsilon^3\big), \tag{5.75b}$$

where the integration constants have been set to zero. Using the variable transformations (5.56) and the rescalings (5.57) the leading order solutions for $S^{\pm}$ read

$$S^+(t,x,0,0;v) \approx -\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\left[\left(\tilde{\Delta}_1 - \tilde{\xi}_1 \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}}\right) \sum_{n=1}^{\infty} t^n v_n^{(1)} + \tilde{\xi}_1 \sum_{n=1}^{\infty}\left(n t^{n-1} x v_n^{(1)} + t^n v_n^{(2)}\right)\right], \tag{5.76a}$$

$$S^-(1,0,0,0;\bar{v}) \approx -\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\left[\left(\tilde{\Delta}_3 - \tilde{\xi}_3 \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}}\right) \sum_{n=1}^{\infty} \frac{n-1}{n+1}\bar{v}_n^{(1)} + \tilde{\xi}_3 \sum_{n=1}^{\infty} \frac{n-1}{n+1}\bar{v}_n^{(2)}\right]. \tag{5.76b}$$

The above solutions for S^+ and S^- are of order ϵ^2 when evaluated at the saddle point.

Saddle-point Approximation

The oscillator construction formula for the saddle-point approximation of a $\mathfrak{bms}_3$-block at the stationary point $(w,\bar{w})$ is given by equation (5.65) – it is therefore necessary to determine

$$\mathcal{I}(t,x;w,\bar{w}) = -4\sum_{n=1}^{\infty} n\left(w_n^{(1)}\bar{w}_n^{(2)} + w_n^{(2)}\bar{w}_n^{(1)}\right) + S^+(t,x;w) + S^-(\bar{w}). \tag{5.77}$$

The coordinates of the stationary point may be found by substituting the solutions (5.76) into the formulae (5.64), this yields

$$w_m^{(1)} = \frac{1}{4m}\frac{\partial S^-}{\partial \bar{v}_m^{(2)}} = -\frac{1}{4m}\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}\frac{m-1}{m+1}}\tilde{\xi}_3, \tag{5.78a}$$

$$w_m^{(2)} = \frac{1}{4m}\frac{\partial S^-}{\partial \bar{v}_m^{(1)}} = -\frac{1}{4m}\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}\frac{m-1}{m+1}}\left(\tilde{\Delta}_3 - \tilde{\xi}_3\frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}}\right), \tag{5.78b}$$

$$\bar{w}_m^{(1)} = \frac{1}{4m}\frac{\partial S^+}{\partial v_m^{(2)}} = -\frac{1}{4m}\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\tilde{\xi}_1 t^m, \tag{5.78c}$$

$$\bar{w}_m^{(2)} = \frac{1}{4m}\frac{\partial S^+}{\partial v_m^{(1)}} = -\frac{1}{4m}\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}t^m\left(\tilde{\Delta}_1 - \tilde{\xi}_1\frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}} + \tilde{\xi}_1 m\frac{x}{t}\right). \tag{5.78d}$$

It is sensible to consider the terms of (5.77) separately. Plugging the expressions (5.78) into the sum of oscillator variables results in

$$\begin{aligned}&-4\sum_{n=1}^{\infty} n\left(w_n^{(1)}\bar{w}_n^{(2)} + w_n^{(2)}\bar{w}_n^{(1)}\right)\\ &\qquad= -\frac{2}{\tilde{c}_{\mathrm{M}}}\left[\left(\tilde{\Delta}_3\tilde{\xi}_1 + \tilde{\Delta}_1\tilde{\xi}_3 - \tilde{\xi}_3\tilde{\xi}_1\frac{\tilde{c}_{\mathrm{L}}}{\tilde{c}_{\mathrm{M}}}\right)\mathcal{F}(t) + \tilde{\xi}_3\tilde{\xi}_1 x\partial_t\mathcal{F}(t)\right],\end{aligned} \tag{5.79}$$

which follows from the definition

$$\sum_{n=2}^{\infty}\frac{n-1}{n(n+1)}t^n = \frac{t^2}{6}\,{}_2F_1\left(2,2;4;t\right) \equiv \frac{1}{6}\mathcal{F}(t), \tag{5.80}$$

