Smoothness Morrey Spaces and Differences: Characterizations and Applications

Dissertation

zur Erlangung des akademischen Grades

doctor rerum naturalium

vorgelegt dem Rat der Fakultät für Mathematik und Informatik

der Friedrich-Schiller-Universität Jena

von M. Sc. Marc Friedrich Hovemann

geboren am 22.06.1991 in Speyer

Gutachter

1. Prof. Dr. Winfried Sickel (Friedrich-Schiller-Universität Jena)

2. Prof. Dr. Wen Yuan (Beijing Normal University, People's Rebuplic of China)

3. Prof. Dr. Leszek Skrzypczak (Adam Mickiewicz University, Poznan, Poland)

Tag der öffentlichen Verteidigung : 14.5.2021

Contents

Chapter 1

Zusammenfassung entsprechend § 8 der Promotionsordnung

Im Rahmen dieser Abhandlung beschäftigen wir uns mit Triebel-Lizorkin-Morrey Räumen, Besov-Morrey Räumen und Besov-type Räumen. Sie alle gehören zu den so genannten Glattheits Morrey Räumen. Die Theorie dieser Funktionenräume baut auf jener der heutzutage gut bekannten Besov Räume $B^s_{p,q}(\mathbb{R}^d)$ und Triebel-Lizorkin Räume $F^s_{p,q}(\mathbb{R}^d)$ auf. Im Prinzip erhält man unsere Glattheits Morrey Räume relativ einfach aus den Definitionen für die Skalen $B^s_{p,q}(\mathbb{R}^d)$ und $F^s_{p,q}(\mathbb{R}^d)$, indem man die Lebesguenorm dort durch eine Morreynorm ersetzt. Aus diesem Grund ist die Definition der Morrey Räume sowie der dazugehörigen Quasinorm für diese Abhandlung von zentraler Bedeutung. Sie lautet wie folgt.

Zentrale Definition 1. ***Morrey Räume.***
Es gelte $0 < p \leq u < \infty$. Dann definieren wir die Morrey Räume $\mathscr{M}^u_p(\mathbb{R}^d)$ als Menge aller Funktionen $f \in L^{loc}_p(\mathbb{R}^d)$, welche

$$\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| := \sup_{y\in\mathbb{R}^d, r>0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \left(\int_{B(y,r)} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty$$

erfüllen. Hierbei wird das Supremum über alle Kugeln $B(y,r) \subset \mathbb{R}^d$ mit Mittelpunkt y und Radius r genommen.

Die Morrey Räume stellen eine Verallgemeinerung der klassischen Lebesgueräume dar und bilden die Grundlage für die Definition unserer Glattheits Morrey Räume. Für diese benötigen wir einige Begriffe aus der Fourier-Analysis. So bezeichnen wir mit $\mathscr{F}$ die Fourier-Transformation. $\mathscr{F}^{-1}$ ist die dazu inverse Transformation. Darüber hinaus arbeiten wir mit einer glatten dyadischen Zerlegung der Eins, abgekürzt durch $(\varphi_k)_{k\in\mathbb{N}_0}$. Basierend auf diesen Begrifflichkeiten ist es nun möglich, die Besov-Morrey Räume $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ sowie die Triebel-Lizorkin-Morrey Räume $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ zu erklären.

Zentrale Definition 2. ***Glattheits Morrey Räume.***
Es gelte $s \in \mathbb{R}$, $0 < p \leq u < \infty$ und $0 < q \leq \infty$. Mit $(\varphi_k)_{k\in\mathbb{N}_0}$ bezeichnen wir eine glatte dyadische Zerlegung der Eins.

(i) Dann definieren wir die Besov-Morrey Räume $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ als Menge aller temperierten Distributionen $f \in \mathcal{S}'(\mathbb{R}^d)$, welche

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| := \Big(\sum_{k=0}^{\infty} 2^{ksq}\|\mathcal{F}^{-1}[\varphi_k\mathcal{F}f]|\mathcal{M}^u_p(\mathbb{R}^d)\|^q\Big)^{\frac{1}{q}} < \infty$$

erfüllen. Im Fall $q=\infty$ sind entsprechende Modifikationen nötig.

(ii) Die Triebel-Lizorkin-Morrey Räume $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ definieren wir als Menge aller Distributionen $f \in \mathcal{S}'(\mathbb{R}^d)$, für die

$$\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| := \Big\|\Big(\sum_{k=0}^{\infty} 2^{ksq}|\mathcal{F}^{-1}[\varphi_k\mathcal{F}f](x)|^q\Big)^{\frac{1}{q}}\Big|\mathcal{M}^u_p(\mathbb{R}^d)\Big\| < \infty$$

gilt. Auch hier sind im Fall $q=\infty$ entsprechende Modifikationen nötig.

Auf ähnliche Weise lassen sich auch die Besov-type Räume $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ definieren. Hier verweisen wir auf Definition 23. Die gerade vorgestellten Glattheits Morrey Räume stellen Verallgemeinerungen der ursprünglichen Besov und Triebel-Lizorkin Räume dar. So beobachten wir $\mathcal{N}^s_{p,p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d)$ und $\mathcal{E}^s_{p,p,q}(\mathbb{R}^d) = F^s_{p,q}(\mathbb{R}^d)$. Blickt man zurück auf die Anfänge der Besov und Triebel-Lizorkin Räume, so stellt man fest, dass diese ursprünglich mittels Differenzen höherer Ordnung beschrieben wurden. Man betrachte hierfür etwa [89], [4] und [71]. Modernere Differenzencharakterisierungen, wie sie sich zum Beispiel in den Theoremen 1 und 2 finden, sind sehr übersichtlich und beschreiben die Glattheits- und Integrierbarkeitseigenschaften der Skalen $B^s_{p,q}(\mathbb{R}^d)$ und $F^s_{p,q}(\mathbb{R}^d)$ auf anschauliche Art und Weise. Auch haben sie sich als nützliches Hilfsmittel bei zahlreichen Anwendungen erwiesen. Aus all diesen Gründen ist es ein Kernziel dieser Abhandlung, Charakterisierungen mittels Differenzen für die Räume $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ und $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ herzuleiten. Diese sollen möglichst übersichtlich sein und für einen möglichst großen Parameterbereich Gültigkeit besitzen. Die bereits in der Literatur existierenden Resultate, siehe Kapitel 4.3 in [144] sowie [116] und [29], sollen deutlich erweitert werden. Tatsächlich ist es im Rahmen dieser Abhandlung gelungen, die folgenden Charakterisierungen zu beweisen. Für $x,h \in \mathbb{R}^d$ und $N \in \mathbb{N}$ bezeichnen wir mit $\Delta^N_h f(x)$ die Differenz $N-$ter Ordnung für eine Funktion f.

Satz 1. ***Charakterisierungen durch Differenzen.***
Es sei $0 < p \le u < \infty$ und $0 < q \le \infty$. Weiterhin sei $N \in \mathbb{N}$.

(i) Es gelte zusätzlich

$$d \max\left(0, \frac{1}{p} - 1\right) < s < N.$$

Dann gehört eine Funktion $f \in L^{loc}_p(\mathbb{R}^d)$ zu $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ genau dann, wenn $f \in L^{loc}_1(\mathbb{R}^d)$ und

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)} := \|f|\mathcal{M}^u_p(\mathbb{R}^d)\| + \Big(\int_0^{\infty} t^{-sq-dq}\Big\|\int_{B(0,t)}|\Delta^N_h f(x)|dh\Big|\mathcal{M}^u_p(\mathbb{R}^d)\Big\|^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$

endlich ist. Darüber hinaus ist $\|\cdot|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)}$ eine äquivalente Quasinorm. Im Fall $q=\infty$ sind entsprechende Modifikationen nötig.

(ii) Es gelte zusätzlich

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right) < s < N.$$

Dann gehört eine Funktion $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ *zu* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ *genau dann, wenn* $f \in L^{loc}_1(\mathbb{R}^d)$ *und*

$$\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)} := \|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + \left\| \left(\int_0^\infty t^{-sq-dq} \left(\int_{B(0,t)} |\Delta^N_h f(x)| dh \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \middle| \mathscr{M}^u_p(\mathbb{R}^d) \right\|$$

endlich ist. Darüber hinaus ist $\| \cdot |\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)}$ *eine äquivalente Quasinorm. Auch hier sind im Fall* $q = \infty$ *entsprechende Modifikationen nötig.*

Für weitere Informationen sei auf die Theoreme 5 und 7 verwiesen. Ein analoges Resultat für die Räume $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ findet sich in Theorem 9. Besonders interessant an Satz 1 ist, dass er auch für den anspruchsvolleren Quasibanachfall, also $0 < p < 1$ und $0 < q < 1$, übersichtliche Charakterisierungen bereit stellt. Allerdings beobachten wir in diesen Fällen zusätzliche Einschränkungen bezüglich des Parameters s. Diese sind uns von den ursprünglichen Besov und Triebel-Lizorkin Räumen wohl vertraut. Dennoch mag es auf den ersten Blick verwundern, dass der neue Morreyparameter u in den Voraussetzungen an s nicht auftritt. Deshalb haben wir in einem nächsten Schritt untersucht, ob die in Satz 1 auftretenden Voraussetzungen auch notwendig sind. Zu diesem Zweck bezeichnen wir mit $\mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$ die Menge aller $f \in L^{loc}_{\max(p,1)}(\mathbb{R}^d)$, für die $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)} < \infty$ gilt. Analog ist $\mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$ die Menge aller $f \in L^{loc}_{\max(p,1)}(\mathbb{R}^d)$ mit $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)} < \infty$. Benutzen wir diese Notation, so erhalten wir das folgende Resultat bezüglich der Notwendigkeit der Voraussetzungen.

Satz 2. ***Differenzen und notwendige Voraussetzungen.***
Es sei $s \in \mathbb{R}$, $0 < p \le u < \infty$ *und* $0 < q \le \infty$. *Weiterhin sei* $N \in \mathbb{N}$.

(a) Dann gilt $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$, *falls einer der folgenden Fälle vorliegt.*

  (i) Es ist $s \le 0$.

  (ii) Es ist $0 < p < 1$ *und* $s < d\frac{p}{u}(\frac{1}{p} - 1)$.

  (iii) Es ist entweder $N < s$ *mit* $0 < q \le \infty$ *oder* $N = s$ *mit* $0 < q < \infty$.

(b) Es gilt $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$, *falls einer der folgenden Fälle vorliegt.*

  (i) Es ist $s \le 0$.

  (ii) Es ist $0 < p < 1$ *und* $s < d\frac{p}{u}(\frac{1}{p} - 1)$.

  (iii) Es ist $q \le \min(1, p)$ *und* $s \le d(\frac{1}{q} - 1)$.

  (iv) Es ist $N < s$ *mit* $0 < q \le \infty$ *oder* $N = s$ *mit* $0 < q < \infty$.

Für weitere Details betrachte man die Theoreme 10 und 12 sowie das komplette Kapitel 6. Ein analoges Resultat für die Räume $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ findet sich in Theorem 13. Sicherlich beantwortet Satz 2 viele Fragen bezüglich der Notwendigkeit der in Satz 1 auftretenden Voraussetzungen. Dennoch werden auch mit Satz 2 noch nicht alle Parameterkonstellationen abgedeckt. So kann für

$0 < p < 1$ und $d\frac{p}{u}(\frac{1}{p}-1) \leq s \leq d(\frac{1}{p}-1)$ noch nicht in jedem Fall entschieden werden, ob eine Charakterisierung mittels Differenzen möglich ist oder nicht. Gleichwohl ist es uns für die Triebel-Lizorkin-Morrey Räume im Spezialfall $p = q$ gelungen, mit Theorem 11 ein optimales Resultat zu beweisen. Differenzencharakterisierungen für Glattheits Morrey Räume besitzen zahlreiche praktische Anwendungen. Aus diesem Grund stellen wir im zweiten Teil dieser Abhandlung mehrere weiterführende Problemstellungen bezüglich unserer Glattheits Morrey Räume vor. Zu diesen leiten wir dann durch das Arbeiten mit Differenzen neue Ergebnisse her. So haben wir zunächst die Glattheits- und Integrierbarkeitseigenschaften gewisser Testfunktionen untersucht. Exemplarisch können wir die Funktion

$$f_\alpha(x) := \psi(x)\,|x|^{-\alpha}$$

betrachten, wobei ψ eine glatte Abschneidefunktion mit kompaktem Träger im Koordinatenursprung ist. Für diese Funktion erhalten wir die folgende Aussage bezüglich der Zugehörigkeit zu den Räumen $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.

Satz 3. ***Glattheitseigenschaften einer Testfunktion.***
Es sei $0 < p < u < \infty$, $0 < q \leq \infty$ *und* $s > \sigma_{p,q}$. *Dann gilt* $f_\alpha \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ *genau dann, wenn* $\alpha + s \leq \frac{d}{u}$.

Für dieses Resultat sei auf Lemma 42 verwiesen. Weitere Testfunktionen und deren Glattheitseigenschaften finden sich in Kapitel 9. Eine weitere sehr nützliche Anwendung von Differenzencharakterisierungen für Funktionenräume findet sich, wenn wir komplexe Interpolation mit zwei Triebel-Lizorkin-Morrey Räumen durchführen wollen. Genauer gesagt sind wir an einer Beschreibung für

$$[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta$$

interessiert, wobei mit $[\cdot,\cdot]_\Theta$ das Ergebnis von Calderóns erster komplexer Interpolationsmethode bezeichnet wird. Ω ist ein beschränktes Lipschitzgebiet. Im Zusammenhang mit dieser Fragestellung spielen die Räume $\mathring{\mathscr{E}}^s_{u,p,q}(\mathbb{R}^d)$ sowie die Charakterisierung derselben mit Hilfe von Differenzen eine wichtige Rolle. Hierfür sei auf Kapitel 8 und insbesondere auf Theorem 15 verwiesen. Des Weiteren benötigen wir universelle Fortsetzungsoperatoren für Triebel-Lizorkin-Morrey Räume auf Gebieten. Diese haben wir in Kapitel 10 konstruiert, betrachte Theorem 17. Mit diesen Hilfsmitteln ergibt sich dann das folgende Resultat, siehe Theorem 18.

Satz 4. ***Komplexe Interpolation von Triebel-Lizorkin-Morrey Räumen.***
Es sei $\Omega \subset \mathbb{R}^d$ *entweder ein beschränktes Lipschitzgebiet für* $d \geq 2$ *oder ein beschränktes Intervall für* $d = 1$. *Weiterhin sei*

(a) $1 \leq p_0 < p_1 < \infty$, $p_0 \leq u_0 < \infty$, $p_1 \leq u_1 < \infty$;

(b) $1 \leq q_0, q_1 \leq \infty$, $\min(q_0, q_1) < \infty$;

(c) $p_0 u_1 = p_1 u_0$;

(d) $s_0, s_1 \geq 0$; *entweder* $s_0 < s_1$ *oder* $0 < s_0 = s_1$ *mit* $q_1 \leq q_0$;

(e) $0 < \Theta < 1$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$, $s := (1-\Theta)s_0 + \Theta s_1$.

Dann gilt

$$[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta = \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega)\,.$$

Weitere Untersuchungen haben ergeben, dass sich vergleichbare Interpolationsresultate auch für andere Parameterkonstellationen nachweisen lassen. Man betrachte hierfür Theorem 19. Andererseits hat sich gezeigt, dass manche der in Satz 4 zu findenden Voraussetzungen aber auch notwendig sind, vergleiche dazu mit den Propositionen 23 und 24 aus Kapitel 11. Gewinnbringend lassen sich Differenzencharakterisierungen auch einsetzen, wenn es um die Untersuchung von Eigenschaften der Operatoren

$$T^+ f = \max(f,0) \qquad \text{und} \qquad Tf = |f|$$

geht. Hierbei soll f eine reellwertige Funktion aus einem Raum $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ sein. Mit $\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ bezeichnen wir den reellen Anteil von $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Wir wollen dann herausfinden, unter welchen Bedingungen an die Parameter eine von f unabhängige Konstante $C > 0$ existiert, sodass

$$\|T^* f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \le C \|f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$$

für alle $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ gilt. Mit T^* meinen wir entweder T oder T^+. Beim Bearbeiten dieser Fragestellung zeigt sich, dass es einen großen Unterschied macht, ob man mit Dimension $d = 1$ oder $d > 1$ hantiert. Das liegt unter Anderem auch daran, dass die für die Räume $F^s_{p,q}(\mathbb{R}^d)$ wohl bekannte Fubini-Property für die Triebel-Lizorkin-Morrey Räume in vielen Fällen keine Gültigkeit besitzt. Für nähere Informationen zu diesem Thema betrachte man Kapitel 12 und insbesondere Lemma 51. Die Beschränktheit der Operatoren T und T^+ kann im Fall $d = 1$ mit Hilfe einer Version einer Hardy-type Ungleichung bewiesen werden. Für $d > 1$ ist die Situation deutlich komplizierter. Hier kann man auf so genannte Morrey Charakterisierungen für die Räume $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ zurückgreifen. Insgesamt erhalten wir dann das folgende Ergebnis, siehe Theorem 24.

Satz 5. ***Beschränktheit von T und T^+.***
Es sei $1 \le p < u < \infty$, $1 \le q \le \infty$ und $s > 0$. Weiterhin gelte

$$\begin{cases} p \neq 1, q \neq \infty \text{ und } \frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d} & \text{für den Fall} \quad s = 1; \\ \frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d} & \text{für den Fall} \quad 1 < s < \min(1 + \frac{1}{p}, 1 + \frac{d}{u}) \text{ und } d > 1; \\ \frac{u}{p} \le d & \text{für den Fall} \quad s = \min(1 + \frac{1}{p}, 1 + \frac{d}{u}). \end{cases}$$

Dann existiert eine Konstante $C > 0$ unabhängig von $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ sodass

$$\|T^+ f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \le C \|f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$$

gilt, genau dann, wenn

$$s < \min\Big(1 + \frac{1}{p}, 1 + \frac{d}{u}\Big).$$

Die selbe Aussage gilt auch für den Operator T an Stelle von T^+.

Ein analoges Resultat ergibt sich auch für die Besov-Morrey Räume, betrachte Theorem 23. Vergleicht man Satz 5 mit der Situation für die ursprünglichen Triebel-Lizorkin Räume, siehe Theorem 22, so ergibt sich für den Morrey Fall ein deutlich komplizierteres Bild. Allerdings ist die in

Satz 5 auftretende Voraussetzung $\frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d}$ höchstwahrscheinlich nicht notwendig und kann bei Benutzung einer anderen Beweismethode möglicherweise weggelassen werden. Insgesamt werden in der vorliegenden Abhandlung zahlreiche neue Erkenntnisse bezüglich Differenzencharakterisierungen von Glattheits Morrey Räumen sowie deren Anwendungen präsentiert. Dennoch sind zu einigen der hier betrachteten Themen neue Fragestellungen und Probleme sichtbar geworden, die Gegenstand weiterführender Forschungen sein können. Die wichtigsten dieser neu erkannten Aufgabenstellungen wurden als offene Probleme formuliert und an geeigneten Stellen im vorliegenden Text aufgelistet.

Chapter 2

Introduction

2.1 Preface

Consider Triebel-Lizorkin-Morrey spaces, Besov-Morrey spaces and Besov-type spaces. They all are representatives for Smoothness Morrey spaces. And they all are the center of attention in this treatise. Nowadays function spaces, that sort functions and distributions with respect to their smoothness and integrability properties, are quite popular. The topic has a long tradition and is addressed in a large number of papers, articles and books. One important precursor for our Smoothness Morrey spaces are the so-called Besov spaces $B^s_{p,q}(\mathbb{R}^d)$. They showed up the first time between 1950 and 1960 and initially have been investigated by S.M. Nikol'skii and O.V. Besov. For that let us mention the papers [89], [4] and [5]. Some years later around 1970 the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ appeared the first time. They have been introduced by P.I. Lizorkin and H. Triebel and primarily showed up in the articles [71], [72] and [126]. In the subsequent years the properties of the spaces $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ have been studied in detail by H. Triebel. In connection with this he wrote several books, for example [128], [129], [131], [133] and [134]. The Smoothness Morrey spaces we intend to investigate in this treatise, are generalizations of the well-known spaces $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$. Roughly speaking we obtain them by replacing the Lebesgue norm in the definitions of the Besov- and Triebel-Lizorkin spaces by a much more general Morrey norm. Consequently the theory of our Smoothness Morrey spaces is based on that for the original spaces $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$. So in 1994 the Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ have been introduced by H. Kozono and M. Yamazaki, see [64]. Later these spaces have been investigated by A. Mazzucato in [79] in connection with Navier-Stokes equations. The Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ showed up the first time in 2002 in a paper written by A. El Baraka, see [31]. Finally in 2005 also the Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ arised. First investigations concerning these spaces have been made by L. Tang and J. Xu in the article [125]. Later also the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ appeared in connection with Navier-Stokes equations, see for example chapter 8.6. in [68]. In the recent years several mathematicians gave attention to the exploration of the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$, $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. Here let us mention W. Yuan, W. Sickel and D. Yang. In their book [144] from 2010 a first systematical and comprehensive investigation of the properties of the Smoothness Morrey spaces can be found. Later also H. Triebel gave special emphasis to these spaces and collected their properties in his volumes [136] and [137]. Therein he used a different notation and dealt with so-called Local and Hybrid Function Spaces. Moreover, we want to refer to D.D. Haroske and L. Skrzypczak. Sometimes also together with coauthors they investigated the embedding properties of our Smoothness Morrey spaces and presented their results in numerous papers, as for instance [48], [49], [50], [51] and [52]. In connection with the Smoothness Morrey spaces we mentioned above, also some more function scales show up. So in 2008 the Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ have been introduced D. Yang and W. Yuan, see [138] and [140]. These function spaces are closely related to the Triebel-Lizorkin-Morrey spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Moreover, under certain conditions on the parameters both scales even coincide. Usually the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ are defined by using fourieranalytical tools and a smooth dyadic decomposition of the unity. This approach is very useful for proving elementary properties of our Smoothness Morrey spaces. However, when we have a look at the situation for the original Besov and Triebel-Lizorkin spaces, we notice, that their historical roots can be found in characterizations in terms of differences. In fact, in [89] and [4], where the spaces $B^s_{p,q}(\mathbb{R}^d)$ showed up primarily, they were described by higher order differences. And also for the spaces $F^s_{p,q}(\mathbb{R}^d)$ there existed characterizations via differences from the very beginning, see [71]. Later in his books [128], [129] and [133] H. Triebel gave special attention to this topic and proved characterizations in terms of differences for the maximum range of parameters, see also the Theorems 1 and 2. Such characterizations have many advantages. So they are relatively transparent and describe the smoothness and integrability properties of the function spaces under investigation in a very natural way. Characterizations via differences only use function values of a function f. Fourier transformations or a smooth dyadic decomposition of the unity are not required. When dealing with special test functions or in the context of certain applications, this means a significant simplification. When the author started to work on this treatise, there already existed some first characterizations by differences also for the new spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ in the literature. Here we want to mention the chapters 4.3 and 4.5 in [144] as well as the papers [116] and [29]. However, especially for the cases $p < 1$ and $q < 1$ many questions remained open there. For that reason in this treatise it will be one of our main goals, to deduce equivalent characterizations in terms of differences for the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, that are as simple as possible and hold for a preferably large range of the parameters. Actually, in what follows we plan to raise the level of knowledge concerning this topic to that we have for the original spaces $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$. So in this treatise we proved the following characterization in terms of differences for the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Let us mention, that by $\Delta^N_h f$ we denote a difference of order $N \in \mathbb{N}$ and increment $h \in \mathbb{R}^d$ for a function f.

Main Result 1. ***Characterizations via Differences.***
Let $0 < p \le u < \infty$ and $0 < q \le \infty$. Moreover let $N \in \mathbb{N}$.

(i) In addition we assume

$$d \max\left(0, \frac{1}{p} - 1\right) < s < N.$$

Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ belongs to $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, if and only if $f \in L^{loc}_1(\mathbb{R}^d)$ and (modifications for $q = \infty$)

$$\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)} := \|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + \left(\int_0^\infty t^{-sq-dq} \Big\| \int_{B(0,t)} |\Delta^N_h f(x)| dh \Big| \mathscr{M}^u_p(\mathbb{R}^d)\Big\|^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

is finite. Furthermore $\|\cdot|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)}$ is an equivalent quasi-norm.

(ii) Assume

$$d\max\left(0,\frac{1}{p}-1,\frac{1}{q}-1\right)<s<N.$$

Then a function $f\in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ belongs to $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, if and only if $f\in L^{loc}_1(\mathbb{R}^d)$ and (modifications for $q=\infty$)

$$\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)}:=\|f|\mathscr{M}^u_p(\mathbb{R}^d)\|+\left\|\Big(\int_0^\infty t^{-sq-dq}\Big(\int_{B(0,t)}|\Delta^N_h f(x)|dh\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\right\|$$

is finite. Moreover $\|\cdot|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)}$ is an equivalent quasi-norm.

More details concerning this result can be found in the Theorems 5 and 7. A similar characterization for the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is given in Theorem 9. One big advantage of Main Result 1 is, that it also provides a lucid characterization for the more demanding quasi-Banach case, namely $0<p<1$ or $0<q<1$. However, in these cases some additional restrictions concerning the parameter s do appear. They are similar to those we observe for the original Besov and Triebel-Lizorkin spaces. Nevertheless it might be surprising, that the new Morrey parameter u does not show up in the conditions concerning s. Hence in a next step we investigated, whether the conditions that can be found in Main Result 1, are also necessary. For that purpose by $\mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$ we denote the set of all $f\in L^{loc}_{\max(p,1)}(\mathbb{R}^d)$, that fulfill $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)}<\infty$. Similarly $\mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$ is the set of all $f\in L^{loc}_{\max(p,1)}(\mathbb{R}^d)$, such that $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)}<\infty$. Using this notation, we obtain the following result concerning the necessity of the conditions.

Main Result 2. ***Differences and necessary Conditions.***
Let $s\in\mathbb{R}$, $0<p\le u<\infty$ and $0<q\le\infty$. Moreover let $N\in\mathbb{N}$.

(a) Then we have $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\neq\mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$, if we are in one of the following cases.

(i) It is $s\le 0$.

(ii) It is $0<p<1$ and $s<d\frac{p}{u}(\frac{1}{p}-1)$.

(iii) It is either $N<s$ with $0<q\le\infty$ or $N=s$ with $0<q<\infty$.

(b) We have $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\neq\mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$, if we are in one of the following cases.

(i) It is $s\le 0$.

(ii) It is $0<p<1$ and $s<d\frac{p}{u}(\frac{1}{p}-1)$.

(iii) It is $q\le\min(1,p)$ and $s\le d(\frac{1}{q}-1)$.

(iv) It is either $N<s$ with $0<q\le\infty$ or $N=s$ with $0<q<\infty$.

For more details we refer to the Theorems 10 and 12 as well as to chapter 6. A similar result for the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can be found in Theorem 13. Of course Main Result 2 answers a lot of questions concerning the necessity of the conditions, that show up in Main Result 1. But also Main Result 2 does not cover all possible parameter constellations. So for $0<p<1$ and $d\frac{p}{u}(\frac{1}{p}-1)\le s\le d(\frac{1}{p}-1)$ in many cases we do not know, whether a characterization in terms of differences for our spaces is possible or not. Nevertheless for the Triebel-Lizorkin-Morrey spaces in the special case $p = q$ we were able to prove an optimal result, see Theorem 11. Characterizations in terms of differences for Smoothness Morrey spaces have many useful applications. For example, in some cases they give us the possibility, to investigate the smoothness and integrability properties of certain test functions. For instance, we can deal with the function

$$f_\alpha(x) := \psi(x)\,|x|^{-\alpha},$$

where ψ is a smooth cut-off function with compact support at the origin. For this function we obtain the following result concerning its affiliation to the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$.

Main Result 3. ***Properties of a Test Function.***
Let $0 < p < u < \infty$, $0 < q \leq \infty$ and $s > \sigma_{p,q}$. Then we have $f_\alpha \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ if and only if $\alpha + s \leq \frac{d}{u}$.

For more details we refer to Lemma 42. Further results concerning this and other test functions can be found in chapter 9. Another very useful application for our characterizations in terms of differences can be recognized, when we deal with complex interpolation of Triebel-Lizorkin-Morrey spaces. More precisely, we are interested in

$$[\mathcal{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathcal{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta,$$

where $[\cdot,\cdot]_\Theta$ denotes the result of Calderón's first complex interpolation method. Here Ω is a bounded Lipschitz domain. In connection with this topic the spaces $\overset{\diamond}{\mathcal{E}}{}^s_{u,p,q}(\mathbb{R}^d)$ and their characterizations via differences play an important role. For that we refer to chapter 8 and especially to Theorem 15. Moreover, we require universal extension operators for Triebel-Lizorkin-Morrey spaces on bounded Lipschitz domains. We constructed such operators in chapter 10, see Theorem 17. Using these tools we obtain the following result concerning complex interpolation, see Theorem 18.

Main Result 4. ***Complex Interpolation of Triebel-Lizorkin-Morrey Spaces.***
Let $\Omega \subset \mathbb{R}^d$ be either a bounded Lipschitz domain for $d \geq 2$ or a bounded interval for $d = 1$. Moreover let

(a) $1 \leq p_0 < p_1 < \infty$, $p_0 \leq u_0 < \infty$, $p_1 \leq u_1 < \infty$;

(b) $1 \leq q_0, q_1 \leq \infty$, $\min(q_0, q_1) < \infty$;

(c) $p_0 u_1 = p_1 u_0$;

(d) $s_0, s_1 \geq 0$; *either* $s_0 < s_1$ *or* $0 < s_0 = s_1$ *with* $q_1 \leq q_0$;

(e) $0 < \Theta < 1$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$, $s := (1-\Theta)s_0 + \Theta s_1$.

Then we have

$$[\mathcal{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathcal{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta = \overset{\diamond}{\mathcal{E}}{}^s_{u,p,q}(\Omega)\,.$$

Similar interpolation results also can be proved for other parameter constellations. For that we refer to Theorem 19. On the other hand it turned out, that at least some of the conditions, that can be found in Main Result 4, are also necessary, see the Propositions 23 and 24 from chapter 11.

Another advantageous application for our characterizations in terms of differences can be found, when we investigate the properties of the operators

$$T^+ f = \max(f, 0) \qquad \text{and} \qquad Tf = |f|.$$

Here f is a real valued function from a space $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. With $\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ we denote the real part of $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Then we want to find out, under what conditions on the parameters, there exists a constant $C > 0$ independent of f, such that we have

$$\|T^* f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$$

for all $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$. The symbol T^* stands either for T or T^+. When we deal with this question, it turns out, that it makes a big different, whether we work with dimension $d = 1$ or $d > 1$. One reason for this is the fact, that the Fubini-property, a well-known tool for the original spaces $F^s_{p,q}(\mathbb{R}^d)$, in many cases does not hold for the Triebel-Lizorkin-Morrey spaces. More details concerning this topic can be found in chapter 12 and especially in Lemma 51. In the case $d = 1$ the boundedness of the operators T and T^+ can be proved by using a Morrey version of a Hardy-type inequality. For $d > 1$ the situation is much more complicated. Here we can apply so called Morrey characterizations for the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. All in all we obtain the following result, see Theorem 24.

Main Result 5. ***The Boundedness of*** T ***and*** T^+**.**
Let $1 \leq p < u < \infty$, $1 \leq q \leq \infty$ *and* $s > 0$. *Moreover assume*

$$\begin{cases} p \neq 1, q \neq \infty \text{ and } \frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d} & \text{for the case} \quad s = 1; \\ \frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d} & \text{for the case} \quad 1 < s < \min(1 + \frac{1}{p}, 1 + \frac{d}{u}) \text{ and } d > 1; \\ \frac{u}{p} \leq d & \text{for the case} \quad s = \min(1 + \frac{1}{p}, 1 + \frac{d}{u}). \end{cases}$$

Then there is a constant $C > 0$ *independent of* $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$, *such that we have*

$$\|T^+ f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|,$$

if and only if

$$s < \min\left(1 + \frac{1}{p}, 1 + \frac{d}{u}\right).$$

The same assertion is true with T *instead of* T^+.

A similar result for the Besov-Morrey spaces can be found in Theorem 23. When we compare Main Result 5 with the outcome for the original Triebel-Lizorkin spaces, see Theorem 22, we observe, that the situation is much more complicated in the Morrey case. On the other hand the condition $\frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d}$ in Main Result 5 seems to be of technical nature only. Maybe it can be left away, when we use another method for the proof. Altogether in this treatise numerous new findings concerning characterizations by differences for Smoothness Morrey spaces and their applications are presented. But nevertheless there are still some open problems concerning that topic, that may be subject of future research. Some of these unsolved issues are formulated and listed in the course of this treatise.

2.2 A short Summary of the Chapters

This treatise consists of a number of chapters. Most of them are divided into sections and subsections. For convenience of the reader in what follows we will give a short summary for all chapters.

Chapter 3. In this introducing chapter we provide the notation, we need for all later considerations. For that purpose we also collect some well-known classical function spaces. This includes the definitions of the original Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ as well.

Chapter 4. Here it is our main intention, to give the definitions for the Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and the Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. Moreover, also the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and the Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ are defined. For all these spaces some basic properties are collected. Most of them are already known, but will be important for the proofs in the later sections. At the end of chapter 4 also some Hardy-Littlewood maximal inequalities as well as some multiplier theorems are recalled.

Chapter 5. In this chapter it is our main goal, to prove characterizations in terms of differences for the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. For that purpose in a first step we recall an abstract theory concerning function spaces developed by L.I. Hedberg and Y.V. Netrusov, see [53]. It turns out, that our Smoothness Morrey spaces fit into this theory. With this in mind we obtain a characterization in terms of generalized ball means for the Triebel-Lizorkin-Morrey spaces. In fact, we observe, that under the condition

$$d \max\left(0, \frac{1}{p}-1, \frac{1}{q}-1, \frac{1}{p}-\frac{1}{v}, \frac{1}{q}-\frac{1}{v}\right) < s < N \tag{2.1}$$

the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ can be described via the equivalent quasi-norm

$$\|f|\mathcal{M}^u_p(\mathbb{R}^d)\| + \left\|\left(\int_0^\infty t^{-sq}\left(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\right)^{\frac{q}{v}} \frac{dt}{t}\right)^{\frac{1}{q}} \Big| \mathcal{M}^u_p(\mathbb{R}^d)\right\|,$$

see Theorem 5 for more details. For the Besov-Morrey spaces and the Besov-type spaces we deduced similar results, see Theorems 7 and 9. Moreover, as a byproduct we also obtained so-called Stein characterizations for the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, see Theorem 6.

Chapter 6. This very technical chapter attends to the question, whether the conditions that can be found in (2.1) are also necessary. For that purpose at first we prove the necessity of $s > 0$. Here as tools also dilation operators and characterizations in terms of the Haar wavelet are used. In a second step due to some embedding theorems we obtain the need of $s \geq d\frac{p}{u}(\frac{1}{p}-1)$ and $s \geq d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$. To prove the necessity of $s > d(\frac{1}{q}-\frac{1}{v})$, we investigate the properties of a random function constructed by M. Christ and A. Seeger in [25]. From that we can derive results for some of the other conditions as well. Finally we show the importance of $s < N$. Here we work with special test functions and apply some ideas from P. Oswald, see Proposition 17.

Chapter 7. This short chapter serves as a first summary of the results we obtained so far. To this end we combine the sufficient conditions from chapter 5 with the necessary ones from chapter 6. In this context for some special cases we also obtain optimal results concerning the characterization of our Smoothness Morrey spaces in terms of differences, see the Theorems 11 and 14. To illustrate our findings, some $(\frac{1}{p}, s)$ - diagrams are presented in this chapter as well.

Chapter 8. In this chapter the subject of interest are the diamond spaces $\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)$ associated to the Triebel-Lizorkin-Morrey spaces, see Definition 32. For them we collect some basic properties. In doing so we also prove an alternative characterization of $\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)$ in terms of Littlewood-Paley decompositions. Moreover, a characterization by differences is provided, see Theorem 15. Finally we study intersections of Triebel-Lizorkin-Morrey spaces. In this context the diamond spaces show up in a very natural way.

Chapter 9. Here it was our intention to investigate the properties of some test functions. More precisely we dealt with

$$f_\alpha(x) := \psi(x)\,|x|^{-\alpha} \qquad \text{and} \qquad h_\alpha(x) := (1-\psi(x))\,|x|^{-\alpha},$$

where ψ is a smooth cut-off function, that is compactly supported in a ball centered at the origin. For these functions we figured out, under what conditions on the parameters they belong to our Smoothness Morrey spaces. So for the Triebel-Lizorkin-Morrey spaces we observed

$$f_\alpha \in \mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d) \quad \text{iff} \quad \alpha + s \leq \frac{d}{u} \qquad \text{and} \qquad h_\alpha \in \mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d) \quad \text{iff} \quad \frac{d}{u} \leq \alpha,$$

see Lemmas 42 and 46 for more details and explanations. Concerning the spaces $\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ there are similar results. For most of the proofs in this chapter the key tool was an appropriate characterization in terms of differences.

Chapter 10. In this paragraph we constructed linear and bounded extension operators E with

$$E: \quad \mathscr{E}^{s}_{u,p,q}(\Omega) \quad \rightarrow \quad \mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d),$$

where Ω is a Lipschitz domain. In fact, we were able to establish an extension operator E, which works for all admissible parameter constellations simultaneously, see Theorem 17. Such an E is called universal. For the proofs in this chapter we used some ideas from V.S. Rychkov, see [102].

Chapter 11. In this chapter we deal with complex interpolation of Triebel-Lizorkin-Morrey spaces on domains. Actually, when $[\cdot,\cdot]_\Theta$ denotes the result of Calderón's first complex interpolation method, we want to study the behavior of $[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta$, where Ω is a bounded Lipschitz domain. For that purpose we concentrate on those cases, where the Lemarié-Rieusset condition $p_0 u_1 = p_1 u_0$ is satisfied. Moreover, we accept the condition $\min(q_0, q_1) < \infty$ as well as some additional restrictions concerning the parameters s_0 and s_1, see the Theorems 18 and 19 for more details. Then for $0 < \Theta < 1$ and

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}, \quad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}, \quad s := (1-\Theta)s_0 + \Theta s_1$$

we find

$$[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta = \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega). \tag{2.2}$$

The main ingredient for the proof are the results concerning intersections of Triebel-Lizorkin-Morrey spaces from chapter 8. In the rear part of chapter 11 we also proved the necessity of some of the conditions from Theorems 18 and 19. So for example we were able to show, that (2.2) does not hold for $\Omega = \mathbb{R}^d$, see Proposition 24.

Chapter 12. This very short paragraph is completely devoted to the Fubini property. For the original Triebel-Lizorkin spaces $F^{s}_{p,q}(\mathbb{R}^d)$ the Fubini property is a well-known and very useful tool to reduce high dimensional problems to $d = 1$, see Lemma 50. However, we have to realize, that for the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ in most of the cases there is no direct counterpart. So we can prove, that for $d \geq 2$ and

$$p \leq \frac{d-1}{d} u$$

the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ do not have the Fubini property, see Lemma 51.

Chapter 13. Finally we investigated the behavior of the operators $T^+ f = \max(f,0)$ and $Tf = |f|$ in the context of real valued Triebel-Lizorkin-Morrey spaces. More precisely we asked, under what conditions on the parameters there is a fixed $C > 0$, such that we have

$$\| T^* f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \| \leq C \| f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \| \tag{2.3}$$

for all real valued $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Here T^* stands either for T or T^+. To answer this question, we dealt with the cases $d = 1$ and $d > 1$ separately. For $d = 1$ we proved, that (2.3) holds, if

$$s < 1 + \frac{1}{u}$$

is fulfilled, whereby for the special case $s = 1$ some additional conditions need to be accepted, see the Propositions 26 and 29. Our main tool to show this is a Hardy-type inequality. In the case $d > 1$ we obtained (2.3) for

$$s < \min\left(1 + \frac{1}{p}, 1 + \frac{d}{u}\right) \qquad \text{and} \qquad \frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d},$$

at what the second condition seems to be of technical nature only. Here for the proof we used so-called Morrey characterizations for the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. For more details we refer to Proposition 28. We also deduced similar results for the Besov-Morrey spaces, see Theorem 23. Moreover, we observed, that some of the conditions we just mentioned are also necessary, see Theorem 24 for more details.

2.3 A Comment concerning Publications of the Author

Most of the results, that are presented in this treatise, already have been published. So in the last few years the author wrote several papers, some of them also together with coauthors. We want to mention at least the following publications.

(1) M. Hovemann, Besov-Morrey spaces and differences, Math. Reports, in press. arXiv:2010.10856 [math.FA]. See also [55].

Some results from this paper can be found in the chapters 5, 6 and 7.

(2) M. Hovemann, Triebel-Lizorkin-Morrey spaces and Differences, Math. Nachr., in press. See also [56].

This article motivated the content of the chapters 5, 6 and 7.

(3) M. Hovemann, Truncation in Besov-Morrey and Triebel-Lizorkin-Morrey spaces, Nonlinear Analysis **204** (2021). See also [57].

This manuscript contains the ideas for our chapters 12 and 13.

(4) M. Hovemann, W. Sickel, Besov-Type Spaces and Differences, Eurasian Math. J. **11**(1) (2020), 25-56. See also [58].

The results from this paper became part of the chapters 5, 6 and 7.

(5) M. Hovemann, W. Sickel, Stein characterizations of Lizorkin-Triebel spaces, work in progress, Jena, 2020. See also [59].

This manuscript is a modified version of the authors master thesis.

(6) C. Zhuo, M. Hovemann, W. Sickel, Complex Interpolation of Lizorkin-Triebel-Morrey Spaces on Domains, Anal. Geom. Metr. Spaces **8**(1) 2020, 268-304. See also [150].

The findings from this paper gained access to our chapters 8, 9, 10 and 11.

Chapter 3

A short Collection of classical Function Spaces

In this treatise our subject of interest are the Triebel-Lizorkin-Morrey spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, the Besov-Morrey spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. When we want to work with such Smoothness Morrey spaces, we need a lot of notions from functional analysis, Fourier analysis and the theory of function spaces. Moreover, we have to use terms from fundamental analysis, measure theory and even from stochastic. Hence in this chapter we collect and explain some notations and concepts, that will be important for us later. Moreover, at the very end of this treatise in chapter 14 you can find a list with all symbols and abbreviations, that are used. For most of them also further explanations or references are given there.

3.1 Basic Notations and Concepts

Let us start to recall some basic notations. As usual $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0$ the set of all natural numbers and 0, $\mathbb{Z}$ the set of all integers and $\mathbb{R}$ the set of all real numbers. By $\mathbb{C}$ we mean the set of all complex numbers. In this treatise $d \in \mathbb{N}$ always stands for the dimension. Then $\mathbb{R}^d$ denotes the d-dimensional Euclidean space. For $x \in \mathbb{R}^d$ and $t > 0$ we put

$$B(x,t) := \{y \in \mathbb{R}^d : \quad |x-y| < t\}. \tag{3.1}$$

Here $|a|$ is the Euclidean norm of $a \in \mathbb{R}^d$. For a set $A \subset \mathbb{R}^d$ by $\overline{A}$ we denote the closure of A in $\mathbb{R}^d$. By ∂A we indicate the boundary of A. The symbol $A^{\complement}$ means $\mathbb{R}^d \setminus A$. For two sets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$ we write $\operatorname{dist}(A,B) = \inf_{x\in A, y \in B} |x-y|$. Moreover, for a set $A \subset \mathbb{R}^d$ we denote the n-dimensional Lebesgue measure with $n \in \mathbb{N}$ and $n \leq d$ by $\lambda_n(A)$. In the special case $n = d$ often we just write $\lambda_d(A) = |A|$. The symbol $\emptyset$ refers to the empty set.
In the context of function spaces some special numbers show up. So for all $p \in (0,\infty]$ and $q \in (0,\infty]$ we write

$$\sigma_p := d \max\left(0, \frac{1}{p} - 1\right) \qquad \text{and} \qquad \sigma_{p,q} := d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right). \tag{3.2}$$

Moreover, for any $s \in \mathbb{R}$ the symbol $[s]$ denotes the integer part of s.

In this treatise many functions will appear. Unless otherwise stated, all functions that show up, are assumed to be complex-valued. That means, we consider functions $f:\ \mathbb{R}^d \to \mathbb{C}$. Often we have to deal with equivalence classes of functions with respect to almost everywhere equality. However, if such an equivalence class contains a continuous representative, then usually we work with this representative and also call the equivalence class a continuous function. Relatively often indicator functions of sets will show up. So for $A \subset \mathbb{R}^d$ we define

$$\chi_A(x) := \begin{cases} 1 & \text{if} \quad x \in A; \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

Sometimes we will work with sequences of numbers or functions. For a sequence $(a_1, a_2, a_3, \ldots)$ often we write $\{a_j\}_{j=1}^{\infty}$ or $\{a_j\}_{j\in\mathbb{N}}$ or just $\{a_j\}_j$ for short. By $\mathbb{C}^{\mathbb{N}}$ we denote the set of all sequences of complex numbers. The sequence spaces l_p are defined in the following way.

Definition 1. ***Sequence Spaces.***
Let $0 < p < \infty$. Then the sequence space l_p is a subspace of $\mathbb{C}^{\mathbb{N}}$, that consists of all sequences $\{a_j\}_{j\in\mathbb{N}}$, that fulfill

$$\|\{a_j\}_{j\in\mathbb{N}}|l_p\| := \Big(\sum_{j=1}^{\infty} |a_j|^p\Big)^{\frac{1}{p}} < \infty.$$

Moreover, the sequence space l_∞ is a subspace of $\mathbb{C}^{\mathbb{N}}$, that contains all sequences $\{a_j\}_{j\in\mathbb{N}}$ with

$$\|\{a_j\}_{j\in\mathbb{N}}|l_\infty\| := \sup_{j\in\mathbb{N}} |a_j| < \infty.$$

For each quasi-normed space X by $\|\cdot|X\|$ we denote the corresponding quasi-norm. For two quasi-Banach spaces X and Y we write $X \hookrightarrow Y$, if $X \subset Y$ and the natural embedding of X into Y is continuous. In what follows sometimes we also will deal with (linear) operators. Given two quasi-Banach spaces X and Y, the operator norm of a linear operator $T : X \to Y$ is denoted by $\|T|\mathscr{L}(X \to Y)\|$. The symbol $\mathscr{L}(X \to Y)$ itself stands for the set of all linear and bounded operators from X to Y. In the special case $X = Y$ sometimes we write $\mathscr{L}(X)$ and $\|T|\mathscr{L}(X)\|$ for short.
In this treatise many proofs will be based on inequalities. In connection with them often symbols like $C, C_1, c, c_1 \ldots$ show up. They stand for positive constants, that only depend on fixed parameters and probably on auxiliary functions. Unless otherwise stated, their values may vary from line to line. Sometimes we use the symbol $\lesssim$ instead of $\leq$. Then $A \lesssim B$ means, that there exists a positive constant C, such that $A \leq CB$. In connection with that the symbol $A \sim B$ will be used as an abbreviation for $A \lesssim B \lesssim A$.
Hereinafter many terms and symbols concerning function spaces will play an important role. For convenience of the reader most of them will be collected and explained in the next few sections.

3.2 Integrability, Smoothness and Derivatives

In this section we recall some basic terms concerning function spaces in connection with integrability and smoothness. Most of the following notions should be well-known. Let us start our repetition with the Lebesgue spaces.

Definition 2. ***Lebesgue Spaces.***
Let $0 < p < \infty$. Then we define the Lebesgue spaces $L_p(\mathbb{R}^d)$ as the collection of all measurable functions f, that fulfill

$$\|f|L_p(\mathbb{R}^d)\| := \Big(\int_{\mathbb{R}^d} |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

In the special case $p = \infty$ the space $L_\infty(\mathbb{R}^d)$ is the set of all measurable functions f, such that

$$\|f|L_\infty(\mathbb{R}^d)\| := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)| < \infty.$$

There also exists a local version of the Lebesgue spaces.

Definition 3. ***Local Lebesgue Spaces.***
Let $0 < p < \infty$. Then a measurable function f belongs to the local Lebesgue space $L_p^{loc}(\mathbb{R}^d)$, if

$$\int_K |f(x)|^p dx < \infty$$

for all compact sets $K \subset \mathbb{R}^d$.

For two functions $f, g \in L_1(\mathbb{R}^d)$ we define the convolution $f * g$ by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)dy. \tag{3.3}$$

Moreover, sometimes we use the abbreviation

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(x)g(x)dx. \tag{3.4}$$

Consequently we also can write $(f * g)(x) = \langle f, g(x-\cdot)\rangle$. Now let us continue our repetition with function spaces that consist of smooth functions.

Definition 4. ***The Space $C(\mathbb{R}^d)$.***
The space $C(\mathbb{R}^d)$ is the collection of all uniformly continuous functions f, that fulfill

$$\|f|C(\mathbb{R}^d)\| := \sup_{x\in\mathbb{R}^d} |f(x)| < \infty.$$

Let $k \in \mathbb{N}$ and $i \in \{1, 2, \dots, d\}$. Then by $\partial_i^k f$ we denote the $k-$th derivative in direction i of a function f. Now let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$ be a multi-index. We write $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$ and introduce the abbreviation

$$D^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_d^{\alpha_d} f. \tag{3.5}$$

Definition 5. ***The Space $C^m(\mathbb{R}^d)$.***
Let $m \in \mathbb{N}$. Then the space $C^m(\mathbb{R}^d)$ consists of all functions $f \in C(\mathbb{R}^d)$, that have classical derivatives $D^\alpha f \in C(\mathbb{R}^d)$ up to the order of $|\alpha| \le m$ and fulfill

$$\|f|C^m(\mathbb{R}^d)\| := \sum_{|\alpha|\le m} \|D^\alpha f|C(\mathbb{R}^d)\| < \infty.$$

In the case $m = \infty$ we define

$$C^\infty(\mathbb{R}^d) := \bigcap_{n\in\mathbb{N}} C^n(\mathbb{R}^d).$$

With the symbol $C_0^\infty(\mathbb{R}^d)$ we denote the space of all functions $f \in C^\infty(\mathbb{R}^d)$, that are compactly supported. Often such functions are called test functions. Sometimes we write $\mathscr{D}(\mathbb{R}^d)$ instead of $C_0^\infty(\mathbb{R}^d)$. Strongly connected with $C_0^\infty(\mathbb{R}^d)$ is the following function space.

Definition 6. ***The Schwartz Space.***
The Schwartz space $\mathscr{S}(\mathbb{R}^d)$ is the collection of all functions $f \in C^\infty(\mathbb{R}^d)$, such that for all $\alpha, \beta \in \mathbb{N}_0^d$ we have

$$\sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty.$$

It is not difficult to see, that $C_0^\infty(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d)$. Very popular in the theory of function spaces are the so-called Sobolev spaces. They contain functions having both smoothness and integrability properties.

Definition 7. ***Sobolev Spaces.***
Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then the Sobolev space $W_p^m(\mathbb{R}^d)$ is the collection of all functions $f \in L_p(\mathbb{R}^d)$, such that for the distributional derivatives we find $D^\alpha f \in L_p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$. We write

$$\|f|W_p^m(\mathbb{R}^d)\| := \sum_{|\alpha| \leq m} \|D^\alpha f|L_p(\mathbb{R}^d)\|.$$

The Sobolev spaces are the initial point for many interesting function spaces. Some of them are collected in the next section.

3.3 Function Spaces defined by Fourier Transform

When we work with Smoothness Morrey spaces, the Fourier transform is a very important tool. For a function $f \in \mathscr{S}(\mathbb{R}^d)$ the Fourier transform $\mathscr{F} f$ and its inverse transform $\mathscr{F}^{-1} f$ are given by

$$\mathscr{F} f(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx \qquad \text{and} \qquad \mathscr{F}^{-1} f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(\xi) e^{ix\xi} d\xi.$$

For $f \in \mathscr{S}(\mathbb{R}^d)$ we observe $\mathscr{F}\mathscr{F}^{-1} f = f$ and $\mathscr{F}^{-1}\mathscr{F} f = f$. Moreover, the Fourier transform is a linear isomorphism for the Schwartz space. In what follows we want to define the Fourier transform for a much larger class of objects. Therefore we introduce the space of the tempered distributions $\mathscr{S}'(\mathbb{R}^d)$.

Definition 8. ***Tempered Distributions.***
The space of the tempered distributions $\mathscr{S}'(\mathbb{R}^d)$ is the collection of all continuous linear functionals $T : \mathscr{S}(\mathbb{R}^d) \to \mathbb{C}$.

The space $\mathscr{S}'(\mathbb{R}^d)$ is the topological dual of $\mathscr{S}(\mathbb{R}^d)$. For $T \in \mathscr{S}'(\mathbb{R}^d)$ we can define the Fourier transform by

$$(\mathscr{F} T)(\phi) := T(\mathscr{F} \phi)$$

for all $\phi \in \mathscr{S}(\mathbb{R}^d)$. $\mathscr{F}$ and $\mathscr{F}^{-1}$ are continuous bijections on $\mathscr{S}'(\mathbb{R}^d)$. We can split up the space $\mathscr{S}'(\mathbb{R}^d)$ in two groups of distributions.

Definition 9. ***Regular and Singular Distributions.***
A distribution $T \in \mathscr{S}'(\mathbb{R}^d)$ is called regular, if there exists a function $f \in L_1^{loc}(\mathbb{R}^d)$, such that

$$T(\phi) = \int_{\mathbb{R}^d} f(x)\phi(x)dx$$

for all $\phi \in \mathscr{S}(\mathbb{R}^d)$. A tempered distribution, that is not regular, is called singular.

Definition 9 allows us to identify regular distributions with functions. Many function spaces are subsets of $\mathscr{S}'(\mathbb{R}^d)$. Some of them are defined by using the Fourier transform. One example are the so-called Bessel-potential spaces.

Definition 10. ***Bessel-Potential Spaces.***
Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then the Bessel-potential space $H_p^s(\mathbb{R}^d)$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^d)$, such that $\mathscr{F}^{-1}[(1+|\xi|^2)^{s/2}\mathscr{F}f(\xi)](\cdot)$ is a regular distribution and

$$\|f|H_p^s(\mathbb{R}^d)\| := \left\|\mathscr{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\mathscr{F}f(\xi)](\cdot)\middle|L_p(\mathbb{R}^d)\right\| < \infty.$$

In some sense the Bessel-potential spaces are generalizations of the Sobolev spaces. So for $1 < p < \infty$ and $m \in \mathbb{N}$ we find $H_p^m(\mathbb{R}^d) = W_p^m(\mathbb{R}^d)$ in the sense of equivalent norms, see the theorem in chapter 2.5.6. in [128]. Therefore the spaces $H_p^s(\mathbb{R}^d)$ sometimes are called fractional Sobolev spaces. Two important classes of function spaces, that are defined via Fourier transform, are the Besov spaces and the Triebel-Lizorkin spaces. To give a precise definition we need a so-called smooth dyadic decomposition of the unity.

Definition 11. ***Smooth dyadic Decomposition of the Unity.***
Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function (called generator), such that $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq \frac{3}{2}$. Then we define a family of functions $(\varphi_k)_{k\in\mathbb{N}_0}$ (called smooth dyadic decomposition of the unity) by $\varphi_0 = \psi$ and for $k \in \mathbb{N}$ by

$$\varphi_k(x) := \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^d.$$

A family of functions $(\varphi_k)_{k\in\mathbb{N}_0}$ like it is defined in Definition 11 has some special properties. So for the supports of the involved functions we observe

$$\operatorname{supp}\varphi_k \subset \left\{x \in \mathbb{R}^d : 2^{k-1} \leq |x| \leq 3\cdot 2^{k-1}\right\}, \qquad k \in \mathbb{N}. \tag{3.6}$$

Moreover, there is the identity

$$\sum_{k=0}^{\infty} \varphi_k(x) = 1, \qquad x \in \mathbb{R}^d. \tag{3.7}$$

The formulas (3.6) and (3.7) explain the name smooth dyadic decomposition of the unity. Later it will be very important for us to deal with functions of the form $\mathscr{F}^{-1}[\varphi_k\mathscr{F}f]$ with $f \in \mathscr{S}'(\mathbb{R}^d)$. By the Paley-Wiener-Schwarz Theorem we find, that $\mathscr{F}^{-1}[\varphi_k\mathscr{F}f]$ is a smooth function for any $f \in \mathscr{S}'(\mathbb{R}^d)$ and for all $k \in \mathbb{N}_0$. There is the following useful decomposition.

Proposition 1. ***Littlewood-Paley Decomposition.***
Let $f \in \mathscr{S}'(\mathbb{R}^d)$ and $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then we have

$$f = \sum_{k=0}^{\infty} \mathscr{F}^{-1}[\varphi_k\mathscr{F}f]$$

with convergence in $\mathscr{S}'(\mathbb{R}^d)$.

Now we are ready to define the Besov spaces.

Definition 12. ***Besov Spaces.***
Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $(\varphi_k)_{k \in \mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Besov space $B^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^d)$, such that

$$\|f|B^s_{p,q}(\mathbb{R}^d)\| := \Big(\sum_{k=0}^{\infty} 2^{ksq} \|\mathscr{F}^{-1}[\varphi_k \mathscr{F} f]|L_p(\mathbb{R}^d)\|^q \Big)^{\frac{1}{q}} < \infty.$$

In the special case $q = \infty$ the sum is replaced by a supremum.

Similar-looking, but a bit more complicated, is the Definition of the Triebel-Lizorkin spaces.

Definition 13. ***Triebel-Lizorkin Spaces.***
Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Let $(\varphi_k)_{k \in \mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^d)$, such that

$$\|f|F^s_{p,q}(\mathbb{R}^d)\| := \Big\| \Big(\sum_{k=0}^{\infty} 2^{ksq} |\mathscr{F}^{-1}[\varphi_k \mathscr{F} f](x)|^q \Big)^{\frac{1}{q}} \Big| L_p(\mathbb{R}^d) \Big\| < \infty.$$

In the special case $q = \infty$ the sum is replaced by an essential supremum.

In some sense the Triebel-Lizorkin spaces are generalizations of the fractional Sobolev spaces. So for $s \in \mathbb{R}$ and $1 < p < \infty$ we observe $H^s_p(\mathbb{R}^d) = F^s_{p,2}(\mathbb{R}^d)$, see the theorem in chapter 2.5.6. in [128]. The Besov spaces and the Triebel-Lizorkin spaces are strongly connected. In fact, for $s \in \mathbb{R}$ and $0 < p < \infty$ we find $B^s_{p,p}(\mathbb{R}^d) = F^s_{p,p}(\mathbb{R}^d)$, see Proposition 2 in chapter 2.3.2. in [128].

Remark 1. ***The History of $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$.***
The Besov spaces have been developed between 1951 and 1961. They were introduced by Nikol'skii and Besov, see [89] and [5]. An early systematic collection of the properties of the spaces $B^s_{p,q}(\mathbb{R}^d)$ written by Peetre can be found in [94]. The Triebel-Lizorkin spaces appeared the first time around 1970. They were introduced by Lizorkin and Triebel, see [71], [72] and [126]. Early studies concerning these spaces also have be done by Peetre. Here we refer to [95]. Later both scales $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ have been investigated in detail in the famous books of Triebel, see [128], [129] and [133].

3.4 Function Spaces on Domains

Sometimes we will deal with function spaces that are defined on domains. Domains Ω are open subsets of $\mathbb{R}^d$. Many function spaces, that are defined on $\mathbb{R}^d$, have counterparts for domains. For example, the Lebesgue spaces on domains are defined as follows.

Definition 14. ***Lebesgue Spaces on Domains.***
Let $0 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ be open. Then we define the space $L_p(\Omega)$ as the collection of all functions f, that are measurable in Ω and fulfill

$$\|f|L_p(\Omega)\| := \Big(\int_{\Omega} |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

Consistent with Definition 5 the symbol $C_0^\infty(\Omega) = \mathscr{D}(\Omega)$ stands for the collection of all complex-valued infinitely often differentiable functions with compact support in Ω. With $\mathscr{D}'(\Omega)$ we denote the dual space of $\mathscr{D}(\Omega)$, that consists of distributions. In what follows for all domains $\Omega \subset \mathbb{R}^d$ and $g \in \mathscr{S}'(\mathbb{R}^d)$ by $g_{|\Omega}$ we denote the restriction of g to Ω. Now we are able to define function spaces on domains for many spaces simultaneously.

Definition 15. ***Function Spaces on Domains.***
Let $X(\mathbb{R}^d)$ be a quasi-normed space of tempered distributions, such that $X(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$. Let Ω denote an open, nontrivial subset of $\mathbb{R}^d$. Then $X(\Omega)$ is defined as the collection of all $f \in \mathscr{D}'(\Omega)$, such that there exists a distribution $g \in X(\mathbb{R}^d)$ satisfying

$$f(\varphi) = g(\varphi) \qquad \textit{for all} \qquad \varphi \in \mathscr{D}(\Omega)\,.$$

Here $\varphi \in \mathscr{D}(\Omega)$ is extended by zero on $\mathbb{R}^d \setminus \Omega$. We put

$$\| f \,| X(\Omega)\| := \inf\Big\{ \| g \,| X(\mathbb{R}^d)\| : \quad g_{|\Omega} = f \Big\}.$$

Popular examples for function spaces on domains are Besov spaces on domains and Triebel-Lizorkin spaces on domains. Extensive explanations concerning that topic can be found in [133], see chapter 1.11. Often it is enough to work with domains, that have boundaries, that fulfill some smoothness properties. So nearly every domain that appears in this text is a so-called Lipschitz domain. For the definition we follow Stein, see [122, VI.3.2].

Definition 16. ***Lipschitz Domains.***
By a Lipschitz domain we mean either a special or a bounded Lipschitz domain.

(i) A special Lipschitz domain is an open set $\Omega \subset \mathbb{R}^d$ lying above the graph of a Lipschitz function $\omega : \mathbb{R}^{d-1} \to \mathbb{R}$, namely

$$\Omega := \{(x', x_d) \in \mathbb{R}^d : \; x_d > \omega(x')\},$$

where ω satisfies, that for all x', $y' \in \mathbb{R}^{d-1}$ we have

$$|\omega(x') - \omega(y')| \le A|x' - y'|$$

with a positive constant A independent of x' and y'.

(ii) A bounded Lipschitz domain is a bounded domain $\Omega \subset \mathbb{R}^d$, whose boundary $\partial\Omega$ can be covered by a finite number of open balls B_k, such that for each $k \in \mathbb{N}$ after a suitable rotation $\partial\Omega \cap B_k$ is a part of the graph of a Lipschitz function.

For simplicity we shall use the convention, that a bounded Lipschitz domain in $\mathbb{R}$ is just a bounded interval. Sometimes we work with domains having a boundary, that is much more smooth, so-called $C^\infty-$ domains. Roughly speaking a $C^\infty-$ domain is a domain, whose boundary everywhere locally can be described in terms of $C^\infty-$ functions. For a precise definition one may consult chapter 3.2.1. in [128].

Chapter 4

Smoothness Morrey Spaces: Definitions and basic Properties

In this chapter we define our Smoothness Morrey spaces, namely the Triebel-Lizorkin-Morrey spaces $\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)$, the Besov-Morrey spaces $\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)$ and the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Furthermore, for all these spaces we will collect some basic properties. Most of the knowledge, that can be found in this chapter, is already well-known. But nevertheless it will be very important for our later considerations.

4.1 Morrey Spaces

Smoothness Morrey spaces are function spaces, that are built upon Morrey spaces. Therefore in this section we give a definition for the Morrey spaces and collect some basic properties of them.

Definition 17. ***Morrey Spaces.***
Let $0 < p \leq u < \infty$. Then the Morrey space $\mathscr{M}^u_p(\mathbb{R}^d)$ is defined to be the set of all functions $f \in L^{loc}_p(\mathbb{R}^d)$, such that

$$\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| := \sup_{y\in\mathbb{R}^d, r>0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

The Morrey spaces have been introduced in 1938 by Morrey, see [83]. They are quasi-Banach spaces and also Banach spaces in the case of $p \geq 1$. Notice, that we obtain an equivalent quasi-norm in Definition 17, when we replace the balls by dyadic cubes there. The Morrey spaces are generalizations of the Lebesgue spaces. So for $0 < p < \infty$ we observe $\mathscr{M}^p_p(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. Moreover, for $0 < p_2 \leq p_1 \leq u < \infty$ we have

$$L_u(\mathbb{R}^d) = \mathscr{M}^u_u(\mathbb{R}^d) \hookrightarrow \mathscr{M}^u_{p_1}(\mathbb{R}^d) \hookrightarrow \mathscr{M}^u_{p_2}(\mathbb{R}^d). \tag{4.1}$$

However, for $p \neq u$ the spaces $\mathscr{M}^u_p(\mathbb{R}^d)$ are much more complicated than the Lebesgue spaces. Then we observe the following somehow unpleasant properties for the Morrey spaces:
• they do not have $C^\infty_0(\mathbb{R}^d)$ as a dense subspace;
• they are not separable;

• they are not reflexive;
• they are not included in $L_1(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)$.
For the first two items we refer to Proposition 2.16 in [137]. The other facts can be found in Example 5.2 in [110] and in section 6 in [43]. Much more knowledge concerning Morrey spaces can be found in the comprehensive books [106] and [107].

4.2 Besov-Morrey Spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and Triebel-Lizorkin-Morrey Spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$

4.2.1 Definitions and historical Remarks

In this section we want to define the Besov-Morrey and the Triebel-Lizorkin-Morrey spaces. They are the main subjects of this treatise. Already from the definitions we can see, that they are generalizations of the Besov spaces and the Triebel-Lizorkin spaces.

Definition 18. ***Besov-Morrey Spaces.***
Let $s\in\mathbb{R}$, $0<p\le u<\infty$ and $0<q\le\infty$. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Besov-Morrey space $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f\in\mathcal{S}'(\mathbb{R}^d)$, such that

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| := \Big(\sum_{k=0}^{\infty} 2^{ksq}\|\mathcal{F}^{-1}[\varphi_k\mathcal{F}f]|\mathcal{M}^u_p(\mathbb{R}^d)\|^q\Big)^{\frac{1}{q}} < \infty.$$

In the case $q=\infty$ the usual modifications are made.

Definition 19. ***Triebel-Lizorkin-Morrey Spaces.***
Let $s\in\mathbb{R}$, $0<p\le u<\infty$ and $0<q\le\infty$. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f\in\mathcal{S}'(\mathbb{R}^d)$, such that

$$\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| := \Big\|\Big(\sum_{k=0}^{\infty} 2^{ksq}|\mathcal{F}^{-1}[\varphi_k\mathcal{F}f](x)|^q\Big)^{\frac{1}{q}}\Big|\mathcal{M}^u_p(\mathbb{R}^d)\Big\| < \infty.$$

In the case $q=\infty$ the usual modifications are made.

Sometimes we use the symbol $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ with $\mathcal{A}\in\{\mathcal{N},\mathcal{E}\}$. By this abbreviation we mean either the Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ or the Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. In what follows we will give a short overview about the history of both scales.

Remark 2. ***The History of the Spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$.***
The Besov-Morrey spaces have been introduced by Kozono and Yamazaki in 1994, see [64]. Later they were studied by Mazzucato, see [79]. In both papers the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ appeared in connection with Navier-Stokes equations. The Triebel-Lizorkin-Morrey spaces have been introduced by Tang and Xu in 2005, see [125]. Later with a different notation they also showed up as Triebel-Lizorkin-type spaces, see [138] and [140], and as Hybrid spaces, see [137]. An extensive and systematically treatise concerning Smoothness Morrey spaces can be found in [144].

Consistent with Definition 7 there also exist Morrey versions of the Sobolev spaces, the so-called Sobolev-Morrey spaces. They are defined in the following way.

Definition 20. ***Sobolev-Morrey Spaces.***
Let $m \in \mathbb{N}$ and $0 < p \leq u < \infty$. Then the Sobolev-Morrey space $W^m\mathcal{M}^u_p(\mathbb{R}^d)$ is the collection of all functions $f \in \mathcal{M}^u_p(\mathbb{R}^d)$, such that all distributional derivatives $D^\alpha f$ of order $|\alpha| \leq m$ belong to $\mathcal{M}^u_p(\mathbb{R}^d)$. We put

$$\| f \,| W^m\mathcal{M}^u_p(\mathbb{R}^d)\| := \sum_{|\alpha|\leq m} \| D^\alpha f \,| \mathcal{M}^u_p(\mathbb{R}^d)\| \, .$$

Later it will be convenient to use $W^0\mathcal{M}^u_p(\mathbb{R}^d) := \mathcal{M}^u_p(\mathbb{R}^d)$. In some sense the Sobolev-Morrey spaces are special cases of the Triebel-Lizorkin-Morrey spaces. In fact, there is the following identity.

Lemma 1. ***Sobolev-Morrey Spaces as Triebel-Lizorkin-Morrey Spaces.***
Let $1 < p \leq u < \infty$ and $m \in \mathbb{N}_0$. Then $\mathcal{E}^m_{u,p,2}(\mathbb{R}^d) = W^m\mathcal{M}^u_p(\mathbb{R}^d)$ in the sense of equivalent norms.

Proof. This result can be found in [116], see Lemma 3.6. and Theorem 3.1. ■

4.2.2 Basic Properties and Embeddings

In this subsection we collect some basic properties of the Besov-Morrey and Triebel-Lizorkin-Morrey spaces. Most of them are already well-known and will be very important for our later considerations. Let us start with the following elementary observations.

Lemma 2. ***Basic Properties of $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Then the following assertions are true.

(i) The spaces $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ are independent of the chosen smooth dyadic decomposition of the unity in the sense of equivalent quasi-norms.

(ii) The spaces $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ are quasi-Banach spaces. For $p \geq 1$ and $q \geq 1$ they are Banach spaces.

(iii) Let $\theta = \min(1, p, q)$. Then we have

$$\|f + g|\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)\|^\theta \leq \|f|\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)\|^\theta + \|g|\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)\|^\theta$$

for all $f, g \in \mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$.

(iv) It holds $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

(v) We have $\mathcal{N}^s_{p,p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{p,p,q}(\mathbb{R}^d) = F^s_{p,q}(\mathbb{R}^d)$.

(vi) For $1 < p \leq u < \infty$ it holds $\mathcal{E}^0_{u,p,2}(\mathbb{R}^d) = \mathcal{M}^u_p(\mathbb{R}^d)$.

Proof. (i) was proved in [125], see Theorem 2.8. The proofs of (ii) and (iii) are standard. We refer to Corollary 2.6. in [64] and to Lemma 2.1. in [144]. (iv) was proved in [108], see Theorem 3.2. and with slightly different formulation in [144], see Proposition 2.3. (v) is obvious, see Proposition 3.6. in [108]. (vi) was proved in [78], see Proposition 4.1. ■

Both the Besov-Morrey and the Triebel-Lizorkin-Morrey spaces have the so-called Fatou property. It reads as follows.

Lemma 3. ***The Fatou Property.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Suppose that $(f_k)_{k\in\mathbb{N}_0}$ is a bounded sequence in $\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$. The limit $f = \lim_{k\to\infty} f_k$ exists in $\mathscr{S}'(\mathbb{R}^d)$. Then we have $f \in \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$ and

$$\|f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\| \leq C \sup_{k\in\mathbb{N}_0} \|f_k|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|.$$

Proof. This result can be found in [108], see Lemma 3.5. ■

The parameter s in the definitions of the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ refers to the smoothness of the involved functions. Roughly speaking larger s means higher smoothness. In the spirit of this observation, there is the following equivalent characterization in terms of distributional derivatives.

Lemma 4. ***Characterization by Derivatives.***
Let $m \in \mathbb{N}$, $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Then we have $f \in \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$, if and only if the tempered distribution f and its distributional derivatives $\partial^m_j f$ with $j \in \{1,2,\dots,d\}$ belong to $\mathscr{A}^{s-m}_{u,p,q}(\mathbb{R}^d)$. Furthermore the quasi-norms $\| f \,|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|$ and

$$\| f \,|\mathscr{A}^{s-m}_{u,p,q}(\mathbb{R}^d)\| + \sum_{j=1}^{d} \|\partial^m_j f \,|\mathscr{A}^{s-m}_{u,p,q}(\mathbb{R}^d)\|$$

are equivalent.

Proof. This characterization can be found in Theorem 2.15 in [125] and in Corollary 3.4 from [108]. ■

It is very interesting to know, whether a function or distribution that belongs to a space $\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$ has some integrability properties or belongs to a Morrey space. For that question beside from s also the parameters u and p play a very important role. There are the following observations.

Lemma 5. ***Subsets of $L^{loc}_1(\mathbb{R}^d)$. Non-Limiting Case I.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Then we find

$$\mathscr{A}^s_{u,p,q}(\mathbb{R}^d) \subset L^{loc}_1(\mathbb{R}^d) \qquad \text{if} \qquad s > \frac{p}{u}\sigma_p$$

and

$$\mathscr{A}^s_{u,p,q}(\mathbb{R}^d) \not\subset L^{loc}_1(\mathbb{R}^d) \qquad \text{if} \qquad s < \frac{p}{u}\sigma_p.$$

Proof. Lemma 5 can be found in [48], see Theorem 3.3. ■

For the limiting case Lemma 5 can be supplemented as follows.

Lemma 6. ***Subsets of $L^{loc}_1(\mathbb{R}^d)$. Limiting Case I.***
Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ and $s = \frac{p}{u}\sigma_p$.

(a) Then for the Besov-Morrey spaces the following assertions are equivalent.

(i) *We have* $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d) \subset L_1^{loc}(\mathbb{R}^d)$.

(ii) *We have* $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathscr{M}^{\frac{u}{\min(p,1)}}_{\max(p,1)}(\mathbb{R}^d)$.

(iii) *We have* $0 < q \leq \min(\max(p,1),2)$.

(b) *For the Triebel-Lizorkin-Morrey spaces the following is equivalent.*

(i) *We have* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \subset L_1^{loc}(\mathbb{R}^d)$.

(ii) *We have* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathscr{M}^{\frac{u}{\min(p,1)}}_{\max(p,1)}(\mathbb{R}^d)$.

(iii) *We have either* $p \geq 1$ *with* $q \leq 2$ *or* $0 < p < 1$.

Proof. For that we refer to [49], see Theorem 3.2 and Theorem 3.4. ■

There exist related results for Smoothness Morrey spaces on domains. Let $\Omega \subset \mathbb{R}^d$ be an open set. Then it is possible to define the spaces $\mathscr{N}^s_{u,p,q}(\Omega)$ and $\mathscr{E}^s_{u,p,q}(\Omega)$ like it is described in Definition 15. For these spaces congenial to Lemma 5 and Lemma 6 we observe the following.

Lemma 7. ***Embeddings in*** $L_v(\Omega)$**.**
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $1 < v < \infty$ *with* $p < v$. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded* $C^\infty-$*domain.*

(i) *Then the embedding* $\mathscr{N}^s_{u,p,q}(\Omega) \hookrightarrow L_v(\Omega)$ *implies* $s \geq d\frac{p}{u}\left(\frac{1}{p} - \frac{1}{v}\right)$.

(ii) *The embedding* $\mathscr{E}^s_{u,p,q}(\Omega) \hookrightarrow L_v(\Omega)$ *implies either*

$$s > d\frac{p}{u}\left(\frac{1}{p} - \frac{1}{v}\right) \qquad \text{or} \qquad s = d\frac{p}{u}\left(\frac{1}{p} - \frac{1}{v}\right) \text{ and } q \leq 2.$$

Proof. This result can be found in [51], see Proposition 5.3, and in [52], see Corollary 4.4. The special case $p = u$ was treated in Corollary 2 of chapter 2.2.4 in [101], see also [120]. ■

The parameter q in the definition of the spaces $\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$ sometimes is called the fine index. It seems to be less important than the other parameters. However, it may play a role, when you investigate some limiting cases. There are elementary embeddings, which tell us, how the fine index q behaves.

Lemma 8. ***Elementary Embeddings.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $\varepsilon > 0$. *Then there are the following embeddings.*

(i) *We have* $\mathscr{A}^{s+\varepsilon}_{u,p,r}(\mathbb{R}^d) \hookrightarrow \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$ *for* $0 < r \leq \infty$.

(ii) *We have* $\mathscr{A}^s_{u,p,q_1}(\mathbb{R}^d) \hookrightarrow \mathscr{A}^s_{u,p,q_2}(\mathbb{R}^d)$ *for* $0 < q_1 \leq q_2 \leq \infty$.

(iii) *We have* $\mathscr{N}^s_{u,p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathscr{N}^s_{u,p,\infty}(\mathbb{R}^d)$.

Proof. These embeddings are standard and follow from the definitions. We refer to Proposition 3.6 in [108]. ■

Many more embedding results are known. For them we refer to [52]. For the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ there exist some useful multiplier theorems. So on the one hand there is the following pointwise multiplier theorem.

Lemma 9. ***Pointwise Multipliers.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Let $m \in \mathbb{N}$ be sufficiently large. Then there exists a positive constant $C(m)$, such that for all $g \in C^m(\mathbb{R}^d)$ and for all $f \in \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$ we have

$$\|f \cdot g|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\| \leq C(m)\Big(\sum_{|\alpha|\leq m}\|D^\alpha g|L_\infty(\mathbb{R}^d)\|\Big)\,\|f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|.$$

Proof. This outcome can be found in [51], see Theorem 2.6. For more details one may consult [103]. A related result is in [144], see Theorem 6.1. ■

On the other hand there is the following Fourier multiplier theorem.

Lemma 10. ***Fourier Multipliers.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. If $m \in \mathbb{N}$ is sufficiently large, then there exists a constant $C > 0$, such that for all $g \in C^\infty(\mathbb{R}^d)$ and $f \in \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$ we have

$$\|\mathscr{F}^{-1}[g\mathscr{F}f]|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\| \leq C \sup_{|\gamma|\leq m}\sup_{x\in\mathbb{R}^d}(1+|x|^2)^{\frac{|\gamma|}{2}}|D^\gamma g(x)|\ \ \|f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|.$$

Proof. To prove this finding, we follow the proof of Theorem 2.3.7. in [128]. Here the assertion was proved for the special case $p = u$. Fortunately almost everything that is done there also can be used for $p < u$. Instead of formula (2.3.6.20) from [128] we apply Proposition 2.12 from [125]. Then the desired outcome follows in the same way as in [128]. Notice, that a result similar to Lemma 10 also can be found in [125], see Proposition 2.14. One may also consult [137], see formula (3.259) from chapter 3.5.2. and Theorem 3.50. A forerunner of Lemma 10 for Morrey spaces can be found in [75]. ■

Sometimes it is helpful to know alternative characterizations for the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$. One of them in terms of atoms can be found in the next subsection.

4.2.3 Atomic Decompositions for $\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$

It is possible to describe the spaces $\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$ by means of atomic decompositions. Details concerning this topic can be found in [97] and in [108]. A short summary of the main ideas also is given in [48]. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ we define the dyadic cube $Q_{j,k} = 2^{-j}([0,1)^d + k)$. By $\chi_{j,k}$ we denote the characteristic function of the cube $Q_{j,k}$. For $0 < u < \infty$ we put $\chi^{(u)}_{j,k} = 2^{\frac{jd}{u}}\chi_{j,k}$. Now we are able to explain what atoms are.

