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Energy Shaping and Partial Feedback Linearization of Mechanical Systems with Kinematic Constraints

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Abstract: Traditionally, the energy shaping for mechanical systems requires the elimination of holonomic and nonholonomic constraints. In recent years, it was argued that such elimination might be unnecessary, leading to a possible simplification of the matching conditions in energy shaping. On the other hand, the partial feedback linearization (PFL) approach has been widely applied to unconstrained mechanical systems, but there is no general result for the constrained case. In this regard, this paper formalizes the PFL for mechanical systems with kinematic constraints and extends the energy shaping of such systems by including systems with singular inertia matrix and non-workless constraint forces, which can arise from the coordinate selection and PFL. We validated the proposed methodology on a 5-DoF portal crane via simulation.

Keywords: Energy shaping, Partial feedback linearization, Mechanical systems, Kinematic constraints, Implicit systems, Crane systems.

1. INTRODUCTION

Shaping the energy of mechanical systems to design control algorithms has been the subject of long-term research. In particular, there are three leading methods that achieve stabilization of an admissible equilibrium by restricting the desired energy function to the sum of potential energy (independent of velocities) and kinetic energy (quadratic in velocities). These methods are the IDA-PBC (Ortega and García-Canseco, 2004), the controlled Lagrangians (Blankenstein et al., 2002), and the Lyapunov direct method for mechanical systems (White et al., 2008). However, Chang et al. (2002) and Donaire et al. (2016b) demonstrate that those approaches are actually equivalent for unconstrained mechanical systems as long as the desired energy remains within the same class.

When modeling mechanical systems for control, we usually aim at having an explicit representation, i.e., we search for generalized (or constraint-free) coordinates such that the obtained model is described by ordinary differential equations (ODEs). If, in addition, the system possesses nonholonomic constraints, then also a change of variables in the velocities is used so as to eliminate these constraints and obtain a system of ODEs (Delgado, 2016). On the other hand, as argued by Blankenstein (2002) for the case with nonholonomic constraints and by Castaños and Gromov (2016) for the holonomic situation, the controller design with energy shaping does not necessarily require the elimination of kinematic constraints, i.e., we may work with the implicit representation given by differential algebraic equations (DAEs). In fact, Cieza and Reger (2019) show that we can take advantage of the implicit representation to simplify, for a class of systems, the matching conditions of IDA-PBC arising in underactuated mechanical systems that otherwise in explicit representation might be too demanding. These matching conditions are essential for energy shaping, and they constitute a system of quasilinear partial differential equations (PDEs).

The celebrated partial feedback linearization (PFL) for underactuated mechanical systems without kinematic constraints was introduced by Spong (1994) and employed by several authors as an essential step or a simplifying tool in the controller design. Although this technique has been applied to the cart-pole and 4-DoF portal crane in implicit representations (Vidal et al., 2020; Huamán et al., 2021), the methodology is still vague and the results are not general. Therefore, in this work, we formalize the PFL for constrained mechanical systems.

Furthermore, we extend the energy shaping of Cieza and Reger (2019) by including systems that may possess a singular inertia matrix (non-regular Lagrangian) and systems whose constraint forces are not necessarily workless. The latter case may result from a preliminary feedback as PFL, while the former is a consequence of the selected coordinates. We refrain from the port-Hamiltonian representation used in IDA-PBC as it does not admit a singular inertia matrix and can be inconvenient for PFL. Finally, we test our results on the portal crane with 5-DoF via simulation.

The paper is organized as follows. In Section 2 we identify the class of mechanical systems. Sections 3 and 4 formalize the PFL and present our energy shaping for constrained systems. To compare the implicit and explicit frameworks, we provide a reduction in Section 5. In Section 6 we check our findings on the 5-DoF portal crane and draw the conclusions in Section 7.

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Notation: We denote by $\mathbb{R}^{n\times m}$ the set of $n\times m$ real matrices, and $\mathbb{R}_d^{n\times m}$ the set of $n\times m$ real matrices with rank d. For compactness of notation, we shall write $\frac{\partial h}{\partial q} = \partial_q h$ and $\left(\frac{\partial h}{\partial q}\right)^\top = \partial_q^\top h$ for any vector-valued function h, and $\frac{\partial^2 f}{\partial q^2} = \partial_{qq} f$ for any scalar-valued function f. The identity matrix of size n is defined as I_n. Given a matrix A, we denote by A^{g}, $A_\perp$, $A_{\top}$, Colsp A and Rowsp A, to the generalized inverse, the left annihilator, the right annihilator, the column space and the row space of A, respectively. We consider the annihilators to be full rank unless they are a zero matrix. We write $\mathbf{V}\big|_{\mathcal{R}}$ to represent the restriction of $\mathbf{V}$ to the set $\mathcal{R}$, and $\mathrm{vec}(x_1, x_2, \dots)$ for the vertical concatenation of any scalars or column vectors $\{x_1, x_2, \dots\}$. In particular, we use bold letters to denote functions from implicit models and non-bold letters for everything else. To avoid cumbersome notation, we also omit the arguments of previously defined functions.

2. CLASS OF SYSTEMS

We consider mechanical systems described by

$$\dot r = \mathbf{J}(r)v, \tag{1a}$$

$$\mathbf{M}(r)\dot v + \boldsymbol{\Gamma}(r,v)v + \mathbf{d}(r) = \bar{\mathbf{B}}(r)\lambda + \mathbf{G}(r)\tau \tag{1b}$$

with smooth holonomic constraints

$$0 = \Phi(r) \tag{1c}$$

and smooth velocity level constraints [2]

$$0 = \mathbf{B}^\top(r)v, \tag{1d}$$

where $r \in \mathcal{R} \subset \mathbb{R}^{n_r}$ are coordinates, $v \in \mathbb{R}^{n_r}$ is a linear transformation of the velocities $\dot r$ with $\mathbf{J} : \mathcal{R} \to \mathbb{R}_{n_r}^{n_r\times n_r}$, $\tau \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the input, $\mathbf{G} : \mathcal{R} \to \mathbb{R}^{n_r\times n_u}$ is the input matrix, $\mathbf{M} : \mathcal{R} \to \mathbb{R}^{n_r\times n_r}$ is the inertia matrix, $\boldsymbol{\Gamma} : \mathcal{R}\times\mathbb{R}^{n_r} \to \mathbb{R}^{n_r\times n_r}$, $\mathbf{d} : \mathcal{R} \to \mathbb{R}^{n_r}$, $\Phi : \mathcal{R} \to \mathbb{R}^{n_\Phi}$, $\mathbf{B} : \mathcal{R} \to \mathbb{R}^{n_r\times n_\lambda}$, and $\bar{\mathbf{B}}(r)\lambda$ are the constraint forces [3] with $\bar{\mathbf{B}} : \mathcal{R} \to \mathbb{R}^{n_r\times n_\lambda}$ and implicit variables $\lambda \in \mathbb{R}^{n_\lambda}$ verifying $n_r > n_\lambda \geq 1$ to discard the situations with no constraints and $v(\cdot) = 0$. Here, $\boldsymbol{\Gamma}$, $\mathbf{d}$ and $\bar{\mathbf{B}}$ do not necessarily satisfy $\dot{\mathbf{M}} = \boldsymbol{\Gamma} + \boldsymbol{\Gamma}^\top$, $\mathbf{B} = \bar{\mathbf{B}}$ or $\mathbf{d} = \mathbf{J}^\top\partial_r^\top\mathbf{V}$ with potential energy $\mathbf{V} : \mathcal{R} \to \mathbb{R}$. Besides, $\mathbf{M}$ does not need to be positive definite or even symmetric. This distinction is fundamental since (1) can include Lagrangian dynamical systems with preliminary feedback and change of variables. We identify the class by imposing the assumptions below.

