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Original published in:	Proceedings in applied mathematics and mechanics. - Weinheim [u.a.] : Wiley-VCH. - 20 (2021), 1, art. e202000197, 4 pp.
Conference:	Annual Meeting of the International Association of Applied Mathematics and Mechanics (GAMM) ; 91 (Kassel) : 2020.03.16-20
Original published:	2021-01-25
ISSN:	1617-7061
DOI:	10.1002/pamm.202000197
[*Visited:*	2022-03-25]

Received: 14 July 2020 | Accepted: 6 November 2020

DOI: 10.1002/pamm.202000197

Lagrangian analysis of long-term dynamics of turbulent superstructures

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In Rayleigh-Bénard convection, turbulent superstructures are large-scale patterns of circulation rolls created by hot ascending and cold descending thermal plumes. The evolution of these large-scale patterns happens on very large time scales τ [1]. Spectral clustering applied to Lagrangian particle trajectories on time intervals smaller than τ can be used to create clusters displaying a structure similar to the patterns detected in the Eulerian frame of reference [2]. However, this technique is unfeasible for the analysis of the evolution of turbulent superstructures due to turbulent dispersion. Therefore, we test the application of concepts of evolutionary spectral clustering [3] on Lagrangian particle trajectories to analyze the long-term dynamics of turbulent superstructures in the Lagrangian frame of reference.

1 Introduction

Despite the fact that fluid motion in a horizontally extended thermal convection layer is typically highly turbulent, the formation of slowly evolving large-scale patterns is observed (e.g. [4]). These patterns, named turbulent superstructures of convection, can be separated from small-scale variations through a time average of the flow field. They follow from circulation rolls, created by hot ascending and cold descending thermal plumes. In ref. [2] it was shown that a spectral clustering approach applied to Lagrangian particle trajectories creates clusters which resemble the turbulent superstructures revealed by the time-averaged temperature field of the midplane. This analysis is based on a three-dimensional simulation data set of Rayleigh-Bénard convection (RBC) with aspect ratio $\Gamma = 16$, Prandtl number Pr $= 0.7$ and Rayleigh number Ra $= 10^5$. Lagrangian tracer particles are initiated uniformly in a plane parallel to the bottom plate. In this setting, the tracer particles first assemble along hot thermal plumes before they rise to the top plate and subsequently descend along cold thermal plumes. At this time, the tracer particles are spread across the full convection cell. Using a spectral clustering algorithm as in [5], Lagrangian coherent sets are interpreted by subsets of trajectories which have a small time-averaged distance in a given time interval.

This is one example for the application of a spatio-temporal clustering algorithm for the identification of Lagrangian coherent sets. Related approaches include [6–9]. In general, the study of Lagrangian coherence comprises different geometric and probabilistic methods. An overview and discussions of different methods can e.g. be found in [10, 11].

The investigation of the long-term dynamics of turbulent superstructures requires very long time spans of t > τ. Due to turbulent dispersion the time-averaged distance of any pair of particle trajectories for such a long time interval will be large. Therefore, standard methods for the definition of coherent structures are deficient in this setting. In other words, for very long time spans a circulation roll or its center will not contain any Lagrangian coherent set or structure. The Lagrangian analysis of the slow evolution of turbulent superstructures thus cannot solely focus on coherent sets. Instead, we here aim at finding subsets of particles which move in a single convection roll for a short time span by means of a spatio-temporal clustering algorithm in consecutive time steps using the same data set as [2]. A leaking Lagrangian coherent set is defined as a temporal sequence of such subsets which remain at a similar position and consist to a significant proportion of the same particles for consecutive time steps. This means that (i) individual paricles are allowed to leave and enter a leaking Lagrangian coherent set and (ii) leaking Lagrangian coherent sets can merge or split up. For this purpose we apply an evolutionary spectral clustering algorithm which is suitable for the observation of long-term evolution of clusters [3]. We compare the pattern created by leaking Lagrangian coherent sets with the turbulent superstructures. To the best of our knowledge this is the first time that evolutionary spectral clustering is applied to convection data.