with the hypergeometric function ${}_2F_1\left(2,2;4;t\right)$. Moreover, $\mathcal{F}(t)$ satisfies the identity

$$\mathcal{F}(t) = 6\left(\frac{t-2}{t}\ln(1-t) - 2\right). \tag{5.81}$$

Evaluating $S^+(t,x;v)$ and $S^-(\bar{v})$ at the stationary point $(w,\bar{w})$ yields the same result for both functions,

$$S^+(t,x;w) = S^-(\bar{w}) = \frac{2}{\tilde{c}_{\text{M}}}\left[\left(\tilde{\Delta}_3\tilde{\xi}_1 + \tilde{\Delta}_1\tilde{\xi}_3 - \tilde{\xi}_3\tilde{\xi}_1\frac{\tilde{c}_{\text{L}}}{\tilde{c}_{\text{M}}}\right)\mathcal{F}(t) + \tilde{\xi}_3\tilde{\xi}_1 x\partial_t\mathcal{F}(t)\right]. \quad (5.82)$$

The expression for the perturbatively heavy vacuum $\mathfrak{bms}_3$-block is found by using (5.79) and (5.82) in (5.77) and plugging the expression into (5.65); the result reads

$$\begin{aligned}&\mathcal{B}_{\Delta_{1,3},\xi_{1,3};0,0}(t,x)\\ &\quad\approx t^{-2\Delta_1}\exp\left[-2\xi_1\frac{x}{t} + \frac{2}{c_{\text{M}}}\left(\left(\Delta_3\xi_1 + \Delta_1\xi_3 - \xi_3\xi_1\frac{c_{\text{L}}}{c_{\text{M}}}\right)\mathcal{F}(t) + \xi_3\xi_1 x\partial_t\mathcal{F}(t)\right)\right],\end{aligned} \quad (5.83)$$

where factors of μ have been absorbed by the non-tilde quantities.

5.3.3 Heavy-light Vacuum $\mathfrak{bms}_3$-block

The second example is the heavy-light vacuum $\mathfrak{bms}_3$-block, which has two heavy and two light external operators. It it still appropriate to pairwise identify $\Delta_1 = \Delta_2$, $\xi_1 = \xi_2$ and $\Delta_3 = \Delta_4$, $\xi_3 = \xi_4$, and the vacuum block means that $\Delta = \xi = 0$. The light operators have infinitesimal scaling dimensions and rapidities of order ϵ, as in section 5.3.2, while those of the heavy operators are of order one. Δ_1, ξ_1 are assigned to the light operators and Δ_3, ξ_3 to the heavy operators; the quantities which are kept fixed in the semiclassical limit share the order of these quantities.

Differential Equations

The heavy-light vacuum $\mathfrak{bms}_3$-block will be determined in a similar fashion as in section 5.3.2. The analysis of equations containing the infinitesimal $\tilde{\Delta}_1$ and $\tilde{\xi}_1$ is unchanged from that of section 5.3.2; hence the solution for S^+ is still given by (5.76a). However, the finite nature of $\tilde{\Delta}_3$ and $\tilde{\xi}_3$ means that the discussion which leads to the linear differential equations (5.71) no longer holds; the differential equations and solution for S^- must thus be re-considered.

An appropriate ansatz for $S^-(\bar{\sigma},\bar{\kappa})$ will be motivated by its behaviour at the saddle-point, and may be found by identifying the lowest order contributions to $\mathcal{I}(t,x;w,\bar{w})$ given by equation (5.77). The stationary-point coordinates $\bar{w}_m^{(i)}$ are given by (5.78c) and (5.78d) and are of infinitesimal order ϵ, while the coordinates $w_m^{(i)}$ will be of order one; hence the sum of oscillator-variable products in $\mathcal{I}(t,x;w,\bar{w})$ is of order ϵ. Similarly, from the solution (5.76a) one concludes that $S^+(t,x;w)$ is of order ϵ. Therefore $S^-(\bar{w})$ may only contribute with terms which are at most order ϵ since higher orders are subleading in the saddle-point approximation. The variables $\bar{\kappa}_n$ and $\bar{\sigma}_n$ are of order ϵ when evaluated at the stationary point and hence $S^-(\bar{\sigma},\bar{\kappa})$ should at most be linear in its variables – a suitable ansatz thus reads[8]

