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Dissertation

Indirect Adaptive Higher-Order Sliding-Mode Control Using the Certainty-Equivalence Principle

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Abstract

Since the late 50s, huge efforts have been made to improve the control algorithms that are capable of compensating for uncertainties and disturbances. Adaptive controllers that adjust their parameters continuously have been used from the beginning to solve this task. This adaptation of the controller allows to maintain a constant performance even under changing conditions.

A different idea is proposed by variable structure systems, in particular by the so-called sliding-mode control. The idea is to employ a very fast switching signal to compensate for disturbances or uncertainties.

This thesis deals with a combination of these two rather different approaches while preserving the advantages of each method. The design of a sliding-mode controller normally does not demand sophisticated knowledge about the disturbance, while the controller's robustness against state-dependent uncertainties might be poor. On the other hand, adaptive controllers are well suited to compensate for parametric uncertainties while unstructured influence may result in a degraded performance.

Hence, the objective of this work is to design sliding-mode controllers that use as much information about the uncertainty as possible and exploit this knowledge in the design. An important point is that the design procedure is based on a rigorous proof of the stability of the combined approach. Only recent results on Lyapunov theory in the field of sliding-mode made this analysis possible. It is shown that the Lyapunov function of the nominal sliding-mode controller has a direct impact on the adaptation law. Therefore, this Lyapunov function has to meet certain conditions in order to allow a proper implementation of the proposed algorithms.

The main contributions of this thesis are sliding-mode controllers, extended by an adaptive part using the certainty-equivalence principle. It is shown that the combination of both approaches results in a novel controller design that is able to solve a control task even in the presence of different classes of uncertainties. In addition to the theoretical analysis, the advantages of the proposed method are demonstrated in a selection of simulation examples and on a laboratory test-bench. The experiments show that the proposed control algorithm delivers better performance in regard to chattering and robustness compared to classical sliding-mode controllers.

Kurzfassung

Seit den 50er Jahren werden große Anstrengungen unternommen, Algorithmen zu entwickeln, welche in der Lage sind Unsicherheiten und Störungen in Regelkreisen zu kompensieren. Früh wurden hierzu adaptive Verfahren, die eine kontinuierliche Anpassung der Reglerparameter vornehmen, genutzt, um die Stabilisierung zu ermöglichen. Die fortlaufende Modifikation der Parameter sorgt dabei dafür, dass strukturelle Änderungen im Systemmodell sich nicht auf die Regelgüte auswirken.

Eine deutlich andere Herangehensweise wird durch strukturvariable Systeme, insbesondere die sogenannte Sliding-Mode Regelung, verfolgt. Hierbei wird ein sehr schnell schaltendes Stellsignal für die Kompensation auftretender Störungen und Modellunsicherheiten so genutzt, dass bereits ohne besonderes Vorwissen über die Störeinflüsse eine beachtliche Regelgüte erreicht werden kann.

Die vorliegende Arbeit befasst sich mit dem Thema, diese beiden sehr unterschiedlichen Strategien miteinander zu verbinden und dabei die Vorteile der ursprünglichen Umsetzung zu erhalten. So benötigen Sliding-Mode Verfahren generell nur wenige Informationen über die Störung, zeigen jedoch Defizite bei Unsicherheiten, die vom Systemzustand abhängen. Auf der anderen Seite können adaptive Regelungen sehr gut parametrische Unsicherheiten kompensieren, wohingegen unmodellierte Störungen zu einer verschlechterten Regelgüte führen.

Ziel dieser Arbeit ist es daher, eine kombinierte Entwurfsmethodik zu entwickeln, welche die verfügbaren Informationen über die Störeinflüsse bestmöglich ausnutzt. Hierbei wird insbesondere Wert auf einen theoretisch fundierten Stabilitätsnachweis gelegt, welcher erst durch Erkenntnisse der letzten Jahre im Bereich der Lyapunov-Theorie im Zusammenhang mit Sliding-Mode ermöglicht wurde.

Anhand der gestellten Anforderungen werden Regelalgorithmen entworfen, die eine Kombination von Sliding-Mode Reglern höherer Ordnung und adaptiven Verfahren darstellen. Neben den theoretischen Betrachtungen werden die Vorteile des Verfahrens auch anhand von Simulationsbeispielen und eines Laborversuchs nachgewiesen. Es zeigt sich hierbei, dass die vorgeschlagenen Algorithmen eine Verbesserung hinsichtlich der Regelgüte als auch der Robustheit gegenüber den konventionellen Verfahren erzielen.

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Contents

Acronyms

AC absolutely continuous. 10, 11, 63, 64, 67, 95

AGSTA adaptive-gain super-twisting algorithm. 3, 82, 83, 101, 114, 115, 126

ALE algebraic Lyapunov equation. 66

CE certainty equivalence. 5, 6, 88, 90, 95, 101, 112, 127

CESTA certainty-equivalence based super-twisting algorithm. 90, 102, 114, 115, 121–124, 126, 150

DC direct current. 120

FOSM first-order sliding-mode. 4, 53, 57, 61–63, 68, 71, 74, 75, 77, 104, 149, 151

HOSM higher-order sliding-mode. 4, 5, 24, 46, 78, 85, 89, 104, 117, 118, 127, 129

MRAC model reference adaptive control. 3, 30, 32, 40, 41, 102, 118, 149

PE persistent excitation. 43, 44, 110, 129

PLC programmable logic controller. 70

PWM pulse-width modulation. 45

SMC sliding-mode control. 45, 46, 55–57, 61, 62, 68, 72, 80, 84, 101, 103

SOSM second-order sliding-mode. 57, 62, 89

STA super-twisting algorithm. x, 3, 5, 6, 65, 68, 77, 89–93, 95, 97, 99, 101–104, 109, 112, 114, 115, 122–126, 128, 150, 151

VGSTA variable-gain super-twisting algorithm. 126

VS-MRAC variable structure model reference adaptive control. 3

VSC variable structure control. 47, 48, 149

Mathematical Conventions

Symbol	**Description**	**Notes**
$\mathbb{I}_n$	The identity matrix of the size $\mathbb{R}^{n\times n}$	n a natural number
$\mathbb{R}$	Set of real numbers	
$\mathbb{R}^n$	The real Euclidan space of dimension n	n a natural number
$\mathbb{R}^{n\times m}$	The real Euclidan space of dimension n	n and m natural numbers
$\lambda_i(A)$	The i-th eigenvalue of the matrix A	$A \in \mathbb{R}^{n\times n}$, $i \in \{1, ..., n\}$
$\lambda_{\min/\max}(A)$	The minimum/maximum eigenvalue of the matrix A	$A \in \mathbb{R}^{n\times n}$
$x^\top$	The transpose of a column vector	$x \in \mathbb{R}^n$
x_i	The i-th element of the vector x	$x \in \mathbb{R}^n$, $i \in \{1, ..., n\}$
$\bar{x}_i$	The vector of the first i elements of x	$x \in \mathbb{R}^n$, $i \in \{1, ..., n\}$

1. Introduction

The ongoing evolution in technological products lead to increased requirements with respect to the control theory that is needed to handle those systems. Moreover, a fully autonomous operation demands a control approach that is able to eliminate the influence of disturbances. It is significantly important that the control system is able to fulfill its task even in the presence of possible uncertainties. Furthermore, it should be ensured that changes in the system, due to wear for example, have no influence on the performance of the control loop.

The purpose of a control-loop is primarily ensuring certain pre-defined properties of a technical system under various kinds of influences. These could be either external disturbances like wind gusts on an airplane or internal effects coming from the features of the system itself that have not been considered during the design of the feedback.

One possible way is to use a so-called robust controllers. The idea is to design a feedback in such a way that the stability is ensured even in the case of unknown model parameters or external disturbances [113]. A well known design approach is the $\mathcal{H}_\infty$ method that provides an optimal controller regarding specifications in the frequency domain [152]. However, these controllers are not able to adjust themselves in case of changing condition. As a result, many of the approaches are relatively conservative as the uncertainty has to be specified during the design of the controller [62]. So for the conventional robust control approaches, there is always a trade-off between robustness and performance.

To address this issue of performance and robustness requirements, adaptive controllers are highly appealing [1, 73]. The parameters of the controller are continuously adapted in order to fulfill a given performance criterion [74]. This procedure serves multiple tasks like the compensation of wear in a mechanical system, self-tuning during setpoint-changes and the invariance against parameter changes. However, conventional adaptive controllers lack in the robustness against uncertainties that cannot be modeled, see [61].

It is stated that control-loops with conventional adaptive controllers tend to become even unstable if a so-called unstructured uncertainty acts on the system.

The field of sliding-mode introduces a completely different perspective on the design of a controller [140]. The idea is to use a very fast switching signal to force the system to a specified behavior despite the presence of disturbances or uncertainties. As it turned out, this is a very powerful and yet simple to implement approach that shows very good robustness properties as the closed loop system is invariant towards bounded, matched phenomenon [36]. Unfortunately, these approaches often suffer from a phenomena called chattering [132, 141] which is caused by unmodeled input dynamics or discrete time implementation. The amplitude of the chattering depends mainly on the gains of the discontinuous part, such that it is advisable to reduce them as much as possible. On the other hand, a minimum gain is needed to ensure the robustness properties.

Furthermore, sliding-mode control mainly focuses on uncertainties that are bounded [36, 132, 141]. While this assumption seems reasonable with respect to external disturbance, it does not include uncertainties depending on the state of the plant in general. In that case, it is not possible to estimate the upper bound on the uncertainty during the design phase of the control algorithm since an upper bound of the system state cannot be assumed before the actual stability proof.

Combining adaptive and sliding-mode control seems to be an ideal fusion that should benefit from the advantages of both approaches. The sliding-mode part handles external disturbances very well while the controller is continuously adjusted by an adaptive part. However, the design should always include a proper stability proof since the controller is often described as a dynamical system itself. Indeed, many adaptive control designs use methods from the Lyapunov stability theory to show the boundeness or even convergence of the system and controller states [122, 123, 153].

1.1. State of the Art

One of the first combinations of adaptive control and ideas from the variable-structure domain is given in [55] and is called variable structure model reference adaptive control (VS-MRAC). Therein the authors present a modification of the classical model reference adaptive control (MRAC)-scheme by using a discontinuous parameter adaptation law. However, this requires *a priori* specification about the bounds of the parameters that should be estimated. Furthermore, the approach is restricted to linear systems with a relative degree of one, which is relaxed to an arbitrary relative degree in [53, 54]. Nevertheless, these approaches use variable-structure methods mainly for the parameter estimation and not for the control itself. Compared to the standard MRAC approach, the robustness toward unmodeled uncertainties could be improved as shown in [33,56].

The super-twisting algorithm (STA) [77] is one of the most popular sliding-mode controllers since it allows the compensation of Lipschitz continuous disturbances with a continuous control law. Moreover, it converges in finite time to a specified manifold in the state-space. The controller design involves parameters that have to be selected properly and they depend in general on an upper bound of the disturbance or its derivative. Unfortunately, these bounds are often unknown which leads in general to an overestimation. In [30, 125], the authors present a method called adaptive-gain super-twisting algorithm (AGSTA) that allows the controller parameters to be adjusted during run-time to avoid this overestimation. The basic idea is to start with little gains and increase them if it is necessary such that high gains are only obtained when the disturbances are large. The idea was extended in [69, 126, 127] by the reduction of the gains whenever a certain domain around the sliding-manifold is reached, allowing the gains to decrease as well. However, these approaches do not incorporate the structure of the uncertainty. As a result, state-dependent uncertainties cannot by handled by this method in general.

The same idea is followed by the adaptive continuous twisting algorithm (ACTA) which has been introduced [95]. In the contrast to the adaptive-gain super-twisting algorithm (AGSTA), this approach allows to be applied to systems with a relative degree of two, while the AGSTA is restricted to systems with a relative degree of one.

For systems given in *parametric strict-feedback-form* and *parametric pure-feedback-form*, the so-called backstepping procedure, i.e. [66, 70], known from the nonlinear control theory can be applied. To improve the robustness against matched uncertainties, in

[115, 128] the last step of the backstepping involves a sliding-mode design. Furthermore, the adaptive backstepping is combined with a second-order sliding-mode approach in [18, 19] such that a continuous control signal is ensured. Moreover, it is shown in [71, 72] that the backstepping procedure enables the possibility to handle unmatched uncertainties as well.

So far, the presented methods use mostly first-order sliding-mode (FOSM). Since the stabilization of a manifold in the state-space is restricted to a relative-degree one condition. As we will see in the process of this thesis, we rely on available Lyapunov functions for the nominal controller design. However, the first stability proof for sliding-mode approaches of a higher order were obtained by geometric reasoning [76–78], such that the intended design procedure cannot be applied. Fortunately, in the last decade the stability proof of sliding-mode controllers with the help of a Lyapunov function became more and more popular. The research mainly focuses on the super-twisting [35, 92, 93] and was extended to the twisting-controller as well [120].

Just recently, by the use of homogeneity, Lyapunov functions for arbitrary-order sliding-mode have been discovered [34, 94], allowing higher-order sliding-mode (HOSM) controllers to be combined with approaches that rely on a nominal Lyapunov function.

1.2. Objectives and Contribution of this Thesis

The main goal of this thesis is to find a novel kind adaptive sliding-mode controllers. One major drawback of the existing approaches is the fact that structural properties about the plant are rarely used. However, this information is in many cases available for free since the design of a controller involves normally the modeling of the plant as well. It is expected that the usage of this information leads to a better performance of the closed-loop system. Furthermore, the class of admissible disturbances should be extended towards explicitly state-dependent uncertainties which are in general not covered by sliding-mode approaches.

In contrast to the approaches presented so far, we do not indent to modify the gains of the sliding-mode part of the controller. This is motivated by the fact that large uncertainties would require larger gains which will result in a larger amplitude of the chattering signal which is highly unintended.

From this considerations, a combination of adaptive and sliding-mode is proposed that relies on the certainty equivalence (CE) principle. It is assumed that the overall uncertainty can be separated into a parametric or structured part and an unstructured term. A key element of the structured uncertainty is that its described by unknown parameters entering the uncertainty linearly and a known regressor. The remaining part of the uncertain term is then covered by the sliding-mode controller.

One major objective of this thesis is a design that guarantees stability in the adaptive case as well. Thus, the certainty equivalence (CE) principle uses the Lyapunov theory in order to obtain the adaptation law. As a result, the stability of the closed loop is ensured by the design itself. However, this procedure generates Lyapunov functions that are negative semi-definite which occurs on a regular basis in the design of adaptive controllers. Unfortunately, the statements that ensure the converge of the system states to the origin cannot be applied in that case since they require mostly the right hand side to be somehow continuous. To the best knowledge to the author, there exists no statement about the stability for a non-autonomous system with a discontinuous right-hand side. Hence, a slightly modified version of an existing theorem is proposed that solves this task allowing the application of the indented design procedure.

Fortunately, as explained in the previous section, Lyapunov functions for higher-order sliding-mode have just recently been presented. Starting with these Lyapunov function, this thesis provides novel adaptive controllers using the super-twisting algorithm (STA) and arbitrary HOSM approaches.

Finally, the proposed approaches are compared to existing algorithms in simulation examples and by implementing the proposed controller on the laboratory test-bench.

1.3. Outline of this Thesis

This thesis is organized in seven chapters, starting with this introduction.

Chapter 2 introduces the mathematical tools which are needed in the other chapters. Since this thesis deals with dynamical systems with a discontinuous right-hand side, it is important to understand the characteristics of these systems. This includes the definition of the solution in the sense of Filippov. We further discuss the stability of a discontinuous system with the focus on Lyapunov theory since the rest of the thesis relies on this concept.

Chapter 3 provides a brief introduction into the field of adaptive control. Here, the main approaches towards adaptation are discussed to get a general overview about the possible implementations. Based on a simple example, the main advantages of an adaptive controller are presented.

Chapter 4 give an insight on sliding-mode control. We start with the fundamental ideas of the first-order controllers and smooth the way to the higher-order approaches. The robustness properties are studied and it is clarified what helps these discontinuous controller to achieve such a remarkable performance. Furthermore we discuss the drawbacks that a discontinuous controller entails.

Chapter 5 contains the main contribution of this thesis. Namely a novel approach towards the combination of adaptive and higher-order sliding-mode control. By having a look at the existing approaches, we are able to formulate the motivation behind the presented approach more detailed. In the first place, we combine the certainty equivalence (CE) principle with the super-twisting algorithm (STA) and investigate the influence of the Lyapunov function for the combined system. After that, we have a look at the latest higher-order sliding-mode controllers for which a Lyapunov function was published just recently.

Chapter 6 deals with the implementation of the controllers from the previous chapter. The advantages of the proposed method are discussed based on simulation examples and on a mechanical laboratory test-bench.

Chapter 7 summarizes the work and gives a brief outline about the possible next steps which have not been investigated yet.

2. Mathematical Preliminaries

As mentioned before, the objective of this thesis is to combine the ideas of adaptive and discontinuous controllers. Both methods have their individual advantages with respect to robustness and performance of the controller design. It should be noted that the people dealing with each kind of design procedure have a different way of approaching the problem. Hence, the mathematical tools for analyzing an adaptive controller are different than for discontinuous ones. This is a strong motivation for having a closer look at the theory of discontinuous systems.

The first issue addressed in this chapter is the definition of a solution of a dynamical system with discontinuous right-hand side. In Section 2.2 we will have a closer look at the common *Filippov* solution concept for a specified system class.

Next to those elementary characteristics, the most important attribute of a closed-loop system is stability. This topic is examined in Section 2.3. It will be discussed whether important concepts from the continuous case can be transferred to the discontinuous case.

2.1. Notations and elementary definitions

Before we can have a look on the basics of discontinuous systems, we have to introduce some fundamental functions, notations and definitions that will be used in the whole thesis.

Sign Function We start off with the sign function. One can find several definitions in the literature depending on the actual context it is used. In this thesis we will use the following definition

$$\operatorname{sign}(a) := \begin{cases} 1 & \text{if} \quad a > 0 \\ \sigma & \text{if} \quad a = 0 \\ -1, & \text{if} \quad a < 0 \end{cases} \tag{2.1}$$

with $\sigma \in \mathbb{R} : -1 \leq \sigma \leq 1$ as it can be found in [110]. The main difference of all the possible versions is the treatment in case when the argument a is equal to zero. Other definitions may assign a function value of 1 or 0. However, in the context of uncertain systems, the definition (2.1) has be proven to be a valid concept to understand the properties of a discontinuous controller. The reason for the set-valued definition at the point 0 will be enlightened in the upcoming sections and especially Chapter 4.

Signed Power A particular notation will be used primarily in the context of homogeneity introduced in Section 2.4 at the end of this chapter. We consider a real variable $a \in \mathbb{R}$ and a real number $b \in \mathbb{R}$, then

$$\lceil a \rfloor^b := \operatorname{sign}(a)\ |a|^b \tag{2.2}$$

is an abbreviation for the sign preserving power. From this notation, we can construct some properties: The sign function (2.1) can be written as

$$\lceil a \rfloor^0 = \operatorname{sign}(a)$$

such that $\lceil 0 \rfloor^0 \in [-1, 1]$ and

$$\lceil a \rfloor^1 = a.$$

On the other hand, the result for even b may differ since

$$\lceil a \rfloor^2 = \operatorname{sign}(a)\,|a|^2 \neq a^2$$

contrary to a first intuitive guess. Furthermore, we can write the product of two terms of (2.2) as

$$\lceil a \rfloor^b \, \lceil a \rfloor^c = |a|^{b+c}$$

with $c \in \mathbb{R}$. The calculation of the derivative can be simplified to

$$\frac{\mathrm{d}}{\mathrm{d}a} \lceil a \rfloor^b = b\,|a|^{b-1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}a} |a|^b = b\,\lceil a \rfloor^{b-1}$$

almost everywhere for a.

Continuous Functions In the further course of this thesis we have to proof stability of a discontinuous dynamic system. Unfortunately most of the classical results on dynamic systems rely on differential equations with a continuous right-hand side. Hence, in Section 2.3 we will have a closer look at the stability conditions. Some of the proofs for those conditions require a certain kind of continuity.

In the following we provide some definitions for continuous functions and some interesting properties as well as the relation between the different kinds of continuity.

Definition 2.1 (Continuous Function, [17, p. 120])**.** *A function $f : \mathbf{D} \to \mathbb{R}$ with $\mathbf{D} \subseteq \mathbb{R}$ is called continuous at point $b \in \mathbf{D}$ if for every $\epsilon > 0$ a $\delta > 0$ exists, such that*

$$\left\| f(a) - f(b) \right\| < \epsilon$$

holds for all $a \in \mathbf{D}$ with

$$\|a - b\| < \delta.$$

Definition 2.2 (Uniformly Continuous Function [17, p. 137]). *A function $f : \mathbf{D} \to \mathbb{R}$ with $\mathbf{D} \subseteq \mathbb{R}$ is called uniformly continuous if, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\|f(a) - f(b)\| < \epsilon$$

for every $a, b \in \mathbf{D}$ with

$$\|a - b\| < \delta.$$

One can easily see that every uniformly continuous function is a continuous function where the contrary implication not necessarily has to be fulfilled.

Definition 2.3 (absolutely continuous (AC) Function [27, p. 337]). *A function f on an interval $[a, b]$ is called absolutely continuous if, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} \Big\| f(a_i) - f(b_i) \Big\| < \epsilon$$

for every finite collection of n pairwise disjoint intervals (a_i, b_i) in $[a, b]$ with

$$\sum_{i=1}^{n} \|a_i - b_i\| < \delta.$$

Hence, every absolutely continuous function is uniformly continuous as well. Furthermore, according to [27, Corollary 5.3.3] it follows that if f and g are two absolutely continuous functions on $[a, b]$, the sum $f + g$ and the difference $f - g$ are absolutely continuous functions as well.

Definition 2.4 (Lipschitz Continuous Function). *A function f on an interval $[a, b]$ is Lipschitz continuous if there exists $L \geq 0$ such that*

$$\Big\| f(a) - f(b) \Big\| \leq L\|a - b\|$$

is fulfilled.