Definition 21. ***(K,L) - Atoms.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Let $K \in \mathbb{N}_0$ and $L \in \mathbb{N}_0 \cup \{-1\}$. Then for $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$ a collection of $L_\infty(\mathbb{R}^d)$-functions $a_{j,k}$ is a family of (K,L) - atoms, if there are constants $C_1 > 1$ and $C_2 > 0$, such that the following properties are fulfilled.

(i) We have $\operatorname{supp} a_{j,k} \subset C_1 Q_{j,k}$.

(ii) *For* $|\alpha| \leq K$ *all classical derivatives* $D^\alpha a_{j,k}$ *exist and we have* $\|D^\alpha a_{j,k}|L_\infty(\mathbb{R}^d)\| \leq C_2 2^{j|\alpha|}$.

(iii) *For* $|\beta| \leq L$ *we have* $\int_{\mathbb{R}^d} x^\beta a_{j,k}(x)dx = 0$. *In the case* $L = -1$ *this condition is empty.*

Moreover, we want to define the following sequence spaces $\mathbf{a}^s_{u,p,q}(\mathbb{R}^d)$, which are connected with the spaces $\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$.

Definition 22. ***Sequence Spaces*** $\mathbf{a}^s_{u,p,q}(\mathbb{R}^d)$.
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ *and* $0 < q \leq \infty$.

(i) *Then the sequence space* $\mathbf{n}^s_{u,p,q}(\mathbb{R}^d)$ *is defined to be the set of all sequences* $\lambda = \{\lambda_{j,k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}^d} \subset \mathbb{C}$, *such that*

$$\|\lambda|\mathbf{n}^s_{u,p,q}(\mathbb{R}^d)\| := \Big(\sum_{j=0}^\infty 2^{jq(s-\frac{d}{u})}\Big\|\sum_{k\in\mathbb{Z}^d}|\lambda_{j,k}|\chi^{(u)}_{j,k}(x)\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^q\Big)^{\frac{1}{q}} < \infty.$$

In the case $q = \infty$ *the usual modifications have to be made.*

(ii) *The sequence space* $\mathbf{e}^s_{u,p,q}(\mathbb{R}^d)$ *is defined to be the set of all sequences* $\lambda = \{\lambda_{j,k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}^d} \subset \mathbb{C}$, *such that*

$$\|\lambda|\mathbf{e}^s_{u,p,q}(\mathbb{R}^d)\| := \Big\|\Big(\sum_{j=0}^\infty 2^{jq(s-\frac{d}{u})}\sum_{k\in\mathbb{Z}^d}|\lambda_{j,k}|^q[\chi^{(u)}_{j,k}(x)]^q\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\| < \infty.$$

In the case $q = \infty$ *the usual modifications have to be made.*

We write $\mathbf{a}^s_{u,p,q}(\mathbb{R}^d)$, when we mean either $\mathbf{n}^s_{u,p,q}(\mathbb{R}^d)$ or $\mathbf{e}^s_{u,p,q}(\mathbb{R}^d)$. Using this notation, we can formulate the following result, that can be found in [97], see the Theorems 2.30 and 2.36. One may also consult [108], see Corollary 4.10 and Theorem 4.12.

Lemma 11. ***Atomic Decompositions for*** $\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$.
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ *and* $0 < q \leq \infty$. *Let* $K \in \mathbb{N}_0$ *and* $L \in \mathbb{N}_0 \cup \{-1\}$, *such that we have* $K \geq \max(0, s+1)$. *Let*

$$L \geq \max(-1, \sigma_p - s)$$

for the $\mathscr{N}$ *- case and*

$$L \geq \max(-1, \sigma_{p,q} - s)$$

for the $\mathscr{E}$ *- case. Then for each* $f \in \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$, *there exists a family* $\{a_{j,k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}^d}$ *of (K,L) - atoms and a sequence* $\lambda = \{\lambda_{j,k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}^d} \in \mathbf{a}^s_{u,p,q}(\mathbb{R}^d)$, *such that*

$$f = \sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^d}\lambda_{j,k}a_{j,k}$$

with convergence in $\mathscr{S}'(\mathbb{R}^d)$ *and*

$$\|\lambda|\mathbf{a}^s_{u,p,q}(\mathbb{R}^d)\| \leq C_1\|f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|,$$

where $C_1 > 0$ *is independent of* λ *and* f. *Conversely there exists a constant* $C_2 > 0$, *such that for all families* $\{a_{j,k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}^d}$ *of (K,L) - atoms and all sequences* $\lambda = \{\lambda_{j,k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}^d} \in \mathbf{a}^s_{u,p,q}(\mathbb{R}^d)$ *we have*

$$\Big\|\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^d}\lambda_{j,k}a_{j,k}\Big|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\Big\| \leq C_2\|\lambda|\mathbf{a}^s_{u,p,q}(\mathbb{R}^d)\|.$$

In other words Lemma 11 tells us, that it is possible to describe the spaces $\mathscr{A}^{s}_{u,p,q}(\mathbb{R}^d)$ only in terms of atoms and without using the Definitions 18 and 19. To end this section let us mention, that there also exist wavelet characterizations for the spaces under consideration. So descriptions of $\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)$ by smooth compactly supported Daubechies wavelets can be found in Theorem 4.1. in [144] or in Theorem 3.26. in [137]. For characterizations in terms of the Haar wavelet we refer to Theorem 3.41. in [137].

4.3 Besov-type Spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and Triebel-Lizorkin-type Spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$

4.3.1 Definitions and historical Remarks

Hereinafter we want to define the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and the Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$. For that purpose we have to work with dyadic cubes. Let $\mathscr{Q}$ be the collection of all dyadic cubes in $\mathbb{R}^d$, more precisely

$$\mathscr{Q} := \{Q_{j,k} \;:\; Q_{j,k} = 2^{-j}([0,1)^d + k) \text{ with } j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^d\}.$$

The symbol $l(P)$ denotes the side-length of a cube $P \in \mathscr{Q}$. We write $j_P := -\log_2(l(P))$.

Definition 23. ***Besov-type Spaces.***
Let $s \in \mathbb{R}$, $0 \le \tau < \infty$ and $0 < p,q \le \infty$. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f \in \mathscr{S}'(\mathbb{R}^d)$, such that

$$\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| := \sup_{P\in\mathscr{Q}} \frac{1}{|P|^\tau}\left(\sum_{k=\max(j_P,0)}^{\infty} 2^{ksq}\left(\int_P |\mathscr{F}^{-1}[\varphi_k \mathscr{F} f](x)|^p dx\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

In the cases $p=\infty$ and/or $q=\infty$ the usual modifications have to be made.

Definition 24. ***Triebel-Lizorkin-type Spaces.***
Let $s \in \mathbb{R}$, $0 \le \tau < \infty$, $0 < p < \infty$ and $0 < q \le \infty$. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Triebel-Lizorkin-type space $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f \in \mathscr{S}'(\mathbb{R}^d)$, such that

$$\|f|F^{s,\tau}_{p,q}(\mathbb{R}^d)\| := \sup_{P\in\mathscr{Q}} \frac{1}{|P|^\tau}\left(\int_P\left(\sum_{k=\max(j_P,0)}^{\infty} 2^{ksq}|\mathscr{F}^{-1}[\varphi_k \mathscr{F} f](x)|^q\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} < \infty.$$

In the case $q=\infty$ the usual modifications have to be made.

Sometimes we write $A^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $A \in \{B,F\}$, when we mean either $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ or $F^{s,\tau}_{p,q}(\mathbb{R}^d)$.

Remark 3. ***The History of the Spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^d)$.***
The Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ have been introduced by El Baraka around 2002, see [31], [32] and [33]. The Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ appeared the first time around 2008 in the papers of Yang and Yuan, see [138] and [140]. A first systematic and comprehensive study of the properties of both scales can be found in [144]. Later the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ also showed up in [137] in connection with Hybrid function spaces.

It turns out, that the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ are well-known, if $\tau > \frac{1}{p}$. So there is the following observation.

Lemma 12. ***The Spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ as Hölder-Zygmund Spaces.***
Let $s \in \mathbb{R}$.

(i) Let $0 < p \leq \infty$. Let either $0 < q < \infty$ and $\tau \in (\frac{1}{p}, \infty)$ or $q = \infty$ and $\tau \in [\frac{1}{p}, \infty)$. Then we have

$$B^{s,\tau}_{p,q}(\mathbb{R}^d) = B^{s+d(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^d)$$

in the sense of equivalent quasi-norms.

(ii) Let $0 < p < \infty$. Let either $0 < q < \infty$ and $\tau \in (\frac{1}{p}, \infty)$ or $q = \infty$ and $\tau \in [\frac{1}{p}, \infty)$. Then we have

$$F^{s,\tau}_{p,q}(\mathbb{R}^d) = B^{s+d(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^d)$$

in the sense of equivalent quasi-norms.

Proof. This result can be found in [141]. We also refer to Proposition 3.5. in [116]. ■

In the case $0 \leq \tau < \frac{1}{p}$ the spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ coincide with the Triebel-Lizorkin-Morrey spaces. More precisely we know the following.

Lemma 13. ***The Coincidence of the Spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Then we have

$$F^{s,\frac{1}{p}-\frac{1}{u}}_{p,q}(\mathbb{R}^d) = \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$$

with equivalent quasi-norms.

Proof. We refer to [144], see Corollary 3.3. ■

For the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ the situation is different. For $0 \leq \tau < \frac{1}{p}$ in most of the cases they do not coincide with the Besov-Morrey spaces. But then the definition of the Besov-type spaces can be simplified a bit.

Lemma 14. ***Simplified Definition of $B^{s,\tau}_{p,q}(\mathbb{R}^d)$.***
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \leq \tau < \frac{1}{p}$ and $0 < q \leq \infty$. Let $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Then the Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is the set of all distributions $f \in \mathscr{S}'(\mathbb{R}^d)$, such that

$$\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\sharp)} := \sup_{P\in\mathscr{Q}} \frac{1}{|P|^\tau} \Big(\sum_{k=0}^{\infty} 2^{ksq} \Big(\int_P |\mathscr{F}^{-1}[\varphi_k \mathscr{F} f](x)|^p dx \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} < \infty.$$

Moreover $\| \cdot |B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\| \cdot |B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\sharp)}$ are equivalent quasi-norms on $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. In the case $q = \infty$ the usual modifications have to be made.

Proof. A proof can be found in [116], see Proposition 3.1. ■

Let us finish this subsection with some comments concerning so-called Local and Hybrid function spaces. Both scales are strongly connected with $A^{s,\tau}_{p,q}(\mathbb{R}^d)$.

Remark 4. ***Local Function Spaces.***
Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $-d/p \leq r < \infty$. Then in [136] Triebel developed the so-called Local spaces $\mathscr{L}^r A^s_{p,q}(\mathbb{R}^d)$, see chapter 1.3.1 in [136] for a definition. These spaces are related to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^d)$. So for $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $-d/p \leq r < \infty$ and $\tau = 1/p + r/d$ we have $A^{s,\tau}_{p,q,unif}(\mathbb{R}^d) = \mathscr{L}^r A^s_{p,q}(\mathbb{R}^d)$, see [145] and Theorem 2.57 in [136]. The space $A^{s,\tau}_{p,q,unif}(\mathbb{R}^d)$ is defined to be the collection of all distributions $f \in \mathscr{S}'(\mathbb{R}^d)$, such that

$$\sup_{\lambda \in \mathbb{R}^d} \| f(\cdot)\Psi(\cdot - \lambda) | A^{s,\tau}_{p,q}(\mathbb{R}^d)\| < \infty.$$

Here $\Psi \in C_0^\infty(\mathbb{R}^d)$ is a non-negative function, such that $\Psi(0) \neq 0$.

Remark 5. ***Hybrid Function Spaces.***
Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $-d/p \leq r < \infty$. In [137] Triebel introduced the so-called Hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^d)$, see chapter 3.3.1 in [137] for a definition. These spaces are strongly connected with $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^d)$. So for $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $-d/p \leq r < \infty$ and $\tau = 1/p + r/d$ we have $L^r A^s_{p,q}(\mathbb{R}^d) = A^{s,\tau}_{p,q}(\mathbb{R}^d)$, see Theorem 3.38. in [137].

4.3.2 Basic Properties and Embeddings

Hereafter we collect some basic properties of the spaces $A^{s,\tau}_{p,q}(\mathbb{R}^d)$. Most of them are already well-known and will be important for us later.

Lemma 15. ***Basic Properties of*** $A^{s,\tau}_{p,q}(\mathbb{R}^d)$.
Let $s \in \mathbb{R}$, $\tau \geq 0$, $0 < p < \infty$, $0 < q \leq \infty$ and $\varepsilon > 0$. Then the following assertions are true.

- *(i) The spaces $A^{s,\tau}_{p,q}(\mathbb{R}^d)$ are independent of the chosen smooth dyadic decomposition of the unity in the sense of equivalent quasi-norms.*
- *(ii) The spaces $A^{s,\tau}_{p,q}(\mathbb{R}^d)$ are quasi-Banach spaces.*
- *(iii) Let $\theta := \min(1, p, q)$. Then we have*

$$\|f + g | A^{s,\tau}_{p,q}(\mathbb{R}^d)\|^\theta \leq \|f | A^{s,\tau}_{p,q}(\mathbb{R}^d)\|^\theta + \|g | A^{s,\tau}_{p,q}(\mathbb{R}^d)\|^\theta$$

  for all $f, g \in A^{s,\tau}_{p,q}(\mathbb{R}^d)$.
- *(iv) We have $\mathscr{S}(\mathbb{R}^d) \hookrightarrow A^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d)$.*
- *(v) The scale $A^{s,\tau}_{p,q}(\mathbb{R}^d)$ is monotone with respect to q, namely if $q_1 \leq q_2$, then $A^{s,\tau}_{p,q_1}(\mathbb{R}^d) \hookrightarrow A^{s,\tau}_{p,q_2}(\mathbb{R}^d)$.*
- *(vi) The scale $A^{s,\tau}_{p,q}(\mathbb{R}^d)$ is monotone with respect to s, namely for all $q_1, q_2 \in (0, \infty]$ we have $A^{s+\varepsilon,\tau}_{p,q_1}(\mathbb{R}^d) \hookrightarrow A^{s,\tau}_{p,q_2}(\mathbb{R}^d)$.*
- *(vii) It holds $A^{s,0}_{p,q}(\mathbb{R}^d) = A^s_{p,q}(\mathbb{R}^d)$.*

Proof. For most of the proofs we refer to [144]. In particular, (i) can be found in Corollary 2.1, for (ii) and (iii) we refer to Lemma 2.1 and (iv) is proved in Proposition 2.3. Parts (v) and (vi) can be found in Proposition 2.1. Part (vii) is obvious. ■

In view of Lemma 12 and Lemma 13 in what follows we concentrate on the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $\tau \le 1/p$. Later it will be interesting to know, under which restrictions on the parameters a space $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ contains singular distributions and under which conditions it does not. The following result was proved in [48], see Theorem 3.6.

Lemma 16. ***Subsets of $L^{loc}_1(\mathbb{R}^d)$. Non-Limiting Case II.***
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \le \tau \le \frac{1}{p}$ and $0 < q \le \infty$. Then the following assertions are true.

(i) Let either $s > 0$ and $p \ge 1$ or $s > d(\frac{1}{p} - 1) - d\tau(1-p)$ and $p < 1$. Then we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) \subset L^{loc}_1(\mathbb{R}^d)$.

(ii) Let either $s < 0$ and $p \ge 1$ or $s < d(\frac{1}{p} - 1) - d\tau(1-p)$ and $p < 1$. Then we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) \not\subset L^{loc}_1(\mathbb{R}^d)$.

There also is a result concerning the limiting cases, see Theorem 3.8. in [48].

Lemma 17. ***Subsets of $L^{loc}_1(\mathbb{R}^d)$. Limiting Case II.***
Let $s = 0$, $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$ and $0 < q \le \infty$. Then we observe $B^{0,\tau}_{p,q}(\mathbb{R}^d) \not\subset L^{loc}_1(\mathbb{R}^d)$, if we are in one of the following cases.

(i) We have $p \ge 2$ and $q > 2$.

(ii) We have $1 \le p < 2$ and $q > p \max(1, \frac{1}{d(1-p\tau)})$.

For $p < 1$ this can be supplemented as follows.

Lemma 18. ***Subsets of $L^{loc}_1(\mathbb{R}^d)$. Limiting Case III.***
Let $0 < q \le p < 1$, $0 \le \tau < \frac{1}{p}$ and $s = d(\frac{1}{p} - 1) - d\tau(1-p)$. Then there exists a constant $C > 0$, such that

$$\| f \,|L_1(\mathbb{R}^d)\| \le C \| f \,|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| \tag{4.2}$$

for all $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $\operatorname{supp} f \subset [-1,1]^d$.

Proof. In Theorem 3.8 in [48] it is proved, that under the given restrictions we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) \subset L^{loc}_1(\mathbb{R}^d)$. Looking into the details of the proof we find, that one can sharpen this result as stated above. ■

A short summary for the Lemmas 16, 17 and 18 reads as follows. We observe, that the line $s = s(p,\tau)$ with

$$s(p,\tau) := \begin{cases} d\left(\frac{1}{p} - 1\right) - d\tau(1-p) & \text{if} \quad 0 < p < 1; \\ 0 & \text{if} \quad 1 \le p < \infty; \end{cases}$$

represents the barrier for singular distributions within the scale $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. To extend this observation we can formulate the following result for Besov-type spaces $B^{s,\tau}_{p,q}(\Omega)$ on domains. They are defined as it is described in Definition 15.

Lemma 19. ***$B^{s,\tau}_{p,q}(\Omega)$ and $L_v(\Omega)$.***
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$ and $0 < q \le \infty$. Let $\max(p,1) < v < \infty$. Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^\infty-$ domain. Then the following assertions are true.

(i) If $s > d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v})$, then we have $B^{s,\tau}_{p,q}(\Omega) \hookrightarrow L_v(\Omega)$.

(ii) If $s < d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v})$, then we observe $B^{s,\tau}_{p,q}(\Omega) \not\subset L_v(\Omega)$.

Proof. This outcome can be found in [37]. ■

Like for the spaces $\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$ also for the Besov-type spaces there exist some useful multiplier theorems. For example, there is the following observation for pointwise multipliers.

Lemma 20. ***Pointwise Multipliers for $B^{s,\tau}_{p,q}(\mathbb{R}^d)$.***
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$ and $0 < q \le \infty$. Let $m \in \mathbb{N}$ be sufficiently large. Then there exists a positive constant $C(m)$, such that for all $g \in C^m(\mathbb{R}^d)$ and all $f \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ we have

$$\| f \cdot g \,|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| \le C(m) \, \| g \,|C^m(\mathbb{R}^d)\| \; \| f \,|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|.$$

Proof. A proof of this multiplier theorem can be found in [144], see Theorem 6.1. ■

4.4 A Collection of useful Inequalities

In this section we collect some inequalities, that can be used in order to work with Smoothness Morrey spaces. At first we want to have a look at so-called Hardy-Littlewood maximal inequalities. For that purpose below we define the Hardy-Littlewood maximal function.

Definition 25. ***Hardy-Littlewood Maximal Function.***
Let $f \in L^{loc}_1(\mathbb{R}^d)$. Then the Hardy-Littlewood maximal function is defined by

$$\boldsymbol{M} f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

for all $x \in \mathbb{R}^d$.

Using this notation we can formulate the Hardy-Littlewood maximal inequalities for Smoothness Morrey spaces.

Lemma 21. ***Hardy-Littlewood Maximal Inequalities.***

(i) Let $1 < p \le u < \infty$. Then there is a constant $C_1 > 0$ independent of $f \in L^{loc}_1(\mathbb{R}^d)$, such that

$$\|\boldsymbol{M} f|\mathscr{M}^u_p(\mathbb{R}^d)\| \le C_1 \|f|\mathscr{M}^u_p(\mathbb{R}^d)\|.$$

(ii) Let $1 < p \le u < \infty$ and $1 < q \le \infty$. Moreover $(f_j)_{j=0}^\infty$ is a sequence of locally Lebesgue-integrable functions on $\mathbb{R}^d$. Then there is a constant $C_2 > 0$ independent of $(f_j)_{j=0}^\infty$, such that

$$\Big\| \Big(\sum_{j=0}^\infty |\boldsymbol{M} f_j|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\| \le C_2 \Big\| \Big(\sum_{j=0}^\infty |f_j|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|.$$

In the case $q = \infty$ the usual modifications have to be made.

(iii) *Let* $1 < p < \infty$, $0 \le \tau < \frac{1}{p}$ *and* $0 < q \le \infty$. *Let* $(f_j)_{j=0}^{\infty}$ *be a sequence of locally Lebesgue-integrable functions on* $\mathbb{R}^d$. *Then there is a constant* $C_3 > 0$ *independent of* $(f_j)_{j=0}^{\infty}$, *such that*

$$\sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}} \Big(\sum_{j=0}^{\infty} \Big(\int_P |(\mathbf{M}(f_j))(x)|^p dx \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} \le C_3 \sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}} \Big(\sum_{j=0}^{\infty} \Big(\int_P |f_j(x)|^p dx \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}}.$$

In the case $q = \infty$ *the usual modifications have to be made.*

Proof. (i) can be found in [27], see also Theorem 6.19 in [105]. (ii) was proved in Lemma 2.5 in [125]. For (iii) we refer to Proposition 2.3 in [151]. ■

Another group of interesting inequalities are multiplier theorems for functions with a compactly supported Fourier transform. Recall, that $H_2^s(\mathbb{R}^d)$ are the Bessel-potential spaces defined in Definition 10.

Lemma 22. ***Multiplier Theorem for band-limited Functions.***
Let $0 < p \le u < \infty$ *and* $0 < q \le \infty$. *Let* $R > 0$ *and* $f \in \mathscr{M}_p^u(\mathbb{R}^d) \cap \mathscr{S}'(\mathbb{R}^d)$ *with* $\operatorname{supp} \mathscr{F} f \subset B(0,R)$.

(i) *Let* $\eta > 0$ *and* $\nu > \frac{1}{\eta} + \frac{d}{2}$. *Let* $h \in H_2^{\nu}(\mathbb{R}^d)$. *Then there is a constant* $C_1 > 0$ *independent of* R, h *and* f, *such that*

$$\int_{\mathbb{R}^d} |\mathscr{F}^{-1} h(x-y) f(y)| dy \le C_1 \, \|h(R\cdot)|H_2^{\nu}(\mathbb{R}^d)\| \, ((\mathbf{M}|f|^{\eta})(x))^{\frac{1}{\eta}}$$

for all $x \in \mathbb{R}^d$.

(ii) *Let* $\nu > \frac{d}{\min(1,p)} + \frac{d}{2}$ *and* $h \in H_2^{\nu}(\mathbb{R}^d)$. *Then there is a constant* $C_2 > 0$ *independent of* R, h *and* f, *such that*

$$\Big\| (2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \mathscr{F}^{-1} h(x-y) f(y) dy \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| \le C_2 \|h(R\cdot)|H_2^{\nu}(\mathbb{R}^d)\| \, \|f|\mathscr{M}_p^u(\mathbb{R}^d)\|.$$

(iii) *Let* $\nu > \frac{d}{\min(1,q,p)} + \frac{d}{2}$ *and* $h_j \in H_2^{\nu}(\mathbb{R}^d)$ *for each* $j \in \mathbb{N}_0$. *Let* $R_j > 0$ *for each* $j \in \mathbb{N}_0$ *and* $(f_j)_{j=0}^{\infty}$ *be a sequence of functions with* $(f_j)_{j=0}^{\infty} \subset \mathscr{M}_p^u(\mathbb{R}^d) \cap \mathscr{S}'(\mathbb{R}^d)$ *and* $\operatorname{supp} \mathscr{F} f_j \subset B(0,R_j)$ *for each* $j \in \mathbb{N}_0$. *Then there is a constant* $C_3 > 0$ *independent of any* R_j, h_j *and* f_j, *such that*

$$\Big\| \Big(\sum_{j=0}^{\infty} \Big| (2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \mathscr{F}^{-1} h_j(x-y) f_j(y) dy \Big|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|$$
$$\le C_3 \Big\| \Big(\sum_{j=0}^{\infty} |f_j|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| \quad \sup_{j\in\mathbb{N}_0} \|h_j(R_j\cdot)|H_2^{\nu}(\mathbb{R}^d)\|.$$

In the case $q = \infty$ *the usual modifications have to be made.*

Proof. This result can be found in [108], see Theorem 2.4. One may also consult Theorem 2.7 in [125]. ■

4.5 Some further related Function Spaces

To complete chapter 4 we want to mention, that in the last few years several modifications of the original Morrey spaces from Definition 17 showed up in the literature. At first let us refer to two monographs written by Besov, Il'in and Nikol'skii, see [7] and [8], and to a paper from Netrusov, see [87]. There the authors modified the Morrey spaces in such a way, that they replaced

$$\sup_{y\in\mathbb{R}^d, r>0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p dx \Big)^{\frac{1}{p}}$$

by

$$\sup_{y\in\mathbb{R}^d, r>0} \min\Big(1, |B(y,r)|\Big)^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p dx \Big)^{\frac{1}{p}}.$$

Another modification, the so-called local Morrey-type space, was investigated in a work from Burenkov and Nursultanov, see [20], as well as in a paper from Burenkov, Chigambayeva and Nursultanov, see [21]. For generalized Morrey spaces, where $|B(y,r)|^{\frac{1}{u}-\frac{1}{p}}$ was replaced by a function $\varphi(y,r)$, we refer to the works written by Mizuhara [82], Nakai [85] and Nakamura, Noi and Sawano [86]. Much more knowledge concerning generalized Morrey spaces also can be found in [107]. All the modified Morrey spaces we just mentioned allow to define versions of Smoothness Morrey spaces, what enlarges the number of existing function spaces a lot. But nevertheless in what follows most of the time we will deal with the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$.

Chapter 5

Equivalent Characterizations via Differences

In this chapter it is our main goal, to prove equivalent characterizations in terms of differences for the spaces $\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)$, $\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. For that purpose we will use an abstract approach to function spaces developed by Hedberg and Netrusov, see [53]. It turns out, that our Smoothness Morrey spaces fit into this theory. With that in mind, we are able to prove characterizations in terms of generalized ball means of differences for the function spaces under consideration. Moreover, as a special feature we will obtain so-called Stein characterizations for the Triebel-Lizorkin-Morrey spaces. Let us mention, that most of the results from this chapter also can be found in the author's papers [55], [56] and [58].

5.1 Differences: Definition and classical Results

One central term in this treatise are differences of first and higher order. Therefore in this section we collect some general knowledge concerning this topic. Let us start with a definition.

Definition 26. ***Differences.***
Let $f : \mathbb{R}^d \to \mathbb{C}$ be a function and let $x, h \in \mathbb{R}^d$. Then the difference of the first order is defined as

$$\Delta^1_h f(x) := f(x+h) - f(x).$$

For $N \in \mathbb{N}$ with $N \geq 2$ the difference of order N is given by

$$\Delta^N_h f(x) := \left(\Delta^1_h \left(\Delta^{N-1}_h f\right)\right)(x).$$

Often it is more convenient, to work with an explicit formula for the differences of higher order. The following result is well-known and can be proved by induction, see also chapter 2.7 in [28].

Lemma 23. ***Explicit Formula for Differences.***
Let $f : \mathbb{R}^d \to \mathbb{C}$ be a function. Let $x, h \in \mathbb{R}^d$ and $N \in \mathbb{N}$. Then we have

$$\Delta^N_h f(x) = \sum_{k=0}^{N} (-1)^{N-k} \binom{N}{k} f(x+kh).$$

If f is a smooth function, the difference of order N is strongly connected with the derivatives of order N. So there is the following very useful observation.

Lemma 24. ***Differences and Derivatives.***
Let $x,h \in \mathbb{R}^d$ and $N \in \mathbb{N}$. Let $f \in C^N(\mathbb{R}^d)$. Then there is a constant $C > 0$ independent of f,x and h, such that

$$|\Delta_h^N f(x)| \leq C|h|^N \max_{|\gamma|=N} \sup_{|x-y|\leq N|h|} |D^\gamma f(y)|.$$

Proof. This estimate is well-known and can be proved using the Mean Value Theorem. It appears in the literature many times, see for example formula (4.28) on page 91 in [144]. ■

As already mentioned differences are a suitable tool to describe our Smoothness Morrey spaces. So for the original Besov spaces and the Triebel-Lizorkin spaces characterizations in terms of differences are known since many years. For $B^s_{p,q}(\mathbb{R}^d)$ there is the following result.

Theorem 1. ***Besov Spaces and Differences.***
Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $N \in \mathbb{N}$. Assume

$$d \max\left(0, \frac{1}{p} - 1\right) < s < N. \tag{5.1}$$

Then $B^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in L_{\max(p,1)}(\mathbb{R}^d)$, such that

$$\|f|L_p(\mathbb{R}^d)\| + \left(\int_{\mathbb{R}^d} |h|^{-sq} \, \|\Delta_h^N f\,|L_p(\mathbb{R}^d)\|^q \frac{dh}{|h|^d}\right)^{\frac{1}{q}} < \infty$$

with equivalent quasi-norms. In the case $q = \infty$ the usual modification has to be made.

Proof. For the proof we refer to [88, 4.3.4], chapter 4 in [8], Theorem 2.5.12 in [128] and Theorem 3.5.3 in [129]. ■

The corresponding result for the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ reads as follows.

Theorem 2. ***Triebel-Lizorkin Spaces and Differences.***
Let $0 < p < \infty$, $0 < q \leq \infty$ and $N \in \mathbb{N}$. Assume

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right) < s < N. \tag{5.2}$$

Then $F^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in L_{\max(p,1)}(\mathbb{R}^d)$, such that

$$\|f|L_p(\mathbb{R}^d)\| + \left\|\left(\int_0^\infty t^{-sq}\left(t^{-d}\int_{B(0,t)} |(\Delta_h^N f)(\cdot)|dh\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \Big|L_p(\mathbb{R}^d)\right\| < \infty$$

with equivalent quasi-norms. In the case $q = \infty$ the usual modification has to be made.

Proof. This characterization can be found in section 2.5.11 in [128], in Theorem 3.5.3 in [129] and in chapter 1.11.9 in [133]. ■

Let us mention, that except from some limiting cases, the conditions concerning the parameters, that can be found in the Theorems 1 and 2, are known to be necessary. Many more information to this are given in chapter 6. In what follows we want to find generalizations for the Theorems 1 and 2, to describe the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. For that purpose roughly speaking we have to replace the Lebesgue quasi-norm by the (maybe somehow modified) Morrey quasi-norm. More details concerning that can be found in the next sections.

5.2 The Theory of Hedberg and Netrusov

There exist a lot of different methods, to prove characterizations in terms of differences for Smoothness Morrey spaces. Here we want to use an abstract approach from Hedberg and Netrusov. In [53] they developed a general theory to describe function spaces, that are related to Besov and Triebel-Lizorkin spaces. This approach can be applied, to deduce characterizations for Smoothness Morrey spaces in terms of differences. To sketch the main ideas of Hedberg and Netrusov, below we will give a short summary of chapter 1 from [53]. The starting point for the theory of Hedberg and Netrusov are quasi-Banach spaces of sequences of functions, denoted by E.

Definition 27. ***Sequence Spaces*** E.
Let E be a quasi-Banach space of sequences of Lebesgue-measurable functions on $\mathbb{R}^d$. Then on E we define a non-negative function $\|\cdot\|_E$, which satisfies the following conditions.

(i) $\|\cdot\|_E$ has the same properties as a norm, except for the triangle inequality, which is replaced by the following property. There exist constants κ with $0<\kappa\le 1$ and $C_E\ge 1$, such that for any family $\{F_i\}_{i=0}^{j}$ of elements in E and any $j\in\mathbb{N}$ one has the inequality

$$\Big\|\sum_{i=0}^{j} F_i\Big\|_E^{\kappa} \le C_E \sum_{i=0}^{j} \|F_i\|_E^{\kappa}.$$

(ii) The metric space $(E,\|\cdot\|_E)$ is complete.

(iii) If $\{f_i\}_{i=0}^{\infty}\in E$ and $\{g_i\}_{i=0}^{\infty}$ is a sequence of measurable functions, such that $|g_i|\le|f_i|$ almost everywhere for all $i\in\mathbb{N}_0$, it follows, that $\{g_i\}_{i=0}^{\infty}\in E$ and $\|\{g_i\}_{i=0}^{\infty}\|_E\le\|\{f_i\}_{i=0}^{\infty}\|_E$.

Based on this definition Hedberg and Netrusov introduced the classes $S(\varepsilon_+,\varepsilon_-,r)$ of spaces E with $\varepsilon_+,\varepsilon_-\in\mathbb{R}$ and $0<r<\infty$. To describe them we need some additional notation. For a sequence of functions $\{f_i\}_{i=0}^{\infty}$ we define the left shift S_+ and the right shift S_- by

$$S_+\left(\{f_i\}_{i=0}^{\infty}\right) := \{f_{i+1}\}_{i=0}^{\infty} \qquad \text{and} \qquad S_-\left(\{f_i\}_{i=0}^{\infty}\right) := \{f_{i-1}\}_{i=0}^{\infty} \tag{5.3}$$

with $f_{-i}=0$ for all $i\in\mathbb{N}$. Moreover, for $0<r<\infty$ and $t\ge 0$ we define the maximal function $M_{r,t}f$ and the operator $\hat{M}_{r,t}$ by

$$\hat{M}_{r,t}\left(\{f_i\}_{i=0}^{\infty}\right) := \{M_{r,t}f_i\}_{i=0}^{\infty} := \left\{ \sup_{a>0}\left(a^{-d}\int_{B(0,a)} \frac{|f_i(\cdot+y)|^r}{(1+|y|)^{rt}}dy\right)^{\frac{1}{r}} \right\}_{i=0}^{\infty}. \tag{5.4}$$

Now we are able to give the following definition.

Definition 28. ***The Classes*** $S(\varepsilon_+,\varepsilon_-,r,t)$.
Let $\varepsilon_+,\varepsilon_-\in\mathbb{R}$, $0<r<\infty$ *and* $t\geq 0$. *We say, that a space* $(E,\|\cdot\|_E)$, *which has all the properties from Definition 27 belongs to the class* $S(\varepsilon_+,\varepsilon_-,r,t)$, *if the following conditions are satisfied.*

(i) The linear operators S_+ *and* S_- *are continuous on* E *and there are constants* $C_1,C_2>0$ *independent of* $j\in\mathbb{N}$, *such that we have*

$$\|(S_+)^j|\mathscr{L}(E)\|\leq C_1 2^{-j\varepsilon_+} \qquad \textit{and} \qquad \|(S_-)^j|\mathscr{L}(E)\|\leq C_2 2^{j\varepsilon_-}.$$

(ii) The operator $\hat{M}_{r,t}$ *is bounded on* E *and there is a constant* $C>0$ *independent of* $\{f_i\}_{i=0}^\infty$, *such that* $\left\|\{M_{r,t}f_i\}_{i=0}^\infty\right\|_E\leq C\,\|\{f_i\}_{i=0}^\infty\|_E$.

We put $S(\varepsilon_+,\varepsilon_-,r)=\bigcup_{t\geq 0}S(\varepsilon_+,\varepsilon_-,r,t)$.

In a next step Hedberg and Netrusov introduced function spaces denoted by $Y(E)$, that are built upon the sequence spaces E. They are defined in the following way.

Definition 29. ***The Spaces*** $Y(E)$.
Let $\varepsilon_+,\varepsilon_-\in\mathbb{R}$ *and* $r>0$. *Moreover let* $E\in S(\varepsilon_+,\varepsilon_-,r)$. *Then the space* $Y(E)$ *consists of all distributions* $f\in\mathscr{S}'(\mathbb{R}^d)$, *which have a representation* $f=\sum_{i=0}^\infty f_i$ *converging in* $\mathscr{S}'(\mathbb{R}^d)$, *such that we have* $\|\{f_i\}_{i=0}^\infty\|_E<\infty$ *and* $\operatorname{supp}\mathscr{F}f_0\subset B(0,2)$ *as well as* $\operatorname{supp}\mathscr{F}f_i\subset B(0,2^{i+1})\setminus B(0,2^{i-1})$ *for all* $i\in\mathbb{N}$.

We put

$$\|f\|_{Y(E)}:=\inf\|\{f_i\}_{i=0}^\infty\|_E\ ,$$

where the infimum is taken over all admissible representations of f as described in Definition 29. Then $\|f\|_{Y(E)}$ is a quasi-norm and $Y(E)$ becomes a quasi-normed space. It was one of the main goals in [53], to prove equivalent characterizations for the spaces $Y(E)$. In this context Hedberg and Netrusov also found a characterization for $Y(E)$ in terms of differences. The following very important theorem is only one part of a much more comprehensive result.

Theorem 3. $Y(E)$ ***and Differences.***
Let $\varepsilon_+,\varepsilon_->0$, $0<r\leq\infty$ *and* $E\in S(\varepsilon_+,\varepsilon_-,r)$. *Let* $0<v\leq\infty$, $N\in\mathbb{N}$ *and suppose*

$$d\max\left(0,\frac{1}{r}-1,\frac{1}{r}-\frac{1}{v}\right)<\varepsilon_+ \qquad \textit{and} \qquad \varepsilon_-<N.$$

Then a function $f\in L_r^{loc}(\mathbb{R}^d)$ *belongs to* $Y(E)$, *if and only if* $f\in L_v^{loc}(\mathbb{R}^d)$ *and the functions*

$$g_0(x)=\left(\int_{B(x,1)}|f(y)|^v dy\right)^{\frac{1}{v}} \quad \textit{and} \quad g_i(x)=2^{\frac{di}{v}}\left(\int_{B(0,2^{-i})}|\Delta_z^N f(x)|^v dz\right)^{\frac{1}{v}},\quad i\in\mathbb{N},$$

satisfy $\|\{g_i\}_{i=0}^\infty\|_E<\infty$. *The quasi-norms* $\|f\|_{Y(E)}$ *and* $\|\{g_i\}_{i=0}^\infty\|_E$ *are equivalent on* $L_r^{loc}(\mathbb{R}^d)$. *In the case* $v=\infty$ *the usual modifications should be made.*

Proof. This result is a combination of Theorem 1.1.14. and Proposition 1.1.12. from [53]. ■

Later the following technical remark will be of some importance for us.

Remark 6. ***A Version of Theorem 3.***
A detailed study of the proof of Theorem 1.1.14. in [53] shows, that it is possible to replace

$$g_0(x) = \Big(\int_{B(x,1)} |f(y)|^{\nu} dy \Big)^{\frac{1}{\nu}} \qquad by \qquad \tilde{g}_0(x) = |f(x)|$$

in the formulation of Theorem 3. To prove this, fortunately almost everything in the proof of Theorem 1.1.14. in [53] can be used unchanged. Only in the Steps 3 and 6 some minor modifications are necessary. The changes in Step 3 are trivial. In Step 6 a combination of Lemma 1.1.4. and Lemma 1.1.3. from [53] delivers the desired result.

In [53] Hedberg and Netrusov proved many more properties of the spaces $Y(E)$. Moreover, they deduced some more equivalent characterizations for those spaces, for example characterizations in terms of atoms or characterizations via approximation by polynomials. For details we refer to [53].

5.3 The Hedberg-Netrusov Approach to Smoothness Morrey Spaces

In this section we want to investigate, how the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ are connected with the theory of Hedberg and Netrusov we described before. So the following few lines are very much in the spirit of chapter 4.5 in [144]. The next definition serves as starting point for our considerations.

Definition 30. ***The Spaces $\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)$, $l^s_q(\mathscr{M}^u_p(\mathbb{R}^d))$ and $\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)$.***
Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. By $\{f_j\}_{j=0}^{\infty}$ we denote sequences of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

(i) Let in addition $p \leq u < \infty$. Then we define a space of sequences of locally Lebesgue-integrable functions on $\mathbb{R}^d$ given by

$$\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d) = \Big\{ \{f_j\}_{j=0}^{\infty} : \quad \|\{f_j\}_{j=0}^{\infty}\|_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)} := \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |f_j|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\| < \infty \Big\}.$$

(ii) Again let $p \leq u < \infty$. We define

$$l^s_q(\mathscr{M}^u_p(\mathbb{R}^d)) = \Big\{ \{f_j\}_{j=0}^{\infty} : \quad \|\{f_j\}_{j=0}^{\infty}\|_{l^s_q(\mathscr{M}^u_p(\mathbb{R}^d))} := \Big(\sum_{j=0}^{\infty} 2^{jsq} \|f_j|\mathscr{M}^u_p(\mathbb{R}^d)\|^q \Big)^{\frac{1}{q}} < \infty \Big\}.$$

(iii) Let in addition $0 \leq \tau < \frac{1}{p}$. Then we define

$$\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d) = \Big\{ \{f_j\}_{j=0}^{\infty} : \|\{f_j\}_{j=0}^{\infty}\|_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} := \sup_{P \in \mathscr{Q}} \frac{1}{|P|^{\tau}} \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big(\int_P |f_j(x)|^p dx \Big)^{\frac{q}{p}} \Big)^{\frac{1}{q}} < \infty \Big\}.$$

Always for $q = \infty$ we have to make the usual modifications.

It is not difficult to see, that the spaces from Definition 30 are valid examples for the space E from the theory of Hedberg and Netrusov, see Definition 27. So there is the following simple observation.

Lemma 25. ***Examples for*** $(E, \|\cdot\|_E)$.
Let $s \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \leq \infty$.

(i) Let in addition $p \leq u < \infty$. *Then the pair* $(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d), \|\cdot\|_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)})$ *fulfills all the properties from Definition 27.*

(ii) Again let $p \leq u < \infty$. *The pair* $(l^s_q(\mathscr{M}^u_p(\mathbb{R}^d)), \|\cdot\|_{l^s_q(\mathscr{M}^u_p(\mathbb{R}^d))})$ *fulfills all the properties from Definition 27.*

(iii) Let $0 \leq \tau < \frac{1}{p}$. *Then the pair* $(\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d), \|\cdot\|_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)})$ *fulfills all the properties from Definition 27.*

Proof. The proof of Lemma 25 is almost obvious. Therefore we will be rather short here.
Proof of (i). It is not difficult to see, that the spaces $\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)$ are quasi-Banach spaces. Furthermore, with $\theta = \min(1, p, q)$ we have

$$\|\{f_j + g_j\}_{j=0}^\infty\|^\theta_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)} \leq \|\{f_j\}_{j=0}^\infty\|^\theta_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)} + \|\{g_j\}_{j=0}^\infty\|^\theta_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)}$$

for all sequences $\{f_j\}_{j=0}^\infty$ and $\{g_j\}_{j=0}^\infty$ of locally Lebesgue-integrable functions. The proof of the lattice property is trivial. More details concerning the spaces $\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)$ can be found in chapter 4.5.1 of [144].
Proofs of (ii) and (iii). These can be done in the same way as for (i). In both cases we can choose $\theta = \min(1, p, q)$ to find

$$\|\{f_j + g_j\}_{j=0}^\infty\|^\theta_{l^s_q(\mathscr{M}^u_p(\mathbb{R}^d))} \leq \|\{f_j\}_{j=0}^\infty\|^\theta_{l^s_q(\mathscr{M}^u_p(\mathbb{R}^d))} + \|\{g_j\}_{j=0}^\infty\|^\theta_{l^s_q(\mathscr{M}^u_p(\mathbb{R}^d))}$$

and

$$\|\{f_j + g_j\}_{j=0}^\infty\|^\theta_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} \leq \|\{f_j\}_{j=0}^\infty\|^\theta_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} + \|\{g_j\}_{j=0}^\infty\|^\theta_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)}.$$

For more details we refer to chapter 4.5.1 in [144] and to Lemma 2.1. in [144]. ■

In a next step we want to investigate, under which conditions on the parameters the spaces from Definition 30 belong to the classes $S(\varepsilon_+, \varepsilon_-, r, t)$ from Definition 28. There is the following result.

Lemma 26. ***Some Elements of*** $S(s, s, r, t)$.
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ *and* $t \geq 0$.

(i) Let in addition $p \leq u < \infty$ *and* $0 < r < \min(p, q)$. *Then* $(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d), \|\cdot\|_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)}) \in S(s, s, r, t)$.

(ii) Let $p \leq u < \infty$ *and* $0 < r < p$. *Then we have* $(l^s_q(\mathscr{M}^u_p(\mathbb{R}^d)), \|\cdot\|_{l^s_q(\mathscr{M}^u_p(\mathbb{R}^d))}) \in S(s, s, r, t)$.

(iii) Let $0 \leq \tau < \frac{1}{p}$ *and* $0 < r < p$. *Then we find* $(\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d), \|\cdot\|_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)}) \in S(s, s, r, t)$.

Proof. *Proof of (i). Step 1.* At first we have to deal with the shift operator S_+. By definition for all $j \in \mathbb{N}$ we find

$$(S_+)^j(\{f_i\}_{i=0}^\infty) = \{f_{i+j}\}_{i=0}^\infty.$$

For each $i \in \mathbb{N}_0$ we can write

$$2^{is}|f_{i+j}| = 2^{-js}2^{(i+j)s}|f_{i+j}|.$$

Consequently it is not difficult to see, that

$$\|(S_+)^j(\{f_i\}_{i=0}^\infty)\|_{\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d)} \le 2^{-js}\|\{f_i\}_{i=0}^\infty\|_{\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d)}.$$

With a view to Definition 28 that means $\varepsilon_+ = s$. For the shift operator S_- we can use similar arguments to find

$$\|(S_-)^j(\{f_i\}_{i=0}^\infty)\|_{\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d)} \le 2^{js}\|\{f_i\}_{i=0}^\infty\|_{\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d)}.$$

This results in $\varepsilon_- = s$.

Step 2. Now we have to deal with the maximal function $M_{r,t}f$. We observe

$$\begin{aligned}\|\{M_{r,t}f_i\}_{i=0}^\infty\|_{\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d)} &= \Big\|\Big(\sum_{i=0}^\infty 2^{isq}|M_{r,t}f_i|^q\Big)^{\frac{1}{q}}\Big|\mathscr{M}_p^u(\mathbb{R}^d)\Big\| \\ &\le C_1\Big\|\Big(\sum_{i=0}^\infty (\mathbf{M}(2^{is}f_i)^r)^{\frac{q}{r}}\Big)^{\frac{r}{q}}\Big|\mathscr{M}_{\frac{p}{r}}^{\frac{u}{r}}(\mathbb{R}^d)\Big\|^{\frac{1}{r}}.\end{aligned}$$

Here $\mathbf{M}$ denotes the Hardy-Littlewood maximal operator. Now we use part (ii) of Lemma 21. This is possible because of $0 < r < \min(p,q)$. We obtain

$$\begin{aligned}\|\{M_{r,t}f_i\}_{i=0}^\infty\|_{\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d)} &\le C_2\Big\|\Big(\sum_{i=0}^\infty 2^{isq}|f_i|^q\Big)^{\frac{r}{q}}\Big|\mathscr{M}_{\frac{p}{r}}^{\frac{u}{r}}(\mathbb{R}^d)\Big\|^{\frac{1}{r}} \\ &= C_2\|\{f_i\}_{i=0}^\infty\|_{\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d)}.\end{aligned}$$

Therefore under the given restrictions on the parameters we find, that the operator $\hat{M}_{r,t}$ is bounded, see Definition 28.

Proofs of (ii) and (iii). Here we can proceed in the same way as for the proof of (i). To show (ii), we can use Lemma 21(i). To prove (iii), we can apply Lemma 21(iii). We omit the details. ■

Now we are ready to figure out, how the spaces $Y(E)$ from Definition 29 are connected with our Smoothness Morrey spaces $\mathscr{E}_{u,p,q}^s(\mathbb{R}^d)$, $\mathscr{N}_{u,p,q}^s(\mathbb{R}^d)$ and $B_{p,q}^{s,\tau}(\mathbb{R}^d)$. We observe the following.

Proposition 2. ***Some Examples for*** $Y(E)$**.**
Let $s \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \le \infty$.

(i) Let in addition $p \le u < \infty$ *and* $0 < r < \min(p,q)$. *Then we have*

$$Y(\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d)) = \mathscr{E}_{u,p,q}^s(\mathbb{R}^d).$$

Moreover $\|\cdot|\mathscr{E}_{u,p,q}^s(\mathbb{R}^d)\|$ *and* $\|\cdot\|_{Y(\mathscr{M}_p^u(l_q^s)(\mathbb{R}^d))}$ *are equivalent quasi-norms.*

(ii) Let $p \le u < \infty$ *and* $0 < r < p$. *Then*

$$Y(l_q^s(\mathscr{M}_p^u(\mathbb{R}^d))) = \mathscr{N}_{u,p,q}^s(\mathbb{R}^d).$$

Moreover $\|\cdot|\mathscr{N}_{u,p,q}^s(\mathbb{R}^d)\|$ *and* $\|\cdot\|_{Y(l_q^s(\mathscr{M}_p^u(\mathbb{R}^d)))}$ *are equivalent quasi-norms.*

(iii) Let $0 \le \tau < \frac{1}{p}$ *and* $0 < r < p$. *Then we have*

$$Y(\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)) = B^{s,\tau}_{p,q}(\mathbb{R}^d).$$

Moreover $\|\cdot|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ *and* $\|\cdot\|_{Y(\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d))}$ *are equivalent quasi-norms.*

Proof. *Proof of (i).*
Step 1. At first we prove $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow Y(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d))$. Therefore we take $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and show, that all properties, that can be found in Definition 29, are fulfilled. Because of part (iv) in Lemma 2 we find $f \in \mathscr{S}'(\mathbb{R}^d)$. If $(\varphi_j)_{j\in\mathbb{N}_0}$ is a smooth dyadic decomposition of the unity, we get $f = \sum_{j=0}^{\infty} \mathscr{F}^{-1}[\varphi_j \cdot \mathscr{F} f]$ with convergence in $\mathscr{S}'(\mathbb{R}^d)$. Now the rest is just a consequence of Definition 19 and the properties of the functions $(\varphi_j)_{j\in\mathbb{N}_0}$, see Definition 11. We observe

$$\|f\|_{Y(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d))} \le \Big\| \big\{\mathscr{F}^{-1}[\varphi_j \cdot \mathscr{F} f]\big\}_{j=0}^{\infty} \Big\|_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)} = \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|. \tag{5.5}$$

It follows $f \in Y(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d))$.
Step 2. Now we prove $Y(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)) \hookrightarrow \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Let $f \in Y(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d))$. So we have a representation $f = \sum_{i=0}^{\infty} f_i$, that fulfills all the properties written down in Definition 29. We put $f_{-1} = 0$ and take $\theta = \min(1,p,q)$. Then we conclude

$$\begin{aligned}
\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^\theta &= \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f](\cdot)|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^\theta \\
&= \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big| \mathscr{F}^{-1}\Big[\varphi_j \mathscr{F} \sum_{i=0}^{\infty} f_i\Big](\cdot) \Big|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^\theta \\
&\le 3 \max_{i\in\{-1,0,1\}} \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f_{j+i}](\cdot)|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^\theta \\
&\le C_1 \max_{i\in\{-1,0,1\}} \Big\| \Big(\sum_{j=0}^{\infty} \Big| \int_{\mathbb{R}^d} \mathscr{F}^{-1}\varphi_j(\cdot - y) 2^{js} f_{i+j}(y)dy \Big|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^\theta .
\end{aligned}$$

Now we use Lemma 22. Choose $\nu > \frac{d}{\min(1,q,p)} + \frac{d}{2}$. Since $\varphi_j \in \mathscr{S}(\mathbb{R}^d)$ we have $\varphi_j \in H^\nu_2(\mathbb{R}^d)$, whereby $H^\nu_2(\mathbb{R}^d)$ denotes a Bessel-potential space. We obtain

$$\begin{aligned}
\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^\theta &\le C_2 \max_{i\in\{-1,0,1\}} \Big\| \Big(\sum_{j=0}^{\infty} |2^{js} f_{i+j}(\cdot)|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^\theta \quad \sup_{k\in\mathbb{N}_0} \|\varphi_k(2^{k+2}\cdot)|H^\nu_2(\mathbb{R}^d)\|^\theta \\
&\le C_3 \Big\| \{f_j\}_{j=0}^{\infty} \Big\|^\theta_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)} \quad \sup_{k\in\mathbb{N}_0} \|\varphi_k(2^{k+2}\cdot)|H^\nu_2(\mathbb{R}^d)\|^\theta .
\end{aligned}$$

Because of $f \in Y(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d))$ we observe $\|\{f_j\}_{j=0}^{\infty}\|_{\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)} < \infty$. The fact that $\sup_{k\in\mathbb{N}_0} \|\varphi_k(2^{k+2}\cdot)|H^\nu_2(\mathbb{R}^d)\|$ is finite is well-known, see for example page 46 in [128]. Consequently we get $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Moreover, the calculations we did are correct for any admissible representation $f = \sum_{i=0}^{\infty} f_i$ and so also for the infimum over all such representations. Hence we get $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \le C\|f\|_{Y(\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d))}$.
Proof of (ii). This can be done in the same way as the proof of (i). Therefore we will omit the details.

Proof of (iii).
Step 1. Here we prove $B^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow Y(\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d))$. To do so, we proceed like it is described in Step 1 of the proof of (i). We only want to mention, that to obtain the counterpart of formula (5.5) we have to use Definition 23 and Lemma 14.
Step 2. Now we prove $Y(\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^d)$. The main idea for the proof is the same as in Step 2 of the proof of (i). However, when we look at the details, some modifications will become necessary here. Therefore we will be more precise in what follows. Let $f \in Y(\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d))$. So there is a representation $f = \sum_{i=0}^{\infty} f_i$ in the sense of Definition 29. We take $\theta = \min(1,p,q)$ and find

$$\begin{aligned}
&\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{\theta}\\
&\quad\le \left[\sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty} 2^{ksq}\Big(\int_P |\mathscr{F}^{-1}[\varphi_k \mathscr{F} f](x)|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}\\
&\quad= \left[\sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty} 2^{ksq}\Big(\int_P |\mathscr{F}^{-1}[\varphi_k \mathscr{F} \sum_{i=0}^{\infty} f_i](x)|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}\\
&\quad\le C_1 \max_{i\in\{-1,0,1\}} \left[\sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty} 2^{ksq}\Big(\int_P \Big|\int_{\mathbb{R}^d} \mathscr{F}^{-1}\varphi_k(x-y) f_{k+i}(y)dy\Big|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}.
\end{aligned}$$

Now we use part (i) of Lemma 22. Choose $0<\eta<p$ and $\nu > \frac{1}{\eta}+\frac{d}{2}$. Since $\varphi_k \in \mathscr{S}(\mathbb{R}^d)$, we have $\varphi_k \in H^{\nu}_2(\mathbb{R}^d)$ for all $k\in\mathbb{N}_0$. Moreover, because of $\|\{f_k\}_{k=0}^{\infty}\|_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} < \infty$ we find $f_k \in \mathscr{M}^u_p(\mathbb{R}^d)$ for all $k\in\mathbb{N}_0$ and $\tau = \frac{1}{p}-\frac{1}{u}$. We obtain

$$\begin{aligned}
&\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{\theta}\\
&\quad\le C_2 \max_{i\in\{-1,0,1\}} \left[\sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty} 2^{ksq}\Big(\int_P \Big(\|\varphi_k(2^{k+2}\cdot)|H^{\nu}_2(\mathbb{R}^d)\|\cdot((\mathbf{M}|f_{k+i}|^{\eta})(x))^{\frac{1}{\eta}}\Big)^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}\\
&\quad\le C_2 \sup_{j\in\mathbb{N}} \|\varphi_j(2^{j+2}\cdot)|H^{\nu}_2(\mathbb{R}^d)\|^{\theta}\\
&\qquad\quad\times \max_{i\in\{-1,0,1\}} \left[\sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty} 2^{ksq}\Big(\int_P \Big(((\mathbf{M}|f_{k+i}|^{\eta})(x))^{\frac{1}{\eta}}\Big)^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}.
\end{aligned}$$

By definition we have $\varphi_k(\xi) = \varphi_1(2^{-k+1}\xi)$ for $k\in\mathbb{N}$ and $\xi\in\mathbb{R}^d$. This implies, that $\sup_{j\in\mathbb{N}_0}\|\varphi_j(2^{j+2}\cdot)|H^{\nu}_2(\mathbb{R}^d)\| < \infty$. Now since $\eta<p$ we can use part (iii) of Lemma 21 and find

$$\begin{aligned}
\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{\theta} &\le C_3 \max_{i\in\{-1,0,1\}} \left[\sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}}\Big(\sum_{k=0}^{\infty} 2^{ksq}\Big(\int_P |f_{k+i}(x)|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\right]^{\theta}\\
&\le C_4 \|\{f_k\}_{k=0}^{\infty}\|^{\theta}_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)}.
\end{aligned}$$

Therefore since $\|\{f_k\}_{k=0}^{\infty}\|_{\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)} < \infty$ we get $f\in B^{s,\tau}_{p,q}(\mathbb{R}^d)$. The calculations we did hold true for any admissible representation $f=\sum_{i=0}^{\infty} f_i$ and thus also for the infimum over all such representations. Consequently we find $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| \le C\|f\|_{Y(\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d))}$. ■

Now we are prepared to formulate a version of Theorem 3 for the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. It reads as follows.

Theorem 4. ***Smoothness Morrey Spaces and Differences. Version I.***
Let $0 < p < \infty$, $0 < q \le \infty$, $0 < v \le \infty$ *and* $N \in \mathbb{N}$.

(i) Let in addition $p \le u < \infty$ *and*

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ *belongs to* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, *if and only if* $f \in L^{loc}_v(\mathbb{R}^d)$ *and (modifications if* $q = \infty$ *and / or* $v = \infty$*)*

$$\underbrace{\Big\| \Big[\Big(\int_{B(x,1)} |f(y)|^v dy \Big)^{\frac{q}{v}} + \sum_{j=1}^{\infty} 2^{jq(s+\frac{d}{v})} \Big(\int_{B(0,2^{-j})} |\Delta^N_z f(x)|^v dz \Big)^{\frac{q}{v}} \Big]^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|}_{\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)} :=} < \infty.$$

The quasi-norms $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$ *and* $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$ *are equivalent for* $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$.

(ii) Let $p \le u < \infty$ *and*

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ *belongs to* $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, *if and only if* $f \in L^{loc}_v(\mathbb{R}^d)$ *and (modifications if* $q = \infty$ *and / or* $v = \infty$*)*

$$\underbrace{\Big(\Big\| \Big(\int_{B(x,1)} |f(y)|^v dy \Big)^{\frac{1}{v}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^q + \sum_{j=1}^{\infty} 2^{jq(s+\frac{d}{v})} \Big\| \Big(\int_{B(0,2^{-j})} |\Delta^N_z f(x)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}}_{\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)} :=}$$

is finite. Moreover the quasi-norms $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|$ *and* $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$ *are equivalent for* $f \in L^{loc}_p(\mathbb{R}^d)$.

(iii) Let $0 \le \tau < \frac{1}{p}$ *and*

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ *belongs to* $B^{s,\tau}_{p,q}(\mathbb{R}^d)$, *if and only if* $f \in L^{loc}_v(\mathbb{R}^d)$ *and (modifications if* $q = \infty$ *and / or* $v = \infty$*)*

$$\underbrace{\sup_{P \in \mathscr{Q}} \frac{1}{|P|^\tau} \Big[\Big(\int_P \Big(\int_{B(x,1)} |f(y)|^v dy \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} + \sum_{j=1}^{\infty} 2^{jsq} \Big(\int_P 2^{\frac{djp}{v}} \Big(\int_{B(0,2^{-j})} |\Delta^N_z f(x)|^v dz \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}}_{\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\clubsuit)} :=}$$

is finite. Moreover the quasi-norms $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ *and* $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$ *are equivalent for* $f \in L^{loc}_p(\mathbb{R}^d)$.

Proof. *Proof of (i).* The proof of (i) is just a combination of Theorem 3 and part (i) of Proposition 2. In a first step we get the above result not for functions $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$, but for $f \in L^{loc}_r(\mathbb{R}^d)$ with

$$\max\left(\frac{d}{s+d}, \frac{d}{s+\frac{d}{v}}\right) < r < \min(p,q).$$

We have $L^{loc}_{\min(p,q)}(\mathbb{R}^d) \subset L^{loc}_r(\mathbb{R}^d)$. So $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ implies $f \in L^{loc}_r(\mathbb{R}^d)$ and the result can be obtained.
Proofs of (ii) and (iii). These can be done in the same way as it is described in the proof of (i). In both cases we have to combine Theorem 3 and Proposition 2. ■

So Theorem 4 provides a first characterization in terms of differences for our Smoothness Morrey spaces. In what follows we will prove alternative versions of Theorem 4, to obtain some further equivalent quasi-norms, that are more easy to handle. For this purpose the next result will be important. It is similar to Theorem 4, but has the advantage, that in the first parts of the equivalent quasi-norms one integral is left away.

Corollary 1. ***Smoothness Morrey Spaces and Differences. Version II.***
Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < v \leq \infty$ and $N \in \mathbb{N}$.

(i) Let in addition $p \leq u < \infty$ and

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ belongs to $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, if and only if $f \in L^{loc}_v(\mathbb{R}^d)$ and (modifications if $q = \infty$ and / or $v = \infty$)

$$\underbrace{\left\| \left[|f(x)|^q + \sum_{j=1}^{\infty} 2^{jq(s+\frac{d}{v})} \left(\int_{B(0,2^{-j})} |\Delta^N_z f(x)|^v dz \right)^{\frac{q}{v}} \right]^{\frac{1}{q}} \middle| \mathcal{M}^u_p(\mathbb{R}^d) \right\|}_{\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} :=} < \infty.$$

The quasi-norms $\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|$ and $\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$ are equivalent for $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$.

(ii) Let $p \leq u < \infty$ and

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ belongs to $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$, if and only if $f \in L^{loc}_v(\mathbb{R}^d)$ and (modifications if $q = \infty$ and / or $v = \infty$)

$$\underbrace{\left(\|f|\mathcal{M}^u_p(\mathbb{R}^d)\|^q + \sum_{j=1}^{\infty} 2^{jq(s+\frac{d}{v})} \left\| \left(\int_{B(0,2^{-j})} |\Delta^N_z f(x)|^v dz \right)^{\frac{1}{v}} \middle| \mathcal{M}^u_p(\mathbb{R}^d) \right\|^q \right)^{\frac{1}{q}}}_{\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} :=} < \infty.$$

The quasi-norms $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|$ and $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$ are equivalent for $f \in L^{loc}_p(\mathbb{R}^d)$.

(iii) Let $0 \le \tau < \frac{1}{p}$ *and*

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L_p^{loc}(\mathbb{R}^d)$ *belongs to* $B^{s,\tau}_{p,q}(\mathbb{R}^d)$*, if and only if* $f \in L_v^{loc}(\mathbb{R}^d)$ *and (modifications if* $q = \infty$ *and / or* $v = \infty$*)*

$$\underbrace{\sup_{P \in \mathcal{Q}} \frac{1}{|P|^{\tau}} \Big[\Big(\int_P |f(x)|^p dx \Big)^{\frac{q}{p}} + \sum_{j=1}^{\infty} 2^{jsq} \Big(\int_P 2^{\frac{djp}{v}} \Big(\int_{B(0,2^{-j})} |\Delta_z^N f(x)|^v dz \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \Big]^{\frac{1}{q}}}_{\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)} :=} < \infty.$$

The quasi-norms $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ *and* $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$ *are equivalent for* $f \in L_p^{loc}(\mathbb{R}^d)$.

Proof. This result can be proved in the same way as Theorem 4. We have to combine Theorem 3, Remark 6 and Proposition 2. ■

Notice, that characterizations similar to Theorem 4 and Corollary 1 also can be found in section 4.5.2 in [144].

5.4 A Characterization of $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ by generalized Ball Means

In this section we prove some more characterizations in terms of differences for the Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. Therefore we will concentrate on equivalent characterizations by so-called generalized ball means of differences. That means, we are interested in Morrey versions of Theorem 2. For a function $f \in L_1^{loc}(\mathbb{R}^d)$ a ball mean of a difference of order $N \in \mathbb{N}$ is given by

$$\frac{1}{|B(0,t)|} \int_{B(0,t)} |\Delta_h^N f(x)| dh$$

with $0 < t < \infty$ and $x \in \mathbb{R}^d$. Such means appear in the literature in connection with the original Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$, see for example section 2.5.11. in [128]. To explain what a generalized ball mean of a difference (sometimes also called v-mean) is, we have to introduce an additional parameter $0 < v \le \infty$. Then the v-mean of a difference of order $N \in \mathbb{N}$ is given by

$$\Big(\frac{1}{|B(0,t)|} \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}}$$

with $0 < t < \infty$ and $x \in \mathbb{R}^d$. In the case $v = \infty$ the usual modifications have to be made. Also the v-means show up in the literature concerning $F^s_{p,q}(\mathbb{R}^d)$, see the chapters 3.5.2 and 3.5.3 in [129]. Now let us turn to the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. In a first step we will prove the following result. In some sense it is an advancement of Theorem 4 (i), where the sum is replaced by an integral.

Proposition 3. ***Generalized Ball Means for*** $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$**.**
Let $0 < p \le u < \infty$, $0 < q \le \infty$, $0 < v \le \infty$ *and* $N \in \mathbb{N}$. *Let*

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ *belongs to* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, *if and only if* $f \in L^{loc}_v(\mathbb{R}^d)$ *and (modifications if* $q = \infty$ *and / or* $v = \infty$*)*

$$\underbrace{\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + \left\|\left(\int_0^1 t^{-sq}\left(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\right)^{\frac{q}{v}} \frac{dt}{t}\right)^{\frac{1}{q}} \middle| \mathscr{M}^u_p(\mathbb{R}^d)\right\|}_{\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)} :=} < \infty.$$

The quasi-norms $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$ *and* $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)}$ *are equivalent for* $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$.

Proof. *Step 1.* At first we prove, that there is a constant $C > 0$ independent of $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$, such that $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} \le C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)}$. For this purpose we use the (quasi-)triangle inequality and get

$$\begin{aligned}\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} &= \left\|\left(|f(x)|^q + \sum_{j=1}^{\infty} 2^{jq(s+\frac{d}{v})}\left(\int_{B(0,2^{-j})} |\Delta^N_z f(x)|^v dz\right)^{\frac{q}{v}}\right)^{\frac{1}{q}} \middle| \mathscr{M}^u_p(\mathbb{R}^d)\right\| \\ &\le C_1\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + C_1\left\|\left(\sum_{j=1}^{\infty} 2^{jq(s+\frac{d}{v})}\left(\int_{B(0,2^{-j})} |\Delta^N_z f(x)|^v dz\right)^{\frac{q}{v}}\right)^{\frac{1}{q}} \middle| \mathscr{M}^u_p(\mathbb{R}^d)\right\|.\end{aligned}$$

Now because of the monotonicity of $\int_{B(0,t)} |\Delta^N_z f(x)|^v dz$ in t we observe

$$\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} \le C_2\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + C_2\left\|\left(\int_0^1 t^{-sq}\left(t^{-d}\int_{B(0,t)} |\Delta^N_z f(x)|^v dz\right)^{\frac{q}{v}} \frac{dt}{t}\right)^{\frac{1}{q}} \middle| \mathscr{M}^u_p(\mathbb{R}^d)\right\|.$$

This is exactly what we want. So step 1 of the proof is complete.
Step 2. Second we will prove, that for $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of f, such that $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)} \le C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$. Because of Corollary 1 we already know $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} < \infty$. Of course we have

$$\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| = \|(|f|^q)^{\frac{1}{q}}|\mathscr{M}^u_p(\mathbb{R}^d)\| \le \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}.$$

Now again using the monotonicity of $\int_{B(0,t)} |\Delta^N_h f(x)|^v dh$ in t we get

$$\begin{aligned}&\left\|\left(\int_0^1 t^{-sq}\left(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\right)^{\frac{q}{v}} \frac{dt}{t}\right)^{\frac{1}{q}} \middle| \mathscr{M}^u_p(\mathbb{R}^d)\right\| \\ &\quad = \left\|\left(\sum_{j=1}^{\infty}\int_{2^{-j}}^{2^{-j+1}} t^{-sq}\left(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\right)^{\frac{q}{v}} \frac{dt}{t}\right)^{\frac{1}{q}} \middle| \mathscr{M}^u_p(\mathbb{R}^d)\right\| \\ &\quad \le C_1\left\|\left(\sum_{j=0}^{\infty} 2^{jsq}2^{jd\frac{q}{v}}\left(\int_{B(0,2^{-j})} |\Delta^N_h f(x)|^v dh\right)^{\frac{q}{v}}\right)^{\frac{1}{q}} \middle| \mathscr{M}^u_p(\mathbb{R}^d)\right\| \\ &\quad \le C_2\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} + C_2\left\|\left(\int_{B(0,1)} |\Delta^N_h f(x)|^v dh\right)^{\frac{1}{v}} \middle| \mathscr{M}^u_p(\mathbb{R}^d)\right\|.\end{aligned}$$

Next we want to apply the formula from Lemma 23. When we use it in combination with a transformation of the coordinates, we obtain

$$\Big\| \Big(\int_{B(0,1)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|$$
$$\leq C_3 \Big\| \sum_{k=1}^{N} \Big(\int_{B(0,1)} |f(x+kh)|^v dh \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| + C_3 \| f | \mathscr{E}_{u,p,q}^s(\mathbb{R}^d) \|^{(\spadesuit)}$$
$$\leq C_4 \Big\| \Big(\int_{B(x,N)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| + C_4 \| f | \mathscr{E}_{u,p,q}^s(\mathbb{R}^d) \|^{(\spadesuit)}.$$

In what follows we want to cover the ball $B(x,N)$ with $(2N+1)^d$ small balls with radius one. Let $i \in \{1,2,\ldots,(2N+1)^d\}$ and w_i appropriate displacement vectors, such that

$$\bigcup_{i=1}^{(2N+1)^d} B(x+w_i,1) \supset B(x,N). \tag{5.6}$$

Then because of the translation-invariance of the Morrey spaces we observe

$$\Big\| \Big(\int_{B(x,N)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| \leq C_5 \sum_{i=1}^{(2N+1)^d} \Big\| \Big(\int_{B(x+w_i,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|$$
$$\leq C_6 \Big\| \Big(\int_{B(x,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|$$
$$\leq C_7 \| f | \mathscr{E}_{u,p,q}^s(\mathbb{R}^d) \|^{(\clubsuit)} \leq C_8 \| f | \mathscr{E}_{u,p,q}^s(\mathbb{R}^d) \|^{(\spadesuit)}.$$

In the last step we used Theorem 4 and Corollary 1. Now in view of Corollary 1 the proof is complete. ■

There also exists a more general version of Proposition 3 with an additional parameter a. For each $1 \leq a \leq \infty$ we will obtain an equivalent quasi-norm to describe the Triebel-Lizorkin-Morrey spaces.

Theorem 5. ***Generalized Ball Means with additional Parameter.***
Let $0 < p \leq u < \infty$, $0 < q \leq \infty$, $0 < v \leq \infty$, $N \in \mathbb{N}$ and $1 \leq a \leq \infty$. Let

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ belongs to $\mathscr{E}_{u,p,q}^s(\mathbb{R}^d)$, if and only if $f \in L_v^{loc}(\mathbb{R}^d)$ and (modifications if $q = \infty$ and / or $v = \infty$)

$$\underbrace{\| f | \mathscr{M}_p^u(\mathbb{R}^d) \| + \Big\| \Big(\int_0^a t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{q}{v}} \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|}_{\| f | \mathscr{E}_{u,p,q}^s(\mathbb{R}^d) \|^{(vma)} :=} < \infty.$$

The quasi-norms $\| f | \mathscr{E}_{u,p,q}^s(\mathbb{R}^d) \|$ and $\| f | \mathscr{E}_{u,p,q}^s(\mathbb{R}^d) \|^{(vma)}$ are equivalent for $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$.

Remark 7. ***The Abbreviation (vma).***
Let $0 < v \leq \infty$ and $1 \leq a \leq \infty$. Then the letters v and a in the abbreviation (vma) indicate the dependence of the concrete quasi-norm $\| \cdot | \mathscr{E}_{u,p,q}^s(\mathbb{R}^d) \|^{(vma)}$ on these parameters. The letter m stands for "mean".

Proof. During this proof we will work with the case $a=\infty$. Then the case $1<a<\infty$ is a simple consequence of Proposition 3 and the case $a=\infty$.

Step 1. At first we prove, that there is a constant $C>0$ independent of $f\in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$, such that $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)}\le C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)}$. But of course this is obvious.

Step 2. Second we prove, that for $f\in\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ there is a constant $C>0$ independent of f, such that $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)}\le C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)}$. Because of Proposition 3 we know, that $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)}$ is finite. We put $\mu=\min(p,q,v)$. At first we split up the integral concerning the variable t. We get

$$\Big\|\Big(\int_0^\infty t^{-sq}\Big(t^{-d}\int_{B(0,t)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^\mu$$
$$\le C_1\Big(\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)}\Big)^\mu+C_1\Big\|\Big(\int_1^\infty t^{-sq}\Big(t^{-d}\int_{B(0,t)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^\mu.$$

Now like in the proof of Proposition 3 we use the monotonicity of $\int_{B(0,t)}|\Delta_h^N f(x)|^v dh$ in t and obtain

$$\Big\|\Big(\int_1^\infty t^{-sq}\Big(t^{-d}\int_{B(0,t)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^\mu$$
$$\le C_2\Big\|\Big(\sum_{j=1}^\infty 2^{-jsq}2^{-jd\frac{q}{v}}\Big(\int_{B(0,2^j)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^\mu.$$

Next we apply the formula from Lemma 23. Moreover, we make use of the convergence of the series $\sum_{j=1}^\infty 2^{-jsq}$. We get

$$\Big\|\Big(\sum_{j=1}^\infty 2^{-jsq}2^{-jd\frac{q}{v}}\Big(\int_{B(0,2^j)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^\mu$$
$$\le C_3\|f|\mathscr{M}^u_p(\mathbb{R}^d)\|^\mu+C_3\Big\|\Big(\sum_{j=1}^\infty 2^{-jsq}2^{-jd\frac{q}{v}}\Big(\int_{B(x,N\cdot 2^j)}|f(z)|^v dz\Big)^{\frac{q}{v}}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^\mu.$$

We want to cover the ball $B(x,N\cdot 2^j)$ with $(2N\cdot 2^j+1)^d$ small balls with radius one. Let $i\in\{1,2,\dots,(2N\cdot 2^j+1)^d\}$ and w_i appropriate displacement vectors, such that

$$\bigcup_{i=1}^{(2N\cdot 2^j+1)^d} B(x+w_i,1)\supset B(x,N\cdot 2^j).$$

We obtain

$$\Big\|\Big(\sum_{j=1}^\infty 2^{-jsq}2^{-jd\frac{q}{v}}\Big(\int_{B(x,N\cdot 2^j)}|f(z)|^v dz\Big)^{\frac{q}{v}}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^\mu$$
$$\le\Big\|\Big(\sum_{j=1}^\infty 2^{-jsq}2^{-jd\frac{q}{v}}\Big(\sum_{i=1}^{(2N\cdot 2^j+1)^d}\int_{B(x+w_i,1)}|f(z)|^v dz\Big)^{\frac{q}{v}}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^\mu.$$

In what follows we use the triangle inequality as well as

$$\Big(\sum_{k=1}^K a_k\Big)^\beta\le\sum_{k=1}^K a_k^\beta\,,\qquad K\in\mathbb{N},$$

with $\beta \in (0,1]$ and $a_k \geq 0$ for all k. We will apply, that we have $\frac{\mu}{q} \leq 1$ and $\frac{p}{\mu} \geq 1$ and $\frac{\mu}{v} \leq 1$ in this order. At the end we use $\frac{p}{\mu} \geq 1$ a second time. We reach

$$\Big\| \Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big(\sum_{i=1}^{(2N\cdot 2^j+1)^d} \int_{B(x+w_i,1)} |f(z)|^v dz \Big)^{\frac{q}{v}} \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^{\mu}$$

$$= \Big\| \Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big(\sum_{i=1}^{(2N\cdot 2^j+1)^d} \int_{B(x+w_i,1)} |f(z)|^v dz \Big)^{\frac{\mu}{v}\frac{q}{\mu}} \Big)^{\frac{\mu}{q}} \Big| \mathscr{M}^{\frac{u}{\mu}}_{\frac{p}{\mu}}(\mathbb{R}^d) \Big\|$$

$$\leq C_4 \sum_{j=1}^{\infty} \sum_{i=1}^{(2N\cdot 2^j+1)^d} 2^{-js\mu} 2^{-jd\frac{\mu}{v}} \Big\| \Big(\int_{B(x+w_i,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^{\mu}$$

$$\leq C_5 \sum_{j=1}^{\infty} 2^{-js\mu} 2^{-jd\frac{\mu}{v}} 2^{jd} \Big\| \Big(\int_{B(x,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^{\mu}.$$

Since $\mu = \min(p,q,v)$ and $s > d \max\left(0, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right)$, the series converges. Finally we observe

$$\Big\| \Big(\int_{B(x,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\| \leq C_6 \| f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \|^{(\clubsuit)} \leq C_7 \| f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \|^{(vm1)}.$$

In the last step we used Theorem 4 and Proposition 3. So in view of Proposition 3 the proof is complete. ■

Theorem 5 delivers a handy equivalent quasi-norm to describe the Triebel-Lizorkin-Morrey spaces. Especially when we work with concrete functions, it might be an advantage to use the quasi-norm $\| \cdot | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \|^{(vma)}$. In most of the cases it is not necessary to work with general parameters v and a. Often it is enough to put $v = 1$ or $v = \infty$. But we have to notice, that changing the parameter v leads to different conditions concerning s in the formulation of Theorem 5. For the parameter a in many cases $a = 1$ or $a = \infty$ is used. We want to remark, that for any $\varepsilon > 0$ it is also possible to prove a version of Theorem 5 with $\varepsilon \leq a \leq \infty$. One practicable way to see this, consists of an appropriate modification of Theorem 3 in combination with some subsequent changes in the proofs of Theorem 4 and Theorem 5.

Remark 8. ***Further Characterizations of $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ via Ball Averages.***
Let us mention, that in the literature there exist more characterizations in terms of ball averages for the Triebel-Lizorkin-Morrey spaces. For example, one of them can be found in [151], see Theorem 3.4. Moreover, characterizations of $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ in terms of the Lusin-area function are known, see [148]. Here also ball averages are involved.

5.5 Stein Characterizations for Triebel-Lizorkin-Morrey Spaces

Hereinafter we want to present one very special characterization in terms of differences for the Triebel-Lizorkin-Morrey spaces, namely the so-called Stein characterization. It is especially easy to handle. The main idea is to apply Theorem 5 with $v = q$ and $a = \infty$. Then we want to replace

$$\int_0^{\infty} \int_{B(0,t)} \ldots \, dh \, dt \qquad \text{by} \qquad \int_{\mathbb{R}^d} \ldots \, dh$$

in the equivalent quasi-norm, to obtain a simple characterization for $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. In detail the outcome reads as follows.

Theorem 6. ***Stein Characterization.***
Let $0 < p \le u < \infty$, $0 < q \le \infty$ and $N \in \mathbb{N}$. Let

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{q}\right) < s < N.$$

Then $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ belongs to $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, if and only if

$$\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(st)} := \|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + \Big\|\Big(\int_{\mathbb{R}^d} |h|^{-sq}|\Delta^N_h f(x)|^q \frac{dh}{|h|^d}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\| < \infty.$$

The quasi-norms $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$ and $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(st)}$ are equivalent for $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$. In the case $q = \infty$ the usual modifications have to be made.