$$S^-(\bar{\sigma},\bar{\kappa}) = \sum_{n=1}^{\infty} C_n \bar{\sigma}_n + \sum_{n=1}^{\infty} D_n \bar{\kappa}_n, \tag{5.84}$$

where the coefficients C_n and D_n depend on $\tilde{\xi}_3$, $\tilde{\Delta}_3$. Plugging the above ansatz into the differential equations (5.70) yields

$$\begin{aligned} 0 = &\sum_{n=1}^{\infty} n \left(\bar{\sigma}_n \left(C_{k+n} - C_n\right) + \bar{\kappa}_n \left(D_{k+n} - D_n\right)\right) - \frac{1}{4}\sum_{n=1}^{k-1} C_n D_{k-n} \\ &+ \tilde{A}_k^- C_k + \tilde{B}_k^- D_k - \tilde{\Delta}_3(k-1)\,, \end{aligned} \tag{5.85a}$$

$$0 = \sum_{n=1}^{\infty} n\bar{\sigma}_n \left(D_{k+n} - D_n\right) - \frac{1}{8}\sum_{n=1}^{k-1} D_{k-n} D_n + \tilde{A}_k^- D_k - \tilde{\xi}_3(k-1). \tag{5.85b}$$

The above equations have the leading order one and the variables $\bar{\sigma}_n$, $\bar{\kappa}_n$ give rise to sub-leading terms of order ϵ at the stationary point – thus terms containing these variables may be dropped. The remaining set of equations takes the form of recurrence relations, i.e.

$$0 = -\frac{1}{4}\sum_{n=1}^{k-1} C_n D_{k-n} - (k+1)\sqrt{\frac{\tilde{c}_{\mathrm{M}}}{48}} C_k - (k+1)\frac{\tilde{c}_{\mathrm{L}}}{2\sqrt{48\tilde{c}_{\mathrm{M}}}} D_k - \tilde{\Delta}_3(k-1) + \mathcal{O}\big(\epsilon\big), \tag{5.86a}$$

$$0 = -\frac{1}{8}\sum_{n=1}^{k-1} D_{k-n} D_n - (k+1)\sqrt{\frac{\tilde{c}_{\mathrm{M}}}{48}} D_k - \tilde{\xi}_3(k-1) + \mathcal{O}\big(\epsilon\big), \tag{5.86b}$$

where the omitted terms are indicated by their order at the stationary point $(w,\bar{w})$ and $\tilde{A}_k^-$ and $\tilde{B}_k^-$ are given by (5.66b). Note that for $k=1$ the above recurrence relations fix the initial values $C_1 = D_1 = 0$.

The method of generating functions may be used to turn the recurrence relations (5.86) into differential equations. To this end, the appropriate ansätze are

$$C(\tau) = \sum_{n=1}^{\infty} C_n \tau^n \quad \text{and} \quad D(\tau) = \sum_{n=1}^{\infty} D_n \tau^n\,. \tag{5.87}$$

As a consequence of the above ansätze $C(0) = D(0) = 0$ and the constraints $C_1 = D_1 = 0$ become the boundary conditions $C'(0) = D'(0) = 0$.

[8] One may again ignore constant terms in the ansatz for S^- since those contribute as insignificant multiplicative factors to the $\mathfrak{bms}_3$-block.

The following steps are needed to express the recurrence relation (5.86b) as a differential equation. First, (5.86b) is multiplied across by $\sum_{k=2}^{\infty}\tau^k$, which is allowed since the equation must hold for all values of k; this gives,

$$-\frac{1}{8}\sum_{k=1}^{\infty}\sum_{n=1}^{k-1}D_{k-n}D_n\tau^k-\sqrt{\frac{\tilde{c}_{\mathrm{M}}}{48}}\sum_{k=1}^{\infty}(k+1)D_k\tau^k-\sum_{k=1}^{\infty}(k-1)\tilde{\xi}_3\tau^k=0\,. \tag{5.88}$$