2 Data set

The data set is obtained from a three-dimensional numerical simulation solving the Boussinesq equations of motion in the dimensionless form. The dimensionless temperatures of the top and bottom plate are held constant at $T = 1$ and $T = 0$, respectively. The cell with height H has an aspect ratio of 16:16:1. We use a subsample of $N_p = 128^2$ Lagrangian tracer particles which are seeded uniformly at $z = 0.03$ and advected individually. Further details on the simulation are given in [2].

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3 Method

3.1 Spectral clustering

In order to identify Lagrangian coherent sets in a particle data set a graph based approach can be used as in [5]. The graph is constructed interpreting individual particle trajectories as nodes and edge weights w_{ij} are inversely proportional to the time-averaged distance of trajectories. This graph is sparsified using an ϵ-neighborhood criterion, i.e. edges with weights $w_{ij} < \epsilon$ are discarded. The non-normalized graph Laplacian matrix L is derived as $L = D - W$, where W is the adjacency matrix and D is the degree matrix of the graph with $D_{ii} = \sum_{j=1}^{n} W_{ij}$. Using spectral graph theory, [12] show that the graph can be cut under the normalized cut criterion based on the eigenvectors to the $n-1$ eigenvalues $\lambda_2, \ldots, \lambda_n$ of the generalized eigenvalue problem $Ly = \lambda Dy$, where $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The clustering of the data set is performed in a post-processing step by the application of a clustering algorithm, e.g. k-means, on the chosen eigenvectors.

3.2 Evolutionary spectral clustering

Evolutionary clustering algorithms are developed in order to partition dynamical data sets at consecutive time steps such that the new partition accounts for the changes in the data and simultaneously incorporates the previous partition. The underlying assumption is that in most dynamical systems the changes in the data are smooth. This condition is taken into account by a quantity called temporal smoothness. In [3] two algorithms are established in which the temporal smoothness is incorporated in the cost function of a spectral clustering algorithm. The basic cost function is given by

$$cost = \alpha \cdot cost_s + \beta \cdot cost_t,$$

where $cost_s$ and $cost_t$ stand for snapshot and temporal cost, respectively. The parameters α and β have to be chosen by the user from $\alpha, \beta \in [0\ 1]$ with $\beta = 1 - \alpha$. The temporal cost may be calculated via one of the two frameworks termed Preserving Cluster Quality (PCQ) and Preserving Cluster Membership (PCM). In this study we will focus on PCM. For a description of PCQ the reader is referred to [3].

The idea of PCM is to punish deviations of the current partition to the previous partition. This means that the temporal cost of two partitions which might have equal snapshot cost is smaller for the partition which deviates less from the previous partition. In accordance with [2] we choose the solution for PCM under consideration of the normalized cut. This is based on the R-way normalized cut by [13]. At time $t \geq 1$ the cost function is minimized by the matrix X_t whose columns are the eigenvectors to the top-k eigenvalues of the matrix $\alpha D_t^{-1/2} W_t D_t^{-1/2} + \beta X_{t-1} X_{t-1}^T$. Here, D_t and W_t are the degree and adjacency matrix of the graph at time t and X_{t-1} is the solution at $t-1$. For $t = 0$ the matrix X_0 consists of the eigenvectors to the top-k eigenvalues of the matrix $D_0^{-1/2} W_0 D_0^{-1/2}$.

3.3 Choice of parameters

In order to ensure that the variations of data points are small for consecutive time steps we choose a small integration time of $\Delta t = 1.97$ free-fall times (T_f) which corresponds to $0.1\bar{\tau}_{to}^L$, where $\bar{\tau}_{to}^L$ is the mean Lagrangian turnover time, see [2]. For each time step we create a sparse unweighted network where links are created when the minimum distance of trajectories is smaller than a cutoff radius ϵ, as in [8]. This calculation is faster than the computation of time-averaged distances while the distribution of links is almost identical for this short time interval. In the presented analysis we choose $\epsilon = 0.75$. The value of α is set to 0.15. We have tested other combinations of ϵ and α (not shown here). When using larger values of α, the spectral gap is less prominent but still clearly visible at the same position. With decreasing values of ϵ, restricted to the case of creating a connected graph, spectral gaps at higher n become more prominent which indicates that the creation of more (and therefore smaller) clusters becomes more significant. However, the orientation of these clusters remains similar to the orientation of clusters in the case of $\epsilon = 0.75$. Further analyses also including variations of Δt and a qualitative evaluation of α are in process.