Assumption 1. The matrix

$$\mathbf{P}(r) := \begin{bmatrix}\mathbf{M} & -\bar{\mathbf{B}}\\ \mathbf{B}^\top & 0\end{bmatrix}$$

is nonsingular for all $r \in \mathcal{R}_\Phi := \{r \in \mathcal{R} \mid 0 = \Phi(r)\}$.

Assumption 2. The initial conditions $(r(t_0), v(t_0))$ are consistent, i.e.,

$$(r(t_0), v(t_0)) \in \mathcal{X}_\mathrm{c} := \left\{(r,\rho) \in \mathcal{R}_\Phi \times \mathbb{R}^{n_r} \;\middle|\; 0 = \mathbf{B}^\top v\right\}.$$

Lemma 3. (Rank identity). Given $[B\ C] \in \mathbb{R}^{n\times m}$, then

$$\mathrm{rank}\,[B\ C] = \mathrm{rank}(C_\perp B) + \mathrm{rank}\,C.$$

Proposition 4. The matrix $\mathbf{P}(r)$ is nonsingular iff

$$\mathrm{rank}\,\bar{\mathbf{B}}_\perp\mathbf{M}\mathbf{B}_\perp^\top = n_r - n_\lambda, \qquad \mathrm{rank}\,\mathbf{B} = \mathrm{rank}\,\bar{\mathbf{B}} = n_\lambda.$$

Proof. Applying Lemma 3 twice on $\mathbf{P}$ results in

$$\begin{aligned}\mathrm{rank}(\mathbf{P}) &= \mathrm{rank}(\begin{bmatrix}\mathbf{M}^\top\bar{\mathbf{B}}_\perp^\top & \mathbf{B}\end{bmatrix}) + \mathrm{rank}(\bar{\mathbf{B}})\\ &= \mathrm{rank}(\mathbf{B}_\perp\mathbf{M}^\top\bar{\mathbf{B}}_\perp^\top) + \mathrm{rank}(\mathbf{B}) + \mathrm{rank}(\bar{\mathbf{B}}),\end{aligned} \tag{2}$$

and the necessity is evident. Now, suppose $\mathbf{P}(r)$ has $n_r + n_\lambda$ linearly independent rows and columns, i.e., $\mathbf{P}(r)$ is nonsingular, then $\mathrm{rank}\,\mathbf{B} = \mathrm{rank}\,\bar{\mathbf{B}} = n_\lambda$, and from (2), $\mathrm{rank}\,\bar{\mathbf{B}}_\perp\mathbf{M}\mathbf{B}_\perp^\top = n_r - n_\lambda$.

Proposition 5. (Well-posedness). Consider system (1) verifying Assumptions 1 and 2. Then, *i)* $\mathcal{R}_\Phi$ and $\mathcal{X}_c$ are regular (or embedded) submanifold of $\mathcal{R}$ and $\mathcal{R}\times\mathbb{R}^{n_r}$ with dimensions $n_r - n_\Phi$ and $2n_r - n_\lambda - n_\Phi$, respectively, *ii)* λ has a unique solution for every triplet $\{r, v, \tau\}$, and *iii)* the DAE system (1) is equivalent to an ODE on $\mathcal{X}_c$.

Proof. Since $\partial_r\Phi\mathbf{J}$ is contained in $\mathbf{B}^\top$ by definition of (1d), and $\mathbf{B}$ is full rank from Assumption 1 and Proposition 4, it follows that the Jacobian of $\begin{bmatrix}\Phi(r)\\ \mathbf{B}^\top(r)v\end{bmatrix}$ is full rank. Now, the proof of statement *i* is a direct application of Lee (2013, Corollary 5.14). To show statements *ii* and *iii*, we write (1b) together with the hidden constraints as [4]

$$\mathbf{P}\begin{bmatrix}\dot v\\ \lambda\end{bmatrix} + \begin{bmatrix}\boldsymbol{\Gamma}v + \mathbf{d}\\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{B}^\top)v\end{bmatrix} = \begin{bmatrix}\mathbf{G}\\ 0\end{bmatrix}\tau. \tag{3}$$

From the nonsingularity of $\mathbf{P}$, we deduce that (1a) and (3) can be written as an ODE with a unique solution to λ. The equivalence is obtained by using consistent initial conditions (Assumption 2) and observing that such an ODE possesses a vector field on $\mathcal{X}_\mathrm{c}$.

Mechanical systems with holonomic constraints do not necessarily have an inertia matrix $\mathbf{M}$ that is positive definite as it is for mechanical systems with generalized (or constraint-free) coordinates, see Section 6.1. Proposition 5 relaxes the conditions of the Prop. 1 of Cieza and Reger (2019) by requiring the nonsingularity of $\bar{\mathbf{B}}_\perp\mathbf{M}\mathbf{B}_\perp^\top$ (see Proposition 4) instead of the one of $\mathbf{M}$.

3. PARTIAL FEEDBACK LINEARIZATION

The proposition below formalizes the PFL for constrained mechanical systems, extending the result of Spong (1994).

Proposition 6. (PFL). Given a well-posed system (1), define the output $y := \mathbf{Y}(r)v$ and function

$$\boldsymbol{\Lambda}(r) := [\mathbf{Y}\ 0]\,\mathbf{P}^{-1}\begin{bmatrix}\mathbf{G}\\ 0\end{bmatrix}$$

with $\mathbf{Y} : \mathcal{R} \to \mathbb{R}^{n_u\times n_r}$ in $\mathcal{C}^1$. If $\boldsymbol{\Lambda}$ is nonsingular for all $r \in \mathcal{R}_\Phi$, then

$$\mathrm{rank}(\begin{bmatrix}\bar{\mathbf{B}} & \mathbf{G}\end{bmatrix}) = \mathrm{rank}(\begin{bmatrix}\mathbf{B} & \mathbf{Y}^\top\end{bmatrix}) = n_\lambda + n_u, \tag{4}$$

and the state feedback law

$$\tau = \boldsymbol{\Lambda}^{-1}\left(u - \dot{\mathbf{Y}}v + [\mathbf{Y}\ 0]\,\mathbf{P}^{-1}\begin{bmatrix}\boldsymbol{\Gamma}v + \mathbf{d}\\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{B}^\top)\,\dot r\end{bmatrix}\right) \tag{5}$$