Proof. To prove Theorem 6, we use Theorem 5 with $v = q$ and $a = \infty$. Then we obtain, that in the case $s > d\max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{q}\right)$ a function $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ belongs to $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, if and only if

$$\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(qm\infty)} = \|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + \Big\|\Big(\int_0^\infty t^{-sq-d}\int_{B(0,t)} |\Delta^N_h f(x)|^q dh \frac{dt}{t}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|$$

is finite (with the usual modifications if $q = \infty$). Now we have to transform the quasi-norm $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(qm\infty)}$ into $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(st)}$. For that purpose we use Fubini's Theorem and get

$$\begin{aligned}
&\Big\|\Big(\int_0^\infty t^{-sq-d}\int_{B(0,t)} |\Delta^N_h f(x)|^q dh \frac{dt}{t}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\| \\
&\qquad = \Big\|\Big(\int_{\mathbb{R}^d}\int_{|h|}^\infty t^{-sq-d-1}dt\; |\Delta^N_h f(x)|^q dh\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\| \\
&\qquad = C_1\Big\|\Big(\int_{\mathbb{R}^d} |h|^{-sq}|\Delta^N_h f(x)|^q \frac{dh}{|h|^d}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|.
\end{aligned}$$

So the proof of Theorem 6 is complete. ■

Theorem 6 was inspired by the ideas of Stein. In [121] he formulated a characterization for the fractional Sobolev spaces in terms of differences, that can be seen as a forerunner of Theorem 6. More precisely Stein dealt with the case $0 < s < 2$, $p > 1$, $\frac{2d}{d+2s} < p = u < \infty$ and $q = 2$, see Theorem 1 and the corresponding remarks in [121]. A few years later in [124] Strichartz refined the characterizations of Stein. Afterwards also Triebel proved Stein characterizations for the original Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$, see chapter 2.5.10. in [128]. There for the parameters he allowed $0 < p = u < \infty$, $0 < q \le \infty$ and $s > d/\min(p,q)$. The advantage of the characterization, that can be found in our Theorem 6 is, that the quasi-norm $\|\cdot|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(st)}$ is a bit more lucid than the quasi-norm $\|\cdot|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$ from Theorem 5. In fact, there is one less integral we have to handle with. On the other hand in the formulation of the Stein characterization for $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, there is the additional condition

$$\frac{d}{p} - \frac{d}{q} < s. \tag{5.7}$$

When we work with the quasi-norm $\| \cdot |\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$ from Theorem 5, it is possible to avoid, that such a condition shows up by choosing the parameter v in a clever way. So for example we can put $v = 1$ and the additional restriction (5.7) will completely vanish. That means, Theorem 5 can be used for a larger range of parameters than Theorem 6. Summarizing we can say, that Stein characterizations are a bit more simple than ball mean characterizations, but if we want to work with them, we have to accept an additional condition concerning s.

5.6 A Characterization of $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ by generalized Ball Means

In this section it will be our main goal, to prove characterizations for the Besov-Morrey spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ in terms of generalized ball means of differences. That means, we are interested in a counterpart of Theorem 5. There is the rule of thumb, that many results for $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ have complements for the Besov-Morrey spaces with more simple proofs. At least for the case of the ball mean characterizations, this rule is correct. So the following result has many similarities with Proposition 3 and Theorem 5.

Theorem 7. ***Generalized Ball Means for*** $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$**.**
Let $0 < p \leq u < \infty$ *and* $0 < q \leq \infty$. *Let* $0 < v \leq \infty$, $1 \leq a \leq \infty$ *and* $N \in \mathbb{N}$. *Moreover we have*

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N.$$

Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ *belongs to* $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, *if and only if* $f \in L^{loc}_v(\mathbb{R}^d)$ *and (modifications if* $q = \infty$ *and / or* $v = \infty$*)*

$$\underbrace{\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + \Big(\int_0^a t^{-sq-d\frac{q}{v}} \Big\| \Big(\int_{B(0,t)} |\Delta^N_h f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|^q \frac{dt}{t} \Big)^{\frac{1}{q}}}_{\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)} :=} < \infty.$$

The quasi-norms $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|$ *and* $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$ *are equivalent for* $f \in L^{loc}_p(\mathbb{R}^d)$.

Remark 9. ***The Abbreviation (vma).***
Let $0 < v \leq \infty$ *and* $1 \leq a \leq \infty$. *Then the letters* v *and* a *in the abbreviation* (vma) *indicate the dependence of the concrete quasi-norm* $\| \cdot |\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$ *on these parameters. The letter* m *stands for "mean".*

Proof. To prove Theorem 7 we can apply the ideas, that can be found in the proofs of Proposition 3 and Theorem 5. Almost all techniques, that are described there, also can be used here. Only some minor modifications have to be made.
Step 1. At first we will deal with the case $a = 1$.
Substep 1.1. We prove, that there is a constant $C > 0$ independent of $f \in L^{loc}_p(\mathbb{R}^d)$, such that $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} \leq C\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)}$. But this is just a consequence of the monotonicity of $\int_{B(0,t)} |\Delta^N_h f(x)|^v dh$ in t.
Substep 1.2. Next we will prove, that for $f \in \mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of f, such that $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)} \leq C\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$. To show this again at first we use the

monotonicity of $\int_{B(0,t)} |\Delta_h^N f(x)|^v dh$ in t. Moreover, we apply the formula from Lemma 23. Then we obtain

$$\Big(\int_0^1 t^{-sq-d\frac{q}{v}} \Big\| \Big(\int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\leq C_1 \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} + C_1 \Big\| \Big(\int_{B(0,1)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|$$
$$\leq C_2 \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} + C_2 \Big\| \Big(\int_{B(x,N)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|.$$

Next we cover the ball $B(x,N)$ with $(2N+1)^d$ small balls with radius one, like it is described in the proof of Proposition 3, see formula (5.6). Moreover, we use the translation-invariance of the Morrey spaces. Then we observe

$$\Big\| \Big(\int_{B(x,N)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\| \leq C_3 \Big\| \Big(\int_{B(x,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|$$
$$\leq C_4 \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}.$$

In the last step we applied Theorem 4 and Corollary 1. This completes the proof for the case $a=1$.
Step 2. Now we will deal with the case $a=\infty$.
Substep 2.1. Here at first we prove, that there is a constant $C>0$ independent of $f \in L_p^{loc}(\mathbb{R}^d)$, such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)} \leq C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)}$. But of course this is obvious.
Substep 2.2. Next we will prove, that for $f \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ there is a constant $C>0$ independent of f, such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)} \leq C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(vm1)}$. To show this for a start because of the monotonicity of $\int_{B(0,t)} |\Delta_h^N f(x)|^v dh$ in t and the formula from Lemma 23, we obtain

$$\Big(\int_1^\infty t^{-sq-d\frac{q}{v}} \Big\| \Big(\int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\leq C_1 \|f|\mathcal{M}^u_p(\mathbb{R}^d)\| + C_1 \Big(\sum_{j=1}^\infty 2^{-jsq} 2^{-jd\frac{q}{v}} \Big\| \Big(\int_{B(x,N2^j)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}.$$

Now like in the proof of Theorem 5 we want to cover the ball $B(x,N2^j)$ with $(2N\cdot 2^j+1)^d$ small balls with radius one. Let $i \in \{1,2,\ldots,(2N\cdot 2^j+1)^d\}$ and w_i appropriate displacement vectors, such that

$$\bigcup_{i=1}^{(2N\cdot 2^j+1)^d} B(x+w_i,1) \supset B(x,N2^j).$$

When we use such a covering, we get

$$\Big(\sum_{j=1}^\infty 2^{-jsq} 2^{-jd\frac{q}{v}} \Big\| \Big(\int_{B(x,N2^j)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}$$
$$\leq \Big(\sum_{j=1}^\infty 2^{-jsq} 2^{-jd\frac{q}{v}} \Big\| \Big(\sum_{i=1}^{(2N\cdot 2^j+1)^d} \int_{B(x+w_i,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}.$$

Next we put $\mu = \min(p,v)$. Recall, that we have $s > d\max\Big(0,\frac{1}{p}-\frac{1}{v}\Big)$. Then since the Morrey spaces are invariant under translation, by the same arguments as in the proof of Theorem 5, we find

$$\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big\| \Big(\sum_{i=1}^{(2N\cdot 2^j+1)^d} \int_{B(x+w_i,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}$$

$$\leq C_2 \Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} 2^{jd\frac{q}{\mu}} \Big\| \Big(\int_{B(x,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}$$

$$\leq C_3 \Big\| \Big(\int_{B(x,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| \leq C_4 \|f | \mathscr{N}_{u,p,q}^s(\mathbb{R}^d)\|^{(vm1)}.$$

In the last step we used Theorem 4, Corollary 1 and the result from Step 1 of this proof. This completes the proof for the case $a = \infty$.

Step 3. At last we look at the case $1 < a < \infty$. But here the proof is just a simple consequence of the things we did before. ■

When we compare the ball mean characterization for the Besov-Morrey spaces with that for $\mathscr{E}_{u,p,q}^s(\mathbb{R}^d)$, we will observe a very important distinction. So in the formulation of Theorem 7, where we dealt with $\mathscr{N}_{u,p,q}^s(\mathbb{R}^d)$, we can find the condition

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s.$$

On the other hand when we work with the Triebel-Lizorkin-Morrey spaces, in Theorem 5 there is the restriction

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s.$$

Whereas the parameter q plays no role in the case of the Besov-Morrey spaces, it causes some additional conditions, when we deal with the spaces $\mathscr{E}_{u,p,q}^s(\mathbb{R}^d)$. This observation supports the rule of thumb, that the Triebel-Lizorkin-Morrey spaces are a bit more complicated, than the spaces $\mathscr{N}_{u,p,q}^s(\mathbb{R}^d)$.

5.7 Besov-Morrey Spaces and Moduli of Smoothness

Hereafter we want to describe an alternative way to characterize the Besov-Morrey spaces in terms of differences. The main idea will be, to work with a Morrey version of the so-called modulus of smoothness. Let $t > 0$, $0 < p \leq \infty$ and $N \in \mathbb{N}$. Then the original modulus of smoothness $\omega_N(f,t)_p$ for a function $f \in L_p(\mathbb{R}^d)$ is defined as

$$\omega_N(f,t)_p = \sup_{|h| \leq t} \|\Delta_h^N f | L_p(\mathbb{R}^d)\|. \tag{5.8}$$

For the classical Besov spaces $B_{p,q}^s(\mathbb{R}^d)$ characterizations in terms of $\omega_N(f,t)_p$ are known since many years. Here for example we can refer to chapter 4 in [8], to 4.3.4 in [88] and to chapter 2.5.12. in [128]. When we want to characterize the spaces $\mathscr{N}_{u,p,q}^s(\mathbb{R}^d)$ via moduli of smoothness, it is not difficult to see, how (5.8) can be modified accordingly. So we have to replace the Lebesgue quasi-norm by a Morrey quasi-norm. In fact, there is the following result.

Theorem 8. ***Morrey Versions for Moduli of Smoothness and*** $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$**.**
Let $0 < p \le u < \infty$, $0 < q \le \infty$ *and* $N \in \mathbb{N}$. *Let*

$$d \max\left(0, \frac{1}{p} - 1\right) < s < N.$$

Then a function $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d)$ *belongs to* $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, *if and only if*

$$\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)} := \|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + \Big(\int_0^\infty t^{-sq}\Big[\sup_{|h|\le t}\|\Delta^N_h f|\mathscr{M}^u_p(\mathbb{R}^d)\|\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty.$$

The quasi-norms $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|$ *and* $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)}$ *are equivalent for* $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d)$. *In the case* $q = \infty$ *the usual modifications have to be made.*

Proof. *Step 1.* At first we prove, that for $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of f, such that $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\| \le C\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)}$. To prove this because of $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d) \subset L^{loc}_p(\mathbb{R}^d)$ and $s > d\max(0, \frac{1}{p} - 1)$ we can apply Theorem 7 with $a = \infty$ and $v = p$. Then we obtain

$$\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\| \le C_1\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| + C_1\Big(\int_0^\infty t^{-sq-d\frac{q}{p}}\Big\|\Big(\int_{B(0,t)}|\Delta^N_h f(x)|^p dh\Big)^{\frac{1}{p}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^q \frac{dt}{t}\Big)^{\frac{1}{q}}.$$

Next we use Fubini's Theorem and get

$$\begin{aligned}
&\Big(\int_0^\infty t^{-sq-d\frac{q}{p}}\Big[\sup_{y\in\mathbb{R}^d, r>0}|B(y,r)|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{B(y,r)}\int_{B(0,t)}|\Delta^N_h f(x)|^p dh\, dx\Big)^{\frac{1}{p}}\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\quad = \Big(\int_0^\infty t^{-sq-d\frac{q}{p}}\Big[\sup_{y\in\mathbb{R}^d, r>0}|B(y,r)|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{B(0,t)}\int_{B(y,r)}|\Delta^N_h f(x)|^p dx\, dh\Big)^{\frac{1}{p}}\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\quad \le C_2\Big(\int_0^\infty t^{-sq}\Big[\sup_{y\in\mathbb{R}^d, r>0}|B(y,r)|^{\frac{1}{u}-\frac{1}{p}}\Big(\sup_{|h|\le t}\int_{B(y,r)}|\Delta^N_h f(x)|^p dx\Big)^{\frac{1}{p}}\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\quad \le C_3\Big(\int_0^\infty t^{-sq}\Big[\sup_{|h|\le t}\|\Delta^N_h f|\mathscr{M}^u_p(\mathbb{R}^d)\|\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}}.
\end{aligned}$$

So Step 1 of the proof is complete.
Step 2. Now we prove, that for $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d) \cap \mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of f, such that $\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)} \le C\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|$. To show this at first we can apply Theorem 7 with $v = p$ and get

$$\|f|\mathscr{M}^u_p(\mathbb{R}^d)\| \le \|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(pm\infty)} \le C_1\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|.$$

To deal with the term

$$\Big(\int_0^\infty t^{-sq}\Big[\sup_{|h|\le t}\|\Delta^N_h f|\mathscr{M}^u_p(\mathbb{R}^d)\|\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$

in what follows we will use some ideas from Triebel, see chapter 2.5.11. in [128]. For a start we transform the integral concerning t into a sum. We obtain

$$\Big(\int_0^\infty t^{-sq}\Big[\sup_{|h|\le t}\|\Delta^N_h f|\mathscr{M}^u_p(\mathbb{R}^d)\|\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}} \le C_1\Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\|\Delta^N_h f|\mathscr{M}^u_p(\mathbb{R}^d)\|\Big]^q\Big)^{\frac{1}{q}}.$$

Now let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity, see Definition 11. We put $\varphi_j = 0$ for $j < 0$. Then because of $s > d\max(0, \frac{1}{p}-1)$ and $f \in \mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ for every $k \in \mathbb{Z}$ we have

$$f = \sum_{m=-\infty}^{\infty} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f]$$

with convergence not only in $\mathscr{S}'(\mathbb{R}^d)$, but also in $\mathscr{M}^u_p(\mathbb{R}^d)$. Let $\theta = \min(1,p,q)$. We observe

$$\begin{aligned}
&\Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\Big[\sup_{|h|\le 2^{-k}} \|\Delta^N_h f|\mathscr{M}^u_p(\mathbb{R}^d)\|\Big]^q\Big)^{\frac{\theta}{q}}\\
&\quad= \Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\Big[\sup_{|h|\le 2^{-k}} \Big\|\Delta^N_h\Big(\sum_{m=-\infty}^{\infty}\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f]\Big)\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}\\
&\quad\le \sum_{m=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\Big[\sup_{|h|\le 2^{-k}} \Big\|\Delta^N_h \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f]\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}.
\end{aligned}$$

Next we split up the outer sum and obtain

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\Big[\sup_{|h|\le 2^{-k}} \Big\|\Delta^N_h \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f]\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}\\
&\quad= \sum_{m=-\infty}^{-1}\Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\Big[\sup_{|h|\le 2^{-k}} \Big\|\Delta^N_h \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f]\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}\\
&\qquad+ \sum_{m=0}^{\infty}\Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\Big[\sup_{|h|\le 2^{-k}} \Big\|\Delta^N_h \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f]\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}.
\end{aligned}$$

To continue the proof at first we will deal with the case $m < 0$. Here we have to start with some preliminary considerations, that also can be found in chapter 2.5.11. in [128]. So for every $|h| \le 1$ and $x \in \mathbb{R}^d$, there is a constant $C_2 > 0$ independent of f and x, such that

$$|(\Delta^N_{2^{-k}h}\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f])(x)| \le C_2 2^{-kN} \sup_{|x-y|\le N2^{-k}} \sum_{|\alpha|=N} |(D^\alpha \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f])(y)|.$$

Moreover, for $j \in \mathbb{Z}$ and $a > 0$ we define the function

$$(\varphi^*_j f)(x) = \sup_{y\in\mathbb{R}^d} \frac{|(\mathscr{F}^{-1}[\varphi_j\mathscr{F} f])(x-y)|}{1+(2^{j+2}|y|)^a}.$$

Notice, that for $j < 0$ because of $\varphi_j = 0$ we also have $\varphi^*_j f = 0$. Then for $|\alpha| = N$ and $y \in \mathbb{R}^d$ there is a constant $C_3 > 0$ independent of f and y, such that

$$|(D^\alpha \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f])(y)| \le C_3 2^{(k+m)N}(\varphi^*_{k+m} f)(y).$$

When we use these estimates, because of the properties of the function $\varphi^*_j f$, we obtain

$$\begin{aligned}
&\sum_{m=-\infty}^{-1}\Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\Big[\sup_{|h|\le 2^{-k}} \Big\|\Delta^N_h \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F} f]\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}\\
&\quad\le C_4 \sum_{m=-\infty}^{-1}\Big(\sum_{k=-\infty}^{\infty} 2^{ksq}2^{mNq}\Big\|\sup_{|x-y|\le N2^{-k}} (\varphi^*_{k+m} f)(y)\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|^q\Big)^{\frac{\theta}{q}}\\
&\quad\le C_5 \sum_{m=-\infty}^{-1} 2^{(N-s)m\theta}\Big(\sum_{k=-\infty}^{\infty} 2^{(k+m)sq}\|\varphi^*_{k+m} f|\mathscr{M}^u_p(\mathbb{R}^d)\|^q\Big)^{\frac{\theta}{q}}.
\end{aligned}$$

Next we put $k+m=j$. Then since $N>s$ we find

$$\sum_{m=-\infty}^{-1}\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\Big\|\Delta_h^N\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f]\Big|\mathscr{M}_p^u(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}$$
$$\le C_5\sum_{m=-\infty}^{-1}2^{(N-s)m\theta}\Big(\sum_{j=-\infty}^{\infty}2^{jsq}\|\varphi_j^*f|\mathscr{M}_p^u(\mathbb{R}^d)\|^q\Big)^{\frac{\theta}{q}}$$
$$\le C_6\Big(\sum_{j=0}^{\infty}2^{jsq}\|\varphi_j^*f|\mathscr{M}_p^u(\mathbb{R}^d)\|^q\Big)^{\frac{\theta}{q}}.$$

Now let $a>\frac{d}{p}$. Then a modification of Lemma 1.1.7. from [53] in combination with part (i) of our Lemma 21 (see also Theorem 1.1. in [139]) yields

$$\sum_{m=-\infty}^{-1}\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\Big\|\Delta_h^N\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f]\Big|\mathscr{M}_p^u(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}$$
$$\le C_7\Big(\sum_{j=0}^{\infty}2^{jsq}\|\mathscr{F}^{-1}[\varphi_j\mathscr{F}f]|\mathscr{M}_p^u(\mathbb{R}^d)\|^q\Big)^{\frac{\theta}{q}}$$
$$=C_7\|f|\mathscr{N}_{u,p,q}^s(\mathbb{R}^d)\|^\theta.$$

It remains to deal with the case $m\ge 0$. Here at first we can use

$$(\Delta_h^N\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)=\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f](x+lh),$$

see Lemma 23. Recall the translation-invariance of the Morrey spaces. We observe

$$\sum_{m=0}^{\infty}\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\Big\|\Delta_h^N\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f]\Big|\mathscr{M}_p^u(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}$$
$$\le C_8\sum_{m=0}^{\infty}\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\sum_{l=0}^{N}\Big\|\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f](x+lh)\Big|\mathscr{M}_p^u(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}$$
$$\le C_9\sum_{m=0}^{\infty}2^{-ms\theta}\Big(\sum_{k=-\infty}^{\infty}2^{ksq}2^{msq}\|\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f]|\mathscr{M}_p^u(\mathbb{R}^d)\|^q\Big)^{\frac{\theta}{q}}.$$

Now we put $k+m=j$. Then since $s>0$ we obtain

$$\sum_{m=0}^{\infty}\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\Big\|\Delta_h^N\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f]\Big|\mathscr{M}_p^u(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\theta}{q}}$$
$$\le C_9\sum_{m=0}^{\infty}2^{-ms\theta}\Big(\sum_{j=0}^{\infty}2^{jsq}\|\mathscr{F}^{-1}[\varphi_j\mathscr{F}f]|\mathscr{M}_p^u(\mathbb{R}^d)\|^q\Big)^{\frac{\theta}{q}}$$
$$\le C_{10}\Big(\sum_{j=0}^{\infty}2^{jsq}\|\mathscr{F}^{-1}[\varphi_j\mathscr{F}f]|\mathscr{M}_p^u(\mathbb{R}^d)\|^q\Big)^{\frac{\theta}{q}}$$
$$=C_{10}\|f|\mathscr{N}_{u,p,q}^s(\mathbb{R}^d)\|^\theta.$$

So the proof is complete. ■

It is interesting to compare the above result with Theorem 7. Let us apply Theorem 7 with $v=\infty$ and $a=\infty$. Then we find, that for

$$\frac{d}{p} < s < N$$

we can describe the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ in terms of the equivalent quasi-norm

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\infty m \infty)} := \|f|\mathcal{M}^u_p(\mathbb{R}^d)\| + \Big(\int_0^\infty t^{-sq} \Big\| \sup_{|h|\le t} |\Delta^N_h f(x)| \Big| \mathcal{M}^u_p(\mathbb{R}^d)\Big\|^q \frac{dt}{t}\Big)^{\frac{1}{q}}.$$

In the case $q=\infty$ the usual modifications have to be made. A comparison of the quasi-norms $\|\cdot|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\infty m \infty)}$ and $\|\cdot|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)}$ shows, that for large enough s the supremum can be both outside and inside of the Morrey quasi-norm. However, the condition $\frac{d}{p} < s$ is much more restrictive than those from Theorem 8. Therefore in many cases we prefer to work with $\|\cdot|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)}$ instead of $\|\cdot|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\infty m \infty)}$.

5.8 Besov-type Spaces and Differences

Hereinafter it is our main goal, to describe the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ in terms of differences. For that purpose we will prove characterizations by generalized ball means of differences. It turns out, that we can proceed in the same way as for the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ or $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. So the following result has many similarities with Theorem 5 and Theorem 7.

Theorem 9. ***Generalized Ball Means for*** $B^{s,\tau}_{p,q}(\mathbb{R}^d)$**.**
Let $0<p<\infty$, $0\le \tau < \frac{1}{p}$, $0<q\le\infty$, $0<v\le\infty$, $1\le a\le\infty$ *and* $N\in\mathbb{N}$. *Let*

$$d \max\Big(0, \frac{1}{p}-1, \frac{1}{p}-\frac{1}{v}\Big) < s < N.$$

Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ *belongs to* $B^{s,\tau}_{p,q}(\mathbb{R}^d)$*, if and only if* $f\in L^{loc}_v(\mathbb{R}^d)$ *and*

$$\underbrace{\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P |f(x)|^p dx\Big)^{\frac{1}{p}} + \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_0^a t^{-sq}\Big(\int_P \Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|^v dh\Big)^{\frac{p}{v}} dx\Big)^{\frac{q}{p}} \frac{dt}{t}\Big)^{\frac{1}{q}}}_{\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)} :=} < \infty$$

(with the usual modifications if $q=\infty$ *and / or* $v=\infty$*). The quasi-norms* $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ *and* $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)}$ *are equivalent for* $f\in L^{loc}_p(\mathbb{R}^d)$.

Remark 10. ***The Abbreviation (vma).***
Let $0<v\le\infty$ *and* $1\le a\le\infty$. *Then the letters* v *and* a *in the abbreviation* (vma) *indicate the dependence of the concrete quasi-norm* $\|\cdot|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)}$ *on these parameters. The letter* m *stands for "mean".*

Proof. For the proof we can proceed in the same way, as it is described in the proof of Theorem 7. Only some minor modifications have to be made. Therefore in what follows we merely recall the main ideas. In a first step we deal with the case $a=1$. For that purpose we use the techniques, which are described in Step 1 of the proof of Theorem 7. We apply the covering argument from formula (5.6) and benefit from part (*iii*) of Theorem 4 and Corollary 1. In a second step we have to investigate the case $a=\infty$. For that purpose we use the ideas from Step 2 of the proof of Theorem 7. We reuse the covering from there and apply part (*iii*) of Theorem 4. At the end to complete the proof we have to look at the case $1<a<\infty$. But here the desired result follows from the things we did before. ■

Remark 11. ***A detailed Proof of Theorem 9.***
In [58] the author together with his co-author W. Sickel investigated in detail, how the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can be described in terms of differences. In this context we also developed a detailed proof for Theorem 9. One may consult Proposition 3.4. and Theorem 3.1. in [58].

In the literature there exist some more characterizations for $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ in terms of differences. They look slightly different. For example, we want to refer to Theorem 4.7. in [144] and Theorem 4.2. in [29]. Let us introduce the following notation. We say, that a Lebesgue-measurable function f belongs to $L^{\tau}_p(\mathbb{R}^d)$, if

$$\|f\,|L^{\tau}_p(\mathbb{R}^d)\| := \sup_{P\in\mathscr{Q},\ |P|\geq 1} \frac{1}{|P|^{\tau}}\Big(\int_P |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty. \tag{5.9}$$

Using this notation we can formulate the next result.

Proposition 4. ***Moduli of Smoothness and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$.***
Let $0<q\leq\infty$ and $N\in\mathbb{N}$.

(i) *Let $1\leq p<\infty$, $0\leq\tau<\frac{1}{p}$ and $0<s<N$. Then a function $f\in L^{loc}_p(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$, if and only if $f\in L^{\tau}_p(\mathbb{R}^d)$ and*

$$\underbrace{\|f\,|L^{\tau}_p(\mathbb{R}^d)\| + \sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}}\Big(\int_0^{2\min(l(P),1)} t^{-sq} \sup_{|h|\leq t} \Big\|\Delta^N_h f\,\Big|L_p(P)\Big\|^q \frac{dt}{t}\Big)^{\frac{1}{q}}}_{\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\omega)} :=} < \infty$$

(with standard modifications if $q=\infty$). The quasi-norms $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\omega)}$ are equivalent for $f\in L^{\tau}_p(\mathbb{R}^d)\cap\mathscr{S}'(\mathbb{R}^d)$.

(ii) *Let $0<p<1$, $0\leq\tau<\frac{1}{p}$ and $d\max(0,\frac{1}{p}-1)<s<N$. Let $\sigma_p<s_0<s$. Then a function $f\in L^{loc}_p(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$, if and only if*

$$\sup_{P\in\mathscr{Q},\ |P|\geq 1} \frac{\|f\,|B^{s_0}_{p,\infty}(2P)\|}{|P|^{\tau}} < \infty$$

and $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\omega)}<\infty$. The quasi-norms $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$ and

$$\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\omega)} + \sup_{P\in\mathscr{Q},\ |P|\geq 1} \frac{\|f\,|B^{s_0}_{p,\infty}(2P)\|}{|P|^{\tau}}$$

are equivalent for $f\in L^{\tau}_p(\mathbb{R}^d)\cap\mathscr{S}'(\mathbb{R}^d)$.

Proof. For a proof we refer to Theorem 4.7. and Theorem 4.9. in [144]. ■

Part (i) of Proposition 4 is in some sense satisfactory. So it is quite close to the classical characterization of $B^s_{p,q}(\mathbb{R}^d)$ by means of the modulus of smoothness. Part (ii) was understood as a first attempt to characterize Besov-type spaces by differences in the case $p < 1$. Here we have to deal with an additional parameter s_0 and Besov spaces on domains. Both things make the equivalent quasi-norm, that shows up in part (ii) of Proposition 4, a little less transparent than that from Theorem 9. Summarizing we can say, that above all the benefit of Theorem 9 consists of providing a lucid characterization in terms of differences for the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ also for the case $0 < p < 1$.

Chapter 6

Smoothness Morrey Spaces and Differences: Necessary Conditions

Let us have a closer look at our main results concerning Smoothness Morrey spaces and differences, we obtained so far, namely the Theorems 5, 7 and 9. In each of them we can find some conditions concerning the parameter s. So in Theorem 5, where we proved a characterization for the Triebel-Lizorkin-Morrey spaces in terms of generalized ball means of differences, there is the restriction

$$d \max\left(0, \frac{1}{p}-1, \frac{1}{q}-1, \frac{1}{p}-\frac{1}{v}, \frac{1}{q}-\frac{1}{v}\right) < s < N. \tag{6.1}$$

In the Theorems 7 and 9, where we received characterizations by differences for the Besov-Morrey spaces $\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)$ and the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$, we can find the condition

$$d \max\left(0, \frac{1}{p}-1, \frac{1}{p}-\frac{1}{v}\right) < s < N. \tag{6.2}$$

In this chapter it will be our main goal to investigate, whether these conditions are not only sufficient, but also necessary for the validity of the theorems in which they appear. We want to prove, that the assertions in the Theorems 5, 7 and 9 become wrong, when we weaken the conditions concerning s too much. For that purpose the following definition will be very important.

Definition 31. ***Smoothness Morrey Spaces defined by Differences.***
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $0 < v \leq \infty$, $1 \leq a \leq \infty$ and $N \in \mathbb{N}$.

(i) Let in addition $0 < p \leq u < \infty$. Then $\mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ is the collection of all $f \in L^{loc}_{\max(p,v)}(\mathbb{R}^d)$, such that $\|f|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$ is finite. We use the notation from Theorem 5.

(ii) Again let in addition $0 < p \leq u < \infty$. Then the space $\mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ is the collection of all $f \in L^{loc}_{\max(p,v)}(\mathbb{R}^d)$, such that $\|f|\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$ is finite. The notation is as in Theorem 7.

(iii) Furthermore let $0 \leq \tau < \frac{1}{p}$. Then $\mathbf{B}^{s,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$ is the collection of all $f \in L^{loc}_{\max(p,v)}(\mathbb{R}^d)$, such that $\|f|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)}$ is finite. We use the notation from Theorem 9.

In what follows we intend to investigate, under which conditions on the parameters we have

$$\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d) \quad \text{and} \quad \mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d) \quad \text{and} \quad B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$$

in the sense of unequal sets. To answer this question a lot of different techniques will be used. Therefore it is the most convenient way, to deal with each condition, that shows up in (6.1) or in (6.2), separately. Most of the results from this chapter also can be found in the author's papers [55], [56] and [58].

6.1 The Necessity of $s > 0$

In this section we want to explain the relevance of the condition $s > 0$, that appears in the Theorems 5, 7 and 9. It will turn out, that it is impossible to describe the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ in terms of differences, if $s < 0$. And also for the limiting case $s = 0$ for many parameter constellations such a characterization does not exist. In detail for the Triebel-Lizorkin-Morrey spaces and the Besov-Morrey spaces there is the following observation.

Proposition 5. ***Differences and*** $s \leq 0$***.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$, $0 < q \leq \infty$, $0 < v \leq \infty$ *and* $N \in \mathbb{N}$ *with* $N > s$.

(a) Then for the Triebel-Lizorkin-Morrey spaces the following assertions are true.

- *(i) Let* $s < 0$ *and* $1 \leq a \leq \infty$*. Then we have* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.
- *(ii) Let* $s = 0$ *and* $a = \infty$*. Then we have* $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{0,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$.
- *(iii) Let* $s = 0$, $1 \leq a \leq \infty$ *and either* $2 < q \leq \infty$ *or* $q = 2$ *in combination with* $1 < p < \infty$*. Then we have* $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{0,N,a}_{u,p,q,v}(\mathbb{R}^d)$.

(b) For the Besov-Morrey spaces we know the following.

- *(i) Let* $s < 0$ *and* $1 \leq a \leq \infty$*. Then we have* $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.
- *(ii) Let* $s = 0$, $1 \leq a \leq \infty$ *and* $p \geq 2$ *with* $q > 2$*. Then we have* $\mathscr{N}^0_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{0,N,a}_{u,p,q,v}(\mathbb{R}^d)$.
- *(iii) Let* $s = 0$, $1 \leq a \leq \infty$ *and* $1 \leq p < 2$ *with* $q > p$*. Then we have* $\mathscr{N}^0_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{0,N,a}_{u,p,q,v}(\mathbb{R}^d)$.
- *(iv) Let* $s = 0$ *and* $a = \infty$*. Then we have* $\mathscr{N}^0_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{0,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$.

Proof. *Step 1.* At first we prove the results for the Triebel-Lizorkin-Morrey spaces.
Substep 1. We start with the proof of (i). In the case $s < 0$ the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ contain singular distributions, see Lemma 5. So a characterization of $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ in terms of differences is not possible.
Substep 2. Now we look at the case $s = 0$ and $a = \infty$. In the case $q = \infty$ the spaces $\mathscr{E}^0_{u,p,\infty}(\mathbb{R}^d)$ contain singular distributions, see Lemma 6. So in what follows we can assume $0 < q < \infty$. Let $f \in C_0^\infty(\mathbb{R}^d)$ with $f(x) = 1$ for $|x| \leq 1$ and $f(x) = 0$ for $|x| > 2$. Then because of Lemma 2 we have $f \in \mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$. But we are able to show $f \notin \mathbf{E}^{0,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$. To see this at first we observe

$$\begin{aligned}\|f|\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)} &\geq C_1\Big(\int_{B(0,1)}\Big(\int_0^\infty\Big(t^{-d}\int_{B(0,t)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}\\ &\geq C_1\Big(\int_{B(0,1)}\Big(\int_5^\infty\Big(t^{-d}\int_{B(0,t)\setminus B(0,4)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}.\end{aligned}$$

Now we use the formula from Lemma 23. For $x \in B(0,1)$ we have $f(x) = 1$. For $x \in B(0,1)$, $|h| \geq 4$ and $k \geq 1$ we observe $|kh + x| \geq ||kh| - |x|| \geq 3$ and so $f(x+kh) = 0$. Hence we obtain $|\Delta_h^N f(x)| = 1$. Consequently we get

$$\|f|\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)} \geq C_1 \Big(\int_{B(0,1)} \Big(\int_5^\infty \Big(t^{-d} \int_{B(0,t)\setminus B(0,4)} 1\, dh \Big)^{\frac{q}{v}} \frac{dt}{t} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}}$$
$$\geq C_2 \Big(\int_5^\infty t^{-1}\, dt \Big)^{\frac{1}{q}} = \infty.$$

Therefore we find $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{0,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$.
Substep 3. Now we deal with $s = 0$ and $1 \leq a < \infty$. In the case $2 < q \leq \infty$ the spaces $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$ contain singular distributions, see Lemma 6. So again a characterization of $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$ in terms of differences is not possible. In the case $s = 0$ with $q = 2$ and $1 < p \leq u < \infty$ we can use an idea from Besov, see the proof of Theorem 2 in [6]. We construct a sequence $(f_n)_{n=1}^\infty$ of indicator functions with $f_n \in \mathscr{E}^0_{u,p,2}(\mathbb{R}^d) = \mathscr{M}^u_p(\mathbb{R}^d)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|f_n|\mathscr{E}^0_{u,p,2}(\mathbb{R}^d)\|^{(vma)} = \infty$. Further explanations can be found in [59].
Step 2. Now we prove the results for the Besov-Morrey spaces, see (b).
Substep 1. At first we prove (i), (ii) and (iii). In each of these cases the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ contain singular distributions, see the Lemmas 5 and 6. So a characterization of $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ in terms of differences is not possible.
Substep 2. Now we look at the case $s = 0$ and $a = \infty$. In the case $q = \infty$ the spaces $\mathscr{N}^0_{u,p,\infty}(\mathbb{R}^d)$ contain singular distributions, see Lemma 6. So we can assume $0 < q < \infty$. Let $f \in C_0^\infty(\mathbb{R}^d)$ with $f(x) = 1$ for $|x| \leq 1$ and $f(x) = 0$ for $|x| > 2$. Then because of Lemma 2 we have $f \in \mathscr{N}^0_{u,p,q}(\mathbb{R}^d)$. But we are able to show $f \notin \mathbf{N}^{0,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$. To prove this, we can use the same arguments as described in Substep 2 from Step 1 of this proof. Only some minor modifications must be done. Therefore we omit the details. The proof is complete. ■

For the limiting case $s = 0$ there exist some more methods to prove, that the Triebel-Lizorkin-Morrey spaces or the Besov-Morrey spaces can not be described in terms of differences. One of these techniques is using dilation operators. In what follows we want to explain the main ideas of this method for the spaces $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$. For that purpose for $f \in \mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$ and $k \in \mathbb{N}$ we consider dilation operators of the form

$$T_k f(x) = f(2^k x) \tag{6.3}$$

with $x \in \mathbb{R}^d$. For the original spaces $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ the behavior of these operators is well-known, see chapter 2.3.1 in [30] and [111] as well as [112]. For us it is interesting to know, how the dilation operators act for $f \in \mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$. We are able to prove the following counterpart of Theorem 2.1. from [112].

Lemma 27. ***Dilation Operators and*** $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$**.**
Let $1 < p \leq u < \infty$ and $0 < q \leq \infty$. Then there are two constants $C_1, C_2 > 0$ independent of $k \in \mathbb{N}$, such that

$$C_1 2^{-k\frac{d}{u}} k^{\frac{1}{q} - \frac{1}{\max(q,2)}} \leq \|T_k|\mathscr{L}(\mathscr{E}^0_{u,p,q}(\mathbb{R}^d))\| \leq C_2 2^{-k\frac{d}{u}} k^{\frac{1}{q} - \frac{1}{\max(q,2)}}.$$

Proof. To prove this result, we can recreate the proof of Theorem 2.1. from [112] step by step. Since the proof of Theorem 2.1. is mainly based on making use of the support properties of some functions, it does not matter, whether we work with the Lebesgue norm or with the Morrey norm. Only a few extra considerations have to be made. So in the proof of Theorem 2.1. the Littlewood-Paley Decomposition Theorem for the spaces $L_p(\mathbb{R}^d)$ is used. Fortunately for $1 < p \leq u < \infty$ such a theorem also exists for the Morrey case, see [60]. Second at the end of Step 1 from the proof of Theorem 2.1. in [112], Young's convolution inequality is used. But also for that a counterpart for the Morrey case exists, see [23]. Everything else we have to change in the proof of Theorem 2.1. is obvious. ■

Now we can apply Lemma 27, to present an alternative proof for the following result, we already know from Proposition 5.

Corollary 2. ***Differences and*** $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$.
Let $s = 0$, $1 < p \leq u < \infty$, $0 < q < 2$, $0 < v \leq \infty$, $a = \infty$ *and* $N \in \mathbb{N}$. *Then the quasi-norms* $\| \cdot |\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\|$ *and* $\| \cdot |\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)}$ *are not equivalent.*

Proof. Let $f \in \mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$ and $k \in \mathbb{N}$. Then on the one hand it is easy to see, that there is a $C > 0$ independent of f and k, such that

$$\|f(2^k \cdot)|\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)} \leq C 2^{-k\frac{d}{u}} \|f|\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)}. \tag{6.4}$$

To prove this, we only need some transformations of the coordinates. On the other hand because of Lemma 27 there exist two constants $C_1, C_2 > 0$ independent of $k \in \mathbb{N}$, such that

$$C_1 2^{-k\frac{d}{u}} k^{\frac{1}{q}-\frac{1}{2}} \leq \|T_k|\mathscr{L}(\mathscr{E}^0_{u,p,q}(\mathbb{R}^d))\| \leq C_2 2^{-k\frac{d}{u}} k^{\frac{1}{q}-\frac{1}{2}}. \tag{6.5}$$

Now we proceed indirectly and follow the proof of Corollary 3.11. in [112]. We assume, that there are two constants $C_3, C_4 > 0$ independent of f, such that for all $f \in \mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$ we have

$$\|f|\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\| \leq C_3 \|f|\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\|^{(vm\infty)} \leq C_4 \|f|\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\|. \tag{6.6}$$

When we combine (6.6) with (6.4) and (6.5), we arrive at

$$k^{\frac{1}{q}-\frac{1}{2}} \leq C_5, \qquad k \in \mathbb{N}.$$

But since $0 < q < 2$ this is not possible for all $k \in \mathbb{N}$. So there is a contradiction. ■

When we look at Proposition 5 and Corollary 2, we find, that for the limiting case $s = 0$ we only have a complete answer, if $a = \infty$. Up to now for the case $1 \leq a < \infty$ there remain some parameter constellations, for which we do not know, whether the spaces $\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{N}^0_{u,p,q}(\mathbb{R}^d)$ can be described in terms of differences or not. To fill this gap, we have to make some extra considerations, that are very technical. So one possible way is to study the properties of some functions, that are linear combinations of indicator functions. In what follows we want to describe this method in detail for the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Of course the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can not be characterized in terms of differences, if $s < 0$, see Lemma 16. To deal with the limiting case $s = 0$, the next result will be a very important step. It was inspired by Besov, see [6].

Lemma 28. ***Differences and*** $B^{0,\tau}_{p,q}(\mathbb{R}^d)$. ***Part I.***
Let $s = 0$, $0 < p < \infty$, $0 < q < \infty$, $0 \leq \tau < \frac{1}{p}$, $0 < v \leq \infty$, $1 \leq a \leq \infty$ *and* $N \in \mathbb{N}$. *Then there exists a sequence* $(g_j)_{j \in \mathbb{N}}$ *of functions with the following properties.*

(i) *We have* $\operatorname{supp} g_j \subset [0,1]^d$ *for all* $j \in \mathbb{N}$.

(ii) *We find* $\|g_j|L_\infty(\mathbb{R}^d)\| \le 1$ *for all* $j \in \mathbb{N}$.

(iii) *We observe* $\sup_{j\in\mathbb{N}} \| g_j \,|B^{0,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)} = \infty$.

Moreover for the case $q \ge \max(p,2)$ *we have* $B^{0,\tau}_{p,q}(\mathbb{R}^d) \ne \mathbf{B}^{0,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$.

Proof. *Step 1.* At first we prove $(i)-(iii)$ simultaneously. Let $\chi_{[0,1)^d}$ denote the characteristic function of the unit cube $[0,1)^d$. We put

$$g_j(x) := \sum_{k_1=0}^{2^{j-1}} \cdots \sum_{k_d=0}^{2^{j-1}} \frac{1-(-1)^{K(k)}}{2}\, \chi_{[0,1)^d}(2^j x - k)\,, \qquad x \in \mathbb{R}^d\,, \quad j \in \mathbb{N}\,, \tag{6.7}$$

where the function K is defined as

$$K(k) := K(k_1,\ldots,k_d) = \sum_{i=1}^{d} k_i\,, \qquad k \in \mathbb{Z}^d.$$

In the case $d = 2$ a picture of this function looks like a checkerboard (if those parts where g_j has value 1 are printed in black). Obviously we have $\operatorname{supp} g_j \subset [0,1]^d$ for all $j \in \mathbb{N}$. Moreover, all coefficients are either 1 or 0 and hence

$$\| g_j \,|L_\infty(\mathbb{R}^d)\| \le 1 \qquad \text{for all} \qquad j \in \mathbb{N}\,. \tag{6.8}$$

Let us prove (iii). Therefore we recall some properties of the functions g_j we already know from [59]. There exist sets $X_j \subset [0,1]^d$ and $H_j \subset \mathbb{R}^d$, such that

$$\Delta_h^N g_j(x) = 1 \qquad \text{if} \qquad (x,h) \in X_j \times H_j\,. \tag{6.9}$$

Furthermore, we have $|X_j| \sim 1$ and

$$C_1 t^d \le |B(0,t) \cap H_j| \le C_2 t^d \qquad \text{for} \qquad 2^{-j} < t < 1\,,$$

where the constants only depend on N and d. Based on these properties it is easy to derive the estimate

$$\begin{aligned} \| g_j \,|B^{0,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)} &\ge \left(\int_0^1 \left(\int_{X_j} \left(t^{-d} \int_{B(0,t)\cap H_j} |\Delta_h^N g_j(x)|^v dh \right)^{\frac{p}{v}} dx \right)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\ge \left(\int_{2^{-j}}^1 \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\ge C_3\, j^{\frac{1}{q}}\,. \end{aligned}$$

When j tends to infinity, this proves (iii). So Step 1 of the proof is complete.

Step 2. Now we prove the result concerning the inequality of the function spaces. Let $L^*_\infty(\mathbb{R}^d)$ be the set of all functions $g \in L_\infty(\mathbb{R}^d)$, such that $\operatorname{supp} g \subset [0,1]^d$. Then from the first step of this proof and the Theorem of Banach-Steinhaus (in the variant of the uniform boundedness principle, valid also with target space being a quasi-Banach space, see [54] for an appropriate version), we obtain

$$L^*_\infty(\mathbb{R}^d) \not\subset \mathbf{B}^{0,\tau,N,a}_{p,q,v}(\mathbb{R}^d). \tag{6.10}$$

Obviously we have

$$L_\infty^*(\mathbb{R}^d) \hookrightarrow \mathscr{M}_p^u(\mathbb{R}^d).$$

For $u = \frac{1}{\frac{1}{p}-\tau}$ this can be complemented by

$$\mathscr{M}_p^u(\mathbb{R}^d) = \mathscr{E}_{u,p,2}^0(\mathbb{R}^d) \hookrightarrow B_{p,\max(p,2)}^{0,\tau}(\mathbb{R}^d)$$

if $1 < p < \infty$, see [116]. Next we shall use the wavelet characterization of $B_{p,q}^{0,\tau}(\mathbb{R}^d)$ obtained in [69] and [70]. For functions with support in $[0,1]^d$ it follows the monotonicity of the quasi-norm with respect to p. So for $p_0 \le p_1$ we have

$$\| f \,| B_{p_0,q}^{0,\tau}(\mathbb{R}^d)\| \le C_1 \| f \,| B_{p_1,q}^{0,\tau}(\mathbb{R}^d)\| ,$$

see [69]. Altogether for $0 < p < \infty$ we obtain

$$L_\infty^*(\mathbb{R}^d) \hookrightarrow B_{p,\max(p,2)}^{0,\tau}(\mathbb{R}^d)\,. \tag{6.11}$$

Hence for $q \ge \max(p,2)$ a combination of (6.10) and (6.11) completes the proof. ∎

In a next step we have to improve Lemma 28 to cover the other values of q. For that purpose we may use the Haar wavelet characterization for the spaces $B_{p,q}^{0,\tau}(\mathbb{R}^d)$ from [147]. Let us introduce some additional notation. By ψ_H we denote the Haar wavelet, that looks like

$$\psi_H(t) := \begin{cases} 1 & \text{if} \quad 0 \le t < \frac{1}{2}; \\ -1 & \text{if} \quad \frac{1}{2} \le t < 1; \\ 0 & \text{otherwise.} \end{cases} \tag{6.12}$$

We put $\Psi(x) := \prod_{j=1}^d \psi_H(x_j)$ with $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Recall, that $\chi_{[0,1)^d}$ is the characteristic function of the unit cube $[0,1)^d$. For $k \in \mathbb{Z}^d$ we put $\chi_{0,k}(x) := \chi_{[0,1)^d}(x-k)$. Let $\chi_{[0,1)}$ denote the characteristic function of the interval $[0,1)$. Then we put

$$h_{i,j,k}(x) := 2^{\frac{jd}{2}} \Big(\prod_{n \in I_1} \chi_{[0,1)}(2^j x_n - k_n) \Big) \Big(\prod_{n \in I_2} \psi_H(2^j x_n - k_n) \Big), \qquad x \in \mathbb{R}^d\,, \tag{6.13}$$

with $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$. Here I_1 and I_2 depend on $i \in \{1, \dots, 2^d-1\}$ and have the properties $I_1 \cup I_2 = \{1,2,\dots,d\}$, $I_1 \cap I_2 = \emptyset$ and $I_2 \ne \emptyset$. This yields $2^d - 1$ possibilities. $\langle f, \chi_{0,k} \rangle$ and $\langle f, h_{i,j,k} \rangle$ denote scalar products (Fourier coefficients of f with respect to the Haar system). For all sequences $t := \{t_{i,j,m}\}_{i \in \{1,\dots,2^d-1\}, j \in \mathbb{N}_0, m \in \mathbb{Z}^d} \subset \mathbb{C}$ a further abbreviation, which will be used, is given by

$$\|t \,| b_{p,q}^{0,\tau}(\mathbb{R}^d)\| := \sup_{P \in \mathscr{Q}} \frac{1}{|P|^\tau} \Big(\sum_{j=\max(j_P,0)}^{\infty} 2^{j(\frac{d}{2}-\frac{d}{p})q} \sum_{i=1}^{2^d-1} \Big[\sum_{m:\ Q_{j,m} \subset P} |t_{i,j,m}|^p \Big]^{\frac{q}{p}} \Big)^{\frac{1}{q}}. \tag{6.14}$$

In the case $q = \infty$ the usual modifications have to be made. Now we are prepared to formulate the following characterization of $B_{p,q}^{0,\tau}(\mathbb{R}^d)$.

Proposition 6. ***Haar Wavelet Characterization for $B_{p,q}^{0,\tau}(\mathbb{R}^d)$.***
Let $1 < p < \infty$, $0 < q \le \infty$ and $0 \le \tau < \frac{1}{p}$. Let $f \in \mathscr{S}'(\mathbb{R}^d)$. Then $f \in B_{p,q}^{0,\tau}(\mathbb{R}^d)$, if and only if f can be represented in $\mathscr{S}'(\mathbb{R}^d)$ as

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \chi_{0,k} \rangle \, \chi_{0,k} + \sum_{i=1}^{2^d-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \langle f, h_{i,j,k} \rangle \, h_{i,j,k} \tag{6.15}$$

with convergence in $\mathscr{S}'(\mathbb{R}^d)$ *and*

$$\|\mu(f)\,|b^{0,\tau}_{p,q}(\mathbb{R}^d)\| := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\sum_{m:\ Q_{0,m}\subset P} |\langle f,\chi_{0,m}\rangle|^p\Big)^{\frac{1}{p}} + \big\|\{\langle f,h_{i,j,m}\rangle\}_{i,j,m}\big|b^{0,\tau}_{p,q}(\mathbb{R}^d)\big\| < \infty.$$

Moreover, the mapping

$$J:\ f\mapsto \{\langle f,\chi_{0,m}\rangle\}_m \cup \{\langle f,h_{i,j,k}\rangle\}_{i,j,k}$$

is an isomorphic map of $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ *onto* $l_p\times b^{0,\tau}_{p,q}(\mathbb{R}^d)$*. In other words* $\|\mu(f)\,|b^{0,\tau}_{p,q}(\mathbb{R}^d)\|$ *is equivalent to* $\|f|B^{0,\tau}_{p,q}(\mathbb{R}^d)\|$*.*

Proof. This result can be found in [147]. One may also consult Theorem 3.41. in [137]. ■

Now we have everything we need, to improve Lemma 28. Let us consider the functions

$$\tilde g_j(x) := \sum_{k_1=0}^{2^{j-1}}\cdots\sum_{k_d=0}^{2^{j-1}} \frac{1-(-1)^{K(k)}}{2}\,\Psi(2^jx-k)\,,\qquad x\in\mathbb{R}^d\,,\qquad j\in\mathbb{N}. \tag{6.16}$$

The function K is the same as in the proof of Lemma 28. For the functions $\tilde g_j$ we observe the following properties.

Lemma 29. ***Differences and*** $B^{0,\tau}_{p,q}(\mathbb{R}^d)$***. Part II.***
Let $s=0$, $0<p<\infty$, $0<q<\infty$, $0\le\tau<\frac{1}{p}$, $0<v\le\infty$, $1\le a\le\infty$ *and* $N\in\mathbb{N}$*. Then the sequence* $(\tilde g_j)_{j\in\mathbb{N}}$ *has the following properties.*

(i) We have $\operatorname{supp}\tilde g_j\subset[0,1]^d$ *for all* $j\in\mathbb{N}$*.*

(ii) We observe $\|\tilde g_j|L_\infty(\mathbb{R}^d)\|\le 1$ *for all* $j\in\mathbb{N}$*.*

(iii) We find $\sup_{j\in\mathbb{N}}\|\,\tilde g_j\,|B^{0,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)}=\infty$*.*

(iv) We have $\sup_{j\in\mathbb{N}}\|\,\tilde g_j\,|B^{0,\tau}_{p,q}(\mathbb{R}^d)\|<\infty$*.*

Proof. The proof of the properties $(i)-(iii)$ is the same as in Lemma 28. So it will be enough, to prove (iv). Therefore we shall use the wavelet characterizations of $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ obtained in [70], [69], [135] and [147]. From the characterization by smooth and compactly supported wavelets and the property (i) it follows the monotonicity of the quasi-norm with respect to p. So for $p_0\le p_1$ we have

$$\|\,\tilde g_j\,|B^{0,\tau}_{p_0,q}(\mathbb{R}^d)\|\le C_1\|\,\tilde g_j\,|B^{0,\tau}_{p_1,q}(\mathbb{R}^d)\|\,,$$

see [69] and [70]. Hence it will be enough to deal with $1<p<\infty$ in (iv). In this situation we may use the Haar wavelet characterization from Proposition 6. It is easy to see, that $\tilde g_j$ is given by its Fourier-Haar series. The frequency level is j and it has 2^{jd} non-zero coefficients, which are all equal to $2^{-\frac{jd}{2}}$. This yields for all $j\in\mathbb{N}$

$$\begin{aligned}\big\|\{\langle\tilde g_j,h_{i,n,m}\rangle\}_{i,n,m}\big|b^{0,\tau}_{p,q}(\mathbb{R}^d)\big\| &\le \sup_{\substack{P\in\mathcal{Q}\\ P\subset[0,1]^d,|P|\ge 2^{-jd}}} \frac{1}{|P|^\tau}2^{j(\frac{d}{2}-\frac{d}{p})}\Big[\sum_{m:\ Q_{j,m}\subset P}|\langle\tilde g_j,h_{i_0,j,m}\rangle|^p\Big]^{\frac{1}{p}}\\ &\le \max_{\ell\in\{0,1,\dots,j\}} 2^{\ell d\tau}2^{j(\frac{d}{2}-\frac{d}{p})}2^{-\frac{jd}{2}}2^{(j-\ell)\frac{d}{p}}\le 1\end{aligned}$$

since $\tau<\frac{1}{p}$. This in combination with Proposition 6 proves (iv). The proof is complete. ■

Now we are ready to prove the following exhaustive result.

Corollary 3. ***Differences and*** $B^{0,\tau}_{p,q}(\mathbb{R}^d)$. ***Part III.***
Let $s=0$, $0<p<\infty$, $0<q\le\infty$, $0\le\tau<\frac{1}{p}$, $0<v\le\infty$, $1\le a\le\infty$ *and* $N\in\mathbb{N}$. *Then we have*

$$B^{0,\tau}_{p,q}(\mathbb{R}^d)\neq \mathbf{B}^{0,\tau,N,a}_{p,q,v}(\mathbb{R}^d).$$

Proof. *Step 1.* At first we look at the case $q=\infty$. Here the result is a consequence of Lemma 16 and Lemma 17.
Step 2. Next we have to deal with the case $\max(p,2)\le q<\infty$. Here the result already has been proved in Lemma 28.
Step 3. Now we assume $1\le q<\max(p,2)$. Then the desired outcome follows from the Theorem of Banach-Steinhaus (in the variant of the uniform boundedness principle) and Lemma 29.
Step 4. At last we have to investigate the case $0<q<1$. We argue by contradiction and assume $B^{0,\tau}_{p,q}(\mathbb{R}^d)=\mathbf{B}^{0,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$. Then $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ can not contain singular distributions and $B^{0,\tau}_{p,q}(\mathbb{R}^d)\subset L^{loc}_1(\mathbb{R}^d)$ follows. Now let $(f_j)_{j\in\mathbb{N}}\subset B^{0,\tau}_{p,q}(\mathbb{R}^d)$ be a convergent sequence with limit $f\in B^{0,\tau}_{p,q}(\mathbb{R}^d)$. In addition we assume, that $(f_j)_{j\in\mathbb{N}}$ converges with respect to the quasi-norm $\| \cdot |B^{0,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)}$ to some function $g\in\mathbf{B}^{0,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$. Convergence in $B^{0,\tau}_{p,q}(\mathbb{R}^d)$ in combination with $B^{0,\tau}_{p,q}(\mathbb{R}^d)\subset L^{loc}_1(\mathbb{R}^d)$ also implies

$$\lim_{j\to\infty}\int_{Q_{0,m}}|f(x)-f_j(x)|\,dx=0\qquad\text{for all}\qquad m\in\mathbb{Z}^d\,.$$

Hence a subsequence converges almost everywhere. On the other hand convergence with respect to the quasi-norm $\| \cdot |B^{0,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)}$ implies convergence in $L^{\tau}_p(\mathbb{R}^d)$, see (5.9). Again there is a subsequence, which must converge almost everywhere. This yields $f=g$ almost everywhere. Hence the embedding $Id:\mathbf{B}^{0,\tau,N,a}_{p,q,v}(\mathbb{R}^d)\to B^{0,\tau}_{p,q}(\mathbb{R}^d)$ is a closed operator. The Closed Graph Theorem yields continuity of this embedding, but by Lemma 29 this is wrong. So the assumption also must be wrong. The proof is complete. ■

Now let us compare Theorem 9, where we described the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ in terms of differences, with Corollary 3. Then we find, that the condition $s>0$ in the formulation of Theorem 9 is not only sufficient, but also necessary. So in the case $s\le 0$ it is not possible to characterize the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ with the quasi-norm $\| \cdot |B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)}$.

6.2 Conditions concerning the Parameters s and p

6.2.1 The Condition $s>d(1/p-1)$

In this section we plan to investigate, whether the condition

$$s>d\Big(\frac{1}{p}-1\Big), \tag{6.17}$$

that can be found in the Theorems 5, 7 and 9, is also necessary. When we have a closer look at these theorems, it becomes clear, that the restriction (6.17) only plays a role for $0<p<1$. For the original Besov and Triebel-Lizorkin spaces it is not difficult, to explain the importance of the number $d(\frac{1}{p}-1)$. So for $s<d(\frac{1}{p}-1)$ both $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ contain singular distributions. For the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ with $p<u$ the situation is much different. Actually, when we only apply arguments, that are connected with singular distributions, we solely obtain the following result.

Proposition 7. ***Differences and*** $s<\frac{p}{u}\sigma_p$.
Let $0<p\le u<\infty$, $0<q\le\infty$, $s\ge 0$, $0<v\le\infty$ *and* $1\le a\le\infty$. *We have* $N\in\mathbb{N}$ *with* $N>s$. *Let* $0<p<1$.

(a) Let in addition $s<d\frac{p}{u}(\frac{1}{p}-1)$. *Then for the Triebel-Lizorkin-Morrey spaces we have* $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\neq\mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.

(b) For the Besov-Morrey spaces the following assertions are true.

  (i) Let $s<d\frac{p}{u}(\frac{1}{p}-1)$. *Then we have* $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\neq\mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.

  (ii) Let $s=d\frac{p}{u}(\frac{1}{p}-1)$ *and* $q>1$. *Then we have* $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\neq\mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.

Proof. *Step 1.* At first we prove the result for the Triebel-Lizorkin-Morrey spaces. In the case $s<d\frac{p}{u}(\frac{1}{p}-1)$ they contain singular distributions, see Lemma 5. So a characterization of $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ in terms of differences is not possible.
Step 2. Now we prove the outcome for the Besov-Morrey spaces. In both cases the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ contain singular distributions, see Lemma 5 and Lemma 6. Therefore again a characterization in terms of differences is not possible. ■

When we compare the Theorems 5 and 7 with Proposition 7 and Lemma 5, it turns out, that for

$$d\frac{p}{u}\Big(\frac{1}{p}-1\Big)<s\le d\Big(\frac{1}{p}-1\Big) \tag{6.18}$$

there is a large area, where the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ do not contain singular distributions and also no characterization in terms of differences is known. The question is, how to fill this gap. For the Besov-Morrey spaces a first step towards the answer reads as follows.

Proposition 8. ***Differences and*** $s>\frac{p}{u}\sigma_p$.
Let $s>0$, $0<p<u<\infty$, $0<q\le\infty$ *and* $N\in\mathbb{N}$ *with* $N>s$. *Let* $0<p<1$ *and*

$$d\frac{p}{u}\Big(\frac{1}{p}-1\Big)<s\le d\Big(\frac{1}{p}-1\Big).$$

Then for all $f\in\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ *there is a* $C>0$ *independent of* f, *such that*

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|\le C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m\infty)}.$$

Proof. To prove this result, we can use the techniques, that are described in Step 2 of the proof from Theorem 2.5.10. in [128]. The main tool for the proof is a classical construction from approximation theory, that can be found in [88], see chapter 5.2.1. Here we apply the version from [115], see the proof of Lemma 10. Notice, that it is important to have $s>d\frac{p}{u}(\frac{1}{p}-1)$, which guarantees $f\in L^{loc}_1(\mathbb{R}^d)\cap\mathcal{S}'(\mathbb{R}^d)$, see Lemma 5. ■

On the other hand for the Triebel-Lizorkin-Morrey spaces in some special cases it is possible to prove, that they can not be characterized in terms of differences, if the parameter s is from the gap given in (6.18). However, for doing so, a new technique needs to be introduced, that also can be used to investigate the importance of the parameter q. Hence we will postpone this. Instead we want to have a look at the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Here we observe the following.

Proposition 9. ***Differences and*** $s < d(\frac{1}{p}-1) - d\tau(1-p)$.
Let $s \in \mathbb{R}$, $0 < p < 1$, $0 < q \leq \infty$, $0 \leq \tau < \frac{1}{p}$, $0 < v \leq \infty$, $1 \leq a \leq \infty$ *and* $N \in \mathbb{N}$ *with* $N > s$. *Let*

$$s < d\Big(\frac{1}{p}-1\Big) - d\tau(1-p).$$

Then we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$.

Proof. In the case $s < d(\frac{1}{p}-1) - d\tau(1-p)$ with $0 < p < 1$ the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ contain singular distributions, see Lemma 16. So a characterization in terms of differences is not possible. Therefore $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$. ■

Let us compare Proposition 9 with Theorem 9 and Lemma 16. It turns out, that for

$$d\Big(\frac{1}{p}-1\Big) - d\tau(1-p) < s \leq d\Big(\frac{1}{p}-1\Big) \tag{6.19}$$

there is a large area, where we have not obtained a characterization in terms of differences, although the Besov-type spaces do not contain singular distributions there. We already observed the same phenomenon for the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, see (6.18). It might be of some interest, that the lower bounds in (6.18) and (6.19) are the same in a certain manner. So if we take the natural choice $\tau = \frac{1}{p} - \frac{1}{u}$, we observe

$$d\Big(\frac{1}{p}-1\Big) - d\tau(1-p) = d\Big(\frac{1}{p}-1\Big) - d\Big(\frac{1}{p}-\frac{1}{u}\Big)(1-p) = d\frac{p}{u}\Big(\frac{1}{p}-1\Big).$$

This underlines the strong connection between the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. More information concerning the gap we described in (6.18) and (6.19) also can be found in [48].

6.2.2 The Condition $s > d(1/p - 1/v)$

Hereinafter we intend to investigate, whether the condition

$$s > d\Big(\frac{1}{p}-\frac{1}{v}\Big), \tag{6.20}$$

that can be found in the Theorems 5, 7 and 9, is also necessary. Looking at these theorems, it becomes clear, that the restriction (6.20) only plays a role for the case $v > p$. At first let us focus on the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$. When we want to explain the importance of the condition (6.20), the following result can be seen as a starting point.

Proposition 10. ***Differences and*** $s < d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$.
Let $s \geq 0$, $0 < p \leq u < \infty$ *and* $0 < q \leq \infty$. *Let* $0 < p < v < \infty$, $1 \leq a \leq \infty$ *and* $N \in \mathbb{N}$ *with* $N > s$.

(a) *Then for the Triebel-Lizorkin-Morrey spaces the following assertions are true.*

(i) *Let* $s < d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$. *Then we have* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.

(ii) *Let* $s = d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$ *with* $q > 2$ *and* $1 < v < \infty$. *Then we observe* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.

(b) *Let* $s < d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$. *Then for the Besov-Morrey spaces we find* $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.

Proof. *Step 1.* At first we prove the result for the Triebel-Lizorkin-Morrey spaces.
Substep 1. We start with the case $0 < v \leq 1$. Here $s < d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$ implies $s < d\frac{p}{u}(\frac{1}{p}-1)$. But from Proposition 7 we know, that in this case it is not possible to describe the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ in terms of differences.
Substep 2. Now we deal with the case $1 < v < \infty$. We have either $s < d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$ or $s = d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$ with $q > 2$. We will argue by contradiction. Our first assumption is $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$. Then $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ can not contain singular distributions and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \subset L_1^{loc}(\mathbb{R}^d)$ follows. Our second assumption is a sharpening of the first one. We assume, that the identity $Id : \mathbf{E}^{s,N,a}_{u,p,q,v}(B(0,\frac{1}{8N})) \to \mathscr{E}^s_{u,p,q}(B(0,\frac{1}{8N}))$ is a continuous operator. Here $\mathbf{E}^{s,N,a}_{u,p,q,v}(B(0,\frac{1}{8N}))$ is defined to be the set of all $f \in \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ satisfying $\operatorname{supp} f \subset B(0,\frac{1}{8N})$. We will first disprove assumption two, afterwards assumption one.
Substep 2.1. Let $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B(0,\frac{1}{4N})$. We will prove, that there is a $C > 0$ independent of f, such that

$$\|f|L_v(\mathbb{R}^d)\| \leq C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|. \tag{6.21}$$

Because of our assumption we can start with

$$\begin{aligned}
\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| &\geq C_1\Big(\int_{B(0,\frac{N+1}{4})}\Big(\int_0^a t^{-sq}\Big(t^{-d}\int_{B(0,t)}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}\\
&\geq C_1\Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}}\Big(\int_{\frac{N+2}{4N}}^1 t^{-sq}\Big(t^{-d}\int_{B(\frac{-x}{N},\frac{1}{4N})}|\Delta_h^N f(x)|^v dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}\\
&\geq C_2\Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}}\Big(\int_{B(\frac{-x}{N},\frac{1}{4N})}|\Delta_h^N f(x)|^v dh\Big)^{\frac{p}{v}}dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

In the second line we used, that for $\frac{N}{4} < |x| < \frac{N+1}{4}$ and $t \geq \frac{N+2}{4N}$ we have $B(\frac{-x}{N},\frac{1}{4N}) \subset B(0,t)$. Next we apply the formula from Lemma 23. For $\frac{N}{4} < |x| < \frac{N+1}{4}$ and $h \in B(\frac{-x}{N},\frac{1}{4N})$ since $\operatorname{supp} f \subset B(0,\frac{1}{4N})$ we obtain $f(x+kh) = 0$ with $k \in \{0,1,\ldots,N-1\}$. So we get

$$\begin{aligned}
\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| &\geq C_2\Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}}\Big(\int_{B(\frac{-x}{N},\frac{1}{4N})}|f(x+Nh)|^v dh\Big)^{\frac{p}{v}}dx\Big)^{\frac{1}{p}}\\
&\geq C_3\Big(\int_{\frac{N}{4}<|x|<\frac{N+1}{4}}\Big(\int_{B(0,\frac{1}{4})}|f(z)|^v dz\Big)^{\frac{p}{v}}dx\Big)^{\frac{1}{p}}\\
&\geq C_4\|f|L_v(\mathbb{R}^d)\|.
\end{aligned}$$

Consequently the proof of inequality (6.21) is complete.

Substep 2.2. In this substep we will work with function spaces on smooth and bounded domains, see also Definition 15. As domain we choose the ball $B(0,\frac{1}{8N})$. We want to prove, that we have the continuous embedding

$$\mathscr{E}^s_{u,p,q}\Big(B(0,\frac{1}{8N})\Big) \hookrightarrow L_v\Big(B(0,\frac{1}{8N})\Big). \tag{6.22}$$

With this in mind we take $f \in \mathscr{E}^s_{u,p,q}(B(0,\frac{1}{8N}))$. Because of the definition there is a $g \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ with $f = g$ on $B(0,\frac{1}{8N})$. In consequence of our assumption it is possible to characterize the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ in terms of the quasi-norm $\| \cdot |\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$. So locally on $B(0,\frac{1}{8N})$ the function f can be understood as a pointwise defined function. Now we take a sequence $(h_l)_{l=1}^\infty \subset \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ with $h_l = f$ on $B(0,\frac{1}{8N})$ for every $l \in \mathbb{N}$, such that

$$\lim_{l\to\infty} \|h_l|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| = \|f|\mathscr{E}^s_{u,p,q}\Big(B(0,\frac{1}{8N})\Big)\|.$$

Moreover, we define a smooth cut-off function $\Psi \in C_0^\infty(\mathbb{R}^d)$ with $\Psi(x) = 1$ on $B(0,\frac{1}{8N})$ and $\Psi(x) = 0$ for every x with $|x| \geq \frac{1}{4N}$. Then for each $l \in \mathbb{N}$ we obtain

$$\|f|L_v\Big(B(0,\frac{1}{8N})\Big)\| \leq \|h_l \cdot \Psi|L_v(\mathbb{R}^d)\|.$$

We have $h_l \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\Psi \in C_0^\infty(\mathbb{R}^d)$. Because of Lemma 9 we find $h_l \cdot \Psi \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Of course it is $\operatorname{supp}(h_l \cdot \Psi) \subset B(0,\frac{1}{4N})$. Now we can apply formula (6.21) and get

$$\|f|L_v\Big(B(0,\frac{1}{8N})\Big)\| \leq C_1 \|h_l \cdot \Psi|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|.$$

Here C_1 is independent of f, h_l and Ψ. Next we use Lemma 9 again. Then for $m \in \mathbb{N}$ sufficiently large we observe

$$\begin{aligned}\|f|L_v\Big(B(0,\frac{1}{8N})\Big)\| &\leq C_2 \|\Psi|C^m(\mathbb{R}^d)\|\, \|h_l|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \\ &\leq C_3 \|h_l|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|.\end{aligned}$$

If l tends to infinity, we find

$$\|f|L_v\Big(B(0,\frac{1}{8N})\Big)\| \leq C_4 \|f|\mathscr{E}^s_{u,p,q}\Big(B(0,\frac{1}{8N})\Big)\|.$$

This proves (6.22).
Substep 2.3. Now we will disprove assumption two. For that purpose we want to use Lemma 7. From the substep before we know $\mathscr{E}^s_{u,p,q}(B(0,\frac{1}{8N})) \hookrightarrow L_v(B(0,\frac{1}{8N}))$. Then Lemma 7 tells us, that we have either

$$s > \max\left(0, d\frac{p}{u}\left(\frac{1}{p} - \frac{1}{v}\right)\right) \qquad \text{or} \qquad s = \max\left(0, d\frac{p}{u}\left(\frac{1}{p} - \frac{1}{v}\right)\right) \text{ with } q \leq 2.$$

But earlier in the proof we said, that we have either $s < d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$ or $s = d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$ with $q > 2$. This is a contradiction. So our assumption on the continuity of the identity must have been wrong.
Substep 2.4. Next we will disprove assumption one. Let $(f_j)_{j\in\mathbb{N}}$ be a convergent sequence in $\mathbf{E}^{s,N,a}_{u,p,q,v}(B(0,\frac{1}{8N}))$ with limit $f \in \mathbf{E}^{s,N,a}_{u,p,q,v}(B(0,\frac{1}{8N}))$. In addition we assume $\lim_{j\to\infty} f_j = g$ in

$\mathcal{E}^{s}_{u,p,q}(B(0,\frac{1}{8N}))$. The first fact implies convergence in $L_p(\mathbb{R}^d)$, see Theorem 5 and Definition 31. This yields convergence almost everywhere for an appropriate subsequence $(f_{j_\ell})_{\ell\in\mathbb{N}}$. Now recall, that we have assumed $\mathcal{E}^{s}_{u,p,q}(\mathbb{R}^d) = \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$. Then $\mathcal{E}^{s}_{u,p,q}(\mathbb{R}^d)$ can not contain singular distributions and $\mathcal{E}^{s}_{u,p,q}(\mathbb{R}^d) \subset L_1^{loc}(\mathbb{R}^d)$ follows. But then we also find $\mathcal{E}^{s}_{u,p,q}(B(0,\frac{1}{8N})) \hookrightarrow L_1(B(0,\frac{1}{8N}))$. Now because of $\lim_{j\to\infty} f_j = g$ in $\mathcal{E}^{s}_{u,p,q}(B(0,\frac{1}{8N}))$ we obtain

$$\lim_{\ell\to\infty} \|f_{j_\ell} - g|L_1\Big(B(0,\frac{1}{8N})\Big)\| \leq C \lim_{\ell\to\infty} \|f_{j_\ell} - g|\mathcal{E}^{s}_{u,p,q}\Big(B(0,\frac{1}{8N})\Big)\| = 0.$$

By switching to a further subsequence if necessary we conclude $f = g$ almost everywhere. So we have proved, that the identity $Id : \mathbf{E}^{s,N,a}_{u,p,q,v}(B(0,\frac{1}{8N})) \to \mathcal{E}^{s}_{u,p,q}(B(0,\frac{1}{8N}))$ is a closed linear operator. The Closed Graph Theorem, which remains to hold for quasi-Banach spaces (see Theorem 2.15 in [98] for instance), yields that Id must be continuous. But this contradicts our previous conclusion. Therefore also our assumption concerning the equality of the sets must be wrong. This proves $\mathcal{E}^{s}_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ as claimed.

Step 2. Now we have to prove the result for the Besov-Morrey spaces. For that purpose we can use the same techniques as for the Triebel-Lizorkin-Morrey spaces. Only some obvious modifications have to be made. So again the main tool for the proof is Lemma 7. We omit the details and refer to [55]. Here some more explanations can be found. ■

Remark 12. ***A Forerunner of Proposition 10.***
Another method to investigate the sense of the condition $s > d(\frac{1}{p} - \frac{1}{v})$, is described in Remark 3.8 in [96]. When we proceed like there, we obtain, that it is not possible to characterize the spaces $\mathcal{E}^{s}_{u,p,q}(\mathbb{R}^d)$ in terms of the quasi-norm $\|\cdot|\mathcal{E}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$, if $s < \frac{d}{u} - \frac{d}{v}$. But because of $p \leq u$ this result is much weaker than Proposition 10.

When we compare Proposition 10 with Theorem 5 or Theorem 7, it turns out, that for

$$d\frac{p}{u}\Big(\frac{1}{p} - \frac{1}{v}\Big) < s \leq d\Big(\frac{1}{p} - \frac{1}{v}\Big) \tag{6.23}$$

there is a large area, where we have no result. For the special case $v = 1$ here we recover the same gap, we have already described in (6.18). Later we will see, that in some special cases for the Triebel-Lizorkin-Morrey spaces we can obtain results, that are much stronger than Proposition 10. Then we will be able to close the gap from (6.23). On the other hand for the Besov-Morrey spaces we can make the following simple observation.

Remark 13. ***The Spaces $\mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ for $s \leq d(\frac{1}{p} - \frac{1}{v})$.***
Let $0 < p \leq u < \infty$, $s > 0$, $0 < q \leq \infty$, $1 \leq a \leq \infty$ and $N \in \mathbb{N}$ with $N > s$. Let $0 < v_1 < v_2 < \infty$. Then we find

$$\mathbf{N}^{s,N,a}_{u,p,q,\infty}(\mathbb{R}^d) \hookrightarrow \mathbf{N}^{s,N,a}_{u,p,q,v_2}(\mathbb{R}^d) \hookrightarrow \mathbf{N}^{s,N,a}_{u,p,q,v_1}(\mathbb{R}^d).$$

So for $v > \max(1,p)$ and $d\max(0,\frac{1}{p}-1) < s \leq d(\frac{1}{p} - \frac{1}{v})$ because of Theorem 7 we have

$$\mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d) \hookrightarrow \mathbf{N}^{s,N,a}_{u,p,q,1}(\mathbb{R}^d) = \mathcal{N}^{s}_{u,p,q}(\mathbb{R}^d).$$

Now let us have a look at the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Here we obtain the following result, which is a counterpart of Proposition 10.

Proposition 11. ***Differences and*** $s < d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v})$.
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $0 \leq \tau < \frac{1}{p}$, $1 \leq a \leq \infty$ *and* $N \in \mathbb{N}$ *with* $s < N$. *Moreover let* $\max(p,1) < v < \infty$. *Furthermore we assume that*

$$s < d\Big(\frac{1}{p} - \frac{1}{v}\Big) - d\tau\Big(1 - \frac{p}{v}\Big).$$

Then $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$ *follows.*

Proof. This result can be proved in the same way, as it is described in the proof of Proposition 10. All techniques, that are described there, also can be used here. Only the usual modifications have to be made. So the main tool for the proof is Lemma 19. Moreover, we have to use Lemma 20. We omit the details and refer to [58]. ■

We observe, that for

$$d\Big(\frac{1}{p} - \frac{1}{v}\Big) - d\tau\Big(1 - \frac{p}{v}\Big) \leq s \leq d\Big(\frac{1}{p} - \frac{1}{v}\Big) \tag{6.24}$$

we neither can apply Proposition 11 nor Theorem 9. We described a comparable phenomenon in (6.23). Later we will see, that in some special cases it is possible to close the gap, that is described in (6.24).

6.3 Conditions concerning the Parameters s and q

In this section we plan to investigate, whether the condition

$$d\max\left(\frac{1}{q} - 1, \frac{1}{q} - \frac{1}{v}\right) < s, \tag{6.25}$$

that can be found in Theorem 5, is also necessary. For that purpose we will use some ideas developed by Christ and Seeger. In [25] and [26] they constructed a random function and investigated the properties of this function, to learn something about the importance of the condition (6.25) for the original Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$. It turns out, that we also can apply the techniques described in [25] and [26], to deal with the Triebel-Lizorkin-Morrey spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ with $p < u$. Thereto in a first step we have to recall the main results as well as some details of the proofs, that can be found in the two papers, we just mentioned. Moreover, we have to make some modifications, to obtain an even stronger result. All this will be done in the next section.

6.3.1 A random Construction of Christ and Seeger

Hereinafter we recall some ideas from [25] and [26]. Here Christ and Seeger developed a method to investigate the importance of the condition (6.25). They dealt with the original Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ and concentrated on the case $v = \infty$. They worked with

$$\mathfrak{D}^{s,q}_{\infty,N} f(x) := \Big(\int_0^1 t^{-sq} \sup_{|h|<t} |\Delta^N_h f(x)|^q \frac{dt}{t}\Big)^{\frac{1}{q}}, \qquad x \in \mathbb{R}^d,$$

and investigated the behavior of the number

$$\mathfrak{A}^{s,q}_{\infty,N}(w) := \sup\Big\{\|\mathfrak{D}^{s,q}_{\infty,N}f|L_p(\mathbb{R}^d)\| : \ \|f|F^s_{p,q}(\mathbb{R}^d)\| \le 1\,, \ \operatorname{supp}\mathscr{F}f \subset B(0,w)\Big\}, \ w > 100\,.$$

Using this notation, the following result was proved.