4 Results

As the turbulent superstructures evolve extremely slowly in this simulation we choose a time-averaged temperature in the midplane with averaging time $N_t \Delta t = 10.4 T_f$ for comparison with the clustering results. This field is plotted in Fig. 1.

For the first time interval [0, Δt] the pattern is best reproduced by 22 clusters created by k-means clustering of the eigenvectors to the 22 leading eigenvalues. The spectrum of eigenvalues and the extracted clusters are shown in Fig. 2a) and b).

After 12 steps of evolutionary spectral clustering, corresponding to the time interval [$23.6T_f$, $25.5T_f$], we still observe a good agreement between the Eulerian pattern in Fig. 1 and the resulting Lagrangian clusters in Fig. 2d). The spectrum of eigenvalues, Fig. 2c), displays a clear spectral gap wich is a result of our choice of α. Same color in Fig. 2b) and d) indicate the same leaking Lagrangian coherent set.

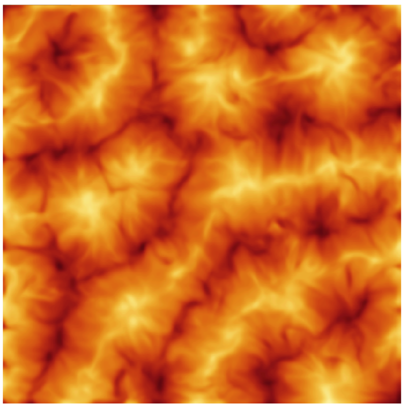


Fig. 1: Time-averaged temperature in the midplane with averaging time $N_t \Delta t = 10.4 T_f$.