Lemma 2.5 (Heine-Cantor theorem [50])**.** *Let $f : \mathbf{D} \to \mathbb{R}$ be a continuous function with $\mathbf{D} \subseteq \mathbb{R}$. If $\mathbf{D}$ is a compact set, then f is uniformly continuous.*

The proof of this lemma can be found in [17, p. 143]

Lemma 2.6 (Comp. of a Lipschitz and absolutely continuous Function [27, p. 391])**.** *Let f be a Lipschitz continuous function on $\mathbb{R}$ and g be absolutely continuous on the interval $[a, b]$, then the composition $f \circ g$ is absolutely continuous on $[a, b]$.*

Proof. According to the Definition 2.4, there exists a constant $L \geq 0$ such that

$$\Big\| f(x) - f(y) \Big\| \leq L\|x - y\|$$

is achieved $\forall x, y \in$ R. For the absolutely continuous function g and $\epsilon > 0$, chose $\delta > 0$, such that every finite collection of n pairwise disjoint intervals (a_i, b_i) in $[a, b]$ satisfies

$$\sum_{i=1}^{n} \|a_i - b_i\| < \delta$$

which implies

$$\sum_{i=1}^{n} \Big\| g(a_i) - g(b_i) \Big\| < \frac{\epsilon}{L} \; .$$

The composition $f(g(x))$ yields

$$\Big\| f(g(a_i)) - f(g(b_i)) \Big\| \leq L \Big\| g(a_i) - g(b_i) \Big\|$$

for each $i = \{1, 2, ..., n\}$ such that

$$\sum_{i=1}^{n} \Big\| f(g(a_i)) - f(g(b_i)) \Big\| < L \sum_{i=1}^{n} \Big\| g(a_i) - g(b_i) \Big\| < \epsilon$$

which completes the proof. □

Lemma 2.7. *The composition $h = f \circ g$ of two uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous as well.*

Proof. Since f is uniformly continuous, for every given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|f(a) - f(b)\| < \epsilon$ whenever $\|a - b\| < \delta$. For this δ, due to the uniform continuity of g, one can find a number $\eta > 0$ such that $\|g(a) - g(b)\| < \delta$ whenever $\|a - b\| < \eta$. Thus, $\|g(a) - g(b)\| < \delta$ holds if $\|a - b\| < \eta$ and hence $\|f(g(a)) - f(g(b))\| < \epsilon$. Therefore, the composition $h = f \circ g$ is uniformly continuous. □

2.2. Systems with piece-wise continuous right-hand side

In the following we consider a dynamical system in the form

$$\dot{x} = f(t, x) \tag{2.3}$$

where $f(t, x)$ might be a piece-wise continuous right-hand side depending on the state vector $x \in \mathbb{R}^n$ and the time variable $t \in \mathbb{R}$. At this point we do not focus on those $f(t, x)$ that are only discontinuous in t. In a control system this may represent the change of an external parameter, for example a reference signal. However, the solution of such a differential equation with a right-hand side discontinuous system can be constructed in a piece-wise manner in the classical sense.

The more interesting case from the viewpoint of a control engineer is, when f is discontinuous in the state x. Unfortunately in that case, the classical (or Carathéodory) solution cannot be applied.

For the intended usage we shall assume some restrictions on the behavior of $f(t, x)$: The right-hand side of (2.3) is piece-wise continuous if $f(t, x)$ consists of a finite number of domains $\mathcal{G}_j \subset \mathbb{R}^{n+1}, j = 1...N$ with disjoint interiors and boundaries $\partial\mathcal{G}_j$ of measure zero. Within these domains $\mathcal{G}_j$, the solution of (2.3) shall be defined in the classical manner.

In the following, a regularization technique will be introduced that allows to give a solution to (2.3) along the boundaries $\partial\mathcal{G}_j$. Those solutions are normally referred as *sliding modes*.

2.2.1. Solution Concept

A suitable solution concept for the system (2.3) in the context of this thesis is given by Filippov in [42, 43] for differential equations with a discontinuous right-hand side.

Definition 2.8 (Filippov Solution [42])**.** *Given the differential equation* (2.3)*, for each point* $(t, x) \in \mathbb{R}\times\mathbb{R}^n$ *we introduce* $F(t, x)$ *as the smallest convex closed set which contains all the limit points of* $f(t, x^*)$ *as* $x^* \Rightarrow x, t = \text{const}$ *and* $(t, x^*) \in \mathbb{R}^{n+1} \setminus \left(\bigcup_{j=1}^{N} \partial\mathcal{G}_j\right)$.

An abslutely continuous function $x(\cdot)$ defined on an Interval $\mathcal{I}$ is said to be a Filippov solution of (2.3) *if it satisfies the differential inclusion*

$$\dot{x} \in F(t,x) \tag{2.4}$$

almost everywhere on $\mathcal{I}$.

According to [42], we can obtain at least locally a solution for system (2.3) for any initial point $x(t_0) = x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$ on a time interval $[t_0, t_1)$. In general, this solution may be not unique.

Now it has to be determined how $F(t,x)$ is constructed: Inside of each domain $\mathcal{G}_j$, i.e. $(t,x) \in \bigcup_{j1}^{N} \mathcal{G}_j$ the set $F(t,x)$ consists only of the point $f(t,x)$. This means that the solution according to Definition 2.8 satisfies (2.3) in the classical sense.

At the boundaries of the intersection of the domains we assume that f is discontinuous on a smooth surface $\mathcal{S}$ given by a function $s(x) = 0$. We will show in Chapter 4 that this is a valid assumption in context of this thesis in the field of sliding-mode control. The surface $\mathcal{S}$ separates the state space x in two domains $\mathcal{G}^- = \{x \in \mathbb{R}^n : s(x) < 0\}$ and $\mathcal{G}^+ = \{x \in \mathbb{R}^n : s(x) > 0\}$. Now, for a state $x^\star$ approaching any point $x \in \mathcal{S}$ and constant time t, the function $f(t, x^\star)$ has the limit values

$$\lim_{\substack{x^\star \in \mathcal{G}^- \\ x^\star \to x}} f(t, x^\star) = f^-(t,x) \tag{2.5a}$$

$$\lim_{\substack{x^\star \in \mathcal{G}^+ \\ x^\star \to x}} f(t, x^\star) = f^+(t,x). \tag{2.5b}$$

Using this definition, one can construct $F(t,x)$ as the closed convex combination of f^- and f^+ for a certain point (t,x). The following Figure 2.1 displays two possible cases for the vectors f^- and f^+: The first picture 2.1a shows the case when both vectors at (t,x) point towards one side of the tangential plane $\mathcal{P}$ of the surface $\mathcal{S}$ for a fixed t. In that case the Filippov solution will pass the surface $\mathcal{S}$ from $\mathcal{G}^-$ to $\mathcal{G}^+$. We can easily check whether this case is set in by checking the following inequalities

$$\nabla s(x) f^-(t,x) < 0 \tag{2.6a}$$

$$\nabla s(x) f^+(t,x) > 0 \tag{2.6b}$$

with $\nabla s(x) = \left(\frac{\partial s(x)}{\partial x_1}, \ldots, \frac{\partial s(x)}{\partial x_n}\right)$. If either (2.6a) or (2.6b) is fulfilled, the system behaves as shown in Figure 2.1a.

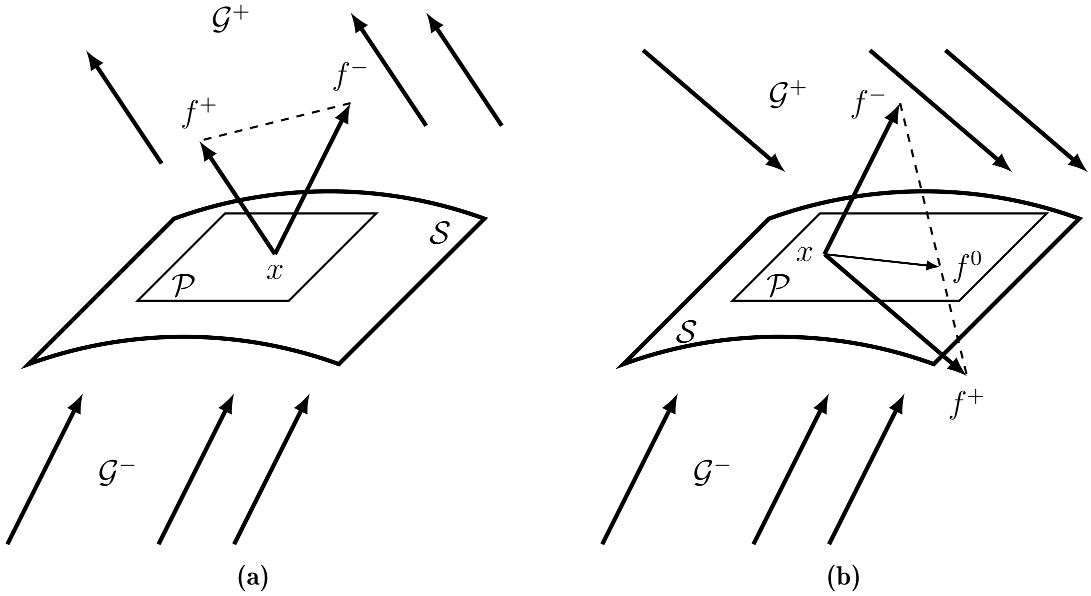


Figure 2.1.: Possible Discontinuous Surfaces [42]

On the other hand if

$$\nabla s(x) f^-(t,x) > 0 \quad \wedge \quad \nabla s(x) f^+(t,x) < 0 \tag{2.7}$$

the linear segment between f^- and f^+ intersects the plane $\mathcal{P}$ as shown in Figure 2.1b. In that case a so called *sliding motion* or *sliding mode* occurs along the surface $\mathcal{S}$. The vector $f^0(t,x)$ denotes the intersection point with the tangential plane $\mathcal{P}$. If this intersection occurs at whole discontinuity, the solution $x(t)$ satisfies the equation

$$\dot{x} = f^0(t,x) \tag{2.8}$$

once it reaches the discontinuous surface $\mathcal{S}$. The vector $f^0(t,x)$ can be determined by simple geometric consideration. Since f^0 is the intersection of $f^-(t,x)$ and $f^+(t,x)$ with the tangential plane, it can be constructed as a linear combination of $f^-(t,x)$ and $f^+(t,x)$

$$f^0(t,x) = \Gamma(t,x) f^+(t,x) + (1 - \Gamma(t,x))\, f^-(t,x) \tag{2.9}$$

with $\Gamma(t,x) \in [0,1]$, a value that denotes the ratio between the two vectorfields that is needed to construct $f^0(t,x)$. From the condition that $f^0(t,x)$ lies in the tangential plane $\mathcal{P}$ we know

$$\nabla s(x)\left[\Gamma(t,x)f^{+}(t,x)+(1-\Gamma(t,x))\,f^{-}(t,x)\right]=0 \tag{2.10}$$

which allows us to calculate the ratio $\Gamma(t,x)$ as

$$\Gamma(t,x)=\frac{\nabla s(x)f^{-}(t,x)}{\nabla s(x)(f^{-}(t,x)-f^{+}(t,x))}. \tag{2.11}$$

From (2.11) we can see that $\Gamma(t,x)$ is defined uniquely and $f^0(t,x)$ according to (2.9) as well. This statement holds unless the denominator

$$\nabla s(x)(f^{-}(t,x)-f^{+}(t,x))=0 \tag{2.12}$$

in (2.11) which makes $\Gamma(t,x)$ disappearing from (2.10) and a unique value for $\Gamma(t,x)$ cannot be identified. From the geometrical viewpoint, this means that both vectors are located in the tangential plane and therefore the closed convex combination as well. Since $f^0(t,x)$ is not defined uniquely in this chase, the system (2.8) is also not defined in a unique way. Consequently, the Filippov approach does not give a unique solution. For a more detailed discussion of the uniqueness properties of the Filippov solution, refer to [42].

Now we shall have a closer look at (2.8) shown in 2.1b. If it can be ensured that $f^{-}(t,x)$ and $f^{+}(t,x)$ point in different directions on the whole surface for arbitrary $x \in \mathbb{R}^n$, the system would evolve only on the surface $\mathcal{S}$. Hence, describing the motion of the system can be done by using (2.8) only. In Chapter 4 it will be explained how this observation can be used in order to design robust control algorithms by the use of discontinuous systems.

2.3. Stability of Discontinuous Systems

The closed loop stability is the essential property of any control system. Hence it is necessary to have some mathematical tools in order to be able to proof stability of the overall control system.

The most common way to analyze the stability of nonlinear systems is via Lyapunov theory. However, discontinuous systems show some worth-noting properties that have to be considered.

2.3.1. Lyapunov Stability

First of all we have to provide the basic stability definitions used in the theory of discontinuous dynamical systems. This thesis will focus on stability in the sense of Lyapunov of the system

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{2.13}$$

with the state vector $x \in \mathbb{R}^n$ and the time variable t and a piecewise discontinuous vector-field $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The stability of discontinous systems has been the topic of a large variety of studies, e.g. [64, 76, 78, 131]. Nevertheless, the understanding of these systems does not reach the same level as for continuous systems. Especially the use of Lyapunov theory is not that common in the field of discontinuous control since the first results rely on geometric reasoning, e.g. [76]. However, in the last decade the application of Lyapunov theory has become more and more popular also in the field of discontinuous systems [93, 96, 119]. In the following we have a look at some of the major theorems that will be later used to combine adaptive and switching controllers.

As explained in the previous section, the solution of a class of nonlinear systems can be also seen as a solution of the differential inclusion

$$\dot{x} \in F(t, x), \quad x(t_0) = x_0 \tag{2.14}$$

where $F(t, x)$ is the smallest convex close set that contains the limit values of $f(t, x)$ from (2.13). So first of all we have to provide some stability definitions of the trajectories of the differential inclusion (2.14) which are possibly not uniquely defined.

We assume that $x = 0$ is the equilibrium point of (2.14) and the solution shall be referred as $x(t, t_0, x_0)$ starting from initial time t_0 and initial point $x_0 = x(t_0)$.

Definition 2.9 (Stability in the Sense of Lyapunov)**.** *The equilibrium point $x = 0$ of the differential inclusion* (2.14) *is said to be stable if for each $t_0 \in \mathbb{R}, \epsilon > 0$ there is $\delta = \delta(\epsilon, t_0) > 0$, depended on ϵ and t_0, such that each solution $x(t, t_0, x_0)$ of* (2.14) *with*

$x(t_0) = x_0 \in \mathbf{B}_\delta$ within the ball $\mathbf{B}_\delta$ centered at the origin with the radius δ exists for all $t \geq t_0$ and satisfies the inequality

$$||x(t, t_0, x_0)|| < \epsilon, t_0 \leq t < \infty.$$

Definition 2.10 (Asymptotic Stability). *The equilibrium point $x = 0$ of the differential inclusion* (2.14) *is said to be asymptotically stable if it is stable and the convergence*

$$\lim_{t \to \infty} ||x(t, t_0, x_0)|| = 0$$

holds for all solutions of (2.14).

Definition 2.11 (Exponential Stability). *The equilibrium point $x = 0$ of the differential inclusion* (2.14) *is said to be exponentially stable if there exist positive constants δ, k and λ such that the inequality*

$$||x(t, t_0, x_0)|| \leq k||x_0||e^{-\lambda(t-t_0)} \tag{2.15}$$

holds for all solutions of (2.14) *initialized within ball $\mathbf{B}_\delta$ and globally exponentially stable if* (2.15) *holds for any initial condition.*

Definition 2.12 (Uniform Stability). *The equilibrium point $x = 0$ of the differential inclusion* (2.14) *is said to be uniformly stable if it is stable and δ in Definition 2.9 does not depend on the initial time, i.e. $\delta = \delta(\epsilon)$.*

Definition 2.13 (Uniform Asymptotic Stability). *The equilibrium point $x = 0$ of the differential inclusion* (2.14) *is said to be uniformly asymptotically stable if it is uniformly stable and there exists a ball $\mathbf{B}_r$ with the radius r not depending on the initial time $r \neq r(t_0)$, such that all solutions $x(t, t_0, x_0)$ starting within $\mathbf{B}_r$ converge uniformly to the origin.*

Theorem 2.14 (Uniform Stability [106]). *Suppose that in a domain ($x \in \mathbf{B}_\delta, t \in \mathbb{R}$) there exists a Lipschitz continuous, positive definite, radially unbounded, decrescent function $V(t, x)$ such that its time derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t), t) = \frac{\mathrm{d}}{\mathrm{d}h} V(t + h, x(t) + h\dot{x}(t))\Big|_{h=0} \tag{2.16}$$

computed along the trajectories $x(t)$ *of* (2.13)*, which are initialized within* $\mathbf{B}_\delta$ *is negative semidefinite almost everywhere, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) \leq 0 \tag{2.17}$$

for almost all t*. Then* (2.13) *is uniformly stable.*

Proof. The proof of this theorem can be obtained in a similar way as the conventional Lyapunov theorem. If the composite Lyapunov function $V(t,x(t))$ is absolute continuous and the time derivative can be expressed by (2.16) almost everywhere and if (2.17) is fulfilled it can be reasoned that $V(t,x(t))$ does not increase along the trajectories of (2.13). Consequently, the system (2.13) has to be uniformly stable. □

The convergence to the origin can be shown with the help of a Lyapunov function the time derivative of which is negative definite according to the following theorem.

Theorem 2.15 (Uniform Asympotic Stability [12])**.** *Let* $F : [\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n]$ *be a set-valued map such that the (local) existence of solutions of* (2.14) *is ensured. Assume that there exists a strict Lyapunov function* $V = V(t,x)$ *such that for some functions* $a, b, c \in \mathcal{K}_o^\infty$ [1] *the inequalities*

$$a(||x||) \leq V(t,x) \leq b(||x||) \quad \forall t \in \mathbb{R} : t \geq 0, x \in \mathbb{R}^n \tag{2.18}$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}v \leq -c(||x||) \quad \forall t \in \mathbb{R} : t \geq 0, x \in \mathbb{R}^n, v \in F(t,x) \tag{2.19}$$

are satisfied. Then the origin of (2.14) *is uniformly globally asymptotically stable.*

Proof. The validity of Theorem 2.15 is quite similar to the uniform stability in Theorem 2.14. The major difference is that the time derivative of $V(t,x)$ has to be upper-bounded by a negative definite function instead of a negative semidefinite one. Hence $V(t,x(t))$ is decreasing along the trajectories except for the time it reaches the origin. □

[1] The definition of a $\mathcal{K}_o^\infty$ function can be found in appendix A.2

2.3.2. Invariance Principle

As explained in the introduction, the aim of this thesis is to combine the advantages of adaptive and discontinuous controllers. This union will be created in Chapter 5 with the help of a Lyapunov function. A very common phenomenon in the field of adaptive controllers and Lyapunov functions is that the time derivative along the system trajectories is negative semidefinite. According to Theorem 2.15 we will not be able to proof asymptotic stability in that case for the whole system. Nevertheless we show at least the convergence to a certain subdomain of the state-space which shall be the topic of the following section.

Therefore we shall have a look at the autonomous system in the first place

$$\dot{x} = \varphi(x), \quad x(t_0) = x_0 \tag{2.20}$$

which is assumed to be sufficiently continuous to have a solution that is continuously continuable to the right. Then we can make use of the following theorem that is commonly known as *LaSalle's invariance principle* [68], *Barbashin-Krasovskii-LaSalle principle* [47] or *Krasovskii-LaSalle principle*.

Theorem 2.16 (Krasovskii-LaSalle's Invariance Principle [75]). *Let $\Omega \in D$ be a compact set that is positively invariant with respect to* (2.20)*. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let $\mathcal{E}$ be the set of all points in Ω where $\dot{V}(x) = 0$. Let $\mathcal{M}$ be the largest invariant set in $\mathcal{E}$. Then every solution starting in Ω approaches $\mathcal{M}$ as $t \to \infty$.*

For a proof, recall for instance [68, p. 128]. According to [106] and [88], the classical invariance principle cannot be applied to discontinuous systems in general. As explained in Section 7.3 in [88], Theorem 2.16 requires a unique solution of the dynamical system. However, dynamical systems described by a differential inclusion may not satisfy this assumption in general.

This open question provided the motivation to find a modified invariance principle that holds for autonomous systems

$$\dot{x} = f(x), \quad x(t_0) = x_0 \tag{2.21}$$

where $f(x)$ is a piecewise continuous function. Furthermore, it shall be assumed that any solution of (2.21) is uniquely continuable to the right, sufficient conditions can be found in [42]. Under this assumption, Theorem 2.16 is extended in [106] to a class of autonomous systems with discontinuous right-hand side:

Theorem 2.17 (Invariance Principle for discontinuous Systems [106, p. 49])**.** *Consider an autonomous system* (2.21)*, each solution of which is uniquely continuable to the right. Suppose there exists a Lipschitz continuous, radially unbounded, positive definite function* $V(x)$ *such that its time derivative along the trajectories of* (2.21) *is negative semidefinite. Let* $\mathcal{M}$ *be the largest positively invariant subset of the manifold where* $\dot{V}(x) = 0$ *and let* $V(x) \to \infty$ *as* $\operatorname{dist}(x, \mathcal{M}) \to \infty$*. Then all the trajectories* $x(t)$ *of* (2.21) *converge to* $\mathcal{M}$ *that is*

$$\lim_{t \to \infty} \operatorname{dist}(x(t), \mathcal{M}) = 0$$

For the proof consider [106].

2.3.3. Extension for non-autonomous Systems

In the previous section we have seen that the invariance principle holds also for systems with a right-hand side piece-wise discontinuous in the state x. Unfortunately the invariance principle from Theorem 2.17 is only applicable to autonomous systems. In the context of a control system we might have external signals like disturbances or varying reference signals. Thus, the system description will be non-autonomous.

In the continuous control theory, there exist several invariance-like theorems, see e.g. [68, p. 323] for non-autonomous systems. However, one condition is always that the right-hand side of the system is Lipschitz-continuous in x, which in our case is not given. In [104] an extended invariance principle is presented which is dealing with discontinuous non-autonomous systems having a Lyapunov function whose time-derivative is negative semidefinite. However, Theorem 1 in this contribution aims for asymptotic stability, which in the context of adaptive control is not strictly necessary. Hence, the following Theorem 2.19 is a modified version of the one given in [104] without the need of asymptotic convergence of all the states.

The invariance principle from the previous section shows that the trajectories of the system (2.13) approach a given set $\mathcal{E}$ where the derivative of the Lyapunov function is equal to zero. For the non-autonomous case it is not obvious how to define such a set, since the time derivative of the Lyapunov function might be time-dependent. If we are able to show that

$$\dot{V}(t,x) \leq -W(x) \tag{2.22}$$

with $W(x)$ being a positive semidefinite function, we expect the trajectories of (2.13) to converge to the set of points where $W(x) = 0$. This will be the statement of the following theorem. At first we shall have a look a lemma that is necessary for the proof and known as *Barbalat's Lemma*.

Lemma 2.18 (Barbalat's Lemma [68])**.** *Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $[0,\infty)$. Suppose that $\lim\limits_{t\to\infty} \int_0^t \Psi(\tau)d\tau$ exists and is finite. Then,*

$$\lim_{t\to\infty} \Psi(t) = 0 \tag{2.23}$$

The proof can be found in [68, p.323].

Theorem 2.19. *Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$ which is supposed to be an equilibrium point of (2.13) at $t = 0$. Let furthermore the right hand side of (2.13) be uniformly bounded in t and let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a positive definite, Lipschitz continuous function such that*

$$\frac{\partial V(x)}{\partial x} f(t,x) \leq -W(x) \tag{2.24}$$

$\forall x \in \mathcal{D}, \forall t \geq 0$, where $W(x)$ is a continuous positive semidefinite function on $\mathcal{D}$. Choose $r > 0$ such that $B_r \subset \mathcal{D}$ and let $\rho \leq \min_{||x||=r} V(x)$. Then all solutions of (2.13) with $x(t_0) \in \{x \in B_r | V(x) \leq \rho\}$ are bounded and satisfy

$$\lim_{t\to\infty} W(x(t)) = 0 \tag{2.25}$$

If $V(x)$ is radially unbounded and the assumptions hold on the domain $\mathcal{D} = \mathbb{R}^n$, the statement can be applied globally.