transforms (1b) into

$$\mathbf{M}_1(r)\dot z + \mathbf{b}_1(r,y,z) + \mathbf{d}_1(r) = \bar{\mathbf{B}}_1(r)\lambda - \mathbf{G}_1(r)u, \tag{6a}$$

$$\dot y = u, \tag{6b}$$

such that the closed-loop (1a), (6) with constraints (1c)–(1d) is well-posed. Here, $[\mathbf{M}_1\ \mathbf{G}_1] = \mathbf{G}_\perp\mathbf{M}\left[\begin{smallmatrix}\mathbf{Z}\\ \mathbf{Y}\end{smallmatrix}\right]^{-1}$,

$$\mathbf{b}_1(r,y,z) = \mathbf{G}_\perp\left(\boldsymbol{\Gamma} - \mathbf{M}\begin{bmatrix}\mathbf{Z}\\ \mathbf{Y}\end{bmatrix}^{-1}\begin{bmatrix}\dot{\mathbf{Z}}\\ \dot{\mathbf{Y}}\end{bmatrix}\right)v,\ \mathbf{d}_1 = \mathbf{G}_\perp\mathbf{d},$$

[2] The velocity level constraints include the nonholonomic constraints and the time derivative of the holonomic constraints (1c).

[3] The constraint forces $\bar{\mathbf{B}}(r)\lambda$ are said to be workless if $0 = v^\top\bar{\mathbf{B}}(r)\lambda$ for all trajectories. Clearly, $\bar{\mathbf{B}}(r)\lambda$ is workless if Colsp $\bar{\mathbf{B}}$ = Colsp $\mathbf{B}$.

[4] The hidden constraints of (1) are obtained by differentiating (1d) along the system's trajectories.

$\bar{\mathbf{B}}_1 = \mathbf{G}_\perp \bar{\mathbf{B}}$, $z = \mathbf{Z}(r)v$, and $\mathbf{Z} : \mathcal{R} \to \mathbb{R}^{(n_r - n_u)\times n_r}$ is an arbitrary $\mathcal{C}^1$ function for which $\left[\begin{smallmatrix}\mathbf{Z}\\ \mathbf{Y}\end{smallmatrix}\right]$ is nonsingular.

Proof. From Assumption 1 and Proposition 4, $\bar{\mathbf{B}}_\perp \mathbf{M}\mathbf{B}_\perp^\top$ is nonsingular and $\mathbf{\Lambda}$ can be written as

$$\mathbf{Y}\mathbf{B}_\perp^\top \left(\bar{\mathbf{B}}_\perp \mathbf{M}\mathbf{B}_\perp^\top\right)^{-1} \bar{\mathbf{B}}_\perp \mathbf{G}.$$

Clearly, $\mathrm{rank}(\bar{\mathbf{B}}_\perp \mathbf{G}) = \mathrm{rank}(\mathbf{B}_\perp \mathbf{Y}^\top) = n_u$ is a necessary condition for the nonsingularity of $\mathbf{\Lambda}$ that is equivalent to (4) under Lemma 3. Since $n_r > n_u$, define

$$\bar{\Lambda}(r) := \begin{bmatrix} \mathbf{G}_\perp & 0 \\ [\mathbf{Y}\ 0]\,\mathbf{P}^{-1} & \\ 0 & I_{n_\lambda} \end{bmatrix}, \qquad \hat{\Lambda}(r) := \bar{\Lambda}\mathbf{P}\begin{bmatrix} \left[\begin{smallmatrix}\mathbf{Z}\\ \mathbf{Y}\end{smallmatrix}\right]^{-1} & 0 \\ 0 & I_{n_\lambda}\end{bmatrix},$$

and note from Lemma 3 that $\bar{\Lambda}$ and $\hat{\Lambda}$ are nonsingular iff so is $\mathbf{\Lambda}$. Left multiplying (3) by $\bar{\Lambda}$, changing the coordinates with $v = \left[\begin{smallmatrix}\mathbf{Z}\\ \mathbf{Y}\end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix}z\\ y\end{smallmatrix}\right]$, and using the feedback (5) leads to

$$\hat{\Lambda}\begin{bmatrix}\dot{z}\\ \dot{y}\\ \lambda\end{bmatrix} + \begin{bmatrix}\mathbf{\Gamma}_1 v + \mathbf{d}_1 \\ 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{B}^\top \left[\begin{smallmatrix}\mathbf{Z}\\ \mathbf{Y}\end{smallmatrix}\right]^{-1}\right)\begin{bmatrix}z\\ y\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0\\ I_{n_u}\\ 0\end{bmatrix}u.$$

Consequently, the closed-loop is well-posed (see Proposition 5) and can be written as (1a), (6) with (1c)–(1d).

Proposition 6 states that system (1) can be partially linearized by the state feedback (5) whenever the system is well-posed and $\mathbf{\Lambda}(r)$ is nonsingular in the configuration manifold $\mathcal{R}_\Phi$. For this, we require an appropriate selection of the output y and linear independence of the columns of $\left[\bar{\mathbf{B}}(r)\ \mathbf{G}(r)\right]$. The latter is a restriction on the class that prevents the input forces from being dominated by the constraints forces. The former involves picking a $\mathcal{C}^1$ function $\mathbf{Y}$, whose rows are linearly independent of the columns of $\mathbf{B}$. This is not surprising, as we should be able to steer $\dot{y}$ arbitrarily without conflicting with the velocity constraints (1d). Finally, to obtain the partially linearized system, we select the variables $z = \mathbf{Z}(r)v$ such that $\left[\begin{smallmatrix}\mathbf{Z}\\ \mathbf{Y}\end{smallmatrix}\right]$ is nonsingular and continuously differentiable. Furthermore, even if the constraint forces of the nominal system obey the Lagrange-d'Alembert principle, i.e., they are workless, they may lose this feature after the PFL, see Section 6.2.

4. ENERGY SHAPING

Following the results of Cieza and Reger (2019) with IDA-PBC, we present the modified energy shaping for mechanical systems with kinematic constraints.