Proposition 12. ***Differences and necessary Conditions concerning*** q***. Part I.***
Let $0 < s < N$ *and* $0 < q < p < \infty$*. Then we have*

$$\begin{aligned}
C_1(\log w)^{\frac{d}{q}-s} \le \mathfrak{A}^{s,q}_{\infty,N}(w) &\le C_2(\log w)^{\frac{d}{q}-s} && \text{if} \quad \frac{d}{p} < s < \frac{d}{q};\\
C_3(\log\log w)^{\frac{1}{q}} \le \mathfrak{A}^{\frac{d}{q},q}_{\infty,N}(w) &\le C_4(\log\log w)^{\frac{1}{q}} && \text{if} \quad \frac{d}{p} < s = \frac{d}{q} \quad \text{and} \quad 0 < q \le 1;\\
\frac{1}{C_5}(\log\log w)^{\frac{1}{q}} \le \mathfrak{A}^{\frac{d}{q},q}_{\infty,N}(w) &\le C_5\, \log\log w && \text{if} \quad \frac{d}{p} < s = \frac{d}{q} \quad \text{and} \quad 1 < q < p\,.
\end{aligned}$$

Proof. This result is proved in [25] and in [26], see Theorem 1.3. and chapter 6. ■

In a remark at the end of chapter 6 in [26] it is mentioned, that it is possible to prove a similar result also for $v < \infty$. For that purpose we define

$$\mathfrak{D}^{s,q}_{v,N}f(x) := \Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{|h|<t}|\Delta^N_h f(x)|^v\,dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{1}{q}}, \qquad x \in \mathbb{R}^d\,.$$

Again we are interested in the behavior of the number

$$\mathfrak{A}^{s,q}_{v,N}(w) := \sup\Big\{\|\mathfrak{D}^{s,q}_{v,N}f|L_p(\mathbb{R}^d)\| : \ \|f|F^s_{p,q}(\mathbb{R}^d)\| \le 1\,, \ \operatorname{supp}\mathscr{F}f \subset B(0,w)\Big\}, \ w > 100\,.$$

Then we obtain the following.

Proposition 13. ***Differences and necessary Conditions concerning*** q***. Part II.***
Let $0 < p < \infty$*,* $0 < s < N$*,* $0 < v < \infty$ *and* $0 < q < v < \infty$*. Let* $q \le p$*. Then we have*

$$\begin{aligned}
C_1(\log w)^{\frac{d}{q}-\frac{d}{v}-s} &\le \mathfrak{A}^{s,q}_{v,N}(w) && \text{if} \quad s < \frac{d}{q}-\frac{d}{v};\\
C_2(\log\log w)^{\frac{1}{q}} &\le \mathfrak{A}^{\frac{d}{q}-\frac{d}{v},q}_{v,N}(w) && \text{if} \quad s = \frac{d}{q}-\frac{d}{v}.
\end{aligned}$$

Proof. To prove this result, we will follow Christ and Seeger, see chapter 6.2. in [25] and in [26] as well. For our purpose almost everything, what is done in [25] and [26], can be taken over unchanged. Only a few modifications have to be made.
Step 1. In their proof Christ and Seeger constructed a random function and made estimations for the expected value of the different quasi-norms of this function. Here we will work with the same random function. Moreover, we will use the same notation, as it can be found in [25] and [26]. Let us consider a function $\eta \in \mathscr{S}(\mathbb{R}^d)$, that fulfills $\operatorname{supp}\mathscr{F}\eta \subset \{\xi \in \mathbb{R}^d \,:\, \frac{1}{2} < |\xi| < 1\}$ and $|\eta(x)| \ge 1$ for $|x| \le 2^{-M+d+2}$. Here $M \in \mathbb{N}$ is chosen later. Moreover, we deal with a function $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi \subset [-2^{-2M-4}, 2^{-2M-4}]^d$, such that

$$|\phi * \eta(z)| \ge C(M) > 0 \qquad \text{for} \qquad |z| \le 2^{-M+d+1}. \tag{6.26}$$

Let $R \in \mathbb{N}$ be arbitrary large and choose a large number $W \in \mathbb{N}$. For $k \in \mathbb{N}$ we define $n_k = kR$ and $r_k = 2^{n_k - M}$. Furthermore, we define the functions

$$\phi_k = r_k^d \phi(r_k \cdot) \qquad \text{and} \qquad \eta_k = r_k^d \eta(r_k \cdot).$$

Put $\alpha = 2^{-Wd} = L^{-1}$. For $n \in \mathbb{N}_0$ let $\mathscr{Q}(n)$ be the set of all dyadic cubes with side length 2^{-n}, that are located in $[0,1)^d$. For every cube Q let χ_Q be the corresponding indicator function. Let Ω be a probability space with probability measure μ. There is a family $\{\theta_{Q,\alpha}\}$ of independent random variables indexed by the dyadic cubes Q. Each of these random variables takes the value 1 with probability α and the value 0 with probability $1-\alpha$. We consider random functions

$$h_k^{\omega,\alpha}(x) = \sum_{Q \in \mathscr{Q}(n_k)} \theta_{Q,\alpha}(\omega) \chi_Q(x).$$

These functions are supported on $[0,1]^d$ and for all x we have $h_k^{\omega,\alpha}(x) \in \{0,1\}$. Now we define

$$g_k^{\omega,\alpha} = \eta_k * h_k^{\omega,\alpha} \qquad \text{and} \qquad G_k^{\omega,\alpha}(x) = 2^{-n_k s} g_k^{\omega,\alpha}(x) \qquad \text{and} \qquad G^{\omega,\alpha}(x) = \sum_{k=1}^{2^{Wd}} G_k^{\omega,\alpha}(x).$$

Further information concerning the used notation can be found in chapter 2, chapter 4 and chapter 6.2. from [26].

Step 2. Now we show, that there is a constant $C_1 > 0$, that only depends on p,q and d, such that

$$\Big(\int_\Omega \| G^{\omega,\alpha} | F^s_{p,q}(\mathbb{R}^d) \|^p d\mu(\omega) \Big)^{\frac{1}{p}} \leq C_1.$$

But this has already been proved by Christ and Seeger. So we can find the following result, see Lemma 6.2.1. in [26].

Lemma 30. ***A random Construction of Christ and Seeger. Some Properties.***
Let $0 < p < \infty$ and $0 < q < \infty$. Then there are constants $C_1, C_2, C_3 > 0$, that only depend on p,q,N and d, such that

$$\Big(\int_\Omega \| G^{\omega,\alpha} | F^s_{p,q}(\mathbb{R}^d) \|^p d\mu(\omega) \Big)^{\frac{1}{p}} \leq C_1$$

and

$$\Big(\int_\Omega \int_{\mathbb{R}^d} \Big[\int_0^1 t^{-sq-1} \sup_{|h| \leq t} \sum_{l \,:\, t2^{n_l} \leq 1} \Big| \Delta_h^N G_l^{\omega,\alpha}(x) \Big|^q dt \Big]^{\frac{p}{q}} dx \, d\mu(\omega) \Big)^{\frac{1}{p}} \leq C_2$$

and

$$\Big(\int_\Omega \int_{\mathbb{R}^d} \Big[\int_0^1 t^{-sq-1} \Big(\sum_{l \,:\, t2^{n_l} \geq 1} |G_l^{\omega,\alpha}(x)| \Big)^q dt \Big]^{\frac{p}{q}} dx \, d\mu(\omega) \Big)^{\frac{1}{p}} \leq C_3.$$

So thanks to Lemma 30 Step 2 is complete.

Step 3. Now we prove, that for large $W \in \mathbb{N}$ and very large $R \in \mathbb{N}$, there is a constant $C > 0$, that only depends on s,p,q and d, such that

$$\Big(\int_\Omega \| \mathfrak{D}^{s,q}_{v,N} G^{\omega,\alpha} | L_p(\mathbb{R}^d) \|^p d\mu(\omega) \Big)^{\frac{1}{p}} \geq C \max \Big(2^{W(-s-\frac{d}{v}+\frac{d}{q})}, W^{\frac{1}{q}} \Big). \tag{6.27}$$

To show this, we will follow the strategy that can be found in chapter 6.2. in [26].
Substep 3.1. At first we have to introduce some additional notation. For sufficiently large $W \in \mathbb{N}$ we write

$$I_{k,j} = [2^{-n_k+j+2d}, 2^{-n_k+j+2d+1}] \cap [0,1]$$

with $k \in \{1,2,\dots,2^{Wd}\}$ and $j \in \{d+M, d+M+1,\dots,W-M-d\}$. Moreover, we use the notation

$$\tilde{\Delta}_h^N G_l^{\omega,\alpha}(x) = \Delta_h^N G_l^{\omega,\alpha}(x) - (-1)^N G_l^{\omega,\alpha}(x)$$

where $G_l^{\omega,\alpha}$ is defined as in Step 1. Furthermore, we will work with the abbreviation

$$\Gamma_{k,N}^l = \Gamma_{k,N}^l(x,h,\omega) = \int_{\mathbb{R}^d} \phi_k(y) \tilde{\Delta}^N_{\frac{h-y}{N}} G_l^{\omega,\alpha}(x)dy$$

where ϕ_k is the same function as in Step 1. Using this notation, the main goal of Substep 3.1. is to prove, that there is a constant $C>0$, that only depends on s,p,q and d, such that

$$\begin{aligned}
&\Big(\int_\Omega \|\mathfrak{D}^{s,q}_{v,N} G^{\omega,\alpha} | L_p(\mathbb{R}^d)\|^p d\mu(\omega)\Big)^{\frac{1}{p}} \\
&\geq C\Big(\int_\Omega \int_{[\frac14,\frac34]^d} \sum_{k=1}^{2^{Wd}} \sum_{j=d+M}^{W-M-d} 2^{(n_k-j)(sq+d\frac{q}{v})} \Big(\int_{B(0,N2^{-n_k+j+d})} \Big|\Gamma^k_{k,N}(x,h,\omega)\Big|^v dh\Big)^{\frac{q}{v}} dx\, d\mu\Big)^{\frac{1}{q}} \\
&\quad - C\Big(\int_\Omega \int_{[\frac14,\frac34]^d} \sum_{k=1}^{2^{Wd}} \sum_{j=d+M}^{W-M-d} 2^{(n_k-j)(sq+d\frac{q}{v})} \Big(\int_{B(0,N2^{-n_k+j+d})} \Big|\sum_{l=k+1}^{2^{Wd}} \Gamma^l_{k,N}\Big|^v dh\Big)^{\frac{q}{v}} dx\, d\mu\Big)^{\frac{1}{q}} - C.
\end{aligned}$$

To prove this, we restrict the domain of integration concerning the variable x to the cube $[\frac14,\frac34]^d$. After this we apply Hölder's inequality and obtain

$$\begin{aligned}
&\Big(\int_\Omega \|\mathfrak{D}^{s,q}_{v,N} G^{\omega,\alpha} | L_p(\mathbb{R}^d)\|^p d\mu(\omega)\Big)^{\frac{1}{p}} \\
&\qquad\geq C_1\Big(\int_\Omega \int_{[\frac14,\frac34]^d} \int_0^1 t^{-sq-d\frac{q}{v}-1} \Big(\int_{B(0,t)} |\Delta_h^N G^{\omega,\alpha}(x)|^v dh\Big)^{\frac{q}{v}} dt\, dx\, d\mu\Big)^{\frac{1}{q}}.
\end{aligned}$$

Next for $t \in [0,1]$ we have

$$\Delta_h^N G^{\omega,\alpha}(x) = \sum_{l=1\dots 2^{Wd}, l\,:\, t2^{n_l}\geq 1} \Delta_h^N G_l^{\omega,\alpha}(x) + \sum_{l=1\dots 2^{Wd}, l\,:\, t2^{n_l}\leq 1} \Delta_h^N G_l^{\omega,\alpha}(x).$$

So we can write

$$\begin{aligned}
&\Big(\int_\Omega \|\mathfrak{D}^{s,q}_{v,N} G^{\omega,\alpha} | L_p(\mathbb{R}^d)\|^p d\mu(\omega)\Big)^{\frac{1}{p}} \\
&\geq C_2\Big(\int_\Omega \int_{[\frac14,\frac34]^d} \int_0^1 t^{-sq-d\frac{q}{v}-1} \Big(\int_{B(0,t)} \Big|\sum_{l=1\dots 2^{Wd}, l\,:\, t2^{n_l}\geq 1} \Delta_h^N G_l^{\omega,\alpha}(x)\Big|^v dh\Big)^{\frac{q}{v}} dt\, dx\, d\mu\Big)^{\frac{1}{q}} \\
&\quad - C_2\Big(\int_\Omega \int_{[\frac14,\frac34]^d} \int_0^1 t^{-sq-d\frac{q}{v}-1} \Big(\int_{B(0,t)} \Big|\sum_{l=1\dots 2^{Wd}, l\,:\, t2^{n_l}\leq 1} \Delta_h^N G_l^{\omega,\alpha}(x)\Big|^v dh\Big)^{\frac{q}{v}} dt\, dx\, d\mu\Big)^{\frac{1}{q}}.
\end{aligned}$$

Now Lemma 30 tells us, that the second term is bounded by a constant. Next we apply

$$\sum_{l=1\dots 2^{Wd}, l\,:\, t2^{n_l}\geq 1} \Delta_h^N G_l^{\omega,\alpha} = \sum_{l=1\dots 2^{Wd}, l\,:\, t2^{n_l}\geq 1} \tilde{\Delta}_h^N G_l^{\omega,\alpha} + \sum_{l=1\dots 2^{Wd}, l\,:\, t2^{n_l}\geq 1} (-1)^N G_l^{\omega,\alpha}.$$

With that we obtain

$$\begin{aligned}&\Big(\int_\Omega \|\mathfrak{D}^{s,q}_{v,N} G^{\omega,\alpha}|L_p(\mathbb{R}^d)\|^p d\mu(\omega)\Big)^{\frac{1}{p}}\\
&\quad\geq C_3\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\int_0^1 t^{-sq-d\frac{q}{v}-1}\Big(\int_{B(0,t)}\Big|\sum_{l=1\dots 2^{Wd},l\,:\,t2^{n_l}\geq 1}\tilde{\Delta}^N_h G^{\omega,\alpha}_l(x)\Big|^v dh\Big)^{\frac{q}{v}} dt\,dx\,d\mu\Big)^{\frac1q}\\
&\quad -C_3\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\int_0^1 t^{-sq-d\frac{q}{v}-1}\Big(\int_{B(0,t)}\Big|\sum_{l=1\dots 2^{Wd},l\,:\,t2^{n_l}\geq 1} G^{\omega,\alpha}_l(x)\Big|^v dh\Big)^{\frac{q}{v}} dt\,dx\,d\mu\Big)^{\frac1q}-C_3.\end{aligned}$$

Again Lemma 30 tells us, that the second term is bounded by a constant. Now we split up the interval $[0,1]$ using the small intervals $I_{k,j}$. Moreover, we make a transformation of the coordinates. When we write $\sum_{k,j}$ for $\sum_{k=1}^{2^{Wd}}\sum_{j=d+M}^{W-M-d}$, we get

$$\begin{aligned}&\Big(\int_\Omega \|\mathfrak{D}^{s,q}_{v,N} G^{\omega,\alpha}|L_p(\mathbb{R}^d)\|^p d\mu(\omega)\Big)^{\frac{1}{p}}\\
&\quad\geq C_4\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\sum_{k,j}\int_{I_{k,j}} t^{-sq-d\frac{q}{v}-1}\Big(\int_{B(0,tN)}\Big|\sum_{l=k}^{2^{Wd}}\tilde{\Delta}^N_{\frac{h}{N}} G^{\omega,\alpha}_l(x)\Big|^v dh\Big)^{\frac{q}{v}} dt\,dx\,d\mu\Big)^{\frac1q}-C_4\\
&\quad\geq C_5\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\sum_{k,j}2^{(n_k-j)(sq+d\frac{q}{v})}\Big(\int_{B(0,N2^{-n_k+j+2d})}\Big|\sum_{l=k}^{2^{Wd}}\tilde{\Delta}^N_{\frac{h}{N}} G^{\omega,\alpha}_l(x)\Big|^v dh\Big)^{\frac{q}{v}} dx\,d\mu\Big)^{\frac1q}-C_5.\end{aligned}$$

Now we use the function $\phi\in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi\subset[-2^{-2M-4},2^{-2M-4}]^d$ from Step 1. It was $\phi_k=r_k^d\phi(r_k\cdot)$ with $r_k=2^{kR-M}$. We can choose ϕ, such that we have $\int_{\mathbb{R}^d}\phi(x)dx=1$. So ϕ_k can be interpreted as an approximation of the identity with respect to R. Moreover, we can find a radial majorant for ϕ. So for k and l as above, because of Theorem 2.1 from [123] we get

$$\lim_{R\to\infty}\phi_k*\tilde{\Delta}^N_{\frac{h}{N}}G^{\omega,\alpha}_l=\tilde{\Delta}^N_{\frac{h}{N}}G^{\omega,\alpha}_l\qquad\text{pointwise almost everywhere.}$$

It is one of the main ideas of this proof, to send R to infinity. Because of this we are able to choose R very large and obtain

$$\begin{aligned}&\Big(\int_\Omega \|\mathfrak{D}^{s,q}_{v,N} G^{\omega,\alpha}|L_p(\mathbb{R}^d)\|^p d\mu(\omega)\Big)^{\frac{1}{p}}\\
&\quad\geq C_6\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\sum_{k,j}2^{(n_k-j)(sq+d\frac{q}{v})}\Big(\int_{B(0,N2^{-n_k+j+d})}\Big|\sum_{l=k}^{2^{Wd}}\phi_k*\tilde{\Delta}^N_{\frac{h}{N}} G^{\omega,\alpha}_l\Big|^v dh\Big)^{\frac{q}{v}} dx\,d\mu\Big)^{\frac1q}-C_6\\
&\quad= C_6\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\sum_{k,j}2^{(n_k-j)(sq+d\frac{q}{v})}\Big(\int_{B(0,N2^{-n_k+j+d})}\Big|\sum_{l=k}^{2^{Wd}}\Gamma^l_{k,N}(x,h,\omega)\Big|^v dh\Big)^{\frac{q}{v}} dx\,d\mu\Big)^{\frac1q}-C_6.\end{aligned}$$

Now we split up

$$\sum_{l=k}^{2^{Wd}}\Gamma^l_{k,N}(x,h,\omega)=\Gamma^k_{k,N}(x,h,\omega)+\sum_{l=k+1}^{2^{Wd}}\Gamma^l_{k,N}(x,h,\omega)$$

and obtain the formula that we stated at the beginning of Substep 3.1. So this substep is complete. *Substep 3.2.* Next we will prove, that for large $W\in\mathbb{N}$, there is a $C>0$, that only depends on q,s,N and d, such that

$$\begin{aligned}&\Big(\int_\Omega\int_{[\frac14,\frac34]^d}\sum_{k=1}^{2^{Wd}}\sum_{j=d+M}^{W-M-d}2^{(n_k-j)(sq+d\frac{q}{v})}\Big(\int_{B(0,N2^{-n_k+j+d})}\Big|\sum_{l=k+1}^{2^{Wd}}\Gamma^l_{k,N}(x,h,\omega)\Big|^v dh\Big)^{\frac{q}{v}} dx\,d\mu\Big)^{\frac1q}\\
&\qquad\leq C2^{-R}2^{W\left(\frac{d}{q}-s\right)}.\end{aligned}$$

To see this, we use the ideas from the proof of Lemma 6.2.2. in [26]. Fortunately it turns out, that we can copy the proof of Lemma 6.2.2. step by step. Only minor modifications have to be made. After some steps of calculation the variable h disappears. So we can calculate the integral $\int_{B(0,N2^{-n_k+j+d})} 1dh$ exactly. A little later because of $2^{(n_k-j)d\frac{q}{v}}2^{(-n_k+j)d\frac{q}{v}} = 1$ also the variable v disappears. So we are exactly in the setting of the proof from Lemma 6.2.2. and can proceed like there. This finishes Substep 3.2.

Substep 3.3. Next we prove, that for large $W \in \mathbb{N}$, there is a constant $C > 0$, that only depends on q, s, N and d, such that

$$\Big(\int_\Omega \int_{[\frac{1}{4},\frac{3}{4}]^d} \sum_{k=1}^{2^{Wd}} \sum_{j=d+M}^{W-M-d} 2^{(n_k-j)(sq+d\frac{q}{v})} \Big(\int_{B(0,N2^{-n_k+j+d})} \Big|\Gamma_{k,N}^k(x,h,\omega)\Big|^v dh\Big)^{\frac{q}{v}} dx\, d\mu\Big)^{\frac{1}{q}}$$
$$\geq C \max\left(2^{W(-s-\frac{d}{v}+\frac{d}{q})}, W^{\frac{1}{q}}\right) - C.$$

To see this, we use the ideas from the proof of Lemma 6.2.3. in [26]. We fix $x \in \left[\frac{1}{4},\frac{3}{4}\right]^d$ and $1 \leq k \leq 2^{Wd}$. Let $V_k^W(x)$ be the union of all dyadic cubes with side-length 2^{-n_k+W+1}, whose boundaries intersect the boundary of the closure of the unique dyadic cube of side-length 2^{-n_k+W+1}, that contains x. Let $\mathscr{V}_k^W(x)$ be the set of all cubes $Q \in \mathscr{Q}(n_k)$, that are contained in the closure of $V_k^W(x)$. By $\Omega(k,x,Q)$ we denote the event, that $\theta_Q(\omega) = 1$, but $\theta_{Q'}(\omega) = 0$ for all $Q' \in \mathscr{V}_k^W(x) \setminus \{Q\}$. Let $\mathscr{W}(k,j,x)$ be the set of all cubes $Q \in \mathscr{Q}(n_k)$, for which we have $2^{-n_k+j} \leq \operatorname{dist}(x,Q) \leq 2^{-n_k+j+1}$. For $Q \in \mathscr{W}(k,j,x)$ by y_Q we denote the center of this cube. Moreover, we put $h_{Q,x} = y_Q - x$. Then we have $|h_{Q,x}| \leq c_1 2^{-n_k+j+1}$. Furthermore, for a suitable $c_2 < 1$ we define

$$H_{Q,x,k,j} = B(0,N2^{-n_k+j+d}) \cap \{h \in \mathbb{R}^d \,:\, |h - h_{Q,x}| \leq c_2 2^{-n_k}\}.$$

Now for $Q \in \mathscr{W}(k,j,x)$, $\omega \in \Omega(k,x,Q)$ and $h \in H_{Q,x,k,j}$ we define

$$I_v^\omega = I_v^\omega(k,x,Q,h) = \int_{\mathbb{R}^d} \phi_k(y)\eta_k * \chi_Q(x + \frac{v}{N}(h-y))dy \tag{6.28}$$

and

$$II^\omega = II^\omega(k,x,Q,h) = \sum_{Q' \in \mathscr{Q}(n_k), Q' \notin \mathscr{V}_k^W(x)} \theta_{Q'}(\omega) \int_{\mathbb{R}^d} \phi_k(y)\tilde{\Delta}^N_{\frac{h-y}{N}} \eta_k * \chi_{Q'}(x)dy. \tag{6.29}$$

Moreover, we write

$$III^\omega = III^\omega(k,x,Q,h) = \sum_{v=1}^{N-1} (-1)^{N-v} \binom{N}{v} I_v^\omega(k,x,Q,h) + II^\omega(k,x,Q,h).$$

When we use these abbreviations, we obtain

$$2^{n_k s}\Gamma_{k,N}^k(x,h,\omega) = I_N^\omega(k,x,Q,h) + III^\omega(k,x,Q,h).$$

So we get

$$\Big(\int_\Omega \int_{[\frac14,\frac34]^d} \sum_{k=1}^{2^{Wd}} \sum_{j=d+M}^{W-M-d} 2^{(n_k-j)(sq+d\frac{q}{v})} \Big(\int_{B(0,N2^{-n_k+j+d})} \Big|\Gamma^k_{k,N}(x,h,\omega)\Big|^v dh\Big)^{\frac{q}{v}} dx\, d\mu\Big)^{\frac1q}$$
$$\geq \Big(\sum_{k,j} 2^{-j(sq+d\frac{q}{v})} 2^{n_k d\frac{q}{v}} \int_{[\frac14,\frac34]^d} \sum_{Q\in\mathscr{W}(k,j,x)} \int_{\Omega(k,x,Q)} \Big(\int_{H_{Q,x,k,j}} \Big|2^{n_k s}\Gamma^k_{k,N}\Big|^v dh\Big)^{\frac{q}{v}} d\mu\, dx\Big)^{\frac1q}$$
$$\geq C_8 \Big(\sum_{k,j} 2^{-j(sq+d\frac{q}{v})} 2^{n_k d\frac{q}{v}} \int_{[\frac14,\frac34]^d} \sum_{Q\in\mathscr{W}(k,j,x)} \int_{\Omega(k,x,Q)} \Big(\int_{H_{Q,x,k,j}} \Big|I^\omega_N\Big|^v dh\Big)^{\frac{q}{v}} d\mu\, dx\Big)^{\frac1q}$$
$$- C_8 \Big(\sum_{k,j} 2^{-j(sq+d\frac{q}{v})} 2^{n_k d\frac{q}{v}} \int_{[\frac14,\frac34]^d} \sum_{Q\in\mathscr{W}(k,j,x)} \int_{\Omega(k,x,Q)} \Big(\int_{H_{Q,x,k,j}} \Big|III^\omega\Big|^v dh\Big)^{\frac{q}{v}} d\mu\, dx\Big)^{\frac1q}.$$

Next we will prove a lower bound for the first term. For $\omega \in \Omega(k,x,Q)$ and $h \in H_{Q,x,k,j}$ we get

$$I^\omega_N(k,x,Q,h) = \int_{\mathbb{R}^d} r_k^d (\phi * \eta)(r_k(x+h-z))\chi_Q(z)dz.$$

Furthermore, for $z \in Q$ we have

$$|r_k(x+h-z)| \leq 2^{n_k-M}(|x+h-y_Q| + |y_Q - z|) \leq (\sqrt{d}+1)2^{-M}.$$

Because of this we can apply (6.26) and obtain $|I^\omega_N(k,x,Q,h)| \geq C'(M) > 0$. Consequently we find

$$\Big(\sum_{k,j} 2^{-j(sq+d\frac{q}{v})} 2^{n_k d\frac{q}{v}} \int_{[\frac14,\frac34]^d} \sum_{Q\in\mathscr{W}(k,j,x)} \int_{\Omega(k,x,Q)} \Big(\int_{H_{Q,x,k,j}} \Big|I^\omega_N(k,x,Q,h)\Big|^v dh\Big)^{\frac{q}{v}} d\mu\, dx\Big)^{\frac1q}$$
$$\geq C_9 \Big(\sum_{k=1}^{2^{Wd}} \sum_{j=d+M}^{W-M-d} 2^{-j(sq+d\frac{q}{v})} \int_{[\frac14,\frac34]^d} \sum_{Q\in\mathscr{W}(k,j,x)} \int_{\Omega(k,x,Q)} 1\, d\mu\, dx\Big)^{\frac1q}$$
$$\geq C_{10} \max\Big(2^{W(-s-\frac{d}{v}+\frac{d}{q})}, W^{\frac1q}\Big).$$

In the last step we used the calculations of Christ and Seeger, see formula (6.43) in the proof of Lemma 6.2.3. from [26]. Now it remains to show, that there is a $C > 0$, that only depends on q, s, N and d, such that

$$\Big(\sum_{k,j} 2^{-j(sq+d\frac{q}{v})} 2^{n_k d\frac{q}{v}} \int_{[\frac14,\frac34]^d} \sum_{Q\in\mathscr{W}(k,j,x)} \int_{\Omega(k,x,Q)} \Big(\int_{H_{Q,x,k,j}} \Big|III^\omega\Big|^v dh\Big)^{\frac{q}{v}} d\mu\, dx\Big)^{\frac1q} < C.$$

To prove this, on the one hand for $1 \leq \nu \leq N-1$ we have to deal with $I^\omega_\nu(k,x,Q,h)$, see formula (6.28). Notice, that for $\omega \in \Omega(k,x,Q)$, $Q \in \mathscr{W}(k,j,x)$, $y \in \operatorname{supp}\phi_k$ and $h \in H_{Q,x,k,j}$ we observe

$$\Big|x + \frac{\nu}{N}(h-y) - y_Q\Big| \geq \Big|x + \frac{\nu}{N}(h_{Q,x}-y) - y_Q\Big| - \Big|\frac{\nu}{N}(h - h_{Q,x})\Big| \geq C2^{-n_k}(2^j - 1).$$

Here we have $j \geq d+M$. Because of the support properties of the involved functions for every $\rho > 0$ there is a $C_\rho > 0$, such that we get $|\eta_k * \chi_Q(x + \frac{\nu}{N}(h-y))| \leq C_\rho 2^{-j\rho}$. So like in the proof of Lemma 6.2.3. from [26] we find $|I^\omega_\nu(k,x,Q,h)| \leq C_{M,\rho}\mathfrak{M}_{\rho,2^{-n_k}}[h^\omega_k]$, whereby we use the abbreviation

$$\mathfrak{M}_{\rho,2^{-n_k}}[h^\omega_k] = \sup_{y\in\mathbb{R}^d} \frac{\sum_{Q\in\mathscr{Q}(n_k)} \theta_Q(\omega)\chi_Q(x+y)}{(1+2^{-n_k}|y|)^\rho}.$$

With similar arguments for the terms $II^{\omega}(k,x,Q,h)$ we can prove $|II^{\omega}(k,x,Q,h)| \le C_{M,\rho}\mathfrak{M}_{\rho,2^{-n_k}}[h_k^{\omega}]$, whereby we have $\omega \in \Omega(k,x,Q)$ and $h \in H_{Q,x,k,j}$. An explanation for that can be found on page 10 in [26]. Now we obtain

$$\Big(\sum_{k,j} 2^{-j(sq+d\frac{q}{v})} 2^{n_k d\frac{q}{v}} \int_{[\frac{1}{4},\frac{3}{4}]^d} \sum_{Q\in\mathscr{W}(k,j,x)} \int_{\Omega(k,x,Q)} \Big(\int_{H_{Q,x,k,j}} \Big|III^{\omega}(k,x,Q,h)\Big|^v dh\Big)^{\frac{q}{v}} d\mu\, dx\Big)^{\frac{1}{q}}$$
$$\le C_{11}\Big(\sum_{k=1}^{2^{Wd}} \sum_{j=d+M}^{W-M-d} 2^{-j(sq+d\frac{q}{v})} \int_{\mathbb{R}^d}\int_{\Omega} \mathfrak{M}_{\rho,2^{-n_k}}[h_k^{\omega}]^q d\mu\, dx\Big)^{\frac{1}{q}} < C_{12}.$$

Here the last estimate is a consequence of formula (6.44) on page 23 in [26]. So Substep 3.3 is complete.

Substep 3.4. Now we are able to finish Step 3 of the proof. Until now we know, that there is a constant $C > 0$, that only depends on s,p,q and d, such that

$$\Big(\int_{\Omega} \|\mathfrak{D}^{s,q}_{v,N} G^{\omega,\alpha}|L_p(\mathbb{R}^d)\|^p d\mu(\omega)\Big)^{\frac{1}{p}} \ge C\max\Big(2^{W(-s-\frac{d}{v}+\frac{d}{q})}, W^{\frac{1}{q}}\Big) - C2^{-R}2^{W(\frac{d}{q}-s)} - C.$$

If R is chosen very large, the term in the middle of the right hand side will vanish. So for large W only the first term is important. Because of this, the proof of formula (6.27) is complete.

Step 4. To finish the proof, we combine the results from Step 2 and Step 3. As it is described on page 3 in [26], we have to use an application of the Banach-Steinhaus Theorem (see the Theorems 2.5 and 2.6 from [98]) to complete the proof. We choose $w = 2^{R2^{Wd}}$ and obtain the stated lower bounds. ■

6.3.2 The Conditions $s > d(1/q - 1)$ and $s > d(1/q - 1/v)$

Now we are able to prove some very strong results concerning the necessity of the conditions, which can be found in (6.25). For that purpose we will use the techniques from Christ and Seeger, see the Propositions 12 and 13. So for the Triebel-Lizorkin-Morrey spaces we observe the following.

Proposition 14. ***The Necessity of*** $s > d(\frac{1}{q} - \frac{1}{v})$**.**
Let $s > 0$, $0 < q \le p \le u < \infty$, $0 < q \le v < \infty$, $1 \le a \le \infty$ *and* $N \in \mathbb{N}$ *with* $N > s$. *Let*

$$s \le d\Big(\frac{1}{q} - \frac{1}{v}\Big).$$

Then we have $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.

Proof. *Step 1.* To prove this result, we use the same random function and the same notation as in the proof of Proposition 13. At first we show, that there is a constant $C > 0$, that only depends on p,u,q and d, such that

$$\Big(\int_{\Omega} \|G^{\omega,\alpha}|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^u d\mu(\omega)\Big)^{\frac{1}{u}} \le C.$$

To see this, we use Lemma 30. Then because of the embedding $F^s_{u,q}(\mathbb{R}^d) = \mathscr{E}^s_{u,u,q}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ we can complete Step 1.

Step 2. Next we prove, that for large $W \in \mathbb{N}$, there is a $C > 0$, such that

$$\Big(\int_\Omega (\|G^{\omega,\alpha}|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)})^u d\mu(\omega)\Big)^{\frac{1}{u}} \geq C \max\Big(2^{W(-s-\frac{d}{v}+\frac{d}{q})}, W^{\frac{1}{q}}\Big).$$

To show this, we are only interested in $x \in [\frac{1}{4},\frac{3}{4}]^d$. Therefore we can get rid of the Morrey quasi-norm, when we choose for the ball $B(y,r)$ a small ball, that covers the cube $[\frac{1}{4},\frac{3}{4}]^d$. Moreover, since $\frac{u}{p} \geq 1$, we can apply the reverse Hölder inequality and obtain

$$\begin{aligned}&\Big(\int_\Omega (\|G^{\omega,\alpha}|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)})^u d\mu(\omega)\Big)^{\frac{1}{u}}\\ &\quad\geq C_1\Big(\int_\Omega\int_{[\frac{1}{4},\frac{3}{4}]^d}\Big(\int_0^1 t^{-sq-d\frac{q}{v}-1}\Big(\int_{B(0,t)}|\Delta_h^N G^{\omega,\alpha}(x)|^v dh\Big)^{\frac{q}{v}}dt\Big)^{\frac{p}{q}}dx\,d\mu\Big)^{\frac{1}{p}}.\end{aligned}$$

But now we are in the same situation as at the beginning of Step 3 of the proof from Proposition 13. So we can use the techniques, that are described there, to get the desired result.
Step 3. To finish the proof, we have to combine the results from Step 1 and Step 2. We can argue like it is described in Step 4 of the proof from Proposition 13. The proof is complete. ■

As a byproduct of Proposition 14 we obtain an improved version of Proposition 10 for the special case $p = q$. So we get the following result.

Corollary 4. ***The Necessity of*** $s > d(\frac{1}{p}-\frac{1}{v})$ ***for*** $p = q$***. Part I.***
Let $s > 0$, $0 < p \leq u < \infty$, $0 < p \leq v < \infty$, $1 \leq a \leq \infty$ *and* $N \in \mathbb{N}$ *with* $N > s$. *Let*

$$s \leq d\Big(\frac{1}{p}-\frac{1}{v}\Big).$$

Then we have $\mathscr{E}^s_{u,p,p}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,a}_{u,p,p,v}(\mathbb{R}^d)$.

Proof. To prove this, it is enough to use Proposition 14 with $p = q$. ■

Notice, that for $p = q$ Corollary 4 allows us to close the gap, we described in (6.23). So under the conditions given in Corollary 4, it is not possible to describe the spaces $\mathscr{E}^s_{u,p,p}(\mathbb{R}^d)$ in terms of the quasi-norm $\|\cdot|\mathscr{E}^s_{u,p,p}(\mathbb{R}^d)\|^{(vma)}$. Furthermore, when we assume $v = 1$, we also can use Corollary 4, to fill in the gap, we found in (6.18) at least for the Triebel-Lizorkin-Morrey spaces. On the other hand we also can apply the techniques developed by Christ and Seeger, to learn something more about the Besov-type spaces for $p = q$. There is the following result.

Proposition 15. ***The Necessity of*** $s > d(\frac{1}{p}-\frac{1}{v})$ ***for*** $p = q$***. Part II.***
Let $s > 0$, $0 < q = p \leq v < \infty$, $0 \leq \tau < \frac{1}{p}$ *and* $1 \leq a \leq \infty$. *Let* $N \in \mathbb{N}$ *with* $N > s$. *Furthermore we have*

$$s \leq d\Big(\frac{1}{p}-\frac{1}{v}\Big).$$

Then $B^{s,\tau}_{p,p}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,a}_{p,p,v}(\mathbb{R}^d)$.

Proof. For the proof we will use the same notation as in Proposition 13.
Step 1. Let $u = \frac{1}{\frac{1}{p}-\tau}$. We will prove, that there is a constant $C_1 > 0$ independent of R and W, such that

$$\Big(\int_\Omega \|G^{\omega,\alpha}|B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^u d\mu(\omega)\Big)^{\frac{1}{u}} < C_1.$$

We can use Lemma 30. Since $u = \frac{1}{\frac{1}{p}-\tau}$, we have $\tau = \frac{1}{p} - \frac{1}{u}$ and $p \leq u$. By Lemma 13 we obtain

$$F^s_{u,p}(\mathbb{R}^d) = \mathscr{E}^s_{u,u,p}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^s_{u,p,p}(\mathbb{R}^d) = B^{s,\tau}_{p,p}(\mathbb{R}^d).$$

Thus we get

$$\Big(\int_\Omega \| G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^u d\mu(\omega) \Big)^{\frac{1}{u}} \leq C_1 \Big(\int_\Omega \| G^{\omega,\alpha} | F^{s}_{u,p}(\mathbb{R}^d)\|^u d\mu(\omega) \Big)^{\frac{1}{u}} \leq C_2.$$

Step 2. Next we prove, that for large $W \in \mathbb{N}$, there is a $C > 0$, such that

$$\Big(\int_\Omega (\| G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^{(vma)})^u d\mu(\omega) \Big)^{\frac{1}{u}} \geq C \max\Big(2^{W(-s-\frac{d}{v}+\frac{d}{p})}, W^{\frac{1}{p}} \Big).$$

To see this, at first we replace the supremum with respect to all dyadic cubes by choosing the specific dyadic cube $P^* = [0,1)^d$. Then we find

$$\begin{aligned}
&\Big(\int_\Omega (\| G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^{(vma)})^u d\mu(\omega) \Big)^{\frac{1}{u}} \\
&\quad \geq \Big(\int_\Omega \Big(\frac{1}{|P^*|^\tau} \Big(\int_0^1 t^{-sp} \int_{P^*} \Big(t^{-d} \int_{B(0,t)} |\Delta^N_h G^{\omega,\alpha}(x)|^v dh \Big)^{\frac{p}{v}} dx \frac{dt}{t} \Big)^{\frac{1}{p}} \Big)^u d\mu(\omega) \Big)^{\frac{1}{u}} \\
&\quad \geq \Big(\int_\Omega \Big(\int_{[\frac{1}{4},\frac{3}{4}]^d} \int_0^1 t^{-sp} \Big(t^{-d} \int_{B(0,t)} |\Delta^N_h G^{\omega,\alpha}(x)|^v dh \Big)^{\frac{p}{v}} \frac{dt}{t} dx \Big)^{\frac{u}{p}} d\mu(\omega) \Big)^{\frac{1}{u}}.
\end{aligned}$$

Next since $\frac{u}{p} \geq 1$, we can apply Hölder's inequality and obtain

$$\begin{aligned}
&\Big(\int_\Omega (\| G^{\omega,\alpha} | B^{s,\tau}_{p,p}(\mathbb{R}^d)\|^{(vma)})^u d\mu(\omega) \Big)^{\frac{1}{u}} \\
&\quad \geq C_2 \Big(\int_\Omega \int_{[\frac{1}{4},\frac{3}{4}]^d} \int_0^1 t^{-sp} \Big(t^{-d} \int_{B(0,t)} |\Delta^N_h G^{\omega,\alpha}(x)|^v dh \Big)^{\frac{p}{v}} \frac{dt}{t} \, dx \, d\mu(\omega) \Big)^{\frac{1}{p}}.
\end{aligned}$$

But now we are in the same situation as at the beginning of Step 3 of the proof from Proposition 13. So we can use the techniques, that are described there, to get the desired result.
Step 3. To complete the proof, we have to combine the results from Step 1 and Step 2. This can be done in the same way as described before. ■

Notice, that for the special case $p = q$ Proposition 15 is an essential improvement of Proposition 11. So it helps us to learn something about the gap we described in (6.24). Moreover, with $v = 1$ we can use Proposition 15, to improve Proposition 9.

6.4 The Necessity of $s < N$

Below we will confirm the necessity of the condition $s < N$, that can be found in the Theorems 5, 7 and 9. With other words we want to explain, why it is not possible to describe the Smoothness Morrey spaces via differences, if the order of the difference is lower than the smoothness parameter s. For that purpose we will investigate the properties of some test functions. So for the Triebel-Lizorkin-Morrey spaces and the Besov-Morrey spaces we can observe the following.

Proposition 16. ***Differences and*** $N \le s$. ***Part I.***
Let $0 < p \le u < \infty$, $s \ge 0$ *and* $0 < q \le \infty$. *Let* $0 < v \le \infty$, $1 \le a \le \infty$ *and* $N \in \mathbb{N}$.

(a) *Then for the Triebel-Lizorkin-Morrey spaces we have* $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$, *if we are in one of the following cases.*

 (i) *We have* $N < s$ *and* $0 < q \le \infty$.

 (ii) *We have* $N = s$ *and* $0 < q < \infty$.

(b) *For the Besov-Morrey spaces we have* $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$, *if we are in one of the following cases.*

 (i) *We have* $N < s$ *and* $0 < q \le \infty$.

 (ii) *We have* $N = s$ *and* $0 < q < \infty$.

Proof. *Step 1.* At first we prove the result for the Triebel-Lizorkin-Morrey spaces. Thereto we work with a function $f \in C_0^\infty(\mathbb{R}^d)$, that has a support in $B(0,3N+3)$. In $B(0,2N+2)$ this function looks like

$$f(x_1,x_2,\dots,x_d) = e^{x_1+x_2+x_3+\dots+x_d}. \tag{6.30}$$

Then of course we find $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. In what follows we want to prove, that we have $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)} = \infty$. Let $0 < \varepsilon < 1$. We define

$$H^d_+ = \{h = (h_1,h_2,\dots,h_d) \in \mathbb{R}^d \ : \ h_1 \ge 0, h_2 \ge 0, \dots, h_d \ge 0\}.$$

Using this notation, we get

$$\begin{aligned}
&\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)} \\
&\qquad \ge C_1\Big(\int_{B(0,1)}\Big(\int_\varepsilon^1 t^{-sq}\Big(t^{-d}\int_{(B(0,t)\setminus B(0,\frac{t}{2}))\cap H^d_+}|\Delta^N_h f(x)|^v dh\Big)^{\frac{q}{v}}\frac{dt}{t}\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

We have $h \in (B(0,t)\setminus B(0,\frac{t}{2}))\cap H^d_+$ and so $|h| \ge \frac{t}{2} \ge \frac{\varepsilon}{2} > 0$. Therefore because of the Mean Value Theorem in several variables there exists a $\zeta \in \mathbb{R}^d$ on the line that connects x and $x+h$, such that for $h = (h_1,h_2,\dots,h_d)$ we obtain

$$\begin{aligned}
|\Delta^N_h f(x)| &= |\Delta^{N-1}_h f(x+h) - \Delta^{N-1}_h f(x)| \\
&= \Big|\frac{\partial \Delta^{N-1}_h f}{\partial y_1}(\zeta)h_1 + \frac{\partial \Delta^{N-1}_h f}{\partial y_2}(\zeta)h_2 + \dots + \frac{\partial \Delta^{N-1}_h f}{\partial y_d}(\zeta)h_d\Big|.
\end{aligned}$$

Now let $k \in \{1,2,\dots,d\}$. We have $|\zeta + Nh| \le 2+N$ and so $\zeta + Nh \in B(0,2N+2)$. Then because of the definition of the function f, see (6.30), for $\zeta = (\zeta_1,\zeta_2,\dots,\zeta_d)$ we observe

$$\begin{aligned}
\frac{\partial \Delta^{N-1}_h f}{\partial y_k}(\zeta) &= \frac{\partial}{\partial y_k}\Big(\sum_{l=0}^{N-1}(-1)^{N-1-l}\binom{N-1}{l}e^{y_1+y_2+\dots+y_d+lh_1+lh_2+\dots+lh_d}\Big)(\zeta) \\
&= \sum_{l=0}^{N-1}(-1)^{N-1-l}\binom{N-1}{l}e^{\zeta_1+\zeta_2+\dots+\zeta_d+lh_1+lh_2+\dots+lh_d} \\
&= \Delta^{N-1}_h f(\zeta).
\end{aligned}$$

Next since $h \in H_+^d$ we obtain

$$|\Delta_h^N f(x)| = \left|\Delta_h^{N-1} f(\zeta)h_1 + \Delta_h^{N-1} f(\zeta)h_2 + \ldots + \Delta_h^{N-1} f(\zeta)h_d\right| \geq \left|\Delta_h^{N-1} f(\zeta)\right| \, |h|.$$

By iteration we can find an $\eta \in B(0,N+1)$, such that

$$|\Delta_h^N f(x)| \geq |f(\eta)| \, |h|^N \geq C(N)|h|^N.$$

Because of (6.30) we have $C(N) > 0$. When we use this, we get

$$\begin{aligned} \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)} &\geq C_2 \Big(\int_{B(0,1)} \Big(\int_\varepsilon^1 t^{-sq} \Big(t^{-d} \int_{(B(0,t)\setminus B(0,\frac{t}{2}))\cap H_+^d} |h|^{Nv} dh \Big)^{\frac{q}{v}} \frac{dt}{t} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \\ &\geq C_3 \Big(\int_\varepsilon^1 t^{q(N-s)-1} dt \Big)^{\frac{1}{q}}. \end{aligned}$$

Now in the case $N < s$ we observe

$$\int_\varepsilon^1 t^{q(N-s)-1} dt = \frac{1}{q(N-s)} \left(1 - \varepsilon^{q(N-s)}\right).$$

For $N = s$ we obtain

$$\int_\varepsilon^1 t^{-1} dt = -\ln(\varepsilon).$$

In both cases we observe, that if ε tends to zero, then $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$ tends to infinity. But we know $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. So this step of the proof is complete.
Step 2. Next we have to prove the result for the Besov-Morrey spaces. For that purpose we can use the same test function and the same methods as we described in Step 1. Only some obvious modifications must be done. We omit the details. ■

In the formulation of Proposition 16 the special case $N = s$ with $q = \infty$ is excluded. Here the proof of necessity becomes much more technical. Below we want to present a method, that also covers this special case, at least for the original Besov spaces $B^s_{p,q}(\mathbb{R}^d)$. The following result was essentially proved by Oswald, see [91].

Proposition 17. ***Differences for*** $s = N$ ***and*** $q = \infty$.
Let $0 < p = u < \infty$, $s = N \in \mathbb{N}$ *and* $q = \infty$. *Let* $1 \leq v \leq \infty$ *and* $1 \leq a \leq \infty$. *Then we have* $\mathscr{N}^N_{p,p,\infty}(\mathbb{R}^d) \neq \mathbf{N}^{N,N,a}_{p,p,\infty,v}(\mathbb{R}^d)$.

Proof. For the proof we will use some ideas from Oswald, see [91]. Although we assume $u = p$, we will keep both numbers different in notation, to point out, why the assumption $u = p$ is needed. We fix $r \in \mathbb{N}$ with $r > 4$, such that $2^{r+1} \geq N + 4$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a function with $\operatorname{supp}\phi \subset B(0,1) \cap [0,1)^d$, such that $\phi(\cdot - 32^{r-2}(1,1,\ldots,1)^T)$ fulfills moment conditions up to order $L \in \mathbb{N}_0 \cup \{-1\}$ with $L \geq \max(-1, \sigma_p - N)$. Moreover, there is a set $D \subset \operatorname{supp}\phi$ with $|D| > \frac{|\operatorname{supp}\phi|}{2}$ on that for all $x \in D$ and $|\gamma| \leq N$ we have $|D^\gamma \phi(x)| > C > 0$. There is a set $\tilde{D} \subset D$, such that for all $x \in \partial\tilde{D}$ we have $2^{-10} > \operatorname{dist}(x, \partial D) > 2^{-20}$. For $k \in \mathbb{N}$ we put $n_k = r(k-1) + 2$ and $x(r) = 32^{r-2}(1,1,\ldots,1)^T$. Moreover, we put $a_k = 2^{n_k(\frac{d}{u} - N)}$. We define the function

$$f(x) = \sum_{k=1}^\infty a_k \phi(2^{n_k} x - x(r)). \tag{6.31}$$

Then for $k \in \mathbb{N}$ we have $\operatorname{supp} \phi(2^{n_k} \cdot -x(r)) \subset B(2^{-n_k}x(r), 2^{-n_k}) \cap Q_{n_k,x(r)}$. Because of this we find $\operatorname{supp} f \subset B(0, 4\sqrt{d} \cdot 32^{r-2} + 4)$. For $k,t \in \mathbb{N}$ with $k \neq t$ we observe

$$\operatorname{supp} \phi(2^{n_k} \cdot -x(r)) \cap \operatorname{supp} \phi(2^{n_t} \cdot -x(r)) = \emptyset. \tag{6.32}$$

Moreover, when we fix a large $l \in \mathbb{N}$ and $h \in \mathbb{R}^d$ with $|h| \leq 2^{-n_{l+1}}2^{-rl}$, for $k,t \in \mathbb{N}$ with $0 < k < t < l-4$ we have

$$\operatorname{supp} \phi(2^{n_k} \cdot + N2^{n_k}h - x(r)) \cap \operatorname{supp} \phi(2^{n_t} \cdot + N2^{n_t}h - x(r)) = \emptyset. \tag{6.33}$$

Now we want to prove $\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\| < \infty$. For that purpose we will use Lemma 11. We already know, that for $k \in \mathbb{N}$ we have $\operatorname{supp} \phi(2^{n_k} \cdot -x(r)) \subset Q_{n_k,x(r)}$. For $|\alpha| \leq K \in \mathbb{N}$ with $K > N+1$ since $\phi \in C_0^\infty(\mathbb{R}^d)$ we find $\|D^\alpha \phi(2^{n_k} \cdot -x(r))|L_\infty(\mathbb{R}^d)\| \leq C_1 2^{n_k|\alpha|}$. For $|\beta| \leq L \in \mathbb{N}_0 \cup \{-1\}$ with $L \geq \max(-1, \sigma_p - N)$ we observe

$$\int_{\mathbb{R}^d} x^\beta \phi(2^{n_k}x - x(r))dx = 2^{-n_k|\beta|}2^{-n_k d} \int_{\mathbb{R}^d} x^\beta \phi(x - x(r))dx = 0.$$

Hence Lemma 11 can be applied and we obtain

$$\begin{aligned}
\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\| &= \Big\| \sum_{k=1}^{\infty} a_k \phi(2^{n_k} \cdot -x(r)) \Big| \mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d) \Big\| \\
&\leq C_1 \sup_{k \in \mathbb{N}} 2^{n_k(N-\frac{d}{u})} \Big\| |a_k| \chi^{(u)}_{n_k,x(r)} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\| \\
&= C_1 \sup_{k \in \mathbb{N}} \Big\| \chi^{(u)}_{n_k,x(r)} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|.
\end{aligned}$$

Now we use $\|\chi^{(u)}_{n_k,x(r)}|\mathcal{M}^u_p(\mathbb{R}^d)\| = 1$, see the remark after Definition 2.9 in [48]. We get

$$\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\| \leq C_1 < \infty.$$

Next we will prove, that $\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\|^{(vma)} = \infty$. To show this, at first we fix a large number $l \in \mathbb{N}$ with $l > 10$. Then because of the disjoint supports of the involved functions we obtain

$$\begin{aligned}
&\Big\| \Big(\int_{B(0,t)} |\Delta^N_h f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^p \\
&\qquad \geq \Big\| \Big(\int_{|h| \leq \min(t, 2^{-n_{l+1}}2^{-rl})} \Big(\sum_{k=1}^{l-6} a_k |\Delta^N_h(\phi(2^{n_k} \cdot -x(r)))(x)| \Big)^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^p.
\end{aligned}$$

Now we use, that for fixed h with $|h| \leq \min(t, 2^{-n_{l+1}}2^{-rl}) = t(r,l)$ and $k \in \mathbb{N}$ we have $\operatorname{supp} \Delta^N_h(\phi(2^{n_k} \cdot -x(r))) \subset B(0, \sqrt{d}32^{r-2}(4+N)+4)$. Therefore instead of the supremum of the Morrey quasi-norm we can choose the ball $B(0, \sqrt{d}32^{r-2}(4+N)+4)$. Then since $v \geq 1$ we find

$$\begin{aligned}
&\Big\| \Big(\int_{B(0,t)} |\Delta^N_h f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^p \\
&\qquad \geq C_1 \Big\| \Big(\int_{|h| \leq t(r,l)} \Big(\sum_{k=1}^{l-6} a_k |\Delta^N_h(\phi(2^{n_k} \cdot -x(r)))(x)| \Big)^v dh \Big)^{\frac{1}{v}} \Big| L_p(\mathbb{R}^d) \Big\|^p \\
&\qquad \geq C_2 t(r,l)^{dp(\frac{1}{v}-1)} \Big\| \int_{|h| \leq t(r,l)} \sum_{k=1}^{l-6} a_k |\Delta^N_h(\phi(2^{n_k} \cdot -x(r)))(x)| dh \Big| L_p(\mathbb{R}^d) \Big\|^p.
\end{aligned}$$

Next we observe

$$\Big\| \Big(\int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|^p \\ \geq C_2 t(r,l)^{dp(\frac{1}{v}-1)} \sum_{k=1}^{l-6} a_k^p 2^{-n_k d} \Big\| \int_{|h| \leq t(r,l)} |\Delta_{2^{n_k} h}^N (\phi(\cdot))(x)| dh \Big| L_p(\mathbb{R}^d) \Big\|^p .$$

Let $\eta = \frac{h}{|h|}$ and $\theta \in (0,1)$. Because of the properties of the sets D and $\tilde{D}$ we obtain

$$\Big\| \Big(\int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|^p \\ \geq C_2 t(r,l)^{dp(\frac{1}{v}-1)} \sum_{k=1}^{l-6} a_k^p 2^{-n_k d} 2^{n_k N p} \Big\| \int_{|h| \leq \frac{t(r,l)}{N}} \Big| \frac{\partial^N \phi}{\partial \eta^N} (x + \theta N 2^{n_k} h) \Big| |h|^N dh \Big| L_p(\tilde{D}) \Big\|^p \\ \geq C_3 t(r,l)^{dp(\frac{1}{v}-1)} \sum_{k=1}^{l-6} a_k^p 2^{-n_k d} 2^{n_k N p} \Big\| \int_{|h| \leq \frac{t(r,l)}{N}} |h|^N dh \Big| L_p(\tilde{D}) \Big\|^p .$$

When we use $a_k = 2^{n_k(\frac{d}{u}-N)}$, we get

$$\Big\| \Big(\int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|^p \geq C_4 \sum_{k=1}^{l-6} 2^{n_k p \frac{d}{u}} 2^{-n_k d} t(r,l)^{Np + \frac{dp}{v}} .$$

Now since $r > 4$ and $l > 10$ we have $2^{-n_{l+1}} 2^{-rl} < 1$. In the special case $p = u$ this leads to

$$\Big(\|f | \mathscr{N}_{u,p,\infty}^N(\mathbb{R}^d)\|^{(vma)} \Big)^p \\ \geq C_5 \sum_{k=1}^{l-6} 2^{n_k p \frac{d}{u}} 2^{-n_k d} \sup_{t \in [0,1]} t^{-Np - \frac{dp}{v}} \min(t, 2^{-n_{l+1}} 2^{-rl})^{Np + \frac{dp}{v}} \\ \geq C_5 \sum_{k=1}^{l-6} 2^{n_k p \frac{d}{u}} 2^{-n_k d} 2^{(-n_{l+1} - rl)(-Np - \frac{dp}{v})} 2^{(-n_{l+1} - rl)(Np + \frac{dp}{v})} \\ = C_5 \sum_{k=1}^{l-6} 2^{n_k p \frac{d}{u}} 2^{-n_k d} = C_5 \sum_{k=1}^{l-6} 1 = C_5 (l-6).$$

So if l tends to infinity, also $\|f | \mathscr{N}_{u,p,\infty}^N(\mathbb{R}^d)\|^{(vma)}$ tends to infinity. The proof is complete. ■

The method we just presented does not work for the case $p < u$. Here to give a complete proof is even more difficult. So we do not attack this problem at this point. Instead we want to formulate a counterpart of Proposition 16 for the Besov-type spaces $B_{p,q}^{s,\tau}(\mathbb{R}^d)$. There is the following result.

Proposition 18. ***Differences and*** $N \leq s$***. Part II.***
Let $s \geq 0$, $0 < p < \infty$, $0 \leq \tau < \frac{1}{p}$ and $N \in \mathbb{N}$. Let $0 < v \leq \infty$ and $1 \leq a \leq \infty$. In addition we assume, that we are in one of the following two cases.

(i) We have $N < s$ and $0 < q \leq \infty$.

(ii) We have $N = s$ and $0 < q < \infty$.

Then we find $B_{p,q}^{s,\tau}(\mathbb{R}^d) \neq \mathbf{B}_{p,q,v}^{s,\tau,N,a}(\mathbb{R}^d)$.

Proof. This result can be proved in the same way, as it is described in the proof of Proposition 16. We can use the same test function and the same strategy, to obtain the desired result. Therefore we omit the details. We refer to [58]. ■

Again we have to mention, that the technique we used in the proof of Proposition 18, does not work for the special case $N = s$ and $q = \infty$. Here some extra considerations have to be made.

Chapter 7

Smoothness Morrey Spaces and Differences: Optimal Results and open Problems

In this chapter we want to sum up, what we learned so far about characterizations in terms of differences for our Smoothness Morrey spaces. For that purpose we will formulate some compound results, where all sufficient and necessary conditions can be found at the same time. For some special cases we also will be able to present optimal results. Moreover, we want to make the situation a bit more transparent by the help of some $(\frac{1}{p}, s)$ - diagrams. Let us start with the Triebel-Lizorkin-Morrey spaces.

7.1 Compound Results for Triebel-Lizorkin-Morrey Spaces

By now we learned a lot about differences in connection with Triebel-Lizorkin-Morrey spaces. So in chapter 5 we proved, that under some restrictions on the parameters it is possible to describe the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ in terms of v-means of differences, see Theorem 5. Later in chapter 6 we recognized, that some of the conditions we just mentioned are also necessary. In the following compound result we will collect both sufficient and necessary conditions at the same time. That allows us to get a better overview concerning the state of knowledge we have up to now. For that purpose let us use the notation from Definition 31. To make things more transparent, we assume $a = \infty$.

Theorem 10. ***Differences for*** $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$***. Compound Result.***
Let $s \in \mathbb{R}$*,* $0 < p \leq u < \infty$ *and* $0 < q \leq \infty$*. Let* $0 < v < \infty$ *and* $N \in \mathbb{N}$*.*

(a) Let in addition

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1, \frac{1}{p} - \frac{1}{v}, \frac{1}{q} - \frac{1}{v}\right) < s < N.$$

Then we have $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{E}^{s,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$*.*

(b) We have $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$, if we are in one of the following cases.

(i) We have $s \leq 0$.

(ii) We have $s < d\frac{p}{u}(\frac{1}{p} - 1)$ with $0 < p < 1$.

(iii) We have $s < d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$ with $0 < p < v < \infty$.

(iv) We have $s \leq d(\frac{1}{q} - \frac{1}{v})$ with $q \leq p$ and $q < v$.

(v) We have either $N < s$ with $0 < q \leq \infty$ or $N = s$ with $0 < q < \infty$.

Proof. This result is just a combination of Theorem 5 with the Propositions 5, 7, 10, 14 and 16. ■

Theorem 10 shows, that there are some gaps between sufficient and necessary conditions. So for

$$d\frac{p}{u}\Big(\frac{1}{p} - 1\Big) < s \leq d\Big(\frac{1}{p} - 1\Big) \qquad \text{and} \qquad d\frac{p}{u}\Big(\frac{1}{p} - \frac{1}{v}\Big) < s \leq d\Big(\frac{1}{p} - \frac{1}{v}\Big)$$

in many cases we do not know, whether the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ can be described in terms of v-means of differences, see also (6.18) and (6.23). Especially the first of these two gaps is remarkable. It tells us, that there is an area, where the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ do not contain singular distributions, but nevertheless a characterization in terms of differences is not available for us at the moment. In what follows, we try to make the situation a bit more transparent, by the help of an $(\frac{1}{p}, s)$ - diagram. For that purpose we assume $p < q$ for every p. We only look at the case $v = 1$. For all parameter constellations where we have $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$, we color the corresponding area in green. This is sector A. When we have $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$, the related area is red. This is sector B. Sometimes we do not know, whether we have $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$ or $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$. Then we color the corresponding area in yellow and call it C. In the following diagram we assume $u = 1$ if $p < 1$. The influence of the parameter q is hidden.

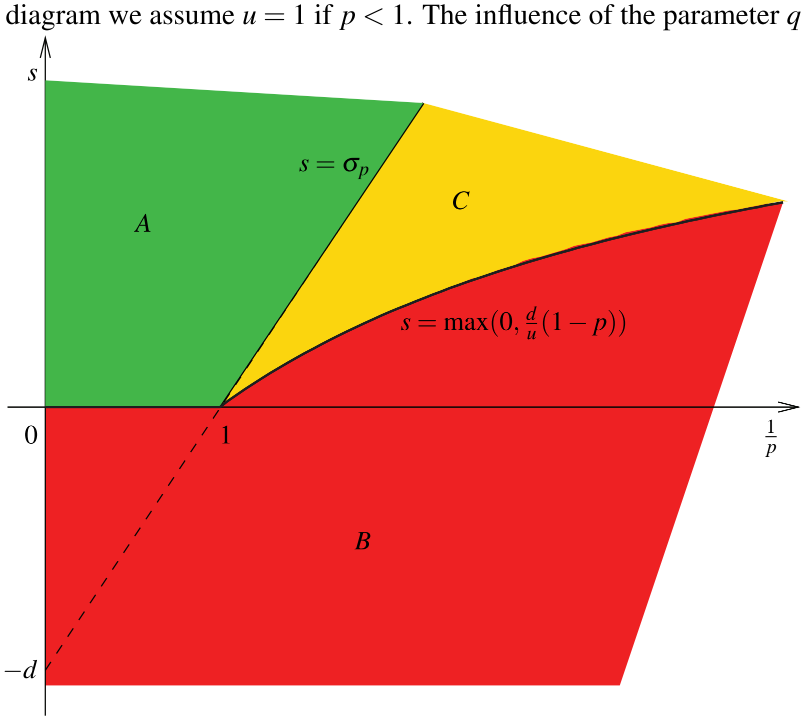

Figure 1. *Ball mean characterization for $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ with $0 < p < u < \infty$ and $p < q \leq \infty$.*

A : $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$

B : $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$

C : open problem

Special attention should be given to the exceptional case $p = q$. Here we were able to prove an optimal result. It reads as follows.

Theorem 11. ***Differences for*** $\mathcal{E}^s_{u,p,p}(\mathbb{R}^d)$***. Optimal Result.***
Let $s \in \mathbb{R}, 0 < p = q \leq u < \infty$ *and* $N \in \mathbb{N}$*. Then* $\mathcal{E}^s_{u,p,p}(\mathbb{R}^d)$ *is the collection of all* $f \in L^{loc}_{\max(p,1)}(\mathbb{R}^d)$*, such that*

$$\|f|\mathcal{M}^u_p(\mathbb{R}^d)\| + \left\| \left(\int_0^\infty t^{-sp} \left(t^{-d} \int_{B(0,t)} |\Delta^N_h f(x)| dh \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \right\|$$

is finite, if and only if

$$d \max\left(0, \frac{1}{p} - 1\right) < s < N.$$

Proof. This result is a special case of Theorem 10 with $q = p$ and $v = 1$. ■

One important advantage of Theorem 11 is, that here the gap we observed in (6.18) disappears. With other words for the spaces $\mathcal{E}^s_{u,p,p}(\mathbb{R}^d)$ we observe the surprising new phenomenon, that there is a large area, where these spaces do not contain singular distributions, but also can not be described in terms of ball means of differences. For the original Triebel-Lizorkin spaces such a problem does not show up. Let us illustrate the result from Theorem 11 by another $(\frac{1}{p}, s)$ - diagram. This time the yellow area does vanish and becomes red (the colors have the same meaning as described before). In the following diagram we assume $p = q$ and $u = 1$ if $p < 1$.

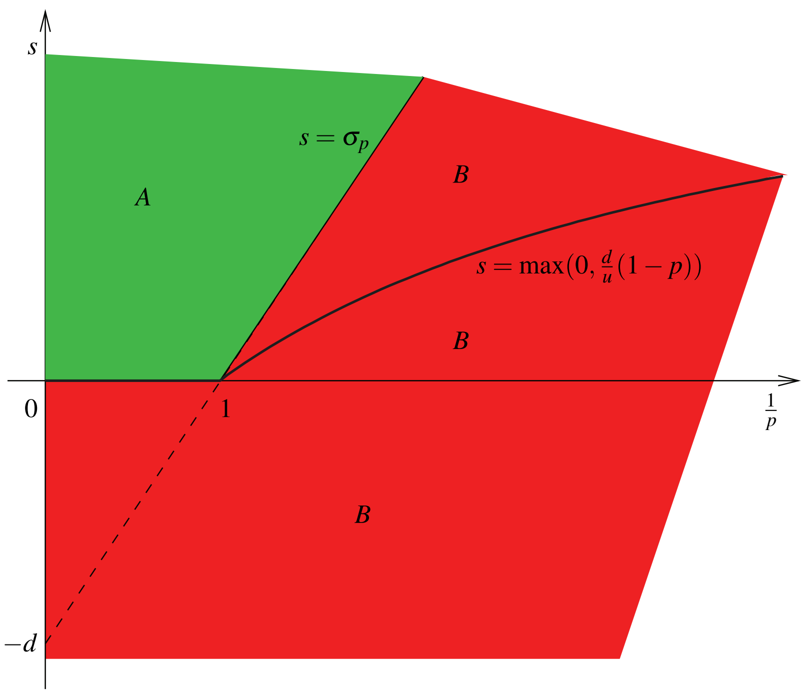

Figure 2. *Ball mean characterization for* $\mathcal{E}^s_{u,p,p}(\mathbb{R}^d)$ *with* $0 < p < u < \infty$.

A : $\mathcal{E}^s_{u,p,p}(\mathbb{R}^d) = \mathbf{E}^{s,N,\infty}_{u,p,p,1}(\mathbb{R}^d)$

B : $\mathcal{E}^s_{u,p,p}(\mathbb{R}^d) \neq \mathbf{E}^{s,N,\infty}_{u,p,p,1}(\mathbb{R}^d)$

At the end of this section we want to present a list of open problems. Here we collect those parameter constellations, that are not covered by Theorem 10. For them at the moment we do not know, whether the Triebel-Lizorkin-Morrey spaces can be described in terms of v-means of differences. So further investigations are necessary.

Open Problem 1. ***Triebel-Lizorkin-Morrey Spaces and Differences.***
Let $s \in \mathbb{R}$, $0 < p \le u < \infty$, $0 < q \le \infty$, $0 < v \le \infty$, $1 \le a \le \infty$ and $N \in \mathbb{N}$ with $N \ge s$. Then at the moment we do not know, whether we have $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ or not, if we are in one of the following situations.

(i) We have $d\frac{p}{u}(\frac{1}{p}-1) \le s \le d(\frac{1}{p}-1)$ with $p \le q < 1$ and $0 < v < 1$.

(ii) We have $\max(0, d(\frac{1}{q}-\frac{1}{v}), d\frac{p}{u}(\frac{1}{p}-1)) \le s \le d(\frac{1}{q}-1)$ with $q \le p < 1$ and $0 < v < 1$.

(iii) We have $d\frac{p}{u}(\frac{1}{p}-\frac{1}{v}) \le s \le d(\frac{1}{p}-\frac{1}{v})$ with $p < q$ and $v > \max(1,p)$.

(iv) We have $s = N$ and $q = \infty$.

7.2 Compound Results for Besov-Morrey Spaces

Now let us turn to the Besov-Morrey spaces. Also for these spaces we obtained characterizations in terms of v-means of differences, see Theorem 7. And we also proved results concerning the necessity of the conditions on the parameters showing up in Theorem 7. In the following theorem, which is a counterpart of Theorem 10, we present both sufficient and necessary conditions simultaneously.

Theorem 12. ***Differences for $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$. Compound Result.***
Let $s \in \mathbb{R}$, $0 < p \le u < \infty$ and $0 < q \le \infty$. Let $0 < v < \infty$ and $N \in \mathbb{N}$.

(a) Let in addition

$$d \max\left(0, \frac{1}{p}-1, \frac{1}{p}-\frac{1}{v}\right) < s < N.$$

Then we have $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{N}^{s,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$.

(b) We have $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$, if we are in one of the following cases.

(i) We have $s \le 0$.

(ii) We have $0 < p < 1$ and

(ii.1) either $s < d\frac{p}{u}(\frac{1}{p}-1)$

(ii.2) or $s = d\frac{p}{u}(\frac{1}{p}-1)$ and $q > 1$.

(iii) We have $s < d\frac{p}{u}(\frac{1}{p}-\frac{1}{v})$ with $0 < p < v < \infty$.

(iv) We have

(iv.1) either $N < s$ and $0 < q \le \infty$

(iv.2) or $N = s$ and $0 < q < \infty$

(iv.3) or $N = s$ with $q = \infty$ and $u = p$ and $v \ge 1$.

Proof. This result is a combination of Theorem 7 with the Propositions 5, 7, 10, 16 and 17. ■

Again there are some gaps between sufficient and necessary conditions. So like in the case of the Triebel-Lizorkin-Morrey spaces, for (6.18) and (6.23) we do not know, whether the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ can be described in terms of v-means of differences. But this time in contrast to the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ for the Besov-Morrey spaces we do not have an optimal result for the special case $p = q$. So there is no counterpart for Theorem 11. The reason for this is, that one of the main tools for the proof of Theorem 11 is the theory of Christ and Seeger, we explained after Proposition 13. The methods from there are specially tailored for the Triebel-Lizorkin spaces and do not work for Besov-Morrey spaces. Let us illustrate the situation in a $(\frac{1}{p}, s)$ - diagram. Therefore we put $v = 1$. For convenience in the following diagram we assume $u = 1$ if $0 < p < 1$. The influence of the parameter q is hidden.

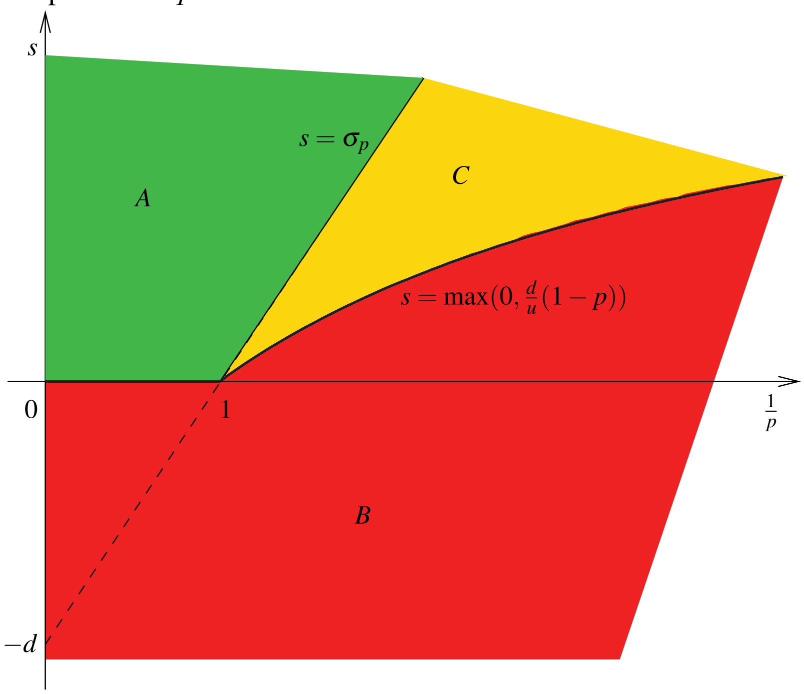

Figure 3. *Ball mean characterization for the Besov-Morrey spaces with $0 < p < u < \infty$.*

A : $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$

B : $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$

C : open problem

Notice, that Figure 3 has many similarities with Figure 1, where we dealt with the Triebel-Lizorkin-Morrey spaces. So in both cases the areas A, B and C are exactly the same. Moreover, for each of these regions we have a similar outcome concerning a characterization in terms of ball means of differences. This observation points out, how close the relationship between the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ is. To complete this section, again we want to collect a list of open problems. So the following parameter constellations are not covered by Theorem 12.

Open Problem 2. ***Besov-Morrey Spaces and Differences.***
Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$, $0 < q \leq \infty$, $0 < v \leq \infty$, $1 \leq a \leq \infty$ and $N \in \mathbb{N}$ with $N \geq s$. Then up to now we do not know, whether we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ or not, if we are in one of the following situations.

(i) We have $s = 0$ with $1 \leq a < \infty$ and either $p \geq 2$ and $q \leq 2$ or $1 \leq p < 2$ and $q \leq p$.

(ii) We have $d\frac{p}{u}(\frac{1}{p} - 1) < s \leq d(\frac{1}{p} - 1)$ with $0 < p < 1$ and $0 < v < 1$.

(iii) We have $d\frac{p}{u}(\frac{1}{p} - \frac{1}{v}) \leq s \leq d(\frac{1}{p} - \frac{1}{v})$ with $v > \max(1, p)$.

(iv) We have $N = s$ with $q = \infty$ and $p < u$.

7.3 Compound Results for Besov-type Spaces

To complete this chapter, let us collect, what we know for the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Also for them we obtained characterizations in terms of v-means of differences, see Theorem 9. In the following compound result one can find sufficient and necessary conditions on the parameters concerning that topic.

Theorem 13. ***Differences for $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Compound Result.***
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 \le \tau < \frac{1}{p}$ and $0 < q \le \infty$. Let $0 < v \le \infty$ and $N \in \mathbb{N}$.

(a) Let in addition

$$d \max\left(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}\right) < s < N.$$

Then we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) = \mathbf{B}^{s,\tau,N,\infty}_{p,q,v}(\mathbb{R}^d)$.

(b) We have $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,\infty}_{p,q,v}(\mathbb{R}^d)$, if we are in one of the following cases.

(i) We have $s \le 0$.

(ii) We have $s < d(\frac{1}{p} - 1) - d\tau(1-p)$ and $0 < p < 1$.

(iii) We have $s < d(\frac{1}{p} - \frac{1}{v}) - d\tau(1 - \frac{p}{v})$ and $\max(p,1) < v < \infty$.

(iv) We have $s \le d(\frac{1}{p} - \frac{1}{v})$ and $q = p \le v < \infty$.

(v) We have either $N < s$ with $0 < q \le \infty$ or $N = s$ with $0 < q < \infty$.

Proof. This result is a combination of Theorem 9 with Corollary 3 and the Propositions 9, 11, 15 and 18. ∎

When we study Theorem 13, we can observe, that there are some gaps between sufficient and necessary conditions. So for

$$d\left(\frac{1}{p} - 1\right) - d\tau(1-p) < s \le d\left(\frac{1}{p} - 1\right) \qquad \text{and} \qquad d\left(\frac{1}{p} - \frac{1}{v}\right) - d\tau\left(1 - \frac{p}{v}\right) \le s \le d\left(\frac{1}{p} - \frac{1}{v}\right)$$

in many cases we do not know, whether the spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can be described by v-means of differences, see also (6.19) or (6.24). Let us illustrate the situation by an $(\frac{1}{p}, s)$ - diagram. For that purpose at first we look at the case $p \neq q$. We put $v = 1$. Moreover, we assume $\tau = \frac{1}{p} - 1$ for every p if $p < 1$. Then in the following diagram in area A a characterization for $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ in terms of differences is known. For B we can prove, that such a characterization can not exist. For zone C we do not know, whether a characterization by differences is possible. Notice, that in both areas A and C the Besov-type spaces do not contain singular distributions. In some sense that observations are similar to those, we made for the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$. Consequently the following diagram also can be seen as a counterpart of Figure 1 and Figure 3.

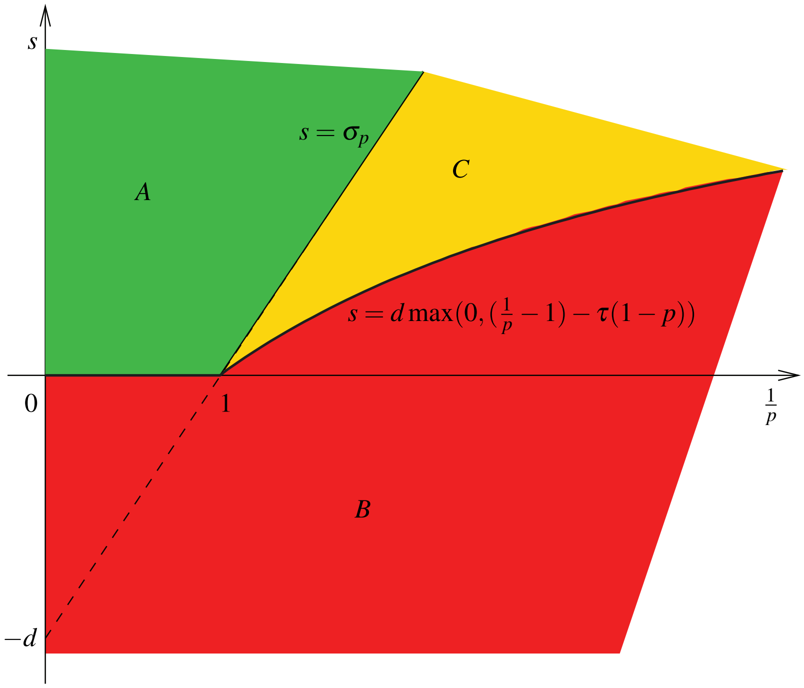

Figure 4. *Characterization in terms of differences for $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $q \neq p$ and $\tau = \frac{1}{p} - 1$ if $p < 1$.*

A: $B^{s,\tau}_{p,q}(\mathbb{R}^d) = \mathbf{B}^{s,\tau,N,\infty}_{p,q,1}(\mathbb{R}^d)$

B: $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,\infty}_{p,q,1}(\mathbb{R}^d)$

C : open problem

For the special case $p = q$ the gap we called area C in Figure 4 disappears. Here we can formulate an optimal result. It reads as follows.