Proof. Theorem 1 in [42, p. 87] ensures that a solution of (2.13) is locally defined for an arbitrary initial point $x_0 = x(t_0) \in \mathcal{D}$. Since $W(x)$ is positive semidefinite, we can ensure that every possible solution of (2.13) starts in a bounded set $\{x \in B_r : V(x(t)) \leq V_r\}$ parameterized by constant values r and $V_r < \min_{||x||=r} V(x)$ fulfills the following condition

$$\sup_{t\in[t_0,\infty)} V(x(t)) \leq V_r. \tag{2.26}$$

Thus, every solution starting in B_r remains bounded. Now we have to show that (2.25) is fulfilled independent of the initial point $x(t_0)$ of (2.13). We achieve this by integrating (2.24) along the solutions of (2.13) that are initialized at $t_0 \in \mathbb{R}$ within a compact set

$$\mathcal{D}_0 = \{x \in \mathcal{D} : V(x) \leq V_0\} \tag{2.27}$$

with $V_0 = V(x(t_0))$. Hence we obtain

$$\int_{t_0}^{t} W(x(\tau))\,\mathrm{d}\tau \leq -\int_{t_0}^{t} \dot{V}(x(\tau))\,\mathrm{d}\tau = V_0 - V(x(t)) \tag{2.28}$$

by integrating (2.26). If we consider the case when $t \to \infty$ for (2.28), we get

$$\int_{t_0}^{\infty} W(x(\tau))\,\mathrm{d}\tau \leq -\int_{t_0}^{\infty} \dot{V}(x(\tau))\,\mathrm{d}\tau \leq V_0 \tag{2.29}$$

which allows combination with (2.26) to conclude, that for every solution $x(t)$ of (2.13) the integrand of $W(x(t))$ is uniformly bounded in t.

Now we have to show that $W(x(t))$ is a uniformly continuous function in t which might not be obvious in the case of a system with a discontinuous right-hand side. As already mentioned, all possible solutions $x(t)$ stay in a bounded set B_r which lets us conclude, together with Lemma 2.5, that $W(x)$ is uniformly continuous in x. We have seen in Section 2.2 that the Filippov solutions are absolutely continuous in t and hence uniformly continuous as well. From Lemma 2.7 we know that the composition of two uniformly continuous functions is uniformly continuous as well, i.e. $W(x(t))$ is uniformly continuous on $[0, \infty)$.

Thus, we can apply Lemma 2.18 to the integral inequality (2.28) which lets us conclude that (2.25) is achieved. Hence, the system states of (2.13) converge to the set

$\{x \in \mathcal{D} : W(x) = 0\}$. If the above assumptions are valid for an arbitrary initial point from the whole state-space, i.e. $x_0 \in \mathbb{R}^n$, the statement can be applied globally. □

2.4. Homogeneity

Another useful tool helping with the analysis of nonlinear dynamical systems is a property called homogeneity. There is a long tradition in using homogeneous functions in the stability analysis [25, 26, 116], since those function imply some interesting and helpful properties as we will see in the following.

Definition 2.20 (Dilation Operator)**.** *Given a vector* $z = (z_1, \ldots, z_n)^\top \in \mathbb{R}^n$ *the dilation operator is given by*

$$\varDelta_\epsilon^r \, z := (\epsilon^{r_1} z_1, \ldots, \epsilon^{r_n} z_n)^\top$$

with $r = (r_1, \ldots, r_n)^\top \in \mathbb{R}^n$ *being a vector of* n *weights.*

Definition 2.21 (r-homogeneous Function [154])**.** *A function* $V : \mathbb{R}^n \to \mathbb{R}$ *fulfilling the identity* $V(\varDelta_\epsilon^r z) = \epsilon^l V(z)$ *for every* $\epsilon > 0$*, is called r-homogeneous of degree* l *for some* $l \in \mathbb{R}$.

Definition 2.22 (r-homogeneous System)**.** *A dynamical system described by the differential equation* $\dot{z} = f(z)$ *(or the differential inclusion* $\dot{z} = F(z)$*) with the state* $z \in \mathbb{R}^n$ *is called r-homogeneous or homogeneous if its vector field* $f(z)$ *(or set-valued vector field* $F(z)$*) satisfies the identity* $f(\varDelta_\epsilon^r z) = \epsilon^l \varDelta_\epsilon^r z f(z)$ *for every* $\epsilon > 0$ *(i.e. the vector field* $f(z)$ *or the set-valued vector field* $F(z)$ *is r-homogeneous).*

Definition 2.23 (Homogeneous Unit Sphere)**.** *The set* $\mathcal{S} = \{z \in \mathbb{R}^n : ||z||_{r,p} = 1\}$ *with* $||z||_{r,p} := \left(\Sigma_{i=1}^n |z_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}}$ *is called a r-homogeneous unit sphere.*

From the definitions above, we can derive some very powerful properties that have been proven to be useful in the field of HOSM. Definition 2.21 makes it possible, together with Definition 2.23, to obtain every value of an homogeneous function just by knowing the value of the function on the homogeneous unit sphere. Hence it is intuitive that for homogeneous systems according to Definition 2.22 local stability implies global stability, see i.e. [12, 79].

3. Basics of Adaptive Controllers

The major objective of any control system is to inject a desired behavior into a given plant. This task involves a certain robustness or insensitivity against uncertainties caused by the plant itself or external factors. In many cases a suitable controller can be found that solves this task very well.

Unfortunately, almost all existing systems will change their behavior more or less over time caused either by wear of the plant itself or changed external conditions. A controller designed using conventional methods might be able to compensate those variations, i.e. by the usage of large stability margins in the initial design. However, there is always a trade-off between robustness and stability on the one hand, and performance on the other hand. Hence it is obvious that it might be useful that the controller is able to handle such situations: it *adapts* to the changed conditions.

In the following chapter we discuss the basic ideas of *adaptive controllers* by having a look at the fundamental differences between conventional and adaptive controllers. We also will have a look at different approaches that are commonly used to design an adaptive controller.

3.1. Conventional Controller Design

Before we may understand the principles of an adaptive control design, we shall have a basic understanding how a control system is obtained. In general, a methodological procedure begins with a dynamical model of the plant that should be controlled. This model is then used in the controller design which makes it necessary to match the model structure accordingly.

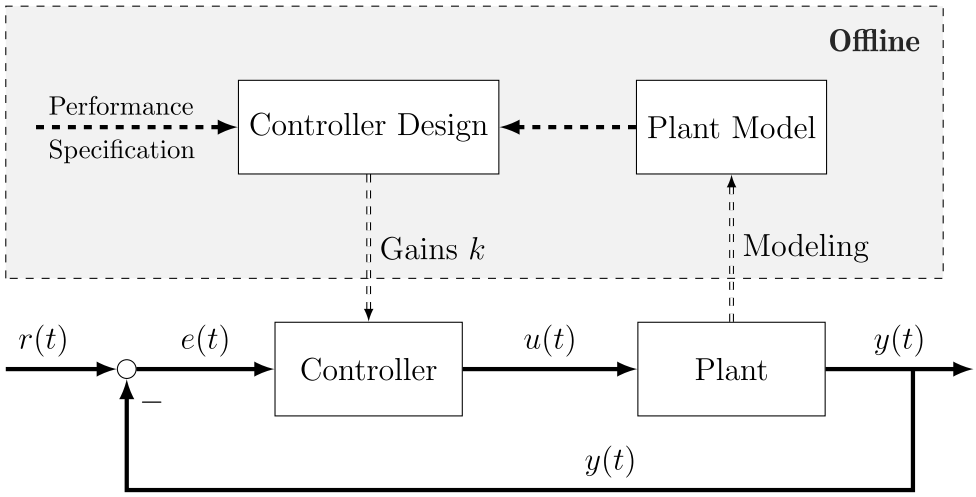


Figure 3.1.: Conventional Controller Design Procedure

Figure 3.1 contains the scheme of a conventional procedure. The next step consists in an adequate controller design based on the model that has been obtained before. As shown in Figure 3.1, both of these steps, the modeling as well as the design, are carried out offline. Finally, the controller is implemented in a suitable environment in order to solve the given task.

However, in many practical applications, the model is not able to fully reflect the actual behavior of the plant. This can be caused by many reasons: One issue might be the complexity of the system that requires some simplifications to be made in order to obtain a model that is suitable for the controller design. Hence the model will be only valid in specific modes of operation of the plant, i.e. the states of the system are in a certain domain. Another source of discrepancies between the dynamical model and the plant might be parameter uncertainties caused by various reasons. During the run-time, many engineering setups will face the problem of regular wear in its components. An example is the furthermore, as soon as an technical system is produced in higher quantities, one has to assume that the system components will be slightly different due to production quality. At this point we might think of capacitors or inductances in an electric circuit. In consequence, the system response to a same control signal might be slightly different.

The question that arises at this point is how to handle those issues. If the parameter uncertainties are known during the design process, we may include the parameter deviation in the design process. In general, this procedure is known as *robust control*. The objective is to find a controller that stabilizes the system at a certain operation point or at a given trajectory despite the presence of uncertainties in the plant.

A well-known method is the so-called $\mathcal{H}_\infty$ controller design or the μ-synthesis approach, see for example [153]. The general idea is to have an estimation about the upper or lower bound of the uncertainty or affiliation to a certain function space and prove that the stability is ensured for all possible cases within the bounds or the function space. However, a worst-case analysis comes often at the price of performance. As we will see in a later example, while maintaining the stability, a robust controller may violate the performance specification that is fulfilled in the ideal case.

At first, we shall start with a definition of an adaptive control system.

Definition 3.1 (Adaptive Control System, [74])**.** *An adaptive control system measures a certain performance index of the control system using the inputs, the states, the outputs and the known disturbances. From the comparison of the measured performance index and a set of given ones, the adaptation mechanism modifies the parameters of the adjustable controller and/or generates an auxiliary control in order to maintain the performance index of the control system close to the set of given ones (i.e., within the set of acceptable ones).*

3.2. Different Approaches to Adaptive Control

According to Definition 3.1, the main difference to conventional (robust) controllers is that the parameters of the control law may change online according to predefined performance criteria. There exist several different approaches how to obtain an adaptive controller that will be discussed in the following.

3.2.1. Open-Loop Adaptive Control

The first controller that could be seen as an adaptive approach is called open-loop adaptive control. A well-know representative is the gain-scheduling method. The

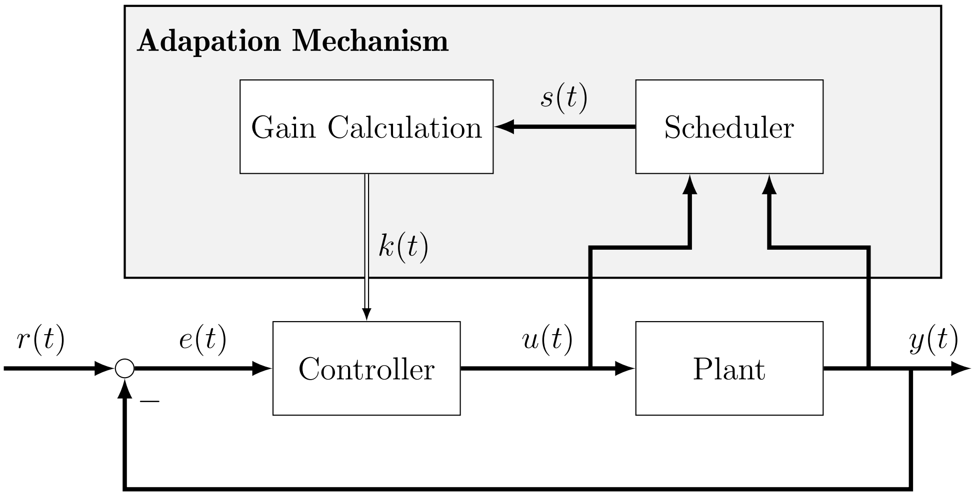


Figure 3.2.: Scheme of an Open-Loop Adaptive Control System with time-varying Controller Gains $k(t)$

main idea can be obtained from Figure 3.2 that shows a typical open-loop adaptation mechanism. It should be noticed that open-loop refers to the variation of the controller gains and does not imply that there is no feedback path in the system at all. The inputs and/or outputs of the plants are used to compute the so-called *scheduling variable* s which is used to obtain the time-variant controller gains. This can be achieved by either using an algorithm which is based on a predefined formula or a look-up table that is created during the controller design process. Hence, the gain-scheduling is actually an open-loop method since it is not supervised if the controller parameter changes have a positive effect on the performance of the closed-loop system as it can be seen in Figure 3.2. This method is applicable when the variation of the plant dynamic is already known during the controller design and has been applied in various technical applications [60, 144].

3.2.2. Estimator Based Design

Adaptive controllers can be set-up by the usage of an estimator to overcome parameter changes in the plant. The main idea can be obtained from Figure 3.3: During the run-time of the controller, an estimator is used to identify unknown or changing parameters

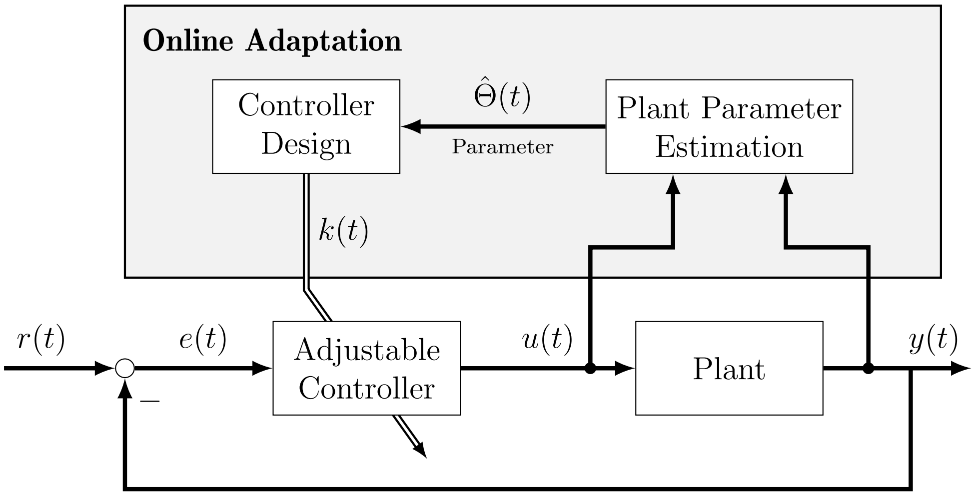


Figure 3.3.: Schematic of an Estimator Based Adaptive Control Design [74]

Θ of the plant. There exists plenty of different approaches to estimate plant parameters, starting from gradient based [2, 148], least-squares [65], nonlinear methods [117] as well as using the principle of homogeneity [118], just to name a few. The estimated plant parameters $\hat{\Theta}$ are then fed into the controller allowing to react in a proper manner to the changed values.

This approach bears a not to be underestimated advantage: Due to the separation of the parameter estimation from the controller, as it can be seen in (3.3), it is possible to modularize the controller architecture. Thanks to this *modular design*, it is possible to exchange the controller or the estimator without a redesign of the corresponding part. Furthermore, the properties of each module can be selected independently.

The major drawback of this procedure is the lack of a stability proof in the design step. Due to the modular design, it is in general difficult to proof stability for the complete closed-loop system.

3.2.3. Lyapunov Based Approaches

The first adaptive controllers where applied in the early 1960's mainly to flight control systems [29] in order to improve the overall performance. At this point, most of the approaches based on the so-called M.I.T. rule [107], which was not governed by a thorough stability analysis in the first place. However, after a crash with an X-15 aircraft [37], the acceptance towards adaptive control decreased. It was obvious that the adaptation loop with its stability properties where not understood well enough at that time.

However, at the same period, several people came up with the idea of using the stability theory of Lyapunov in order to obtain the adaptation law, see [91, 108, 122, 123]. In general, one could classify adaptive schemes as *direct* and *indirect* adaptation approaches. In this context, indirect means that the controller adapts the parameter of the plant, whereas for direct approaches the controller parameters are modified during runtime.

Remark 3.2. *The classification of direct and indirect adaptive control is not consistent in the literature. Some authors would classify estimator-based approaches as indirect, since the plant parameters are identified* [74]. *However, in this thesis, similar to other authors* [73], *we will assign the terminus indirect to the adaptation of the plant parameter in the controller itself instead of using a designated estimator.*

Most of the approaches specify a reference model that should represent the ideal closed-loop performance. Hence, these method is formerly known as model reference adaptive control (MRAC). In Figure 3.4, the typical block diagram of such an approach

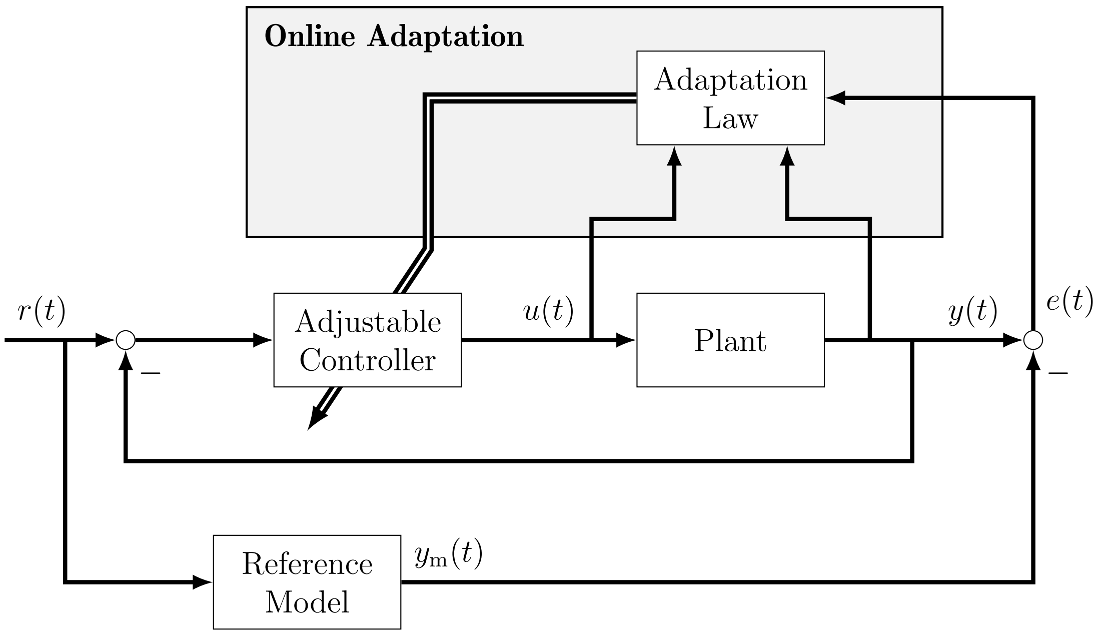


Figure 3.4.: Design Principle of a model reference adaptive control (MRAC) Approach

is depicted. We can see that the error $e(t)$ between the reference model and the plant is fed into the adaptation mechanism which modifies the control law.

In order to obtain the adaptation law, many designs rely on the so called *certainty equivalence principle*. The main idea is to assume a set of nominal parameters that allows a proof of the stability via the Lyapunov method, see Chapter 2. In a consecutive step, the assumption of known parameters is dropped in favor of an estimate of the actual parameters. The Lyapunov function for the nominal case is then extended by the estimation error. Finally, the adaptation law is obtained by eliminating the influence of the estimation error on the time derivative of the Lyapunov function. We will discuss this procedure in the following section with the help of a brief example.

3.3. Exemplary Design Procedure

As we have seen in the previous section, there are too many different approaches towards an adaptive controller design in order to present them all in this thesis. Instead we focus on one scheme that introduces the main ideas behind adaptive control and demonstrates the advantages and drawbacks of the principle.

Let us start with a linear plant of the form

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,\left(u(t) + \Delta(x(t))\right) \\ y(t) &= C\,x(t) \end{aligned} \tag{3.1}$$

with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$ where n defines the number of states. The initial state of (3.1) shall be defined by $x(0) = x_0$. Furthermore, the system (3.1) is affected by the term

$$\Delta : \mathbb{R}^n \to \mathbb{R} \tag{3.2}$$

acting as a matched uncertainty to the system. We may assume that (3.2) can be expressed by

$$\Delta(x) = \Theta^\top \omega(x(t)) \tag{3.3}$$

where $\Theta \in \mathbb{R}^p$ is an unknown but constant parameter vector and $\omega : \mathbb{R}^n \to \mathbb{R}^p$ a known function, sometimes also referred as *regressor*. We may also assume that the pair (A, B) from (3.1) is controllable and that the state $x(t)$ is available for the control algorithm.

The objective is to find a controller that is able to track a reference signal $r(t) \in \mathbb{R}$ despite the presence of the uncertainty $\Delta(t)$. As mentioned in Definition 3.1, an important feature of an adaptive controller is the evaluation of a performance index during run-time. Often, this can be represented by a dynamic system that covers the behavior of the closed loop, so it seems intuitive to specify a so-called *reference model*

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}} x_{\mathrm{m}}(t) + B_{\mathrm{m}}\, r(t) \tag{3.4}$$

with the matrices $A_{\mathrm{m}} \in \mathbb{R}^{n\times n}$ and $B_{\mathrm{m}} \in \mathbb{R}^{n\times n}$ and an initial value $x_{\mathrm{m}}(0) = x_{\mathrm{m},0}$, such that (3.4) describes the demanded tracking performance. Hence, the procedure that we are discussing here is also known as model reference adaptive control (MRAC). It shall be noticed that A_{m} has to be a Huwitz matrix, otherwise the state x_{m} may diverge.

Now we have to find a controller that forces the trajectories of the plant (3.1) to the ones represented by (3.4). In other words, the error $e(t) \in \mathbb{R}^n$ defined as

$$e(t) := x(t) - x_{\mathrm{m}}(t) \tag{3.5}$$

shall reach zero asymptotically. Therefore we will rely on the following structure

$$u(t) = u_{\mathrm{nom}}(t) + u_{\mathrm{adapt}}(t) \tag{3.6}$$

as control law. Hereby the term u_{nom} provides a nominal control that enforces (3.4) without the influence of the parametric uncertainty (3.3), whereas u_{adapt} is the adaptive part of the controller that should handle the uncertain part of the system (3.1). Hence, the controller design can be split in two major steps

1. Find a control law $u_{\mathrm{nom}}(t)$ which will ensure the the desired dynamics (3.4) for the closed loop while assuming perfectly known plant parameters.

2. Apply an adaptive control $u_{\mathrm{adapt}}(t)$ such that potential parameter deviations are compensated.

In general, this procedure is known as *certainty-equivalence* principle [73, p.2].

Proposition 3.3. *The controller*

$$u(t) = K_1\, x(t) + K_2\, r(t) - \hat{\Theta}^\top(t)\omega(x(t)) \tag{3.7}$$

with the adaptation law

$$\dot{\hat{\Theta}}(t) = \Gamma\, \omega(x(t))\, e^{\top}(t)\, P\, B, \quad \hat{\Theta}(0) = \hat{\Theta}_0 \tag{3.8}$$

with $\Gamma \in \mathbb{R}, \Gamma > 0$ *and* $P \in \mathbb{R}^{n \times n}, P = P^{\top} > 0$, *stabilizes the origin of* (3.1) *despite the unknown constant parameter* Θ.

Proof. As mentioned before, in the first step we may assume that the parameters Θ in (3.1) are perfectly known, i.e. $\hat{\Theta} = \Theta$. Using this assumption and applying the control law (3.7) to the system, we obtain

$$\dot{x}(t) = (A + B\, K_1)\; x(t) + B\, K_2\, r(t) \tag{3.9}$$

and with the substitutions

$$A_{\mathrm{m}} = A + B\, K_1, \quad B_{\mathrm{m}} = B\, K_2 \tag{3.10}$$

the closed-loop dynamics read as

$$\dot{x}(t) = A_{\mathrm{m}}\, x(t) + B_{\mathrm{m}}\, r(t) \tag{3.11}$$

that should cover the desired closed-loop performance specification. Since the pair (A, B) is controllable according to the assumptions, we can be sure that there exits a solution for K_1, K_2 and A_{m} is Hurwitz. Therefore it is possible to find a matrix $P > 0$ for any $Q > 0$ such that the Lyapunov equation

$$0 = A_{\mathrm{m}}^{\top} P + P\, A_{\mathrm{m}} + Q \tag{3.12}$$

holds for any P according to the proposition. If we combine (3.11) and (3.4), we can express the error dynamics by

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}}\, e(t) \tag{3.13}$$

which allows us to conclude that the error $e(t)$ asymptotically converges to the origin since A_{m} is Hurwitz.