Proposition 7. Let system (1) verify Assumption 2. Suppose $\mathbf{M}$ as symmetric, $\mathbf{B}$ of full rank, $\mathrm{Colsp}\,\bar{\mathbf{B}} = \mathrm{Colsp}\,\mathbf{B}$ and $\mathbf{d} = \mathbf{J}^\top \partial_r^\top \mathbf{V}$ with $\mathbf{V} : \mathcal{R} \to \mathbb{R}$ in $\mathcal{C}^1$. Let

$$\mathbf{V}(r) - \mathbf{V}(r_\mathrm{d}) > 0 \qquad \forall r \in \mathcal{R}_\Phi - \{r_\mathrm{d}\}, \tag{7a}$$

$$\mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r) \succ 0 \qquad \forall r \in \mathcal{R}_\Phi, \tag{7b}$$

$$v^\top\left(\tfrac{1}{2}\dot{\mathbf{M}} - \mathbf{\Gamma}\right)v \le 0 \qquad \forall (r,v) \in \mathcal{X}_\mathrm{c}. \tag{7c}$$

Then system (1) with zero input ($\tau = 0$) is stable in the equilibrium $(r_\mathrm{d}, 0)$. Asymptotic stability is given whenever the largest invariant set of (1) contained in

$$\Omega = \left\{(r,v) \in \mathcal{X}_\mathrm{c} \mid v^\top\left(\tfrac{1}{2}\dot{\mathbf{M}} - \mathbf{\Gamma}\right)v = 0\right\}$$

possesses no other solution than $(r(t), v(t)) \equiv (r_\mathrm{d}, 0)$.

Proof. By Proposition 4, the conditions (7b), $\mathrm{Colsp}\,\bar{\mathbf{B}} = \mathrm{Colsp}\,\mathbf{B}$ and $\mathrm{rank}\,\mathbf{B} = n_\lambda$ are sufficient to guarantee Assumption 1, meaning that the system at hand is well-posed. Define

$$\mathbf{E} := \tfrac{1}{2}v^\top \mathbf{M} v + \mathbf{V}.$$

Its time derivative along the system's trajectories reads

$$\dot{\mathbf{E}} = v^\top\left(\tfrac{1}{2}\dot{\mathbf{M}} - \mathbf{\Gamma}\right)v,$$

which is obtained from $\mathbf{d} = \mathbf{J}^\top \partial_r^\top \mathbf{V}$, $\tau = 0$ and the workless feature of the constraint forces imposed by $\mathrm{Colsp}\,\bar{\mathbf{B}} = \mathrm{Colsp}\,\mathbf{B}$. Since $\mathcal{X}_\mathrm{c}$ is a regular manifold, let $(\mathcal{N}, \Psi)$ be a coordinate chart on $\mathcal{X}_\mathrm{c}$ with local coordinates $\bar{x}$ s.t. $(r_\mathrm{d}, 0) \in \mathcal{N}$. Then, (1) can be locally expressed as an ODE with coordinates $\bar{x} \in \Psi(\mathcal{N}) \subset \mathbb{R}^{2n_r - n_\phi - n_\lambda}$. Define $\bar{E}(\bar{x}) := \mathbf{E}(\Psi^{-1}(\bar{x})) - \mathbf{E}(r_\mathrm{d}, 0)$. From Lyapunov's Theorem, the point $\bar{x}_\mathrm{d} := \Psi(r_\mathrm{d}, 0)$ is a stable equilibrium of the local ODE if $\bar{E}$ is positive definite and $\dot{\bar{E}}$ is negative semidefinite, both about $\bar{x}_\mathrm{d}$. For this, rewrite

$$\mathcal{X}_\mathrm{c} = \left\{(r, v \mid r \in \mathcal{R}_\Phi,\, \bar{v} \in \mathbb{R}^{n_r - n_\lambda},\, v = \mathbf{B}_\perp^\top(r)\bar{v}\right\},$$

and note that $\mathbf{E}|_{\mathcal{X}_\mathrm{c}}$ can be expressed as

$$\mathbf{E}|_{\mathcal{X}_\mathrm{c}} = \tfrac{1}{2}v^\top \mathbf{M} v\big|_{\mathcal{X}_\mathrm{c}} + \mathbf{V}\big|_{\mathcal{R}_\Phi} = \tfrac{1}{2}\bar{v}^\top \mathbf{B}_\perp \mathbf{M}\mathbf{B}_\perp^\top \bar{v}\big|_{\mathcal{R}_\Phi} + \mathbf{V}\big|_{\mathcal{R}_\Phi}.$$

Thus, conditions (7) are sufficient for $\mathbf{E}\big|_\mathcal{N}$ being positive definite about $(r_\mathrm{d}, 0)$ and $\dot{\mathbf{E}}\big|_\mathcal{N}$ being negative semidefinite, i.e., $\bar{x}_\mathrm{d}$ is a stable equilibrium of the local ODE and $(r_\mathrm{d}, 0)$ is a stable equilibrium of (1). Finally, asymptotic stability follows from a standard application of LaSalle's invariance principle on the underlying ODE, obtained from (1a) and (3), with a positively invariant set contained in $\mathcal{N}$.

Proposition 7 offers sufficient conditions to guarantee (asymptotic) stability in the equilibrium $(r_\mathrm{d}, 0)$. The subsequent proposition shapes the total energy of mechanical systems ensuring (asymptotic) stabilization of an admissible equilibrium.

Lemma 8. Given $A \in \mathbb{R}^{n\times s}$ and $G \in \mathbb{R}^{n\times m}$, the equation

$$0 = A + GK$$

has a solution $K \in \mathbb{R}^{m\times s}$ if and only if $0 = G_\perp A$. If a solution exists, they are all of the form $K = -G^\mathrm{g}A + G_\mp \nu$, where ν is arbitrary and of adequate size.

Proof. See Piziak and Odell (2007).

Proposition 9. Let system (1) with $\mathbf{B}(r)$ of full rank verify Assumption 2. Suppose there exist $\mathcal{C}^1$ functions $\mathbf{X} : \mathcal{R} \to \mathbb{R}^{n_r\times n_r}_{n_r}$ and $\mathbf{V}_\mathrm{d} : \mathcal{R} \to \mathbb{R}$ with $\mathbf{X}\mathbf{M}^\top = \mathbf{M}\mathbf{X}^\top$ s.t.

$$\mathbf{N}_\perp\left(\mathbf{d} - \mathbf{X}\mathbf{J}^\top \partial_r^\top \mathbf{V}_\mathrm{d}\right) = 0 \qquad \forall r \in \mathcal{R}_\Phi, \tag{8a}$$

$$\mathbf{N}_\perp \mathbf{X}\mathbf{B} = 0 \qquad \forall r \in \mathcal{R}_\Phi, \tag{8b}$$

with $\mathbf{N} = \left[\mathbf{G}\ \bar{\mathbf{B}}\right]$. The feedback

$$\tau = [I_{n_u}\ 0]\,\mathbf{N}^\mathrm{g}\left(\mathbf{d} - \mathbf{X}\mathbf{J}^\top \partial_r^\top \mathbf{V}_\mathrm{d} + \mathbf{X}\mathbf{B}\lambda_\mathrm{d} + \mathbf{N}\mathbf{K}\mathbf{X}^{-\top}v\right) \tag{9}$$

with $\mathbf{K} : \mathcal{R} \times \mathbb{R}^{n_r} \to \mathbb{R}^{n_r\times n_r}$ transforms (1b) into

$$\mathbf{X}^{-1}\mathbf{M}\dot{v} + \mathbf{X}^{-1}\left(\mathbf{\Gamma} - \mathbf{N}\mathbf{K}\mathbf{X}^{-\top}\right)v + \mathbf{J}^\top \partial_r^\top \mathbf{V}_\mathrm{d} = \mathbf{B}\lambda_\mathrm{d}, \tag{10}$$