Theorem 14. ***Differences for $B^{s,\tau}_{p,p}(\mathbb{R}^d)$. Optimal Result.***
Let $s \in \mathbb{R}, 0 < p = q < \infty$, $0 \leq \tau < \frac{1}{p}$ and $N \in \mathbb{N}$. Then $B^{s,\tau}_{p,p}(\mathbb{R}^d)$ is the collection of all functions $f \in L^{loc}_{\max(p,1)}(\mathbb{R}^d)$, such that

$$\sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_P |f(x)|^p dx\Big)^{\frac{1}{p}} + \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau}\Big(\int_0^\infty t^{-sp}\int_P \Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)| dh\Big)^p dx \frac{dt}{t}\Big)^{\frac{1}{p}}$$

is finite, if and only if

$$d\max\left(0, \frac{1}{p} - 1\right) < s < N.$$

Proof. This result is a special case of Theorem 13 with $p = q$ and $v = 1$. ■

Theorem 14 corresponds to Theorem 11, where we obtained a similar result for the Triebel-Lizorkin-Morrey spaces. In fact, the assertion in both theorems is almost the same. To see this, recall, that for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 \leq \tau < \frac{1}{p}$ we have

$$B^{s,\tau}_{p,p}(\mathbb{R}^d) = F^{s,\tau}_{p,p}(\mathbb{R}^d). \tag{7.1}$$

We refer to Proposition 2.1.(iii) in [144]. When we combine this observation with Lemma 13, we find, that Theorem 14 is just a reformulation of Theorem 11. Also for Theorem 14 it is possible to illustrate the situation in a $(\frac{1}{p}, s)$ - diagram. We assume $p = q$ for each p and $\tau = \frac{1}{p} - 1$ if $p < 1$. Moreover, we work with $v = 1$.

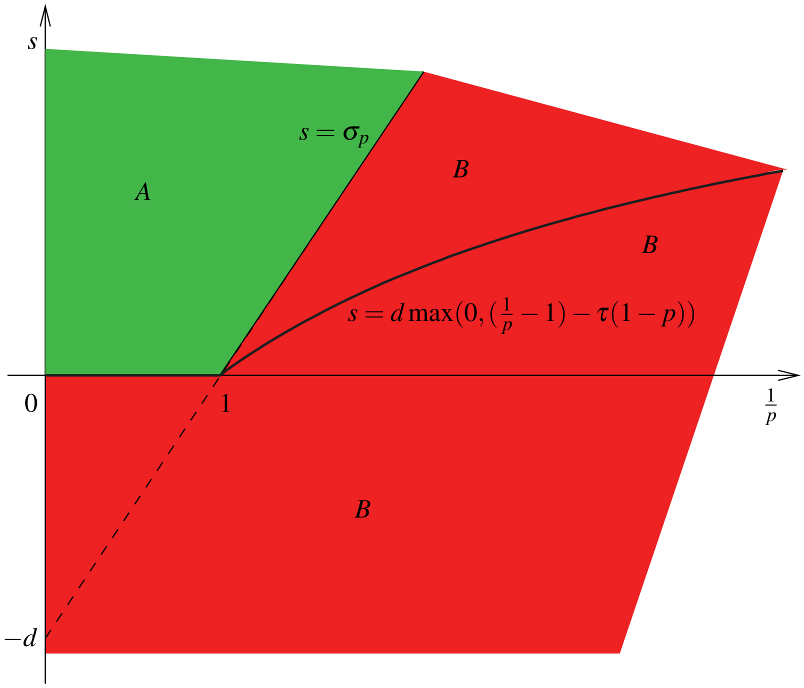

Figure 5. *Characterization in terms of differences for $B^{s,\tau}_{p,p}(\mathbb{R}^d)$ in the special case $q = p$ with $\tau = \frac{1}{p} - 1$ if $p < 1$.*

A: $B^{s,\tau}_{p,p}(\mathbb{R}^d) = \mathbf{B}^{s,\tau,N,\infty}_{p,p,1}(\mathbb{R}^d)$

B: $B^{s,\tau}_{p,p}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,\infty}_{p,p,1}(\mathbb{R}^d)$

In view of (7.1) it is not surprising, that Figure 2 and Figure 5 look almost identically. To complete this section, let us collect a list of open problems showing up in Theorem 13. So there are some parameter constellations, for that we do not know, whether we can describe the Besov-type spaces in terms of differences.

Open Problem 3. ***Besov-type Spaces and Differences.***
Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $0 \leq \tau < \frac{1}{p}$ and $N \in \mathbb{N}$ with $N \geq s$. Let $0 < v \leq \infty$ and $1 \leq a \leq \infty$. Then we do not know, whether we have $B^{s,\tau}_{p,q}(\mathbb{R}^d) = \mathbf{B}^{s,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$ or $B^{s,\tau}_{p,q}(\mathbb{R}^d) \neq \mathbf{B}^{s,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$, if we are in one of the following cases.

(i) We have $d(\frac{1}{p}-\frac{1}{v}) - d\tau(1-\frac{p}{v}) \leq s \leq d(\frac{1}{p}-\frac{1}{v})$ and $q \neq p < v < \infty$ with $v > 1$.

(ii) We have $d(\frac{1}{p}-1) - d\tau(1-p) \leq s \leq d(\frac{1}{p}-1)$ with $0 < p < 1$ and $0 < v < 1$.

(iii) We have $N = s$ and $q = \infty$.

Chapter 8

The Diamond Spaces associated to $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$

Characterizations in terms of differences for the Triebel-Lizorkin-Morrey spaces have many useful applications. For example, they can be used, when we want to learn something about complex interpolation of two Triebel-Lizorkin-Morrey spaces. In connection with this topic so-called diamond spaces play an important role. Hence this chapter is devoted to the diamond spaces associated to $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. They are denoted by $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$. One of the main goals in this chapter will be, to prove characterizations in terms of differences for these spaces. Let us remark, that a major part of the theory that is presented in this chapter, also can be found in the author's article [150].

8.1 Diamond Spaces: Definitions and basic Properties

In this section we give a rigorous definition for the so-called diamond spaces. Moreover, we collect some elementary properties of them. Roughly speaking the diamond spaces are subspaces of function spaces, that fulfill some additional smoothness properties. For a given function space X we denote the associated diamond space by $\overset{\diamond}{X}$. There is the following precise definition, see also Definition 2.23. in [146].

Definition 32. ***Diamond Spaces.***
Let X be a quasi-Banach space of distributions or functions.

(i) By $\overset{\diamond}{X}$ we denote the closure in X of the set of all infinitely often differentiable functions $f \in X$, that fulfill $D^\alpha f \in X$ for all $\alpha \in \mathbb{N}_0^d$.

(ii) Let $C_0^\infty(\mathbb{R}^d) \hookrightarrow X$. Then by $\mathring{X}$ we denote the closure of $C_0^\infty(\mathbb{R}^d)$ in X.

In what follows we want to look at some concrete examples for diamond spaces. Let us start with the spaces associated to the Morrey spaces $\mathscr{M}^u_p(\mathbb{R}^d)$, namely $\mathring{\mathscr{M}}{}^u_p(\mathbb{R}^d)$ and $\overset{\diamond}{\mathscr{M}}{}^u_p(\mathbb{R}^d)$. For these spaces there are the following explicit descriptions.

Lemma 31. ***Diamond Spaces associated to Morrey Spaces.***
Let $1 \le p < u < \infty$.

(i) $\overset{\diamond}{\mathcal{M}}{}^u_p(\mathbb{R}^d)$ *is equal to the collection of all $f \in \mathcal{M}^u_p(\mathbb{R}^d)$, that fulfill*

$$\lim_{r \downarrow 0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p \, dx \Big)^{\frac{1}{p}} = 0 \tag{8.1}$$

and

$$\lim_{r \to \infty} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p \, dx \Big)^{\frac{1}{p}} = 0 \tag{8.2}$$

both uniformly in $y \in \mathbb{R}^d$ and

$$\lim_{|y| \to \infty} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p \, dx \Big)^{\frac{1}{p}} = 0 \tag{8.3}$$

uniformly in $r \in (0,\infty)$.

(ii) $\overset{\diamond}{\mathcal{M}}{}^u_p(\mathbb{R}^d)$ *is equal to the collection of all $f \in \mathcal{M}^u_p(\mathbb{R}^d)$, such that* (8.1) *holds true uniformly in $y \in \mathbb{R}^d$.*

Proof. This result can be found in [146], see Lemma 2.33. ■

A second example are the diamond spaces associated to the Sobolev-Morrey spaces, recall Definition 20. It is not difficult to see, that the spaces $\overset{\diamond}{W}{}^m \mathcal{M}^u_p(\mathbb{R}^d)$ can be described in the following way.

Lemma 32. ***Diamond Spaces associated to Sobolev-Morrey Spaces.***
Let $1 \le p < u < \infty$ and $m \in \mathbb{N}$. Then $\overset{\diamond}{W}{}^m \mathcal{M}^u_p(\mathbb{R}^d)$ is equal to the collection of all $f \in W^m \mathcal{M}^u_p(\mathbb{R}^d)$, such that for any $\beta \in \mathbb{N}_0^d$ with $|\beta| \le m$ we have

$$\lim_{r \downarrow 0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |D^\beta f(x)|^p \, dx \Big)^{\frac{1}{p}} = 0$$

uniformly in $y \in \mathbb{R}^d$.

Proof. This result is a consequence of Definition 20 and Lemma 31. ■

Most important for us will be the diamond spaces associated to the Triebel-Lizorkin-Morrey spaces. They are defined as described in Definition 32 and have the symbol $\overset{\diamond}{\mathcal{E}}{}^s_{u,p,q}(\mathbb{R}^d)$. It turns out, that under certain conditions on the parameters the spaces $\overset{\diamond}{\mathcal{E}}{}^s_{u,p,q}(\mathbb{R}^d)$ coincide with some other function spaces. So we can make the following observation.

Lemma 33. ***Diamond Spaces associated to Triebel-Lizorkin Spaces.***
Let $s \in \mathbb{R}$, $0 < p \le u < \infty$ and $0 < q \le \infty$. Then the following assertions are true.

(i) We have $\overset{\diamond}{\mathcal{E}}{}^s_{u,p,q}(\mathbb{R}^d) = \overset{\circ}{\mathcal{E}}{}^s_{u,p,q}(\mathbb{R}^d)$ if and only if $u = p$.

(ii) We have $\overset{\diamond}{\mathcal{E}}{}^s_{u,p,q}(\mathbb{R}^d) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ if and only if $u = p$ and $0 < q < \infty$.

Proof. This result is proved in [146], see Lemma 2.25. and Lemma 2.26. ■

In the subsequent sections it is our main goal, to learn more about the properties of the spaces $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$. For that purpose sometimes we also have to deal with the following function spaces, that have many connections to the diamond spaces.

Definition 33. ***The Spaces*** $E^s_{u,p,q}(\mathbb{R}^d)$.
Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $s \geq 0$. *Then the set* $E^s_{u,p,q}(\mathbb{R}^d)$ *is the collection of all functions* $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, *such that* $D^\alpha f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ *for all* $\alpha \in \mathbb{N}_0^d$.

It is not difficult to see, that we have

$$\overline{E^s_{u,p,q}(\mathbb{R}^d)}^{\|\cdot|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|} = \overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d).$$

There also exists a counterpart of Definition 33 for domains. Let Ω be a Lipschitz domain, see Definition 16. Let $s \geq 0$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Then we put

$$E^s_{u,p,q}(\Omega) := \Big\{ f \in \mathscr{D}'(\Omega) : \ \exists\, g \in E^s_{u,p,q}(\mathbb{R}^d) \quad \text{such that} \quad f = g \quad \text{on} \quad \Omega \Big\}. \tag{8.4}$$

When we use the definition, we find, that we also can write

$$E^s_{u,p,q}(\Omega) = \Big\{ f \in \mathscr{E}^s_{u,p,q}(\Omega) : \ D^\alpha f \in \mathscr{E}^s_{u,p,q}(\Omega) \quad \text{for all} \quad \alpha \in \mathbb{N}_0^d \Big\}. \tag{8.5}$$

But we know even more. So it turns out, that the spaces $E^s_{u,p,q}(\Omega)$ are independent of the parameters. There is the following result.

Lemma 34. ***The Spaces*** $E^s_{u,p,q}(\Omega)$ ***are independent of the Parameters.***
Let $\Omega \subset \mathbb{R}^d$ *be either a bounded Lipschitz domain if* $d \geq 2$ *or a bounded interval if* $d = 1$. *Let* $s \geq 0$, $0 < p \leq u < \infty$ *and* $0 < q \leq \infty$. *Then the set* $E^s_{u,p,q}(\Omega)$ *is independent of the parameters* s, u, p *and* q. *Indeed, it holds*

$$E^s_{u,p,q}(\Omega) = \Big\{ f \in C^\infty(\Omega) : \ D^\alpha f \in L_\infty(\Omega) \quad \textit{for all} \quad \alpha \in \mathbb{N}_0^d \Big\}.$$

Proof. For the proof we can follow the ideas from Proposition 4.21. in [134]. One may also consult the proof of Theorem 2.45. on page 1895 in [146]. ■

8.2 Characterizations for Diamond Spaces

In this section we want to learn more about the properties of the spaces $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$. For that purpose we will prove some equivalent characterizations for them.

8.2.1 Characterizations using a Littlewood-Paley Decomposition

Hereinafter we are going to prove a characterization of $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$ using the Littlewood-Paley decomposition. For that purpose, we have to introduce an additional notation. Let $(\varphi_j)_{j=0}^\infty$ be a smooth dyadic decomposition of the unity. Then for $f \in \mathscr{S}'(\mathbb{R}^d)$ we put

$$S^N f(x) := \sum_{j=0}^{N} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f](x)\,, \qquad N \in \mathbb{N}_0. \tag{8.6}$$

Of course by the Paley-Wiener-Schwarz Theorem $S^N f$ are smooth functions. But we know even more. So we can make the following observations.

Lemma 35. ***Some Properties of** $S^N f$**.***
Let $0 < p \le u < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Then the sequence $(S^N f)_{N=0}^\infty$ has the following properties.

(i) We have $S^N f \in \mathscr{E}^\sigma_{u,p,q}(\mathbb{R}^d)$ for all $\sigma \in \mathbb{R}$.

(ii) For all $\alpha \in \mathbb{N}_0^d$ we have $D^\alpha(S^N f) \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.

(iii) For all $\alpha \in \mathbb{N}_0^d$ we have $D^\alpha(S^N f) \in L_\infty(\mathbb{R}^d)$.

(iv) It holds the identity

$$S^N f(x) = \mathscr{F}^{-1}[\varphi_0(2^{-N}\cdot)\mathscr{F} f](x)\,, \qquad x \in \mathbb{R}^d, \quad N \in \mathbb{N}_0.$$

(v) There exists a constant c independent on f, such that

$$\sup_{N \in \mathbb{N}_0} \| S^N f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \le c \, \| f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|\,. \tag{8.7}$$

Proof. Part (i) is a consequence of the estimate

$$\| S^N f \,|\mathscr{E}^\sigma_{u,p,q}(\mathbb{R}^d)\| \le c \left\| \Big(\sum_{j=0}^{N+1} 2^{j\sigma q} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f](\cdot)|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \right\|$$

with some c independent of f and $N \in \mathbb{N}_0$, see Lemma 22. From (i) we derive, that $S^N f \in \mathscr{E}^{s+m}_{u,p,q}(\mathbb{R}^d)$ with $m \in \mathbb{N}_0$. Next we use Lemma 4, to obtain $D^\alpha(S^N f) \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ for $|\alpha| = m$. To show (iii), it is enough to apply Proposition 2.6 from [144]. The next part (iv) is an elementary conclusion of the definition of the functions φ_j with $j \in \{1, 2, \ldots, N\}$. Finally (v) follows from Lemma 22. We observe

$$\| S^N f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| = \| \mathscr{F}^{-1}[\varphi_0(2^{-N}\cdot)\,\mathscr{F} f] \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \le c_1 \| f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|\,.$$

So the proof is complete. ∎

Now we are prepared to prove a characterization of $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$ using the Littlewood-Paley decomposition.

Proposition 19. ***Littlewood-Paley Characterization for*** $\mathring{\mathscr{E}}^s_{u,p,q}(\mathbb{R}^d)$**.**
Let $0 < p \le u < \infty$, $0 < q \le \infty$ *and* $s \in \mathbb{R}$. *Then* $\mathring{\mathscr{E}}^s_{u,p,q}(\mathbb{R}^d)$ *is the collection of all* $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, *such that*

$$\lim_{N\to\infty} \| f - S^N f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| = 0 \,. \tag{8.8}$$

Proof. It is not difficult to see, that if (8.8) holds, we have $f \in \mathring{\mathscr{E}}^s_{u,p,q}(\mathbb{R}^d)$. Therefore in what follows we suppose that $f \in \mathring{\mathscr{E}}^s_{u,p,q}(\mathbb{R}^d)$. We want to prove (8.8). For that purpose by $(f_\ell)_{\ell\in\mathbb{N}}$ we denote a sequence in $E^s_{u,p,q}(\mathbb{R}^d)$, such that

$$\lim_{\ell\to\infty} \| f - f_\ell \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| = 0 \,.$$

Without loss of generality we may assume

$$\| f - f_\ell \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| < \frac{1}{\ell} \qquad \text{for all} \qquad \ell \in \mathbb{N} \,.$$

Let $\sigma \in \mathbb{R}$ with $\sigma > s$. We use a standard Fourier multiplier assertion from Lemma 22. Then we obtain

$$\begin{aligned}
\| f_\ell - S^N f_\ell \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| &= \Big\| \sum_{j=N+1}^{\infty} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f_\ell](\,\cdot\,) \Big| \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\Big\| \\
&\le c_1 \Big\| \Big(\sum_{j=N}^{\infty} 2^{jsq} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f_\ell](\,\cdot\,)|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d)\Big\| \\
&\le c_2\, 2^{N(s-\sigma)} \Big\| \Big(\sum_{j=N}^{\infty} 2^{j\sigma q} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f_\ell](\,\cdot\,)|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d)\Big\| \\
&\le c_2\, 2^{N(s-\sigma)} \, \| f_\ell \,|\mathscr{E}^\sigma_{u,p,q}(\mathbb{R}^d)\| \,.
\end{aligned}$$

This shows, that

$$\lim_{N\to\infty} \| f_\ell - S^N f_\ell \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| = 0 \qquad \text{for any} \qquad \ell \in \mathbb{N}.$$

Hence for $\ell \in \mathbb{N}$ there exists some $N(\ell) \in \mathbb{N}$, such that

$$\| f_\ell - S^{N(\ell)} f_\ell \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| < \frac{1}{\ell} \,.$$

This yields

$$\begin{aligned}
&\| f - S^{N(\ell)} f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \\
&\quad \le c_3 \| f - f_\ell \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| + c_3 \| f_\ell - S^{N(\ell)} f_\ell \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| + c_3 \| S^{N(\ell)} f_\ell - S^{N(\ell)} f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \\
&\quad \le c_3 \frac{2}{\ell} + c_3 \| S^{N(\ell)} (f_\ell - f) \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \\
&\quad \le c_3 \frac{2+c}{\ell} \,,
\end{aligned}$$

where c is the constant from (8.7). Consequently we have the convergence of an appropriate subsequence $(S^{N(\ell)} f)_{\ell=1}^\infty$. It remains to switch from the subsequence to the whole sequence. Therefore we assume, that our sequence $(N(\ell))_{\ell\in\mathbb{N}}$ satisfies

$$N(\ell+1) - N(\ell) > 5 \qquad \text{for all} \qquad \ell \in \mathbb{N}.$$

Furthermore, we will use the identity

$$\begin{aligned}\Big\| \sum_{j=M}^{N} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f] \Big| \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\Big\| = \Big\|\Big(\sum_{m=M+1}^{N-1} & 2^{msq} |\mathscr{F}^{-1}[\varphi_m \mathscr{F} f](\cdot)|^q \\ & + 2^{Msq}|\mathscr{F}^{-1}[\varphi_M(\varphi_M + \varphi_{M+1})\mathscr{F} f](\cdot)|^q \\ & + 2^{(M-1)sq}|\mathscr{F}^{-1}[\varphi_{M-1}\varphi_M \mathscr{F} f](\cdot)|^q \\ & + 2^{Nsq}|\mathscr{F}^{-1}[\varphi_N(\varphi_{N-1} + \varphi_N)\mathscr{F} f](\cdot)|^q \\ & + 2^{(N+1)sq}|\mathscr{F}^{-1}[\varphi_{N+1}\varphi_N \mathscr{F} f](\cdot)|^q\Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d)\Big\|,\end{aligned}$$

valid for all natural numbers M and N such that $2 \le M+1 < N-1$. This follows from

$$\varphi_m \cdot \Big(\sum_{j=M}^{N} \varphi_j\Big) = \begin{cases} \varphi_m & \text{if} \quad M < m < N; \\ \varphi_{M-1}\varphi_M & \text{if} \quad m = M-1; \\ \varphi_M(\varphi_M + \varphi_{M+1}) & \text{if} \quad m = M; \\ \varphi_N(\varphi_{N-1} + \varphi_N) & \text{if} \quad m = N; \\ \varphi_{N+1}\varphi_N & \text{if} \quad m = N+1; \\ 0 & \text{otherwise}. \end{cases}$$

From Lemma 22 we know

$$\| \mathscr{F}^{-1}[\varphi_j \varphi_\ell \mathscr{F} f](\cdot) | \mathscr{M}^u_p(\mathbb{R}^d)\| \le c_4 \| \mathscr{F}^{-1}[\varphi_\ell \mathscr{F} f](\cdot) | \mathscr{M}^u_p(\mathbb{R}^d)\|.$$

Altogether this shows

$$\begin{aligned} & \Big\|\Big(\sum_{m=M+1}^{N-1} 2^{msq}|\mathscr{F}^{-1}[\varphi_m \mathscr{F} f](\cdot)|^q\Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d)\Big\| \\ & \qquad \le \Big\| \sum_{j=M}^{N} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f] \Big| \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\Big\| \\ & \qquad \le c_5 \Big\|\Big(\sum_{m=M-1}^{N+1} 2^{msq}|\mathscr{F}^{-1}[\varphi_m \mathscr{F} f](\cdot)|^q\Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d)\Big\| \end{aligned} \tag{8.9}$$

with some constant c_5 independent of M, N and f. Let

$$1 \le N(\ell) \le M-3 < N-3 < N+2 \le N(\ell+1).$$

Then (8.9) implies

$$\begin{aligned} \| S^N f - S^{M-1} f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| & = \Big\| \sum_{j=M}^{N} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f] \Big| \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\Big\| \\ & \le c_5 \Big\|\Big(\sum_{m=M-1}^{N+1} 2^{msq}|\mathscr{F}^{-1}[\varphi_m \mathscr{F} f]|^q\Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d)\Big\| \\ & \le c_5 \Big\| \sum_{j=N(\ell)+1}^{N(\ell+1)} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f] \Big| \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\Big\| \\ & = c_5 \| S^{N(\ell+1)} f - S^{N(\ell)} f | \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|. \end{aligned}$$

Repeating the argument, we conclude, that

$$\| S^N f - S^{M-1} f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq c_6 \| S^{N(\ell+2)} f - S^{N(\ell-1)} f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$$

for all M,N such that $N(\ell) \leq M \leq N \leq N(\ell+1)$ with c_6 independent of ℓ. Consequently $(S^N f)_{N=0}^\infty$ is a Cauchy sequence in $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. This proves the claim. ■

Actually, Proposition 19 is not new. We refer to [47], see Theorem 1.1. However, the proof we gave here, is slightly different and covers some more cases. Next we want to use Proposition 19 to prove an embedding result, that connects Triebel-Lizorkin-Morrey spaces and diamond spaces. So we can observe the following.

Corollary 5. ***Embedding in Diamond Spaces.***
Let $0 < p \leq u < \infty$, $0 < q_0, q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$ with $s_1 < s_0$. Then we have the continuous embedding

$$\mathscr{E}^{s_0}_{u,p,q_0}(\mathbb{R}^d) \hookrightarrow \overset{\diamond}{\mathscr{E}}{}^{s_1}_{u,p,q_1}(\mathbb{R}^d)\,.$$

Proof. Let $f \in \mathscr{E}^{s_0}_{u,p,q_0}(\mathbb{R}^d)$. Lemma 22 yields

$$\begin{aligned} \|f - S^N f \,|\mathscr{E}^{s_1}_{u,p,q_1}(\mathbb{R}^d)\| &\leq c_1 \Big\| \Big(\sum_{j=N}^{\infty} 2^{js_1q_1} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f]|^{q_1} \Big)^{\frac{1}{q_1}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\| \\ &\leq c_2\, 2^{-(s_0-s_1)N} \Big\| \sup_{j \geq N} 2^{js_0} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f]| \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\| \end{aligned}$$

with constants c_1 and c_2 independent of f and N. Since $\mathscr{E}^{s_0}_{u,p,q_0}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{s_0}_{u,p,\infty}(\mathbb{R}^d)$ for $N \to \infty$ this implies (8.8) and therefore by Proposition 19 we find $f \in \overset{\diamond}{\mathscr{E}}{}^{s_1}_{u,p,q_1}(\mathbb{R}^d)$. The proof is complete. ■

8.2.2 Characterizations in Terms of Differences

Hereafter we want to prove characterizations in terms of differences for the spaces $\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)$. We will divide this task into two parts. So on the one hand we can observe the following.

Lemma 36. ***Diamond Spaces and Differences. Part I.***
Let $0 < p < u < \infty$, $0 < q \leq \infty$ and $s > \sigma_{p,q}$. Let $N \in \mathbb{N}$ with $s < N$. Then $\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)$ is contained in the set of all $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, that fulfill

$$\lim_{r \downarrow 0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p \, dx \Big)^{\frac{1}{p}} = 0 \tag{8.10}$$

and

$$\lim_{r \downarrow 0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big[\int_{B(y,r)} \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta^N_h f(x)| \, dh \Big)^q \frac{dt}{t} \Big)^{\frac{p}{q}} dx \Big]^{\frac{1}{p}} = 0\,, \tag{8.11}$$

both uniformly in $y \in \mathbb{R}^d$.

Proof. *Step 1.* In a first step we deal with functions f belonging to $E^s_{u,p,q}(\mathbb{R}^d)$. Clearly those functions are uniformly Lipschitz continuous on $\mathbb{R}^d$. To see (8.10) in this situation we argue as follows. Obviously we have

$$\Big(\int_{B(y,r)} |f(x)|^p\, dx\Big)^{\frac{1}{p}} \le \ \|f\,|L_\infty(\mathbb{R}^d)\| \ \ |B(y,r)|^{\frac{1}{p}}.$$

Multiplying this inequality by $|B(y,r)|^{1/u-1/p}$, it follows for $u<\infty$, that the right-hand side tends to 0 (uniformly in y) if $r\downarrow 0$. The argument for deriving (8.11) is quite similar. Recall, that for a smooth function with $N\in\mathbb{N}$ we have

$$|\Delta^N_h f(x)| \le c_1 \Big(\max_{|\alpha|\le N}\sup_{y\in\mathbb{R}^d} |D^\alpha f(y)|\Big)\,|h|^N \qquad \text{with} \qquad x,h\in\mathbb{R}^d$$

and with a constant c_1 independent of f,x and h. Hence we find

$$\Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|\, dh\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \le c_2 \Big(\int_0^1 t^{-sq} t^{Nq}\,\frac{dt}{t}\Big)^{\frac{1}{q}} \le c_3 < \infty$$

for some c_3 independent of x. This implies

$$|B(y,r)|^{\frac{1}{u}-\frac{1}{p}}\Big[\int_{B(y,r)}\Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|\, dh\Big)^q \frac{dt}{t}\Big)^{\frac{p}{q}} dx\Big]^{\frac{1}{p}} \le c_3\,|B(y,r)|^{\frac{1}{u}}$$

and therefore the claim follows.
Step 2. Now we turn to the general case. Let $f\in \overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$ and let $\varepsilon>0$ be given. Then with $M\in\mathbb{N}$ it follows

$$\begin{aligned}
&|B(y,r)|^{\frac{1}{u}-\frac{1}{p}}\Big[\int_{B(y,r)}\Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{B(0,t)} |\Delta^N_h f(x)|\, dh\Big)^q \frac{dt}{t}\Big)^{\frac{p}{q}} dx\Big]^{\frac{1}{p}}\\
&\qquad \le c_1 \|f - S^M f\,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|\\
&\qquad\qquad + c_1 |B(y,r)|^{\frac{1}{u}-\frac{1}{p}}\Big[\int_{B(y,r)}\Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{B(0,t)} |\Delta^N_h (S^M f)(x)|\, dh\Big)^q \frac{dt}{t}\Big)^{\frac{p}{q}} dx\Big]^{\frac{1}{p}}.
\end{aligned}$$

Here we used Theorem 5. Now the second term on the righ-hand side becomes smaller than $\varepsilon>0$, if $r\le r_0(\varepsilon)$, since $S^M f\in E^s_{u,p,q}(\mathbb{R}^d)$ and therefore we may use Step 1. The first term on the right-hand side will be smaller than $\varepsilon>0$, if $M\ge M_0(\varepsilon)$ thanks to Proposition 19. Both statements hold uniformly in y. This proves (8.11). The convergence in (8.10) can be proved in a similar way. ∎

Now we turn to the converse part of Lemma 36. Here we have to accept some more restrictive conditions.

Lemma 37. ***Diamond Spaces and Differences. Part II.***
Let $1\le p<u<\infty$, $1\le q<\infty$ and $s>0$. Let $N\in\mathbb{N}$ with $s<N$. Let $f\in\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ be a function with compact support and such that (8.10) *as well as* (8.11) *hold uniformly in $y\in\mathbb{R}^d$. Then we have $f\in\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$.*

Proof. Because of Proposition 19, it is enough to prove

$$\lim_{M\to\infty} \|f - S^M f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| = 0.$$

Using Theorem 5 with $a=1$ and $v=1$, this can be reduced to show

$$\lim_{M\to\infty} \|f - S^M f|\mathscr{M}^u_p(\mathbb{R}^d)\| = 0 \tag{8.12}$$

and

$$\lim_{M\to\infty} \left\| \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N (f - S^M f)|\, dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \right\| = 0. \tag{8.13}$$

Step 1. We shall show (8.12). For that purpose let $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Let $M \in \mathbb{N}$ and $0 < \sigma < s$. Then we find

$$\begin{aligned}
\| f - S^M f \,|\mathscr{M}^u_p(\mathbb{R}^d)\| &\leq \Big\| \sum_{j=M+1}^{\infty} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f](\cdot)| \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\| \\
&\leq c_1 2^{-M\sigma} \Big\| \sum_{j=M+1}^{\infty} 2^{j\sigma} |\mathscr{F}^{-1}[\varphi_j \mathscr{F} f](\cdot)| \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\| \\
&\leq c_1 2^{-M\sigma} \| f \,| \mathscr{E}^\sigma_{u,p,1}(\mathbb{R}^d)\| \\
&\leq c_2 2^{-M\sigma} \| f \,| \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| .
\end{aligned}$$

Here c_2 is independent of f and $M \in \mathbb{N}$. So since $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, if M tends to infinity, (8.12) follows.

Step 2. Next we prove (8.13). Let B be a ball in $\mathbb{R}^d$. Since f satisfies (8.11), for every $\varepsilon > 0$ we find some $\delta > 0$, such that

$$\sup_{|B|<\delta} |B|^{\frac{1}{u}-\frac{1}{p}} \left[\int_B \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \leq \varepsilon.$$

Due to $p \geq 1$ and $q \geq 1$, the generalized Minkowski inequality in combination with a standard convolution inequality yield

$$\begin{aligned}
&\left[\int_B \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N S^M f(x)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \\
&\qquad \leq c_3 \left[\int_B \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}
\end{aligned}$$

with c_3 independent of B and f. Consequently we get

$$\begin{aligned}
&\left\| \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N (f - S^M f)(\cdot)|\, dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \right\| \\
&\quad \leq c_4\, \varepsilon + \sup_{|B|\geq\delta} |B|^{\frac{1}{u}-\frac{1}{p}} \left[\int_B \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N (f - S^M f)(x)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}.
\end{aligned} \tag{8.14}$$

Since $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ the supremum on the right-hand side is finite. By the definition of the supremum there exists a sequence of balls $B_j := B(y_j, r_j)$ with $j \in \mathbb{N}$ and $|B(y_j,r_j)| \geq \delta$, such that

$$\begin{aligned}
&\sup_{|B|\geq\delta} |B|^{\frac{1}{u}-\frac{1}{p}} \left[\int_B \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N (f - S^M f)(x)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \\
&\quad \leq \frac{1}{j} + |B_j|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B_j} \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N (f - S^M f)(x)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \\
&\quad \leq \frac{1}{j} + |B_j|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{\mathbb{R}^d} \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N (f - S^M f)(x)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{p}{q}} dx \right]^{\frac{1}{p}} \\
&\quad \leq \frac{1}{j} + c_5\, \delta^{\frac{1}{u}-\frac{1}{p}}\, \|f - S^M f | F^s_{p,q}(\mathbb{R}^d)\|. \qquad (8.15)
\end{aligned}$$

Here in the last step we used Theorem 5 for the original Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$.
Substep 2.1. We claim, that a function $f \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ with compact support belongs to $F^s_{p,q}(\mathbb{R}^d)$ as well. We may assume $\operatorname{supp} f \subset B(0,R)$ for some $R > 1$. Based on Theorem 5 we observe that

$$\begin{aligned}
&\|f|L_p(\mathbb{R}^d)\| + \left\| \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(\cdot)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| L_p(\mathbb{R}^d) \right\| \\
&\quad = \|f|L_p(B(0,R))\| + \left\| \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(\cdot)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| L_p(B(0,R+N)) \right\| \\
&\quad \leq |B(0,R)|^{-\frac{1}{u}+\frac{1}{p}} \|f|\mathcal{M}^u_p(\mathbb{R}^d)\| \\
&\qquad + |B(0,R+N)|^{-\frac{1}{u}+\frac{1}{p}} \left\| \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(\cdot)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \right\| \\
&\quad \leq c_6\, (R+N)^{d(\frac{1}{p}-\frac{1}{u})} \|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|. \qquad (8.16)
\end{aligned}$$

Hence we find $f \in F^s_{p,q}(\mathbb{R}^d)$.
Substep 2.2. Next we shall use Lemma 33 and Proposition 19. Since $f \in F^s_{p,q}(\mathbb{R}^d) = \mathcal{E}^s_{p,p,q}(\mathbb{R}^d) = \overset{\diamond}{\mathcal{E}}{}^s_{p,p,q}(\mathbb{R}^d)$ and $1 \leq p, q < \infty$, we get

$$\lim_{M\to\infty} \|f - S^M f | F^s_{p,q}(\mathbb{R}^d)\| = 0. \qquad (8.17)$$

Here the restriction $q < \infty$ is essential, see also the remark in chapter 2.3.3 in [128]. Finally we collect (8.14)-(8.17) together and find, that for fixed ε and associated δ

$$\begin{aligned}
&\left\| \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N (f - S^M f)(\cdot)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \right\| \\
&\qquad \leq c_4\, \varepsilon + \frac{1}{j} + c_5\, \delta^{\frac{1}{u}-\frac{1}{p}}\, \|f - S^M f | F^s_{p,q}(\mathbb{R}^d)\| \leq c_7\, \varepsilon + \frac{1}{j},
\end{aligned}$$

if M is chosen large enough. So if j tends to infinity, this proves (8.13). The proof is complete. ■

In Lemma 37 we assumed, that the function under investigation has compact support. Hence the following definition will be important for us.

Definition 34. ***The Spaces*** $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d;B)$**.**
Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $s \geq 0$. *Let* B *be a ball in* $\mathbb{R}^d$. *Then* $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d;B)$ *is the collection of all* $f \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ *satisfying* $\operatorname{supp} f \subset B$.

Using this notation, we can formulate the next theorem, which is the main result of this section.

Theorem 15. ***Diamond Spaces and Differences. Part III.***
Let $1 \leq p < u < \infty$, $1 \leq q < \infty$ and $s > 0$. Let B be a ball in $\mathbb{R}^d$. Then $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d;B)$ belongs to $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$, if and only if (8.10) and (8.11) hold uniformly in $y \in \mathbb{R}^d$.

Proof. This result is a combination of Lemma 36 and Lemma 37. ■

8.2.3 Characterizations by Mollifiers

It is also possible, to describe the spaces $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$ by using mollifiers. In this section we will briefly collect the main ideas concerning that topic. For that purpose we need some more notation. Therefore let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a function satisfying

$$\int_{\mathbb{R}^d} \rho(x)\, dx = 1 \qquad \text{and} \qquad \operatorname{supp}\rho \subset B(0,1)\,.$$

We put $\rho_j(x) := 2^{jd}\rho(2^j x)$ with $x \in \mathbb{R}^d$ and $j \in \mathbb{N}$. Such a function ρ sometimes is called a mollifier. Mollifiers have the special property, that their convolution with a distribution is a smooth function. For a quasi-Banach space X that is continuously embedded into $\mathscr{S}'(\mathbb{R}^d)$, we define X^{loc} as the collection of all $f \in \mathscr{S}'(\mathbb{R}^d)$, such that the pointwise product fulfills $\varphi \cdot f \in X$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. Convergence of a sequence $\{f_j\}_{j=1}^\infty$ with limit f in X^{loc} is defined as

$$\lim_{j\to\infty} \| f\,\varphi - f_j\,\varphi|X\| = 0 \qquad \text{for all} \qquad \varphi \in C_0^\infty(\mathbb{R}^d)\,.$$

The next lemma tells us, what happens, when we convolute our mollifier ρ with a distribution $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.

Lemma 38. ***Mollifiers and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.***
Let $1 \leq p \leq u < \infty$, $1 \leq q \leq \infty$ and $s > 0$. Let $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Then the sequence $\{f * \rho_j\}_{j=1}^\infty$ has the following properties.

- *(i) For all $\alpha \in \mathbb{N}_0^d$ and all $j \in \mathbb{N}$ we have $D^\alpha(f * \rho_j) \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Moreover we have $f * \rho_j \in E^s_{u,p,q}(\mathbb{R}^d)$.*
- *(ii) For all $\alpha \in \mathbb{N}_0^d$ and all $j \in \mathbb{N}$ we have $D^\alpha(f * \rho_j) \in L_\infty(\mathbb{R}^d)$.*
- *(iii) For all $j \in \mathbb{N}$ we have $f * \rho_j \in C^\infty(\mathbb{R}^d)$.*
- *(iv) For all $j \in \mathbb{N}$ we have $f * \rho_j \in \mathscr{E}^\sigma_{u,p,q}(\mathbb{R}^d)$ for all $\sigma \in \mathbb{R}$.*
- *(v) There exists a constant c independent of f, such that*

$$\sup_{j\in\mathbb{N}} \| f * \rho_j \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq c\, \| f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|\,.$$

Proof. Essentially all of Lemma 38 is known. So we skip the proof. ■

Let us compare Lemma 38 with Lemma 35, where we collected the properties of $S^N f$. We observe, that the sequences $\{f * \rho_j\}_{j=1}^\infty$ and $\{S^N f\}_{N=0}^\infty$ have many properties in common. So in both cases the involved functions are very smooth. In Proposition 19 we learned, that Littlewood-Paley decompositions can be used, to describe the spaces $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$. Therefore it is not surprising, that also convolutions with mollifiers can be applied to characterize the spaces $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$. The following result can be seen as a counterpart of Proposition 19.

Proposition 20. ***Mollifiers and*** $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$.
Let $1 \le p \le u < \infty$, $1 \le q \le \infty$ *and* $s > 0$. *Let* $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. *Then the following assertions are equivalent.*

(i) We have $f \in \overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$.

(ii) We have $\lim_{j\to\infty} \| f * \rho_j - f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| = 0$.

Proof. We will not use Proposition 20 in what follows. Therefore we will drop the proof. ■

8.3 Diamond Spaces and Intersections of Triebel-Lizorkin-Morrey Spaces

In this section we intend to investigate the properties of intersections of Triebel-Lizorkin-Morrey spaces. Later they will play an important role, when we are looking for descriptions of interpolation spaces. For a start we want to prove the following lemma.

Lemma 39. ***Intersections of Triebel-Lizorkin-Morrey Spaces. Part I.***
Let $\Theta \in (0,1)$ *and* $i \in \{0,1\}$. *Let* $s_i \in \mathbb{R}$, $p_i \in (0,\infty)$, $q_i \in (0,\infty]$ *and* $u_i \in [p_i,\infty)$. *Moreover we put* $s = (1-\Theta)s_0 + \Theta s_1$,

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1} \qquad \text{and} \qquad \frac{1}{u} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

Then we have

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^s_{u,p,q}(\mathbb{R}^d).$$

Proof. Because of our assumptions and Hölder's inequality we have

$$\| (2^{js} a_j)_{j=0}^\infty | l_q \| \le \| (2^{js_0} a_j)_{j=0}^\infty | l_{q_0} \|^{1-\Theta} \, \| (2^{js_1} a_j)_{j=0}^\infty | l_{q_1} \|^{\Theta}.$$

This will be applied with $a_j := \mathscr{F}^{-1}[\varphi_j \mathscr{F} f]$ and $j \in \mathbb{N}_0$. We continue by a further application of Hölder's inequality and find

$$\begin{aligned} &\Big\| \, \| (2^{js} a_j)_{j=0}^\infty | l_q \| \, \Big| L_p(B(y,r)) \Big\| \\ &\qquad \le \Big\| \, \| (2^{js_0} a_j)_{j=0}^\infty | l_{q_0} \|^{1-\Theta} \, \| (2^{js_1} a_j)_{j=0}^\infty | l_{q_1} \|^{\Theta} \, \Big| L_p(B(y,r)) \Big\| \\ &\qquad \le \Big\| \, \| (2^{js_0} a_j)_{j=0}^\infty | l_{q_0} \| \, \Big| L_{p_0}(B(y,r)) \Big\|^{1-\Theta} \, \Big\| \, \| (2^{js_1} a_j)_{j=0}^\infty | l_{q_1} \| \, \Big| L_{p_1}(B(y,r)) \Big\|^{\Theta}. \end{aligned}$$

This proves the claim. ■

In a next step we prove a version of Lemma 39 for functions with compact supports. For that purpose we work with the spaces $\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d;B)$ from Definition 34. When we do so, the diamond spaces $\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)$ show up in a very natural way.

Lemma 40. ***Intersections of Triebel-Lizorkin-Morrey Spaces. Part II.***
Let $\Theta \in (0,1)$, $0 \leq s_0 \leq s_1$, $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1 \leq \infty$, $\min(q_0,q_1) < \infty$, $p_0 < u_0$, $p_1 < u_1$ *and* $u_0 < u_1$, *such that* $s = (1-\Theta)s_0 + \Theta s_1$,

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1} \qquad \text{and} \qquad \frac{1}{u} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

In addition we assume either $s_0 < s_1$ *or* $0 < s_0 = s_1$ *and* $q_1 \leq q_0$. *Let* B *be a ball in* $\mathbb{R}^d$. *Then we have*

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d;B) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d;B) \hookrightarrow \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d).$$

Proof. By Lemma 39 we already know, that

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d).$$

Now we want to employ Theorem 15. This is possible, because we have $s > 0$ and $q < \infty$. Let $f \in \mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d;B) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d;B)$. Using $p_0 < p < p_1$ and Hölder's inequality, we find

$$\begin{aligned}
|B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{B(y,r)} |f(x)|^p dx \Big)^{\frac{1}{p}} &\leq |B(y,r)|^{\frac{1}{u}-\frac{1}{p_1}} \Big(\int_{B(y,r)} |f(x)|^{p_1} dx \Big)^{\frac{1}{p_1}} \\
&= |B(y,r)|^{\frac{1}{u}-\frac{1}{u_1}} |B(y,r)|^{\frac{1}{u_1}-\frac{1}{p_1}} \Big(\int_{B(y,r)} |f(x)|^{p_1} dx \Big)^{\frac{1}{p_1}} \\
&\leq |B(y,r)|^{\frac{1}{u}-\frac{1}{u_1}} \| f | \mathscr{M}^{u_1}_{p_1}(\mathbb{R}^d) \|. \qquad (8.18)
\end{aligned}$$

This tends to zero, if $r \to 0$ due to $u_0 < u < u_1$. Now we proceed similarly with the term

$$I(f,y,r,s,u,p,q) := |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(y,r)} \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(x)| \, dh \Big)^q \frac{dt}{t} \Big)^{\frac{p}{q}} dx \right]^{\frac{1}{p}}$$

with $N > s$. Using $p_0 < p < p_1$, we observe by the Hölder inequality

$$\begin{aligned}
I(f,y,r,s,u,p,q) &\leq |B(y,r)|^{\frac{1}{u}-\frac{1}{p_1}} \left\| \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(\cdot)| \, dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| L_{p_1}(B(y,r)) \right\| \\
&\leq |B(y,r)|^{\frac{1}{u}-\frac{1}{u_1}} \left\| \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N f(\cdot)| \, dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| \mathscr{M}^{u_1}_{p_1}(\mathbb{R}^d) \right\| \\
&\leq c_1 \, |B(y,r)|^{\frac{1}{u}-\frac{1}{u_1}} \, \| f | \mathscr{E}^{s}_{u_1,p_1,q}(\mathbb{R}^d) \| \\
&\leq c_2 \, |B(y,r)|^{\frac{1}{u}-\frac{1}{u_1}} \, \| f | \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d) \|. \qquad (8.19)
\end{aligned}$$

Here we used Theorem 5 and the elementary embedding $\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{s}_{u_1,p_1,q}(\mathbb{R}^d)$, see Proposition 2.1 in [144]. As in (8.18), it is obvious that the right-hand side tends to zero for $r \to 0$ uniformly in y. Hence by Theorem 15, (8.18) and (8.19) we finally proved $f \in \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)$. ■

Lemma 40 will be very important for us later, when we deal with complex interpolation of Triebel-Lizorkin-Morrey spaces. It already gives us a good impression, how the conditions on the parameters will look like in typical results concerning that topic. However, before we are ready to prove assertions on complex interpolation, we need some more preparations. So for example we have to investigate the smoothness properties of some test functions. This will be done in the next chapter.

Chapter 9

Some Test Functions for $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$

In this chapter we shall investigate some families of test functions. So on the one hand we will deal with functions, that have a local singularity at the origin. On the other hand we have a look at radial symmetric functions with a certain decay at infinity. For such functions we want to figure out, under what conditions on the parameters they belong to our Smoothness Morrey spaces. There are several reasons, why this is an interesting problem. First of all it allows us to get a better feeling for the spaces under consideration. Second, when we know, that under certain conditions on the parameters some special test functions do not belong to our function spaces, this sometimes gives us the possibility, to prove the necessity of some conditions in our main results. In the proofs in this chapter we often will work with the characterizations in terms of differences, we obtained earlier in this treatise. So the next pages will help to demonstrate, how differences can be used, to investigate the properties of special test functions. It turns out, that if the function under investigation has a certain structure, it has many advantages, to apply an equivalent quasi-norm, that uses differences. Let us say, that some calculations that are done in this chapter, also can be found in [150].

9.1 Functions with a local Singularity

Hereinafter we investigate the properties of a family of test functions, that have a local singularity at the origin. For that purpose at first we define a smooth cut-off function ψ. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a radial-symmetric and real-valued function with $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}^d$. Moreover, we have $\psi(x) = 1$, if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 3/2$. In what follows we are interested in the family of functions given by

$$f_\alpha(x) := \psi(x)\,|x|^{-\alpha}, \qquad x \in \mathbb{R}^d \setminus \{0\}, \qquad \alpha > 0. \tag{9.1}$$

We want to know, under what conditions on the parameters the functions f_α belong to some Smoothness Morrey spaces. For the Morrey spaces themselves it is already well-known, that for

$0 < p < u < \infty$ we have

$$f_\alpha \in \mathscr{M}^u_p(\mathbb{R}^d) \qquad \text{if and only if} \qquad \alpha \le \frac{d}{u}.$$

This result can be found in [146], see page 1849. There we also find $f_{d/u} \notin \mathring{\mathscr{M}}^u_p(\mathbb{R}^d)$. Now let us turn to function spaces of higher smoothness. For a start in a first step we investigate, under which conditions on the parameters the functions f_α belong to the Sobolev-Morrey spaces, see Definition 20.

Lemma 41. ***Sobolev-Morrey Spaces and Functions with a local Singularity.***
Let $0 < p < u < \infty$, $m \in \mathbb{N}$ and $m < \frac{d}{u}$. Then the following assertions are true.

(i) We have $f_\alpha \in W^m\mathscr{M}^u_p(\mathbb{R}^d)$ if and only if $m + \alpha \le \frac{d}{u}$.

(ii) Let in addition $p \ge 1$. Then we have $f_\alpha \notin \mathring{W}^m\mathscr{M}^u_p(\mathbb{R}^d)$ if $m + \alpha = \frac{d}{u}$.

Proof. *Step 1.* At first we prove sufficiency in (i). Let $\beta \in \mathbb{N}_0^d$ with $|\beta| \le m$. It follows from the Leibniz rule and the smoothness of ψ, that

$$|D^\beta f_\alpha(x)| \le C_{\alpha,\beta}\, |x|^{-(\alpha+|\beta|)}\,, \qquad |x| < \frac{3}{2}\,,$$

with an appropriate constant $C_{\alpha,\beta}$. Hence with $m + \alpha \le \frac{d}{u}$ we find $D^\beta f_\alpha \in \mathscr{M}^u_p(\mathbb{R}^d)$ and therefore $f_\alpha \in W^m\mathscr{M}^u_p(\mathbb{R}^d)$.
Step 2. Now we prove necessity in (i). Therefore let $f_\alpha \in W^m\mathscr{M}^u_p(\mathbb{R}^d)$. We fix $\beta := (m,0,\ldots,0)$. We need to distinguish m even and m odd. If $m = 2m'$, then

$$D^\beta f_\alpha(x) = D^\beta(|x|^{-\alpha}) = \sum_{i=0}^{m'} c_i \frac{x_1^{2i}}{|x|^{\alpha+2m'+2i}}\,, \qquad |x| < 1\,,$$

where $\{c_i\}_{i=0}^{m'}$ are appropriate constants independent of x. If $m = 2m' + 1$, then

$$D^\beta f_\alpha(x) = D^\beta(|x|^{-\alpha}) = \sum_{i=0}^{m'} d_i \frac{x_1^{2i+1}}{|x|^{\alpha+2m'+1+2i}}\,,$$

where $\{d_i\}_{i=0}^{m'}$ are appropriate constants independent of x. Observe, that the terms $\frac{x_1^j}{|x|^{\alpha+2m'+j}}$ are ordered. So we have

$$\frac{|x_1|^{j+2}}{|x|^{\alpha+2m'+j+2}} \le \frac{|x_1|^j}{|x|^{\alpha+2m'+j}}\,.$$

Now we choose a subset A of $\mathbb{R}^d$ and a constant $c > 0$ by

$$A := \Big\{x \in \mathbb{R}^d : \ |x| < 1\,,\ |x_1| \ge \frac{\max(|x_2|,\ldots,|x_d|)}{c}\Big\}.$$

Let E denote the minimum of those constants $c_0,\ldots,c_{m'},d_0,\ldots,d_{m'}$ which are all positive. Then $c \ge 1$ is chosen in such a way, that

$$\Big|\sum_{i=0}^{m'} c_i \frac{x_1^{2i}}{|x|^{\alpha+2m'+2i}}\Big| \ge \frac{E}{2}\, \frac{|x_1^{2m'}|}{|x|^{\alpha+4m'}}\,, \qquad x \in A\,,$$

if $m = 2m'$ and

$$\Big|\sum_{i=0}^{m'} d_i \frac{|x_1|^{2i+1}}{|x|^{\alpha+2m'+2i+1}}\Big| \geq \frac{E}{2}\,\frac{|x_1|^{2m'+1}}{|x|^{\alpha+4m'+1}}\,, \qquad x\in A\,,$$

if $m = 2m'+1$. Then for $r\in(0,1)$ and β as above we have

$$\begin{aligned}\|f_\alpha|W^m\mathscr{M}^u_p(\mathbb{R}^d)\| &\geq |B(0,r)|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{B(0,r)\cap A}|D^\beta(|x|^{-\alpha})|^p\,dx\Big)^{\frac{1}{p}}\\ &\geq E_1\,|B(0,r)|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{B(0,r)\cap A}|x|^{-(\alpha+m)p}\,dx\Big)^{\frac{1}{p}}\\ &\geq E_2\,r^{\frac{d}{u}-(\alpha+m)}\end{aligned}$$

for appropriate positive constants E_1 and E_2 independent of r. On the one hand this yields necessity of $\alpha+m\leq\frac{d}{u}$ in (i). On the other hand we get $f_{\frac{d}{u}-m}\notin \mathring{W}^m\mathscr{M}^u_p(\mathbb{R}^d)$, see Lemma 32. ■

Now we turn to the Triebel-Lizorkin-Morrey spaces. Here the proof will be a bit more technical.

Lemma 42. ***Triebel-Lizorkin-Morrey Spaces and Functions with a local Singularity.***
Let $0<p<u<\infty$ and $0<q\leq\infty$. Let $s>\sigma_{p,q}$. Then the following assertions are true.

(i) We have $f_\alpha\in\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ if and only if $\alpha+s\leq\frac{d}{u}$.

(ii) We have $f_\alpha\notin\mathring{\mathscr{E}}^s_{u,p,q}(\mathbb{R}^d)$ if $\alpha+s=\frac{d}{u}$.

Proof. *Step 1.* We start with the proof of (i). Thereto we will use Theorem 5 with $v=1$ and $a=1$.
Substep 1.1. Sufficiency. By means of the elementary embedding $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\hookrightarrow\mathscr{E}^s_{u,p,\infty}(\mathbb{R}^d)$ we may restrict us to the case $q<\infty$. The membership of f_α in Morrey spaces already has been investigated. Therefore it remains to deal with

$$\Big\|\Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{B(0,t)}|\Delta^N_h f_\alpha(\cdot)|\,dh\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|.$$

Here we assume $\alpha+s\leq d/u$ and $N>s$. Because of the compact support it will be enough to deal with small balls in the Morrey quasi-norm. Furthermore, because of the radial symmetry it will be sufficient to study the balls $B(0,r)$ with $0<r<1$. So we are interested in

$$\sup_{0<r<1} r^{d(\frac{1}{u}-\frac{1}{p})}\Big(\int_{|x|<r}\Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{B(0,t)}|\Delta^N_h|x|^{-\alpha}|\,dh\Big)^q\frac{dt}{t}\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}. \tag{9.2}$$

We split the integral with respect to h into three parts, namely
(a) $|h|<|x|/(2N)$,
(b) $|x|/(2N)\leq|h|<2|x|$ and
(c) $|h|\geq 2|x|$.
Case (a). Here for $1\leq l\leq N$ we have $|x+lh|\geq|x|-N|h|\geq|x|/2$. Moreover, the Mean Value Theorem yields

$$|\Delta^N_h|x|^{-\alpha}|\leq C_{\alpha,N}|h|^N\max_{|\gamma|=N}\sup_{|x-y|\leq N|h|}|D^\gamma|y|^{-\alpha}|\leq c_1\,|h|^N\,|x|^{-\alpha-N}.$$

Consequently we find

$$\int_0^{|x|} t^{-(s+d)q}\Big(\int_{\substack{|h|<t,\\ |h|<|x|/(2N)}} |\Delta_h^N |x|^{-\alpha}|dh\Big)^q \frac{dt}{t} \le c_2|x|^{-(\alpha+N)q}\int_0^{|x|} t^{(N-s)q}\frac{dt}{t} \le c_3|x|^{-(\alpha+s)q}.$$

In the case $0<|x|<t<1$ we use the trivial estimate

$$|\Delta_h^N |x|^{-\alpha}| \le 2^N \max_{0\le l\le N} |x+lh|^{-\alpha} \le C|x|^{-\alpha}$$

and obtain

$$\int_{|x|}^1 t^{-sq}\Big(t^{-d}\int_{\substack{|h|<t,\\ |h|<|x|/(2N)}} |\Delta_h^N |x|^{-\alpha}|\,dh\Big)^q \frac{dt}{t} \le c_4\,|x|^{-\alpha q}\int_{|x|}^1 t^{-sq}\frac{dt}{t} \le c_5\,|x|^{-(\alpha+s)q}.$$

Combining both estimates in Case (a) we get

$$\int_0^1 t^{-sq}\Big(t^{-d}\int_{\substack{|h|<t,\\ |h|<|x|/(2N)}} |\Delta_h^N |x|^{-\alpha}|\,dh\Big)^q \frac{dt}{t} \le c_6\,|x|^{-(\alpha+s)q}. \tag{9.3}$$

Case (c). Next we look at the case $2|x|\le|h|<t\le 1$. Here we observe

$$\begin{aligned}
&\int_0^1 t^{-sq}\Big(t^{-d}\int_{2|x|\le|h|<t} |\Delta_h^N |x|^{-\alpha}|\,dh\Big)^q \frac{dt}{t}\\
&\qquad\le c_7\int_{2|x|}^1 t^{-sq}\Big(t^{-d}\int_{2|x|\le|h|<t} (|x|^{-\alpha}+|x+Nh|^{-\alpha})\,dh\Big)^q \frac{dt}{t}\\
&\qquad\le c_8\left[\int_{2|x|}^1 t^{-sq}\Big(t^{-d}\int_{2|x|\le|h|<t} |x+Nh|^{-\alpha}\,dh\Big)^q \frac{dt}{t} + |x|^{-(\alpha+s)q}\right].
\end{aligned}$$

Now since $|x+Nh|\ge N|h|-|x|\ge c_9|h|$ we obtain

$$\begin{aligned}
&\int_0^1 t^{-sq}\Big(t^{-d}\int_{2|x|\le|h|<t} |\Delta_h^N |x|^{-\alpha}|\,dh\Big)^q \frac{dt}{t}\\
&\qquad\le c_{10}\left[|x|^{-\alpha q}\int_{2|x|}^1 t^{-sq}\frac{dt}{t} + |x|^{-(\alpha+s)q}\right] \le c_{11}\,|x|^{-(\alpha+s)q}.
\end{aligned} \tag{9.4}$$

Case (b). It remains to deal with $|x|/(2N)\le|h|<2|x|$. Temporarily we assume $2|x|<1$. In analogy to Case (c) we find

$$\begin{aligned}
&\int_0^1 t^{-sq}\Big(t^{-d}\int_{\substack{|h|<t,\\ |x|/(2N)\le|h|<2|x|}} |\Delta_h^N |x|^{-\alpha}|\,dh\Big)^q \frac{dt}{t}\\
&\qquad\le c_{12}\int_{|x|/(2N)}^1 t^{-sq}\Big(t^{-d}\int_{\substack{|h|<t,\\ |x|/(2N)\le|h|<2|x|}} (|x|^{-\alpha}+|x+Nh|^{-\alpha})\,dh\Big)^q \frac{dt}{t}\,.
\end{aligned}$$

It is not difficult to see, that

$$\int_{|x|/(2N)}^1 t^{-sq}\Big(t^{-d}\int_{\substack{|h|<t,\\ |x|/(2N)\le|h|<2|x|}} |x|^{-\alpha}\,dh\Big)^q \frac{dt}{t} \le c_{13}\,|x|^{-(\alpha+s)q}\,.$$

On the other hand we observe

$$\begin{aligned}
&\int_{|x|/(2N)}^{1} t^{-sq}\Big(t^{-d}\int_{\substack{|h|<t,\\ |x|/(2N)\le|h|<2|x|}} |x+Nh|^{-\alpha}\,dh\Big)^{q}\frac{dt}{t}\\
&\quad\le \int_{|x|/(2N)}^{1} t^{-sq}\Big(t^{-d}\int_{|y|<\min(|x|+Nt,(2N+1)|x|)} |y|^{-\alpha}\,dy\Big)^{q}\frac{dt}{t}\\
&\quad\le c_{14}\int_{|x|/(2N)}^{1} t^{-sq}t^{-dq}(\min(|x|+Nt,(2N+1)|x|))^{(-\alpha+d)q}\,\frac{dt}{t}\\
&\quad= c_{14}\int_{|x|/(2N)}^{2|x|} t^{-sq}t^{-dq}(|x|+Nt)^{(-\alpha+d)q}\,\frac{dt}{t} + c_{14}\int_{2|x|}^{1} t^{-sq}t^{-dq}((2N+1)|x|)^{(-\alpha+d)q}\frac{dt}{t}\\
&\quad= c_{14}(I_1+I_2),
\end{aligned}$$

where we used $\alpha<d$. Since

$$I_1 \le c_{15}\,|x|^{-(s+\alpha)q} \qquad\text{and}\qquad I_2\le c_{16}\,|x|^{-(s+\alpha)q}$$

it also follows for the case $|x|/(2N)\le|h|<2|x|$, that

$$\int_{0}^{1} t^{-sq}\Big(t^{-d}\int_{\substack{|h|<t,\\ |x|/(2N)\le|h|<2|x|}} |\Delta_h^N|x|^{-\alpha}|\,dh\Big)^{q}\frac{dt}{t}\le c_{17}\,|x|^{-(s+\alpha)q}. \tag{9.5}$$

The modifications for the case $|x|\le 1<2|x|$ are obvious. Now we are well prepared to deal with (9.2). When we combine (9.3) - (9.5), we obtain

$$\begin{aligned}
&\sup_{0<r<1} r^{d(\frac1u-\frac1p)}\Big(\int_{|x|<r}\Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{B(0,t)}|\Delta_h^N|x|^{-\alpha}|\,dh\Big)^q\frac{dt}{t}\Big)^{\frac pq}dx\Big)^{\frac1p}\\
&\qquad\le c_{18}\sup_{0<r<1} r^{d(\frac1u-\frac1p)}\Big(\int_{|x|<r}|x|^{-(\alpha+s)p}dx\Big)^{\frac1p}\\
&\qquad\le c_{19}\sup_{0<r<1} r^{d(\frac1u-\frac1p)}\Big(\int_0^r t^{-(\alpha+s)p}t^{d-1}dt\Big)^{\frac1p}.
\end{aligned}$$

Since $\alpha+s\le d/u<d/p$ this integral exists and we find

$$\begin{aligned}
&\sup_{0<r<1} r^{d(\frac1u-\frac1p)}\Big(\int_{|x|<r}\Big(\int_0^1 t^{-sq}\Big(t^{-d}\int_{B(0,t)}|\Delta_h^N|x|^{-\alpha}|\,dh\Big)^q\frac{dt}{t}\Big)^{\frac pq}dx\Big)^{\frac1p}\\
&\qquad\qquad\le c_{20}\sup_{0<r<1} r^{d(\frac1u-\frac1p)}r^{-(\alpha+s)}r^{\frac dp} = c_{20}\sup_{0<r<1} r^{\frac du-\alpha-s}<\infty.
\end{aligned}$$

This proves $f_\alpha\in\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ in the case $\alpha+s\le d/u$.
Substep 1.2. Necessity. Let $\alpha+s>d/u$. By means of the elementary embedding $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\hookrightarrow\mathscr{E}^s_{u,p,\infty}(\mathbb{R}^d)$ it will be enough, to consider the case $q=\infty$. We claim, that

$$\sup_{0<r<1} r^{d(\frac1u-\frac1p)}\Big(\int_{|x|<r}\Big(\sup_{0<t<1} t^{-s-d}\int_{B(0,t)}|\Delta_h^N|x|^{-\alpha}|\,dh\Big)^p dx\Big)^{\frac1p}=\infty.$$

Write $x=(x_1,x_2,\ldots,x_d)\in\mathbb{R}^d$ and $h=(h_1,h_2,\ldots,h_d)\in\mathbb{R}^d$. We put

$$\Omega(t):=\Big\{h\in\mathbb{R}^d:\ |h|<t\,,\ \frac{t}{2\sqrt d}\le\min_k h_k\Big\}\,,\qquad t>0\,.$$

Then for all $0 < t \le 1$ it follows the existence of a positive constant c_{21}, such that

$$|\Omega(t)| \ge c_{21}\, t^d\,. \tag{9.6}$$

Let $2^{-j-1} \le |x| \le 2^{-j}$ for some $j \in \mathbb{N}$ and $\min_k x_k \ge 0$. Moreover, we assume $2^{-i} \le t < 2^{-i+1}$ for some $i \in \mathbb{N}$ with $1 \le i < j - L'$, where $L' \in \mathbb{N}$ will be chosen later. Now let $h \in \Omega(2^{-i})$. Then for $k \in \{1,2,\ldots,d\}$ since $j - i > L'$ we observe

$$2x_k h_k \ge x_k \frac{2^{-i}}{\sqrt{d}} \ge x_k^2\, \frac{2^{j-i}}{\sqrt{d}} \ge x_k^2\, 2^{L'}\, \frac{1}{\sqrt{d}}\,.$$

Let $L \in \mathbb{N}$ such that $2^{L'} \frac{1}{\sqrt{d}} \ge 2^L$. Hence $(x_k + h_k)^2 \ge 2^L x_k^2$ and therefore

$$|x+h|^\alpha \ge 2^{\alpha \frac{L}{2}} |x|^\alpha\,.$$

The restrictions $x_k, h_k \ge 0$ for all $k \in \{1,2,\ldots,d\}$ also imply

$$|x+\ell h|^\alpha \ge 2^{\alpha\frac{L}{2}}|x|^\alpha \qquad \text{and} \qquad |x|^{-\alpha} \ge 2^{\alpha\frac{L}{2}}|x+\ell h|^{-\alpha}$$

for all $\ell \in \{1,\ldots,N\}$. Now we are able to find an appropriate estimate of $|\Delta_h^N f_\alpha|$. Under the constraints collected above, we obtain

$$|\Delta_h^N f_\alpha(x)| \ge |x|^{-\alpha} - \Big(\sum_{\ell=0}^{N-1} \binom{N}{\ell} |x+(N-\ell)h|^{-\alpha}\Big) \ge |x|^{-\alpha}(1 - 2^{N-\alpha\frac{L}{2}}).$$

Now we choose $L \in \mathbb{N}$ as small as possible, such that $1 - 2^{N-\alpha L/2} \ge 1/2$ is fulfilled. Then L only depends on N and α and we get

$$|\Delta_h^N f_\alpha(x)| \ge \frac{1}{2}\, |x|^{-\alpha}.$$

Choose $L' \in \mathbb{N}$ as the smallest number, that fulfills $2^{L'} \frac{1}{\sqrt{d}} \ge 2^L$. Then with $2^{-j-1} \le |x| \le 2^{-j} < 2^{-2-L'}$ and $\min_k x_k \ge 0$ we obtain

$$\begin{aligned}
\sup_{0<t<1} t^{-s-d} \int_{B(0,t)} |\Delta_h^N |x|^{-\alpha}|\, dh \;&\ge\; \sup_{i\in\mathbb{N}} 2^{i(s+d)} \int_{B(0,2^{-i})} |\Delta_h^N |x|^{-\alpha}|\, dh \\
&\ge\; \sup_{i\in\{1,2,\ldots,j-L'-1\}} 2^{i(s+d)} \int_{\Omega(2^{-i})} |\Delta_h^N |x|^{-\alpha}|\, dh \\
&\ge\; c_{22}\, 2^{(j-L'-1)s}\, |x|^{-\alpha}
\end{aligned}$$

for some positive c_{22} (independent of x) by taking into account (9.6). Next since $2^{-j-1} \le |x| \le 2^{-j}$ this can be rewritten as

$$\sup_{0<t<1} t^{-s-d} \int_{B(0,t)} |\Delta_h^N |x|^{-\alpha}|\, dh \ge c_{23}\, |x|^{-(\alpha+s)}\,. \tag{9.7}$$

In what follows we need the notation

$$B_j^+ := \Big(B(0,2^{-j}) \setminus B(0,2^{-j-1})\Big) \cap \Big\{x \in \mathbb{R}^d :\;\; x_k \ge 0 \text{ for all } k = 1,2,\ldots,d\Big\}, \qquad j \in \mathbb{N}.$$

By the construction and (9.7) it follows

$$\begin{aligned}\sup_{0<r<1} r^{d(\frac{1}{u}-\frac{1}{p})}\bigg(\int_{|x|<r}\bigg(\sup_{0<t<1} t^{-(s+d)}\int_{B(0,t)}|\Delta_h^N|x|^{-\alpha}|\,dh\bigg)^p dx\bigg)^{\frac{1}{p}}\\ \geq c_{23}\sup_{0<r<1} r^{d(\frac{1}{u}-\frac{1}{p})}\bigg(\sum_{j=L'+2}^{\infty}\int_{B(0,r)\cap B_j^+}|x|^{-(\alpha+s)p}dx\bigg)^{\frac{1}{p}}\\ \geq c_{24}\sup_{0<r<1} r^{d(\frac{1}{u}-\frac{1}{p})}\bigg(\int_0^{\min(r,2^{-L'-2})} t^{-(\alpha+s)p}t^{d-1}dt\bigg)^{\frac{1}{p}}.\end{aligned}$$

Now there are two possibilities. Either this integral is infinite or it is finite. In the first case our claim follows. In the second case when we have $-(\alpha+s)+d/p>0$, we conclude

$$\begin{aligned}\sup_{0<r<1} r^{d(\frac{1}{u}-\frac{1}{p})}\bigg(\int_{|x|<r}\bigg(\sup_{0<t<1} t^{-(s+d)}\int_{B(0,t)}|\Delta_h^N|x|^{-\alpha}|\,dh\bigg)^p dx\bigg)^{\frac{1}{p}}\\ \geq c_{25}\sup_{0<r<1} r^{d(\frac{1}{u}-\frac{1}{p})}\min(r,2^{-L'-2})^{-(\alpha+s)+\frac{d}{p}}\geq c_{26}\sup_{0<r<2^{-L'-2}} r^{\frac{d}{u}-\alpha-s}.\end{aligned}\tag{9.8}$$

Because of $\alpha+s>d/u$ the right-hand side is not finite and therefore $f_\alpha\notin\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.
Step 2. Next we prove (ii). We fix $\alpha:=d/u-s>0$. By means of the elementary embedding $\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)\hookrightarrow\overset{\diamond}{\mathscr{E}}{}^s_{u,p,\infty}(\mathbb{R}^d)$ it will be enough to concentrate on $q=\infty$. Here we can apply formula (9.8). It follows

$$\lim_{r\downarrow 0} r^{d(\frac{1}{u}-\frac{1}{p})}\bigg(\int_{|x|<r}\bigg(\sup_{0<t<1} t^{-(s+d)}\int_{B(0,t)}|\Delta_h^N|x|^{-\alpha}|\,dh\bigg)^p dx\bigg)^{\frac{1}{p}}\geq c_{26}\lim_{r\downarrow 0} r^{\frac{d}{u}-\alpha-s}=c_{26}>0.$$

Hence by Lemma 36 we find $f_{\frac{d}{u}-s}\notin\overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$. The proof is complete. ■

Notice, that a forerunner of Lemma 42 can be found in [101], see Lemma 2.3.1. Here the behavior of the function f_α has been investigated for the original Besov and Triebel-Lizorkin spaces. So for $0<p<\infty$, $0<q\leq\infty$ and $s>\sigma_p$ we know

$$f_\alpha\in F^s_{p,q}(\mathbb{R}^d)\qquad\text{if and only if}\qquad s<\frac{d}{p}-\alpha.$$

When we compare this with Lemma 42, we observe a different behavior for the limiting case $\alpha=d/p-s$. Here the function belongs to our function spaces in the Morrey case, but not in the Lebesgue case. As a reason for this difference we can identify the supremum in the Morrey quasi-norm. Now we turn to the Besov-Morrey spaces. For them the counterpart of Lemma 42 reads as follows.

Lemma 43. ***Besov-Morrey Spaces and Functions with a local Singularity. Part I.***
Let $0<p\leq u<\infty$, $0<q\leq\infty$ and $s>\sigma_p$. Then we have $f_\alpha\in\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ if and only if we have either $\alpha+s<\frac{d}{u}$ or $\alpha+s=\frac{d}{u}$ and $q=\infty$.

Proof. Essentially this can be proved in the same way as Lemma 42. Therefore we omit the details. We refer to [55], see Lemma 5. ■

When we compare the results for the Triebel-Lizorkin-Morrey spaces and the Besov-Morrey spaces, it turns out, that there is a different outcome for the borderline case $\alpha + s = d/u$, see Lemmas 42 and 43. The reason for this is the different position of the Morrey quasi-norm in the definitions of the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$. Notice, that in most of the cases the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\overset{\diamond}{\mathscr{N}}{}^s_{u,p,q}(\mathbb{R}^d)$ coincide, see Lemma 2.26 in [146]. Hence Lemma 43 also tells us, under what conditions we have $f_\alpha \in \overset{\diamond}{\mathscr{N}}{}^s_{u,p,q}(\mathbb{R}^d)$. Now let us have a look at the Besov-type spaces. For them we observe the following.

Lemma 44. ***Besov-type Spaces and Functions with a local Singularity.***
Let $0 < p < \infty$, $0 < q \leq \infty$ and $0 \leq \tau < 1/p$. Let $s > \sigma_p$. Then the following assertions are true.

(i) We have $f_\alpha \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ if and only if $\alpha + s \leq d(\frac{1}{p} - \tau)$.

(ii) We have $f_\alpha \notin \overset{\diamond}{B}{}^{s,\tau}_{p,q}(\mathbb{R}^d)$ if $\alpha + s = d(\frac{1}{p} - \tau)$.

Proof. This result can be proved with the same techniques as described in the proof of Lemma 42. So our main tool is Theorem 9. A detailed proof can be found in [149]. ∎

Let us compare the Lemmas 42 and 44. When we put $\tau = 1/p - 1/u$, we find, that we have almost the same outcome for the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. Also for the borderline case $\alpha + s = d(1/p - \tau)$ the behavior is similar. The reason for this is the fact, that in the definitions of both scales $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ the supremum of the Morrey quasi-norm stands outside. In contrast to this, when we compare the Lemmas 43 and 44, we observe a different behavior for the limiting case for the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$. This difference is caused by the different position of the supremum in the definitions of both scales. A detailed study of Lemma 43 brings out, that for $\alpha + s = d/u$ in most of the cases the function f_α does not belong to the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$. However, we are able to modify the function f_α in such a way, that the behavior at the critical border changes. For that purpose let $\alpha > 0$, $\delta > 0$ and $\vartheta > 0$ with ϑ very small. $\rho \in C_0^\infty(\mathbb{R}^d)$ is a smooth cut-off function with $\rho(x) = 1$ for $x \in B(0,\vartheta)$ and $\rho(x) = 0$ for $|x| > 2\vartheta$. We put

$$f_{\alpha,\delta}(x) = \rho(x)|x|^{-\alpha}(-\ln|x|)^{-\delta}. \tag{9.9}$$

That means, we want to modify the singularity at the origin a bit by a logarithmic term. When we study the properties of this function $f_{\alpha,\delta}$, we obtain the following result.

Lemma 45. ***Besov-Morrey Spaces and Functions with a local Singularity. Part II.***
Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ and $s > \sigma_p$. Let $\alpha > 0$ and $\delta > 0$. Then we have $f_{\alpha,\delta} \in \mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ if and only if we have either $\alpha + s < \frac{d}{u}$ or $\alpha + s = \frac{d}{u}$ with $\delta q > 1$.

Proof. Basically this result can be proved in the same way as Lemma 42. So our main tool for the proof is Theorem 7. For more details we refer to [55]. ∎

So it turns out, that if δ becomes larger, the regularity of $f_{\alpha,\delta}$ is increasing with respect to the fine index q.

9.2 Functions with a special Behavior at Infinity

In what follows, we plan to investigate a family of smooth and radial symmetric test functions with certain decay at infinity. More precisely for $\alpha > 0$ we will deal with the function

$$h_\alpha(x) := (1-\psi(x))\,|x|^{-\alpha}\,, \qquad x \in \mathbb{R}^d\,. \tag{9.10}$$

Here ψ is the same cut-off function as in (9.1). We want to find out, under what conditions on the parameters the functions h_α belong to some of our Smoothness Morrey spaces. For the Morrey spaces themselves with $0 < p < u < \infty$ we observe

$$h_\alpha \in \mathscr{M}^u_p(\mathbb{R}^d) \qquad \text{if and only if} \qquad \frac{d}{u} \le \alpha.$$

For that we also refer to [146], see page 1849. Now let us have a look at the Triebel-Lizorkin-Morrey spaces. For them we obtain the following result.

Lemma 46. ***Triebel-Lizorkin-Morrey Spaces and Functions with certain Decay.***
Let $0 < p < u < \infty$ and $0 < q \le \infty$.

(i) Let $s > \sigma_{p,q}$. Then we have $h_\alpha \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ if and only if $\frac{d}{u} \le \alpha$.

(ii) For all $s \in \mathbb{R}$ we have $h_{d/u} \in \mathring{\mathscr{E}}^s_{u,p,q}(\mathbb{R}^d)$.

(iii) We have $h_{d/u} \in \mathring{W}^m \mathscr{M}^u_p(\mathbb{R}^d)$ for all $m \in \mathbb{N}$.