Now, the question may arise what will happen if the parameter vector Θ is not known.

The closed loop dynamics will then read as

$$\dot{x}(t) = A_\mathrm{m}\, x(t) + B_\mathrm{m} r(t) - B\, \tilde{\Theta}^\top(t)\omega(x(t)) \tag{3.14}$$

where the parameter vector $\tilde{\Theta}$ is defined as

$$\tilde{\Theta}(t) := \hat{\Theta}(t) - \Theta. \tag{3.15}$$

Consequently the new error dynamics is given by

$$\dot{e}(t) = A_\mathrm{m}\, e(t) - B\, \tilde{\Theta}^\top(t)\omega(x(t)) \tag{3.16}$$

which allows us to conclude that the error may not converge to zero even if A_m is Hurwitz since the term $\tilde{\Theta}^\top(t)\omega(x(t))$ may not vanish. Now the adaptation part of the controller comes into play. Therefore we have to obtain a suitable way in order to modify the estimated parameters $\hat{\Theta}$ during the run-time of the controller.

To solve this task, we will start with a Lyapunov function for the nominal system. For (3.13) a convenient candidate is

$$V_\mathrm{nom}(t) = e^\top(t)\, P\, e(t) \tag{3.17}$$

with P from Proposition 3.3. If we calculate the time derivative of the Lyapunov function (3.17) along the trajectories of (3.13), we end up with

$$\dot{V}_\mathrm{nom}(t) = e^\top(t)\left(A_\mathrm{m}^\top P + P\, A_\mathrm{m}\right) e(t) = -e^\top(t)\, Q\, e(t) < 0 \tag{3.18}$$

which is strictly negative for all error values $e(t) \neq 0$. As mentioned before, we may conclude that the error will converge to the origin asymptotically, see Theorem 2.19.

In order to obtain the adaptation rule, we extend the nominal Lyapunov function (3.17) by a quadratic term of the estimation error

$$V(t) = V_\mathrm{nom}(t) + \frac{1}{\Gamma}\tilde{\Theta}^\top(t)\, \tilde{\Theta}(t) \tag{3.19}$$

where $\Gamma > 0$ denotes a parameter that allows us to tune the adaptation speed. Now, calculating the time derivative of (3.19) along the trajectories of (3.16) gives us

$$\dot{V}(t) = e^\top(t)\left(A_\mathrm{m}^\top P + P\, A_\mathrm{m}\right) e(t) - 2\, \tilde{\Theta}^\top\omega(x(t))e^\top(t)P\, B + \frac{2}{\Gamma}\tilde{\Theta}^\top(t)\dot{\tilde{\Theta}}(t) \tag{3.20}$$

that can be written as

$$\dot{V}(t) = -e^{\top}(t) Q\, e(t) + 2\tilde{\Theta}\left(\frac{1}{\Gamma}\dot{\hat{\Theta}} - \omega(x(t)) e^{\top}(t) P\, B\right) \tag{3.21}$$

since $\dot{\tilde{\Theta}} = \dot{\hat{\Theta}}$ due to the assumption of an unknown but constant parameter vector in (3.1). If we substitute $\dot{\hat{\Theta}}(t)$ by the expression given in (3.8), we can observe that all terms in (3.21) depending on the parameter estimation error $\tilde{\Theta}$ will vanish. Thus, the time derivative of the extended Lyapunov function is similar to (3.18). Therefore we may conclude that (3.21) is negative semi-definite in $e(t)$ and $\tilde{\Theta}$. By applying Theorem 2.19 we can argue that the state error, except for $\hat{\Theta}$, will converge asymptotically to zero. □

3.4. Representative Example

In the following example we shall have a look at some of the key properties of an adaptive controller design. A simple but yet illustrative example is the mechanical

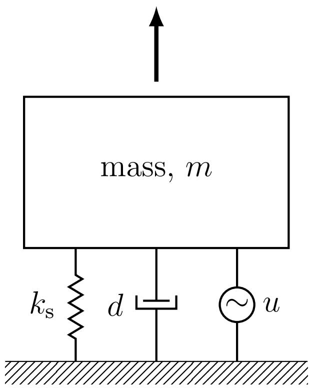


Figure 3.5.: Schematic of a Spring-Mass-Damper system

system shown in Figure 3.5 consisting of a mass m, a spring with the spring rate k_s, a damper with the damping coefficient d and an actuator that induces the force u. By the straightforward analysis of the participating forces, we end up with the following differential equation

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{1}{m}\left(-k_\mathrm{s}\, x_1(t) - d\, x_2(t) + u\right) \end{aligned} \tag{3.22}$$

where x_1 denotes the position and x_2 the velocity of the mass m respectively. The initial values for the system (3.22) are given by $x_1(0) = x_{1,0}$ and $x_2(0) = x_{2,0}$. In the following we shall assume that the parameters m, k_s and d are strictly positive. However, in this example we have a look at the case when k_s and d are unknown and piecewise constant.

This case study may represent a far more complex technical setup like a vehicle suspension system. Hence it might reasonable that the spring rate or the damping coefficient change over the time. This can be caused by various reasons like wear or other influences like temperature variations.

The objective in designing a controller for such a setup is to achieve robustness against those parameter variations. As mentioned in the introduction of this chapter, the classical approach would be to design a robust controller that stabilizes the closed loop system for every possible set of parameters. According to the Definition 3.1, an adaptive controller will involve some kind of performance measurement. If we think about the suspension system, it is easy to understand that in many technical applications, the performance of a closed loop system is similarly important as the stability.

In this particular example we consider a spring-mass-damper system with the following nominal parameters

$$m = 1, \quad d = 3, \quad , k_\mathrm{s} = 5. \tag{3.23}$$

Consequently, the eigenvalues $\lambda_{1,2}$ from A in (3.1) can be computed to be

$$\lambda_{1,2} = -1.5 \pm 1.66. \tag{3.24}$$

At this point we shall notice that a in many applications an oscillating output caused by a complex conjugated pair of eigenvalues is highly undesirable. One may think of a milling machine, where an overshoot of the desired position will damage the work-piece. On the other hand, if we think about the already mentioned vehicle suspension, a weakly damped system may result poor cornering performance.

Controller Design

Due to those requirements, we consider two controller designs. The first control approach is a robust linear feedback which is extended in the second step by an adaptive part.

Linear Controller The control law for the linear controller is given by

$$u_{\text{lin}} = k_1\, x(t) + k_2\, r(t) \tag{3.25}$$

where $x(t)$ represents the stave vector of (3.22) and $r(t)$ is the reference value that should be tracked. Based on the requirements and nominal values (3.23) for the system parameters, we choose the controller gains as $k_1 = (-1, 1)$ and $k_2 = 2$. This selection gives us a nominal closed-loop system similar to (3.4) with

$$A_{\text{m}} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}, \quad B_{\text{m}} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{3.26}$$

which has a multiple eigenvalue at -2 and a steady state gain of 1.

Adaptive Controller As mentioned before, in a next step we design an adaptive version of the controller from the previous paragraph. Hereby we will rely on the same reference model given by the matrices in (3.26) such that a performance comparison of both controllers is possible.

In the introduction of this section it is stated that some parameters of the plant may be uncertain. For the spring-mass-damper system this shall be the damping coefficient d and the spring rate k_{s}. However, for the linear controller we assume some nominal values of these parameters in order to obtain the robust controller gains. From (3.7) we can see that the adaptive control approach will consists of the linear controller and a term that shall inhibit the influence of the variations of the parameters on the closed loop system. It is now possible to tackle this problem from two sides: We can either assume that the parameters d and k_{s} are zero and calculate k_1 and k_2 accordingly. Or we can design the linear control part for the nominal values of the plant parameters as it has been done in the previous paragraph. Hence, an entry in the vector Θ will represent the deviation from the nominal plant parameter instead of its actual value.

In both cases the regressor variable is given by

$$\omega(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \tag{3.27}$$

where x_1 is proportional to the force of the spring and x_2 to the one introduced by the damper. For the calculation of the adaptation law, we use the following symmetric matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix} \tag{3.28}$$

in the derivative of the Lyapunov function (3.18). By using the Lyapunov equation (3.12), we can compute the symmetric matrix in (3.17) to

$$P = \begin{pmatrix} \frac{45}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{41}{32} \end{pmatrix} \tag{3.29}$$

that is needed in the adaptation law (3.8). Additionally, the gain of the adaptive part is selected to $\Gamma = 20$. Finally, we end up with

$$\dot{\hat{\Theta}}(t) = \omega(x(t))\, e^{\top}(t) \begin{pmatrix} \frac{5}{2} \\ \frac{205}{8} \end{pmatrix} \tag{3.30}$$

as differential equation for the estimated parameter. The initial value for $\hat{\Theta}$ is set to zero since we assume that the plant parameters are perfectly known in the beginning.

Results

Now that we have designed two controllers for the spring-mass-damper system, we shall compare both approaches. The objective is to track a reference position $r(t)$ that acts as input to the reference model (3.4). In Figure 3.6 we can locate the given reference signal as solid orange line. To evaluate the performance to a step response, the reference is given as a pulse signal between 0 and 1 where the pulse length is equally distributed. Furthermore, the plant parameters shall change during the simulation from 3 for the damping coefficient at 20 seconds to a value of 0.2 and the spring rate will change from 5 to 6 at 100 seconds.

According to previous analysis, both controllers should be able to stabilize the system with the changed set of parameters. However, a stability analysis in general does not provide any information about the performance of the closed loop system.

The first state of the reference model, i.e. the desired position of the mass m, is shown in the upper graph in Figure 3.6 as a solid green line. On the other hand, the mass

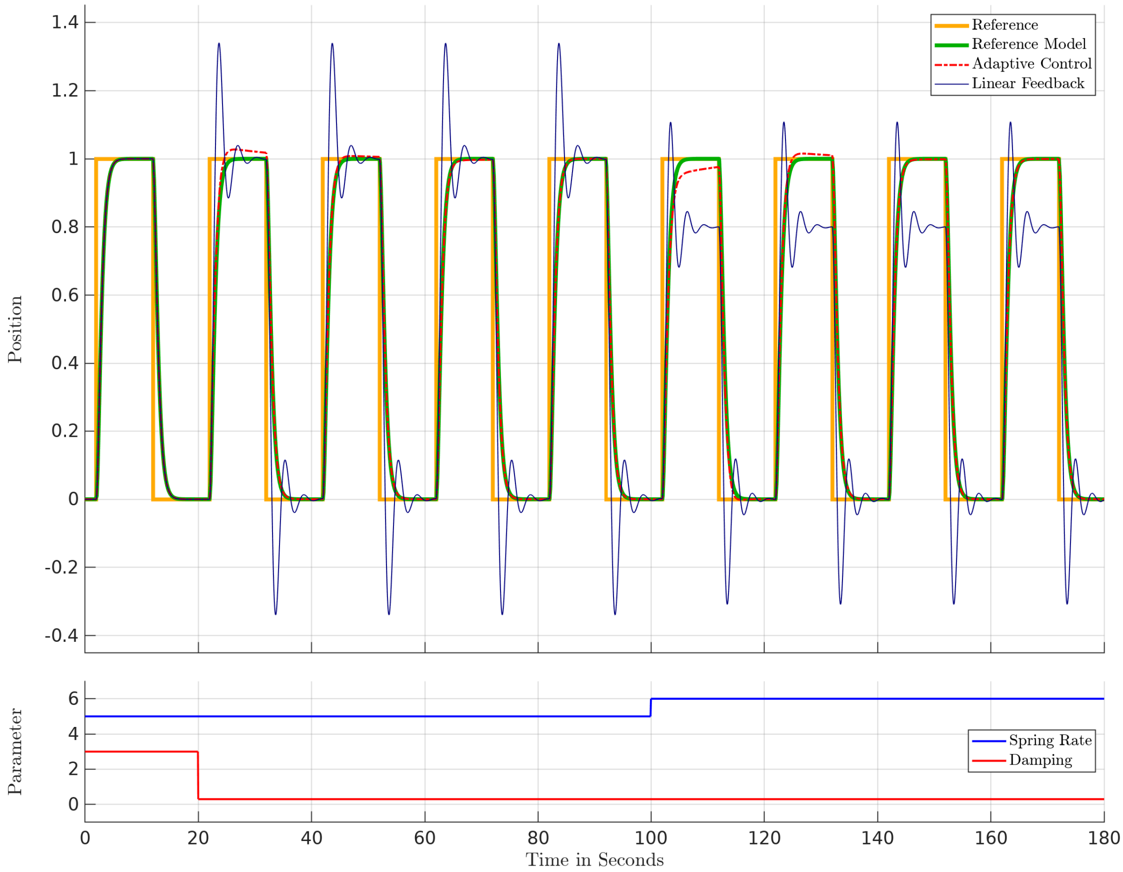


Figure 3.6.: Tracking Performance Comparison of both implemented Controllers

position using the linear feedback is depicted as a thin solid blue line and the step response of the adaptive controller is given by the a dash-dotted red graph.

From the timeseries given by Figure 3.6 we can observe many interesting characteristics of both controllers. As expected, during the first 20 seconds, both control approaches behave exactly the same since the plant parameters are set to their nominal values which makes an adaptation not necessary. From 20 seconds ongoing, we notice that the output of both control loops starts to differ. This is caused by the significant change of the damping parameter of the plant from 3 to 0.2. A short stability analysis reveals that the closed loop with the linear feedback is still stable for this new parameter value. Unfortunately, the tracking performance of the first approach is reduced substantially compared to the first time interval. This observation is underlined by the graphs shown in Figure 3.7 that contains the state errors $e_1(t)$ and $e_2(t)$ of the linear feedback and the adaptive controller with a similar line style as in Figure 3.6. One can clearly see that

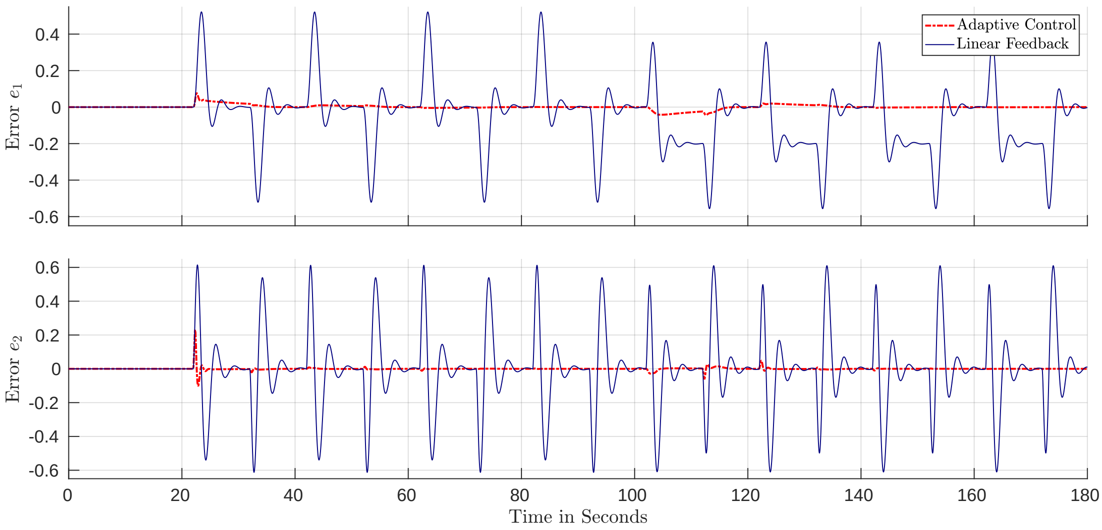


Figure 3.7.: State Error Comparison

the tracking performance is improved drastically by the usage of an adaptive controller compared to the linear feedback since the overshoot is significantly lower. After the first two reference steps with the changed plant parameters, the error in the position tracking is almost completely gone for the adaptive approach while the performance of the conventional linear feedback does not change.

These observations recur when the spring rate of the plant changes at 100 seconds as it can be seen in Figure 3.6. After a short adaptation period, the MRAC algorithms manages to preserve the desired tracking performance. For the linear feedback however, the properties become even worse compared to the previous time interval. Due to the changed spring rate, the conventional controller is not even able to achieve a steady state tracking error close to zeros. As shown in Figure 3.7, the linear feedback shows that same overshoot behavior in addition with a discrepancy of about 0.2 between the actual position and the reference value.

At this point we may conclude that the MRAC approach shows a significant better performance compared to the conventional linear design. While both controllers stabilize the system, the adaptive controller is able to guarantee a predefined tracking performance which cannot be achieved with the classic approach. Adding an integral part to the linear controller would resolve the steady state error but the poor transient behavior would remain the same.

Another important point when comparing different control strategies is the control signal itself. Figure 3.8 contains the output of both implemented controllers for the

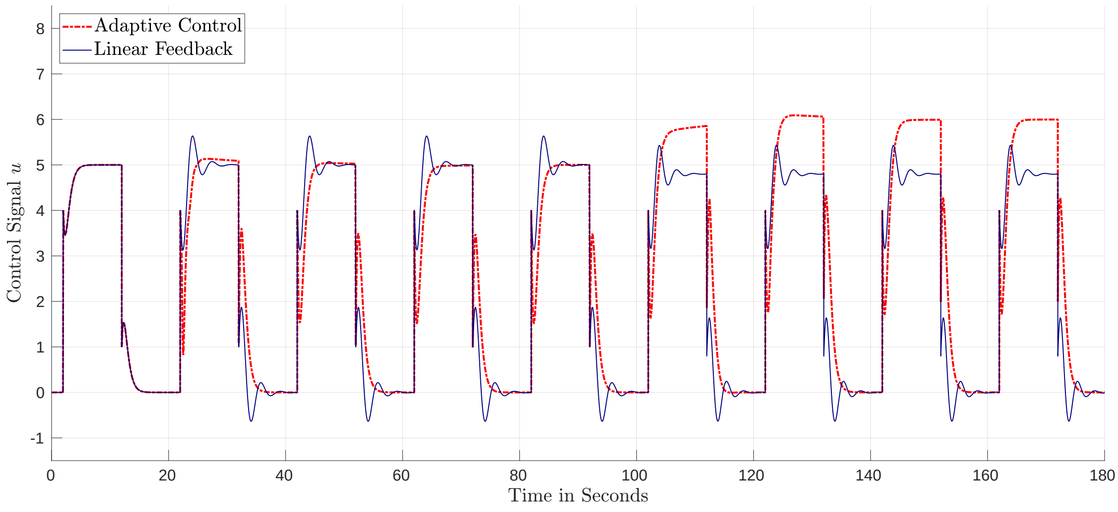


Figure 3.8.: Control Signal Comparison of the Linear Feedback and the Adaptive Controller

given example. In the first section, when the parameters of the plant match the assumed values, both controllers generate obviously the same output signal. However, as soon as the parameter of the plant obtain a different value, we can observe that the adaptive controller starts to behave differently. As we have discussed before, the performance of the adaptive controller is significantly better compared to its linear counterpart. Based on a first intuitive guess one would expect that this comes at the price of higher control effort. A look at Figure 3.8 reveals that this assumption cannot be confirmed in the given example. The signal generated by the adaptive controller contains less over- or undershoot in the changes of the reference signal while the position error in Figure 3.7 is simultaneously smaller as well. This can be explained with a much more purposeful appliance of the control action that compensates the uncertainties almost perfectly. It should be noticed that the higher amplitude of the adaptive controller after about 100 seconds is required to maintain the zero steady state error due to the larger spring rate.

Beside the main objective, the tracking of a reference signal, we shall discuss briefly another interesting feature of the MRAC design, namely the parameter estimation. The time response of the estimated values of the parameters are shown in Figure 3.9 for the example system (3.22) with the adaptation law (3.30). It should be noticed that the value in Figure 3.9 represents the deviation of parameters to their nominal values. The spring rate is shown in red and the damping coefficient in blue. For both parameters,

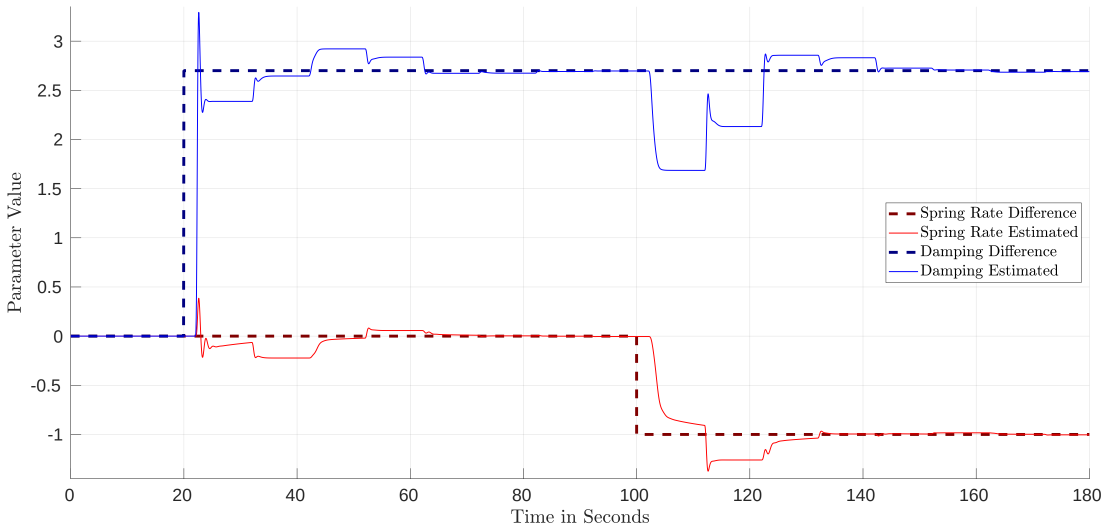


Figure 3.9.: Evolution of the estimated parameters compared to their actual value

the estimated deviation is represented by a thin solid line, whereas the actual deviation is marked by a dashed line.

At 20 seconds, the damping coefficient is changed to a significant lower value. Contrary to the expectation, it takes a few seconds for the adaptation to take place. We can explain this behavior by having a look at the reference signal $r(t)$ in Figure 3.6 that is changing at 24 seconds while being constant before. In a steady state, the damping coefficient has no influence on the system states such that according to (3.30) no adaptation is required. However, in the following 60 seconds, the parameter deviation converges close to the actual value.

A similar procedure can be observed for the spring rate change at 100 seconds. Due to the fact that the position is zero right at the time of the parameter change, there is no influence on the system dynamics. Again, the first reference step causes the adaptation of the second parameter.

3.5. Parameter Convergence

We have seen in the previous example, that the estimated parameters converges close to the actual values. A question that often arises is, whether we can ensure that the estimation error vanishes. For the estimator based adaptive controllers briefly presented in Subsection 3.2.2, it is common to proof the parameter convergence. However, it is not always possible to proof the stability of the closed loop including estimator and controller.

For the Lyapunov-based approaches, the situation is a little bit different. The convergence of the states to the origin is commonly ensured by the design procedure itself. However, in general the parameter convergence cannot be guaranteed as it can be seen from the derivation of the Lyapunov function (3.21).