The powers of τ may be split in the first term, hence

$$-\frac{1}{8}\sum_{k=1}^{\infty}\sum_{n=1}^{k-1}D_{k-n}D_n\tau^k=-\frac{1}{8}\sum_{k=1}^{\infty}\sum_{n=1}^{k-1}\left(D_{k-n}\tau^{k-n}\right)\left(D_n\tau^n\right). \tag{5.89}$$

To extract $D(\tau)$-terms from the above expression the two sums must be rewritten as

$$\sum_{k=1}^{\infty}\sum_{n=1}^{k-1}\equiv\sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty}\,, \tag{5.90}$$

such that the first term in (5.88) becomes

$$-\frac{1}{8}\sum_{n=1}^{\infty}D_n\tau^n\sum_{k=n+1}^{\infty}D_{k-n}\tau^{k-n}=-\frac{1}{8}D(\tau)^2\,. \tag{5.91}$$

Taking out a factor of τ in term proportional to k in the second term of (5.88) it becomes the derivative of $D(\tau)$, i.e.

$$\sqrt{\frac{\tilde{c}_{\mathrm{M}}}{48}}\left(\tau\sum_{k=1}^{\infty}kD_k\tau^{k-1}+\sum_{k=1}^{\infty}D_k\tau^k\right)=\sqrt{\frac{\tilde{c}_{\mathrm{M}}}{48}}\left(\tau\partial_\tau D(\tau)+D(\tau)\right). \tag{5.92}$$

Finally, the last term of (5.88) can be rewritten in terms of the geometric series

$$\tilde{\xi}_3\sum_{k=1}^{\infty}(k-1)\tau^k=\tilde{\xi}_3\sum_{k=1}^{\infty}k\tau^{k-1}\tau^2=\tilde{\xi}_3\frac{\tau^2}{(1-\tau)^2}. \tag{5.93}$$

Combining the above re-expressions one arrives at

$$\partial_\tau\left(\tau\cdot D(\tau)\right)=-\frac{1}{8}\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\left(D(\tau)\right)^2-\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\frac{\tilde{\xi}_3\tau^2}{(1-\tau)^2}+\mathcal{O}(\epsilon^2). \tag{5.94a}$$

Applying the same reasoning to (5.86a) results in a second differential equation

$$\partial_\tau \left(\tau \cdot C(\tau)\right) = -\sqrt{\frac{3}{\tilde{c}_{\mathrm{M}}}} C(\tau) D(\tau) - \sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}} \frac{\tilde{\Delta}_3 \tau^2}{(1-\tau)^2} - \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}} \partial_\tau \left(\tau \cdot D(\tau)\right) + \mathcal{O}(\epsilon^2). \tag{5.94b}$$

The solution procedure of the differential equations (5.94) is detailed in appendix D of [4]; to leading order in infinitesimal quantities the solutions read

$$\begin{aligned} C(\tau) \approx & -\frac{\tau(1-\tau)^{\beta_3-1}\left(24\tilde{\Delta}_3 + \tilde{c}_{\mathrm{L}}(\beta_3^2-1)\right)}{\sqrt{3\tilde{c}_{\mathrm{M}}}\left((1-\tau)^{\beta_3}-1\right)^2} \ln(1-\tau) \\ & - \frac{12\tilde{\Delta}_3 \tau}{\sqrt{3\tilde{c}_{\mathrm{M}}}\beta_3} \left(\frac{(1-\tau)^{\beta_3}+1}{(\tau-1)\left((1-\tau)^{\beta_3}-1\right)} \right) \\ & - \frac{\tilde{c}_{\mathrm{L}}}{2\sqrt{3\tilde{c}_{\mathrm{M}}}(\tau-1)} \left(\tau - 2 - \frac{\tau}{\beta_3} \frac{(1-\tau)^{\beta_3}+1}{\left((1-\tau)^{\beta_3}-1\right)} \right), \end{aligned} \tag{5.95a}$$

$$D(\tau) \approx -\sqrt{\frac{\tilde{c}_{\mathrm{M}}}{3}} \frac{1}{1-\tau} \left(2 - \tau + \beta_3 \tau \left(1 - \frac{2}{1-(1-\tau)^{\beta_3}} \right) \right), \tag{5.95b}$$

with

$$\beta_3 = \sqrt{1 - \tilde{\xi}_3 \frac{24}{\tilde{c}_{\mathrm{M}}}}. \tag{5.96}$$

The ansatz (5.84) is expressed in terms of the coefficients C_m and D_m, which can be extracted from the solutions for $C(\tau)$ and $D(\tau)$ above; however, such an analysis is not necessary to perform the saddle point and to determine the $\mathfrak{bms}_3$-block.