Proof. *Step 1.* We start with the proof of (ii). Here temporarily we assume $s > \sigma_{p,q}$. Clearly $h_{d/u}$ is a $C^\infty-$ function. Let $\alpha \in \mathbb{N}^d_0$ be a multi-index. Then we claim, that $D^\alpha h_{d/u}$ belongs to $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ for all α. That means we claim, that $h_{d/u} \in E^s_{u,p,q}(\mathbb{R}^d) \subset \mathring{\mathscr{E}}^s_{u,p,q}(\mathbb{R}^d)$, see Definition 33. We shall work with Theorem 5. Therefore we have to deal with

$$\|\, D^\alpha h_{d/u}\,|\mathscr{M}^u_p(\mathbb{R}^d)\,\| \tag{9.11}$$

and

$$\left\| \left(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta^N_h D^\alpha h_{d/u}(\cdot)|\, dh \Big)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right| \mathscr{M}^u_p(\mathbb{R}^d) \right\| \tag{9.12}$$

with $N > s$. Let us start with (9.11). Since $h_{d/u}$ is smooth, estimates with respect to small balls are no problem. By means of the radial symmetry and $h_{d/u} \in C^\infty(\mathbb{R}^d)$ elementary calculations show, that it will be sufficient, to estimate

$$I_1 = \sup_{r>2} |B(0,r)|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{2<|x|<r} |D^\alpha |x|^{-\frac{d}{u}}|^p dx \Big)^{\frac{1}{p}}.$$

For any α by induction one can prove the existence of a constant C_α, such that

$$|D^\alpha |x|^{-\frac{d}{u}}| \le C_\alpha\, |x|^{-\frac{d}{u}-|\alpha|}\,, \qquad |x| > 0\,. \tag{9.13}$$

Hence

$$I_1 \le c_1 \sup_{r>2} r^{\frac{d}{u}-\frac{d}{p}} \Big(\int_{2<t<r} t^{-\frac{dp}{u}-|\alpha|p} t^{d-1} dt \Big)^{\frac{1}{p}} < \infty.$$

Therefore (9.11) is finite.
Now we turn to (9.12). Since $h_{d/u}$ is radial symmetric, $h_{d/u} \in C^\infty(\mathbb{R}^d)$ and $\operatorname{supp} h_{d/u} \cap B(0,1) = \emptyset$, again some elementary calculations show, that it will be sufficient to estimate

$$I_2 = \sup_{r>2+N} r^{\frac{d}{u}-\frac{d}{p}} \Big(\int_{2+N<|x|<r} \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N D^\alpha |x|^{-\frac{d}{u}}| \, dh \Big)^q \frac{dt}{t} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}}.$$

Therefore we can apply a consequence of the Mean Value Theorem consisting in

$$|\Delta_h^N D^\alpha |x|^{-\frac{d}{u}}| \leq C_{\alpha,N} |h|^N \max_{|\gamma|=N} \sup_{|x-z|\leq N|h|} |D^\gamma D^\alpha |z|^{-\frac{d}{u}}|$$

for some constant $C_{\alpha,N}$ independent of x with $|x| > 2+N$ and h with $|h| < 1$. Using this and $s < N$, we obtain

$$\begin{aligned} I_2 &\leq c_2 \sup_{r>2+N} r^{\frac{d}{u}-\frac{d}{p}} \Big(\int_{2+N<|x|<r} \Big(\int_0^1 t^{-sq+Nq} \max_{|\gamma|=N} \sup_{|x-z|\leq N} |D^{\gamma+\alpha}|z|^{-\frac{d}{u}}|^q \frac{dt}{t} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \\ &\leq c_3 \sup_{r>2+N} r^{\frac{d}{u}-\frac{d}{p}} \Big(\int_{2+N<|x|<r} \max_{|\gamma|=N} \sup_{|x-z|\leq N} |z|^{-\frac{dp}{u}-p|\gamma|-p|\alpha|} \Big(\int_0^1 t^{-sq+Nq} \frac{dt}{t} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \\ &\leq c_4 \sup_{r>2+N} r^{\frac{d}{u}-\frac{d}{p}} \Big(\int_{2+N<|x|<r} \sup_{|x-z|\leq N} |z|^{-\frac{dp}{u}-pN-p|\alpha|} dx \Big)^{\frac{1}{p}}. \end{aligned}$$

For $2+N < |x| < r$ we define $z' := \frac{|x|-N}{|x|} x$. Then because of

$$|z'| = |x| - N \qquad \text{and} \qquad |z'-x| = N$$

we obtain

$$\sup_{|x-z|\leq N} |z|^{-\frac{dp}{u}-pN-p|\alpha|} = \Big| \frac{|x|-N}{|x|} x \Big|^{-\frac{dp}{u}-pN-p|\alpha|} = (|x|-N)^{-\frac{dp}{u}-pN-p|\alpha|}.$$

Now we insert this in our estimate and use $|\alpha| = M \in \mathbb{N}_0$, to find

$$\begin{aligned} I_2 &\leq c_4 \sup_{r>2+N} r^{\frac{d}{u}-\frac{d}{p}} \Big(\int_{2+N<|x|<r} (|x|-N)^{-\frac{dp}{u}-pN-pM} dx \Big)^{\frac{1}{p}} \\ &\leq c_4 \sup_{r>2} r^{\frac{d}{u}-\frac{d}{p}} \Big(\int_{2<|x|<r} |x|^{-\frac{dp}{u}-pN-pM} dx \Big)^{\frac{1}{p}} \\ &\leq c_5 \sup_{r>2} r^{\frac{d}{u}-\frac{d}{p}} \Big(\int_{2<t<r} t^{-\frac{dp}{u}-pN-pM} t^{d-1} dt \Big)^{\frac{1}{p}}. \end{aligned}$$

But this term is almost the same as in the estimate of I_1. So like before we find $I_2 < \infty$. This proves the claim for $s > \sigma_{p,q}$. In the case $s \leq \sigma_{p,q}$ we may use the continuous embedding $\overset{\diamond}{\mathscr{E}}{}^{s_0}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \overset{\diamond}{\mathscr{E}}{}^{s_1}_{u,p,q}(\mathbb{R}^d)$ with $s_1 < s_0$. This completes the proof of (ii).
Step 2. Next we prove (iii). But this follows from the fact, that (9.11) is finite, see Definition 20 and Definition 32. Moreover, for $p > 1$ the result (iii) is a special case of (ii) anyway, see Lemma 1.
Step 3. Now we prove (i). Here at first we look at the case $\alpha < d/u$. Then we find $h_\alpha \notin \mathscr{M}^u_p(\mathbb{R}^d)$ and so also $h_\alpha \notin \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, see Theorem 5. Next we deal with $\alpha = d/u$. This time from (ii) we know $h_{d/u} \in \overset{\diamond}{\mathscr{E}}{}^s_{u,p,q}(\mathbb{R}^d)$ and so $h_{d/u} \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, see Definition 32. In a last step we look at $\alpha > d/u$. Here we use Theorem 5 and a modification of the calculations from Step 1 to obtain the desired result. ■

Notice, that in the formulation of Lemma 46 the smoothness parameter s does not play an important role. But since h_α is a smooth $C^\infty-$ function, this is not surprising. Let us mention, that it is not difficult to prove counterparts of Lemma 46 for the other Smoothness Morrey spaces. For example, for the Besov-type spaces we observe the following.

Lemma 47. ***Besov-type Spaces and Functions with certain Decay.***
Let $0 < p < \infty$, $\tau \in [0, 1/p)$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *Let* $\alpha = d(\frac{1}{p} - \tau)$. *Then we have* $h_\alpha \in \mathring{B}^{s,\tau}_{p,q}(\mathbb{R}^d)$.

Proof. This result can be proved with the same methods, we also used to show Lemma 46. For details we refer to [149]. ■

Knowledge concerning the smoothness properties of test functions like f_α and h_α will be important for us later, when we deal with complex interpolation of Triebel-Lizorkin-Morrey spaces. Then we can use the results from Lemma 42 and Lemma 46, to prove the necessity of some conditions, that are showing up in connection with this topic. However, for the interpolation results we intend to prove, some more preparations are required. So we need universal extension operators for Triebel-Lizorkin-Morrey spaces on domains. Those operators will be constructed in the next chapter.

Chapter 10

Extension Operators for Triebel-Lizorkin-Morrey Spaces on Domains

In this chapter it is our main goal, to construct linear and bounded extension operators from $\mathscr{E}^s_{u,p,q}(\Omega)$ into $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, where Ω is a Lipschitz domain as defined in Definition 16. Roughly speaking the purpose of such operators is, to transform functions from $\mathscr{E}^s_{u,p,q}(\Omega)$ into functions from Triebel-Lizorkin-Morrey spaces defined on the whole $\mathbb{R}^d$. Thereby on Ω itself the original function shall not be changed. When we construct such operators, in a first step we obtain an extension operator, that depends on the parameters p,q and s. Later in a second step we receive an extension operator, which works for all admissible parameter constellations simultaneously. Sometimes such an operator is called a universal extension operator. To construct our universal extension operator, we use some ideas of Rychkov, see [102]. Special thanks goes to C. Zhuo, see also [150]. He proved most of the results, that can be found in chapter 10.

10.1 Extension Operators from $\mathscr{E}^s_{u,p,q}(\Omega)$ into $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$

Hereafter we construct an extension operator, that maps from $\mathscr{E}^s_{u,p,q}(\Omega)$ into $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. For that purpose at first we need some additional notation. For any function h we use $L_h \in \mathbb{N}_0$ to denote the maximal number, such that h has vanishing moments up to order L_h, namely

$$\int_{\mathbb{R}^d} x^{\alpha} h(x)\, dx = 0 \qquad \text{for all multi-indices } \alpha \text{ with } |\alpha| \le L_h. \tag{10.1}$$

If either no or all moments vanish, we put $L_h = -1$ or $L_h = \infty$. For a given function λ we define $\lambda_j(x) := 2^{jd}\lambda(2^j x)$ with $x \in \mathbb{R}^d$ and $j \in \mathbb{N}$. Using this notation, we can formulate the following characterization for the Triebel-Lizorkin-Morrey spaces.

Lemma 48. ***A Characterization of $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ via Functions with vanishing Moments.***
Let $1 \le p \le u < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$. Let $\lambda_0 \in \mathscr{S}(\mathbb{R}^d)$ be a function, such that

$$\int_{\mathbb{R}^d} \lambda_0(x)\, dx \neq 0 \tag{10.2}$$

and

$$L_\lambda \geq [s] \qquad \textit{with} \qquad \lambda(\cdot) := \lambda_0(\cdot) - 2^{-d}\lambda_0\left(\frac{\cdot}{2}\right) \tag{10.3}$$

are fulfilled. Then the Triebel-Lizorkin-Morrey space $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)$, such that

$$\| f \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|_{\lambda_0} := \left\| \Big(\sum_{k=0}^{\infty} 2^{ksq}\, |\mathscr{F}^{-1}[\lambda_k \mathscr{F} f](\cdot)|^q \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \right\| < \infty$$

in the sense of equivalent norms. In the case $q = \infty$ the usual modifications have to be made.

Proof. In principle the proof follows the same lines as in the case $p = u$, see [17], [18] and Proposition 1.2 in [102]. So we skip the details. ■

In what follows we will deal with Triebel-Lizorkin-Morrey spaces on Lipschitz domains. We refer to the Definitions 15 and 16. For a special Lipschitz domain Ω one can find a narrow vertically directed cone K with vertex at the origin, such that its shifts $x + K$ are in Ω for every $x \in \Omega$. For example, we may take

$$K := \{(x', x_d) \in \mathbb{R}^d : \ |x'| < A^{-1} x_d\}, \tag{10.4}$$

where A denotes the Lipschitz constant of ω. Here ω is the Lipschitz function from Definition 16. Let $-K := \{-x : \ x \in K\}$ be the "reflected" cone. Then for every test function $\gamma \in \mathscr{D}(-K)$ and $f \in \mathscr{D}'(\Omega)$, the convolution $\gamma * f(x) = \langle f, \gamma(x - \cdot)\rangle$ is well defined in Ω, since $\operatorname{supp} \gamma(x - \cdot) \subset \Omega$ for $x \in \Omega$. When we use this notation, we can obtain a decomposition for functions, that are defined on Ω.

Proposition 21. ***Decomposition for Functions defined on Lipschitz Domains. Part I.***
Let $\Omega \subset \mathbb{R}^d$ be a special Lipschitz domain and let K be one associated cone as above. Moreover, let $\varphi_0 \in \mathscr{D}(-K)$ have nonzero integral and let $\varphi(\cdot) := \varphi_0(\cdot) - 2^{-d}\varphi_0(\cdot/2)$. Then for any given $L \in \mathbb{N}_0$ there exist functions $\psi_0,\ \psi \in \mathscr{D}(-K)$, such that $L_\psi \geq L$ and

$$f = \sum_{j=0}^{\infty} \psi_j * \varphi_j * f \tag{10.5}$$

for all $f \in \mathscr{D}'(\Omega)$.

Proof. This result can be found in [102]. ■

In what follows for any $f : \ \Omega \to \mathbb{C}$ by f_Ω we denote its extension from Ω to all of $\mathbb{R}^d$ by zero. In addition for $g : \ \mathbb{R}^d \to \mathbb{C}$ the symbol $g_{|_\Omega}$ denotes the restriction of g to Ω. This notation will also be used for distributions. Now we are able to formulate a first result concerning extension operators from $\mathscr{E}^s_{u,p,q}(\Omega)$ into $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.

Theorem 16. ***An Extension Operator for Triebel-Lizorkin-Morrey Spaces.***
Let $\Omega \subset \mathbb{R}^d$ be a special Lipschitz domain and K its associated cone. Let $s \in \mathbb{R}$, $1 \leq p \leq u < \infty$ and $1 \leq q \leq \infty$. Let $\varphi_0 \in \mathscr{D}(-K)$ satisfy (10.2) *and* (10.3)*. Let $\psi_0,\ \psi \in \mathscr{D}(-K)$ be given by Proposition 21 such that $L_\psi > d/\min(p,q)$. Then the map E defined by*

$$Ef := \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f)_\Omega\,, \qquad f \in \mathscr{D}'(\Omega)$$

induces a linear and bounded extension operator from $\mathscr{E}^s_{u,p,q}(\Omega)$ into the space $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Moreover, for any $f \in \mathscr{D}'(\Omega)$ we have $E(f)_{|_\Omega} = f$.

Proof. For the proof we follow the ideas of Rychkov, see [102]. Only a few modifications have to be made. The parameters s,p,u,q and d are considered to be fixed in what follows.
Step 1. Let $\mathscr{M}^u_p(l^s_q[\mathbf{P}])(\mathbb{R}^d)$ be the space of all sequences $\{g_j\}_{j\in\mathbb{N}_0}$ of locally integrable functions on $\mathbb{R}^d$, such that

$$\|\{g_j\}_{j=0}^\infty\,|\mathscr{M}^u_p(l^s_q[\mathbf{P}])(\mathbb{R}^d)\| := \Big\|\Big(\sum_{j=0}^\infty |2^{js}\mathbf{P}(g_j)|^q\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\| < \infty, \tag{10.6}$$

where $\mathbf{P}(g_j)$ denotes the Peetre maximal function of g_j, namely

$$\mathbf{P}(g_j)(x) := \sup_{y\in\mathbb{R}^d}\frac{|g_j(y)|}{(1+2^j|x-y|)^N}, \qquad x\in\mathbb{R}^d. \tag{10.7}$$

The natural number N will be chosen, such that

$$\frac{d}{\min(p,q)} < N \le L_\psi. \tag{10.8}$$

We claim, that for any $\{g_j\}_{j=0}^\infty \in \mathscr{M}^u_p(l^s_q[\mathbf{P}])(\mathbb{R}^d)$ the series $\sum_{j=0}^\infty \psi_j * g_j$ converges in $\mathscr{S}'(\mathbb{R}^d)$ and

$$\Big\|\sum_{j=0}^\infty \psi_j * g_j\Big|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\Big\| \le C_1\|\{g_l\}_{l=0}^\infty\,|\mathscr{M}^u_p(l^s_q[\mathbf{P}])(\mathbb{R}^d)\|. \tag{10.9}$$

By (2.14) from [102] we know, that if $L_\varphi \ge [s]$ and $L_\psi \ge N$, there exists some $\sigma\in(0,\infty)$, such that

$$2^{ls}|\varphi_l * \psi_j * g_j(x)| \le C_2 2^{-|l-j|\sigma}2^{js}\mathbf{P}(g_j)(x). \tag{10.10}$$

Here the constant C_2 is independent of $x\in\mathbb{R}^d$ and $l,j\in\mathbb{N}_0$ as well as $\{g_j\}_{j=0}^\infty$. By Lemma 48 we may assume, that $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ is equipped with the norm generated by φ_0. Thus for any $j\in\mathbb{N}_0$ we have

$$\|\psi_j * g_j|\mathscr{E}^{s-2\sigma}_{u,p,q}(\mathbb{R}^d)\| \le C_3\Big\|\Big(\sum_{l=0}^\infty 2^{-(2l+|l-j|)\sigma q}[2^{js}\mathbf{P}(g_j)]^q\Big)^{\frac{1}{q}}\Big|\mathscr{M}^u_p(\mathbb{R}^d)\Big\|.$$

From this we conclude, that for any $j\in\mathbb{N}_0$

$$\begin{aligned}\|\psi_j * g_j\,|\mathscr{E}^{s-2\sigma}_{u,p,q}(\mathbb{R}^d)\| &\le C_4\,2^{-j\sigma}\,\|2^{js}\,\mathbf{P}(g_j)|\mathscr{M}^u_p(\mathbb{R}^d)\| \\ &\le C_5\,2^{-j\sigma}\,\|\{\mathbf{P}(g_l)\}_{l=0}^\infty\,|\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)\|.\end{aligned}$$

This implies, that for all $k_1,\ k_2\in\mathbb{N}$ with $k_1<k_2$ we find

$$\Big\|\sum_{j=k_1}^{k_2}\psi_j * g_j\Big|\mathscr{E}^{s-2\sigma}_{u,p,q}(\mathbb{R}^d)\Big\| \le C_6\sum_{j=k_1}^{k_2} 2^{-j\sigma}\,\|\{\mathbf{P}(g_l)\}_{l=0}^\infty\,|\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)\| \le C_7 2^{-k_1\sigma}.$$

Hence $\sum_{j=0}^\infty \psi_j * g_j$ converges in $\mathscr{E}^{s-2\sigma}_{u,p,q}(\mathbb{R}^d)$ and therefore in $\mathscr{S}'(\mathbb{R}^d)$, since $\mathscr{E}^{s-2\sigma}_{u,p,q}(\mathbb{R}^d)\hookrightarrow\mathscr{S}'(\mathbb{R}^d)$. Now we turn to the norm estimate. By (10.10) for any $l\in\mathbb{N}_0$ and any $x\in\mathbb{R}^d$ we also find

$$2^{ls}\Big|\varphi_l * \Big(\sum_{j=0}^\infty \psi_j * g_j\Big)(x)\Big| \le C_8\sum_{j=0}^\infty 2^{-|l-j|\sigma}2^{js}\,\mathbf{P}(g_j)(x).$$

Taking the $\mathscr{M}^u_p(l_q)$-norm on both sides, it is easy to see, that (10.9) holds true.

Step 2. Now we aim to prove, that $f \in \mathscr{E}^s_{u,p,q}(\Omega)$ implies

$$E(f) \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d) \qquad \text{and} \qquad \| E(f) \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \le C_9 \| f \,|\mathscr{E}^s_{u,p,q}(\Omega)\| \,. \tag{10.11}$$

By definition for any $\varepsilon \in (0,\infty)$ there exists $g \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, such that $g_{|\Omega} = f$ in $\mathscr{D}'(\Omega)$ and

$$\| g \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \le \| f \,|\mathscr{E}^s_{u,p,q}(\Omega)\| + \varepsilon .$$

Let $g_j := (\varphi_j * f)_\Omega$ with $j \in \mathbb{N}_0$. We will show, that

$$\| \{(\varphi_j * f)_\Omega\}_{j=0}^\infty \,|\mathscr{M}^u_p(l^s_q[\mathbf{P}])(\mathbb{R}^d)\| \le C_{10} \| g \,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \,. \tag{10.12}$$

Again we apply an inequality due to Rychkov, see page 248 in [102]. We have

$$\sup_{y\in\mathbb{R}^d} \frac{|(\varphi_j * f)_\Omega(y)|}{(1+2^j|x-y|)^N} \le C_{11} \begin{cases} \sup\limits_{y\in\Omega} \frac{|\varphi_j * f(y)|}{(1+2^j|x-y|)^N} & \text{if} \quad x \in \Omega; \\ \sup\limits_{y\in\Omega} \frac{|\varphi_j * f(y)|}{(1+2^j|\widetilde{x}-y|)^N} & \text{if} \quad x \notin \overline{\Omega}. \end{cases}$$

Here $\widetilde{x} := (x', 2\omega(x') - x_d) \in \Omega$ is the point symmetric to $x \notin \overline{\Omega}$ with respect to $\partial\Omega$. Since the convolution of φ_j with f in Ω is only using values in Ω, we obtain

$$\varphi_j * f(x) = \varphi_j * g(x) \qquad\qquad \text{for any } x \in \Omega .$$

Hence

$$\sup_{y\in\mathbb{R}^d} \frac{|(\varphi_j * f)_\Omega(y)|}{(1+2^j|x-y|)^N} \le C_{12} \begin{cases} \mathbf{P}(\varphi_j * g)(x) & \text{if} \quad x \in \Omega; \\ \mathbf{P}(\varphi_j * g)(\widetilde{x}) & \text{if} \quad x \notin \overline{\Omega}. \end{cases}$$

Obviously for any ball $B(z,r) \subset \mathbb{R}^d$ we know, that

$$\begin{aligned} &|B(z,r)|^{\frac{1}{u}-\frac{1}{p}} \Big\| \Big(\sum_{j=0}^\infty \Big| 2^{js} \sup_{y\in\mathbb{R}^d} \frac{(\varphi_j * f)_\Omega}{(1+2^j|\cdot - y|)^N} \Big|^q \Big)^{\frac{1}{q}} \Big| L^p(B(z,r)) \Big\| \\ &\quad \le C_{13} |B(z,r)|^{\frac{1}{u}-\frac{1}{p}} \Big\| \Big(\sum_{j=0}^\infty \Big| 2^{js} \mathbf{P}(\varphi_j * g)(\cdot) \Big|^q \Big)^{\frac{1}{q}} \Big| L^p(B(z,r)\cap\Omega) \Big\| \\ &\quad + C_{13} |B(z,r)|^{\frac{1}{u}-\frac{1}{p}} \Big\| \Big(\sum_{j=0}^\infty \Big| 2^{js} \mathbf{P}(\varphi_j * g)(\widetilde{\cdot}) \Big|^q \Big)^{\frac{1}{q}} \Big| L^p(B(z,r)\cap\overline{\Omega}^\complement) \Big\| \\ &\quad =: \mathrm{I} + \mathrm{II} . \end{aligned}$$

Of course we have

$$\mathrm{I} \le C_{14} \| \{\mathbf{P}(\varphi_j * g)\}_{j=0}^\infty \,|\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)\| \,.$$

Concerning II we argue as follows. Let $x \in B(z,r) \cap \overline{\Omega}^\complement$. Independent of the situation ($z \in \Omega$ or $z \notin \Omega$) we associate to z the vector $\widetilde{z} := (z', 2\omega(z') - z_d)$. Here ω refers to the function occuring in the definition of a special Lipschitz domain, see Definition 16. It follows, that

$$|\widetilde{z} - \widetilde{x}|^2 \le |z' - x'|^2 + (2A\,|z'-x'| + |z_d - x_d|)^2 < \max(2A,1)^2\, r^2 .$$

We have $\widetilde{x} \in B(\widetilde{z}, \max(2A,1)r)$. By Rademacher's Theorem ω is differentiable almost everywhere in $\mathbb{R}^{d-1}$. Using this, we observe, that the transformation $T(x) = \widetilde{x}$ with $x \in \mathbb{R}^d$ has the Jacobi determinant $|\det J_T(x)| = 1$ almost everywhere. Thus it follows from a change of variable formula, see for example [40] or [10], that

$$\int_{B(z,r)\cap\overline{\Omega}^\complement} \Big(\sum_{j=0}^{\infty} \Big|2^{js}\mathbf{P}(\varphi_j * g)(T(x))\Big|^q\Big)^{\frac{p}{q}} dx \\ \leq C_{15}\int_{B(\widetilde{z},\max(2A,1)r)} \Big(\sum_{j=0}^{\infty} \Big|2^{js}\mathbf{P}(\varphi_j * g)(\widetilde{x})\Big|^q\Big)^{\frac{p}{q}} d\widetilde{x}.$$

Applying this inequality, we obtain

$$\begin{aligned} \mathrm{II} &\leq C_{16}|B(z,r)|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{B(\widetilde{z},\max(2A,1)r)} \Big(\sum_{j=0}^{\infty} \Big|2^{js}\mathbf{P}(\varphi_j * g)(\widetilde{x})\Big|^q\Big)^{\frac{p}{q}} d\widetilde{x}\Big)^{\frac{1}{p}} \\ &\leq C_{17}|B(z,\max(2A,1)r)|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{B(\widetilde{z},\max(2A,1)r)} \Big(\sum_{j=0}^{\infty} \Big|2^{js}\mathbf{P}(\varphi_j * g)(\widetilde{x})\Big|^q\Big)^{\frac{p}{q}} d\widetilde{x}\Big)^{\frac{1}{p}} \\ &\leq C_{18}\|\{\mathbf{P}(\varphi_j * g)\}_{j=0}^{\infty}|\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)\|. \end{aligned}$$

From this combined with the characterization of $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ via the Peetre maximal function with $N > d/\min(p,q)$ (see for example subsection 11.2 in [70]), we further deduce the fact, that we have $\mathrm{II} \leq C_{19}\|g|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$. Thus (10.12) is proved. By Step 1, (10.11) and (10.12) we conclude, that

$$\|E(f)|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C_{20}\|\{(\varphi_j * f)_\Omega\}_{j=0}^{\infty}|\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)\| \leq C_{21}\|f|\mathscr{E}^s_{u,p,q}(\Omega)\| + \varepsilon.$$

Letting $\varepsilon \to 0$ we find, that E is a bounded linear operator from $\mathscr{E}^s_{u,p,q}(\Omega)$ into $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.
Step 3. Let $\rho \in \mathscr{D}(\Omega)$. Then

$$\operatorname{supp}\int_{\mathbb{R}^d} \psi_j(x-\cdot)\rho(x)\,dx \subset \overline{\Omega},$$

where we used the fact, that the supports of ψ_0 and ψ are in $-K$. Hence

$$\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\psi_j(x-y)\rho(x)\,dx\Big)(\varphi_j * f)_\Omega(y)\,dy \\ = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\psi_j(x-y)\rho(x)\,dx\Big)(\varphi_j * f)(y)\,dy.$$

Finally from Proposition 21 we conclude

$$E(f)_{|\Omega} = \sum_{j=0}^{\infty}\psi_j * \varphi_j * f = f \qquad \text{in } \mathscr{D}'(\Omega).$$

So the proof is complete. ■

We remark, that the extension operator E in Theorem 16 depends on p, q and s. More precisely we need to have

$$[s] \leq L_\varphi \qquad \text{and} \qquad \min(p,q) > \frac{d}{L_\psi}. \tag{10.13}$$

In the next section we will show, that it is possible to overcome these restrictions.

10.2 Universal Extension Operators for Triebel-Lizorkin-Morrey Spaces

In this section it is our main goal, to construct a universal extension operator for the Triebel-Lizorkin-Morrey spaces. For that purpose we use some ideas from Rychkov, see [102]. Let us introduce some additional notation. Let Ω and K be as in the section before. By $\mathscr{S}'(\Omega)$ we denote the subset of $\mathscr{D}'(\Omega)$ consisting of all distributions, that have finite order and at most polynomial growth at infinity. More precisely we have $f \in \mathscr{S}'(\Omega)$, if and only if the estimate

$$|\langle f, \gamma \rangle| \leq c \sup_{x \in \Omega, |\alpha| \leq M} |D^{\alpha}\gamma(x)|(1+|x|)^M \qquad \text{for all} \qquad \gamma \in \mathscr{D}(\Omega)$$

is true with some constants c and $M \in \mathbb{N}_0$ depending on f. On page 250 in [102] we find, that $f \in \mathscr{S}'(\Omega)$, if and only if there exists a $g \in \mathscr{S}'(\mathbb{R}^d)$, such that $g_{|\Omega} = f$. In particular $\mathscr{E}^s_{u,p,q}(\Omega)$ is a subset of $\mathscr{S}'(\Omega)$. For distributions $f \in \mathscr{S}'(\Omega)$ we know the following decomposition lemma.

Lemma 49. ***Decomposition for Functions defined on Lipschitz Domains. Part II.***
Let $\Omega \subset \mathbb{R}^d$ be a special Lipschitz domain and K its associated cone. Then there exist four functions $\varphi_0,\ \varphi,\ \psi_0,\ \psi \in \mathscr{S}(\mathbb{R}^d)$ supported in $-K$, such that $L_\varphi = L_\psi = \infty$ and (10.5) *holds in $\mathscr{D}'(\Omega)$ for any $f \in \mathscr{S}'(\Omega)$.*

Proof. This result can be found in [102], see Theorem 4.1. ■

Now we are able, to construct a universal extension operator for the Triebel-Lizorkin-Morrey spaces.

Theorem 17. ***A Universal Extension Operator for Triebel-Lizorkin-Morrey Spaces. Part I.***
Let $s \in \mathbb{R}$, $1 \leq p \leq u < \infty$ and $1 \leq q \leq \infty$. Let $\Omega \subset \mathbb{R}^d$ be a special Lipschitz domain. Let $\varphi_0,\ \varphi,\ \psi_0,\ \psi \in \mathscr{S}(\mathbb{R}^d)$ be as in Lemma 49. Then the map E defined by

$$Ef := \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f)_\Omega\,, \qquad f \in \mathscr{S}'(\Omega), \tag{10.14}$$

yields a linear and bounded extension operator from $\mathscr{E}^s_{u,p,q}(\Omega)$ into $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ for all admissible values of $p,\ q,\ u$ and s.

Proof. The proof is based on that of Theorem 16 and similar to that of Theorem 4.1(b) in [102]. Let $f \in \mathscr{E}^s_{u,p,q}(\Omega)$. Then $f \in \mathscr{S}'(\Omega)$ follows. By Lemma 49 we have

$$\sum_{j=0}^{\infty} \psi_j * \varphi_j * f = f$$

in $\mathscr{D}'(\Omega)$. Moreover, since the supports of ψ_0 and ψ are in $-K$, it follows, that

$$E(f)_{|\Omega} = \sum_{j=0}^{\infty} \psi_j * \varphi_j * f = f.$$

It remains to prove, that the series in (10.14) converges in $\mathscr{S}'(\mathbb{R}^d)$ and

$$\| E(f)\,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C_1 \| f\,|\mathscr{E}^s_{u,p,q}(\Omega)\|\,. \tag{10.15}$$

Observe, that for any $l,\ j \in \mathbb{N}_0$ and $x \in \mathbb{R}^d$ we have

$$|\varphi_l * \psi_j * (\varphi_j * f)_\Omega(x)| \le \int_{\mathbb{R}^d} |\varphi_l * \psi_j(z)||(\varphi_j * f)_\Omega(x-z)|\,dz$$
$$\le \mathbf{P}((\varphi_j * f)_\Omega)(x) \int_{\mathbb{R}^d} |\varphi_l * \psi_j(z)|(1+2^j|z|)^N\,dz\,.$$

Here N is chosen as in (10.8). By Lemma 2.1 in [17], see also (4.8) in [102], we know, that for any $M \in \mathbb{N}$ and any $l,\ j \in \mathbb{N}_0$ we have

$$\int_{\mathbb{R}^d} |\varphi_l * \psi_j(z)|(1+2^j|z|)^N\,dz \ \le\ C_2 2^{-|l-j|M}.$$

Thus there is a $\sigma > 0$, such that

$$2^{ls}|\varphi_l * \psi_j * (\varphi_j * f)_\Omega(x)| \ \le\ C_3 2^{-|l-j|\sigma} 2^{js}\mathbf{P}((\varphi_j * f)_\Omega)(x), \qquad x \in \mathbb{R}^d\,.$$

Now by an argument similar to that we used in the proof of Theorem 16 above, we conclude, that the series in (10.14) converges in $\mathscr{S}'(\mathbb{R}^d)$. So (10.15) holds. ■

There also is a version of Theorem 17 for bounded Lipschitz domains.

Corollary 6. ***A Universal Extension Operator for Triebel-Lizorkin-Morrey Spaces. Part II.*** *Let $s \in \mathbb{R}$, $1 \le p \le u < \infty$ and $1 \le q \le \infty$. Let $\Omega \subset \mathbb{R}^d$ be either a bounded Lipschitz domain if $d \ge 2$ or a bounded interval if $d = 1$. Then there exists a linear and bounded extension operator E_Ω, such that*

$$E_\Omega \in \mathscr{L}(\mathscr{E}^s_{u,p,q}(\Omega) \to \mathscr{E}^s_{u,p,q}(\mathbb{R}^d))$$

simultaneously for all admissible values of $p,\ q,\ u$ and s. In addition for any $f \in \mathscr{S}'(\Omega)$ we have $E_\Omega(f)_{|\Omega} = f$ in $\mathscr{D}'(\Omega)$.

Proof. This result is a consequence of Theorem 17. We know, that multiplication by smooth functions from $\mathscr{D}(\Omega)$ preserves $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, see Theorem 6.1 in [144]. So a standard procedure (see [122] or page 244 in [102]) allows us to reduce the case of bounded Lipschitz domains to special Lipschitz domains. ■

We want to mention, that a different extension operator for Smoothness Morrey spaces has been investigated in [90], but restricted to a smaller class of domains.

Remark 14. ***Universal Extension Operators for $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $B^{s,\tau}_{p,q}(\mathbb{R}^d)$.*** *There also exists a universal extension operator for the Besov-Morrey spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, see Proposition 4.13 in [146]. A counterpart for the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can be found in [149].*

Our universal extension operator from Corollary 6 can be used, to prove results concerning complex interpolation of Triebel-Lizorkin-Morrey spaces on bounded Lipschitz domains. This will be done in the next chapter.

Chapter 11

Complex Interpolation of Triebel-Lizorkin-Morrey Spaces on Domains

In this chapter we deal with complex interpolation of Triebel-Lizorkin-Morrey spaces on domains. For that purpose let (X_0,X_1) be an interpolation couple of Banach spaces. Then by $[X_0,X_1]_\Theta$ we denote the result of the complex interpolation of these spaces. Sometimes $[X_0,X_1]_\Theta$ is also called Calderón's first complex interpolation method. We refer to Calderón [24], Bergh, Löfström [3], Kreĭn, Petunin, Semenov [65], Lunardi [74] and Triebel [127] for the basics. One of the most popular formulas in interpolation theory is given by

$$[L_{p_0}(\mathbb{R}^d), L_{p_1}(\mathbb{R}^d)]_\Theta = L_p(\mathbb{R}^d)\,, \tag{11.1}$$

with $1 \le p_0 < p_1 \le \infty$, $0 < \Theta < 1$ and $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$. It turns out, that within the larger family of Morrey spaces, the formula (11.1) is a singular point. Essentially as a result of Lemarié-Rieusset, see [66] and [67], it is known, that

$$[\mathscr{M}^{u_0}_{p_0}(\mathbb{R}^d), \mathscr{M}^{u_1}_{p_1}(\mathbb{R}^d)]_\Theta \neq \mathscr{M}^{u}_{p}(\mathbb{R}^d)\,,$$

except the trivial cases given by either $u_0 = p_0$, $u_1 = p_1$ or $u_0 = u_1$, $p_0 = p_1$. In [146] and [43] different explicit descriptions for the spaces $[\mathscr{M}^{u_0}_{p_0}(\mathbb{R}^d), \mathscr{M}^{u_1}_{p_1}(\mathbb{R}^d)]_\Theta$ can be found. When switching from Lebesgue spaces to Morrey spaces, we add two phenomena, one local and one global, see Definition 17. Hence when turning to spaces defined on bounded domains, the situation is becoming more easy. Here the global condition plays no role anymore. Based on this observation, in [146] one can find the formula

$$[\mathscr{M}^{u_0}_{p_0}([0,1]^d), \mathscr{M}^{u_1}_{p_1}([0,1]^d)]_\Theta = \mathring{\mathscr{M}}^{u}_{p}([0,1]^d) \tag{11.2}$$

with

$$1 \le p_0 < u_0 < \infty, \qquad 1 < p_1 < u_1 < \infty, \qquad p_0 < p_1, \qquad 0 < \Theta < 1$$

and

$$p_0 u_1 = p_1 u_0\,, \qquad \frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\,, \qquad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}\,.$$

The aim of this chapter will consist in an extension of (11.2) to the Triebel-Lizorkin-Morrey spaces $\mathscr{E}^s_{u,p,q}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. For doing that, we only will investigate cases, where the Lemarié-Rieusset condition $p_0 u_1 = p_1 u_0$ is satisfied. The importance of this restriction was proved in [66] and [67], see also the earlier works [100] and [9]. Let us mention, that many results from this chapter also can be found in the paper [150].

11.1 Complex Interpolation and sufficient Conditions

The main goal of this section is, to prove a counterpart of formula (11.2) for the Triebel-Lizorkin-Morrey spaces. For that purpose in a first step we observe, that there is a connection between complex interpolation and intersections of Triebel-Lizorkin-Morrey spaces.

Proposition 22. ***Complex Interpolation and Intersections.***
Let $\Theta \in (0,1)$, $s_i \in \mathbb{R}$, $p_i \in [1,\infty)$, $q_i \in [1,\infty]$ and $u_i \in [p_i,\infty)$ with $i \in \{0,1\}$. Let $p_0\, u_1 = p_1\, u_0$ and $s = (1-\Theta)s_0 + \Theta s_1$ as well as

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1} \qquad \text{and} \qquad \frac{1}{u} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

(i) Then we have

$$\big[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)\big]_\Theta = \overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)}^{\|\cdot|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|}. \tag{11.3}$$

(ii) Let $\Omega \subset \mathbb{R}^d$ be either a bounded Lipschitz domain if $d \geq 2$ or a bounded interval if $d = 1$. Then

$$\big[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)\big]_\Theta = \overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)}^{\|\cdot|\mathscr{E}^s_{u,p,q}(\Omega)\|}$$

holds.

Proof. Essentially (11.3) is proved in [146]. Nevertheless for convenience of the reader we will sketch a proof.
Step 1. Proof of (i). We need to switch to the associated sequence spaces $\mathbf{e}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)$ based on appropriate wavelet isomorphisms. For more details and proofs we refer to [97], [104] and [137]. For a definition of the spaces $\mathbf{e}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)$ recall Definition 22. The advantage of the sequence spaces $\mathbf{e}^s_{u,p,q}(\mathbb{R}^d)$ compared with the function spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ is, that they are Banach lattices. Calderón products $X_0^{1-\Theta} X_1^\Theta$ are well-defined for Banach lattices, see [24]. In [113] and [114] Shestakov has proved the following useful identity. Let (X_0, X_1) be an interpolation couple of Banach lattices and $\Theta \in (0,1)$. Then

$$[X_0, X_1]_\Theta = \overline{X_0 \cap X_1}^{\|\cdot|X_0^{1-\Theta} X_1^\Theta\|}.$$

We have

$$\mathbf{e}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)^{1-\Theta}\, \mathbf{e}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)^{\Theta} = \mathbf{e}^s_{u,p,q}(\mathbb{R}^d),$$

see [142]. So under the same restrictions as in Proposition 22, we find

$$[\mathbf{e}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathbf{e}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]_\Theta = \overline{\mathbf{e}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathbf{e}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)}^{\|\cdot|\mathbf{e}^s_{u,p,q}(\mathbb{R}^d)\|}.$$

Complex interpolation spaces are invariant under isomorphisms. Again based on appropriate wavelet isomorphisms we can turn back to the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. This proves (i).
Step 2. Proof of (ii). Here for the proof we employ a standard method, see for example Theorem 6.4.2 in [3], Theorem 1.2.4 in [127] or [132]. Suppose that E is our universal extension operator with respect to Ω, that was constructed in Corollary 6. Then we have

$$E \in \mathscr{L}(\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega) \to \mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)) \qquad \text{and} \qquad E \in \mathscr{L}(\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega) \to \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d))$$

as well as

$$E \in \mathscr{L}(\mathscr{E}^{s}_{u,p,q}(\Omega) \to \mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)).$$

It follows, that E is a coretraction to the restriction R with respect to Ω. It is $R \circ E = I$. Here I denotes the identity on the space defined on the domain. At the same time E is a linear and continuous extension operator in $\mathscr{L}(X \to Y)$ with

$$X := \overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)}^{\|\cdot|\mathscr{E}^{s}_{u,p,q}(\Omega)\|}$$

and

$$Y := \overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)}^{\|\cdot|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|}.$$

Furthermore, the restriction R applied to Y leads to X. Hence Theorem 1.2.4 in [127] together with Step 1 yield (ii). ■

Remark 15. ***Some Forerunners of Proposition 22.***

(i) *The formula* (11.3) *itself is explicitly stated in [47], see Theorem 1.5, but under slightly more restrictive conditions. Whereas in [47] the formula* (11.3) *was reduced to results on the second complex interpolation method of Calderón and an abstract result of Bergh, see [2], we employed Calderón products and an abstract result of Shestakov, see [113] and [114].*

(ii) *The formula* (11.3) *has several forerunners. So for Morrey spaces it has been used before in [73]. For the classical situation $p = u$ we refer to [118] and [119]. The general case was treated in [146].*

In what follows, we want to prove a characterization of the spaces $[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta$, that is as simple as possible. So it turns out, that these spaces coincide with the diamond spaces associated to the Triebel-Lizorkin-Morrey spaces.

Theorem 18. ***Complex Interpolation of Triebel-Lizorkin-Morrey Spaces. Part I.***
Let $\Omega \subset \mathbb{R}^d$ be either a bounded Lipschitz domain if $d \geq 2$ or a bounded interval if $d = 1$. We assume, that

(a) $1 \leq p_0 < p_1 < \infty$, $p_0 \leq u_0 < \infty$, $p_1 \leq u_1 < \infty$;

(b) $1 \leq q_0, q_1 \leq \infty$, $\min(q_0, q_1) < \infty$;

(c) $p_0 u_1 = p_1 u_0$;

(d) $s_0, s_1 \geq 0$; *either* $s_0 < s_1$ *or* $0 < s_0 = s_1$ *and* $q_1 \leq q_0$;

(e) $0 < \Theta < 1$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$, $s := (1-\Theta)s_0 + \Theta s_1$.

Then it holds

$$[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta = \mathring{\mathscr{E}}^{s}_{u,p,q}(\Omega)\,. \tag{11.4}$$

Proof. *Step 1.* Based on Proposition 22, we have to calculate

$$\overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)}^{\|\cdot|\mathscr{E}^{s}_{u,p,q}(\Omega)\|}\,.$$

Lemma 34 yields

$$E^{s}_{u,p,q}(\Omega) = E^{s_0}_{u_0,p_0,q_0}(\Omega) = E^{s_1}_{u_1,p_1,q_1}(\Omega)$$

and hence

$$E^{s}_{u,p,q}(\Omega) = E^{s_0}_{u_0,p_0,q_0}(\Omega) \cap E^{s_1}_{u_1,p_1,q_1}(\Omega).$$

Therefore just by the definition of the space $\mathring{\mathscr{E}}^{s}_{u,p,q}(\Omega)$, Definition 33 and the trivial embeddings $E^{s_i}_{u_i,p_i,q_i}(\Omega) \hookrightarrow \mathscr{E}^{s_i}_{u_i,p_i,q_i}(\Omega)$ with $i \in \{0,1\}$, we find

$$\mathring{\mathscr{E}}^{s}_{u,p,q}(\Omega) = \overline{E^{s}_{u,p,q}(\Omega)}^{\|\cdot|\mathscr{E}^{s}_{u,p,q}(\Omega)\|} \hookrightarrow \overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)}^{\|\cdot|\mathscr{E}^{s}_{u,p,q}(\Omega)\|}\,.$$

Step 2. Recall, that we have either $0 \leq s_0 < s_1$ or $0 < s_0 = s_1$ and $q_1 \leq q_0$. That means, the conditions in Lemma 40 are satisfied. We claim, that

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega) \hookrightarrow \mathring{\mathscr{E}}^{s}_{u,p,q}(\Omega)\,. \tag{11.5}$$

Let E denote the common extension operator. Let $f \in \mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)$. Then $Ef \in \mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)$. Let ψ be a function in $\mathscr{D}(\mathbb{R}^d)$, such that $\psi(x) = 1$ on $\overline{\Omega}$. Then the operator $h \mapsto \psi \cdot h$ belongs to $\mathscr{L}(\mathscr{E}^{\sigma}_{x,y,z}(\mathbb{R}^d) \to \mathscr{E}^{\sigma}_{x,y,z}(\mathbb{R}^d))$ for all admissible tuples (σ, x, y, z). Hence $g := \psi \cdot Ef \in \mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)$. Obviously we have $(\psi \cdot Ef)_{|\Omega} = f$ in $\mathscr{D}'(\Omega)$. Let B be a ball, such that $\overline{\Omega} \subset \operatorname{supp} \psi \subset B$. Then we find

$$g \in \mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d;B) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d;B) \hookrightarrow \mathring{\mathscr{E}}^{s}_{u,p,q}(\mathbb{R}^d)\,,$$

see Lemma 40. Obviously this means $f \in \mathring{\mathscr{E}}^{s}_{u,p,q}(\Omega)$ and this proves (11.5). So Step 1 and Step 2 combined with Theorem 17 complete the proof. ■

Let us add some remarks concerning the conditions on the parameters, that can be found in Theorem 18. So in (a) we have $p_0 < p_1$. This restriction comes in because of the method we use for the proof. It shows up in Lemma 40, when we apply Hölder's inequality. In (b) we find $\min(q_0, q_1) < \infty$. Maybe also this condition has technical reasons. We need it in Lemma 37, see formula (8.17). The restrictions concerning s_0 and s_1 given in (d) derive from Lemma 40. They are connected with embeddings. When we apply Theorem 18 with $p_0 = u_0$ and $p_1 = u_1$, we recover a well-known formula for the original Triebel-Lizorkin spaces. Then we find

$$[F^{s_0}_{p_0,q_0}(\Omega), F^{s_1}_{p_1,q_1}(\Omega)]_\Theta = \mathring{F}^{s}_{p,q}(\Omega) = F^{s}_{p,q}(\Omega)\,, \tag{11.6}$$

but under the extra condition (d). The Lemarié-Rieusset condition (c) disappears in this case. There is a list of references for (11.6). For example, let us mention Theorem 2.4.2.1 in [127], where the cases, that Ω is the whole $\mathbb{R}^d$ or a bounded $C^\infty-$ domain have been investigated. The situation $\Omega = \mathbb{R}^d$ also has been treated in [35] and in [62]. In [132] Triebel dealt with the case, that Ω is a bounded Lipschitz domain. There also is a version of Theorem 18 for the Sobolev-Morrey spaces. It reads as follows.

Corollary 7. ***Complex Interpolation of Sobolev-Morrey Spaces. Part I.***
Let $0 < \Theta < 1$, $m_0 \in \mathbb{N}_0$, $m_1 \in \mathbb{N}$ and either $m_0 < m_1$ or $0 < m_0 \le m_1$. Let $1 < p_0 < p_1 < \infty$, $p_0 < u_0 < \infty$, $p_1 < u_1 < \infty$ and $p_0\, u_1 = p_1\, u_0$. We define

$$s := (1-\Theta)m_0 + \Theta m_1\,, \qquad \frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \textit{and} \qquad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}\,.$$

Let $\Omega \subset \mathbb{R}^d$ be either a bounded Lipschitz domain if $d \ge 2$ or a bounded interval if $d = 1$. Then we have

$$\big[W^{m_0}\mathscr{M}^{u_0}_{p_0}(\Omega), W^{m_1}\mathscr{M}^{u_1}_{p_1}(\Omega)\big]_\Theta = \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,2}(\Omega). \tag{11.7}$$

In the special case $s = m \in \mathbb{N}$ we have

$$\big[W^{m_0}\mathscr{M}^{u_0}_{p_0}(\Omega), W^{m_1}\mathscr{M}^{u_1}_{p_1}(\Omega)\big]_\Theta = \overset{\diamond}{W}{}^{m}\mathscr{M}^{u}_{p}(\Omega). \tag{11.8}$$

Proof. This result is a direct consequence of Theorem 18 and Lemma 1. ■

Beside the situation we described in Theorem 18, there exist some more parameter constellations, for which we can characterize $[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta$.

Theorem 19. ***Complex Interpolation of Triebel-Lizorkin-Morrey Spaces. Part II.***
Let $\Omega \subset \mathbb{R}^d$ be as in Theorem 18. Let the parameters satisfy the conditions (a), (b), (c) and (e) from Theorem 18. In addition we require

$$\textit{(d')} \qquad s_0, s_1 \in \mathbb{R} \quad \textit{and} \quad s_0 - \frac{d}{u_0} > s_1 - \frac{d}{u_1}\,.$$

Then (11.4) *holds as well.*

Proof. For the proof at first we recall some well-known embedding relations. The new restriction (d') guarantees the continuous embedding

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{t}_{u_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d) \qquad \text{with} \qquad t := s_0 - d\left(\frac{1}{u_0} - \frac{1}{u_1}\right).$$

For that we refer to Corollary 2.2 in [144] and to [52]. In addition we get

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{t_\Theta}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d) \qquad \text{with} \qquad t_\Theta := s_0 - d\left(\frac{1}{u_0} - \frac{1}{u}\right).$$

Here we used $p_0 < p < p_1$ and $u_0 < u < u_1$ as well as

$$s_1 - \frac{d}{u_1} < s - \frac{d}{u} = (1-\Theta)\left(s_0 - \frac{d}{u_0}\right) + \Theta\left(s_1 - \frac{d}{u_1}\right) < s_0 - \frac{d}{u_0}\,.$$

Because of $t_\Theta > s$ we may apply Corollary 5 and obtain

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{t_\Theta}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)\,.$$

Consequently we have

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)\cap\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)=\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)\hookrightarrow \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)\,.$$

Hence we also find

$$\overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)\cap\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)}^{\|\cdot|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|}\hookrightarrow \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)\,. \tag{11.9}$$

Employing the universal extension operator E from Corollary 6, we conclude

$$\overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega)\cap\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)}^{\|\cdot|\mathscr{E}^{s}_{u,p,q}(\Omega)\|}\hookrightarrow \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega)\,.$$

To prove the reverse embedding, we argue as before. By Lemma 34 we have

$$E^{s}_{u,p,q}(\Omega)=E^{s_0}_{u_0,p_0,q_0}(\Omega)\cap E^{s_1}_{u_1,p_1,q_1}(\Omega)\subset\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega)\cap\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega).$$

This yields

$$\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega)\hookrightarrow[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega),\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta\,.$$

The proof is complete. ■

Let us add one remark concerning the new condition (d'). It guarantees the embedding $\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)\hookrightarrow\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)$, see also Theorem 2.1 in [52]. This makes it very simple to calculate the intersection $\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)\cap\mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)$. If condition (d') is violated, the behavior of our interpolation spaces may change. For that we refer to Proposition 23 below. A visualization of restriction (d') can be found in Figure 6. Notice, that in Theorem 19 we always have $s_0>s_1$. So there is no overlap with Theorem 18. Again there is a counterpart of Theorem 19 for the Sobolev-Morrey spaces.

Corollary 8. ***Complex Interpolation of Sobolev-Morrey Spaces. Part II.***
Let $0<\Theta<1$, $1<p_0<p_1<\infty$, $p_0<u_0<\infty$, $p_1<u_1<\infty$ *and* $p_0\,u_1=p_1\,u_0$. *Let* $m_0\in\mathbb{N},m_1\in\mathbb{N}_0$ *and* $m_0-\frac{d}{u_0}>m_1-\frac{d}{u_1}$. *We define*

$$s:=(1-\Theta)m_0+\Theta m_1\,,\qquad \frac{1}{p}:=\frac{1-\Theta}{p_0}+\frac{\Theta}{p_1}\qquad\textit{and}\qquad\frac{1}{u}:=\frac{1-\Theta}{u_0}+\frac{\Theta}{u_1}\,.$$

Let $\Omega\subset\mathbb{R}^d$ *be either a bounded Lipschitz domain if* $d\geq 2$ *or a bounded interval if* $d=1$. *Then* (11.7) *holds. In particular, if* $s=m\in\mathbb{N}$, *then also* (11.8) *is true.*

Proof. This result follows from Theorem 19 and Lemma 1. ■

11.2 Complex Interpolation and necessary Conditions

When we look at the main results concerning complex interpolation of Triebel-Lizorkin-Morrey spaces we proved in the last section, see Theorem 18 and Theorem 19, we find a lot of conditions on the parameters. It turns out, that at least some of these conditions are also necessary. So we can observe the following.

Proposition 23. ***Necessary Conditions for Complex Interpolation. Part I.***
Let $\Omega \subset \mathbb{R}^d$ be a domain. We assume, that

(a) $1 \le p_0 < p_1 < \infty$, $p_0 < u_0 < \infty$, $p_1 < u_1 < \infty$;

(b) $1 \le q_0, q_1 \le \infty$;

(c) $p_0 u_1 = p_1 u_0$.

If $0 < s_0 < \frac{d}{u_0}$, *and if*

$$s_1 := s_0 - d\left(\frac{1}{u_0} - \frac{1}{u_1}\right) > 0\,,$$

then with $0 < \Theta < 1$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$ *and* $s := (1-\Theta)s_0 + \Theta s_1$, *it holds*

$$[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta \not\subset \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega)\,.$$

Proof. For the complex method it is well-known, that $X_0 \cap X_1$ is a dense subset of $[X_0, X_1]_\Theta$, see for example Theorem 4.2.2 in [3] or Theorem 1.9.3 in [127]. Let the restrictions of Proposition 23 with respect to $p_0, p_1, u_0, u_1, q_0, q_1, s_0, s_1$ and Θ be satisfied. The parameters p, u, q and s are then fixed as well. Without loss of generality we may assume, that Ω contains the ball $B(0,2)$. Now we employ Lemma 42. The results from there immediately carry over to the spaces defined on domains. Therefore we choose $\alpha := \frac{d}{u_0} - s_0$. By assumption $\alpha > 0$ and $\alpha = \frac{d}{u_1} - s_1 = \frac{d}{u} - s$. Thus Lemma 42 implies

$$f_\alpha \in \mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega) \qquad \text{and} \qquad f_\alpha \notin \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega).$$

This proves the claim. ∎

Notice, that in general Proposition 23 has no counterpart for the original Triebel-Lizorkin spaces. Here in most of the cases (11.6) also holds for $s_1 = s_0 - d(\frac{1}{p_0} - \frac{1}{p_1})$, see Corollary 1.111 in [133]. So for the spaces $F^s_{p,q}(\Omega)$ the situation is much more transparent. Both Theorems 18 and 19 are formulated for bounded domains. Therefore one may ask, whether there are counterparts for Triebel-Lizorkin-Morrey spaces, that are defined on the whole $\mathbb{R}^d$. However, the answer is negative.

Proposition 24. ***Necessary Conditions for Complex Interpolation. Part II.***

(i) Let s_0 and s_1 be positive real numbers. Let the conditions (a), (b), (c) and (e) from Theorem 18 be satisfied. Moreover assume $p_0 < u_0$ and $p_1 < u_1$. Then

$$\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d) \not\subset [\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]_\Theta\,.$$

(ii) Under the same restrictions as in Theorem 19 we have

$$[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]_\Theta \hookrightarrow \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)\,.$$

Proof. *Step 1.* Proof of (i). It will be enough to show, that there exists a function $h \in \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)$, such that $h \notin [\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]_\Theta$. Therefore we will work with the family of test functions $h_{d/u}$, we investigated in Lemma 46. We have

$$\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d) \subset \mathscr{M}^{u_0}_{p_0}(\mathbb{R}^d) \cap \mathscr{M}^{u_1}_{p_1}(\mathbb{R}^d).$$

Moreover, we observe

$$\overline{\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d) \cap \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)}^{\|\cdot|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|} \subset \overline{\mathscr{M}^{u_0}_{p_0}(\mathbb{R}^d) \cap \mathscr{M}^{u_1}_{p_1}(\mathbb{R}^d)}^{\|\cdot|\mathscr{M}^{u}_{p}(\mathbb{R}^d)\|}.$$

Lemma 46 yields $h_{d/u} \in \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\mathbb{R}^d)$. But we have

$$h_{d/u} \notin \overline{\mathscr{M}^{u_0}_{p_0}(\mathbb{R}^d) \cap \mathscr{M}^{u_1}_{p_1}(\mathbb{R}^d)}^{\|\cdot|\mathscr{M}^{u}_{p}(\mathbb{R}^d)\|}.$$

This has been proved in [146], see page 1891. So the proof of (i) is complete.
Step 2. Proof of (ii). This has been proved in (11.9). ■

Notice, that Proposition 24 (i) has no counterpart for the original Triebel-Lizorkin spaces. For them we know a positive result, see for example Theorem 2.4.7 in [128]. Proposition 24 supplements the knowledge about the Morrey spaces. Here it holds

$$\overset{\diamond}{\mathscr{M}}{}^{u}_{p}(\mathbb{R}^d) \not\subset [\mathscr{M}^{u_0}_{p_0}(\mathbb{R}^d), \mathscr{M}^{u_1}_{p_1}(\mathbb{R}^d)]_\Theta \hookrightarrow \overset{\diamond}{\mathscr{M}}{}^{u}_{p}(\mathbb{R}^d)$$

if $1 \le p_0 < p_1 < \infty$, $p_0 < u_0 < \infty$, $p_1 < u_1 < \infty$ and $p_0 u_1 = p_1 u_0$. For that we refer to Corollary 2.38 in [146]. To summarize the results concerning complex interpolation we obtained up to now, let us compare Theorem 18, Theorem 19 and Proposition 23. For that purpose we shall plot a $(\frac{1}{u}, s)$-diagram, see Figure 6. Here the influence of the parameters p_0, q_0, p_1, q_1 is ignored. First we fixed a point $(1/u_0, s_0)$. Then we have indicated, for which regions in the plane we may apply either Theorem 18 or Theorem 19 or Proposition 23.

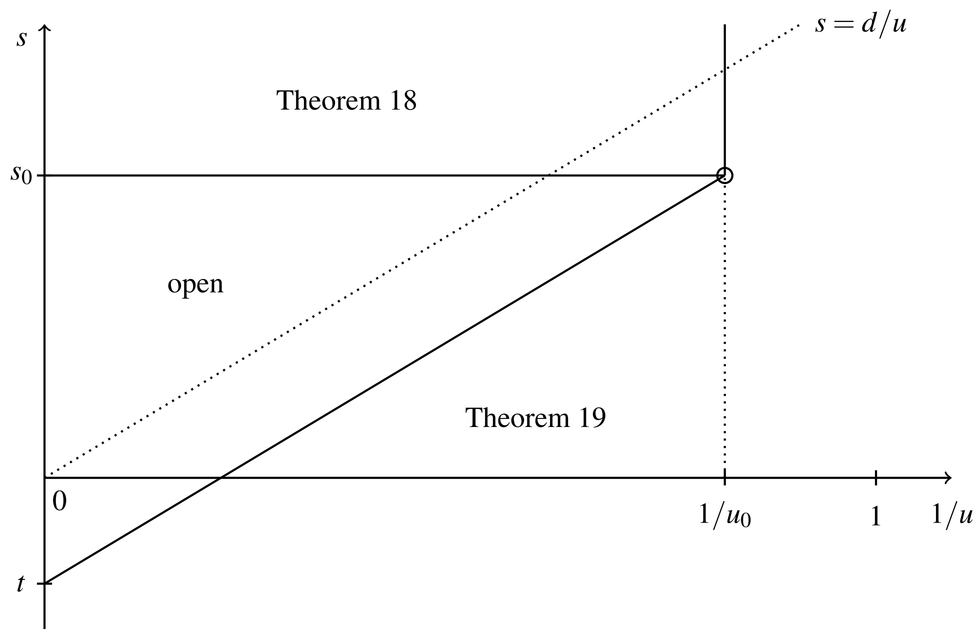

Figure 6. *Complex interpolation of Triebel-Lizorkin-Morrey spaces on domains.*

The point t is given by the Sobolev-type embedding as $t := s_0 - d/u_0$. In the open rectangle $\{(1/u_1, s_1): \ u_0 < u_1\,,\ s_1 > s_0\}$ we can apply Theorem 18. In the open triangle with corner points $(0,t)$, $(1/u_0, s_0)$, $(0, s_0)$ we do not know $[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta$. Below of the line connecting $(0,t)$ and $(1/u_0, s_0)$ we may apply Theorem 19. On this critical line Proposition 23 applies.

11.3 Related Results and open Problems

In what follows we collect some more material concerning interpolation of Morrey spaces and Smoothness Morrey spaces. Let us start with some further results regarding Calderón's first complex interpolation method $[\cdot,\cdot]_\Theta$. At first we want to mention, that the diamond spaces on domains form a scale under complex interpolation. So we have

$$[\overset{\diamond}{\mathscr{E}}{}^{s_0}_{u_0,p_0,q_0}(\Omega), \overset{\diamond}{\mathscr{E}}{}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta = \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega)$$

at least under the restrictions given in Theorem 18 or in Theorem 19. This follows from

$$E^s_{u,p,q}(\Omega) = E^{s_0}_{u_0,p_0,q_0}(\Omega) \cap E^{s_1}_{u_1,p_1,q_1}(\Omega) \subset [\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta = \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega)\,,$$

see Lemma 34, and from

$$\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega) = \overline{E^s_{u,p,q}(\Omega)}^{\|\cdot|\mathscr{E}^s_{u,p,q}(\Omega)\|} \qquad \text{as well as} \qquad \overline{\overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega)}^{\|\cdot|\mathscr{E}^s_{u,p,q}(\Omega)\|} = \overset{\diamond}{\mathscr{E}}{}^{s}_{u,p,q}(\Omega)\,.$$

Next we want to point out, that there is a counterpart of Theorem 18 for the Besov-Morrey spaces on domains. It reads as follows.

Theorem 20. ***Complex Interpolation of Besov-Morrey Spaces.***
Let $\Omega \subset \mathbb{R}^d$ be a bounded interval if $d = 1$ or a bounded Lipschitz domain if $d \geq 2$. Assume, that $0 < p_i \leq u_i < \infty$, $s_i \in \mathbb{R}$ and $q_i \in (0,\infty)$ for $i \in \{0,1\}$. Let

$$s := (1-\Theta)\, s_0 + \Theta\, s_1\,, \qquad \frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\,, \qquad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}.$$

If $u_0 p_1 = u_1 p_0$, then

$$[\mathscr{N}^{s_0}_{u_0,p_0,q_0}(\Omega), \mathscr{N}^{s_1}_{u_1,p_1,q_1}(\Omega)]_\Theta = \overset{\diamond}{\mathscr{N}}{}^{s}_{u,p,q}(\Omega)$$

holds true for all $\Theta \in (0,1)$.

Proof. This result can be found in [146], see Theorem 2.45 and Corollary 2.65. ■

There is a surprising difference to the case of the Triebel-Lizorkin-Morrey spaces. So in Theorem 20 we do not have an influence of the relation between s_0 and s_1. The main reason for this more simple behavior can be found in

$$\overset{\diamond}{\mathscr{N}}{}^{s}_{u,p,q}(\mathbb{R}^d) = \mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d) \qquad \text{if and only if} \qquad q \in (0,\infty),$$

see Lemma 2.26. in [146]. Let us mention, that there also exists a counterpart of Theorem 18 for the Besov-type spaces. Here we observe the following.

Theorem 21. ***Complex Interpolation of Besov-type Spaces.***
Let $\Omega \subset \mathbb{R}^d$ be either a bounded Lipschitz domain if $d \geq 2$ or a bounded interval if $d = 1$. We assume, that we have

(a) $1 \leq p_0 < p_1 < \infty$, $\tau_0 \in [0, 1/p_0)$, $\tau_1 \in [0, 1/p_1)$;

(b) $1 \leq q_0,\ q_1 \leq \infty$ *and* $\min(q_0, q_1) < \infty$;

(c) $p_0 \tau_0 = p_1 \tau_1$;

(d) $s_0,\ s_1 \geq 0$; *either* $s_0 < s_1$ *or* $0 < s_0 = s_1$ *and* $q_1 \leq q_0$;

(e) $0 < \Theta < 1$, $s := (1-\Theta)s_0 + \Theta s_1$, $\tau := (1-\Theta)\tau_0 + \Theta \tau_1$,

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}.$$

Then it holds

$$[B^{s_0,\tau_0}_{p_0,q_0}(\Omega), B^{s_1,\tau_1}_{p_1,q_1}(\Omega)]_\Theta = \mathring{B}^{s,\tau}_{p,q}(\Omega).$$

Proof. Theorem 21 is one of the main results in [149]. ■

In the paper [149] also some more results concerning complex interpolation of Besov-type spaces can be found. Beside $[\cdot,\cdot]_\Theta$ there are some more interpolation methods, that are of certain interest. Two examples are the $\pm$ method of Gustavsson and Peetre, see [39] and [38], and the second complex interpolation method introduced by Calderón. Concerning the $\pm$ method, denoted by $\langle \cdot, \cdot, \Theta \rangle$, Yuan, Sickel and Yang proved in [146], that

$$\langle \mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d), \Theta \rangle = \mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)$$

holds subject to the restrictions

(a) $0 < p_0 < p_1 < \infty$, $p_0 \leq u_0 < \infty$, $p_1 \leq u_1 < \infty$;

(b) $0 < q_0, q_1 \leq \infty$;

(c) $p_0 u_1 = p_1 u_0$;

(d) $s_0, s_1 \in \mathbb{R}$;

(e) $0 < \Theta < 1$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$, $s := (1-\Theta)s_0 + \Theta s_1$.

Concerning the second complex interpolation method, denoted by $[\cdot,\cdot]^\Theta$, Hakim, Nogayama and Sawano proved in [47], that

$$[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]^\Theta = \mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)$$

holds, if (a)-(e) are satisfied and in addition we have $p_0, p_1, q_0, q_1 \in (1, \infty)$. Furthermore, in view of certain subspaces of Triebel-Lizorkin-Morrey spaces, for each of the methods $[\cdot,\cdot]_\Theta$, $\langle \cdot, \cdot, \Theta \rangle$ and $[\cdot,\cdot]^\Theta$ much more is known. In particular the behavior of the following expressions already has been investigated:

- $\langle \mathring{\mathscr{E}}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d), \Theta\rangle$, $\langle \mathring{\mathscr{E}}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathring{\mathscr{E}}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d), \Theta\rangle$;
- $[\mathring{\mathscr{E}}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]_\Theta$, $[\mathring{\mathscr{E}}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathring{\mathscr{E}}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]_\Theta$;
- $[\mathring{\mathscr{E}}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]^\Theta$, $[\mathring{\mathscr{E}}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathring{\mathscr{E}}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]^\Theta$;
- $[\overset{\diamond}{\mathscr{E}}{}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \overset{\diamond}{\mathscr{E}}{}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]^\Theta$;
- $[\overset{\diamond}{\mathscr{E}}{}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \overset{\diamond}{\mathscr{E}}{}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]_\Theta$.

We refer to [109], [73], [142], [146] and [47] for results and further explanations. Similar investigations also have be done for Morrey spaces and certain subspaces of Morrey spaces. Here we want to mention [42], [43], [46], [41], [44] and [45]. For interpolation results concerning homogeneous Sobolev-Morrey spaces we refer to [22]. Interpolation of so-called local Morrey spaces was studied in [19] and [21]. To complete this chapter, we want to present a list of open problems concerning complex interpolation. Thereto we will concentrate on the first complex interpolation method for Triebel-Lizorkin-Morrey spaces on bounded Lipschitz domains.

Open Problem 4. ***Open Problems concerning Complex Interpolation.***

(i) The first question is about the role of the Lemarié-Rieusset condition $u_0\,p_1 = p_0\,u_1$, see Theorem 18. How do the interpolation spaces look like, if this condition is violated? There are some special cases, which one should investigate first like the following. Let $p_0 = p_1$ and $u_0 < u_1$. How do the interpolation spaces

$$[W^{m_0}\mathscr{M}^{u_0}_{p_0}(\Omega), W^{m_1}\mathscr{M}^{u_1}_{p_0}(\Omega)]_\Theta$$

look like in the case $m_0 < m_1$?

(ii) What happens, if $s_0 - d(\frac{1}{u_0} - \frac{1}{u_1}) < s_1 < s_0$ and $u_0\,p_1 = p_0\,u_1$? These cases are not treated in the Theorems 18 and 19. We also refer to Figure 6.

(iii) Find a characterization of $[\mathscr{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d), \mathscr{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)]_\Theta$ for all admissible constellations of the parameters. The answer could become technical.

(iv) In Theorem 18 we had to exclude the case $q_0 = q_1 = \infty$. So the question is about the characterization of $[\mathscr{E}^{s_0}_{u_0,p_0,\infty}(\Omega), \mathscr{E}^{s_1}_{u_1,p_1,\infty}(\Omega)]_\Theta$.

(v) Let us turn to Corollary 7. Here the case $p_0 = 1$ has been left out. What happens if $p_0 = 1$?

(vi) Probably even more difficult is the following question. Is there a wider class of domains, than bounded Lipschitz domains, that allows the validity of Theorem 18?

(vii) In the Theorems 18 and 19 we concentrated on the Banach space case. But of course the Triebel-Lizorkin-Morrey spaces are also defined for values $u, p, q \in (0,1)$, see Definition 19. Extensions of the complex method to quasi-Banach spaces are known as well. Here we refer to [61], [63], [62] and [143]. So the problem is to prove a counterpart of Theorem 18 for $u, p, q \in (0,1)$.

Chapter 12

The Fubini Property

The main subject of this chapter is the so called Fubini property. For the original Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ this property is known since many years. Roughly speaking, it allows us to describe Triebel-Lizorkin spaces, that are defined on the $\mathbb{R}^d$ with $d > 1$, in terms of one-dimensional Triebel-Lizorkin spaces. This offers the advantage, that sometimes we get the possibility, to reduce high-dimensional problems to one-dimensional problems. Therefore the main goal of this chapter will be, to find out, whether also the Triebel-Lizorkin-Morrey spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ with $p < u$ have the Fubini property.

12.1 Triebel-Lizorkin Spaces and the Fubini Property

Hereinafter we briefly recall the Fubini property for the original Triebel-Lizorkin spaces. In the literature there exist various versions of so called Fubini-type theorems. One of them reads as follows.

Lemma 50. ***The Fubini Property for Triebel-Lizorkin Spaces.***
Let $d \geq 2$. Moreover let $0 < p < \infty$, $0 < q \leq \infty$ and $s > \sigma_{p,q}$. Then for all $f \in F^s_{p,q}(\mathbb{R}^d)$ the quasi-norms $\| f | F^s_{p,q}(\mathbb{R}^d) \|$ and

$$\| f | F^s_{p,q}(\mathbb{R}^d) \|^F = \sum_{j=1}^{d} \Big\| \| f(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_d) | F^s_{p,q}(\mathbb{R}) \| \Big| L_p(\mathbb{R}^{d-1}) \Big\|$$

are equivalent.

Proof. This result can be found in [131], see Theorem 4.4. We also refer to [128], see section 2.5.13. ■

Lemma 50 has many useful applications. For example, it can be used, to extend assertions on mapping properties of nonlinear operators from the one-dimensional to the d-dimensional case. Here for instance we can refer to section 5.4.1 in [101]. Because of the advantages of Lemma 50, it would be desirable to have Fubini-type theorems also for the Triebel-Lizorkin-Morrey spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ with $p < u$.

12.2 Do the Spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ have the Fubini Property ?

Now let us investigate, whether there is a counterpart of Lemma 50 for the Triebel-Lizorkin-Morrey spaces. For that purpose we will use the following definition.

Definition 35. ***The Fubini Property.***
Let $s > 0$, $0 < p \le u < \infty$ and $0 < q \le \infty$. Let $d \ge 2$. Then we say, that the space $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ has the Fubini property, if the quasi-norm

$$\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^F = \sum_{j=1}^{d} \Big\| \|f(x_1,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_d)|\mathscr{E}^s_{u,p,q}(\mathbb{R})\| \Big| \mathscr{M}^u_p(\mathbb{R}^{d-1})\Big\| \tag{12.1}$$

is equivalent to $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$.

At the first moment one maybe could hope, that the Triebel-Lizorkin-Morrey spaces have the Fubini property also for $p < u$. But unfortunately for $p \ne u$ there is the following result.

Lemma 51. ***Many Triebel-Lizorkin-Morrey Spaces do not have the Fubini Property.***
Let $0 < p < u < \infty$, $0 < q \le \infty$ and $s > \sigma_{p,q}$. Let $d \ge 2$. Let in addition

$$p \le \frac{d-1}{d} u.$$

Then the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ do not have the Fubini property.

Proof. This result also can be found in the author's paper [57]. To prove it, we investigate the properties of a special test function. Let $f \in C_0^\infty(\mathbb{R}^{d-1})$ be a function $f : \mathbb{R}^{d-1} \to \mathbb{R}$, that depends on $x' = (x_2, x_3, \ldots, x_d)$ and has a support in $[0,1]^{d-1}$. We assume $\int_{[0,1]^{d-1}} |f(x')|^p dx' = 1$. Now we define the function $g : \mathbb{R}^d \to \mathbb{R}$ by $g(x_1, x') = f(x')$. At first we prove $\|g|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^F = \infty$. Of course we have $\mathscr{E}^s_{u,p,q}(\mathbb{R}) \hookrightarrow \mathscr{M}^u_p(\mathbb{R})$, see also Theorem 5. Let $t > 0$. Then we find

$$\begin{aligned}
\|g|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^F &\ge C_1 \Big\| \|g(\cdot,x')|\mathscr{M}^u_p(\mathbb{R})\| \Big| \mathscr{M}^u_p(\mathbb{R}^{d-1})\Big\| \\
&\ge C_2 t^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{[0,1]^{d-1}} \int_0^t |g(x_1,x')|^p dx_1 dx' \Big)^{\frac{1}{p}} \\
&= C_2 t^{\frac{1}{u}-\frac{1}{p}} \Big(\int_0^t \int_{[0,1]^{d-1}} |f(x')|^p dx' dx_1 \Big)^{\frac{1}{p}} = C_2 t^{\frac{1}{u}}.
\end{aligned}$$

If t tends to infinity, we obtain $\|g|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^F = \infty$. Now we prove, that under the given conditions we have $g \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$. Because of Theorem 5, it is enough to show $\|g|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m1)} < \infty$. At first we look at the Morrey quasi-norm. Since g is bounded and smooth, we only have to deal with large cubes $[0,t]^d$ with $t \ge 1$. We observe

$$\sup_{t \ge 1} t^{\frac{d}{u}-\frac{d}{p}} \Big(\int_0^t \int_{[0,1]^{d-1}} |f(x')|^p dx' dx_1 \Big)^{\frac{1}{p}} = \sup_{t \ge 1} t^{\frac{d}{u}-\frac{d-1}{p}} < \infty.$$

In the last step we used $p \le \frac{d-1}{d} u$. Now we have to deal with the second part of the quasi-norm $\|g|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1m1)}$. To do so, we apply the well-known formula

$$|\Delta_h^N g(x)| \le C|h|^N \max_{|\gamma|=N} \sup_{|x-y| \le N|h|} |D^\gamma g(y)|, \tag{12.2}$$

see also Lemma 24. Because of $f \in C_0^\infty(\mathbb{R}^{d-1})$ and $\operatorname{supp} f \subset [0,1]^{d-1}$, we find

$$\begin{aligned}
&\Big\| \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta_h^N g(x)| dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| \\
&\qquad \leq C_3 \Big\| \Big(\int_0^1 t^{-sq+Nq} \chi_{\{|x'| \leq d+Nd\}}(x) \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| \\
&\qquad \leq C_4 \Big\| \chi_{\{|x'| \leq d+Nd\}}(x) \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\|.
\end{aligned}$$

In the last step we used $N > s$. $\chi_{\{|x'| \leq d+Nd\}}$ is a characteristic function, that is zero, if $|x'|$ is large. Since $|\chi_{\{|x'| \leq d+Nd\}}(x)| \leq 1$ for all $x \in \mathbb{R}^d$, the supremum of the Morrey quasi-norm is realized by big cubes $[-t,t]^d$ with $t > d+Nd$. Therefore we obtain

$$\begin{aligned}
\Big\| \chi_{\{|x'| \leq d+Nd\}}(x) \Big| \mathscr{M}_p^u(\mathbb{R}^d) \Big\| &\leq C_5 \sup_{t > d+Nd} t^{\frac{d}{u} - \frac{d}{p}} \Big(\int_{-t}^t \int_{[-d-Nd, d+Nd]^{d-1}} \chi_{\{|x'| \leq d+Nd\}}(x) dx' dx_1 \Big)^{\frac{1}{p}} \\
&\leq C_6 \sup_{t > d+Nd} t^{\frac{d}{u} - \frac{d}{p} + \frac{1}{p}} < \infty.
\end{aligned}$$

In the last step again we used $p \leq \frac{d-1}{d} u$. The proof is complete. ■

So it turns out, that in most of the cases the Triebel-Lizorkin-Morrey spaces do not have the Fubini property. Of course Lemma 51 does not cover all possible cases. Hence it remains the following open problem.

Open Problem 5. ***Open Problem concerning the Fubini Property.***
Let $s > 0$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Let $d \geq 2$. Moreover, we assume

$$p \neq u \qquad \text{and} \qquad p > \frac{d-1}{d} u.$$

Then we want to know, whether the spaces $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ have the Fubini property according to Definition 35.

Chapter 13

Smoothness Morrey Spaces and Truncation

In this chapter we study the mapping properties of the truncation operator T^+ given by

$$(T^+ f)(x) = \max(f(x), 0), \qquad x \in \mathbb{R}^d, \tag{13.1}$$

in which f is a real-valued function from a Besov-Morrey space or a Triebel-Lizorkin-Morrey space. The operator T^+ is one member of a bigger class of operators, that is called composition operators, see chapter 5.3 in [101]. Theory concerning composition operators $T(g) : f \mapsto g \circ f$ or even more general Nemytzkij operators can be found in chapter 5 in [101] and in [1]. One may also consult [12], [14], [15] and [16]. Truncation operators play an important role in the theory of nonlinear partial differential equations, see for example chapter 8 in [36]. Throughout this chapter we will answer several questions concerning the operator T^+. In doing so our main focus will be on looking for sufficient and necessary conditions, under that the operator T^+ is bounded on the Besov-Morrey spaces or the Triebel-Lizorkin-Morrey spaces. It turns out, that the characterizations in terms of differences, we obtained for the spaces $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, see the Theorems 5 and 7, are very useful tools, to prove results concerning the boundedness of the truncation operator. But also some other auxiliaries will be used. T^+ is strongly connected with the operator T given by

$$(Tf)(x) = |f(x)|, \qquad x \in \mathbb{R}^d. \tag{13.2}$$

Both operators T^+ and T have many properties in common. Therefore in what follows we also will study the behavior of T. Notice, that most of the results from this chapter also can be found in the author's article [57].

13.1 Truncations: Classical Results and basic Properties

Now let us start with the investigation of the operators T^+ and T. For that purpose in this section we collect some first basic properties concerning the operators under investigation. The operator T^+ in the version of formula (13.1) only makes sense for real-valued functions. Therefore the following definition will be important for us later.

Definition 36. ***Real-valued Smoothness Morrey Spaces.***
Let $1 \le p \le u < \infty$, $1 \le q \le \infty$ and $s > 0$. Then we define the following function spaces.

(i) The real Morrey space $\mathbb{M}^u_p(\mathbb{R}^d)$ is defined to be the set of all real-valued functions f, that fulfill $f \in \mathscr{M}^u_p(\mathbb{R}^d)$.

(ii) The real Besov-Morrey space $\mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ is the collection of all real-valued functions f, that fulfill $f \in \mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$.

(iii) The real Triebel-Lizorkin-Morrey space $\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ is the collection of all real-valued functions f, such that $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$.

In what follows sometimes we write $\mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$. Then we mean either $\mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ or $\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$. Of course we always have $\mathbb{A}^s_{u,p,q}(\mathbb{R}^d) \subset \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$. Because of this many results we know for the usual Besov-Morrey and Triebel-Lizorkin-Morrey spaces, have obvious counterparts for the spaces $\mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$. So most of the results we found in subsection 4.2.2 also apply for the spaces $\mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$. Moreover, also the spaces $\mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$ can be described in terms of differences, like it is described in the Theorems 5 and 7. Using the notation from Definition 36, we are able to give a precise formulation of the problems we want to solve in the course of this chapter. So we will deal with the following queries. When we write T^*, we mean either T or T^+.

(1) Under which conditions on the parameters s,p,u,q and d do we have $T^*(\mathbb{A}^s_{u,p,q}(\mathbb{R}^d)) \subset \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$? This is the so-called acting property.

(2) Under which conditions on the parameters the operator $T^* : \mathbb{A}^s_{u,p,q}(\mathbb{R}^d) \to \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$ is bounded on $\mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$? With other words we want to know, when we can find a constant $C > 0$ independent of $f \in \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$, such that

$$\|T^* f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\| \le C\|f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|$$

holds for all $f \in \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$.