By using the following state transformation

$$z(t) = \begin{pmatrix} P^{\frac{1}{2}}\, e(t) \\ \Gamma^{-\frac{1}{2}} \Theta(t) \end{pmatrix} \tag{3.31}$$

with $P = P^\top > 0$ from (3.17), we can rewrite the error dynamics (3.16) of the adaptive control loop and the estimation error (3.8) to

$$\dot{z}(t) = J(t)\, z(t) \tag{3.32}$$

where the time variant matrix $J(t)$ is given by

$$J(t) = \begin{pmatrix} P^{\frac{1}{2}} A_{\mathrm{m}} P^{-\frac{1}{2}} & -\Gamma^{\frac{1}{2}} P^{\frac{1}{2}} B \omega^\top(x(t)) \\ \Gamma^{\frac{1}{2}} \omega^\top(x(t)) B^\top P^{\frac{1}{2}} & 0 \end{pmatrix}. \tag{3.33}$$

Now, if we can show that $z(t)$ from (3.32) converges asymptotically to zero, we will show simultaneously that the parameter estimation error vanishes. This problem was studied in many contributions, see e.g. [7, 8, 101]. It turned out that in order to proof the asymptotic convergence, the regressor signal $\omega(x(t))$ has to satisfy the persistent excitation condition.

Definition 3.4 (persistent excitation (PE))**.** *Consider the time interval $\mathcal{I} := [t_0, \infty]$ and a signal $\zeta(t) : \mathcal{I} \to \mathrm{R}^{n\times p}$. The signal is called persistently exciting on a time interval δ, iff there exists $\alpha, \delta > 0$, such that the following inequality*

$$\int_t^{t+\delta} \zeta(\tau)^\top \zeta(\tau)\, \mathrm{d}\tau \geq \alpha\, \mathbb{I}_p \tag{3.34}$$

holds for each $t \in \mathcal{I}$. The interval T and α must be independent from t.

As it has been shown in [99], the trajectories of (3.32) will converge if the regressor satisfies the persistent excitation condition from Definition 3.4 and $\omega(x(t))$ is bounded.

Let us now recall the example from the previous section and the relevance of those findings for this actual problem. The PE condition from Definition 3.4 basically states that the regressor signal $\omega(x(t))$ has to be "rich" enough to ensure the parameter convergence. But what does this actually mean? If we have a look at the time series in Figure 3.9, we can see that the parameter adjustment happens mainly in the time after the steps of the reference signal. For the time between the steps, the estimation almost does not take place. If we consider for example the damping coefficient d from (3.22), we know from the regressor variable (3.27), that the damping force is proportional to the velocity x_2. However, whenever the system is at an equilibrium point, the damping has no effect on the dynamics. Hence no control input is required to compensate the disturbance since it is zero at that point. Consequently, the estimation error of the damping coefficient is irrelevant at that time and makes no adjustment necessary.

This observation is crucial for nearly all adaptive control systems that are designed using the certainty equivalence principle. Here, the main goal is not to identify the actual plant parameters, instead the controller should find a set of parameters that stabilize the closed loop system. As stated before, parameter convergence would require the PE condition to be satisfied which, in general, cannot be ensured since the regressor signal might depend on an external input like the reference for instance. Nevertheless, there are some studies that deal with the absence of the PE condition, see e.g. [31, 102]

4. Sliding Mode Control

In the following chapter a completely different approach of designing a control system, compared to the previous chapter, is presented. Here the focus shall lie on the use of discontinuous control and its advantages and disadvantages compared to conventional and adaptive approaches. We will start with the basic ideas behind sliding-mode control and present some of the latest results in the field of higher-order sliding-mode control.

The concept was first introduced by Utkin, see e.g. [140]. The main idea is to use a discontinuous control signal in order to compensate for external disturbances and/or model uncertainties. It turned out that it is a very powerful method to design a robust controller that is extremely robust and furthermore even able to stabilize the system in finite time. To this point, sliding-mode control has been used in miscellaneous fields of engineering:

Vehicle dynamics require robust strategies for all kinds of problems. Sliding-mode control can be used for on-line estimation of vehicle [137] and road parameters [136], drive train applications [9, 52] and safety critical control [3, 6, 151].

Electric Machines & Converters profit from the switching signal used in sliding-mode control (SMC). While many applications suffer from undesired effects like chattering (see Section 4.3), electrical power systems use methods like pulse-width modulation (PWM) to convert a continuous signal to a switched one in order to increase the efficiency of the electrical circuit. Hence it seems natural to use a switching controller like SMC and apply it directly to various kinds of power converters [129, 130] or electric motors [114] and benefit from the robustness properties. Another advantage is demonstrated in [145] where a sliding-mode approach is efficiently used to reduce the electromagnetic emission of a power converter.

Pneumatic Actuators are highly nonlinear and uncertain systems. In authors of [30, 126] reveal that SMC can handle such complex systems while leaving only a few tuning parameters to the application engineer.

Bio Engineering is a topic where the system dynamics are often partially or even completely unknown. As shown in [51] it turns out that SMC can be used where the relative-degree between input and output is highly uncertain. The authors of [142] use HOSM techniques in order to robustly estimate states of a bioreactor in the presence of measurement noise.

just to name only a few examples.

4.1. General Concept

Sliding-mode control can be seen as part of the more general concept of variable structure control (VSC). The first approaches were presented in the late fifties, see e.g. [41], in order to find a method that is capable handling parameter uncertainties or disturbances. Hence, the idea is to use a controller that changes its structure with respect to certain rules in order to improve the robustness of the closed loop. Figure 4.1

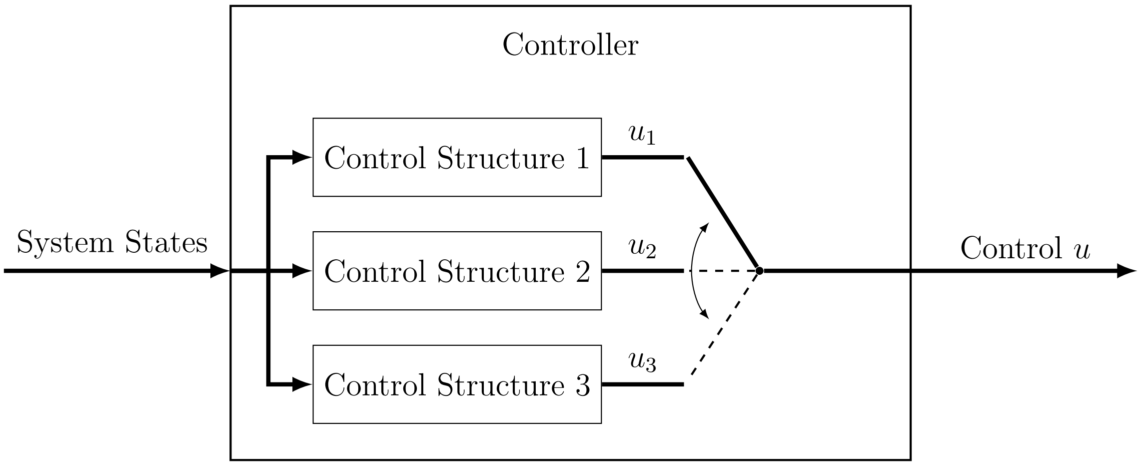


Figure 4.1.: Conceptional Composition of a VSC Controller

shows the principle of a VSC approach with 3 different control structures. The controller alternates between those different control laws based on information like system states or other external signals. It is expected that the usage of different control strategies may improve the performance or robustness of the closed loop in varying conditions.

Sliding-mode control, however, is a specific kind of VSC approach. We consider the following nonlinear dynamical system

$$\dot{x}(t) = f(t, x) + g(t, x)\, u + d(t) \tag{4.1}$$

with the state $x \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, the scalar input $u(t) \in \mathbb{R}$, a disturbance term $d(t) : \mathrm{R} \to \mathbb{R}^n$ and an inital state $x(t_0) = x_0$. In general, the design of a sliding mode controller for such a system can be split up in two major tasks:

1. The choice of an appropriate manifold $s(x) = 0$ with a relative degree of one with respect to the input u that enforces the desired reduced stable dynamics.

2. Find a suitable discontinuous control law that is able to drive the system to the designed manifold beside the presence of disturbances or model uncertainties.

It seems to be a natural choice to use the defined manifold $s(x) = 0$ for the switching of the control structure:

$$u(x) = \begin{cases} u^+(x), & s(x) > 0 \\ u^-(x), & s(x) < 0 \end{cases} \tag{4.2}$$

with two different control structures $u^-(x)$ and $u^+(x)$ in the sense of VSC. Applying the controller (4.2) to the system (4.1), yields

$$\dot{x} = \begin{cases} f(t,x) + g(t,x)u^+ + d(t) =: f^+(t,x), & s(x) > 0 \\ f(t,x) + g(t,x)u^- + d(t) =: f^-(t,x), & s(x) < 0 \end{cases} \tag{4.3}$$

where $f^+(t,x)$ and $f^-(t,x)$ should be understood in the same way as in (2.5). The solution of system (4.3) shall be defined in the sense of Filippov as explained in Chapter 2.

Now the question emerges how the control law should be designed in order to enforce the sliding mode described in Section 2.2. Therefore we calculate the time derivative of the function $s(x)$ that defines the sliding manifold. We obtain

$$\dot{s} = \frac{\partial s}{\partial x}\,\dot{x} = \frac{\partial s}{\partial x} f(t,x) + \frac{\partial s}{\partial x} g(t,x)\,u + \frac{\partial s}{\partial x} d(t) \tag{4.4}$$

using the system definition from (4.1). As mentioned before, we require a relative degree of one of the sliding manifold with respect to the input u. From (4.4) we can see that we have to examine

$$\frac{\partial s}{\partial x} g(t,x) \neq 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n \tag{4.5}$$

to ensure a well-defined relative degree. The uncertainty $d(t)$ has to fulfill the so called matching condition:

Definition 4.1 (Matching condition [124]). *The term $d(t)$ in* (4.1) *is called matched uncertainty if there exits a $\zeta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ that satisfies*

$$d(t) = g(t,x)\,\zeta(t,x) \tag{4.6}$$

and is uniformly radially bounded, i.e.

$$||\zeta(t,x)|| \leq \bar{\zeta} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n \tag{4.7}$$

with a constant $\bar{\zeta} \in \mathbb{R}$.

Loosely speaking, this condition guarantees that the disturbance acts in the same channel as the control input and hence, can be compensated by the control signal.

Proposition 4.2. *The control law*

$$u = -\left(\frac{\partial s}{\partial x} g(t,x)\right)^{-1} \frac{\partial s}{\partial x} f(t,x) - \left(\frac{\partial s}{\partial x} g(t,x)\right)^{-1} K \operatorname{sign}(s(x)) \tag{4.8}$$

stabilizes the equilibrium point $x = 0$ *of the system* (4.1) *asymptotically if the following conditions are met*

- *The origin* $x = 0$ *of the zero dynamics of* $s(x)$ *is asymptotically stable with respect to* u.
- *The relative degree of* $s(x)$ *with respect to* u *is one and well defined, see* (4.5).
- *The disturbance* $d(t)$ *meets the matching condition from Definition 4.1.*
- *The Gain* K *is selected such that*

  $$K > \sup_{t,x} \left\| \frac{\partial s}{\partial x} d(t) \right\| \tag{4.9}$$

  is ensured.

Proof. By applying the control law (4.8) to the sliding dynamics (4.4) we obtain

$$\dot{s} = -K(t,x) \operatorname{sign}(s(x)) + \frac{\partial s}{\partial x} d(t) \tag{4.10}$$

where the terms depending on the vector fields f and g from the primary system description (4.1) are canceled. We may use the following Lyapunov function candidate

$$V(s) = \frac{1}{2} s^2 \tag{4.11}$$

to demonstrate the attractiveness of the sliding manifold. Calculating the time derivative of (4.11) yields

$$\dot{V} = \dot{s}\, s. \tag{4.12}$$

To show that $\dot{V}$ is negative definite, we have to ensure that the following inequality

$$-K\,|s(x)| + \frac{\partial s}{\partial x} d(t)\, s(x) < 0 \tag{4.13}$$

that is obtained by inserting (4.10) into (4.12). Due to the assumption (4.9) we can conclude that the left side of (4.13) is negative for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. This shows that the sliding surface is attractive and trajectories starting outside of it will converge to the domain where $s(x) = 0$. Once the surface is reached, due to the assumption of the stable zero dynamics, we can conclude that the system states converge asymptotically to zero. □

In fact, it is sufficient to show that the sliding surface is (globally) attractive which is often shown by the

Proposition 4.3 (Reaching condition [36]). *The sliding surface $s(x) = 0$ is locally attractive if*

$$\lim_{s \to +0} \dot{s} < 0 \quad \text{and} \lim_{s \to -0} \dot{s} > 0 \tag{4.14}$$

in some domain $\Omega \in \mathbb{R}^n$.

The statement (4.14) can be rewritten as

$$\dot{s}\, s < 0 \tag{4.15}$$

in order to show the attractiveness of the sliding surface.

4.1.1. Illustrative Example

To illustrate the results from above, we shall have a look at a simple example. We consider the second order dynamical system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u + \Delta(t)\end{aligned} \tag{4.16}$$

with the states $x_1, x_2 \in \mathbb{R}$, the control input $u \in \mathbb{R}$ and a disturbance $\Delta(t) : \mathbb{R} \to \mathbb{R}$. From the viewpoint of an engineer this may represent the motion of a single-mass system disturbed by an unknown external force $\Delta(t)$. The control objective would be to steer the state variables to zero under the assumption that the external force is absolutely bounded, i.e.

$$|\Delta(t)| \leq D, \forall t \in \mathbb{R} \tag{4.17}$$

with constant $D > 0$. The controller is designed as described in the previous section with the sliding surface as a linear combination of the states

$$s = c\,x_1 + x_2 \tag{4.18}$$

with $c = 1$. This selection results in the control law

$$u = -c\,x_2 - K\text{sign}\,(s) \tag{4.19}$$

where K can be selected as a constant since $\frac{\partial s}{\partial x}$ gives a constant value as well. Together with the assumption (4.17) we can choose a constant K such that (4.9) is satisfied for any state values and at any time.

System without Disturbance

In the first place we have a look at the system without disturbance, i.e. $\Delta \equiv 0$. The simulation shall start at the initial point $(x_1, x_2) = (1, 1.5)$. Figure 4.2 displays the evolution of the system states x_1, x_2 as well as the sliding variable s calculated using (4.18). We can observe that s converges linearly in time to the sliding surface, i.e. $s = 0$, in about 1.5s. This segment is called *Reaching Phase* and denotes the time interval when s is nonzero. Once the sliding surface is reached, the so-called *Sliding*

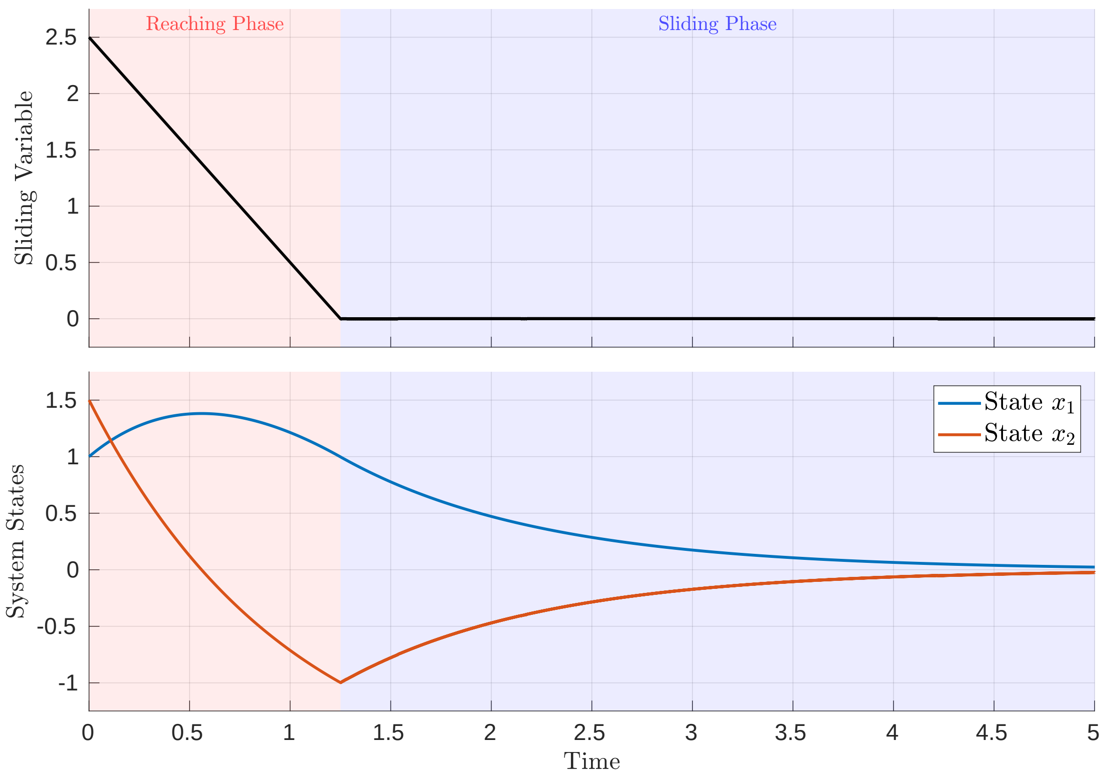


Figure 4.2.: Sliding Mode Control of a 2nd Order System without Disturbance

Phase starts. The origin of this nomenclature is better understood by taking a look at Figure 4.3 which depicts the phase portrait of the solution for the given initial point. In the graph, the gray solid line denotes the sliding manifold, i.e. where $s = 0$. The blue line represents the trajectory of the system system without disturbance. We can observe that those trajectory reaches the manifold and then "slides" along the manifold to the origin. Along this line the sliding variable is zero, hence we may rewrite $s(x) = 0$ to

$$x_2 = -c\,x_1 \tag{4.20}$$

where we can replace x_2 with the first line of (4.16). which yields

$$\dot{x}_1 = -c\,x_1 \tag{4.21}$$

that represents the dynamics of the state x_1 on the sliding surface. The reduced system (4.21) is a first order differential equation, whereas the original system (4.16) is

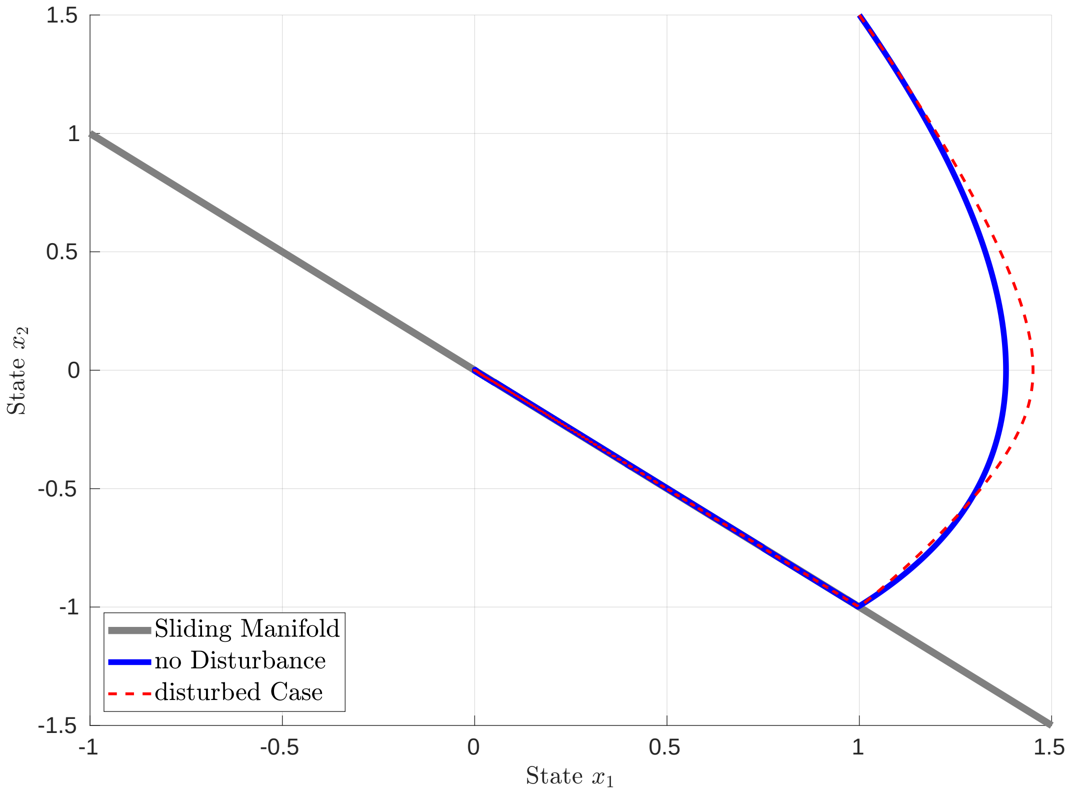


Figure 4.3.: Phase Portrait of a first-order sliding-mode Control System

of order two. This however, means that on the sliding surface we are able to decrease the order of the closed loop system by one compared to the open loop case.

Furthermore the sliding surface can be reached in finite time, which will be demonstrated by the following statement.

Proposition 4.4 (Finite Reaching Time)**.** *System* (4.16) *with the controller* (4.19) *converges in finite time to the sliding surface. The convergence time* t_r *can be calculated by*

$$t_\mathrm{r} = \frac{|s(0)|}{\alpha} \tag{4.22}$$

with $s(0)$ *being the value of* (4.18) *at time* $t = 0$.

Proof. Consider the function V defined as

$$V(s) = s^2 \tag{4.23}$$

with s being the sliding variable (4.18). Calculating the time derivative of (4.23) yields

$$\dot{V} = \frac{\partial V}{\partial s}\dot{s} = 2\,\dot{s}\,s = -2\,K\;|s|\,. \tag{4.24}$$

If we compare (4.24) with (4.23) we conclude that

$$\dot{V} = -2\,K\,V^{\frac{1}{2}} \tag{4.25}$$

holds $\forall s \in \mathbb{R}$. By separation of variables in (4.25) we obtain

$$\int\limits_{V(0)}^{V(t)} W^{-\frac{1}{2}}\;\mathrm{d}W = \int\limits_{0}^{t} -2\,K\;\mathrm{d}\tau \tag{4.26}$$

and integration on both sides yields

$$2V^{\frac{1}{2}}(t) - 2V^{\frac{1}{2}}(0) = -2\,K\,t \tag{4.27}$$

At the reaching time t_r we know that $V(t_\mathrm{r}) = 0$ since $s(t_\mathrm{r}) = 0$. Applying this to (4.27) and solving for t_r, we get

$$t_\mathrm{r} = \frac{V^{\frac{1}{2}}(0)}{K} \tag{4.28}$$

and indeed by replacing $V(0)$ with its definition (4.23), we get (4.22) from the proposition. □

For the given initial value for $x(0)$ we calculate a reaching time of $t_\mathrm{r} = 1.25\,\mathrm{s}$ that coincide with the simulation results in Figure 4.2. It should be mentioned that in most cases, the actual reaching time is less than the calculated value from (4.28) since this is the upper bound.