Implementing the Saddle-point Approximation

In terms of the rescaled oscillator variables (5.57) the solution for S^+ in equation (5.76a) and ansatz for S^- given by (5.84) read

$$S^+(t,x,0,0;v) \approx -\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}} \left[\left(\tilde{\Delta}_1 - \tilde{\xi}_1 \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}} \right) \sum_{n=1}^{\infty} t^n v_n^{(1)} + \tilde{\xi}_1 \sum_{n=1}^{\infty} \left(n t^{n-1} x v_n^{(1)} + t^n v_n^{(2)} \right) \right], \tag{5.97a}$$

$$S^-(1,0,0,0;\bar{v}) = \sum_{n=2}^{\infty} \left(C_n \bar{v}_n^{(1)} + D_n \bar{v}_n^{(2)} \right), \tag{5.97b}$$

where $C_1 = D_1 = 0$ has been used such that the sum in S^- starts at $n = 2$.

The next step in determining the saddle-point approximation of the $\mathfrak{bms}_3$-block is to evaluate $\mathcal{I}(t,x;w,\bar{w})$. Using (5.97) the coordinates of the stationary point take the form

$$w_m^{(1)} = \frac{1}{4m} D_m\,, \qquad w_m^{(2)} = \frac{1}{4m} C_m, \tag{5.98a}$$

$$\bar{w}_m^{(1)} = -\frac{1}{4m}\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}}\tilde{\xi}_1 t^m, \qquad \bar{w}_m^{(2)} = -\frac{1}{4m}\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}} t^m \left(\tilde{\Delta}_1 - \tilde{\xi}_1 \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}} + m\,\tilde{\xi}_1 \frac{x}{t}\right). \tag{5.98b}$$

Inserting the above values into the definition (5.77) yields

$$\mathcal{I}(t,x;w,\bar{w}) = -\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}} \sum_{m=2}^{\infty} \frac{1}{4m} t^m \left(\tilde{\xi}_1 C_m + \tilde{\Delta}_1 D_m - \tilde{\xi}_1 \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}} D_m + m\,\tilde{\xi}_1 \frac{x}{t} D_m\right). \tag{5.99}$$

The terms containing coefficients C_m and D_m, defined in equation (5.87), may be expressed as the following integrals

$$\sum_{m=1}^{\infty} \frac{1}{m} C_m t^m = \int_0^t \mathrm{d}\tau\, \frac{C(\tau)}{\tau} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{1}{m} D_m t^m = \int_0^t \mathrm{d}\tau\, \frac{D(\tau)}{\tau}; \tag{5.100}$$

hence

$$\mathcal{I}(t,x;w,\bar{w}) = -\frac{1}{4}\sqrt{\frac{48}{\tilde{c}_{\mathrm{M}}}} \left(\tilde{\xi}_1 \int_0^t \mathrm{d}\tau \frac{C(\tau)}{\tau} + \tilde{\Delta}_1 \int_0^t \mathrm{d}\tau \frac{D(\tau)}{\tau} - \tilde{\xi}_1 \frac{\tilde{c}_{\mathrm{L}}}{2\tilde{c}_{\mathrm{M}}} \int_0^t \mathrm{d}\tau \frac{D(\tau)}{\tau} + \tilde{\xi}_1 \frac{x}{t} D(t)\right). \tag{5.101}$$