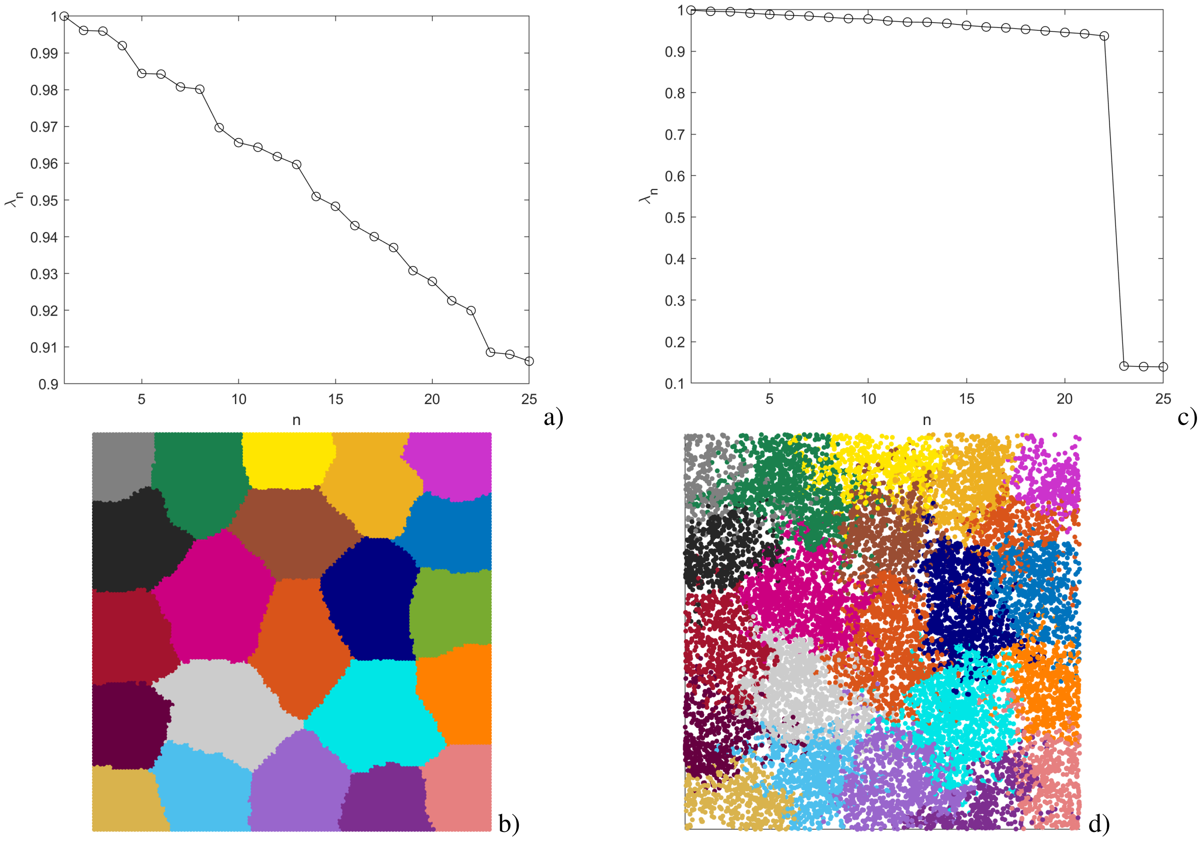


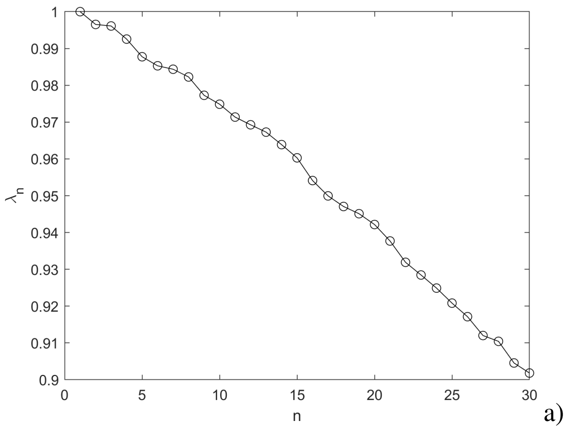
Fig. 2: Spectrum of eigenvalues and k-means clustering result for the time interval $[0,\, \Delta t]$ (**a**) and **b**)) and $[23.6T_f,\, 25.5T_f]$ (**c**) and **d**)). Initial positions at the considered time interval of tracer particles that belong to the same cluster are plotted in the same color.

For comparison, Fig. 3 shows the particles after $23.6T_f$ colored according to the initial clustering, i.e. the advected clusters of Fig. 2b). The clusters are larger and more blurred compared to the corresponding clusters in Fig. 2d). As the clusters created using evolutionary clustering show less filamentation than the advected clusters, they are better candidates for coherent sets in the corresponding time interval.
In contrast, Fig. 4 displays the results for the same time interval $[23.6T_f,\, 25.5T_f]$ without the use of evolutionary spectral clustering. There is no clearly distinguishable spectral gap in the spectrum of eigenvalues. The clustering result on 15 eigenvectors is shown in Fig. 4b) which has less similarity with the Eulerian pattern than the result in Fig. 2d).

5 Discussion

In this work, we have succesfully applied an evolutionary spectral clustering method to convection data. After 12 evolutionary steps, corresponding to more than one free-fall time unit, the extracted clusters using k-means are more consistent with

Fig. 3: Particles colored according to the initial clustering for time interval [0, Δt], plotted at the positions at time $23.6T_f$.



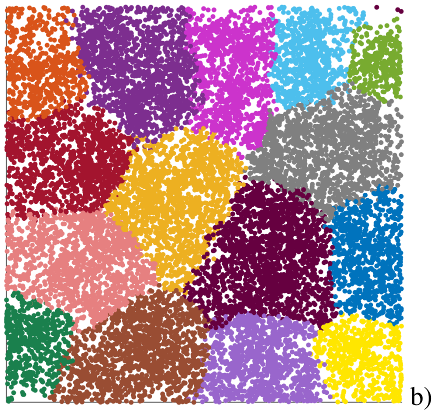


Fig. 4: **a)** Spectrum of eigenvalues and **b)** k-means clustering result for the time interval $[23.6T_f,\ 25.5T_f]$ without the application of evolutionary clustering.

the Eulerian patterns than clusters created without the use of evolutionary clustering. Furthermore, they are less filamented compared to the advected initial clusters. In future work we will study longer times which requires longer integration times for the individual time steps in order to reduce computing time. Furthermore, we will qualitatively analyze the effect of the choice of α on the resulting leaking Lagrangian coherent sets.

Acknowledgements The work of CS is supported by the Priority Programme on Turbulent Superstructures of the Deutsche Forschungsgemeinschaft within Grant No. SPP 1881. Open access funding enabled and organized by Projekt DEAL.

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