It is not difficult to see, that boundedness and the acting property are strongly connected with each other. When we know, that the operator T^* is bounded on a space $\mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$, we also have $T^*(\mathbb{A}^s_{u,p,q}(\mathbb{R}^d)) \subset \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$. Hence in most of the results and proofs we only will speak about boundedness. Another interesting fact is, that the operators T^+ and T have many properties in common. The reason for this is, that for real-valued f we have

$$\max(f(x),0) = \frac{1}{2}f(x) + \frac{1}{2}|f(x)|. \tag{13.3}$$

Because of this formula we learn, that whenever we have

$$\|Tf|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\| \le C_1\|f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|$$

for all $f \in \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$, we also have

$$\|T^+ f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\| \le C_2\|f|\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|$$

for all $f \in \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$. Thus in what follows in some proofs we only will work with the operator T. For the original Besov spaces as well as for the original Triebel-Lizorkin spaces there already exist results concerning the boundedness of truncation operators. Let $\mathbb{B}^s_{p,q}(\mathbb{R}^d)$ be the real part of $B^s_{p,q}(\mathbb{R}^d)$ and $\mathbb{F}^s_{p,q}(\mathbb{R}^d)$ be the real part of $F^s_{p,q}(\mathbb{R}^d)$. Then the following result is known since many years.

Theorem 22. ***Boundedness of Truncation Operators. Classical Case.***
Let $\mathbb{A} \in \{\mathbb{B}, \mathbb{F}\}$. Let $1 \le p < \infty$, $1 \le q \le \infty$ and $0 < s < 1 + 1/p$. For $\mathbb{A} = \mathbb{F}$ in the case $p = 1$ we assume $s \neq 1$. Then there is a constant $C > 0$ independent of $f \in \mathbb{A}^s_{p,q}(\mathbb{R}^d)$, such that

$$\|T^+ f | A^s_{p,q}(\mathbb{R}^d)\| \le C \|f | A^s_{p,q}(\mathbb{R}^d)\|$$

holds for all $f \in \mathbb{A}^s_{p,q}(\mathbb{R}^d)$. Moreover, in the formulation of Theorem 22 one can replace the operator T^+ by T.

Proof. This result was proved in [131], see Theorem 25.8 in chapter 25. For earlier contributions we refer to chapter 5.4.1. in [101] as well as to [11], [13] and [92]. Early findings for Sobolev spaces can be found in [76]. ■

In the next sections we will prove extensions of Theorem 22 for the spaces $\mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ with $p < u$. Thereby it turns out, that both the expected outcome and the methods for the proofs heavily depend on the parameters s, p, u and d. Very comfortable the situation is in the case $0 < s < 1$. Here we obtain the following result with an almost obvious proof.

Proposition 25. ***The Boundedness of T. The Case $0 < s < 1$.***
Let $1 \le p \le u < \infty$, $1 \le q \le \infty$ and $0 < s < 1$. Then there is a constant $C > 0$ independent of $f \in \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$, such that

$$\|T f | \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\| \le C \|f | \mathscr{A}^s_{u,p,q}(\mathbb{R}^d)\|$$

holds for all $f \in \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$.

Proof. *Step 1.* At first we deal with the case $\mathscr{A} = \mathscr{E}$. Since $0 < s < 1$ and $p, q \ge 1$, we can use Theorem 5 with $v = 1$, $a = \infty$ and $N = 1$. So we have to work with

$$\| \, |f| \, |\mathscr{M}^u_p(\mathbb{R}^d)\| + \Big\| \Big(\int_0^\infty t^{-sq} \Big(t^{-d} \int_{B(0,t)} |\Delta^1_h |f|(x)| dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}^d) \Big\|.$$

Now because of the triangle inequality we have

$$|\Delta^1_h |f|(x)| = ||f(x+h)| - |f(x)|| \le |f(x+h) - f(x)| = |\Delta^1_h f(x)|.$$

When we use Theorem 5 again, the proof for the case $\mathscr{A} = \mathscr{E}$ is complete.
Step 2. In the case $\mathscr{A} = \mathscr{N}$ the proof can be done in the same way. Here instead of Theorem 5 we have to use Theorem 7. We omit the details. ■

When we look at the case $s \ge 1$, the situation becomes much more complicated. Here it makes a big different, whether we deal with $d = 1$ or $d > 1$. Therefore in a first step we will concentrate on the one-dimensional case.

13.2 On the Boundedness of T in the Case $s > 1$ and $d = 1$

In this section we study the properties of the mapping $T : f \to |f|$ for functions from Triebel-Lizorkin-Morrey spaces or Besov-Morrey spaces in the case $s > 1$ and dimension $d = 1$. To this end a Hardy-type inequality will be an important tool.

13.2.1 A Morrey Version for Hardy-type Inequalities

In this subsection we prove a Morrey version for Hardy-type inequalities. It will be very important, when we investigate the boundedness of the operator T in the case $d = 1$. In a first step we formulate a Hardy-type inequality, that is well-known since many years and is not especially tailored for Morrey spaces. Let us recall some notation. For $x \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$ we write $\operatorname{dist}(x,A) = \inf_{y \in A} |x-y|$. By $A^{\complement}$ we mean $\mathbb{R}^d \setminus A$. With $\mathbb{S}(\mathbb{R}^d)$ we denote the collection of all real-valued functions f, that fulfill $f \in \mathscr{S}(\mathbb{R}^d)$.

Lemma 52. ***A Hardy-type Inequality.***
Let $1 \le p < \infty$, $1 \le q \le \infty$ and $0 < s < 1/p$. Let $d = 1$. Then there exists a constant $C > 0$, such that

$$\int_I |f(x)|^p \operatorname{dist}(x,I^{\complement})^{-sp} dx \ \le \ C \int_I \Big(\int_0^\infty r^{-sq} \Big(\int_{\substack{-1<h<1 \\ x-rh \in I}} |f(x) - f(x-rh)| dh \Big)^q \frac{dr}{r} \Big)^{\frac{p}{q}} dx$$

holds for all intervals I and all $f \in \mathbb{S}(\mathbb{R})$ satisfying $\int_I f(x)dx = 0$ if I is bounded.

Proof. This result can be found in [13], see Lemma 1. One may also consult [11] or chapter 3.1 in [80]. A detailed proof is given in [101], see Lemma 1 in chapter 5.4.1. ■

Now we want to prove a counterpart of Lemma 52, that also can be applied in the context of Morrey spaces. It reads as follows.

Lemma 53. ***A Hardy-type Inequality. A Morrey Version.***
Let $1 \le p \le u < \infty$, $1 \le q \le \infty$ and $0 < s < 1/u$. Let $d = 1$. Then there exists a constant $C > 0$, such that

$$\sup_{\substack{a,b \in \mathbb{R} \\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{I \cap (a,b)} |f(x)|^p \operatorname{dist}(x,I^{\complement})^{-sp} dx \Big)^{\frac{1}{p}}$$
$$\le C \sup_{\substack{a,b \in \mathbb{R} \\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{I \cap (a,b)} \Big(\int_0^\infty r^{-sq} \Big(\int_{\substack{-1<h<1 \\ x-rh \in I}} |f(x) - f(x-rh)| dh \Big)^q \frac{dr}{r} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}}$$

holds for all intervals I and all $f \in \mathbb{S}(\mathbb{R})$ satisfying $\int_I f(x)dx = 0$ if I is bounded.

Proof. To prove this result, we can use the methods, that are described in the proof of Lemma 1 from chapter 5.4.1. in [101], see also Lemma 1 in [13] and [11]. Only a few modifications have to be made.
Step 1. At first we look at the case $I = (0,\infty)$. For $x > 0$ we put

$$g(x) = \frac{1}{x} \int_0^x \Big(f(x) - f(y) \Big) dy \qquad \text{and} \qquad h(x) = g(x) - \int_x^\infty g(y) \frac{dy}{y}.$$

For these functions, since f is smooth, we observe

$$g'(x) = f'(x) - \Big(-\frac{1}{x^2}\int_0^x f(y)dy + \frac{f(x)}{x}\Big) \qquad \text{and} \qquad h'(x) = g'(x) + \frac{g(x)}{x} = f'(x).$$

Due to $f \in \mathbb{S}(\mathbb{R})$ we find $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = 0$ and therefore can conclude $f = h$. When we use this identity, we obtain

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(a,b)} |f(x)|^p x^{-sp} dx\Big)^{\frac{1}{p}} \\
&\qquad \le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(a,b)} |g(x)|^p x^{-sp} dx\Big)^{\frac{1}{p}} \\
&\qquad\qquad + \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(a,b)} \Big| \int_x^\infty g(y)\frac{dy}{y}\Big|^p x^{-sp} dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Now at first we look at the second term. We put $y = x\xi$ and get

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(a,b)} \Big| \int_x^\infty g(y)\frac{dy}{y}\Big|^p x^{-sp} dx\Big)^{\frac{1}{p}} \\
&\qquad \le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \int_1^\infty \xi^{-1} \Big(\int_{(0,\infty)\cap(a,b)} |g(x\xi)x^{-s}|^p dx\Big)^{\frac{1}{p}} d\xi.
\end{aligned}$$

To continue, we put $x\xi = z$ and obtain

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(a,b)} \Big| \int_x^\infty g(y)\frac{dy}{y}\Big|^p x^{-sp} dx\Big)^{\frac{1}{p}} \\
&\qquad \le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} \int_1^\infty \xi^{-1-\frac{1}{u}+s} |\xi a - \xi b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(\xi a,\xi b)} |g(z)z^{-s}|^p dz\Big)^{\frac{1}{p}} d\xi.
\end{aligned}$$

Next we have to use $s < 1/u$. Then we find

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(a,b)} \Big| \int_x^\infty g(y)\frac{dy}{y}\Big|^p x^{-sp} dx\Big)^{\frac{1}{p}} \\
&\qquad \le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} \sup_{1\le\rho\le\infty} |\rho a - \rho b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(\rho a,\rho b)} |g(z)z^{-s}|^p dz\Big)^{\frac{1}{p}} \int_1^\infty \xi^{-1-\frac{1}{u}+s} d\xi \\
&\qquad \le C_1 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,\infty)\cap(a,b)} |g(z)z^{-s}|^p dz\Big)^{\frac{1}{p}}.
\end{aligned}$$

In what follows we will work with the abbreviation

$$u(r,x) = \int_{\substack{-1<y<1\\ x-ry>0}} |f(x) - f(x-ry)| dy.$$

Then because of

$$g(x) = \frac{y}{x}\int_0^{\frac{x}{y}} \Big(-f(x)+f(x-ry)\Big)\, dr\,, \qquad\qquad y\neq 0\,,$$

we can apply Hölder's inequality to get

$$|g(x)| = 2\Big|\int_{\frac{1}{2}}^1 \frac{y}{x}\int_0^{\frac{x}{y}} \Big(-f(x)+f(x-ry)\Big)\, dr\, dy\Big| \le C_2\Big(\int_0^{2x} |u(r,x)|^q x^{-1} dr\Big)^{\frac{1}{q}}.$$

Using this estimate, we obtain

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,\infty)\cap(a,b)} |g(x)|^p x^{-sp} dx\Big)^{\frac{1}{p}}\\
&\qquad \le C_3 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,\infty)\cap(a,b)} \Big(\int_0^{2x} |u(r,x)|^q x^{-1-sq} dr\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}\\
&\qquad \le C_4 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,\infty)\cap(a,b)} \Big(\int_0^{\infty} |u(r,x)|^q r^{-1-sq} dr\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

In the last step we used $r\le 2x$. So Step 1 of the proof is complete.
Step 2. Next we look at the case $I=(0,1)$. At first we observe

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,1)\cap(a,b)} |f(x)|^p \operatorname{dist}(x,(0,1)^{\complement})^{-sp} dx\Big)^{\frac{1}{p}}\\
&\qquad \le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,1)\cap(a,b)} |f(x)|^p x^{-sp} dx\Big)^{\frac{1}{p}}\\
&\qquad\qquad + \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,1)\cap(a,b)} |f(1-y)|^p y^{-sp} dy\Big)^{\frac{1}{p}}.
\end{aligned}$$

Notice, that we have $f(1-\cdot)\in\mathbb{S}(\mathbb{R})$ and $\int_0^1 f(1-y)dy = \int_0^1 f(y)dy = 0$. So we can proceed for both terms simultaneously. Thanks to a transformation of the coordinates at the end of the calculations we will do now, we can see, that it is enough to deal with

$$\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,1)\cap(a,b)} |f(x)|^p x^{-sp} dx\Big)^{\frac{1}{p}}.$$

Now let $\eta\in C_0^\infty(\mathbb{R})$ be a cut-off function with $0\le\eta(x)\le 1$ for all x and $\eta(x)=1$ if $0\le x\le 1/2$ and $\operatorname{supp}\eta\subset[-\frac{1}{4},\frac{3}{4}]$. Then we find

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,1)\cap(a,b)} |f(x)|^p x^{-sp} dx\Big)^{\frac{1}{p}}\\
&\qquad \le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(0,\frac{1}{2})\cap(a,b)} |f(x)\eta(x)|^p x^{-sp} dx\Big)^{\frac{1}{p}}\\
&\qquad\qquad + C_1 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(\frac{1}{2},1)\cap(a,b)} |f(x)|^p dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Here for the first term because of $(0,\frac{1}{2}) \subset (0,\infty)$ we can use the result from Step 1. When we introduce the abbreviations J_1 and J_2 like it is done below, we obtain

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} |f(x)|^p x^{-sp} dx \Big)^{\frac{1}{p}} \\
&\quad \le C_1 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} |f(x)|^p dx \Big)^{\frac{1}{p}} + C_2 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \\
&\qquad \Big(\int_{(0,\infty)\cap(a,b)} \Big(\int_0^\infty r^{-sq} \Big(\int_{\substack{-1<h<1\\ x-rh>0}} |f(x)\eta(x) - f(x-rh)\eta(x-rh)| dh \Big)^q \frac{dr}{r} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \\
&\quad = C_3(J_1 + J_2).
\end{aligned}$$

When we replace f by $f\chi_{[0,1]}$, we can split up J_2 in the following way:

$$\begin{aligned}
J_2 &\le C_4 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{\substack{(0,1)\\ \cap(a,b)}} \Big(\int_0^\infty r^{-sq} \Big(\int_{\substack{-1<h<1\\ 0<x-rh<1}} |f(x) - f(x-rh)| dh \Big)^q \frac{dr}{r} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \\
&\quad + C_4 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{\substack{(1,\infty)\\ \cap(a,b)}} \Big(\int_0^\infty r^{-sq} \Big(\int_{\substack{-1<h<1\\ 0<x-rh<\frac{3}{4}}} |f(x-rh)\eta(x-rh)| dh \Big)^q \frac{dr}{r} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \\
&\quad + C_4 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{\substack{(0,\infty)\\ \cap(a,b)}} \Big(\int_0^\infty \Big(\int_{\substack{-1<h<1\\ x-rh>0}} |f\chi_{[0,1]}(x)| |\Delta^1_{rh}\eta(x-rh)| dh \Big)^q \frac{dr}{r^{sq+1}} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \\
&\quad = C_4(J_{21} + J_{22} + J_{23}).
\end{aligned}$$

Now J_{21} is what we want to have. When we use $0 < s < 1$ and that η is smooth like in the proof of Lemma 1 from chapter 5.4.1 in [101] for J_{23} we find

$$J_{23} \le C_5 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} |f(x)|^p dx \Big)^{\frac{1}{p}}.$$

For J_{22} with $x - rh = z$ we obtain

$$\begin{aligned}
J_{22} &\le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(1,\infty)\cap(a,b)} \Big(\int_{x-\frac{3}{4}}^\infty r^{-sq} \Big(\int_{\substack{-1<h<1\\ \frac{x}{r}-\frac{3}{4r}<h<\frac{x}{r}}} |f\chi_{[0,1]}(x-rh)| dh \Big)^q \frac{dr}{r} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \\
&\le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(1,\infty)\cap(a,b)} \Big(\int_{x-\frac{3}{4}}^\infty r^{-sq-q-1} dr \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}} \ \Big(\int_0^1 |f(z)| dz \Big) \\
&\le C_6 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(\frac{1}{4},\infty)\cap(a,b)} x^{-(s+1)p} dx \Big)^{\frac{1}{p}} \ \Big(\int_0^1 |f(z)| dz \Big).
\end{aligned}$$

Now because of $0 < s < 1$ and $u \ge 1$, we find $x^{-(s+1)}\chi_{(\frac{1}{4},\infty)}(x) \in \mathbb{L}_u(\mathbb{R})$. Here $\mathbb{L}_u(\mathbb{R})$ is the real part of the Lebesgue space $L_u(\mathbb{R})$. With $\mathbb{L}_u(\mathbb{R}) \hookrightarrow \mathbb{M}^u_p(\mathbb{R})$ we get $x^{-(s+1)}\chi_{(\frac{1}{4},\infty)}(x) \in \mathbb{M}^u_p(\mathbb{R})$. So Hölder's inequality yields

$$J_{22} \le C_7 \Big(\int_0^1 |f(z)|^p dz \Big)^{\frac{1}{p}} \le C_7 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} |f(z)|^p dz \Big)^{\frac{1}{p}}.$$

Next we apply $\int_0^1 f(x)dx = 0$. This leads to

$$\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} |f(z)|^p dz \Big)^{\frac{1}{p}}$$

$$= \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} \Big| f(z) - \int_0^1 f(y)dy \Big|^p dz \Big)^{\frac{1}{p}}$$

$$\le \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} \Big(\int_0^1 |f(z)-f(y)|dy \Big)^p dz \Big)^{\frac{1}{p}}$$

$$\le C_8 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} \Big(\int_1^2 r^{-s-\frac{1}{q}} \Big(\int_0^1 |f(z)-f(y)|dy \Big) dr \Big)^p dz \Big)^{\frac{1}{p}}.$$

When we use $q \ge 1$ and $y = z - rh$, we find

$$\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} |f(z)|^p dz \Big)^{\frac{1}{p}}$$

$$\le C_9 \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(0,1)\cap(a,b)} \Big(\int_1^2 r^{-sq-1} \Big(\int_0^1 |f(z)-f(y)|dy \Big)^q dr \Big)^{\frac{p}{q}} dz \Big)^{\frac{1}{p}}$$

$$\le C_{10} \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{\substack{(0,1)\\ \cap(a,b)}} \Big(\int_0^\infty r^{-sq-1} \Big(\int_{\substack{-1<h<1\\ 0<z-rh<1}} |f(z)-f(z-rh)|dh \Big)^q dr \Big)^{\frac{p}{q}} dz \Big)^{\frac{1}{p}}.$$

So this step of the proof is complete.
Step 3. At last we have to deal with any other interval $I = (c,d)$ with $-\infty \le c < d \le \infty$. But then the desired inequality follows from the model cases $I = (0,\infty)$ as well as $I = (0,1)$ and an appropriate transformation of the coordinates. So the proof is complete. ■

When we compare the Lemmas 52 and 53, it turns out, that the condition $s < 1/p$ from Lemma 52 is replaced by $s < 1/u$ in the formulation of Lemma 53. This observation will be very important for what follows.

13.2.2 The Boundedness of T on $\mathbb{E}^s_{u,p,q}(\mathbb{R})$ for $d = 1$

In this subsection we show, that under certain conditions on the parameters the operator T is bounded on $\mathbb{E}^s_{u,p,q}(\mathbb{R})$ for $d = 1$ also in the case $s \ge 1$. Here for the proof our Morrey version of the Hardy-type inequality, see Lemma 53, will play a key role. The main result of this subsection reads as follows.

Proposition 26. ***The Boundedness of*** T ***on*** $\mathbb{E}^s_{u,p,q}(\mathbb{R})$***. The Case*** $d = 1$.
Let $1 \le p \le u < \infty$, $1 \le q \le \infty$ *and* $1 < s < 1 + 1/u$. *Let* $d = 1$. *Then there is a constant* $C > 0$ *independent of* $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R})$, *such that we have*

$$\|Tf|\mathscr{E}^s_{u,p,q}(\mathbb{R})\| \le C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R})\| \tag{13.4}$$

for all $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R})$.

Proof. To prove this result, we follow the ideas from the proof of the theorem in chapter 5.4.1. in [101], see also Theorem 1 in [13].
Step 1. At first we prove (13.4) for real-valued $f \in C_0^\infty(\mathbb{R})$. We use Lemma 4 with $m = 1$. Then we find

$$\| \, |f| \, |\mathscr{E}^s_{u,p,q}(\mathbb{R})\| \leq C_1 \| \, |f| \, |\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\| + C_1 \|\partial^1 |f| \, |\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\|.$$

In general $\partial^1|f|$ is a distributional derivative. Since $f \in C_0^\infty(\mathbb{R})$, we find, that $|f|$ is a Lipschitz continuous function. So the classical derivative exists almost everywhere and coincides with the distributional one almost everywhere. Hence in our case we can also understand $\partial^1|f|$ as a classical derivative. Let us look at $\| \, |f| \, |\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\|$. Because of $1 < s < 1 + 1/u < 2$ we have $0 < s-1 < 1$. So we can apply Proposition 25 and obtain

$$\| \, |f| \, |\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\| \leq C_2 \|f|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\| \leq C_2 \|f|\mathscr{E}^{s}_{u,p,q}(\mathbb{R})\|.$$

Now we want to work with $\|\partial^1 |f| \, |\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\|$. Since $p \geq 1$ and $q \geq 1$, we can apply Theorem 5 with $v = 1, a = \infty$ and $N = 1$. We get

$$\begin{aligned} &\|\partial^1 |f| \, |\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\| \\ &\quad \leq C_3 \|\partial^1 |f| \, |\mathscr{M}^u_p(\mathbb{R})\| + C_3 \Big\| \Big(\int_0^\infty t^{-(s-1)q-q} \Big(\int_{-t}^t |\Delta_h^1 \partial^1 |f|(x)| dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| \mathscr{M}^u_p(\mathbb{R}) \Big\|. \end{aligned}$$

At first we look at $\|\partial^1 |f| \, |\mathscr{M}^u_p(\mathbb{R})\|$. Because of f is real-valued, it is possible to define the sets $\Omega_f = \{x \in \mathbb{R} : f(x) \geq 0\}$ and

$$\begin{aligned} SC_f = &\{x \in \mathbb{R} : \exists \varepsilon > 0 \text{ with } f(y) < 0 \text{ for } y \in (x-\varepsilon, x) \text{ and } f(y) > 0 \text{ for } y \in (x, x+\varepsilon)\} \\ &\cup \{x \in \mathbb{R} : \exists \varepsilon > 0 \text{ with } f(y) > 0 \text{ for } y \in (x-\varepsilon, x) \text{ and } f(y) < 0 \text{ for } y \in (x, x+\varepsilon)\}. \end{aligned}$$

Now recall, that we have $f \in C_0^\infty(\mathbb{R})$. Hence the set SC_f has Lebesgue measure zero. Therefore if we work with the Morrey norm, we can exclude the set SC_f. Because of this we find

$$\begin{aligned} \|\partial^1 |f| \, |\mathscr{M}^u_p(\mathbb{R})\| \leq & \, C_4 \sup_{\substack{a,b \in \mathbb{R} \\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{((a,b)\cap\Omega_f)\setminus SC_f} |\partial^1 f(x)|^p dx \Big)^{\frac{1}{p}} \\ & + C_4 \sup_{\substack{a,b \in \mathbb{R} \\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{((a,b)\cap\Omega_f^{\complement})\setminus SC_f} |-\partial^1 f(x)|^p dx \Big)^{\frac{1}{p}} \\ \leq & \, C_5 \|\partial^1 f|\mathscr{M}^u_p(\mathbb{R})\|. \end{aligned}$$

Now we can use Theorem 5 and Lemma 4. Then we get

$$\|\partial^1 f|\mathscr{M}^u_p(\mathbb{R})\| \leq C_6 \|\partial^1 f|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\| \leq C_7 \|f|\mathscr{E}^{s}_{u,p,q}(\mathbb{R})\|.$$

So it remains, to deal with

$$\sup_{\substack{a,b \in \mathbb{R} \\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(a,b)} \Big(\int_0^\infty t^{-(s-1)q-q} \Big(\int_{-t}^t |\Delta_h^1 \partial^1 |f|(x)| dh \Big)^q \frac{dt}{t} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}}.$$

Here we follow the ideas from the proof of the theorem in chapter 5.4.1. in [101]. At first for $a < b$ we write $(a,b) = ((a,b)\cap\Omega_f)\cup((a,b)\cap\Omega_f^{\complement})$ and for $t\in\mathbb{R}$

$$(-t,t) = ((-t,t)\cap\{h\in\mathbb{R} : x+h\in\Omega_f\})\cup((-t,t)\cap\{h\in\mathbb{R} : x+h\notin\Omega_f\}).$$

Now we can use a version of the triangle inequality to split up the different cases. Then two different situations show up. On the one hand it is possible, that we have $x\in\Omega_f$ and $x+h\in\Omega_f$. So we find $|\Delta_h^1\partial^1|f|(x)| = |\partial^1 f(x+h)-\partial^1 f(x)| = |\Delta_h^1\partial^1 f(x)|$. But then we have

$$\Big\|\Big(\int_0^\infty t^{-(s-1)q-q}\Big(\int_{-t}^t|\Delta_h^1\partial^1 f(x)|dh\Big)^q\frac{dt}{t}\Big)^{\frac1q}\Big|\mathscr{M}_p^u(\mathbb{R})\Big\|$$
$$\le C_8\|\partial^1 f|\mathscr{E}_{u,p,q}^{s-1}(\mathbb{R})\|\le C_9\|f|\mathscr{E}_{u,p,q}^{s}(\mathbb{R})\|.$$

The case $x\in\Omega_f^{\complement}$ and $x+h\notin\Omega_f$ leads to the same result. On the other hand we have the situation $x\in\Omega_f$ and $x+h\notin\Omega_f$. Here we obtain

$$|\Delta_h^1\partial^1|f|(x)| = |\partial^1 f(x+h)+\partial^1 f(x)|\le 2|\partial^1 f(x)|+|\Delta_h^1\partial^1 f(x)|.$$

With $|\Delta_h^1\partial^1 f(x)|$ we can work like it is described before. So it remains, to deal with

$$\sup_{\substack{a,b\in\mathbb{R}\\ a<b}}|a-b|^{\frac1u-\frac1p}\Big(\int_{(a,b)\cap\Omega_f}\Big(\int_0^\infty t^{-(s-1)q-q}\Big(\int_{\substack{(-t,t)\,\cap\\ \{h\in\mathbb{R}\,:\,x+h\notin\Omega_f\}}}|\partial^1 f(x)|dh\Big)^q\frac{dt}{t}\Big)^{\frac pq}dx\Big)^{\frac1p}.$$

Notice, that the case $x\in\Omega_f^{\complement}$ and $x+h\in\Omega_f$ leads to a similar situation. Next we observe

$$\sup_{\substack{a,b\in\mathbb{R}\\ a<b}}|a-b|^{\frac1u-\frac1p}\Big(\int_{(a,b)\cap\Omega_f}|\partial^1 f(x)|^p\Big(\int_0^\infty t^{-(s-1)q-q}\Big(\int_{\substack{(-t,t)\,\cap\\ \{h\in\mathbb{R}\,:\,x+h\notin\Omega_f\}}}1dh\Big)^q\frac{dt}{t}\Big)^{\frac pq}dx\Big)^{\frac1p}$$
$$\le\sup_{\substack{a,b\in\mathbb{R}\\ a<b}}|a-b|^{\frac1u-\frac1p}\Big(\int_{(a,b)\cap\Omega_f}|\partial^1 f(x)|^p\Big(\int_{\operatorname{dist}(x,\Omega_f^{\complement})}^\infty t^{-(s-1)q-q}\Big(\int_{-t}^t 1dh\Big)^q\frac{dt}{t}\Big)^{\frac pq}dx\Big)^{\frac1p}$$
$$\le C_{10}\sup_{\substack{a,b\in\mathbb{R}\\ a<b}}|a-b|^{\frac1u-\frac1p}\Big(\int_{(a,b)\cap\Omega_f}|\partial^1 f(x)|^p\Big(\int_{\operatorname{dist}(x,\Omega_f^{\complement})}^\infty t^{-(s-1)q}\frac{dt}{t}\Big)^{\frac pq}dx\Big)^{\frac1p}$$
$$\le C_{11}\sup_{\substack{a,b\in\mathbb{R}\\ a<b}}|a-b|^{\frac1u-\frac1p}\Big(\int_{(a,b)\cap\Omega_f}|\partial^1 f(x)|^p\operatorname{dist}(x,\Omega_f^{\complement})^{-(s-1)p}dx\Big)^{\frac1p}.$$

Since $f\in C_0^\infty(\mathbb{R})$, we can find disjoint open intervals I_i, such that $\Omega_f=\bigcup_i\overline{I_i}$. For $|I_i|<\infty$ we can write $I_i=(c_i,d_i)$ with $c_i<d_i<c_{i+1}<d_{i+1}$ and $f(c_i)=f(d_i)=0$. So for $|I_i|<\infty$ we observe $\int_{I_i}\partial^1 f(x)dx = f(d_i)-f(c_i)=0$. Now we get

$$\sup_{\substack{a,b\in\mathbb{R}\\ a<b}}|a-b|^{\frac1u-\frac1p}\Big(\int_{(a,b)\cap\Omega_f}|\partial^1 f(x)|^p\operatorname{dist}(x,\Omega_f^{\complement})^{-(s-1)p}dx\Big)^{\frac1p}$$
$$=\sup_{\substack{a,b\in\mathbb{R}\\ a<b}}|a-b|^{\frac1u-\frac1p}\Big(\sum_i\int_{(a,b)\cap\overline{I_i}}|\partial^1 f(x)|^p\operatorname{dist}(x,I_i^{\complement})^{-(s-1)p}dx\Big)^{\frac1p}.$$

There are 3 different possible cases. First it is possible, that the interval (a,b) only intersects one single interval I_1. Then because of $s-1 < 1/u$ we can apply Lemma 53 and obtain

$$\sup_{\substack{a,b\in\mathbb{R}\\a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\sum_i \int_{(a,b)\cap\overline{I_i}} |\partial^1 f(x)|^p \operatorname{dist}(x, I_i^{\complement})^{-(s-1)p} dx\Big)^{\frac{1}{p}}$$

$$= \sup_{\substack{a,b\in\mathbb{R}\\a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(a,b)\cap\overline{I_1}} |\partial^1 f(x)|^p \operatorname{dist}(x, I_1^{\complement})^{-(s-1)p} dx\Big)^{\frac{1}{p}}$$

$$\leq C_{12} \sup_{\substack{a,b\in\mathbb{R}\\a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_a^b \Big(\int_0^\infty r^{-(s-1)q} \Big(\int_{-1}^1 |\partial^1 f(x) - \partial^1 f(x-rh)| dh\Big)^q \frac{dr}{r}\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}.$$

Second it is possible, that the interval (a,b) intersects two intervals I_1 and I_2. Then we get

$$\sup_{\substack{a,b\in\mathbb{R}\\a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\sum_i \int_{(a,b)\cap\overline{I_i}} |\partial^1 f(x)|^p \operatorname{dist}(x, I_i^{\complement})^{-(s-1)p} dx\Big)^{\frac{1}{p}}$$

$$\leq C_{13} \sum_{i\in\{1,2\}} \sup_{\substack{a,b\in\mathbb{R}\\a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(a,b)\cap\overline{I_i}} |\partial^1 f(x)|^p \operatorname{dist}(x, I_i^{\complement})^{-(s-1)p} dx\Big)^{\frac{1}{p}}.$$

Now again we can apply Lemma 53 with the same result as before. The third case is, that the interval (a,b) intersects $n \in \mathbb{N}$ intervals I_i with $n \geq 3$. But then we have the situation, that (a,b) intersects the intervals I_1 and I_n and completely covers $I_2, I_3, \ldots, I_{n-1}$. So we find

$$\sup_{\substack{a,b\in\mathbb{R}\\I_2,\ldots,I_{n-1}\subset(a,b)}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\sum_i \int_{(a,b)\cap\overline{I_i}} |\partial^1 f(x)|^p \operatorname{dist}(x, I_i^{\complement})^{-(s-1)p} dx\Big)^{\frac{1}{p}}$$

$$\leq C_{14} \sum_{i\in\{1,n\}} \sup_{\substack{a,b\in\mathbb{R}\\a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{(a,b)\cap\overline{I_i}} |\partial^1 f(x)|^p \operatorname{dist}(x, I_i^{\complement})^{-(s-1)p} dx\Big)^{\frac{1}{p}}$$

$$+ C_{14} \sup_{\substack{a,b\in\mathbb{R}\\I_2,\ldots,I_{n-1}\subset(a,b)}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\sum_{i=2}^{n-1} \int_{\overline{I_i}} |\partial^1 f(x)|^p \operatorname{dist}(x, I_i^{\complement})^{-(s-1)p} dx\Big)^{\frac{1}{p}}.$$

Now for the first part again we can use Lemma 53. For the second part because of $s-1 < 1/u \leq 1/p$ we can apply Lemma 52. Using $f(c_2) = \ldots = f(d_{n-1}) = 0$ we obtain

$$\sup_{\substack{a,b\in\mathbb{R}\\I_2,\ldots,I_{n-1}\subset(a,b)}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\sum_{i=2}^{n-1} \int_{\overline{I_i}} |\partial^1 f(x)|^p \operatorname{dist}(x, I_i^{\complement})^{-(s-1)p} dx\Big)^{\frac{1}{p}}$$

$$\leq C_{15} \sup_{\substack{a,b\in\mathbb{R}\\a\leq c_2<d_{n-1}\leq b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{c_2}^{d_{n-1}} \Big(\int_0^\infty r^{-(s-1)q} \Big(\int_{-1}^1 |\Delta_{rh}^1 \partial^1 f(x-rh)| dh\Big)^q \frac{dr}{r}\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}$$

$$\leq C_{15} \sup_{\substack{a,b\in\mathbb{R}\\a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_a^b \Big(\int_0^\infty r^{-(s-1)q} \Big(\int_{-1}^1 |\partial^1 f(x) - \partial^1 f(x-rh)| dh\Big)^q \frac{dr}{r}\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}}.$$

So when we use Theorem 5 and Lemma 4, we find

$$\begin{aligned}
&\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_{(a,b)\cap\Omega_f} |\partial^1 f(x)|^p \operatorname{dist}(x,\Omega_f^{\complement})^{-(s-1)p}dx\Big)^{\frac{1}{p}}\\
&\qquad \le C_{16}\sup_{\substack{a,b\in\mathbb{R}\\ a<b}} |a-b|^{\frac{1}{u}-\frac{1}{p}}\Big(\int_a^b\Big(\int_0^\infty r^{-(s-1)q}\Big(\int_{-1}^1 |\partial^1 f(x)-\partial^1 f(x-rh)|dh\Big)^q\frac{dr}{r}\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}\\
&\qquad \le C_{17}\|\partial^1 f|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R})\| \le C_{18}\|f|\mathscr{E}^{s}_{u,p,q}(\mathbb{R})\|.
\end{aligned}$$

Step 1 of the proof is complete.

Step 2. Now we prove (13.4) for $f\in\mathbb{E}^s_{u,p,q}(\mathbb{R})$. Let $\rho\in\mathbb{S}(\mathbb{R})$ be a real even function with $\rho(x)=1$ if $|x|\le 1$ and $\rho(x)=0$ if $|x|\ge 3/2$. For $j\in\mathbb{N}_0$ and $x\in\mathbb{R}$ we put $\rho^j(x)=\rho(2^{-j}x)$. Moreover, we define $f_j(x)=\rho^j(x)\mathscr{F}^{-1}[\rho^j\mathscr{F}f](x)$, see Step 1 of the proof of Theorem 25.8 in [131]. Then f_j is real and because of the Paley-Wiener-Schwarz Theorem we find $f_j\in C_0^\infty(\mathbb{R})$. So we also have $f_j\in\mathbb{E}^s_{u,p,q}(\mathbb{R})$. We observe

$$\lim_{j\to\infty} f_j = f \qquad \text{and} \qquad \lim_{j\to\infty}|f_j| = |f|$$

with convergence in $\mathscr{S}'(\mathbb{R})$. So we can apply the Fatou property, see Lemma 3, and Step 1 of this proof. Then we get

$$\|\,|f|\,|\mathscr{E}^s_{u,p,q}(\mathbb{R})\| \le C_1\sup_{j\in\mathbb{N}_0}\|\,|f_j|\,|\mathscr{E}^s_{u,p,q}(\mathbb{R})\| \le C_2\sup_{j\in\mathbb{N}_0}\|f_j|\mathscr{E}^s_{u,p,q}(\mathbb{R})\|.$$

Next we use Lemma 9. Let $m\in\mathbb{N}$ be large enough. We find

$$\begin{aligned}
\sup_{j\in\mathbb{N}_0}\|f_j|\mathscr{E}^s_{u,p,q}(\mathbb{R})\| &\le C_3\sup_{j\in\mathbb{N}_0}\Big(\sum_{|\alpha|\le m}\|D^\alpha\rho^j|L_\infty(\mathbb{R})\|\Big)\,\|\mathscr{F}^{-1}[\rho^j\mathscr{F}f]|\mathscr{E}^s_{u,p,q}(\mathbb{R})\|\\
&\le C_4\sup_{j\in\mathbb{N}_0}\|\mathscr{F}^{-1}[\rho^j\mathscr{F}f]|\mathscr{E}^s_{u,p,q}(\mathbb{R})\|.
\end{aligned}$$

Now we apply Lemma 10. Let $N\in\mathbb{N}$ be large enough. Then we obtain

$$\begin{aligned}
\sup_{j\in\mathbb{N}_0}\|\mathscr{F}^{-1}[\rho^j\mathscr{F}f]|\mathscr{E}^s_{u,p,q}(\mathbb{R})\| &\le C_5\sup_{j\in\mathbb{N}_0}\sup_{|\gamma|\le N}\sup_{x\in\mathbb{R}}(1+|x|^2)^{\frac{|\gamma|}{2}}|D^\gamma\rho^j(x)|\ \ \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R})\|\\
&\le C_6\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R})\|.
\end{aligned}$$

So the whole proof is complete. ■

Let us compare the original result Theorem 22 with Proposition 26. In Theorem 22, when we deal with the classical Triebel-Lizorkin spaces, we find, that the operator T is bounded, if $s<1+1/p$. When we switch to the Triebel-Lizorkin-Morrey spaces in Proposition 26, this condition is replaced by $s<1+1/u$. The main reason for this fact is, that a similar change also happened in the step from Lemma 52 to Lemma 53, where we proved a Hardy-type inequality. Of course we also want to investigate the behavior of truncation operators in the context of Triebel-Lizorkin-Morrey spaces in the case of $d>1$. This will be done in the next section.

13.3 Truncation and Smoothness Morrey Spaces in higher Dimensions

In the case of the original Triebel-Lizorkin spaces the step from $d = 1$ to $d > 1$ is very easy, see chapter 5.4.1. in [101]. The reason for this is, that the spaces $F^s_{p,q}(\mathbb{R}^d)$ have the so-called Fubini property, see Lemma 50 and the references that are given there. However, we learned, that for $p < u$ in most of the cases the Triebel-Lizorkin-Morrey spaces do not have the Fubini property. This we proved in Lemma 51. Consequently when we want to prove results concerning the boundedness of the operator T in the context of Triebel-Lizorkin-Morrey spaces for $d \in \mathbb{N}$, we have to look for an innovative strategy.

13.3.1 The Boundedness of T on Triebel-Lizorkin-Morrey Spaces for $d \in \mathbb{N}$

In this subsection we prove one of our main results concerning the boundedness of the truncation operator. For that purpose we use, that in the case of small s for the Triebel-Lizorkin-Morrey spaces there exist so-called Morrey characterizations, see chapter 3.6.3 in [137]. Recall, that for $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^d$ we defined $Q_{j,m} = 2^{-j}m + 2^{-j}[0,1)^d$. For $c > 1$ by $cQ_{j,m}$ we denote a cube concentric with $Q_{j,m}$ that has side-length $c2^{-j}$. Using this notation, we can formulate the following result.

Proposition 27. ***Morrey Characterizations.***
Let $1 \leq p \leq u < \infty$ and $1 \leq q \leq \infty$. Let $0 < s < \min(1/p, d/u)$. Let $f \in \mathscr{S}'(\mathbb{R}^d)$. Then we have $f \in \mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$ if and only if

$$\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{MC} = \sup_{j\in\mathbb{Z}, m\in\mathbb{Z}^d} 2^{j(\frac{d}{p}-\frac{d}{u})} \|f|F^s_{p,q}(2Q_{j,m})\| \tag{13.5}$$

is finite. The norms $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|$ and $\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|^{MC}$ are equivalent.

Proof. This result is a combination of Theorem 3.64. from [137] with Theorem 3.38. from [137] and Corollary 3.3. from [144]. One may also consult Theorem 2.29. in [136]. ∎

By the help of Proposition 27 we can prove the following result concerning the boundedness of the operator T.

Proposition 28. ***The Boundedness of** T **for** $d \in \mathbb{N}$. **Part I.***
Let $1 \leq p < u < \infty$, $1 \leq q \leq \infty$ and $d \in \mathbb{N}$. Let

$$\frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d}$$

and

$$1 < s < \min\Big(1 + \frac{1}{p}, 1 + \frac{d}{u}\Big).$$

Then there is a constant $C > 0$ independent of $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$, such that we have

$$\|Tf|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \tag{13.6}$$

for all $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$.

Proof. Our main tool for the proof will be Proposition 27 in combination with the original result Theorem 22. We divide the proof into several substeps.

Step 1. Preparations and Fatou property.

This step is similar to Step 2 of the proof from Proposition 26. Let $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d)$ be a real function with $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| > 2$. For $j \in \mathbb{N}$ we define $f_j = \mathscr{F}^{-1}[\psi(2^{-j}\cdot)(\mathscr{F}f)]$. Then we have $f_j \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ and $\operatorname{supp}\mathscr{F}f_j \subset B(0,2^{j+1})$. So because of the Paley-Wiener-Schwarz Theorem f_j is a smooth function. Moreover, it is not difficult to see, that for all $\alpha \in \mathbb{N}_0^d$ we have $D^\alpha f_j \in L_\infty(\mathbb{R}^d)$ and $D^\alpha f_j \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$. We observe

$$\lim_{j\to\infty} f_j = f \qquad \text{and} \qquad \lim_{j\to\infty} |f_j| = |f|$$

with convergence in $\mathscr{S}'(\mathbb{R}^d)$. Let us assume, that we know (13.6) for all functions f_j. Then we can apply the Fatou property, see Lemma 3, and find

$$\|\,|f|\,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C_1 \sup_{j\in\mathbb{N}} \|\,|f_j|\,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C_2 \sup_{j\in\mathbb{N}} \|f_j|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C_3 \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|.$$

In the last step we used Lemma 10. Therefore it is enough to prove (13.6) for analytic functions $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$, that fulfill $\operatorname{supp}\mathscr{F}f \subset B(0,2^R)$ for some $R \in \mathbb{N}$. The same trick is used in Step 1 of the proof of Theorem 25.8 in [131].

Step 2. Pick out cubes without zeros.

In what follows $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ is an analytic function with $\operatorname{supp}\mathscr{F}f \subset B(0,2^R)$ for some $R \in \mathbb{N}$. At first we use Lemma 4. Then we find

$$\|\,|f|\,|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \leq C_1 \|\,|f|\,|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\| + C_1 \sum_{i=1}^d \|\partial_i^1 |f|\,|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\|.$$

For the first term because of $0 < s-1 < 1$ we can apply Proposition 25. When we use Lemma 4 again, we get

$$\|\,|f|\,|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\| \leq C_2 \|f|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\| \leq C_2 \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|. \tag{13.7}$$

So in what follows, we have to deal with $\|\partial_i^1|f|\,|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\|$ with $i \in \{1,2,\ldots,d\}$. Because of $0 < s-1 < \min(1/p, d/u)$ we can apply Proposition 27. Then we obtain

$$\|\partial_i^1|f|\,|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\| \leq C_3 \sup_{j\in\mathbb{Z}, m\in\mathbb{Z}^d} 2^{j(\frac{d}{p}-\frac{d}{u})} \|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\|.$$

Now by $\mathscr{Q}_{SC}(f)$ we denote the set of all cubes of the form $2Q_{j,m}$, that have the following property. There exist $y_1, y_2 \in 2Q_{j,m}$, such that we have $f(y_1) < 0 < f(y_2)$. That means, f has a sign change in $2Q_{j,m}$. Using this notation, we can write

$$\begin{aligned}\|\partial_i^1|f|\,|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\| \leq C_3 & \sup_{\substack{j\in\mathbb{Z}, m\in\mathbb{Z}^d \\ 2Q_{j,m}\in\mathscr{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})} \|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\| \\ & + C_3 \sup_{\substack{j\in\mathbb{Z}, m\in\mathbb{Z}^d \\ 2Q_{j,m}\notin\mathscr{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})} \|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\|.\end{aligned}$$

In each cube that fulfills $2Q_{j,m} \notin \mathscr{Q}_{SC}(f)$ the function f is either positive everywhere or negative everywhere. Therefore for $2Q_{j,m} \notin \mathscr{Q}_{SC}(f)$ it does not matter, whether we write

$\|\partial_i^1 |f| \,|F^{s-1}_{p,q}(2Q_{j,m})\|$ or $\|\partial_i^1 f|F^{s-1}_{p,q}(2Q_{j,m})\|$. Hence a combination of Proposition 27 and Lemma 4 yields

$$\begin{aligned}\sup_{\substack{j\in\mathbb{Z},m\in\mathbb{Z}^d\\ 2Q_{j,m}\notin\mathscr{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\| &\le C_4 \sup_{j\in\mathbb{Z},m\in\mathbb{Z}^d} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 f|F^{s-1}_{p,q}(2Q_{j,m})\|\\ &\le C_5\|\partial_i^1 f|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\|\\ &\le C_6\|f|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|.\end{aligned}$$

So hereinafter we only have to deal with cubes, that fulfill $2Q_{j,m}\in\mathscr{Q}_{SC}(f)$. That means, we have to investigate the term

$$\sup_{\substack{j\in\mathbb{Z},m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathscr{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\|. \tag{13.8}$$

Step 3. Pick out cubes of middle size.

To continue, we split up the term (13.8) in the following way.

$$\begin{aligned}&\sup_{\substack{j\in\mathbb{Z},m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathscr{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\|\\ &\qquad\le \sup_{\substack{j\ge R,m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathscr{Q}_{SC}(f)}}\ldots+\sup_{\substack{0<j\le R,m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathscr{Q}_{SC}(f)}}\ldots+\sup_{\substack{j\in\mathbb{Z}\setminus\mathbb{N},m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathscr{Q}_{SC}(f)}}\ldots.\end{aligned}$$

Whereas the first and the last term are like we want, we should have a closer look at the second one. Therefore let $k\in\mathbb{N}$ be a natural number with $0<k\le R$ and let $m\in\mathbb{Z}^d$. We investigate

$$2^{k(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{k,m})\|.$$

Therefore we define the set $Z(f)=\{x\in\mathbb{R}^d : f(x)=0\}$. In addition for each $l\in\mathbb{N}$ we define numbers R_l by $R_l=R+l$. Then for all $l\in\mathbb{N}$ we have $R_l>R$ and we observe $\lim_{l\to\infty}R_l=\infty$. Using this numbers for all $l\in\mathbb{N}$ we define sets

$$ZF_l(f)=\Big\{x\in\mathbb{R}^d\ :\ \operatorname{dist}(x,Z(f))\le\frac{1}{100d}2^{-R_l}\Big\}. \tag{13.9}$$

Then of course we have $2Q_{k,m}=[2Q_{k,m}\cap ZF_l(f)]\cup[2Q_{k,m}\cap ZF_l(f)^{\complement}]$. Moreover, since f is real valued, we can define the sets

$$F^+(f)=\{x\in\mathbb{R}^d : f(x)>0\}\qquad\text{and}\qquad F^-(f)=\{x\in\mathbb{R}^d : f(x)<0\}. \tag{13.10}$$

Now for all $x\in ZF_l(f)^{\complement}$ we can observe $f(x)\neq0$. So it is not difficult to see, that we can write $ZF_l(f)^{\complement}=[ZF_l(f)^{\complement}\cap F^+(f)]\cup[ZF_l(f)^{\complement}\cap F^-(f)]$. Consequently for all $l\in\mathbb{N}$ we obtain the disjoint decomposition

$$\begin{aligned}2Q_{k,m}&=[2Q_{k,m}\cap ZF_l(f)]\cup[2Q_{k,m}\cap[ZF_l(f)^{\complement}\cap F^+(f)]]\cup[2Q_{k,m}\cap[ZF_l(f)^{\complement}\cap F^-(f)]]\\ &=A_1^l\cup A_2^l\cup A_3^l.\end{aligned}$$

(Notice, that it is not really necessary to work with a disjoint decomposition here. Therefore it is also possible to deal with open sets that are overlapping a bit.) Now let $l^*\in\mathbb{N}$ be a fixed large natural number, that will be specified later. Then we find

$$\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{k,m})\|\le\|\partial_i^1|f|\,|F^{s-1}_{p,q}(A_1^{l^*})\|+\|\partial_i^1|f|\,|F^{s-1}_{p,q}(A_2^{l^*})\|+\|\partial_i^1|f|\,|F^{s-1}_{p,q}(A_3^{l^*})\|.$$

Notice, that for all $x \in A_2^{l^*}$ we have $f(x) > 0$. Therefore since $A_2^{l^*} \subset 2Q_{k,m}$, we can write

$$\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(A_2^{l^*})\| = \|\partial_i^1 f|F^{s-1}_{p,q}(A_2^{l^*})\| \leq \|\partial_i^1 f|F^{s-1}_{p,q}(2Q_{k,m})\|.$$

On the other hand for $x \in A_3^{l^*}$ we observe $f(x) < 0$. So because of $A_3^{l^*} \subset 2Q_{k,m}$ we find

$$\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(A_3^{l^*})\| = \|\partial_i^1 f|F^{s-1}_{p,q}(A_3^{l^*})\| \leq \|\partial_i^1 f|F^{s-1}_{p,q}(2Q_{k,m})\|.$$

Next because of $s-1 < \min(1/p, d/u)$ we can apply Proposition 27. Consequently when we use Lemma 4, for all $0 < k \leq R$ we get

$$\begin{aligned}
&2^{k(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(2Q_{k,m})\| \\
&\quad\leq 2^{k(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(A_1^{l^*})\| + C_1 2^{k(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 f|F^{s-1}_{p,q}(2Q_{k,m})\| \\
&\quad\leq 2^{k(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(A_1^{l^*})\| + C_2 \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|.
\end{aligned}$$

Next we observe

$$\lim_{l\to\infty} 2Q_{k,m} \cap ZF_l(f) = 2Q_{k,m} \cap Z(f)$$

as sets. We need some knowledge concerning the zero set $Z(f)$ of real analytic functions $f : \mathbb{R}^d \to \mathbb{R}$ with $f \neq 0$. For this reason we collected everything we need in the appendix section 13.6 at the end of this chapter, see Lemma 55. Let $\sigma = \sigma(p,u,d) > 0$ be a small number, that will be specified later. Then from (iii) in Lemma 55 we learn the following. For each real analytic f with $f \neq 0$ and each cube $2Q_{k,m}$, we find a maybe very large $l^* = l^*(f,R,p,u,d) \in \mathbb{N}$, such that $2^{R_{l^*}-k}$ is much larger than R_{l^*} and $R_{l^*}\, 2^{\sigma(k-R_{l^*})} \leq 1$ and such that the set $2Q_{k,m} \cap ZF_{l^*}(f)$ can be covered by $c(d)R_{l^*}2^{(d-1)(R_{l^*}-k)}$ cubes of the form $2Q_{R_{l^*},n}$ with appropriate $n \in \mathbb{Z}^d$. Here $c(d)$ only depends on d. We used $\lim_{x\to\infty} x2^{\sigma(k-x)} = 0$. Notice, that under the given assumptions we always can choose $l^* = l^*(f,R,p,u,d)$ independent of k and m, see (iii) in Lemma 55. Moreover, we always can ensure that $l^* < \infty$. Again we refer to (iii) in Lemma 55 and its proof. Using such a natural number l^* and the associated covering, we obtain

$$\begin{aligned}
&2^{k(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(2Q_{k,m} \cap ZF_{l^*}(f))\| \\
&\quad\leq 2^{k(\frac{d}{p}-\frac{d}{u})}\sum_n \|\partial_i^1 |f| \, |F^{s-1}_{p,q}(2Q_{R_{l^*},n})\| \\
&\quad\leq C_3 2^{k(\frac{d}{p}-\frac{d}{u})} R_{l^*}\, 2^{(d-1)(R_{l^*}-k)} 2^{-R_{l^*}(\frac{d}{p}-\frac{d}{u})} \sup_n 2^{R_{l^*}(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(2Q_{R_{l^*},n})\| \\
&\quad\leq C_3 R_{l^*}\, 2^{(k-R_{l^*})(\frac{d}{p}-\frac{d}{u}-d+1)} \sup_{\substack{j\geq R, m\in\mathbb{Z}^d \\ 2Q_{j,m}\in\mathscr{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(2Q_{j,m})\|.
\end{aligned}$$

In the last step we used $R_{l^*} = R + l^* > R$. Now we apply the assumption $1/p - 1/u > 1 - 1/d$. Because of $d/p - d/u - d + 1 > 0$ there exists a $\sigma > 0$, such that $d/p - d/u - d + 1 - \sigma > 0$. Then since $k \leq R < R_{l^*}$ we get

$$R_{l^*}\, 2^{(k-R_{l^*})(\frac{d}{p}-\frac{d}{u}-d+1)} = R_{l^*}\, 2^{\sigma(k-R_{l^*})} 2^{(k-R_{l^*})(\frac{d}{p}-\frac{d}{u}-d+1-\sigma)} \leq 1.$$

Therefore we obtain

$$2^{k(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(2Q_{k,m} \cap ZF_{l^*}(f))\| \leq C_3 \sup_{\substack{j\geq R, m\in\mathbb{Z}^d \\ 2Q_{j,m}\in\mathscr{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1 |f| \, |F^{s-1}_{p,q}(2Q_{j,m})\|.$$

Notice, that the right-hand side is independent of l^*. So we do not have to deal with this parameter in what follows. All in all we find

$$\sup_{\substack{j\in\mathbb{Z},m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\| \\ \leq C_4 \sup_{\substack{j\geq R,m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} \ldots \;+ C_4 \sup_{\substack{j\in\mathbb{Z}\setminus\mathbb{N},m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} \ldots \;+ C_4\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|.$$

When we define the set $\mathbb{Z}(R)=\mathbb{Z}\setminus\mathbb{N}\,\cup\,\{j\in\mathbb{N}: j\geq R\}$, that means, in what follows we only have to deal with the term

$$\sup_{\substack{j\in\mathbb{Z}(R),m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\|. \tag{13.11}$$

Step 4. Apply the result for the original Triebel-Lizorkin spaces.
Now for all $j\in\mathbb{Z}$ and all $m\in\mathbb{Z}^d$ there exists a function $f_{j,m}\in\mathbb{F}^s_{p,q}(\mathbb{R}^d)$ with $f_{j,m}(x)=f(x)$ for all $x\in 2Q_{j,m}$, such that $\|f_{j,m}|F^s_{p,q}(\mathbb{R}^d)\|\leq 2\|f|F^s_{p,q}(2Q_{j,m})\|$. Because of the definition of the norm $\|\cdot|F^{s-1}_{p,q}(2Q_{j,m})\|$ we can write

$$\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\| = \|\partial_i^1|f_{j,m}|\,|F^{s-1}_{p,q}(2Q_{j,m})\| \leq \|\partial_i^1|f_{j,m}|\,|F^{s-1}_{p,q}(\mathbb{R}^d)\|.$$

Next we can use a version of Lemma 4 for the original Triebel-Lizorkin spaces. Moreover, because of $1\leq p<\infty$, $1<s<1+1/p$ and the fact that $f_{j,m}\in\mathbb{F}^s_{p,q}(\mathbb{R}^d)$ is real-valued, we can apply the well-known Theorem 22. This leads to

$$\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\|\leq C_1\|\,|f_{j,m}|\,|F^s_{p,q}(\mathbb{R}^d)\|\leq C_2\|f_{j,m}|F^s_{p,q}(\mathbb{R}^d)\|\leq C_3\|f|F^s_{p,q}(2Q_{j,m})\|.$$

When we use this inequality for (13.11), we find

$$\sup_{\substack{j\in\mathbb{Z}(R),m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|\partial_i^1|f|\,|F^{s-1}_{p,q}(2Q_{j,m})\|\leq C_4 \sup_{\substack{j\in\mathbb{Z}(R),m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|f|F^s_{p,q}(2Q_{j,m})\|.$$

Hereafter we have to distinguish between small cubes and large cubes. Therefore we use the definition of the set $\mathbb{Z}(R)$, to write

$$\sup_{\substack{j\in\mathbb{Z}(R),m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|f|F^s_{p,q}(2Q_{j,m})\| \\ \leq \sup_{\substack{j\geq R,m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|f|F^s_{p,q}(2Q_{j,m})\| + \sup_{\substack{j\in\mathbb{Z}\setminus\mathbb{N},\,m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|f|F^s_{p,q}(2Q_{j,m})\|.$$

To continue the proof, we have to deal with both terms separately.
Step 5. Complete the proof for cubes of large and middle size.
Here we have to deal with

$$\sup_{\substack{j\in\mathbb{Z}\setminus\mathbb{N},\,m\in\mathbb{Z}^d\\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})}\|f|F^s_{p,q}(2Q_{j,m})\|.$$

For that purpose we use formula (3.307) from [137], see also Proposition 4.21. in [134] and its proof. There we learn, that for $j \in \mathbb{Z} \setminus \mathbb{N}$ and $m \in \mathbb{Z}^d$, there exists a general constant C_1 independent of j and m, such that

$$\|f|F^s_{p,q}(2Q_{j,m})\| \leq C_1 \|f|F^{s-1}_{p,q}(2Q_{j,m})\| + C_1 \sum_{k=1}^{d} \|\partial^1_k f|F^{s-1}_{p,q}(2Q_{j,m})\|. \tag{13.12}$$

Therefore we obtain

$$\begin{aligned} &\sup_{\substack{j\in\mathbb{Z}\setminus\mathbb{N},\, m\in\mathbb{Z}^d \\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})} \|f|F^s_{p,q}(2Q_{j,m})\| \\ &\qquad \leq C_1 \sup_{j\in\mathbb{Z},m\in\mathbb{Z}^d} 2^{j(\frac{d}{p}-\frac{d}{u})} \|f|F^{s-1}_{p,q}(2Q_{j,m})\| + C_1 \sum_{k=1}^{d} \sup_{j\in\mathbb{Z},m\in\mathbb{Z}^d} 2^{j(\frac{d}{p}-\frac{d}{u})} \|\partial^1_k f|F^{s-1}_{p,q}(2Q_{j,m})\|. \end{aligned}$$

Now because of $s-1 < \min(1/p, d/u)$ we can apply Proposition 27 again. When we use Lemma 4, we find

$$\begin{aligned} \sup_{\substack{j\in\mathbb{Z}\setminus\mathbb{N},\, m\in\mathbb{Z}^d \\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})} \|f|F^s_{p,q}(2Q_{j,m})\| &\leq C_2 \|f|\mathcal{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\| + C_2 \sum_{k=1}^{d} \|\partial^1_k f|\mathcal{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\| \\ &\leq C_3 \|f|\mathcal{E}^{s}_{u,p,q}(\mathbb{R}^d)\|. \end{aligned}$$

So this step of the proof is complete. Notice, that formula (13.12) also holds for $j \in \mathbb{N}$. But then the constant C_1 depends on j and tends to infinity, if j tends to infinity. Whereas this is not a problem for small $j \in \mathbb{N}$, for the case of large $j \in \mathbb{N}$ we have to go another way.

Step 6. The case of small cubes.

Substep 6.1. Construct auxiliary functions.

Here we have to investigate the term

$$\sup_{\substack{j\geq R,m\in\mathbb{Z}^d \\ 2Q_{j,m}\in\mathcal{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})} \|f|F^s_{p,q}(2Q_{j,m})\|. \tag{13.13}$$

Let $j \in \mathbb{N}_0 \cup \{-1\}$ and $c \geq 1$. Since (13.13) is invariant under translation, in what follows we can work with cubes denoted by cQ_j, that have side-length $c \cdot 2^{-j}$ and the center at the origin. Let $h \in C_0^\infty(\mathbb{R}^d)$ be a smooth function with $h(x) = 1$ for all $x \in 2Q_0$ and $h(x) = 0$ for all $x \notin 8Q_0$, that fulfills $\|h|C^2(\mathbb{R}^d)\| \leq C_1$ for some constant $C_1 > 0$. For $j \in \mathbb{N}_0$ we define

$$h_j(x) = h(2^j x) \qquad \text{and} \qquad g_j(x) = f(x) \cdot h_j(x). \tag{13.14}$$

In what follows we will investigate the properties of the functions g_j.

(i) For all $x \in 2Q_j$ we have $g_j(x) = f(x)$. Moreover, we find $\operatorname{supp} g_j \subset 8Q_j$.

(ii) The functions g_j are bounded. More precisely we have

$$\|g_j|L_\infty(\mathbb{R}^d)\| \leq \sup_{x\in 8Q_j} |f(x)|\,|h_j(x)| \leq C_1 \|f|L_\infty(8Q_j)\|.$$

(iii) The functions g_j are smooth. For the first derivatives with $|\alpha| = 1$ we obtain

$$\begin{aligned}\|D^\alpha g_j|L_\infty(\mathbb{R}^d)\| &\le \sup_{x\in 8Q_j}\Big(|D^\alpha f(x)|\,|h_j(x)| + |D^\alpha h_j(x)|\,|f(x)|\Big)\\ &\le C_1\|f|C^1(8Q_j)\| + C_1 2^j \sup_{x\in 8Q_j}|f(x)|.\end{aligned}$$

In view of (13.13) we can assume $2Q_j \in \mathscr{Q}_{SC}(f)$. Therefore there exists a $z \in 2Q_j$, such that $f(z) = 0$. Since f is smooth, for each $y \in 8Q_j$ by the Mean Value Theorem we find, that there exists a constant C_2, that is independent of j and f, such that

$$|f(y)| \le C_2\|f|C^1(8Q_j)\|\,|y-z| \le C_2\|f|C^1(8Q_j)\|2d\cdot 2^{-j}. \tag{13.15}$$

So we can conclude $\|D^\alpha g_j|L_\infty(\mathbb{R}^d)\| \le C_3\|f|C^1(8Q_j)\|$.

(iv) For the second derivatives with $|\alpha| = 1$ and $|\beta| = 1$ when we use $2Q_j \in \mathscr{Q}_{SC}(f)$ again we observe

$$\begin{aligned}&\|D^\alpha D^\beta g_j|L_\infty(\mathbb{R}^d)\|\\ &\qquad\le C_1\|f|C^2(8Q_j)\| + C_4 2^j\|f|C^1(8Q_j)\| + \sup_{x\in 8Q_j}|D^\alpha D^\beta h_j(x)|\,|f(x)|\\ &\qquad\le C_1\|f|C^2(8Q_j)\| + C_4 2^j\|f|C^1(8Q_j)\| + C_5 2^{2j}2^{-j}\|f|C^1(8Q_j)\|\\ &\qquad\le C_6\max(\|f|C^2(8Q_j)\|, 2^j\|f|C^1(8Q_j)\|).\end{aligned}$$

In what follows it will turn out, that the properties (i) - (iv) of the functions g_j are exactly what we need to continue our proof.

Substep 6.2. Use the smoothness properties of f and g_j.

Now we use the functions g_j we have constructed before to deal with the term (13.13). We want to apply Remark 2.12. from [134]. Let $U_\lambda = \{x \in \mathbb{R}^d : |x_r| < \lambda\}$ with $0 < \lambda \le 1$. Then for $s > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$ we find $\|f(\lambda\cdot)|F^s_{p,q}(\mathbb{R}^d)\| \sim \lambda^{s-d/p}\|f|F^s_{p,q}(\mathbb{R}^d)\|$ for $f \in F^s_{p,q}(\mathbb{R}^d)$ with $\operatorname{supp} f \subset U_\lambda$. This is the so called local homogeneity property. When we use it, in our case because of $\operatorname{supp} g_j \subset 8Q_j$ and the definition of the norm $\|\cdot|F^s_{p,q}(8Q_j)\|$ we get

$$2^{j(\frac{d}{p}-\frac{d}{u})}\|f|F^s_{p,q}(2Q_j)\| \le 2^{j(\frac{d}{p}-\frac{d}{u})}\|g_j|F^s_{p,q}(8Q_j)\| \le C_1 2^{j(s-\frac{d}{u})}\|g_j(2^{-j+2}\cdot)|F^s_{p,q}(Q_{-1})\|.$$

Because of $p \ge 1$, $q \ge 1$ and $s > 0$ it is possible to describe the space $\mathbb{F}^s_{p,q}(Q_{-1})$ in terms of differences, see Theorem 1.118. in [133]. Then we obtain

$$\begin{aligned}&\|g_j(2^{-j+2}\cdot)|F^s_{p,q}(Q_{-1})\|\\ &\quad\le C_2\|g_j(2^{-j+2}\cdot)|L_p(Q_{-1})\| + C_2\left\|\left(\int_0^1 t^{-sq}\Big(t^{-d}\int_{V^2}|\Delta_h^2 g_j(2^{-j+2}\cdot)|dh\Big)^q\frac{dt}{t}\right)^{\frac{1}{q}}\Big|L_p(Q_{-1})\right\|\end{aligned}$$

with $V^2 = V^2(x,t) = \big\{h \in \mathbb{R}^d : |h| < t \text{ and } x + \tau h \in Q_{-1} \text{ for } 0 \le \tau \le 2\big\}$. Notice, that the constant C_2 is independent of j. Now on the one hand thanks to (ii) from our construction of g_j and a transformation of the coordinates we find

$$2^{j(s-\frac{d}{u})}\|g_j(2^{-j+2}\cdot)|L_p(Q_{-1})\| \le C_3 2^{j(\frac{d}{p}-\frac{d}{u})}2^{js}\|g_j|L_p(8Q_j)\| \le C_4 2^{-j\frac{d}{u}}2^{js}\|f|L_\infty(8Q_j)\|.$$

Like before, see formula (13.15), for all $y \in 8Q_j$ we observe $|f(y)| \leq C_5 \|f|C^1(8Q_j)\| 2d \cdot 2^{-j}$. Hence we conclude

$$2^{j(s-\frac{d}{u})} \|g_j(2^{-j+2}\cdot)|L_p(Q_{-1})\| \leq C_6 \|f|C^1(8Q_j)\| 2^{-j\frac{d}{u}} 2^{js} 2^{-j} = C_6 \|f|C^1(8Q_j)\| 2^{j(s-1-\frac{d}{u})}.$$

Now we have to deal with the term that contains differences. When we use (iv) from our construction of g_j, thanks to some transformations of the coordinates, we obtain

$$\begin{aligned}
&2^{j(\frac{d}{p}-\frac{d}{u})} 2^{-j(\frac{d}{p}-s)} \Big\| \Big(\int_0^1 t^{-sq} \Big(t^{-d} \int_{V^2(x,t)} |(\Delta_h^2 g_j(2^{-j+2}\cdot))(x)| dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| L_p(Q_{-1}) \Big\| \\
&\leq C_7 2^{j(\frac{d}{p}-\frac{d}{u})} \Big\| \Big(\int_0^{c2^{-j}} t^{-sq} \Big(t^{-d} \int_{|h|<t} |(\Delta_h^2 g_j)(x)| dh \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| L_p(8Q_j) \Big\| \\
&\leq C_8 \max(\|f|C^2(8Q_j)\|, 2^j \|f|C^1(8Q_j)\|)\, 2^{j(\frac{d}{p}-\frac{d}{u})} \Big\| \Big(\int_0^{c2^{-j}} t^{-sq+2q} \frac{dt}{t} \Big)^{\frac{1}{q}} \Big| L_p(8Q_j) \Big\| \\
&\leq C_9 \max(\|f|C^2(8Q_j)\|, 2^j \|f|C^1(8Q_j)\|)\, 2^{j(\frac{d}{p}-\frac{d}{u})} 2^{-j(2-s)} 2^{-j\frac{d}{p}} \\
&= C_{10} \max(2^{-j}\|f|C^2(8Q_j)\|, \|f|C^1(8Q_j)\|)\, 2^{j(s-1-\frac{d}{u})}.
\end{aligned}$$

Up to now we proved, that for all $j \in \mathbb{N}, m \in \mathbb{Z}^d$ and all cubes $2Q_{j,m} \in \mathscr{Q}_{SC}(f)$ we have

$$2^{j(\frac{d}{p}-\frac{d}{u})} \|f|F^s_{p,q}(2Q_{j,m})\| \leq C_{11} \max(2^{-j}\|f|C^2(\mathbb{R}^d)\|, \|f|C^1(\mathbb{R}^d)\|)\, 2^{j(s-1-\frac{d}{u})}. \tag{13.16}$$

Notice, that we have $s < 1 + d/u$ and so $s - 1 - d/u < 0$. So for large $j \in \mathbb{N}$ that tend to ∞, the right hand side tends to zero. In what follows we will make this more precise.

Substep 6.3. Use that $\mathscr{F} f$ has compact support.

Now we have to deal with the terms $\|f|C^1(\mathbb{R}^d)\|$ and $\|f|C^2(\mathbb{R}^d)\|$. Let us start with the first one. Let $|\alpha| \leq 1$ and $(\varphi_k)_{k\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. Since $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ is smooth, we find

$$\|D^\alpha f|L_\infty(\mathbb{R}^d)\| = \Big\| \sum_{k=0}^\infty \mathscr{F}^{-1}[\varphi_k \mathscr{F}(D^\alpha f)] \Big| L_\infty(\mathbb{R}^d) \Big\| \leq \sum_{k=0}^\infty \|\mathscr{F}^{-1}[\varphi_k \mathscr{F}(D^\alpha f)]|L_\infty(\mathbb{R}^d)\|.$$

Now since $\operatorname{supp} \varphi_k \mathscr{F}(D^\alpha f) \subset B(0, 2^{k+1})$, we can use formula (7) from the proof of Corollary 2.3 in [108], see also Proposition 3.7 in [108]. Then we obtain

$$\|D^\alpha f|L_\infty(\mathbb{R}^d)\| \leq C_1 \sum_{k=0}^\infty 2^{k\frac{d}{u}} \|\mathscr{F}^{-1}[\varphi_k \mathscr{F}(D^\alpha f)]|\mathscr{M}^u_p(\mathbb{R}^d)\|.$$

Recall, that we have $\operatorname{supp} \mathscr{F} f \subset B(0, 2^R)$ for some $R \in \mathbb{N}$. By standard arguments we find, that this property carries over to $\mathscr{F}(D^\alpha f)$. Because of this for $\sigma > 0$ we also can write

$$\begin{aligned}
\|D^\alpha f|L_\infty(\mathbb{R}^d)\| &\leq C_1 \sum_{k=0}^{R+1} 2^{k(\frac{d}{u}-s+1)} 2^{k\sigma} 2^{k(s-1-\sigma)} \|\mathscr{F}^{-1}[\varphi_k \mathscr{F}(D^\alpha f)]|\mathscr{M}^u_p(\mathbb{R}^d)\| \\
&\leq C_2 2^{R(\frac{d}{u}-s+1)} 2^{R\sigma} \sum_{k=0}^\infty 2^{k(s-1-\sigma)} \|\mathscr{F}^{-1}[\varphi_k \mathscr{F}(D^\alpha f)]|\mathscr{M}^u_p(\mathbb{R}^d)\| \\
&\leq C_3 2^{R(\frac{d}{u}-s+1)} 2^{R\sigma} \|D^\alpha f|\mathscr{N}^{s-1-\sigma}_{u,p,1}(\mathbb{R}^d)\| \\
&\leq C_4 2^{R(\frac{d}{u}-s+1)} 2^{R\sigma} \|D^\alpha f|\mathscr{E}^{s-1}_{u,p,q}(\mathbb{R}^d)\|.
\end{aligned}$$

In the last steps we used the definition of the Besov-Morrey spaces and Proposition 3.6 from [108]. Let us put $\sigma = 1/R$. Then since $|\alpha| \leq 1$, with Lemma 4 we get

$$\|D^{\alpha} f | L_{\infty}(\mathbb{R}^d)\| \leq C_5 2^{R(\frac{d}{u}-s+1)} \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|. \tag{13.17}$$

Next we have to deal with $\|f|C^2(\mathbb{R}^d)\|$. Therefore we have to investigate $\|D^{\beta} f|L_{\infty}(\mathbb{R}^d)\|$ with $|\beta| = 2$. Since f is smooth, this can be done in the same way as before. One obtains

$$\|D^{\beta} f|L_{\infty}(\mathbb{R}^d)\| \leq C_6 2^{R(\frac{d}{u}-s+1)} 2^R \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|. \tag{13.18}$$

When we combine this with formula (13.16) and (13.17), for all $j \in \mathbb{N}, m \in \mathbb{Z}^d$ and all cubes $2Q_{j,m} \in \mathscr{Q}_{SC}(f)$ we find

$$\begin{aligned} 2^{j(\frac{d}{p}-\frac{d}{u})} \|f|F^s_{p,q}(2Q_{j,m})\| &\leq C_7 \max(2^{-j}\|f|C^2(\mathbb{R}^d)\|, \|f|C^1(\mathbb{R}^d)\|)\, 2^{j(s-1-\frac{d}{u})} \\ &\leq C_8 \max(2^{-j+R}, 1) 2^{R(\frac{d}{u}-s+1)} 2^{j(s-1-\frac{d}{u})} \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|. \end{aligned}$$

Now because of $s-1-d/u < 0$ in view of (13.13) we obtain

$$\begin{aligned} &\sup_{\substack{j \geq R, m \in \mathbb{Z}^d \\ 2Q_{j,m} \in \mathscr{Q}_{SC}(f)}} 2^{j(\frac{d}{p}-\frac{d}{u})} \|f|F^s_{p,q}(2Q_{j,m})\| \\ &\qquad \leq C_9 \sup_{j \geq R, m \in \mathbb{Z}^d} \max(2^{-j+R}, 1) 2^{R(\frac{d}{u}-s+1)} 2^{j(s-1-\frac{d}{u})} \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \\ &\qquad \leq C_{10} \|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|. \end{aligned}$$

So this step and the whole proof are complete. ■

Remark 16. ***A Comment concerning the Conditions in Proposition 28.***
In the formulation of Proposition 28 we can find two different types of conditions. So on the one hand we assume $s < \min(1+1/p, 1+d/u)$. *Later we will see, that this condition is necessary, see Proposition 31. On the other hand there is the restriction* $1/p - 1/u > 1 - 1/d$. *It is possible, that this condition is of technical nature only and can be avoided by using another method for the proof. Notice, that this assumption can be left away, when in addition we assume* $\operatorname{supp} \mathscr{F} f \subset B(0, 2^R)$ *with fixed* $0 < R < \infty$ *in the formulation of Proposition 28.*

Under some additional conditions, it is possible to prove a version of Proposition 28 for the special case $s = 1$. Especially for $1 < p \leq u < \infty$, $q = 2$ and $s = 1$ this is important. Recall, that in this case the Triebel-Lizorkin-Morrey spaces coincide with the Sobolev-Morrey spaces, see Definition 20 and Lemma 1. There is the following result.

Proposition 29. ***The Boundedness of*** T ***for*** $d \in \mathbb{N}$***. Part II.***
Let $1 < p < u < \infty$, $1 \leq q < \infty$ *and* $s = 1$. *Let* $1/p - 1/u > 1 - 1/d$. *Then there is a constant* $C > 0$ *independent of* $f \in \mathbb{E}^1_{u,p,q}(\mathbb{R}^d)$, *such that we have*

$$\|Tf|\mathscr{E}^1_{u,p,q}(\mathbb{R}^d)\| \leq C \|f|\mathscr{E}^1_{u,p,q}(\mathbb{R}^d)\| \tag{13.19}$$

for all $f \in \mathbb{E}^1_{u,p,q}(\mathbb{R}^d)$.

Proof. This result can be proved in the same way as Proposition 28. All arguments that are used there, also hold in the situation of Proposition 29. The reason for this is, that our main tool Theorem 22 also is valid for $s = 1$, when we assume $1 < p < \infty$. When we follow the strategy described in the proof of Proposition 28, because of $s - 1 = 0$ we sometimes have to work with smoothness zero. Let us explain why this is not a problem. At first we mention, that thanks to the Fatou property we can work with smooth $C^\infty-$ functions. Therefore we do not have to deal with singular distributions. Next we should notice, that Lemma 4 also holds for smoothness zero. Our tool Proposition 27 is valid for $s - 1 = 0$ as well, see Theorem 3.64 in [137]. Here the restrictions $p \neq 1$ and $q \neq \infty$ are required. Finally we mention, that because of $\mathscr{E}^1_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^{1/2}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathscr{E}^0_{u,p,q}(\mathbb{R}^d)$ and Proposition 25 instead of formula (13.7) we can write

$$\| \, |f| \, |\mathscr{E}^0_{u,p,q}(\mathbb{R}^d)\| \leq C_1 \| \, |f| \, |\mathscr{E}^{1/2}_{u,p,q}(\mathbb{R}^d)\| \leq C_2 \|f|\mathscr{E}^{1/2}_{u,p,q}(\mathbb{R}^d)\| \leq C_3 \|f|\mathscr{E}^1_{u,p,q}(\mathbb{R}^d)\|.$$

This simple observation completes the proof. ■

Notice, that Proposition 29 does not cover the special case $p = 1$ for $s = 1$. On the other hand for the original Sobolev spaces with $u = p = 1$, $q = 2$ and $s = 1$ Marcus and Mizel proved in [76], that T is a continuous operator, see Theorem 1 in [76].

13.3.2 The Boundedness of T on Besov-Morrey Spaces for $d \in \mathbb{N}$

Hereafter we want to deduce a counterpart of Proposition 28 for the Besov-Morrey spaces. To this end, our main tool will be real interpolation. Let $0 < \Theta < 1$ and $1 \leq q \leq \infty$. Then for an interpolation couple (X_0, X_1) of Banach spaces by $(X_0, X_1)_{\Theta,q}$ we denote the result of the real interpolation of these spaces. For the general background concerning real interpolation we refer to [127] and [3]. Using the notation from above we can observe the following connection between Morrey spaces, Triebel-Lizorkin-Morrey spaces and Besov-Morrey spaces.

Lemma 54. ***Real Interpolation of Smoothness Morrey Spaces.***
Let $0 < \Theta < 1$, $s_1 > 0$, $1 \leq p \leq u < \infty$ and $1 \leq q, q_1 \leq \infty$. Then we have

$$\mathscr{N}^{\Theta s_1}_{u,p,q}(\mathbb{R}^d) = \left(\mathscr{M}^u_p(\mathbb{R}^d), \mathscr{E}^{s_1}_{u,p,q_1}(\mathbb{R}^d) \right)_{\Theta,q}$$

in the sense of equivalent norms.

Proof. This result was proved in [117], see Corollary 2.3. ■

Now we are prepared to show, that under some conditions on the parameters the operator T is bounded on the Besov-Morrey spaces.

Proposition 30. ***The Boundedness of T for** $d \in \mathbb{N}$**. Part III.***
Let $1 \leq p < u < \infty$ and $1 \leq q \leq \infty$. Assume $1/p - 1/u > 1 - 1/d$ in the case of $d > 1$. Let

$$0 < s < \min\left(1 + \frac{1}{p}, 1 + \frac{d}{u}\right).$$

Then there is a constant $C > 0$ independent of $f \in \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$, such that we have

$$\|Tf|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\| \leq C \|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\| \qquad (13.20)$$

for all $f \in \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$.