Evaluating integrals in (5.100) using the solutions (5.95) gives

$$\int_0^t \mathrm{d}\tau \frac{D(\tau)}{\tau} = \sqrt{\frac{\tilde{c}_{\mathrm{M}}}{3}} \left(2\ln\Big(1-(1-t)^{\beta_3}\Big) - (\beta_3 - 1)\ln(1-t) - 2\ln(t)\right), \tag{5.102}$$

and

$$\begin{aligned}\int_0^t \mathrm{d}\tau \frac{C(\tau)}{\tau} &= \frac{\tilde{c}_{\mathrm{L}}}{\sqrt{3\tilde{c}_{\mathrm{M}}}} \left(\ln\Big(1-(1-t)^{\beta_3}\Big) - \ln(t)\right) + \frac{24}{\sqrt{3\tilde{c}_{\mathrm{M}}}\beta_3^2} \left(\tilde{\Delta}_3 - \frac{\tilde{c}_{\mathrm{L}}}{\tilde{c}_{\mathrm{M}}}\tilde{\xi}_3\right) \\ &+ \frac{1}{2\sqrt{3\tilde{c}_{\mathrm{M}}}\beta_3} \left(\frac{24\tilde{\Delta}_3 + \tilde{c}_{\mathrm{L}}(\beta_3 - 1) + (1-t)^{\beta_3}\Big(24\tilde{\Delta}_3 + \tilde{c}_{\mathrm{L}}(\beta_3-1)(1+2\beta_3)\Big)}{1-(1-t)^{\beta_3}}\right) \ln(1-t).\end{aligned} \tag{5.103}$$

Using the above expressions in (5.101) and in turn plugging that into (5.65) the heavy-light vacuum $\mathfrak{bms}_3$-block takes the form

$$\mathcal{B}^{\mathrm{HHLL}}_{\Delta_{1,3},\xi_{1,3};0,0}(t,x) \approx \left(\frac{(1-t)^{\beta_3-1}}{(1-(1-t)^{\beta_3})^2}\right)^{\Delta_1} \exp\left[-\xi_1 x \frac{(1-t)^{\beta_3}(1+\beta_3)+\beta_3-1}{(1-t)(1-(1-t)^{\beta_3})}\right] \times \exp\left[\frac{12\xi_1}{c_{\mathrm{M}}\beta_3}\left(\xi_3\frac{c_{\mathrm{L}}}{c_{\mathrm{M}}}-\Delta_3\right)\left(\frac{1+(1-t)^{\beta_3}}{1-(1-t)^{\beta_3}}\right)\ln(1-t)\right], \tag{5.104}$$

where factors of μ are absorbed by the non-tilde quantities. This result generalises the heavy-light vacuum $\mathfrak{bms}_3$-block presented in [193]. Moreover, (5.104) may be consistently (modulo constant prefactors) transformed into the perturbatively heavy vacuum $\mathfrak{bms}_3$-block (5.83) by choosing Δ_3, ξ_3 to be infinitesimal; expanding β_3 to first order in ξ_3; and keeping terms up to second order in infinitesimal quantities.

6 Discussion and Outlook

The work presented in this thesis has considered aspects of symmetries and symmetry breaking in several contexts: in hydrodynamics, in bottom-up holography, and within the scope of flatspace holography.

Spontaneous breaking of translational invariance was considered from the perspective of hydrodynamics. It is well-known that spontaneous breaking of translational invariance gives rise to phonons and elastic properties; the main focus of this thesis was novel effects associated to the so-called strain pressure. Strain pressure appears in systems that spontaneously break translational invariance but which do not minimise the free energy – due to external strain which may, for instance, be imposed by boundary conditions. The main contribution to the topic by the work presented in this thesis pertains to the temperature derivative of the strain pressure, which was shown to be generically non-zero even for thermodynamically stable systems that minimise the free energy [2]. Strain pressure and its temperature derivative directly contribute to the hydrodynamic dispersion relations in the longitudinal sector of the dynamics, while implicitly contributing to thermodynamic quantities such as the momentum susceptibility. In the conformal limit, which is relevant for applications to holography, the effects of strain pressure only explicitly contribute to the diffusive mode of the longitudinal sector of the hydrodynamics. The temperature derivative of the strain pressure also allows for a bulk elastic modulus in thermodynamically stable, conformally invariant systems.

A superfluid may be described by including the effects of spontaneous $U(1)$ symmetry breaking into a hydrodynamic analysis. This well-established phenomenon was considered in a novel setting by the addition of a small explicit breaking of $U(1)$ symmetry [3]. Such a pseudo-spontaneous regime was achieved via a modification of the conservation equation for the $U(1)$ symmetry current, rendering the current non-conserved. The modification consisted of a term corresponding to a mass for the $U(1)$ Goldstone boson, as well as a term responsible for charge relaxation. Moreover, phase relaxation – which acts as a dampening for the Goldstone boson and which may appear independently of the other explicit breaking effects – was included via the Josephson relation, i.e. the equation which governs the dynamics of the Goldstone. The dynamics of the system were for simplicity, and later applicability, presented for a conformally invariant system.