Influence of an External Disturbance

Now we shall investigate the case when the disturbance is nonzero in order to demonstrate the effectiveness of the SMC approach. Therefore we select the disturbance term as

$$\Delta(t) = \sin(t) \tag{4.29}$$

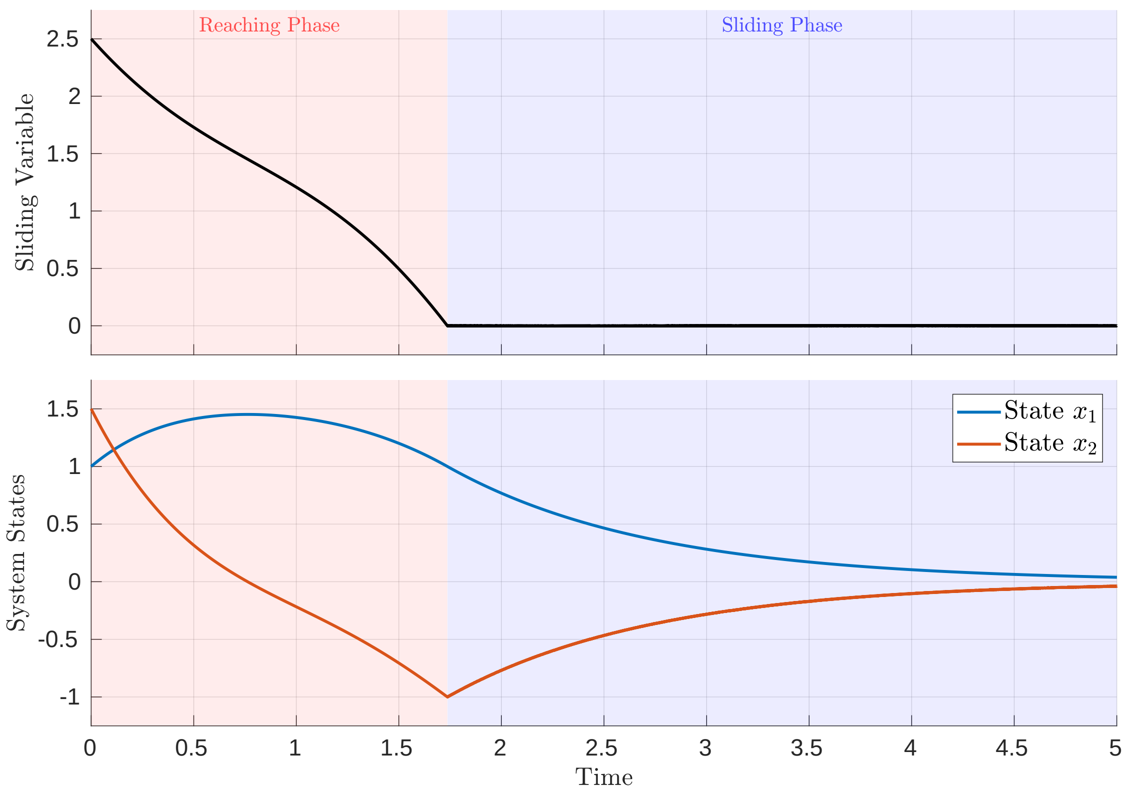


Figure 4.4.: Disturbed 2$^{\text{nd}}$ Order System with Sliding Mode Control

with an upper bound $D = 1$ such that (4.17) is satisfied. The results of the simulation in the new setting are shown in Figure 4.4. Comparing the graphs in Figure 4.4 with the results from the undisturbed case in Figure 4.2, we can see that there is hardly no difference. The system converges to the sliding manifold in a slightly different manner to the sliding manifold witch is caused by the disturbance $\Delta(t)$. However, once the sliding manifold is reached, the trajectories during the sliding phase are identical to the undisturbed case. This observation can be justified by the analysis of the reduced dynamics from the previous paragraph. Since the gain K is selected such that (4.9) is fulfilled, we can ensure the the reaching condition is met. Hence, the system will converge to the sliding manifold where $s(x) = 0$ which guarantees the reduced dynamics given in (4.21).

As mentioned before, the main difference lies in the reaching phase and therefore it seems to be obvious that the estimation of the reaching time may be different.

Proposition 4.5 (Finite Reaching Time under Disturbance)**.** *System* (4.16) *with the* *controller* (4.19) *converges in finite time to the sliding surface despite the influence of the bounded disturbance* $\Delta(t)$ *The convergence time* t_{r} *is bounded as*

$$t_{\mathrm{r}} \leq \frac{|s(0)|}{2(K-D)} \tag{4.30}$$

with $s(0)$ *being the value of* (4.18) *at time* $t = 0$.

Proof. For the proof we shall consider the same function V as given in (4.23). For the disturbed case, the time derivative yields

$$\dot{V}(t) = 2\dot{s}\,s = -2\,K\,|s| + 2\,\Delta(t)\,s \tag{4.31}$$

which can be estimated as

$$\dot{V}(t) \leq -(K-D)\,|s| \tag{4.32}$$

using the assumption (4.17) on the disturbance term $\Delta(t)$. It can be clearly seen that the reaching condition is satisfied if $K > D$. Moreover the statement

$$\dot{V}(t) \leq 2(K-D)\,V^{\frac{1}{2}}(t) \tag{4.33}$$

is true $\forall t \in \mathbb{R}$. By separation of variables and integration similar to the previous paragraph, we get for the reaching time

$$t_{\mathrm{r}} \leq \frac{V^{\frac{1}{2}}(0)}{2(K-D)} \tag{4.34}$$

as upper bound similar to the statement in the proposition. □

This estimation is in general very conservative which can be also seen in the example under consideration. By using $K = 2$ and $D = 1$ we would calculate the reaching time for the given initial value as $t_{\mathrm{r}} \leq 2.5\,\mathrm{s}$ which is significantly bigger than the actual reaching time from the simulation results that is about $1.75\,\mathrm{s}$. Since the estimation (4.32) also holds for the worst case $\Delta \equiv D$, the estimated reaching time may also be the worst case.

To summarize the given results, the simulation gives an idea how powerful the SMC approach can be. Despite the presence of matched disturbances, the system converges in finite time to a given manifold which can be chosen by the control engineer. Once the sliding manifold is reached, the influence of the time varying disturbance to the reduced dynamics is completely neglected which is a significant advantage compared to conventional controller design.

4.1.2. Robustness Analysis and Alternative Solution Concept

In the previous section we have seen that a closed loop with sliding-mode control shows a remarkable robustness against matched uncertainties. Now we discuss by which properties of the SMC, this can be achieved. First of all, it is obvious that the influence of the disturbance has to be compensated somehow by the control action. A close look at (4.10) reveals that at the sliding surface, i.e. $s \equiv 0$, the switching term has to be exactly the opposite of the disturbance.

This leads to an alternative interpretation of the solution of a system with a discontinuous right-hand side. The idea is to use the method of equivalent control which was introduced in [138, 139]. Here, the derivative of the sliding variable (4.10) at the sliding surface is solved for the control variable u. We obtain for the example in the previous section

$$\dot{s} = 0 \quad \Rightarrow u_{\text{eq}}(t) = -\Delta(t) - c\,x_2(t) \tag{4.35}$$

which is exactly the expression that is needed to enforce the sliding mode. It should be noted that applying the control u_{eq} allows to describe the solution of the differential equation in a conventional sense.

However, if we compare the Filippov solution and the one proposed by Utkin, it seems obvious that there has to be a connection between both concepts. If the reaching condition (4.14) is satisfied, we known that the system moves along the tangential plane defined in Section 2.2. By comparing (4.19) and (4.35), it is obvious that the switching control $-K\,\text{sign}\,(s(t))$ has to mimic the disturbance perfectly. We can use this observation in order to reconstruct the input disturbance which can then be used for various applications like fault detection, see [40].

4.2. Higher-Order Sliding-Mode

One issue of the SMC concept presented so far is the limitation to relative-degree one of the sliding manifold with respect to the control input u. Due to this relative-degree one condition, these approaches are often called FOSM, and are obviously able to reduce the system order by one. The more popular the sliding mode approach became, the more the question arised if it is possible to find methods that may extend the relative degree to an arbitrary value up to the system order. The results of these considerations shall be discussed in this section.

4.2.1. Twisting Controller

It seems to be obvious that after a first-order design, the subsequent step is to find a second-order sliding-mode (SOSM) approach. One of the earliest SMC designs for the order of 2 is the so called *twisting* controller, see [85]. If we consider the example from above, the control law for this method reads as

$$u = -k_1 \operatorname{sign}(s) - k_2 \operatorname{sign}(\dot{s}) \tag{4.36}$$

with $s = x_1$ and the gains $k_1, k_2 > 0$ satisfying the following conditions

$$k_2 > D \tag{4.37a}$$

$$k_1 > D + k_2. \tag{4.37b}$$

From the definition of the sliding manifold we can see that the relative degree with respect to the input u is now 2.

Proposition 4.6. *The controller* (4.36) *stabilizes the system* (4.16) *in the presence of the uncertainty* $\Delta(t)$.

Proof. In this case we use a Lyapunov function in order to show that the closed-loop system is asymptotically stable. A suitable candidate is the expression

$$V_1 = k_1 \, |x_1| + \frac{1}{2} x_2^2 \tag{4.38}$$

with k_1 from (4.36). Calculating the time derivative yields

$$\begin{aligned}\dot{V}_1 =& k_1 \operatorname{sign}(x_1)\ \dot{x}_1 + x_2\,\dot{x}_2 \\ =& k_1 \operatorname{sign}(x_1)\ x_2 + x_2\left(-k_1 \operatorname{sign}(x_1) - k_2 \operatorname{sign}(x_2) + \Delta(t)\right) \\ =& x_2\left(-k_2 \operatorname{sign}(x_2) + \Delta(t)\right) \\ \leq& -\left(k_2 - D\right)|x_2| \end{aligned} \tag{4.39}$$

which is, under the assumption (4.37a), negative semi-definite since

$$\dot{V}_1 = 0 \ \ \forall x \ \in \ \{x \in \mathbb{R} | x_2 = 0\}\,.$$

Using Lemma 2.18 we can show that the trajectories of the system converge to a point where $x_2 \equiv 0$. Moreover, this implies that x_1 converges to a constant value which might be nonzero. The convergence to $x_1 \equiv 0$ can be shown by having a closer look at $\dot{x}_2$. By using the assumption given in (4.37b) we can conclude that $\dot{x}_2 = 0$ is achieved if $x_1 \equiv 0$. □

Due to the semi-definiteness of the time-derivative of the given Lyapunov function V_1 we cannot use a similar inequality compared to the first-order case given in (4.25). Hence, an estimation of the convergence time with the same method as in the previous section is not possible for this Lyapunov function.

However, [111] propose a different Lyapunov function that is suitable for finite time estimation as well. The Lyapunov function candidate reads as

$$V_2(x_1, x_2) = \begin{cases} \frac{k^2}{4}\left(\frac{x_2 \operatorname{sign}(x_1)}{\gamma} + k_0\sqrt{|x_1| + \frac{x_2}{2\,\gamma}}\right)^2 & \text{if} \quad x_1\,x_2 \neq 0 \\ \frac{\bar{k}}{4}\,x_2^2 & \text{if} \quad x_1 = 0 \\ \frac{1}{4}\,|x_1| & \text{if} \quad x_2 = 0 \end{cases} \tag{4.40}$$

with

$$\gamma = k_1 + (k_2 - D)\operatorname{sign}(x_1\,x_2) \tag{4.41a}$$

$$k = \sqrt{\frac{\gamma}{2}}\left|\sqrt{2\,\gamma}\mathring{k} - 1\right| > 0 \tag{4.41b}$$

$$k_0 = \sqrt{\frac{2}{\gamma}}\frac{\operatorname{sign}(x_1\,x_2)}{\sqrt{2\,\gamma}\mathring{k} - 1} \tag{4.41c}$$

and $\mathring{k}$ such that

$$\frac{1}{\sqrt{2\,(k_1 + k_2 - D)}} < \mathring{k} < \frac{1}{\sqrt{2\,(k_1 - k_2 + D)}} \tag{4.42}$$

holds. In this particular example, we have

$$k_1 = 4, \quad k_2 = 2, \quad D = 1 \tag{4.43}$$

which simplifies (4.41a) to

$$\gamma = 4 + 3\,\mathrm{sign}\,(x_1\, x_2)\,. \tag{4.44}$$

Furthermore, we select $\mathring{k} = \frac{1}{2\sqrt{2}}$ such that (4.42) is ensured. Finally, we obtain for the Lyapunov function for the given gains k_1, k_2 and the upper bound on the disturbance D the following expression

$$V_2 = \begin{cases} \frac{(0.5\sqrt{\gamma}-1)^2}{8\,\gamma}\left(|x_2| + \frac{\sqrt{2\,\gamma\,|x_1|+x_2^2}}{0.5\sqrt{\gamma}-1}\right)^2 & \text{if} \quad x_1\, x_2 \neq 0 \\ \frac{\mathring{k}}{4}\, x_2^2 & \text{if} \quad x_1 = 0 \\ \frac{1}{4}\,|x_1| & \text{if} \quad x_2 = 0 \end{cases} \tag{4.45}$$

with γ from (4.44). Figure 4.5 shows the surface of the Lyapunov function (4.40) for this particular set of parameters and an example trajectory starting at the point $(x_1,\, x_2) = \left(\frac{1}{2},\, 2\right)$.

Additionally, it is shown in [111] that the Lyapunov candidate (4.45) and its time derivative satisfy the following inequality

$$\dot{V}_2 \leq -k_{\min}\frac{k_1 - k_2 - D}{k_1 - k_2 + D}\sqrt{V_2} \tag{4.46}$$

with $k_{\min} := min(k)$ and k from (4.41b). Note that the expression (4.46) is close to the inequality for the first order algorithm (4.25). Based on (4.46) we can estimate the reaching time t_{r} with

$$t_{\mathrm{r}} \leq \frac{2\,(k_1 - k_2 + D)}{k_{\min}\,(k_1 - k_2 - D)}\sqrt{V_2(x(0))} \tag{4.47}$$

in dependence of the initial state $x(0)$.

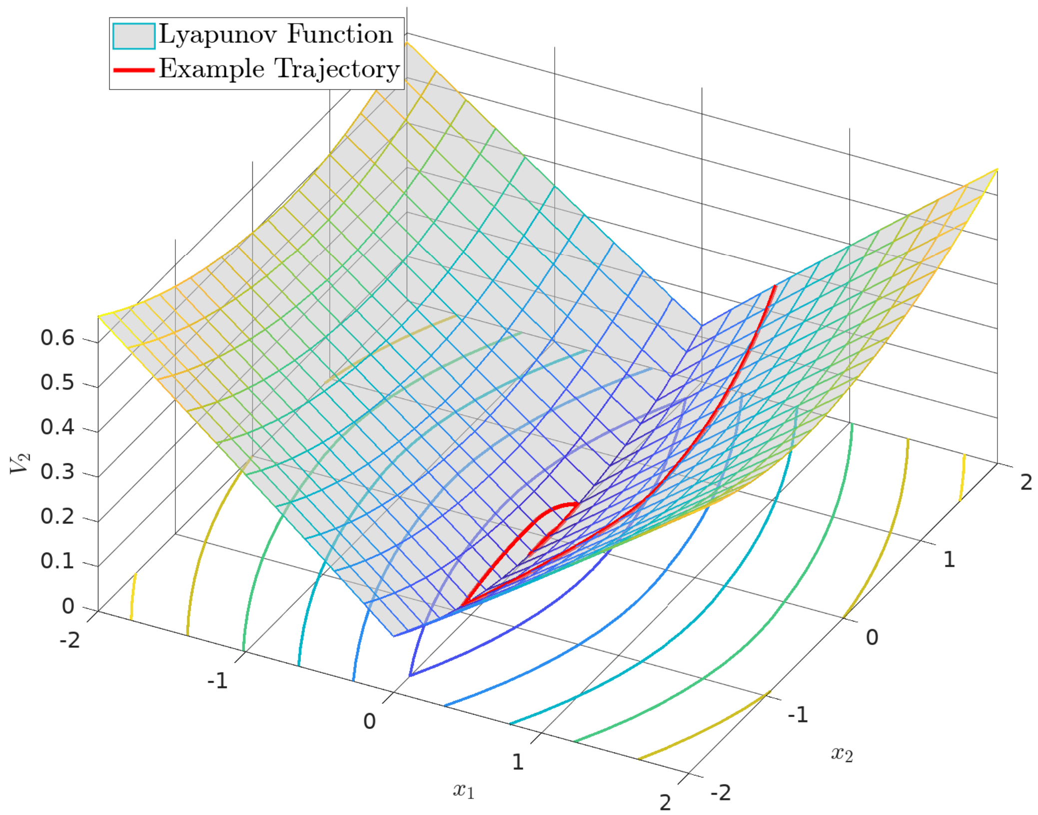


Figure 4.5.: Example Lyapunov Funktion for the Twisting Controller

Remark 4.7. *It should be noticed that in contrast to FOSM the Lyapunov function* (4.45) *is not limited to proof the convergence to a submanifold of the original state space. As shown before, the states of the FOSM approach converge exponentially to the origin. For the twisting controller, however, we can show with the help of* (4.45) *that the state trajectories of the closed loop system converge to the origin in finite time. This of course applies only for second order systems, for higher order system we can proof that the sliding surface* $s = 0$ *and its first derivative* $\dot{s} = 0$ *is reached in finite time.*

The differences of the twisting approach compared to the first-order SMC shall be illustrated by a simulation setup with same external disturbance (4.29) given in the previous example. Figure 4.6 shows the phase portrait and the time evolution of the system states from the initial state $x(0) = (0.5, 2)$. The phase portrait in the upper graph indicates the origin for the name of the *twisting* controller. We can see that the state trajectory *twist* around the origin with a decaying amplitude.

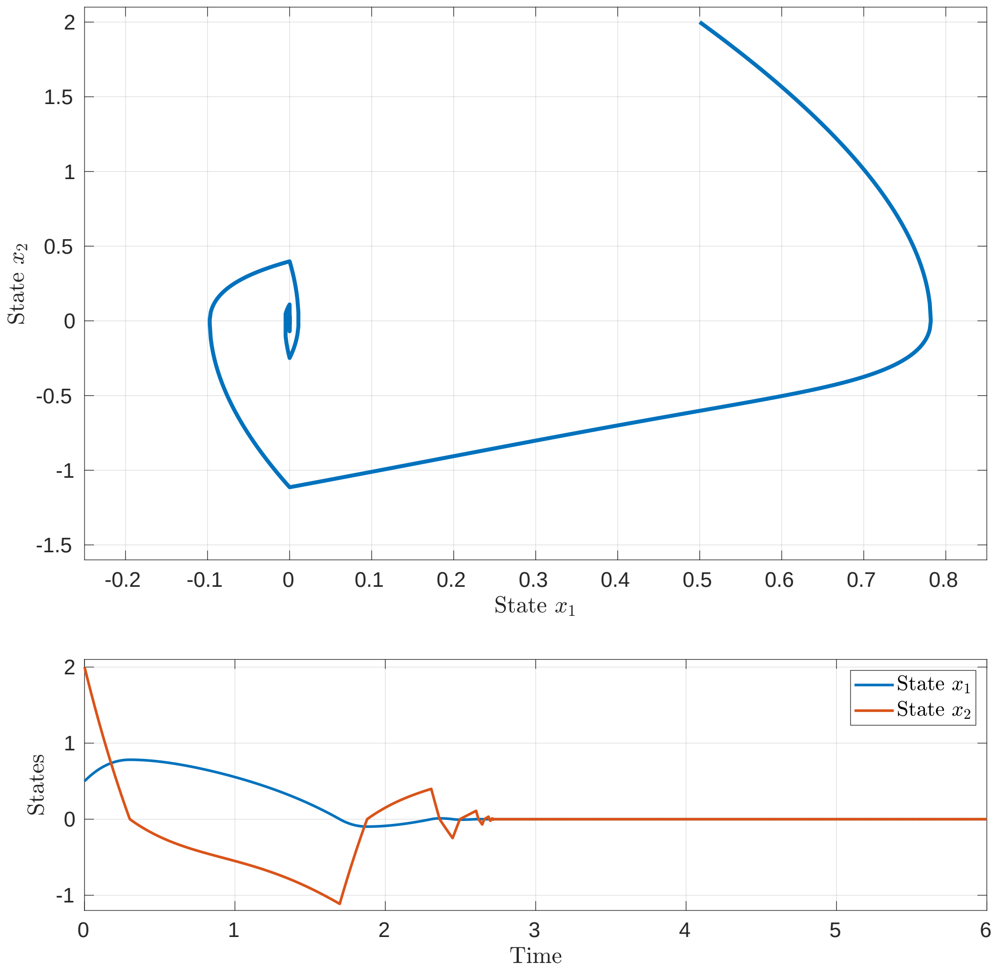


Figure 4.6.: Disturbed 2$^{\text{nd}}$ Order System with Twisting Controller

A closer look at the line where $x_1 = 0$ reveals the points where the control law (4.36) changes its sign. These characteristic points can be used for proofing stability of the closed loop as shown in [76] with some basic geometric consideration. The idea is that the crossing points with the abscissa and ordinate axes are getting nearer and nearer to the origin. Similarly it can be shown that the algorithm converges in finite time to the origin by evaluating the time difference between each crossing with the abscissa axis, see [20].

This statement is underlined by the lower plot in Figure 4.6 showing the evolution of the system states x_1 and x_2. We can see that the trajectory for x_2 crosses abscissa axis in the lower plot in decaying time intervals until both states reaches zero in a finite time. Comparing this results with the ones shown in Figure 4.4, we can clearly observe the difference between the two approaches. The sliding phase is completely missing for the twisting controller which avoids the exponentially convergence to the origin that is caused by the linear sliding surface (4.18). However, the major common property of both approaches is the invariance against external disturbances. As mentioned before, the same disturbance term acts in this simulation as in the example for the FOSM approach. We can see in Figure 4.6 that the trajectories of the closed loop system reach the origin in a finite time and the influence of (4.29) is completely eliminated.

4.2.2. Super-Twisting Controller

So far we have seen the SOSM controllers in combination with a plant that has a relative degree of two which allows the convergence of the system trajectories to the origin in finite time. The main similarity of the sliding mode approaches so far is the appearance of the switching term directly in the control signal which enforces the finite time convergence and ensures the robustness against matched uncertainties. However, this method bears some disadvantages compared to classical approaches, which will be discussed in Section 4.3, mainly caused by the discontinuity of the control signal.

The so-called Super-Twisting controller is a special kind of SOSM approach that was intended to avoid some of the issues coming along with SMC since it generates a continuous control signal. The main difference to the algorithms presented so far is the discrepancy between the relative degree and the sliding mode order. The super-twisting controller is suitable for sliding manifolds with a relative-degree of one with respect to the control input.

Similar to the approaches introduced up to this point, we shall have a look at a second order dynamical system (4.16). Due to the relative-degree-one condition of the sliding manifold with respect to the control input for the super-twisting controller we use the same sliding manifold as for the FOSM approach given in (4.18).

Undisturbed Case

First of all we shall have a look at the stability assuming that the disturbance $\Delta(t)$ is not existent, i.e. $\Delta \equiv 0$. The controller is given by

$$\begin{aligned} u &= -c\,x_2 - k_1 \lceil s \rfloor^{\frac{1}{2}} + v \\ \dot{v} &= -k_2 \operatorname{sign}(s)\,. \end{aligned} \tag{4.48}$$

Theorem 4.8. *The control law* (4.48) *with the gains* $k_1, k_2 > 0$ *and the initial state* $v(0) = v_0$ *applied to system* (4.16) *enforces a sliding mode on the manifold* (4.18) *while* u *is absolutely continuous in* t.

Proof. The discontinuity $\operatorname{sign}(s)$ is now located in the differential equation of the controller state. Hence, the actual control u applied to the system is now absolutely continuous (AC) in t.