where λ_d is the new implicit variable. Suppose

$$\mathbf{V}_\mathrm{d}(r) - \mathbf{V}_\mathrm{d}(r_\mathrm{d}) > 0 \quad \forall r \in \mathcal{R}_\Phi - \{r_\mathrm{d}\}, \tag{11a}$$

$$\mathbf{B}_\perp(r)\mathbf{X}^{-1}\mathbf{M}(r)\mathbf{B}_\perp^\top(r) \succ 0 \quad \forall r \in \mathcal{R}_\Phi, \tag{11b}$$

$$v^\top \mathbf{X}^{-1}\mathbf{W}\mathbf{X}^{-\top}v \le 0 \quad \forall (r,v) \in \mathcal{X}_\mathrm{c} \tag{11c}$$

where $\mathbf{W} = \frac{1}{2}\dot{\mathbf{M}}\mathbf{X}^\top - \frac{1}{2}\dot{\mathbf{X}}\mathbf{M}^\top - \mathbf{\Gamma}\mathbf{X}^\top + \mathbf{N}\mathbf{K}$. Then the closed-loop (1a), (10) with constraints (1c) and (1d) is stable in $(r_\mathrm{d}, 0)$. Furthermore, asymptotic stability is achieved whenever the largest invariant set of the closed-loop contained in

$$\Omega_\mathrm{d} = \left\{(r,v) \in \mathcal{X}_\mathrm{c} \mid v^\top \mathbf{X}^{-1}\mathbf{W}\mathbf{X}^{-\top} v = 0\right\}$$

possesses no other solution than $(r(t), v(t)) \equiv (r_\mathrm{d}, 0)$.

Proof. Multiplying (10) on the left by $\mathbf{X}$ and equating the result with (1b) yields

$$\mathbf{N}\begin{bmatrix}\tau\\ \lambda\end{bmatrix} = \mathbf{d} - \mathbf{X}\mathbf{J}^\top \partial_r^\top \mathbf{V}_\mathrm{d} + \mathbf{X}\mathbf{B}\lambda_\mathrm{d} + \mathbf{N}\mathbf{K}\mathbf{X}^{-\top} v. \qquad (12)$$

Thus, from Lemma 8 and (12), conditions (8) with feedback (9) are sufficient to transform (1b) into (10). Note that λ_d is the new implicit variable of the system, which means that the solution of λ is irrelevant. The stability results are obtained from a straightforward application of Proposition 7 on the closed-loop (1a), (10), where $\mathbf{X}^{-1}\mathbf{M}$ is symmetric from $\mathbf{X}\mathbf{M}^\top = \mathbf{M}\mathbf{X}^\top$.

The actual synthesis with this proposition starts by choosing $\mathbf{N}_\perp$ and r_d. Then, from $\mathbf{X}\mathbf{M}^\top = \mathbf{M}\mathbf{X}^\top$, (8b) and (11b)–(11c), we obtain a solution of $\mathbf{X}$ and $\mathbf{K}$. Imposing $\mathbf{M}\mathbf{X}^\top \succ 0$ is sufficient for (11b) but it requires the nonsingularity of $\mathbf{M}$ and may hinder the solution of $\mathbf{X}$ in a similar way as in Cieza and Reger (2019) for IDA-PBC. Next, we solve $\mathbf{V}_\mathrm{d}$ from (8a) and (11a). Here, (11a)–(11b) are locally equivalent to

$$\partial_r(\mathbf{V}_\mathrm{d} + \mu_\mathrm{d}^\top \Phi)\big|_{r=r_\mathrm{d}} = 0, \qquad (13a)$$

$$\left(\partial_r^\top \Phi\right)_\perp \partial_{rr}(\mathbf{V}_\mathrm{d} + \mu_\mathrm{d}^\top \Phi) \left(\partial_r^\top \Phi\right)_\perp^\top\Big|_{r=r_\mathrm{d}} \succ 0, \qquad (13b)$$

$$\mathbf{B}_\perp \mathbf{X}^{-1}\mathbf{M}\mathbf{B}_\perp^\top\big|_{r=r_\mathrm{d}} \succ 0 \qquad (13c)$$

for some constant $\mu_\mathrm{d} \in \mathbb{R}^{n_\lambda}$.[5] In case $\mathbf{V}_\mathrm{d}$ cannot be obtained, we search for a different $\mathbf{X}$ in the previous step. If $\operatorname{rank}\mathbf{N} = n_r$, the matching conditions (8) are trivial, and there always exist $\mathbf{V}_\mathrm{d}$, $\mathbf{X}$ and $\mathbf{K}$ verifying (11). Note that the matching of the potential energy and constraints, given by (8), are similar to the ones of Cieza and Reger (2019), but the matching of kinetic energy, given by (11c), is not and require the solution of DAEs instead of PDEs. Lastly, we calculate λ_d from (10) and the hidden constraints, select $\mathbf{N}^\mathrm{g}$, and build feedback (9). If $n_u + n_\lambda > \operatorname{rank}\mathbf{N}$, we may add $\mathbf{N}_\top \nu$ to (9) with arbitrary ν, see Lemma 8.

5. EQUIVALENCE BETWEEN REPRESENTATIONS

Since the DAE systems that we discuss are evolving on a regular manifold, there always exists a local ODE that represent its behavior. In other words, we can eliminate the kinematic constraints and provide a reduced system. This is fundamental to compare the results between the implicit and explicit representations.

Proposition 10. (Reduction). Consider the well-posed system (1). Let $(\mathcal{N}, \xi^{-1})$ be a coordinate chart on $\mathcal{R}_\Phi$ with local coordinates $q \in \xi^{-1}(\mathcal{N}) \subset \mathbb{R}^{n_r - n_\Phi}$, Let $T : \xi^{-1}(\mathcal{N}) \to \mathbb{R}^{(n_r - n_\lambda)\times n_r}$ and $\bar{T} : \xi^{-1}(\mathcal{N}) \to \mathbb{R}^{(n_r - n_\lambda)\times n_r}$ be full-rank left annihilators of $\mathbf{B} \circ \xi$ and $\bar{\mathbf{B}} \circ \xi$, respectively, where $T \in \mathcal{C}^1$. Then, for all $r \in \mathcal{N}$, system (1) with $v = T^\top(q)s \in \mathbb{R}^{n_r - n_\lambda}$ and $r = \xi(q)$ can be reduced to

$$\dot{q} = J(q)s, \qquad (14a)$$

$$M(q)\dot{s} + \Gamma(q,s)s + d(q) = G(q)\tau, \qquad (14b)$$

[5] If there are no holonomic constraints, conditions (13a)–(13b) reduce to $\partial_r \mathbf{V}_\mathrm{d}|_{r=r_\mathrm{d}} = 0$ and $\partial_{rr}\mathbf{V}_\mathrm{d}|_{r=r_\mathrm{d}} \succ 0$.