It was found that the Goldstone mass, charge relaxation and phase relaxation all contribute to the dynamics of the system. Without explicit breaking a superfluid supports several propagating sound modes: first and second sound, and a mode – called fourth sound – which appears when the dynamics of the energy-momentum tensor are decoupled (the probe limit). It was shown that – in the presence of explicit $U(1)$ symmetry breaking and phase relaxation – the mode which would constitute first sound turned into a sound mode of a normal conformal fluid, while second and fourth sound gained a complex gap and became quadratically, rather than linearly, dependent on the momentum. The sum of attenuation constants in the purely spontaneous state was found to be equal to the sum of the analogous coefficients in the pseudo-spontaneous phase. Moreover, a Gell-Mann-Oaks-Renner relation for the mass of the pseudo-Goldstone was found (in the probe limit) by allowing for a momentum-dependent susceptibility and imposing symmetry of the off-diagonal retarded Green's functions. In contrast to previous work considering phase relaxation in superfluids, the phase relaxation did not appear in the probe-limit AC conductivity when the $U(1)$ was explicitly broken (without charge relaxation); nevertheless the usual pole at zero frequency was absent – but without a Drude-peak – and the DC conductivity was finite.

The hydrodynamic framework with spontaneously broken translational invariance was examined using a holographic massive gravity model – a bottom-up model which gives rise to spontaneous breaking of translational invariance in the dual field theory [1, 2]. The massive gravity model was considered in two configurations which were called strained and unstrained; the unstrained set-up was constructed such that its free energy was minimised, whereas the strained models did not minimise their free energy. The unstrained configurations had not previously been considered in this context. Holographic formulae for several thermodynamic and hydrodynamic quantities, including the strain pressure and its temperature derivative, were presented. Numerical techniques were utilised to successfully compare the dynamics of specific quasi-normal modes to hydrodynamic dispersion relations, the implications of which were twofold: a discrepancy between holography and previous hydrodynamic frameworks was resolved; and the presence and importance of the temperature derivative of the strain pressure for unstrained configurations was confirmed.

The hydrodynamic framework for pseudo-spontaneous $U(1)$ symmetry breaking was tested against the probe-limit dynamics of two distinctly modified versions of the standard holographic superfluid model [3]. The first modification was realised by introducing a source for the charged operator which is responsible for the superfluid phase transition in the dual field theory; for small values of the source this results in pseudo-spontaneous breaking of $U(1)$ symmetry in the dual field theory. As a consequence of this breaking method the superfluid phase transition became smeared, i.e. no longer sharp at a critical temperature – an effect which grew with the amount of explicit breaking. Moreover, this model displayed a massive pseudo-Goldstone boson and a phase relaxation induced by explicit symmetry breaking, but no charge relaxation. The hydrodynamic derivation for the mass of the pseudo-Goldstone was shown to hold. The appearance of an effective phase relaxation was observed when the $U(1)$ symmetry was explicitly broken; for small amounts of explicit breaking the induced phase relaxation was shown to obey a relation between the pseudo-Goldstone mass and Goldstone diffusivity, adding evidence for a proposed universal. Moreover, the dispersion relations of the lowest quasi-normal modes were successfully matched to hydrodynamic formulae for a range of small explicit breakings. The finite nature of the probe-limit AC and DC conductivities was also confirmed.

The second modification of the holographic superfluid model was made by introducing a mass for the bulk gauge field; for small masses the dual field theory displays pseudo-spontaneous $U(1)$ symmetry breaking, but with substantially different properties compared to the first modification. The phase transition remained sharp, as in a normal superfluid, but the critical temperature and value of the condensate increased with the explicit breaking. This model displayed charge relaxation which (for small breakings) obeyed a first-order formula depending on the mass of the bulk gauge field, but the Goldstone remained massless and phase relaxation was absent. The behaviour of the lowest quasi-normal modes was consistent with the hydrodynamic dispersion relations mentioned previously, for small breakings.