In order to show the convergence to the sliding manifold we may use a slightly modified version of the Lyapunov function (4.38) from the twisting controller

$$V_1 = k_2\,|s| + \frac{1}{2}v^2 \tag{4.49}$$

where the state variables are replaced by the sliding variable and the controller state. The time derivative of (4.49) along the system trajectories yields

$$\begin{aligned} \dot{V}_1 =& k_2 \operatorname{sign}(s)\,\dot{s} + \dot{v}\,v \\ =& k_2 \operatorname{sign}(s)\,\left(-k_1 \lceil s \rfloor^{\frac{1}{2}} + v\right) - k_2 \operatorname{sign}(s)\,v \\ =& -k_1\,k_2\,|s|^{\frac{1}{2}} \end{aligned} \tag{4.50}$$

which is negative semi-definite if $k_1, k_2 > 0$. Together with Theorem 2.19 we may conclude that the system states converge to the sliding manifold, i.e. where s is equal to zero.

The AC property follows directly from the definition of the solution concept in Definition 2.8 which states that trajectories of the system are AC in time t. If we apply Corollary 5.3.3 from [27] and the fact that a composition of an absolutely continuous (AC), monotone function and a AC function is AC as well, see [27, p. 391], to the control law given in (4.48), we can conclude that u is absolutely continuous in time. □

However, similar to the first approach for the twisting-controller, the usage of the Lyapunov function (4.49) only provides a time-derivative that is negative semi-definite. Unfortunately this implies that the finite-time convergence cannot be shown since a relation similar to (4.32) is not possible in that case.

Disturbed Case

This observation leads to the question: Which disturbances can be handled by this control algorithm? By applying the controller (4.48) to the second order system (4.16), we obtain for the closed loop the following differential equation

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -c\,x_2(t) - k_1 \lceil s(t) \rfloor^{\frac{1}{2}} + v(t) + \Delta(t) \\ \dot{v}(t) &= -k_2 \operatorname{sign}(s(t)) \,. \end{aligned} \tag{4.51}$$

We can see from (4.48) that the controller now has two terms that could be used to affect the influence of the disturbance $\Delta(t)$, namely $-k_1 \lceil s \rfloor^{\frac{1}{2}}$ and v. Hence, we shall split up the disturbance into a sum of two terms as well:

$$\Delta(t) = \Delta_1(t) + \Delta_2(t) \tag{4.52}$$

where each part is handled individually by the corresponding section of the controller. Furthermore, we assume the following bounds on the disturbance

$$|\Delta_1(t)| \leq \Omega_1 \, |s(t)|^{\frac{1}{2}} \tag{4.53}$$

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \Delta_2(t) \right| \leq \Omega_2 \tag{4.54}$$

for all $t \in \mathbb{R}, t > 0$ with the known constants $\Omega_1, \Omega_2 > 0$.

So how can we show for the given class of disturbances that the sliding surface $s \equiv 0$ of the super-twisting algorithm (STA) is attractive? First of all, we introduce the following state transformation for the controller state

$$\bar{v}(t) = v(t) + \Delta_2(t) \tag{4.55}$$

such that we can rewrite the dynamics for the sliding variable as

$$\begin{aligned} \dot{s}(t) &= -k_1 \lceil s \rfloor^{\frac{1}{2}} + \bar{v}(t) + \Delta_1(t) \\ \dot{\bar{v}} &= -k_2 \mathrm{sign}\,(s) + \dot{\Delta}_2(t) \end{aligned} \tag{4.56}$$

with the two disturbances from (4.52). If we would now try to apply the Lyapunov function (4.49) to the disturbed case, of course with the transformed state $\bar{v}$ instead of v, we can see easily, that the term $\dot{\Delta}_2(t)$ is not canceled. Hence, we need a different Lyapunov function to proof the asymptotic stability.

A possible solution to this issue is addressed in [96] using the state transformation

$$\begin{aligned} z_1(t) &= \lceil s(t) \rfloor^{\frac{1}{2}} \\ z_2(t) &= \bar{v}(t). \end{aligned} \tag{4.57}$$

Now, if we can show that z_1 and z_2 converge to the origin, it follows directly that s and $\bar{v}$ reach zero as well.

We can express the dynamics of $z = (z_1, z_2)^\top$ (excluding $z_1 = 0$) by calculating the time derivative of (4.57) along the trajectories of (4.56). This results in

$$\dot{z}(t) = \begin{pmatrix} \dfrac{\partial z_1}{\partial s}\dot{s} \\ \dfrac{\partial z_2}{\partial v}\dot{\bar{v}} \end{pmatrix} = \begin{pmatrix} \dfrac{\dot{s}}{2\,|s|^{\frac{1}{2}}} \\ \dot{\bar{v}} \end{pmatrix} \tag{4.58}$$

and by substituting (4.56) we end up with

$$\dot{z}(t) = \frac{1}{|z_1(t)|} \begin{pmatrix} \frac{1}{2}\big(-k_1\, z_1(t) + z_2(t) + \Delta_1(t) \big) \\ -k_2\, z_1(t) + \dot{\Delta}_2(t)\, |z_1(t)| \end{pmatrix} \tag{4.59}$$

which can be rewritten as

$$\dot{z}(t) = \frac{1}{|z_1(t)|} A\, z + \frac{1}{|z_1(t)|}\, d_1(t) + d_2(t) \tag{4.60}$$

with A, d_1 and d_2 defined as

$$A := \begin{pmatrix} -\frac{k_1}{2} & \frac{1}{2} \\ -k_2 & 0 \end{pmatrix},\ d_1(x) := \begin{pmatrix} \frac{1}{2}\,\Delta_1(t) \\ 0 \end{pmatrix},\ d_2(x) := \begin{pmatrix} 0 \\ \dot{\Delta}_2(t) \end{pmatrix}.$$

At this point it is worth mentioning that the differential equation (4.60) has no solution along $z_1 = 0$ since the Filippov solution concept cannot be applied when $z_1 = 0$. This issue will be clarified in the following proof. Note that for brevity, we only recall the proof for the undisturbed case.

Theorem 4.9. *Let* V*, given as*

$$V = z^\top P\, z \tag{4.61}$$

with $P \in \mathbb{R}^{2\times 2}$ *and* $P = P^\top > 0$*. There exists* $k_1, k_2 > 0$ *such that* V *serves as a Lyapunov function for the undisturbed system* (4.56)*.*

Proof. The proof is taken from [97]:

Since we are dealing with the undisturbed case, i.e. $\Delta_1 \equiv 0$ and $\Delta_2 \equiv 0$, the time derivative of (4.61) yields

$$\dot{V} = \dot{z}^\top P\, z + z^\top P\, \dot{z}$$

and by replacing $\dot{z}$ with (4.60), we obtain

$$\dot{V} = \frac{1}{|z_1|} z^\top \left(A^\top P + P\, A\right)\, z\ . \tag{4.62}$$

From the algebraic Lyapunov equation (ALE)

$$A^\top P + P\, A = -Q \tag{4.63}$$

we know that for every symmetric and positive definite Q there exists a symmetric and positive matrix P if and only if A is Hurwitz. Hence we may rewrite (4.62) as

$$\dot{V} = -\frac{1}{|z_1|} z^\top Q\, z \tag{4.64}$$

by using (4.63) and the fact that A for k_1 and k_2 the matrix A is Hurwitz. At the first glance we may conclude that since (4.64) is negative definite, that V from (4.61) is a valid Lyapunov function. However, the Theorem 2.14 requires an at least locally Lipschitz continuous function V. A closer look at (4.64) reveals that this condition cannot be fulfilled on the set $\mathcal{S} = \{(z_1, z_2) \in \mathbb{R}^2 | z_1 = 0\}$. Fortunately, as shown in [97], we can overcome this issue by applying the theorem of Zubov [112, p.391], which was firstly given in [154]. This requires showing that V decreases monotonically along the trajectories of (4.56) and converges to zero.

Let us assume that $V(\zeta(t, s_0, v_0))$ with $\zeta(t, s_0, v_0)$ representing the solution of (4.56), is AC in time. In addition, if and only if $\dot{V}$ is negative definite, we can conclude that $V(\zeta(t, s_0, v_0))$ is a monotonically decreasing function, see [12, p. 207]. According to the Definition 2.8, the solution of (4.56) is an AC function of time as well as the Lyapunov function (4.61). Unfortunately, we cannot ensure that the composition of two absolutely continuous functions is AC as well unless V is Lipschitz or $\zeta(t, s_0, v_0)$ is monotonic, see [27, p. 391] and Lemma 2.6. By applying the state transformation (4.57) to the Lyapunov function V, a term including $\lceil \zeta_1(t, s_0, v_0) \rfloor^{\frac{1}{2}}$ pops up, indicating that V is not Lipschitz at $s = 0$. However, let us have a look at (4.56) when it crosses $\zeta_1(t, s_0, v_0) = 0$. If we assume that $\zeta_2(t, s_0, v_0)$ is nonzero during the crossing, we conclude from the first line of (4.56) that $\zeta_1(t, s_0, v_0)$ is either monotonically decreasing or increasing during a time interval containing the point of the crossing. On the other hand, if $\zeta_2(t, s_0, v_0) = 0$ at the zero-crossing of $\zeta_1(t, s_0, v_0)$, it is obvious that $\zeta_1(t, s_0, v_0)$ will remain zero. Both cases ensure the AC condition given in Definition 2.3. From this observation and the fact that $\dot{V}$ is negative definite almost everywhere, it can be ensured that $V(\zeta(t, s_0, v_0))$ is a monotonically decreasing function according to [12, p. 207] and its derivative can be expressed by (4.64) almost everywhere. □

For the disturbed case, refer to Theorem 2 and 3 in [97].

Simulation Example

Similar to the previous controllers, the STA approach shall be implemented in a brief simulation example that should demonstrate the properties of the design. To ensure the comparability to the other SMC algorithms, we consider the same second order system (4.16) with the same sliding manifold as for the FOSM given in (4.18). Of course, the system is affected by the same disturbance (4.29) as in the earlier examples.

We select the gains k_1, k_2 in the control law (4.48) as

$$k_1 = 1.5\,\sqrt{\Omega_2} \tag{4.65a}$$

$$k_2 = 1.1\,\Omega_2 \tag{4.65b}$$

with the upper bound of the disturbance Ω_2 from (4.7) identified as $\Omega_2 = 2$. This particular selection of gains was firstly proposed in [77]. A stability proof for these set of parameters can be found in [121].

The simulation of the closed loop system with the STA results in a variation of the state variables in time as shown in Figure 4.7. Similar to Figure 4.4, the reaching and sliding phase are highlighted in red and blue, respectively. It is worth noting that the control signal $u(t)$, that is produced by the controller (4.48) is absolute continuous in t in contrast to the FOSM approach. Nevertheless, the sliding manifold is reached in finite time, despite the influence of the disturbance $\Delta(t)$.

An even more important behavior can be observed in the lower graph of Figure 4.7 that contains the controller state $v(t)$ as well as the negative value of the disturbance. We can observe that in the sliding phase of the closed loop, the controller state represents the disturbance perfectly. This is also commonly known as *input reconstruction* which can be also achieved with the FOSM. Instead of low-pass filtering or an equivalent method, which is needed in the FOSM case, the super-twisting algorithm (STA) does not require any of these techniques for the disturbance reconstruction. Hence, the controller state $v(t)$ provides a directly usable estimation of the matched disturbance.

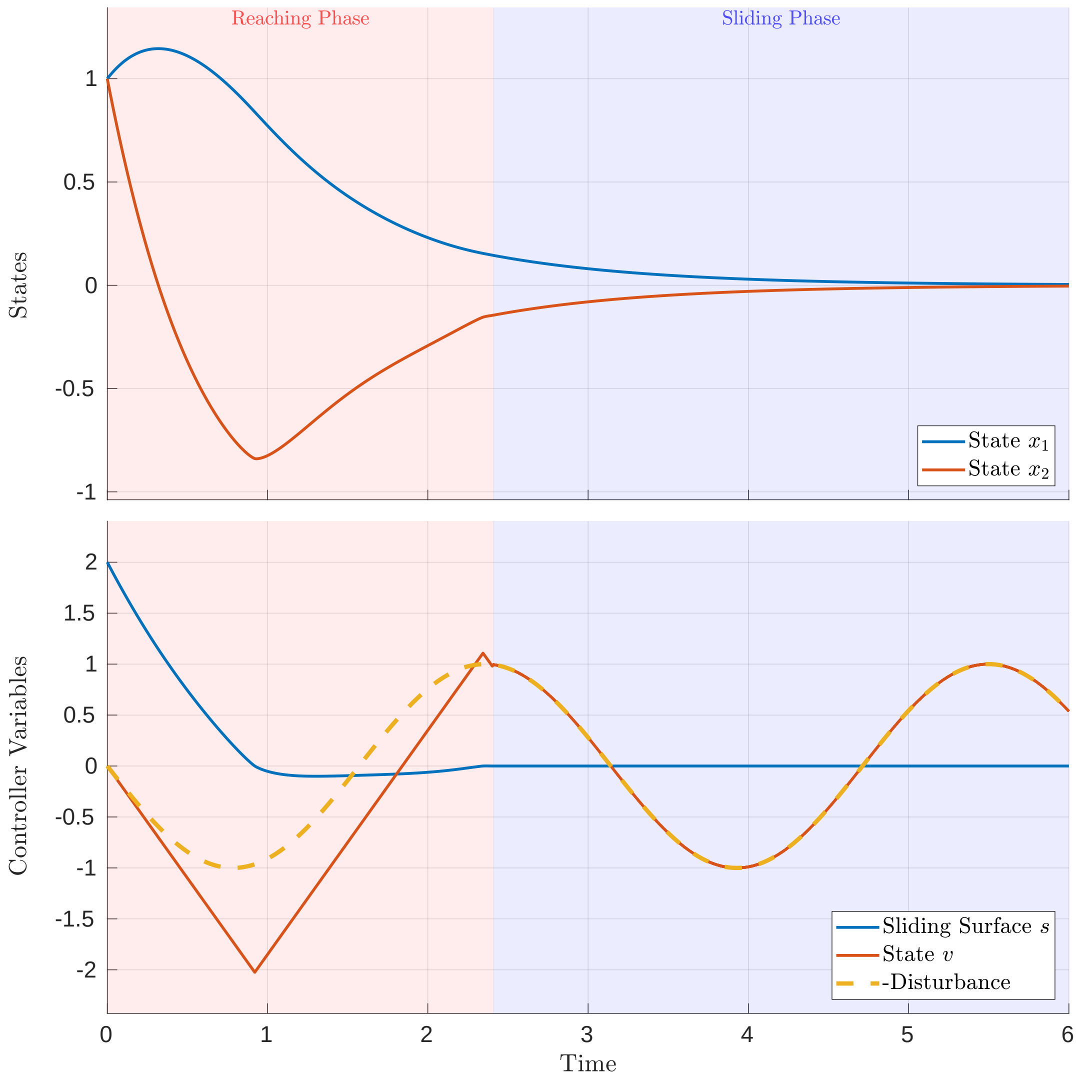


Figure 4.7.: Disturbed 2$^{\text{nd}}$ Order System with Super-Twisting Controller

4.3. Chattering

Up to this point, sliding mode control seems to be the ideal approach in order to tackle various kinds of disturbances or uncertainties. The sliding manifold that achieves a desired behavior of the system is even reached in finite time as shown in the previous sections. However, applying those control approaches to an actual plant will eventually show a different performance than expected. In general this effect is so called *chattering* and is caused by the fast switching of the control signal u on the sliding manifold introduced by the sign-function. An actual system will not be able to reproduce the arbitrary fast switching of the control signal due to the following reasons:

1. A discrete time implementation of the controller on a digital process controller like a micro-controller or programmable logic controller (PLC): The minimal realizable sample time implies an upper bound of the switching frequency that prevents infinite fast switching. This phenomenon occurs it the case of a numerical simulation as well since the time between two steps has to be strictly greater than zero.

2. Unknown or unmodeled actuator dynamics that have a characteristic similar to a low-pass filter: One can think of a mechanical system that is driven by an electric machine. Since the time constants in the current dynamics are much smaller compared to the mechanical part, they are often neglected and a force or torque is assumed to be the actual input to the system.

In general, this behavior is highly undesirable since it affects the overall controller performance in a negative way. First of all, in mechanical systems, the high frequency control signal may stimulate modes of the structure that are not covered by the analytical model and therefore not considered in the controller design. Beside that, the excitement of these modes may result in a high wear of the mechanical system and/or undesired acoustic noises. On a first glance, those effects may not lower the performance and robustness of the sliding mode controller but may result in less acceptance for this approach.

Remark 4.10. *Regarding the input dynamics one could wonder if the actuator behavior is known, why it is not integrated in the system description. There are several reasons for that: One is the fact that sliding mode, similar to many adaptive approaches, requires a matched uncertainty property in the most cases which might be not the case for a dynamic actuator. While this is only a weak argument, the implementation of the control law could be more of an issue. In many mechanical applications the dynamics of widely used electrical actuators are not included in the system description. This could be justified since the time constants of an electrical motor for example, are much smaller than those induced by the mechanical part. However, if the controller is implemented in a discrete time environment, the sample time has to be chosen accordingly to the smallest time constant in the system. If one would include the dynamics of the electrical network in the system description, one would have to select a much smaller one that enhances the requirements regarding the sampling hardware or the processing power.*

Hence, many applications try to avoid the actuator dynamics in order to save resources that could be used for better control algorithms.

In order to illustrate the mentioned drawbacks, the application of the FOSM controller to a first-order system

$$\dot{\sigma} = u \tag{4.66}$$

shall be investigated. Hereby we analyze the two major reasons for the occurrence of chattering in the closed loop:

1. The controller is implemented in discrete time, i.e. the system states are sampled every 10 milliseconds and the output is changed with the same rate. It should be mentioned that the control law (4.66) is a static feedback such that the discrete-time implementation is identical. However, the simulation is carried out with a much smaller sample time of 0.1μs and a forward Euler discretization.

2. The system has unmodeled input or actuator dynamics, in this example this is represented by a second-order transfer function given by

$$G_{\text{in}}(s) = \frac{1}{\tau^2 \, s^2 + \tau \, s + 1} \tag{4.67}$$

   where s denotes the Laplace variable and τ is selected to 0.1 seconds.

A simulation using the control law

$$u = -K \, \text{sign}\,(\sigma) \tag{4.68}$$

with $K = 10$ and an initial value of $\sigma(0) = 1$ shall demonstrate the chattering effect on the closed loop performance. The discrete time version is similar to (4.68) with the difference that the output y is sampled and input u is applied only at points that are multiples of the sample time.

The results of the simulation are shown in Figure 4.8 over a time span of 3 seconds. The upper graph displays the influence of the discrete-time implementation. One can observe that the system converges first into a domain around zero and then performs a zig-zag motion around the origin. This motion is caused by the fact that the control u is not applied continuously and hence the system cannot achieve an actual sliding mode according to Definition 2.8.

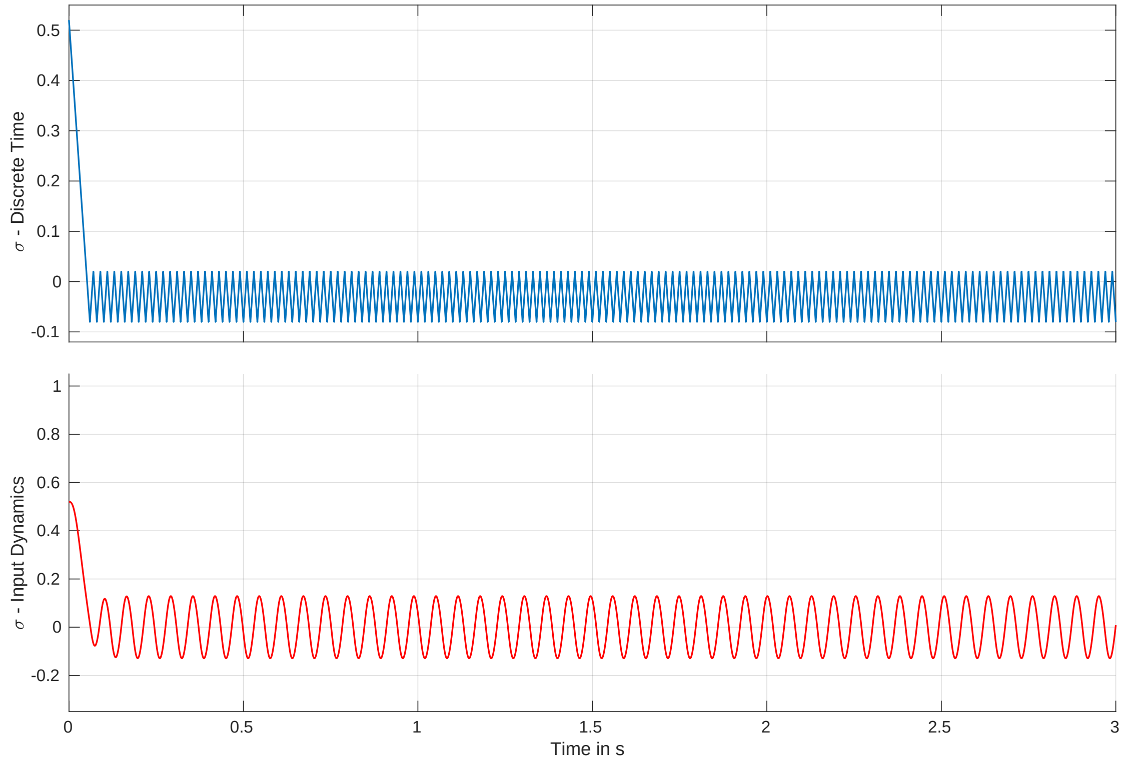


Figure 4.8.: Possible Trajectories for Chattering Effects

The second graph illustrates the effect of an input dynamic that has not been taken into account during the controller design. It can be seen that the controller (4.68) forces the system again to a domain around the origin. On the first view it looks like that the system converges asymptotically to zero, but the magnified view in the lower plot indicates that this is actually not the case. Rather than converging, the system performs a periodic movement with a high frequency around a small domain around the origin.

Comparing the results of those two simulations, the effect seems to be from a similar nature: A periodic movement of the system around the origin or the sliding manifold, respectively. As mentioned before, in an actual physical system this behavior might be highly undesirable since it may cause high wear in a mechanical system and generates annoying noises that reduces the acceptance of the control approach significantly.

Hence, one major topic in the field of sliding mode theory is to find methods that reduce the chattering phenomena while preserving the robustness properties of the SMC approach.

4.3.1. Analysis

Before presenting the countermeasures that are widely used to prevent the chattering effect, we will have a closer look at the correlation between the control law (4.68) and the periodic motion that is generated. The analysis of the chattering phenomena started as soon as people began to implement discontinuous control in a non-ideal environment.

Discrete Time Implementation In order to evaluate the influence of a discrete time implementation of a sliding-mode controller, we consider the closed loop

$$\dot{\sigma} = -K \operatorname{sign}(\sigma) \tag{4.69}$$

which can be obtained if we apply the control law (4.68) to the system (4.66). Figure 4.8 reveals that the discrete time implementation leads to a periodic movement around the origin of (4.69). To describe the chattering effect, we have to figure out which parameters of the closed-loop show an influence on the amplitude and the period of the chattering signal.

The idea behind sliding-mode is to generate an infinitely fast switching signal that compensates any disturbances perfectly. However, in an environment where the controller is implemented in discrete time, this cannot be achieved. If we have a look at the differential equation (4.69) of the closed-loop, we can easily see that the interesting behavior occurs at $\sigma = 0$, where the switching takes places. So for a worst-case estimation of the chattering amplitude, we shall have a look at the point $t_0 > 0$ right before the system trajectories of (4.69) cross the sliding surface.