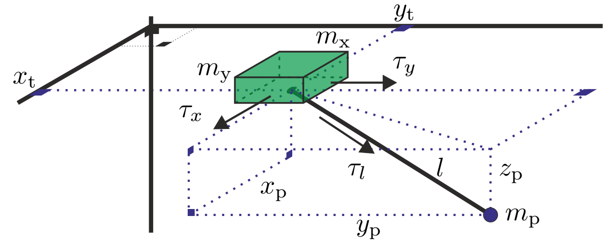

Fig. 1. Diagram of the portal crane.

where $M(q) = \bar{T}\mathbf{M}T^\top$ is nonsingular, $G(q) = \bar{T}\mathbf{G}$, $J(q) = (\partial_q \xi)^\mathrm{g} \mathbf{J}T^\top$, $\Gamma(q,s) = \bar{T}\mathbf{M}\dot{T}^\top s + \bar{T}\mathbf{\Gamma}v$, $v = Ls$, $L(q) = \mathbf{M}T^\top M^{-1}$, and $d(q) = \bar{T}\mathbf{d}$.

Proof. Consider the change of variables

$$r = \hat{\xi}(\hat{q}) := \xi(q) + \left((\partial_q \xi)^\mathrm{g}\right)_\top q_\mathrm{x}, \qquad (15a)$$

$$v = \hat{T}^\top(q)\hat{s} := \left[T^\top(q)\ \mathbf{B}(\xi(q))\right]\hat{s}, \qquad (15b)$$

with $\hat{s} = \operatorname{vec}(s, s_\mathrm{x})$, $\hat{q} = \operatorname{vec}(q, q_\mathrm{x})$, $s_\mathrm{x} \in \mathbb{R}^{n_\lambda}$ and $q_\mathrm{x} \in \mathbb{R}^{n_\Phi}$. By definition of ξ and T, it follows that $\partial_{\hat{q}}\hat{\xi}\big|_{q_\mathrm{x}=0}$ and $\hat{T}$ are nonsingular for all $q \in \xi^{-1}(\mathcal{N})$ and a neighborhood of $q_\mathrm{x} = 0$, i.e., the map $(\hat{q}, \hat{s}) \mapsto (\hat{\xi}(\hat{q}), \hat{T}^{-1}(q)\hat{s})$ is a $\mathcal{C}^1$ diffeomorphism where $q_\mathrm{x}(\cdot) = 0$ for every solution $r(\cdot) \in \mathcal{R}_\Phi$. Besides, substituting (15b) into (1d) shows that every solution $v(\cdot) \in \mathcal{X}_\mathrm{c}$ implies $s_\mathrm{x}(\cdot) = 0$. Now, replacing (15) with $q_\mathrm{x} = 0$ and $s_\mathrm{x} = 0$ into (1a)–(1b), and left multiplying the result by $\left[\bar{T}^\top\ \bar{\mathbf{B}}\right]^\top$, which is full rank in $\mathcal{R}_\Phi$, yields (14b) with

$$\dot{q} = \left(\partial_{\hat{q}}\hat{\xi}\big|_{q_\mathrm{x}=0}\right)^{-1}\mathbf{J}T^\top s, \qquad \bar{\mathbf{B}}^\top \mathbf{M}T^\top \dot{s} = \star + \bar{\mathbf{B}}^\top \bar{\mathbf{B}}\lambda,$$

where '$\star$' shall denote unspecified elements. Since

$$\left(\partial_{\hat{q}}\hat{\xi}\big|_{q_\mathrm{x}=0}\right)^{-1} = \begin{bmatrix}(\partial_q\xi)^\mathrm{g}\\ L^{-1}\left(\partial_q\xi\right)_\perp\end{bmatrix}, \quad L(q) = \left(\partial_q\xi\right)_\perp\left((\partial_q\xi)^\mathrm{g}\right)_\top,$$

$\operatorname{Rowsp}\left(\partial_q\xi\right)_\perp = \operatorname{Rowsp}\partial_r\Phi$, $\operatorname{Rowsp}\left(\partial_r\Phi\mathbf{J}\right) \subset \operatorname{Rowsp}\mathbf{B}^\top$, we can write

$$\left(\partial_{\hat{q}}\hat{\xi}\big|_{q_\mathrm{x}=0}\right)^{-1}\mathbf{J}T^\top = \left[J^\top\ 0\right]^\top,$$

which is consistent with $q_\mathrm{x}(\cdot) = 0$ and (14a). Finally, nonsingularity of $\bar{T}\mathbf{M}T^\top$ is deduced from (2) with a well-posed system (1), giving a unique solution to $\dot{s}$ and λ.

Proposition 10 removes the kinematic constraints of system (1), reducing it from an implicit model of $2n_r$ states and $n_\lambda + n_\Phi$ constraints to an explicit one of $2n_r - n_\lambda - n_\Phi$ states. The new states q represent generalized coordinates while the states s are projection of $\dot{r}$. Note that Proposition 10 offers high flexibility in the representation of the reduced system (14) since ξ, T, $\bar{T}$ and $\left(\partial_q\xi\right)^\mathrm{g}$ are non-unique. However, all reductions are locally equivalent because they rely on the topology of the constrained state space manifold $\mathcal{X}_\mathrm{c}$, meaning that the coordinates we choose to represent the dynamics are immaterial.

6. EXAMPLE ON A 5-DOF PORTAL CRANE

For illustration, let us consider the portal (or overhead) crane, shown in Figure 1. We shall design an energy shaping controller that suppresses the payload swing and stabilizes it in a desired position. Inspired by Vidal et al. (2020) for the case with 4-DoF, we will derive a 5-DoF model, and use the PFL of Proposition 6 followed by the total energy shaping of Proposition 9.

6.1 Implicit Model

The crane system is composed of a bridge of mass $m_\mathrm{x} - m_\mathrm{y}$, a trolley of mass m_y mounted under the bridge, and a hanging payload of mass m_p. The bridge slides along parallel runways in the x-axis and is actuated by the force τ_x, whereas the trolley does so along the bridge in the y-axis and is actuated by the force τ_y. A winch attached to the trolley supports the payload by exerting a force τ_l on the rope of length l. We denote by $(x_\mathrm{t}, y_\mathrm{t}, 0)$ the trolley position in the inertial frame and by $(x_\mathrm{p}, y_\mathrm{p}, z_\mathrm{p})$ the payload position relative to $(x_\mathrm{t}, y_\mathrm{t}, 0)$ that satisfies

$$0 = \Phi(r) := \tfrac{1}{2}\left(x_\mathrm{p}^2 + y_\mathrm{p}^2 + z_\mathrm{p}^2 - l^2\right). \tag{16a}$$

The following assumptions are made: *i)* The payload is a point mass. *ii)* The rope mass is negligible and $l > 0$. *iii)* The gravity of magnitude g_c points downwards (direction $-z$). *iv)* The initial conditions are consistent (Assumption 2 holds). *v)* The Rayleigh dissipation function is of the form $\hat{\mathbf{D}}(r, \dot{r}) := \frac{1}{2}c_x\dot{x}_\mathrm{t}^2 + \frac{1}{2}c_y\dot{y}_\mathrm{t}^2 + \frac{1}{2}c_l\dot{l}^2$.