Formal aspects of Bondi-Meltzer-Sachs (BMS) symmetry have also been considered, motivated by its relevance to a potential holographic duality involving flat spacetimes. A novel representation, in terms of the oscillator formalism, was found for the highest weight representation of the two-dimensional $\mathfrak{bms}_3$ algebra [4] – this was achieved by taking a non-relativistic contraction of a linear-dilaton-like theory. As a test of the representation the Gram matrix was calculated, which was in agreement with previous results using other methods. Some BMS field theory quantities – namely the two point correlation function and two $\mathfrak{bms}_3$-blocks – were calculated by implementing the tool-set which accompanies the oscillator construction. The $\mathfrak{bms}_3$-blocks were considered in the semiclassical limit, which allowed them to be calculated by a saddle-point approximation – as a consequence the exponentiation of $\mathfrak{bms}_3$-blocks in the semiclassical limit was proven. Two examples of semiclassical $\mathfrak{bms}_3$-blocks were computed: the pertubatively heavy, and heavy-light vacuum $\mathfrak{bms}_3$-blocks. The two-point correlation function was in agreement with existing literature while the heavy-light vacuum $\mathfrak{bms}_3$-block generalised previous results; the pertubatively heavy vacuum $\mathfrak{bms}_3$-block was consistent with the heavy-light vacuum $\mathfrak{bms}_3$-block.

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There are several avenues worth exploring further.

Future hydrodynamic work may be approached from both theoretical and practical directions. It would be of great interest to understand the limits of incorporating explicit symmetry breaking into a hydrodynamic framework, with or without spontaneous symmetry breaking. Considering a first-principles approach to this topic – rather than bending the hydrodynamic rules as in this thesis – may be a worthwhile endeavour that could yield a framework with a larger range of applicability than what is currently available. Nevertheless, extending the methodology considered in this thesis to more involved, perhaps more realistic scenarios may prove to be useful for initial explorations. Along such practical lines it would be of interest to search for signatures of a universal phase relaxation in physical systems exhibiting pseudo-spontaneous symmetry breaking. Similarly, understanding the relevance, and detecting the presence, of strain pressure in condensed matter or solid state systems constitutes another compelling continuation of the work presented in this thesis.

The applied holography perspective considered in this thesis will likely continue to provide useful guidance for research into the principles of hydrodynamics, as well as being effective at testing new findings. The investigations of this thesis may be directly extended upon by including backreaction into the analysis of section 4.2, or by finding a thermodynamically stable holographic model for spontaneous translational symmetry breaking which does not exhibit unphysical behaviour of the hydrodynamic modes. Moreover, building a holographic model which displays phase relaxation without explicit symmetry breaking would be of interest. Constructing bottom-up holographic models which capture the full dynamics of realistic systems will however continue to be a difficult task.

It would be interesting to apply the oscillator construction of $\mathfrak{bms}_3$ to compute $\mathfrak{bms}_3$-blocks – semiclassical or otherwise – beyond the vacuum module, as well as blocks containing more than four external operators. Another possibility is to consider $\mathfrak{bms}_3$-blocks for backgrounds other than the plane – for instance on the torus or on the cylinder. Furthermore, including supersymmetry could yield additional constraining powers. Similar considerations would also be of interest for the oscillator construction of the Virasoro algebra. For applications to unitary and ultra-relativistic theories it would be important to find an oscillator construction for the induced representation of $\mathfrak{bms}_3$; this is perhaps the representation which is the most relevant for flatspace holography. Finally, it would be valuable to adapt the oscillator formalism for higher-dimensional incarnations of BMS or conformal symmetry.

Clearly the study of symmetry and symmetry breaking continues to present possibilities to gain many further insights; hopefully the examples discussed above will be returned to in future work.

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Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

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- Martin Ammon, Daniel Areán, Matteo Baggioli, Sebastian Grieninger und Akash Jain – bei den in Kapitel 3 und 4 präsentierten Ergebnissen, basierend auf [1–3]
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Weitere Personen waren an der inhaltlich-materiellen Erstellung der vorliegenden Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs-bzw. Beratungsdiensten (Promotionsberater oder andere Personen) in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen.

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Die geltende Promotionsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt.

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