Proof. For the proof we use the corresponding result for the Triebel-Lizorkin-Morrey spaces. Moreover, we apply a result concerning the real interpolation of Lipschitz continuous operators, that goes back to Peetre, see [93]. One may also consult Proposition 1 in chapter 2.5.4 in [101]. Here for us it is convenient to follow the explanations given in Step 5 of the proof of Theorem 25.8 in [131]. We use the same notation as there. So we put $Tf = |f|$ and $A_0 = \mathbb{M}^u_p(\mathbb{R}^d)$ and $A_1 = \mathbb{E}^{s_1}_{u,p,1}(\mathbb{R}^d)$. For $1 \leq p \leq u < \infty$ and $s_1 > 0$ we have $\mathbb{E}^{s_1}_{u,p,1}(\mathbb{R}^d) \hookrightarrow \mathbb{M}^u_p(\mathbb{R}^d)$. Moreover, from the Propositions 25, 26 and 28 we learn, that for $1 \leq p < u < \infty$ and $0 < s_1 < \min(1+1/p, 1+d/u) \leq 2$ with $s_1 \neq 1$ we have $\|Tf|\mathscr{E}^{s_1}_{u,p,1}(\mathbb{R}^d)\| \leq C\|f|\mathscr{E}^{s_1}_{u,p,1}(\mathbb{R}^d)\|$ for all $f \in \mathbb{E}^{s_1}_{u,p,1}(\mathbb{R}^d)$. Here in the case of $d > 1$ the assumption $1/p - 1/u > 1 - 1/d$ is needed. Furthermore, because of the triangle inequality, we observe

$$\| |f| - |g| |\mathscr{M}^u_p(\mathbb{R}^d)\| \leq \|f - g|\mathscr{M}^u_p(\mathbb{R}^d)\|$$

for all $f, g \in \mathbb{M}^u_p(\mathbb{R}^d)$. Then like it is described in Step 5 of the proof of Theorem 25.8 in [131], for all $0 < \Theta < 1$ and $1 \leq q \leq \infty$ we have

$$\|Tf|(\mathbb{M}^u_p(\mathbb{R}^d), \mathbb{E}^{s_1}_{u,p,1}(\mathbb{R}^d))_{\Theta,q}\| \leq C\|f|(\mathbb{M}^u_p(\mathbb{R}^d), \mathbb{E}^{s_1}_{u,p,1}(\mathbb{R}^d))_{\Theta,q}\|$$

for all $f \in (\mathbb{M}^u_p(\mathbb{R}^d), \mathbb{E}^{s_1}_{u,p,1}(\mathbb{R}^d))_{\Theta,q}$. Now we apply Lemma 54, which tells us, that

$$\mathbb{N}^{\Theta s_1}_{u,p,q}(\mathbb{R}^d) = \Big(\mathbb{M}^u_p(\mathbb{R}^d), \mathbb{E}^{s_1}_{u,p,1}(\mathbb{R}^d)\Big)_{\Theta,q}$$

with $0 < \Theta < 1$ and $1 \leq q \leq \infty$. Because of $0 < s_1 < \min(1+1/p, 1+d/u)$ and $s_1 \neq 1$ we find $0 < \Theta s_1 < \min(1+1/p, 1+d/u)$. So the proof is complete. ■

Notice, that for the original Besov spaces, there exist several different methods, to prove results like Proposition 30. In [13] and in chapter 5.4.1 in [101] a Hardy-type inequality in combination with real interpolation was applied. In [92] some tools from approximation theory for linear splines are used to prove a result for the spaces $\mathbb{B}^s_{p,q}(\mathbb{R})$. A third method using atoms can be found in [131], see Theorem 25.8.

13.4 Truncation and necessary Conditions concerning the Parameters s, p and u

When you look at the Propositions 26, 28 and 30, you always will find the condition

$$s < \min\Big(1 + \frac{1}{p}, 1 + \frac{d}{u}\Big). \tag{13.21}$$

In this section we investigate, whether this condition is also necessary. For that purpose we will deal with some special test functions. Let us start with the Triebel-Lizorkin-Morrey spaces.

Proposition 31. ***Truncations and necessary Conditions. Part I.***
Let $1 \leq p \leq u < \infty$ and $1 \leq q \leq \infty$. Let either $s \geq 1 + 1/p$ or $s > 1 + d/u$. Then there exists a function $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$, such that $Tf \notin \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ and $T^+ f \notin \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$.

Proof. For the proof we will deal with a special test function. Let $(x_1, x_2, \ldots, x_d) = x \in \mathbb{R}^d$. Then we define a real-valued function $f \in C_0^\infty(\mathbb{R}^d)$, such that

$$f(x) = x_1 \quad \text{for} \quad |x| < 10d(s+2) \quad \text{and} \quad f(x) = 0 \quad \text{for} \quad |x| > 11d(s+2). \tag{13.22}$$

Of course we have $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$. In what follows we prove $Tf \notin \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ and $T^+ f \notin \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ simultaneously. For that purpose we write T^* when we mean either T or T^+. Because of $p \geq 1$ and $q \geq 1$ we can apply Theorem 5 with $v = 1$, $a = \infty$ and $N \in \mathbb{N}$ with $s < N < s+2$. Notice, that we always have $N \geq 2$. We write $(h_1, h_2, \ldots, h_d) = h \in \mathbb{R}^d$. Then we find

$$\begin{aligned} &\|T^* f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \\ &\qquad \geq C_1 \sup_{\substack{P \text{ dyadic cube} \\ P \subset [0,1]^d}} |P|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_P \Big(\int_{\frac{3}{2}x_1}^{2x_1} t^{-sq} \Big(t^{-d} \int_{\substack{|h| \leq t \\ h_1 \leq -x_1}} |\Delta_h^N T^* f(x)| dh \Big)^q \frac{dt}{t} \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}}. \end{aligned}$$

Let $x \in [0,1]^d$. Then we have $|f(x)| = x_1$ and $\max(f(x), 0) = x_1$. Moreover, let $3/2\, x_1 < t < 2x_1$ and $|h| \leq t$ with $h_1 \leq -x_1$. Then for $l \in \{1, 2, \ldots, N\}$ we observe $|f(x+lh)| = -x_1 - lh_1$ and $\max(f(x+lh), 0) = 0$. Recall, that for $N \geq 2$ there are the elementary formulas

$$\sum_{l=0}^N (-1)^l \binom{N}{l} = 0 \qquad \text{and} \qquad \sum_{l=0}^N (-1)^l \binom{N}{l} l = 0.$$

Hence for the operators T and T^+ in the case $N \geq 2$ we get

$$|\Delta_h^N |f|(x)| = 2x_1 \qquad \text{and} \qquad |\Delta_h^N T^+ f(x)| = x_1.$$

So we have almost the same outcome for T and T^+. Therefore for both cases we can proceed in the same way now. For $3/2\, x_1 < t < 2x_1$ we have $t^{-d} \geq C_2 x_1^{-d}$. Moreover, there exists a constant C_d, that depends on d, such that $\int_{|h| \leq t,\ h_1 \leq -x_1} 1 dh \geq C_d x_1^d$. So we obtain

$$\|T^* f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \geq C_3 \sup_{\substack{P \text{ dyadic cube} \\ P \subset [0,1]^d}} |P|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_P x_1^{p-sp} dx \Big)^{\frac{1}{p}}.$$

In what follows we are only interested in dyadic cubes, that look like $2^{-j}[0,1)^d$ with $j \in \mathbb{N}_0$. Then we can apply Fubini's Theorem and get

$$\|T^* f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \geq C_4 \sup_{j \in \mathbb{N}_0} 2^{-jd(\frac{1}{u}-\frac{1}{p})} 2^{-j(d-1)\frac{1}{p}} \Big(\int_0^{2^{-j}} x_1^{p-sp} dx_1 \Big)^{\frac{1}{p}}.$$

If we are in the case $s \geq 1 + 1/p$ the integral is infinite and the proof is complete. So in what follows we assume $s < 1 + 1/p$, but $s > 1 + d/u$. Then we find

$$\|T^* f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \geq C_5 \sup_{j \in \mathbb{N}_0} 2^{-jd(\frac{1}{u}-\frac{1}{p})} 2^{-j(d-1)\frac{1}{p}} 2^{-j(1-s+\frac{1}{p})} = C_5 \sup_{j \in \mathbb{N}_0} 2^{-j(\frac{d}{u}-s+1)}.$$

But because of $d/u - s + 1 < 0$ we obtain $\|T^* f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| = \infty$. The proof is complete. ■

Now we turn our attention to the Besov-Morrey spaces. Here also for the critical border

$$s = \min\Big(1 + \frac{1}{p}, 1 + \frac{d}{u}\Big) \tag{13.23}$$

we obtain an almost complete result.

Proposition 32. ***Truncations and necessary Conditions. Part II.***
Let $1 \le p \le u < \infty$ and $1 \le q \le \infty$. Moreover we are in one of the following situations.

(i) We have $s > \min(1+\frac{1}{p}, 1+\frac{d}{u})$.

(ii) We have $s = \min(1+\frac{1}{p}, 1+\frac{d}{u})$ and $q \neq \infty$.

(iii) We have $d = 1$ with $s = 1+\frac{1}{u}$ and $q = \infty$.

Then there exists a function $f \in \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$, such that $Tf \notin \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ and $T^+f \notin \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$.

Proof. *Step 1.* At first we look at the cases $s > 1+d/u$ and $s = 1+d/u$ with $q \neq \infty$. Here we work with the same function $f \in C_0^\infty(\mathbb{R}^d)$ as in the proof of Proposition 31, see formula (13.22). We proceed like there and apply Theorem 7 with $v = 1$, $a = \infty$ and $N \ge 2$. Then we find $Tf \notin \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ and $T^+f \notin \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$. Notice, that for $q \neq \infty$ we also obtain a result for $s = 1+d/u$. The reason for this is, that in the norm, that can be found in Theorem 7, the integral concerning t is outside of the Morrey norm.
Step 2. Now we look at the case $s > 1+1/p$. We work with the same function $f \in C_0^\infty(\mathbb{R}^d)$ as in the proof of Proposition 31. Of course we have $f \in \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$. Let $\varepsilon > 0$ such that $s > s-\varepsilon > 1+1/p$. We have $\mathbb{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathbb{N}^{s-\varepsilon}_{u,p,p}(\mathbb{R}^d) \hookrightarrow \mathbb{E}^{s-\varepsilon}_{u,p,p}(\mathbb{R}^d)$. Now we can apply Proposition 31 and its proof. So we obtain $Tf \notin \mathbb{E}^{s-\varepsilon}_{u,p,p}(\mathbb{R}^d)$ and $T^+f \notin \mathbb{E}^{s-\varepsilon}_{u,p,p}(\mathbb{R}^d)$. Consequently we get $Tf \notin \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ and $T^+f \notin \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$.
Step 3. Next we have to deal with $s = 1+1/p$ and $0 < q < \infty$. Again we work with the function $f \in C_0^\infty(\mathbb{R}^d)$ from the proof of Proposition 31, see formula (13.22). Of course we have $f \in \mathbb{N}^{1+1/p}_{u,p,q}(\mathbb{R}^d)$. In what follows we will prove $Tf \notin \mathbb{N}^{1+1/p}_{u,p,q}(\mathbb{R}^d)$ and $T^+f \notin \mathbb{N}^{1+1/p}_{u,p,q}(\mathbb{R}^d)$ simultaneously. For that purpose we write T^* when we mean either T or T^+. We use Theorem 7 with $a = \infty$, $v = 1$ and $N > 1+1/p$. For the supremum in the Morrey norm we choose the smallest ball B^* with $[0,1]^d \subset B^*$. We write $x' = (x_2, x_3, \dots, x_d) \in \mathbb{R}^{d-1}$. Then we find

$$\begin{aligned}\|T^*f|\mathcal{N}^{1+\frac{1}{p}}_{u,p,q}(\mathbb{R}^d)\| &\ge C_1\Big(\int_0^\infty t^{-sq-dq}\Big\|\Big(\int_{B(0,t)}|\Delta_h^N T^*f(x)|dh\Big)\Big|\mathcal{M}^u_p(\mathbb{R}^d)\Big\|^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\ge C_2\Big(\int_0^1 t^{-q(s+d)}\Big(\int_0^{\frac{t}{2}}\int_{[0,1]^{d-1}}\Big(\int_{\substack{|h|\le t\\ h_1\le -\frac{t}{2}}}|\Delta_h^N T^*f(x)|dh\Big)^p dx'dx_1\Big)^{\frac{q}{p}}\frac{dt}{t}\Big)^{\frac{1}{q}}.\end{aligned}$$

Now for $t \in [0,1]$ and $x \in [0,\frac{t}{2}] \times [0,1]^{d-1}$ we have $|f(x)| = x_1$ and $\max(f(x),0) = x_1$. Moreover, for $|h| \le t$ with $h_1 \le -t/2$ and $l \in \{1,2,\dots,N\}$ we can observe $|f(x+lh)| = -x_1 - lh_1$ and $\max(f(x+lh),0) = 0$. Because of this like in the proof of Proposition 31 for $N \ge 2$ we find

$$|\Delta_h^N|f|(x)| = 2x_1 \qquad \text{and} \qquad |\Delta_h^N T^+f(x)| = x_1.$$

So we have almost the same outcome for T and T^+. Therefore for both cases we can proceed in the same way now. We obtain

$$\begin{aligned}\|T^*f|\mathcal{N}^{1+\frac{1}{p}}_{u,p,q}(\mathbb{R}^d)\| &\ge C_3\Big(\int_0^1 t^{-sq-dq}\Big(\int_0^{\frac{t}{2}} x_1^p\int_{[0,1]^{d-1}}\Big(\int_{\substack{|h|\le t\\ h_1\le -\frac{t}{2}}} 1dh\Big)^p dx'\,dx_1\Big)^{\frac{q}{p}}\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\ge C_4\Big(\int_0^1 t^{-sq+q+\frac{q}{p}-1}dt\Big)^{\frac{1}{q}} = \infty.\end{aligned}$$

In the last step we used $s = 1+1/p$ and $0 < q < \infty$. Hence this step of the proof is complete.
Step 4. Now we look at the case $d = 1$ with $s = 1+1/u$ and $q = \infty$. We will work with a function, that can be found in Lemma 2 in chapter 5.4.1 in [101], see also the proposition in [13]. Let $\varphi \in \mathbb{S}(\mathbb{R})$ be a real-valued and odd function with $\operatorname{supp} \mathscr{F}\varphi \subset [-1,1]$ and $\varphi(x) = x$ for $-1 \leq x \leq 1$. We define

$$g(x) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x). \tag{13.24}$$

From Lemma 2 in chapter 5.4.1 in [101] we learn, that we have $g \in \mathbb{B}^{1+1/u}_{u,\infty}(\mathbb{R})$. Now because of $\mathbb{B}^{1+1/u}_{u,\infty}(\mathbb{R}) = \mathbb{N}^{1+1/u}_{u,u,\infty}(\mathbb{R}) \hookrightarrow \mathbb{N}^{1+1/u}_{u,p,\infty}(\mathbb{R})$ we also find $g \in \mathbb{N}^{1+1/u}_{u,p,\infty}(\mathbb{R})$. In what follows we will prove, that we have $Tg \notin \mathbb{N}^{1+1/u}_{u,p,\infty}(\mathbb{R})$ and $T^+g \notin \mathbb{N}^{1+1/u}_{u,p,\infty}(\mathbb{R})$ simultaneously. For that purpose as before we write T^* when we mean either T or T^+. We use Theorem 7 with $a = \infty$ and $v = \infty$. This is possible because of $1/p \leq 1 < 1+1/u$. Since $1+1/u < 2$ we can put $N = 2$. Then we find

$$\|T^*g|\mathscr{N}^{1+\frac{1}{u}}_{u,p,\infty}(\mathbb{R})\| \geq C_1 \sup_{0 \leq t < \infty} t^{-1-\frac{1}{u}} \sup_{\substack{a<b \\ |a|\leq t, |b| \leq t}} |a-b|^{\frac{1}{u}-\frac{1}{p}} \Big(\int_a^b |\Delta^2_{-x} T^* g(x)|^p dx \Big)^{\frac{1}{p}}.$$

From the proof of Lemma 2 in chapter 5.4.1 in [101] we know $g(0) = 0$ and $|g(x)| = |g(-x)|$. Since g is odd, we observe

$$\Delta^2_{-x}|g|(x) = 2|g(x)| \qquad \text{and} \qquad \Delta^2_{-x} T^+ g(x) = |g(x)|.$$

So we have almost the same outcome for T and T^+. Then in both cases we obtain

$$\begin{aligned} \|T^*g|\mathscr{N}^{1+\frac{1}{u}}_{u,p,\infty}(\mathbb{R})\| &\geq C_2 \sup_{0 \leq t < \infty} t^{-1-\frac{1}{u}} t^{\frac{1}{u}-\frac{1}{p}} \Big(\int_0^t |g(x)|^p dx \Big)^{\frac{1}{p}} \\ &\geq C_3 \sup_{0<t<1} t^{-2} \int_0^t \Big| \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x) \Big| dx. \end{aligned}$$

In the last step we used the Hölder inequality. Now for $0 < t < 1$ let $L(t) \in \mathbb{N}_0$ be the biggest natural number, such that $L(t) < \frac{\ln(\frac{1}{t})}{\ln(2)}$. Then for $j \in \mathbb{N}_0$ with $j \leq L(t)$ and $x \leq t$ we have $2^j x \leq 1$. Now because of the definition of the function φ we obtain

$$\begin{aligned} \|T^*g|\mathscr{N}^{1+\frac{1}{u}}_{u,p,\infty}(\mathbb{R})\| &\geq C_3 \sup_{0<t<1} t^{-2} \int_0^t \Big| \sum_{j=0}^{L(t)-1} x + \sum_{j=L(t)}^{\infty} 2^{-j} \varphi(2^j x) \Big| dx \\ &\geq C_4 \sup_{0<t<1} t^{-2} \Big(L(t) \frac{t^2}{2} - Kt \sum_{j=L(t)}^{\infty} 2^{-j} \Big). \end{aligned}$$

In the last step we used, that $\varphi \in \mathbb{S}(\mathbb{R})$ is bounded by a constant $K < \infty$. Now we calculate $\sum_{j=L(t)}^{\infty} 2^{-j} \leq C_5 2^{-L(t)} \leq C_6 t$. Hence we get

$$\|T^*g|\mathscr{N}^{1+\frac{1}{u}}_{u,p,\infty}(\mathbb{R})\| \geq C_7 \sup_{0<t<1} t^{-2} \Big(L(t) \frac{t^2}{2} - Kt^2 \Big) \geq C_8 \lim_{t \downarrow 0} \Big(\frac{1}{2} \ln \Big(\frac{1}{t} \Big) - K \Big) = \infty.$$

So the proof is complete. ∎

13.5 Compound Results and outstanding Issues concerning Truncations

In this section we want to sum up, what we learned up to now concerning the boundedness of the truncation operators T and T^+. For that purpose we formulate some compound results, that contain both sufficient and necessary conditions on the parameters at the same time. Let us start with the Besov-Morrey spaces. For them we can summarize our findings as follows.

Theorem 23. ***Truncations. Compound Result I.***
Let $1 \le p < u < \infty$, $1 \le q \le \infty$ and $s > 0$. We assume

$$\begin{cases} \frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d} & \text{in the case} \quad 1 \le s < \min(1+\frac{1}{p}, 1+\frac{d}{u}) \text{ and } d > 1; \\ q \neq \infty & \text{in the case} \quad s = \min(1+\frac{1}{p}, 1+\frac{d}{u}) \text{ and } d > 1. \end{cases}$$

Then T^+ acts on $\mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ and there is a constant $C > 0$ independent of $f \in \mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$, such that we have

$$\|T^+ f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\| \le C\|f|\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)\|, \tag{13.25}$$

if and only if

$$s < \min\left(1+\frac{1}{p}, 1+\frac{d}{u}\right). \tag{13.26}$$

Moreover in the formulation of Theorem 23 one can replace the operator T^+ by T.

Proof. This result is just a combination of the Propositions 25, 30 and 32. ■

It turns out, that the critical border $s = 1+1/p$ we know for the spaces $\mathbb{B}^s_{p,q}(\mathbb{R}^d)$, see Theorem 22, is replaced by $s = \min(1+1/p, 1+d/u)$ in the case of the spaces $\mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$. There is the surprising new phenomenon, that for $p < u$ the critical border also depends on the dimension d. For $p = u$ this is not the case. Here we always have $\min(1+1/p, 1+d/u) = 1+1/p$. So we recover the original result. Moreover, for $d = 1$ because of $p \le u$ we obtain $\min(1+1/p, 1+1/u) = 1+1/u$. Hence the condition concerning the parameter s becomes much more easy in this case. As already mentioned in Remark 16, the additional condition $1/p - 1/u > 1 - 1/d$ we need in the case $d > 1$ seems to be of technical nature. Maybe it can be left away using another method for the proof. Now we turn to the Triebel-Lizorkin-Morrey spaces. Here our main outcome reads as follows.

Theorem 24. ***Truncations. Compound Result II.***
Let $1 \le p < u < \infty$, $1 \le q \le \infty$ and $s > 0$. We assume

$$\begin{cases} p \neq 1, q \neq \infty \text{ and } \frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d} & \text{in the case} \quad s = 1; \\ \frac{1}{p} - \frac{1}{u} > 1 - \frac{1}{d} & \text{in the case} \quad 1 < s < \min(1+\frac{1}{p}, 1+\frac{d}{u}) \text{ and } d > 1; \\ \frac{u}{p} \le d & \text{in the case} \quad s = \min(1+\frac{1}{p}, 1+\frac{d}{u}). \end{cases}$$

Then T^+ acts on $\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ and there is a constant $C > 0$ independent of $f \in \mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$, such that we have

$$\|T^+ f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\| \le C\|f|\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)\|, \tag{13.27}$$

if and only if

$$s < \min\left(1+\frac{1}{p}, 1+\frac{d}{u}\right).$$

Moreover in the formulation of Theorem 24 one can replace the operator T^+ by T.

Proof. To show this result, we have to combine the Propositions 25, 26, 28, 29 and 31. ■

So it turns out, that also in the case of the Triebel-Lizorkin-Morrey spaces the critical border $s = 1 + 1/p$ we know for the spaces $\mathbb{F}^s_{p,q}(\mathbb{R}^d)$ is replaced by $s = \min(1+1/p, 1+d/u)$ for $p < u$. All in all we can say, that the boundedness of the operators T and T^+ is at least partly understood now. But one may also ask, whether the truncation operator is continuous or even Lipschitz continuous on $\mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$. In general for $s > 0$ Lipschitz continuity can not be expected. For the special case $p = u$ we refer to [131], see Theorem 25.14. Most likely it is possible to use the ideas from there also for $p < u$. On the other hand to prove a satisfactory result concerning continuity, seems to be a difficult problem. In what follows we will collect some more open problems concerning the operators T^+ and T.

Open Problem 6. ***Open Problems concerning Truncations.***

(i) The first query is related to the mapping properties of T^+ in the case $d > 1$ and $1 \leq s < \min(1+1/p, 1+d/u)$. Is it possible to omit the assumption $1/p - 1/u > 1 - 1/d$ you can find in the main results Theorem 23 and Theorem 24?

(ii) The next question concerns the mapping properties of the operator T^+ in the context of the Triebel-Lizorkin-Morrey spaces on the critical border $s = 1 + d/u$ with $u > dp$, see Theorem 24. Do we have $T^+(\mathbb{E}^{1+d/u}_{u,p,q}(\mathbb{R}^d)) \subset \mathbb{E}^{1+d/u}_{u,p,q}(\mathbb{R}^d)$?

(iii) The spaces $\mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$ are also well-defined for $0 < p \leq u < \infty$ and $0 < q \leq \infty$. One may discuss the mapping properties of T^+ and T in this more general setting. Some results concerning the special case $p = u$ can be found in Theorem 25.8 in [131].

(iv) The next issue concerns the continuity of T^+. Under which conditions on the parameters s, p, u, q and d the operator $T^+ : \mathbb{A}^s_{u,p,q}(\mathbb{R}^d) \to \mathbb{A}^s_{u,p,q}(\mathbb{R}^d)$ is continuous? Notice, that for the special case $p = u$ and $0 < s \leq 1$ some positive results are already known, see [76], [84] and Theorem 3 in chapter 5.5.2 in [101].

13.6 Appendix: The zero Set of real analytic Functions

In this section we collect some facts concerning the zero set of real analytic functions. They are used in the proof of Proposition 28, see section 13.3.1.

Lemma 55. ***Zeros of real analytic Functions.***
Let $f:\mathbb{R}^d \to \mathbb{R}$ be an analytic function. Then we know the following.

(i) Let $f \neq 0$. Then the set $Z(f)=\{x\in\mathbb{R}^d : f(x)=0\}$ is the union of countably many compact sets K_j with $\lambda_{d-1}(K_j)<\infty$ for all $j\in\mathbb{N}$. Here with λ_{d-1} we denote the $(d-1)-$dimensional Lebesgue measure. Moreover, the Hausdorff dimension of $Z(f)$ does not exceed $d-1$.

(ii) Let $f\neq 0$. Suppose that $f(0',x_d)$ has a zero of multiplicity $m\in\mathbb{N}$ at $x_d=0$. Then there exist open intervals $I_1,I_2,\ldots,I_d\subset\mathbb{R}$ centered at 0, such that $f(x',\cdot)$ has for each $x'\in\mathbb{R}^{d-1}$ with $x'\in I_1\times I_2\times\ldots\times I_{d-1}$ not more than m zeros in I_d counted according to their multiplicities. Moreover, the multiplicity m is always finite (after a suitable rotation of f maybe).

(iii) Let $R\in\mathbb{N}$, $v\in\mathbb{Z}^d$ and $k\in\mathbb{N}$ with $0<k\leq R$. Let $Q_{k,v}$ be a dyadic cube and $f\neq 0$. Then there exist
- *a natural number $n(f)\in\mathbb{N}$ that depends on f,*
- *a constant $c(d)$ that only depends on d,*
- *a number $r\in\mathbb{N}$ with r much larger than R and 2^{r-k} much larger than $c(d)n(f)$,*

such that the set $Z(f)\cap 2Q_{k,v}$ can be covered by $c(d)\,n(f)\,2^{(d-1)(r-k)}$ $d-$dimensional cubes with side-length 2^{-r}.

Proof. *Proof of (i) and (ii).* Fact (i) can be deduced from Theorem 14.4.9. in [99]. For the result concerning the Hausdorff dimension we also refer to [81]. A definition of the Hausdorff dimension can be found in Definition 14.4.1. in [99], and also in [34] and [77]. Fact (ii) can be derived from Lemma 14.1.2.(i) in [99]. The result concerning the multiplicity also can be found in the proof of Claim 2 in [81].
Proof of (iii). At first we prove a version of (iii) with a natural number $n(f)=n(f,k,v)$, that also depends on k and v. Later at the end of the proof we explain, why this number $n(f)$ also can be chosen independent from k and v. Let us start with the case $d=1$. Since f is a real analytic function, it is well-known, that $Z(f)\cap 2Q_{k,v}$ consists of a finite number $n_0(f,k,v)\in\mathbb{N}$ of isolated points in $\mathbb{R}$. Therefore for each large $r\in\mathbb{N}$ the set $Z(f)\cap 2Q_{k,v}$ can be covered by $n_0(f,k,v)$ intervals with side-length 2^{-r}.
Now we look at the case $d>1$. Here our strategy will be to construct an algorithm, that delivers a covering, that fulfills all the properties given in Lemma 55 (iii). Let $z_1\in Z(f)\cap\overline{2Q_{k,v}}$ such that the multiplicity $m_1\in\mathbb{N}$ of the zero is as big as possible. From fact (ii) we know $m_1<\infty$ (maybe after a rotation of f). If there exist two or more zeros with the same maximal multiplicity, we choose that one, that allows us to find the biggest cube Q_1 appropriate to the description given now. From (ii) we learn, that there exist open intervals $I_1,I_2,\ldots,I_d\subset\mathbb{R}$ with $|I_i|=2^{-r_1}$ for all i and for some $r_1\geq k$, such that z_1 is in the center of $Q_1=I_1\times I_2\times\ldots\times I_d$ and such that $f(z',\cdot)$ has for each $z'\in\mathbb{R}^{d-1}$ with $z'\in I_1\times I_2\times\ldots\times I_{d-1}=Q_1'$ not more than m_1 zeros in I_d counted according to their multiplicities. In view of (i) that means the set $Z(f)\cap Q_1$ consists of not more than m_1 manifolds of dimension $d-1$, that meet at z_1 and maybe also somewhere else. Choose the cube $Q_1\subset\mathbb{R}^d$ as large as possible. That means for each $\varepsilon>0$ there exists a set $Z_{1,\varepsilon}\subset(1+\varepsilon)Q_1'$

with $\lambda_{d-1}(Z_{1,\varepsilon}) > 0$, such that in $Z_{1,\varepsilon} \times (1+\varepsilon)I_d$ the function $f(z',\cdot)$ has more than m_1 zeros for some $z' \in Z_{1,\varepsilon}$. We find, that the set $(Z(f) \cap 2Q_{k,v}) \cap Q_1$ can be covered by not more than $c(d)m_1 2^{(d-1)(t_1-r_1)}$ cubes of dimension d with side-length 2^{-t_1} for some $t_1 \in \mathbb{N}$ with t_1 much larger than r_1, such that $c(d)m_1$ is much smaller than $2^{t_1-r_1}$. To see this, we can argue as follows. At first assume z_1 has multiplicity one. Then from the Implicit Function Theorem it follows, that the set $Z(f) \cap Q_1$ is the graph of a $C^\infty-$ function of $d-1$ variables, see also Remark 14.1.4. in [99]. Therefore the set $Z(f) \cap Q_1$ can be interpreted as a smooth manifold of dimension $d-1$, see for example chapter 16.1 in [130]. Consequently we also find, that $Z(f) \cap Q_1$ is a so-called $(d-1)-$ set, see Definition 3.1 in [130] and Remark 16.3 in [130]. Hence $Z(f) \cap Q_1$ has Minkowski dimension (box counting dimension) of $d-1$, see Remark 3.5 in [130]. For details concerning the Minkowski dimension we refer to [77], pages 76-81, and to [34]. From the definition of the Minkowski dimension it follows, that $Z(f) \cap Q_1$ can be covered by $c(d)2^{(d-1)(t_1-r_1)}$ cubes of side-length 2^{-t_1} when t_1 is large enough. For multiplicity $m_1 > 1$ we have to decompose the set $Z(f) \cap Q_1$ in not more than m_1 smooth manifolds, that we cover separately. See also Theorem 14.1.3. in [99].

Now take $z_2 \in Z(f) \cap \overline{2Q_{k,v}}$, such that $z_2 \notin Q_1$ and with multiplicity $m_2 \in \mathbb{N}$ as big as possible. Of course we have $m_2 \leq m_1$. We proceed exactly as before and obtain a cube Q_2 and numbers m_2, r_2, t_2 as well as a covering of $(Z(f) \cap 2Q_{k,v}) \cap Q_2$. We choose the cube $Q_2 \subset \mathbb{R}^d$ as large as possible (in the sense described before) but in such a way, that $(Q_1 \cap Z(f)) \cap Q_2 = \emptyset$. This process will be continued till the whole set $Z(f) \cap 2Q_{k,v}$ is covered. If there are two or more zeros with the same multiplicity m_i, we always continue with that one that allows us to find the biggest cube Q_i. At the end we obtain a sequence of cubes $\{Q_i\}_i$ and sequences $\{m_i\}_i, \{r_i\}_i, \{t_i\}_i$. Below in Figure 7 we tried to illustrate one step of the algorithm we just described for $d = 2$. Notice, that the iteration ends after a finite number of $w \in \mathbb{N}$ of steps.

To see this, we can argue as follows. Assume $w = \infty$. Then since $2Q_{k,v}$ is bounded, we find a subsequence $\{Q_{i_l}\}_l$ of cubes, such that $\lim_{l\to\infty} |Q_{i_l}| = 0$. Because of the definition of the algorithm and the fact that $Z(f)$ consists of countably many compact sets, see (i), that implies the following. There exist two sequences of sets $\{A_l\}_l$ and $\{B_l\}_l$ with

- $A_l = (Z(f) \cap \overline{Q_{i_l}})$ and $B_l = (Z(f) \cap (\overline{2Q_{i_l} \setminus Q_{i_l}}))$ for all $l \in \mathbb{N}$;
- $\lambda_{d-1}(A_l) > 0$ and $B_l \neq \emptyset$ for all $l \in \mathbb{N}$;
- for each A_l there is a generating zero z_{i_l} with multiplicity m_{i_l};
- we know $\infty > m_{i_1} \geq \ldots \geq m_{i_l} \geq m_{i_{l+1}} \geq \ldots$ for all $l \in \mathbb{N}$;
- the set $A_l \cup B_l$ is not connected for all large l due to the definition of the algorithm;
- $\lim_{l\to\infty} \operatorname{dist}(A_l, B_l) = 0$.

But in the limiting case $l \to \infty$ the last 3 points and (ii) generate a contradiction. An increase of multiplicity in a late step of the algorithm (forced by the last point and (ii)) is forbidden and an infinite multiplicity does not exist. So our assumption must be wrong and we find $w < \infty$. We tried to illustrate this argument in Figure 8 below for $d = 2$.

Now we have to unify the size of the very small cubes we use for the covering. Therefore because of $\max_{1\leq i\leq w} m_i = m_1$, we put $n(f,k,v) = m_1 w$. Moreover, we choose $t^* \geq \max_{1\leq i\leq w} t_i$, such that 2^{t^*-k} is much bigger than $c(d)m_1 w$. So due to $\min_{1\leq i\leq w} r_i \geq k$ we can cover the set $Z(f) \cap 2Q_{k,v}$ with $c(d)m_1 w 2^{(d-1)(t^*-k)}$ cubes with side-length 2^{-t^*}.

To complete the proof, we show, that the number $n(f) = n(f,k,v)$ also can be chosen independent from k and v. To see this, at first recall, that we have $0 < k \leq R$. Because of this it is enough to work with $k = 1$ to identify a possible number $n(f,k,v)$. To prove the independence from $v \in \mathbb{Z}^d$, we choose $v^* \in \mathbb{Z}^d$, such that $n(f,v^*)$ is maximal. We already explained $n(f,v^*) < \infty$. But of course the number $n(f,v^*)$ works for each $v \in \mathbb{Z}^d$. So $n(f,v) = n(f)$ only depends on f. ■

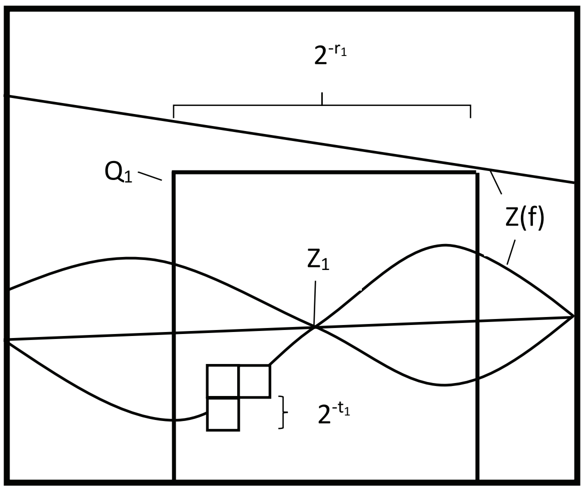

Figure 7. ***One step of the algorithm.*** *In this picture we try to illustrate a typical situation for $d = 2$. The notation is the same as in the proof of (iii). $Z(f)$ is the zero set of a real analytic function. z_1 is a zero of maximal multiplicity, that allows to find a cube Q_1, that is as large as possible apposite to the algorithm. The small cubes with side-length 2^{-t_1} deliver a covering for $Z(f) \cap Q_1$.*

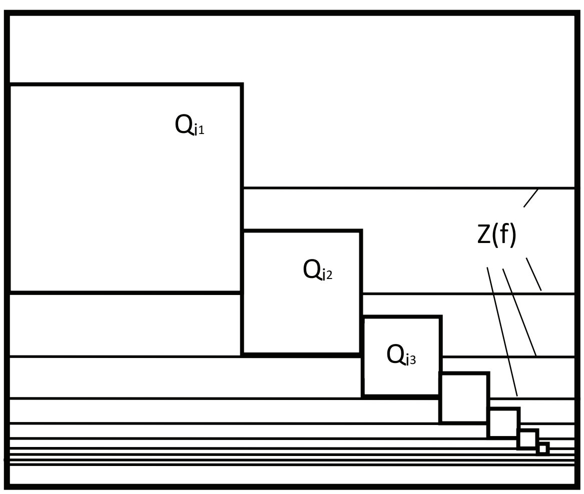

Figure 8. ***The reason for*** $w < \infty$***.*** *In this picture we try to illustrate, why the algorithm must break after a finite number of steps in the case $d = 2$. Assume the number of steps is $w = \infty$. That means, the algorithm produces a subsequence of cubes $\{Q_{i_l}\}_l$, that become smaller and smaller. In the limiting case this implies the existence of $(d-1)-$ dimensional manifolds $A_\infty \subset Z(f)$ and $B_\infty \subset Z(f)$, that are not connected, but have a distance of 0. That either contradicts (ii) or results in $f = 0$.*

Chapter 14

Symbols and Figures

14.1 Tables of Symbols

Here you can find a collection of the symbols and abbreviations, that are used. For most of them short explanations or references are provided. Symbols that show up only once, are not listed.

Numbers and Sets

Symbol	**Explanation**
$\overline{A}$ for $A \subset \mathbb{R}^d$	closure of A
$A^{\complement}$ for $A \subset \mathbb{R}^d$	$\mathbb{R}^d \setminus A$
$B(x,t)$	open ball, see formula (3.1) on page 23
$\mathbb{C}$	complex numbers
d	dimension
∂A for $A \subset \mathbb{R}^d$	boundary of A
L_h	number of vanishing moments for a function h, see formula (10.1)
$\mathbb{N}$, $\mathbb{N}_0$	natural numbers, natural numbers including 0
$Q_{j,k}$	dyadic cube, namely $Q_{j,k} = 2^{-j}([0,1)^d + k)$
$cQ_{j,k}$	cube concentric with $Q_{j,k}$ and side-length $c2^{-j}$
$\mathscr{Q}$	collection of all dyadic cubes
$\mathbb{R}$	real numbers
$\mathbb{R}^d$	d-dimensional Euclidean space
σ_p, $\sigma_{p,q}$	see formula (3.2) on page 23
$[s]$ for $s \in \mathbb{R}$	integer part of s
Ω	often a Lipschitz domain, see Definition 16 on page 29
$\mathbb{Z}$	integer numbers
$Z(f)$	$Z(f) = \{x \in \mathbb{R}^d : f(x) = 0\}$ for a function f

Volume and Length

Symbol	Explanation
$\lvert a\rvert$ for $a \in \mathbb{R}^d$	Euclidean norm of a
$\lvert A\rvert$ for $A \subset \mathbb{R}^d$	d-dimensional Lebesgue measure of A
$\operatorname{dist}(A,B)$ for $A,B \subset \mathbb{R}^d$	Euclidean distance of A and B
$l(P)$	side-length of a cube $P \in \mathscr{Q}$
$\lambda_n(A)$ for $A \subset \mathbb{R}^d$	n-dimensional Lebesgue measure of A
j_P	$j_P := -\log_2(l(P))$

Sequences and Functions

Symbol	Explanation
$\{a_j\}_{j=1}^{\infty}$, $\{a_j\}_{j\in\mathbb{N}}$	sequences
$a_{j,k}$	(K,L)-atoms, see Definition 21 on page 36
$\mathbb{C}^{\mathbb{N}}$	set of all complex sequences
f_α	function with a local singularity, see (9.1) on page 121
$f_{\alpha,\delta}$	function with singularity and logarithm, see (9.9) on page 128
f_Ω	extension of f from Ω to $\mathbb{R}^d$ by zero
$f_{\vert\Omega}$	restriction of f to Ω
$f * g$	convolution of f and g, see formula (3.3) on page 25
$\langle f,g\rangle$	see formula (3.4) on page 25
h_α	function with certain decay at infinity, see (9.10) on page 129
$h_{i,j,k}$	see formula (6.13) on page 76
$\mathbb{N}_0^d$	set of all multi-indices of length d
ψ_H	Haar wavelet, see formula (6.12) on page 76
χ_A for $A \subset \mathbb{R}^d$	indicator function of A
$\chi_{j,k}$	characteristic function of the cube $Q_{j,k}$

Sequence Spaces

Symbol	Explanation
$\mathbf{a}^s_{u,p,q}(\mathbb{R}^d)$	either $\mathbf{n}^s_{u,p,q}(\mathbb{R}^d)$ or $\mathbf{e}^s_{u,p,q}(\mathbb{R}^d)$
$\mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^d)$	see Definition 30 on page 49
$b^{0,\tau}_{p,q}(\mathbb{R}^d)$	see formula (6.14) on page 76
$\mathbf{e}^s_{u,p,q}(\mathbb{R}^d)$	sequence space associated to $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$, see Definition 22 on page 37
l_p, l_∞	Lebesgue sequence spaces, see Definition 1 on page 24
$l^s_q(\mathscr{M}^u_p(\mathbb{R}^d))$	see Definition 30 on page 49
$\mathscr{M}^u_p(l^s_q)(\mathbb{R}^d)$	see Definition 30 on page 49
$\mathscr{M}^u_p(l^s_q[\mathbf{P}])(\mathbb{R}^d)$	see formula 10.6 on page 135
$\mathbf{n}^s_{u,p,q}(\mathbb{R}^d)$	sequence space associated to $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$, see Definition 22 on page 37
$S(\varepsilon_+,\varepsilon_-,r,t)$	classes of sequence spaces, see Definition 28 on page 48

Interpolation Methods

Symbol	Explanation
$[X_0, X_1]_\Theta$	Calderón's first complex interpolation method
$[X_0, X_1]^\Theta$	Calderón's second complex interpolation method
$\langle X_0, X_1, \Theta \rangle$	$\pm$ method of Gustavsson and Peetre
$(X_0, X_1)_{\Theta,q}$	real interpolation method

Operators and Operator Theory

Symbol	Explanation
∂_i^k	$k-$th derivative in direction i
D^α	derivative apposite to a multi-index α, see formula (3.5) on page 25
Δ_h^1, Δ_h^N	differences of order 1 and N, see Definition 26 on page 45
$\mathscr{F}$, $\mathscr{F}^{-1}$	Fourier transform, inverse Fourier transform
$\mathscr{L}(X)$	set of all linear and bounded operators from X to X
$\mathscr{L}(X \to Y)$	set of all linear and bounded operators from X to Y
$\Vert \cdot \vert \mathscr{L}(X) \Vert$	norm of linear operator from X to X
$\Vert \cdot \vert \mathscr{L}(X \to Y) \Vert$	norm of linear operator from X to Y
$\mathbf{M}$	Hardy-Littlewood maximal operator, see Definition 25 on page 42
$\hat{M}_{r,t}$	maximal operator, see formula (5.4) on page 47
$\mathbf{P}$	Peetre maximal operator, see formula (10.7) on page 135
S_+, S_-	left shift and right shift, see formula (5.3) on page 47
S^N	see formula (8.6) on page 110
T	operator associated to $\vert \cdot \vert$, see formula (13.2) on page 157
T^+	truncation operator, see formula (13.1) on page 157
T^*	either T or T^+
T_k	dilation operator, see formula (6.3) on page 73

Function Spaces

Symbol	Explanation
$\mathscr{A}^s_{u,p,q}(\mathbb{R}^d)$	either $\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$ or $\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$
$A^{s,\tau}_{p,q}(\mathbb{R}^d)$	either $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ or $F^{s,\tau}_{p,q}(\mathbb{R}^d)$
$B^s_{p,q}(\mathbb{R}^d)$	Besov spaces, see Definition 12 on page 28
$B^{s,\tau}_{p,q}(\mathbb{R}^d)$	Besov-type spaces, see Definition 23 on page 38
$\mathbf{B}^{s,\tau,N,a}_{p,q,v}(\mathbb{R}^d)$	see Definition 31 on page 71
$C(\mathbb{R}^d)$	space of continuous functions, see Definition 4 on page 25
$C^m(\mathbb{R}^d)$, $C^\infty(\mathbb{R}^d)$	spaces of smooth functions, see Definition 5 on page 25
$C^\infty_0(\mathbb{R}^d)$	contains all functions from $C^\infty(\mathbb{R}^d)$ with compact support
$\mathscr{D}(\mathbb{R}^d)$	$\mathscr{D}(\mathbb{R}^d) = C^\infty_0(\mathbb{R}^d)$
$\mathscr{D}'(\mathbb{R}^d)$	dual space of $\mathscr{D}(\mathbb{R}^d)$
$\mathscr{E}^s_{u,p,q}(\mathbb{R}^d)$	Triebel-Lizorkin-Morrey spaces, see Definition 19 on page 32
$\mathbb{E}^s_{u,p,q}(\mathbb{R}^d)$	real Triebel-Lizorkin-Morrey space, see Definition 36 on page 158
$\mathscr{E}^s_{u,p,q}(\mathbb{R}^d;B)$	see Definition 34 on page 116
$E^s_{u,p,q}(\mathbb{R}^d)$	see Definition 33 on page 109
$\mathbf{E}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$	see Definition 31 on page 71
$F^s_{p,q}(\mathbb{R}^d)$	Triebel-Lizorkin spaces, see Definition 13 on page 28
$F^{s,\tau}_{p,q}(\mathbb{R}^d)$	Triebel-Lizorkin-type spaces, see Definition 24 on page 38
$H^s_p(\mathbb{R}^d)$	Bessel-potential spaces, see Definition 10 on page 27
$L_p(\mathbb{R}^d)$, $L_\infty(\mathbb{R}^d)$	Lebesgue spaces, see Definition 2 on page 25
$L^{loc}_p(\mathbb{R}^d)$	local Lebesgue spaces, see Definition 3 on page 25
$L^*_\infty(\mathbb{R}^d)$	contains all functions from $L_\infty(\mathbb{R}^d)$ with support in $[0,1]^d$
$L^\tau_p(\mathbb{R}^d)$	see formula (5.9) on page 69
$\mathscr{L}^r A^s_{p,q}(\mathbb{R}^d)$	Local Function Spaces, see Remark 4 on page 40
$L^r A^s_{p,q}(\mathbb{R}^d)$	Hybrid Function Spaces, see Remark 5 on page 40
$\mathscr{M}^u_p(\mathbb{R}^d)$	Morrey space, see Definition 17 on page 31
$\mathbb{M}^u_p(\mathbb{R}^d)$	real Morrey space, see Definition 36 on page 158
$\mathscr{N}^s_{u,p,q}(\mathbb{R}^d)$	Besov-Morrey spaces, see Definition 18 on page 32
$\mathbb{N}^s_{u,p,q}(\mathbb{R}^d)$	real Besov-Morrey space, see Definition 36 on page 158
$\mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$	see Definition 31 on page 71
$\mathscr{S}(\mathbb{R}^d)$	Schwartz space, see Definition 6 on page 26
$\mathbb{S}(\mathbb{R}^d)$	real part of the Schwartz space
$\mathscr{S}'(\mathbb{R}^d)$	space of tempered distributions, see Definition 8 on page 26
$W^m_p(\mathbb{R}^d)$	Sobolev spaces, see Definition 7 on page 26
$W^m\mathscr{M}^u_p(\mathbb{R}^d)$	Sobolev-Morrey spaces, see Definition 20 on page 33
$X(\Omega)$	function space X on domain Ω, see Definition 15 on page 29
$\overset{\diamond}{X}$	diamond space associated to X, see Definition 32 on page 107
$\mathring{X}$	see Definition 32 on page 107
$Y(\cdot)$	function spaces from Hedberg and Netrusov, see Definition 29 on page 48

Quasi-Norms for Smoothness Morrey Spaces

Symbol	Explanation
$\|\cdot\|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|$	Fourier analytic characterization, see Definition 23 on page 38
$\|\cdot\|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\sharp)}$	simplified Fourier analytic characterization, see Lemma 14 on page 39
$\|\cdot\|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$	see Theorem 4 on page 54
$\|\cdot\|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$	see Corollary 1 on page 55
$\|\cdot\|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(vma)}$	characterization via generalized ball means, see Theorem 9 on page 68
$\|\cdot\|B^{s,\tau}_{p,q}(\mathbb{R}^d)\|^{(\omega)}$	characterization via moduli of smoothness, see Proposition 4 on page 69
$\|\cdot\|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|$	Fourier analytic characterization, see Definition 19 on page 32
$\|\cdot\|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$	see Theorem 4 on page 54
$\|\cdot\|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$	see Corollary 1 on page 55
$\|\cdot\|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|^{F}$	Triebel-Lizorkin-Morrey-Fubini spaces, see Definition 35 on page 154
$\|\cdot\|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|^{MC}$	Morrey characterizations, see Proposition 27 on page 169
$\|\cdot\|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(st)}$	Stein characterization, see Theorem 6 on page 61
$\|\cdot\|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$	characterization via generalized ball means, see Theorem 5 on page 58
$\|\cdot\|\mathscr{E}^{s}_{u,p,q}(\mathbb{R}^d)\|_{\lambda_0}$	characterization via vanishing moments, see Lemma 48 on page 133
$\|\cdot\|\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)\|$	Fourier analytic characterization, see Definition 18 on page 32
$\|\cdot\|\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$	see Theorem 4 on page 54
$\|\cdot\|\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$	see Corollary 1 on page 55
$\|\cdot\|\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(vma)}$	characterization via generalized ball means, see Theorem 7 on page 62
$\|\cdot\|\mathscr{N}^{s}_{u,p,q}(\mathbb{R}^d)\|^{(\omega)}$	characterization via moduli of smoothness, see Theorem 8 on page 65

14.2 Table of Figures

Here all figures, that show up in this treatise, are listed. Concerning the Figures 1 - 5 special thanks goes to professor Dorothee Haroske.

Chapter 15

Acknowledgement

At first and most of all I would like to thank my supervisor professor Winfried Sickel. He draw my attention to the very interesting topic of function spaces. During my whole time as a doctoral candidate he permanently supported me with many attractive issues, countless useful hints and a large amount of valuable literature. In the course of many discussions with and without blackboard I learned so much from him concerning function spaces and many other interesting topics. Moreover, I would like to thank him for his contribution to the papers [58], [59] and [150].
Next I want to give thanks to Ciqiang Zhuo. He was my coauthor for the paper [150]. Chapter 10 in this treatise, where we dealt with extension operators for Triebel-Lizorkin-Morrey spaces, was mainly written by him.
Furthermore, I would like to thank the State of Thuringia, the Friedrich-Schiller-University Jena and especially the Graduiertenakademie. They funded my doctorate time by a scholarship called Landesgraduiertenstipendium.
Many thanks goes to professor Dorothee Haroske. She designed the diagrams Figure 1 - 5. Moreover, I would like to thank her for the great engagement concerning research, teaching, organizing the function spaces seminar and many other things.
Furthermore, I appreciate my parents Ulrike and Frank Hovemann for their support and encouragement.
At last I would like to thank all my friends and colleagues from the faculty of mathematics at the Friedrich-Schiller-University Jena. Every workday with you is very pleasant.

Bibliography

[1] J. Appell and P. Zabrejko, Nonlinear superposition operators, Cambr. Univ. Press, Cambridge, 1990.

[2] J. Bergh, Relation between the two complex methods of interpolation, Indiana Univ. Math. J. **28**(5) (1979), 775-778.

[3] J. Bergh and J. Löfström, Interpolation spaces. An Introduction, Springer, Berlin, 1976.

[4] O.V. Besov, On a family of function spaces. Embedding theorems and extensions, Dokl. Acad. Nauk SSSR **126** (1959), 1163-1165.

[5] O.V. Besov, On a family of function spaces in connection with embeddings and extensions, Trudy Mat. Inst. Steklov **60** (1961), 42 - 81.

[6] O. V. Besov, On spaces of functions of smoothness zero, Mat. Sbornik **203**(8) (2012), 1077-1090.

[7] O.V. Besov, V.P. Il'in and S.M. Nikol'skii, Integral representations of functions and embedding theorems, Vol. I+II, V. H. Winston & Sons, Washington D.C., 1978/1979, Translated from Russian, Scripta Series in Mathematics, edited by M. H. Taibleson.

[8] O.V. Besov, V.P. Il'in and S.M. Nikol'skii, Integralnye predstavleniya funktsii i teoremy vlozheniya, Russian, Second ed., Fizmatlit "Nauka", Moscow, 1996.

[9] O. Blasco, A. Ruiz and L. Vega, Non-interpolation in Morrey-Campanato and block spaces, Ann. Scuola Norm Sup. Pisa Cl. Sci. **28**(4) (1999), 31-40.

[10] B. Bojarski and T. Iwaniec, Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$. Ann. Acad. Sci. Fenn. Ser. AI Math. **8** (1983), 257-324.

[11] G. Bourdaud, The functional calculus in Sobolev spaces, In: Function spaces, differential operators and nonlinear analysis. Teubner-Texte Math. 133, Teubner, Stuttgart, 1993, 127-142.

[12] G. Bourdaud and M. Lanza de Cristoforis, Regularity of the symbolic calculus in Besov algebras, Studia Math. **184**(3) (2008), 271-298.

[13] G. Bourdaud and Y. Meyer, Fonctions qui opèrent sur les espaces de Sobolev, J. Funct. Anal. **97**(2) (1991), 351-360.

[14] G. Bourdaud and M. Moussai, Continuity of composition operators in Sobolev spaces, Ann. I. H. Poincare - AN (2019).

[15] G. Bourdaud, M. Moussai and W. Sickel, A necessary condition for composition in Besov spaces, Complex Variables and Elliptic Equations **65**(1) (2020), 22-39.

[16] G. Bourdaud and W. Sickel, Composition operators on function spaces with fractional order of smoothness, Harmonic Analysis and Nonlinear Partial Differential Equations, RIMS Kokyuroku Bessatsu, B26, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011, 93-132.

[17] H.-Q. Bui, M. Paluszyński and M.H. Taibleson, A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces, Studia Math. **119**(3) (1996), 219-246.

[18] H.-Q. Bui, M. Paluszyński and M.H. Taibleson, Characterization of the Besov-Lipschitz and Triebel-Lizorkin spaces. The case $q < 1$, J. Fourier Anal. Appl. **3** (1997), 837-846.

[19] V.I. Burenkov, D.K. Darbayeva and E.D. Nursultanov, Description of interpolation spaces for general local Morrey-type spaces, Eurasian Math. J. **4**(1) (2013), 4653.

[20] V.I. Burenkov, E.D. Nursultanov, Description of interpolation spaces for local Morrey-type spaces, Trudy Math. Inst. Steklov **269** (2010), 46-56.

[21] V.I. Burenkov, E.D. Nursultanov and D.K. Chigambayeva, Description of the interpolation spaces for a pair of local Morrey-type spaces and their generalizations, Trudy Mat. Inst. Steklova **284** (2014), 105-137.

[22] V.I. Burenkov, A. Ghorbanalizadeh and Y. Sawano, On the equivalence of the K-functional and the modulus of continuity on the Morrey spaces, J. Approx. Theory **248** (2019), 19 pp.

[23] V.I. Burenkov and T.V. Tararykova, Youngs inequality for convolutions in Morrey-type spaces, Eurasian Math. J. **7**(2) (2016), 92-99.

[24] A.P. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. **24** (1964), 113-190.

[25] M. Christ and A. Seeger, Necessary conditions for vector-valued operator inequalities in harmonic analysis, Proc. London Math. Soc. **93**(3) (2006), 447-473.

[26] M. Christ and A. Seeger, Necessary conditions for vector-valued operator inequalities in harmonic analysis, online version from https://www.math.wisc.edu/ seeger/preprints.html.

[27] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Math. **7** (1987), 273-279.

[28] R.A. DeVore and G. Lorentz, Constructive approximation, Springer, Berlin, 1993.

[29] D. Drihem, Characterizations of Besov-Type and Triebel-Lizorkin-Type Spaces by Differences, Journal Funct. Spaces Appl., 2012, Art. ID 328908, 24 pp.

[30] D.E. Edmunds and H. Triebel, Function Spaces, Entropy Numbers and Differential Operators, Cambridge University Press, Cambridge, 1996.

[31] A. El Baraka, An embedding theorem for Campanato spaces, Electron. J. Differential Equations **66** (2002), 1-17.

[32] A. El Baraka, Function spaces of BMO and Campanato type, Proc. of the 2002 Fez Conference on Partial Differential Equations, 109-115 (electronic), Electron. J. Differ. Equ. Conf. 9, Southwest Texas State Univ., San Marcos, TX, 2002.

[33] A. El Baraka, Littlewood-Paley characterization for Campanato spaces, J. Funct. Spaces Appl. **4** (2006), 193-220.

[34] K. Falconer, Fractal Geometry - Mathematical foundations and applications, 2nd Edition, Wiley, 2003.

[35] M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, J. Funct. Anal. **93** (1990), 34-170.

[36] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Springer, Berlin, 1977.

[37] H.F. Goncalves, D.D. Haroske and L. Skrzypczak, Compact embeddings in Besov-type and Triebel-Lizorkin-type Spaces on bounded domains, Rev. Mat. Complut. (2020).

[38] J. Gustavsson, On interpolation of weighted L^p-spaces and Ovchinnikov's theorem, Studia Math. **72** (1982), 237-251.

[39] J. Gustavsson and J. Peetre, Interpolation of Orlicz spaces, Studia Math. **60** (1977), 33-59.

[40] P.M. Hajlasz, Change of variables formula under minimal assumptions, Colloquium Math. LXIV (1993), 93-100.

[41] D.I. Hakim, Complex interpolation of certain closed subspaces of generalized Morrey spaces, Tokyo J. Math. **41**(2) (2018), 487-514.

[42] D.I. Hakim and Y. Sawano, Interpolation of generalized Morrey spaces, Rev. Mat. Complut. **29**(2) (2016), 295-340.

[43] D.I. Hakim and Y. Sawano, Calderón's First and Second Complex Interpolations of Closed Subspaces of Morrey Spaces, J. Fourier Anal. Appl. **23** (2017), 1195-1226.

[44] D.I. Hakim and Y. Sawano, Complex interpolation of vanishing Morrey spaces, Ann. Funct. Anal. **11** (2020), 643-661.

[45] D.I. Hakim and Y. Sawano, Complex interpolation of various subspaces of Morrey spaces, Sci. China Math. **63** (2020), 937-964.

[46] D.I. Hakim, S. Nakamura and Y. Sawano, Complex interpolation of smoothness Morrey subspaces, Constr. Approx. **46** (2017), 489-563.

[47] D.I. Hakim, T. Nogayama and Y. Sawano, Complex interpolation of smoothness Triebel-Lizorkin-Morrey spaces, Math. J. Okayama Univ. **61** (2019), 99-128.

[48] D.D. Haroske, S.D. Moura and L. Skrzypczak, Smoothness Morrey Spaces of regular distributions, and some unboundedness property, Nonlinear Analysis **139** (2016), 218-244.

[49] D.D. Haroske, S.D. Moura and L. Skrzypczak, Some embeddings of Morrey spaces with critical smoothness, J. Fourier Anal. Appl. **26**(50) (2020).

[50] D.D. Haroske and L. Skrzypczak, Continuous embeddings of Besov-Morrey function spaces, Acta Math. Sinica, Vol. **28** (2012), 1307-1328.

[51] D.D. Haroske and L. Skrzypczak, Embeddings of Besov-Morrey spaces on bounded domains, Studia Math. **218**(2) (2013).

[52] D.D. Haroske and L. Skrzypczak, On Sobolev and Franke-Jawerth embeddings of smoothness Morrey spaces, Rev. Mat. Complut. **27**(2) (2014), 541-573.

[53] L.I. Hedberg and Y.V. Netrusov, An axiomatic approach to function spaces, spectral synthesis, and Luzin approximation, Mem. Amer. Math. Soc. **882** (2007), 97 pp.

[54] F. Hirzebruch and W. Scharlau, Einführung in die Funktionalanalysis, Bibliographisches Institut, Mannheim, 1971.

[55] M. Hovemann, Besov-Morrey spaces and differences, Math. Reports, in press.

[56] M. Hovemann, Triebel-Lizorkin-Morrey spaces and differences, Math. Nachr., in press.

[57] M. Hovemann, Truncation in Besov-Morrey and Triebel-Lizorkin-Morrey spaces, Nonlinear Analysis **204** (2021), 112239.

[58] M. Hovemann and W. Sickel, Besov-type spaces and differences, Eurasian Math. J. **11**(1) (2020), 25-56.

[59] M. Hovemann and W. Sickel, Stein characterizations of Lizorkin-Triebel spaces, preprint, Jena, 2020.

[60] T. Izumi, Y. Sawano and H. Tanaka, Littlewood-Paley theory for Morrey spaces and their preduals, Rev. Mat. Complut. **28**(2) (2015), 411-447.

[61] N. Kalton, Plurisubharmonic functions on quasi-Banach spaces, Studia Math. **84** (1986), 297-324.

[62] N. Kalton, S. Mayboroda and M. Mitrea, Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations. Interpolation Theory and Applications, Contemp. Math. **445** (2007), 121-177.

[63] N. Kalton and M. Mitrea, Stability results on interpolation scales of quasi-Banach spaces and applications, Trans. Amer. Math. Soc. **350** (1998), 3903-3922.

[64] H. Kozono, M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, Comm. Partial Differential Equations **19**(5-6) (1994), 959-1014.

[65] S.G. Kreĭn, Y.I. Petunin and E.M. Semenov, Interpolation of Linear Operators. Moscow: Nauka, 1978, engl. translation AMS, Providence, R.I., 1982.

[66] P.G. Lemarié-Rieusset, Multipliers and Morrey spaces, Potential Anal. **38** (2013), 741-752.

[67] P.G. Lemarié-Rieusset, Erratum to ”Multipliers and Morrey spaces”, Potential Anal. **41** (2014), 1359-1362.

[68] P.G. Lemarié-Rieusset, The Navier-Stokes Problem in the 21st Century, CRC Press, Taylor and Francis Group, Boca Raton, 2016.

[69] Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets, J. Fourier Anal. Appl. **18** (2012), 1067-1111.

[70] Y. Liang, D. Yang, W. Yuan, Y. Sawano and T. Ullrich, A new framework for generalized Besov-type and Triebel-Lizorkin-type spaces, Dissertationes Math. **489** (2013), 1-114.

[71] P.I. Lizorkin, Operators connected with fractional derivatives and classes of differentiable functions, Trudy Mat. Inst. Steklov **117** (1972), 212-243.

[72] P.I. Lizorkin, Properties of functions of the spaces $\Lambda^r_{p,\theta}$, Trudy Mat. Inst. Steklov **131** (1974), 158-181.

[73] Y. Lu, D. Yang and W. Yuan, Interpolation of Morrey spaces on metric measure spaces, Canad. Math. Bull. **57** (2014), 598-608.

[74] A. Lunardi, Interpolation Theory, Lect. Notes. Pisa: Scuola Normale Superiore Pisa, 2009.

[75] D. Maharani, J. Widjaja and M. Wono Setya Budhi, Boundedness of Mikhlin Operator in Morrey Space, J. Phys.: Conf. Ser. **1180** (2019).

[76] M. Marcus and V.J. Mizel, Every superposition operator mapping one Sobolev space into another is continuous, J. Funct. Anal. **33** (1978), 217-229.

[77] P. Mattila, Geometry of sets and measures in euclidean spaces, Cambridge Univ. Press, 1995.

[78] A. Mazzucato, Decomposition of Besov-Morrey spaces. In: Harmonic Analysis at Mount Holyoke 2001. Contemp. Math. **320** (2003), 279-294.

[79] A. Mazzucato, Besov-Morrey spaces: function space theory and applications to non-linear PDE, Trans. Amer. Math. Soc. **355** (2003), 1297-1369.

[80] P. Mironescu, The role of the Hardy type inequalities in the theory of function spaces, Rev. Roumaine Math. Pures Appl. **63**(4) (2018), 447-525.

[81] B.S. Mityagin, The zero set of a real analytic function, arXiv: 1512.07276v1 [math.CA] 22 Dec 2015.

[82] T. Mizuhara, Boundedness of some classical operators on generalized Morrey spaces, Harmonic Anal., Proc. Conf., Sendai/Jap. 1990, ICM-90 Satell. Conf. Proc., 1991, 183-189.

[83] C.B. Morrey, On the solutions of quasi linear elliptic partial differential equations, Trans. Amer. Math. Soc. **43** (1938), 126-166.

[84] R. Musina, A.I. Nazarov, A note on truncations in fractional Sobolev spaces, Bulletin of Mathematical Sciences **9**(1) (2019), 7 pp.

[85] E. Nakai, Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potential on generalized Morrey spaces, Math. Nachr. **166** (1994), 95-103.

[86] S. Nakamura, T. Noi and Y. Sawano, Generalized Morrey spaces and trace operator, Science China Math. **59** (2016), 281-336.

[87] Y.V. Netrusov, Some imbedding theorems for spaces of Besov-Morrey type (Russian). In: Numerical methods and questions in the organisation of calculations, 7, Zapiski Nauchnykh Sem. Leningrad. Otdel. Mat. Inst. Steklov **139** (1984), 139-147.

[88] S.M. Nikol'skii, Approximation of functions of several variables and imbedding theorems, Springer Verlag, Berlin, 1975.

[89] S.M. Nikol'skii, Inequalities for entire analytic functions of finite order and their application to the theory of differentiable functions of several variables, Trudy Mat. Inst. Steklov **38** (1951), 244-278.

[90] S.D. Moura, J.S. Neves and C. Schneider, Spaces of generalized smoothness in the critical case: Optimal embeddings, continuity envelopes and approximation numbers, J. Approx. Theory **187** (2014), 82-117.

[91] P. Oswald. Manuscript. Personally communicated in 2019.

[92] P. Oswald, On the boundedness of the mapping $f \to |f|$ in Besov spaces, Comment. Math. Univ. Carolin. **33**(1) (1992), 57-66.

[93] J. Peetre, Interpolation of Lipschitz operators and metric spaces, Mathematica (Cluj) **12** (1970), 1-20.

[94] J. Peetre, New thoughts on Besov Spaces, Duke university, Mathematics series I, Durham, 1976.

[95] J. Peetre, On spaces of Triebel-Lizorkin type, Ark. Mat. **13** (1975), 123-130.

[96] M. Prats and E. Saksman, A $T(1)$ theorem for fractional Sobolev spaces on domains, J. Geom. Anal. **27** (2017), 2490-2538.

[97] M. Rosenthal, Local means, wavelet bases, representations, and isomorphisms in Besov-Morrey and Triebel-Lizorkin-Morrey spaces, Math. Nachr. **286** (2013), 59-87.

[98] W. Rudin, Functional Analysis, McGraw-Hill Book Company, New York, 1973.

[99] W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[100] A. Ruiz and L. Vega, Corrigenda to "Unique continuation for Schrödinger operators with potential in Morrey spaces" and a remark on interpolation of Morrey spaces, Publ. Mat. **3** (1995), 405-411.

[101] T. Runst, W. Sickel, Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, De Gruyter Series in Nonlinear Analysis and Applications 3, Walter de Gruyter & Co., Berlin, 1996.

[102] V.S. Rychkov, On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains, J. London Math. Soc. **60**(2) (1999), 237-257.

[103] Y. Sawano, Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces on domains, Math. Nachr. **283** (2010), 1456-1487.

[104] Y. Sawano, Wavelet characterization of Besov-Morrey and Triebel-Lizorkin-Morrey spaces, Funct. Approx. Comment Math. **38** (2008), 93-107.

[105] Y. Sawano, Theory of Besov spaces, Springer, Singapore, 2018.

[106] Y. Sawano, G. Di Fazio, D.I. Hakim, Morrey Spaces: Introduction and Applications to Integral Operators and PDE's, Volume I, Chapman and Hall/CRC, Boca Raton, 2020.

[107] Y. Sawano, G. Di Fazio, D.I. Hakim, Morrey Spaces: Introduction and Applications to Integral Operators and PDE's, Volume II, Chapman and Hall/CRC, Boca Raton, 2020.

[108] Y. Sawano and H. Tanaka, Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, Math. Z. **257** (2007), 871-905.

[109] Y. Sawano and H. Tanaka, Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces for non-doubling measures, Math. Nachr. **282** (2009), 1788-1810.

[110] Y. Sawano and H. Tanaka, The Fatou property of block spaces, J. Math. Sci. Univ. Tokyo **22** (2015), 663-683.

[111] C. Schneider, On dilation operators in Besov spaces, Rev. Mat. Complut. **22**(1) (2009), 111-128.

[112] C. Schneider and J. Vybiral, On dilation operators in Triebel-Lizorkin spaces, Funct. Approx. Comment. Math. **41**(2) (2009), 139-162.

[113] V.A. Shestakov, Interpolation of linear operators in spaces of measurable functions., Funktsional Anal. i Prilozhen **8** (1974), 91-92.

[114] V.A. Shestakov, On complex interpolation of Banach spaces of measurable functions, Vestnik Leningrad Univ. **19** (1974), 64-68.

[115] W. Sickel, On pointwise multipliers for $F^s_{p,q}(\mathbb{R}^n)$ in case $\sigma_{p,q} < s < n/p$, Ann. Mat. Pura Appl. **176** (1999), 209-250.

[116] W. Sickel, Smoothness spaces related to Morrey spaces - a survey. I, Eurasian Math. J. **3** (2012), 110-149.

[117] W. Sickel, Smoothness spaces related to Morrey spaces - a survey. II, Eurasian Math. J. **4** (2013), 82-124.

[118] W. Sickel, L. Skrzypczak and J. Vybíral, Complex interpolation of weighted Besov- and Lizorkin-Triebel spaces, Acta Math. Sinica **30** (2014), 1297-1323.

[119] W. Sickel, L. Skrzypczak and J. Vybíral, Complex interpolation of weighted Besov- and Lizorkin-Triebel spaces (extended version). arxiv: 1212.1614.

[120] W. Sickel and H. Triebel, Hölder inequalities and sharp embeddings in function spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type, Z. Anal. Anwendungen **14** (1995), 105-140.

[121] E.M. Stein, The Characterization of Functions Arising as Potentials, Bull. Amer. Math. Soc. **67** (1961), 102-104.

[122] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, 1970.

[123] E.M. Stein and R. Shakarchi, Real Analysis: Measure Theory, Integration, and Hilbert Spaces, Princeton Lectures in Analysis, Vol. III, Princeton Univ. Press, Princeton and Oxford, 2005.

[124] R.S. Strichartz, Multipliers on fractional Sobolev spaces, J. Math. Mechanics **16** (1967), 1031-1060.

[125] L. Tang and J. Xu, Some properties of Morrey type Besov-Triebel spaces, Math. Nachr. **278** (2005), 904-917.

[126] H. Triebel, Spaces of distributions of Besov type on Euclidean *n*-space. Duality, interpolation, Ark. Mat. **11** (1973), 13-64.

[127] H. Triebel, Interpolation theory, function spaces, differential operators, Amsterdam, North-Holland, 1978. (Sec. ed. Heidelberg, Barth, 1995).

[128] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, 1983.

[129] H. Triebel, Theory of Function Spaces II, Birkhäuser, Basel, 1992.

[130] H. Triebel, Fractals and Spectra, Birkhäuser, Basel, 1997.

[131] H. Triebel, The Structure of Functions, Birkhäuser, Basel, 2001.

[132] H. Triebel, Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers, Rev. Mat. Complut. **15**(2) (2002), 475-524.

[133] H. Triebel, Theory of Function Spaces III, Birkhäuser, Basel, 2006.

[134] H. Triebel, Function spaces and wavelets on domains, European Math. Soc. Publishing House, Zürich, 2008.

[135] H. Triebel, Characterizations of some function spaces in terms of Haar wavelets, Comment. Math. **53** (2013), 135-153.

[136] H. Triebel, Local Function Spaces, Heat and Navier-Stokes Equations, in: EMS Tracts in Mathematics, vol. 20, European Mathematical Society, Zürich, 2013.

[137] H. Triebel, Hybrid Function Spaces, Heat and Navier-Stokes Equations, in: EMS Tracts in Mathematics, vol. 24, European Mathematical Society, Zürich, 2014.

[138] D. Yang, W. Yuan, A new class of function spaces connecting Triebel-Lizorkin spaces and *Q* spaces, J. Funct. Anal. **255** (2008), 2760-2809.

[139] D. Yang, W. Yuan, Characterizations of Besov-type and Triebel-Lizorkin-type spaces via maximal functions and local means, Nonlinear Analysis **73** (2010), 3805-3820.

[140] D. Yang, W. Yuan, New Besov-type spaces and Triebel-Lizorkin-type spaces including *Q* spaces, Math. Z. **265** (2010), 451-480.

[141] D. Yang, W. Yuan, Relations among Besov-type spaces, Triebel-Lizorkin-type spaces and generalized Carleson measure spaces, Applicable Analysis, **92**(3) (2013), 549-561.

[142] D. Yang, W. Yuan and C. Zhuo, Complex interpolation on Besov-type and Triebel-Lizorkin-type spaces, Anal. Appl. **11** (2013), 45 pp.

[143] W. Yuan, A note on complex interpolation and Calderón product of quasi-Banach spaces, Taiwanese J. Math. **18**(5) (2014), 1527-1548.

[144] W. Yuan, W. Sickel and D. Yang, Morrey and Campanato meet Besov, Lizorkin and Triebel, Lecture Notes in Mathematics Vol. 2005, Springer, Berlin, 2010.

[145] W. Yuan, W. Sickel and D. Yang, On the coincidence of certain approaches to smoothness spaces related to Morrey spaces, Math. Nachr. **286** (2013), 1571-1584.

[146] W. Yuan, W. Sickel and D. Yang, Interpolation of Morrey-Campanato and Related Smoothness Spaces. Science China Math. **58** (2015), 1835-1908.

[147] W. Yuan, W. Sickel and D. Yang, Regularity properties of the Haar system with respect to Besov-type spaces, Studia Math. (online).

[148] J. Zhang, C. Zhou, D. Yang and Z. He, Littlewood-Paley characterizations of Triebel-Lizorkin-Morrey spaces via ball averages, Nonlinear Analysis **150** (2017), 76-103.

[149] C. Zhou, Complex Interpolation of Besov-type Spaces on Domains, manuscript, 2020.

[150] C. Zhuo, M. Hovemann and W. Sickel, Complex Interpolation of Lizorkin-Triebel-Morrey Spaces on Domains, Anal. Geom. Metr. Spaces **8**(1) 2020, 268-304.

[151] C. Zhou, W. Sickel, D. Yang and W. Yuan, Characterizations of Besov-Type and Triebel-Lizorkin-Type Spaces via Averages on Balls, Canad. Math. Bull. Vol. **60**(3) (2017), 655-672.

Chapter 16

Informationen zum Autor entsprechend § 5 der Promotionsordnung

16.1 Publikationen zur Dissertation (papers)

(1) M. Hovemann, Besov-Morrey spaces and differences, Math. Reports, in press. arXiv:2010.10856 [math.FA].

(2) M. Hovemann, Triebel-Lizorkin-Morrey spaces and Differences, Math. Nachr., in press.

(3) M. Hovemann, Truncation in Besov-Morrey and Triebel-Lizorkin-Morrey spaces, Nonlinear Analysis **204** (2021), 112239.

(4) M. Hovemann, W. Sickel, Besov-Type Spaces and Differences, Eurasian Math. J. **11**(1) (2020), 25-56.

(5) M. Hovemann, W. Sickel, Stein characterizations of Lizorkin-Triebel spaces, work in progress, Jena, 2020.

(6) C. Zhuo, M. Hovemann, W. Sickel, Complex Interpolation of Lizorkin-Triebel-Morrey Spaces on Domains, Anal. Geom. Metr. Spaces **8**(1) 2020, 268-304.

16.2 Wissenschaftliche Fachvorträge zu Themen der Dissertation (talks)

Datum	Ort	Titel
14.5.2018	PhD Seminar Jena	Characterizations of Triebel-Lizorkin spaces by differences
24.7.2018	Seminar Siegmundsburg	Strichartz-Charakterisierungen für $F^s_{p,q}(\mathbb{R}^d)$
2.11.2018	Tag der Fakultät für Mathematik und Informatik Jena	Charakterisierungen von Funktionenräumen mittels Differenzen
23.11.2018	The Prague seminar on function spaces (Prag)	Triebel-Lizorkin-Morrey spaces and differences
11.1.2019	Seminar Funktionenräume Jena	Triebel-Lizorkin-Morrey spaces and differences: sufficient conditions
18.1.2019	Seminar Funktionenräume Jena	Triebel-Lizorkin-Morrey spaces and differences: necessary conditions
14.6.2019	FSDONA (Turku, Finnland)	Triebel-Lizorkin-Morrey spaces and differences
26.8.2019	Seminar Siegmundsburg	Besov-Morrey spaces and differences
27.1.2020	PhD Seminar Jena	Besov-Morrey spaces and differences
12.2.2020	Seminar Funktionenräume Jena	Besov-Morrey spaces and truncations - an introduction
10.7.2020	Seminar Funktionenräume Jena	Truncation in Besov-Morrey and Triebel-Lizorkin-Morrey spaces
16.12.2020	Oberseminar zur Numerik und Optimierung Marburg	Triebel-Lizorkin-Morrey spaces and differences: characterizations and applications

16.3 Lehre (teaching)

Semester	**Lehrveranstalltung**
Sommersemester 2017	Tutor Numerische Mathematik (B.Sc. Informatik)
Wintersemester 2017/2018	Übung Analysis 1 (Lehramt Gymnasium)
Wintersemester 2017/2018	Übung Mathematik 1 (B.Sc. Werkstoffwissenschaften)
Sommersemester 2018	Übung Analysis 2 (Lehramt Gymnasium)
Wintersemester 2018/2019	Übung Mathematik 1 (B.Sc. Werkstoffwissenschaften)
Sommersemester 2019	Übung Numerische Mathematik (B.Sc. Informatik)
Wintersemester 2019/2020	Übung Analysis 1 (B.Sc. Mathematik)
Sommersemester 2020	Übung Analysis 2 (B.Sc. Mathematik)

16.4 Ehrenwörtliche Erklärung

Hiermit erkläre ich,

- dass mir die geltende Promotionsordnung der Fakultät bekannt ist,
- dass ich die Dissertation selbst angefertigt habe, keine Textabschnitte oder Ergebnisse eines Dritten oder eigener Prüfungsarbeiten ohne Kennzeichnung übernommen habe und alle von mir benutzten Hilfsmittel, persönlichen Mitteilungen und Quellen in meiner Arbeit angegeben habe,
- dass ich die Hilfe eines Promotionsberaters nicht in Anspruch genommen habe und dass Dritte weder unmittelbar noch mittelbar geldwerte Leistungen von mir für Arbeiten erhalten haben, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen,
- dass ich die Dissertation noch nicht als Prüfungsarbeit für eine staatliche oder andere wissenschaftliche Prüfung eingereicht habe.

Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts wurde ich von Prof. Dr. Winfried Sickel unterstützt. Teile von Kapitel 10 wurden von Ciqiang Zhuo erstellt. Er ist Koautor bei unserem gemeinsamen Artikel [150]. Die Abbildungen Figure 1 - 5 wurden mir freundlicherweise von Prof. Dr. Dorothee Haroske zur Verfügung gestellt.

Ich habe weder die gleiche, noch eine in wesentlichen Teilen ähnliche, noch eine andere Abhandlung bei einer anderen Hochschule als Dissertation eingereicht.

Jena, den .... Marc Hovemann