By choosing the coordinates

$$r = \mathrm{vec}(x_\mathrm{p}, y_\mathrm{p}, z_\mathrm{p}, x_\mathrm{t}, y_\mathrm{t}, l) \in \mathcal{R} = \mathbb{R}^6,$$

we have the Lagrange equations of motion

$$\hat{\mathbf{M}}\ddot{r} + \partial_{\dot{r}}^\top\hat{\mathbf{D}} + \partial_r^\top\hat{\mathbf{V}} = \partial_r^\top\Phi\hat{\lambda} + \hat{\mathbf{G}}\tau, \tag{16b}$$

where $\tau = \mathrm{vec}(\tau_x, \tau_y, \tau_l)$, $\hat{\mathbf{G}} = [0_{3\times3}\ I_3]^\top$, $\hat{\mathbf{V}} = g_\mathrm{c}m_\mathrm{p}z_\mathrm{p}$,

$$\hat{\mathbf{M}} = \begin{bmatrix} m_\mathrm{p} & 0 & 0 & m_\mathrm{p} & 0 & 0 \\ 0 & m_\mathrm{p} & 0 & 0 & m_\mathrm{p} & 0 \\ 0 & 0 & m_\mathrm{p} & 0 & 0 & 0 \\ m_\mathrm{p} & 0 & 0 & m_\mathrm{p}+m_\mathrm{x} & 0 & 0 \\ 0 & m_\mathrm{p} & 0 & 0 & m_\mathrm{p}+m_\mathrm{y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the system has 5 DoF, the holonomic constraint is given by (16a), and the velocity constraint reads

$$0 = \partial_r\Phi\,\dot{r}. \tag{16c}$$

Since $\mathbf{P} = \left[\begin{smallmatrix} \hat{\mathbf{M}} & -\partial_r^\top\Phi \\ \partial_r\Phi & 0 \end{smallmatrix}\right]$ is nonsingular, Assumption 1 holds and system (16) is well-posed even though $\hat{\mathbf{M}}$ is singular (see Proposition 5). In previous works, the singularity was conveniently avoided by fixing l to be constant, i.e., obtaining a model with 4-DoF (Vidal et al., 2020). Now, let us pick the generalized coordinates $q = \mathrm{vec}(\beta, \alpha, x_\mathrm{t}, y_\mathrm{t}, l)$ verifying $r = \xi(q)$ with

$$x_\mathrm{p} = l\sin\beta,\ \ y_\mathrm{p} = l\cos\beta\sin\alpha,\ \ z_\mathrm{p} = -l\cos\beta\cos\alpha.$$

It is easy to check that the Lagrange equations obtained with coordinates q are equal to the ones of (16) after the reduction of Proposition 10 and $T = \bar{T} = \partial_q^\top\xi$.

6.2 Partial Feedback Linearization

Choosing $y = \mathrm{vec}(\dot{x}_\mathrm{t}, \dot{y}_\mathrm{t}, \dot{l}) = \mathbf{Y}\dot{r}$, we see that

$$\det\mathbf{\Lambda} = \frac{x_\mathrm{p}^2 + y_\mathrm{p}^2 + z_\mathrm{p}^2}{l^2 m_\mathrm{p}m_\mathrm{y}m_\mathrm{x}} = \frac{1}{m_\mathrm{p}m_\mathrm{y}m_\mathrm{x}} > 0, \quad \forall r \in \mathcal{R}_\Phi.$$

Hence, feedback (5) with $\hat{\mathbf{G}}_\perp = m_\mathrm{p}^{-1}[I_3\ 0_{3\times3}]$ and $z = \mathrm{vec}(\dot{x}_\mathrm{p}, \dot{y}_\mathrm{p}, \dot{z}_\mathrm{p}) = \mathbf{Z}\dot{r}$ transforms (16b) into

$$\ddot{r} + \partial_r^\top\mathbf{V} = \bar{\mathbf{B}}\lambda + \mathbf{G}u, \tag{17}$$

where $\mathbf{V} = g_\mathrm{c}z_\mathrm{p}$, $\bar{\mathbf{B}}(r) = \mathrm{vec}(x_\mathrm{p}, y_\mathrm{p}, z_\mathrm{p}, 0, 0, 0)$ and

$$\mathbf{G} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

Given that $\mathrm{Colsp}\,\bar{\mathbf{B}}(r) \neq \mathrm{Colsp}\,\partial_r^\top\Phi$, the constraint forces are not workless for every trajectory of the system, which was also a limitation on previous approaches (Cieza and Reger, 2019; Castaños and Gromov, 2016). Observe that (17) is much simpler than (16b) because we only require the constant gravity parameter g_c. Now, following Vidal et al. (2020), feedback (5) can be approximated by a velocity tracking controller for y with an integrator in its input. Such a velocity controller may be for example a PID plus feedforward, meaning that we may avoid identification of masses, frictions and others parameters. Using Proposition 10 on (16a), (16c), (17) with $T(q) = \partial_q^\top\xi$ and

$$\bar{T}(q) = \begin{bmatrix} l\cos(\beta) & -l\sin(\alpha)\sin(\beta) & l\cos(\alpha)\sin(\beta) & 0 & 0 & 0 \\ 0 & l\cos(\alpha)\cos(\beta) & l\cos(\beta)\sin(\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we obtain the same system as if the PFL was implemented on the Lagrangian equations with generalized coordinates, showing equivalence between the implicit and explicit perspectives. However, note that *i)* the reduced system does not fit into the Lagrangian framework, *ii)* the underactuation degree is greater than one, and *iii)* M depends on the actuated coordinate l. These features prevent us from using standard methods such as the well-known PID-PBC of Donaire et al. (2016a) to shape the total energy which could otherwise be very challenging.

6.3 Total Energy Shaping

To shape the energy of (17), we start by choosing

$$\mathbf{N}_\perp = \begin{bmatrix} -z_\mathrm{p} & 0 & x_\mathrm{p} & -z_\mathrm{p} & 0 & 0 \\ 0 & -z_\mathrm{p} & y_\mathrm{p} & 0 & -z_\mathrm{p} & 0 \end{bmatrix}$$

and $r_\mathrm{d} = \mathrm{vec}(0, 0, -l^\star, x_\mathrm{t}^\star, y_\mathrm{t}^\star, l^\star)$ with $l^\star > 0$. For simplicity, consider $\mathbf{X} = \mathbf{X}^\top$ to be constant and write

$$\mathbf{V}_\mathrm{d} = \mathbf{V} + \frac{k_1}{2}(x_\mathrm{t} - x_\mathrm{t}^\star)^2 + \frac{k_2}{2}(y_\mathrm{t} - y_\mathrm{t}^\star)^2 + \frac{k_3}{2}(l - l^\star + b)^2$$

with constants $b, k_i \in \mathbb{R}$. Hence, conditions (8), (11b) and (11c) are algebraic and can be satisfied with [6]

$$\mathbf{X} = \begin{bmatrix} a_1+1 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & a_3+1 & 0 & 0 & -a_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -a_1 & 0 & 0 & a_1 & 0 & 0 \\ 0 & -a_3 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 \end{bmatrix}, \quad \mathbf{K} = -\mathbf{K}_\mathrm{c}\mathbf{N}^\top,$$

$\mathbf{K}_\mathrm{c} = \mathrm{diag}(c_1, c_2, c_3, 0)$ for any positive constants a_i and c_i. Then, from (13a)–(13b), $\mathbf{V}_\mathrm{d}\big|_{\mathcal{R}_\Phi}$ has a strict local minimum in r_d whenever $\mu_\mathrm{d} = \frac{g_\mathrm{c}}{l^\star}$, $b = \frac{g_\mathrm{c}}{k_3}$, and $k_i > 0$, $i = \{1, 2, 3\}$. Finally, we pick $[I_3\ 0_{3\times1}]\,\mathbf{N}^\mathrm{g} = [0_{3\times3}\ I_3]$, calculate λ_d, and build the stabilizing controller (9) with $\mathbf{N}_\mathcal{I}\nu = 0$. Similarly, the energy shaping can also be applied to the system without PFL, namely (16b) where $\hat{\mathbf{M}}$ is singular.

6.4 Simulation

Figure 2 shows the simulation results for the portal crane in PFL with controller parameters $k_1 = k_2 = 5$, $k_3 = 4$, $a_1 = a_2 = a_3 = 1$, $c_1 = c_2 = c_3 = 5$ and initial

[6] By using Prop. 10, it can be seen that the target inertial matrix of the reduced closed-loop is state dependent even though $\mathbf{X}$ is constant.

conditions $x_{\mathrm{p}}(0) = y_{\mathrm{p}}(0) = x_{\mathrm{t}}(0) = 0\mathrm{m}$, $y_{\mathrm{t}}(0) = 0.8\mathrm{m}$, $l(0) = 1\mathrm{m}$ and $\dot{r}(0) = 0$. The desired position r_{d} is set to *i)* $x_{\mathrm{t}}^{\star} = 0\mathrm{m}$, $y_{\mathrm{t}}^{\star} = 0.8\mathrm{m}$ and $l^{\star} = 1\mathrm{m}$ for $t \in [0, 1[\mathrm{s}$, and *ii)* $x_{\mathrm{t}}^{\star} = 1\mathrm{m}$, $y_{\mathrm{t}}^{\star} = 0.2\mathrm{m}$ and $l^{\star} = 0.5\mathrm{m}$ for $t \in [1, 15[\mathrm{s}$. Here $\mathbf{E}_{\mathrm{d}} := \frac{1}{2}\dot{r}^{\top}\mathbf{X}^{-1}\dot{r} + \mathbf{V}_{\mathrm{d}}$ is the energy of (17) in closed-loop with (9), where $\mathbf{V}_{\mathrm{d}}$ is shifted s.t. $\mathbf{V}_{\mathrm{d}}(r_{\mathrm{d}}) = 0$. Clearly, the controller achieves asymptotic stabilization of r_{d} and $\mathbf{E}_{\mathrm{d}}$ is monotonically decreasing as expected.

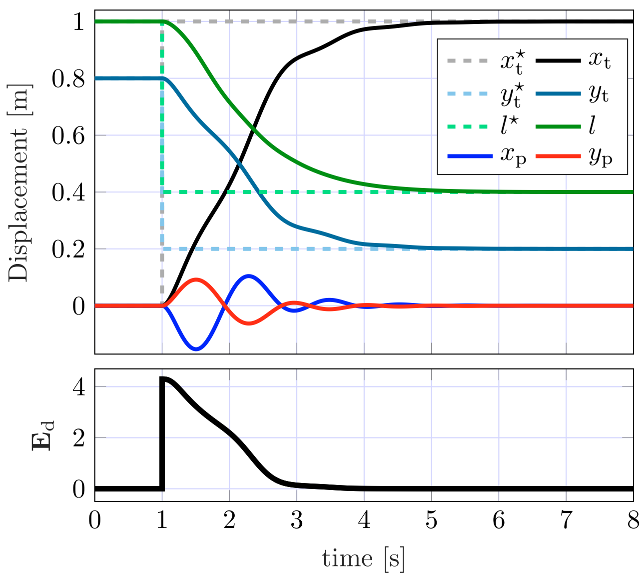

Fig. 2. Closed-loop simulation of the portal crane.

7. CONCLUSION

In this work, we formalize the PFL for constrained mechanical systems. The method requires input forces and constraint forces in different subspaces, and its application may break with the workless feature of the constrained forces. For the energy shaping, we can successfully include systems with PFL, extending previous results. Besides, the obtained matching conditions are similar to the ones of Cieza and Reger (2019) for IDA-PBC. The main difference lies in the kinetic matching, which in our case may require the solution of DAEs instead of PDEs. Both the PFL and energy shaping can be used with singular inertia matrices, which is not feasible with port-Hamiltonian systems without modification.

The approaches are verified on the 5-DoF crane, whose implicit model has a singular inertia matrix. After PFL, the resulting inertia matrix is nonsingular, but the constraint forces are not workless. However, the model is much simpler because it only depends on the gravity constant as a parameter. Finally, for illustration, the total energy shaping is carried out with a constant target inertia matrix and a candidate potential energy that was chosen beforehand.

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Appendix A. PROOF OF LEMMA 3

Given $B \in \mathbb{R}^{n\times s}$ and $C \in \mathbb{R}^{n\times(m-s)}$, we have

$$\begin{aligned}\operatorname{rank}[B\ C] &= \operatorname{rank}\left(\begin{bmatrix}C_{\perp}\\ C^{\top}\end{bmatrix}[B\ C]\begin{bmatrix}I_s & 0\\ F & I_{m-s}\end{bmatrix}\right)\\ &= \operatorname{rank}\begin{bmatrix}C_{\perp}B & 0\\ C^{\top}(B+CF) & C^{\top}C\end{bmatrix}\end{aligned} \quad \text{(A.1)}$$

for some $F \in \mathbb{R}^{(m-s)\times s}$, where $\operatorname{rank}\left[C\ C_{\perp}^{\top}\right] = n$. Since,

$$C^{\top}(B+CF) = 0 \quad \iff \quad \exists P \text{ s.t. } \left[C\ C_{\perp}^{\top}\right]\begin{bmatrix}F\\ P\end{bmatrix} = B,$$

there always exists F verifying $C^{\top}(B+CF) = 0$, and our claim follows easily from (A.1) with such an F.