We shall assume that the sliding surface is reached from the positive domain, i.e.

$$\sigma(t_0) = \sigma_0 = \lim_{\sigma \to 0^+} \sigma(t) \tag{4.70}$$

such that we can reason that the control input is equal to $-K$ for the next sampling period T_s. At the end of this interval, the sliding variable will reach

$$\sigma(t_0 + T_\mathrm{s}) - \sigma(t_0) = \int\limits_{t_0}^{t_0+T_\mathrm{s}} \dot{\sigma}(t)\,\mathrm{d}t = 0 - K\,T_\mathrm{s}. \tag{4.71}$$

The same value, but positive, will be reached if we approach the sliding surface from the negative half of the state-space. Now, for the given example, we can identify the maximum amplitude of the chattering signal to 0.1, with a gain of 10 and $T_s = 10$ms. Having a look at Figure 4.8 reveals, that the trajectories of the closed-loop system in the upper graph lie in the domain of ± 0.1. The frequency is simply twice the sample time, because around the sliding surface the sign of the control law (4.68) changes with every sampling interval.

It should be noticed that this estimation is only valid in the undisturbed case. At the point when an external disturbance is acting on the system, we have to consider worst-case assumptions unless we can rely on more detailed information of the disturbance.

Unmodeled Input Dynamics As we have seen in the example system, another important reason for chattering is the presence of unmodeled input dynamics. Hereby, the analysis of the chattering relies on slightly different thoughts compared to the discrete time case. First of all, we consider the system (4.66) with the control law (4.68) and the unmodeled actuator dynamics (4.67). A schematic of the close-loop system is shown

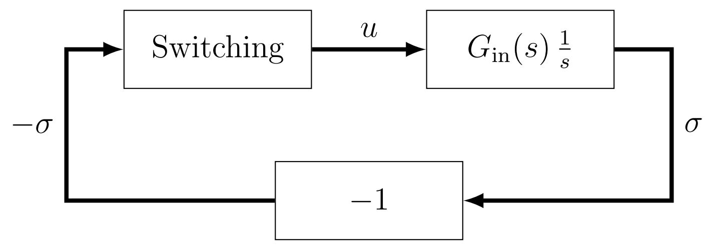


Figure 4.9.: Simplified System Structure for the Chattering Analysis of FOSM

in Figure 4.9.

For the purpose of classifying the effects of the actuator dynamics, we will use so-called describing functions. This approach is widely known in the field of classical nonlinear control [11, 45] and has been adapted for sliding mode in various publications, e.g. see [28, 57, 135].

The idea is, that the whole closed-loop system can be described by a static nonlinear term $F(y)$ and a linear dynamics $W(s)$ as shown in Figure 4.10 that is quite similar to the sliding-mode system in Figure 4.9. We assume that the point-symmetric non-linearity $F(y)$ leads to a periodic, sinusoidal signal

$$-y(t) = A \sin(\omega\, t) \tag{4.72}$$

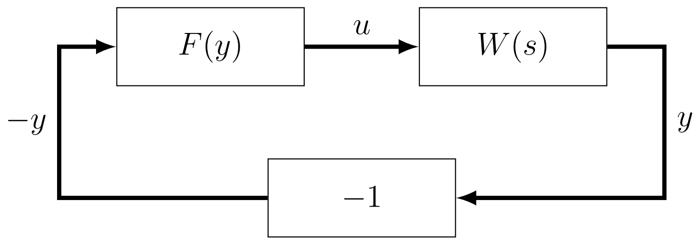


Figure 4.10.: General Nonlinear System for Describing Function Analysis

in the system. Hence, $F(y)$ can be expressed by

$$F(y) = N(A, \omega)\, y \tag{4.73}$$

with $N(A, \omega)$ being a Fourier approximation

$$N(A, \omega) = \frac{\omega}{A\,\pi} \int_0^{\frac{2\pi}{\omega}} F(A\, \sin(\omega\, t)) \sin(\omega\, t)\, \mathrm{d}t + \frac{j\,\omega}{A\,\pi} \int_0^{\frac{2\pi}{\omega}} F(A\, \sin(\omega\, t)) \cos(\omega\, t)\, \mathrm{d}t \tag{4.74}$$

of the nonlinear part of the system. On the other hand, the linear dynamic $W(s)$ is given by

$$W(s) = G_{\text{in}}(s)\, \frac{1}{s}. \tag{4.75}$$

Definition 4.11 (Harmonic Balance [11])**.** *The system given in Figure 4.10 is said to be in Harmonic Balance if one can find a frequency $\omega_{\text{c}} \in \mathbb{R}$ and an amplitude $A_{\text{c}} \in \mathbb{R}$ such that the condition*

$$1 + W(j\, \omega_{\text{c}})\, N(A_{\text{c}}, \omega_{\text{c}}) = 0 \tag{4.76}$$

is satisfied.

By using the harmonic balance condition given in Definition 4.11, we are able to identify the frequency and the amplitude of the sliding variable $\sigma(t)$. In our case, the nonlinear term is simply given by (4.68), such that the evaluation of (4.74) leads to

$$N(A, \omega) = \frac{4\, K}{A\, \pi} \tag{4.77}$$

with $F(y) = K \operatorname{sign}(y)$ and $\omega \neq 0$. It is worth noticing, that the expression (4.77), in the case of FOSM, is independent of the frequency ω of the oscillation.

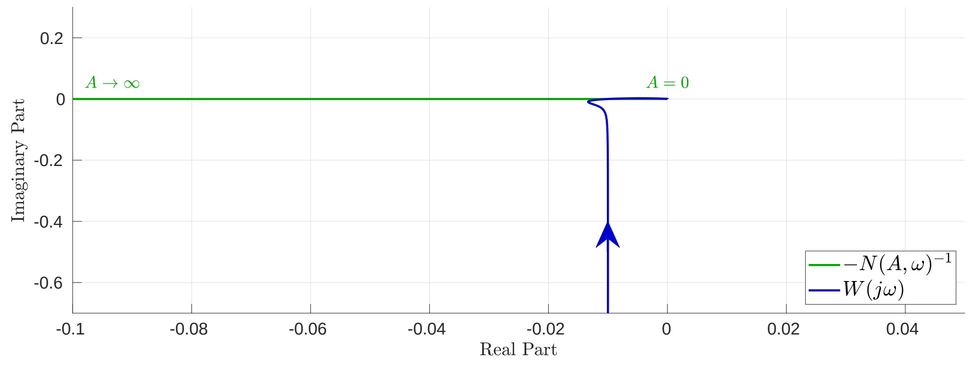


Figure 4.11.: Nyquist Plot of the Input Dynamics $G_{\text{in}}(j\omega)$ and the Describing Function $N(A,\omega)$

For a better understanding, we evaluate the harmonic balance condition graphically. Therefore, Figure 4.11 contains the Nyquist plot of $G_{\text{in}}(j\omega)$ as well as the describing function $N(A,\omega)$. To satisfy (4.76), we have to find the intersection point between the two graphs of $G_{\text{in}}(j\omega)$ and $N(A,\omega)$. Figure 4.12 contains a magnified variant

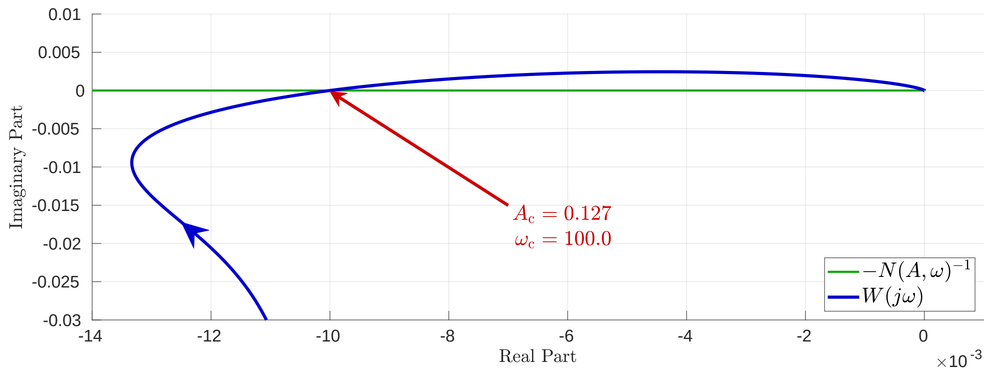


Figure 4.12.: Intersection Point of the Describing Function $N(A,\omega)$ and Nyquist Plot of $W(s)$

of Figure 4.11 showing the domain close to the origin. We can now identify the intersection point (A_c, ω_c) of the describing function and the Nyquist diagram by solving the imaginary part of (4.76) for ω_c. In our case, we obtain

$$\omega_c = \frac{1}{\tau} \tag{4.78}$$

where τ is the time constant from the input dynamic (4.67). If we substitute ω in (4.76) with ω_c from (4.78) we can identify the amplitude as

$$A_c = \frac{4\,K\,\tau}{\pi}. \tag{4.79}$$

At this point, we notice that the amplitude of the chattering signal is directly proportional to the gain K of the sliding-mode controller.

In a similar way, this approach can be also used for second-order sliding-mode control. This has been investigated for example in [28] and [135]. In a similar way compared to

Parameter	FOSM	Twisting	STA
Describing Function $N(A,\omega)$	$\frac{4K}{A\pi}$	$\frac{4}{A\,\omega\pi}(k_2 - j\,k_1)$	$\frac{k_1}{\sqrt{A}} - j\frac{4\,k_2}{\pi\,A\,\omega}$
Amplitude	$\tau\,\frac{2\,K}{\pi}$	$\tau^2\left(\frac{2k_2^4}{\pi\left(\sqrt{k_1^2+k_2^2}-k_1\right)^3}\right)$	$\tau^2\left(\frac{\pi k_1^2 + 16\,k_2}{4\,k_1\pi^2}\right)$
Frequency	$\frac{1}{\tau}$	$\frac{1}{\tau}\,\frac{\sqrt{k_1^2+k_2^2}-k_1}{c_2}$	$\frac{k_1\,\pi\sqrt{k_1^2+\frac{16\,k_2}{\pi}}}{\tau\,(\pi\,k_1^2+16\,k_2)}$

Table 4.1.: Comparison of Chattering Parameters for first-order sliding-mode (FOSM), twisting and super-twisting algorithm (STA)

the procedure that we applied to the first-order sliding-mode system, one can estimate the chattering parameters for the super-twisting algorithm (STA) from (4.48). A listing of the parameters of the chattering can be found in Table 4.1, comparing the first-order approach with the super-twisting.

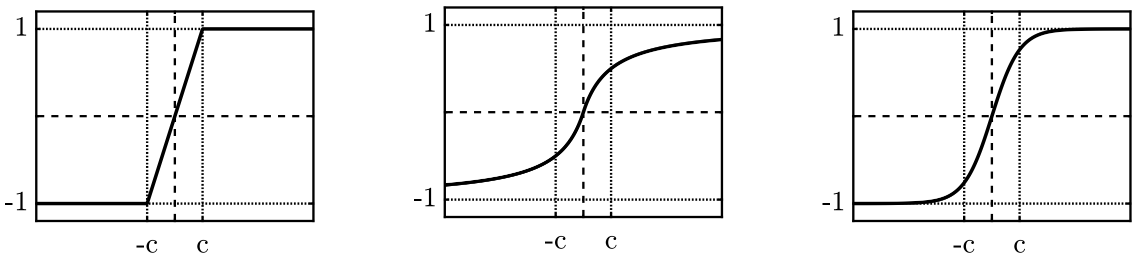
We can see that the rules for calculating the chattering amplitude and frequency differ between the three approaches. Here it should be emphasized that the describing function for the super-twisting is now depending on the frequency as well. The formulas for computing the parameters for the twisting and super-twisting tend to be a bit more complex. However, we can learn from Table 4.1 that the chattering amplitude depends now proportionally on τ^2 instead on a linear correlation as for the first-order implementation.

Hence, we cannot give an ultimate statement about which approach is the better option since this would depend on the actual input dynamics and the disturbance the controller has to face.

4.3.2. Countermeasures

We have seen in the section before that applying discontinuous control to a non-ideal setup might lead to an undesired behavior. Hence, there exist many approaches that try to avoid or reduce the chattering:

Boundary Layer Method The so-called boundary-layer approaches are probably the most widely used methods since they are quite straight-forward. The idea is to replace



(a) $\operatorname{sat}\left(\frac{s}{c}\right)$ **(b)** $\frac{s}{|s|+c}$ **(c)** $\frac{2}{\pi}\arctan\left(\frac{s}{c}\right)$

Figure 4.13.: Various Replacements for the sign-Function

the sign statement in the control law by a continuous function, see [39, p.15] for example. Some possible replacements are shown in Figure 4.13, including the saturation function in 4.13a, a sigmoid function in 4.13b and the arctangent in 4.13c. Here we should point out that the boundary layer removes the discontinuous control at all. Strictly spoken, the sliding-mode is no longer existent and therefore the robustness properties are lost as well.

Higher Order Sliding Mode Control Another idea is to replace first-order sliding-mode with a higher-order algorithm since they may produce an absolutely continuous control signal. When the first higher-order controller appeared, they seemed to be the perfect solution and it was claimed that the chattering could be removed [21, 22]. However, as turned out recently, see e.g. [109, 135], this statement is not true in general since chattering occurs in HOSM approaches as well. We can see from Table 4.1, the characterizing parameters of the chattering depend on the controller gains as well as on the input dynamics. Therefore, for the decision between conventional first-order and higher-order sliding-mode control, the whole system has to be taken into account.

Time Varying Gain Sliding Mode The analysis in the previous section shows that the chattering depends in all cases on the gains of the discontinuous controller so that it is desirable to use the smallest gains possible. However, we known from the controller design that a certain minimum gain is required to eliminate the influence of the disturbance on the closed-loop system. To obtain suitable gains, we have to find an upper bound of the disturbance or its time-derivative that has to be known during the controller design. Unfortunately, in many applications, these upper bounds are often overestimated to ensure stability in all possible cases. This overestimation may lead to unnecessary high gains which will result in a large chattering amplitude. From this observation, one came up with the idea of modifying the gains during the run-time of the controller instead of choosing a fixed set before the implementation.

There are basically two approaches possible: The first idea is to adapt the gains of the sliding-mode controller with an adaptation law, similar to the algorithms shown in Chapter 3. In the classification of this chapter, this can be seen as direct adaptive control. A variant of the twisting that uses adaptation of the gains is shown in [69]. For the super-twisting, this idea has been applied for example in [30, 125].

If it is possible to provide a more detailed estimation of the upper bound of the disturbance in dependency of the system state or time, a memory-less[1] approach can be applied as shown for example in [38] or [103]. Here, the more precise estimation of the disturbance is used to obtain smaller gains compared to the conventional approaches.

In both cases, we can benefit from gains that are just as large as the requirements from the system in order to achieve stability. Hence, smaller gains will directly improve the chattering behavior of the system.

Proper Discrete Time Implementation Up to this point we have focused on methods that target the chattering effect caused by unknown input dynamics. However, the previous section reveals that the discrete time implementation can also have a significant influence on the chattering as well.

[1] Memory-less implies that the controller has no internal states, i.e. it is just a static function between the measurement signal and the controller function.

Nowadays nearly all implementations of a control algorithm run on a digital processors, hence, a discrete time realization cannot be avoided in those cases. As a result, many publications in the last decades focuses on the time discretization of sliding-mode approaches.

One idea is to provide an estimation of the equivalent control, see Subsection 4.1.2 and use this estimation for the disturbances rejection [32, 49, 58, 90, 134]. Hence, the gain of the discontinuous control can be increased which has a direct effect on the chattering. In order to improve the quality of the estimation, adaptive or self-learning algorithms [23, 67, 100] are combined with the conventional SMC.

Exploiting the homogeneity property, a discrete-time HOSM algorithm that preserves the theoretical optimal asymptotic steady-state accuracy under sampling and hold, can be derived by proper discretization of the continuous-time algorithms [24,76,81,83,86,87]. In the former category the interest primarily focus on the analysis of the explicit Euler discretized closed-loop dynamics. Based on geometrical reasoning the first order as well as some second order sliding mode algorithms have been studied and steady state bounds of the output have been provided in [44, 146, 147].

Up to this point, many discrete-time implementation use the Euler forward discretization. Recent results [4,59,143] came up with the idea of an implicit discretization for the SMC controller. This approach is applied to conventional first order sliding mode [4, 143], the Twisting controller [59] and a nested SMC [89]. Furthermore, in [5] the implicit implementation is used in the case of a super-twisting observer. Unlike the explicit Euler discretization, the implicit methods are able to reach the origin in the unperturbed case in finite time.

The presence of the disturbances discrete-time environment will result in asymptotic accuracy for the trajectories which is known from the analysis of continuous SMC systems [76, 82, 84]. The most significant advantage is that the gains of controller that are implemented implicitly have no effect on the precision. Hence, overestimation is less of a problem in that case which is a huge improvement for many applications.

5. Adaptive Sliding-Mode Control

In this chapter, we will have a look at the main result of this thesis that is the combination of sliding-mode and adaptive control. Before we start with the actual contribution, we will briefly analyze some of the approaches in this field which have been presented to this point. Since both control strategies have their individual advantages, it seems obvious that a combination may provide even better robustness or performance properties. On the one hand we have the sliding-mode approach that is capable of dealing with changing disturbances with a memory-less static feedback as shown in Chapter 4, which is quite remarkable. These controllers are also able to force the system to a fixed point or manifold in finite time.

A completely different path is chosen when we decide to use an adaptive controller as presented in Chapter 3. Similar to the sliding-mode approach, we specify a certain performance criterion which is often done by the usage of a reference model. The controller is then designed such that it is capable by *adapt*, i.e. modifying itself, in order to deal with changing model parameters or certain external disturbances. In contrast to the sliding-mode approach, this involves in the most cases some kind of memory which is often implemented by a differential equation that is solved during the run-time of the controller. However, most adaptive controllers require some kind of description or model of the uncertainty in order adapt itself properly. A drawback of this method is that the performance for unmodelled uncertainties is not comparable to the one shown by sliding-mode approaches. In many cases it is only possible reach a small domain around zero of the control error.

After a short presentation of an alternative approach towards adaptive sliding-mode control, we will discuss the basic problem statement that should be covered by the contribution. This leads directly to the motivation of this work and the questions that have not been addressed so far. After that we will discuss the main idea and apply it firstly to the super-twisting controller. At this point we discuss the influence of the Lyapunov function that is used in the design process for the adaptive controller.

Additionally we will apply the proposed method to the latest higher-order sliding-mode controllers design for systems with an arbitrary relative degree.

Adaptive-Gain Super-Twisting Algorithm

Here we summarize briefly the adaptive-gain super-twisting algorithm (AGSTA). Therefore we consider the problem given in Subsection 4.2.2 with an important difference: We do no longer assume that the upper bounds on the disturbances are known. Only their existence has to be ensured.

By using the notation in [127], the AGSTA control law is given by

$$\begin{aligned} u &= -\alpha(t)\ |\sigma|^{\frac{1}{2}}\ \mathrm{sign}\,(\sigma) + v \\ \dot{v} &= -\beta(t)\,\mathrm{sign}\,(\sigma)\ . \end{aligned} \tag{5.1}$$

with the time-variant gains α, β. These gains are continuously modified based on the following differential equation

$$\begin{aligned} \dot{\alpha}(t) &= \begin{cases} \omega_1 \sqrt{\frac{\gamma_1}{2}}\,\mathrm{sign}\,(|\sigma| - \mu)\,, & \text{if } \alpha(t) > \alpha_{\mathrm{m}} \\ \eta, & \text{if } \alpha(t) \leq \alpha_{\mathrm{m}} \end{cases} \\ \beta(t) &= 2\,\epsilon\,\alpha(t) \end{aligned} \tag{5.2}$$

where ω_1, γ_1, μ, $\epsilon > 0$ parameters which can be used for tuning the adaptation rate. We may interpret the constant value α_{m} as a minimum value for the adaptive controller gain $\alpha(t)$.

The Stability of the closed-loop system (4.51), but with the control law replaced by (5.1) and (5.2) respectively, is analyzed using the following Lyapunov equation

$$V_{\mathrm{a}}(z, \alpha, \beta) = z^\top P_{\mathrm{a}}\, z + \frac{1}{2\,\gamma_1}(\alpha(t) - \alpha^*)^2 + \frac{1}{2\,\gamma_2}(\beta(t) - \beta^*)^2 \tag{5.3}$$

where z is defined as in (4.57) and some constant values γ_1, γ_2, α^*, $\beta^* > 0$. Furthermore P_{a} is given by

$$P_{\mathrm{a}} = \begin{pmatrix} \lambda + 4\,\epsilon^2 & -2\,\epsilon \\ -2\,\epsilon & 1 \end{pmatrix}\ . \tag{5.4}$$

and is a positive definite matrix if $\lambda > 0$.

Inserting the control law (5.1) and the adaptation law (5.2), the derivative of the Lyapunov function (5.3) satisfies the following relation uniformly in time:

$$\dot{V}_{\mathrm{a}}(z, \alpha, \beta) \leq -\eta_0 \left(V_{\mathrm{a}}(z, \alpha, \beta)\right)^{\frac{1}{2}} \tag{5.5}$$

where η_0 is a positive constant given in [127].

Before we continue, we shall have a closer look at this approach. First of all, according to the classification in Chapter 3, the AGSTA is a direct adaptive approach since the controller gains are modified directly. The main idea of this adaptive controller is to increase the gains until a vicinity of the sliding surface is reached. Otherwise the gains are reduced to a minimal value. We should point out that this approach does not require any knowledge about an upper bound of the disturbance. Nevertheless, the existence of such a bound must be ensured. As a result, this adaptive sliding-mode controller cannot be applied to system with a state-dependent disturbance as the upper bound may grow with the state.

5.1. Motivation

As already stated in the introduction, the main idea of this work is to combine adaptive control with techniques from variable-structure control. We have seen in the introduction that there are basically two approaches possible: One possible way is to take a conventional adaptive controller and use a switching adaptation law. Another possible path is to extend a classical sliding-mode controller and modify it in such a way that it is capable of dealing with changing conditions beyond their robustness and performance properties.

But first, let us have a look at the different kind of uncertainties a control algorithm has to face in general. We can categorize these uncertainty basically in two groups:

Structured Uncertainties mainly represent uncertainties in the model itself. As the name already implies, there is some information about the structure of the uncertainty available. Hence, in most of the cases it means that we can obtain a model of the uncertainty which is described by a function of the system states and/or time and some possibly unknown parameters. Of course, if the model is perfectly known, one could simply extend the controller by a negative signal of the uncertainty and therefore eliminate its influence. This kind of uncertainty can be found in many control problems: We have already seen one example in Chapter 3 with the spring-mass-damper system where the influence of the particular element is known structurally, but cannot be quantified exactly due to unknown or changing parameters. Other examples in mechanical systems are masses or distances among joins, since the connection between the system states and the force generated by the uncertainty can be quantified.

The occurrence of structured uncertainties is not limited to mechanical systems. Another important field are electrical setups, i.e. in power electronics where the system dynamics can be described by the Kirchhoff's laws. Similar to mechanical systems, these models contain parameters like resistors, inductances or capacitors whose value are unknown or may change over time.

Unstructured Uncertainties are the counterpart to the structured uncertainties. The major difference is, that it is not possible to come up with a suitable mathematical description for this kind of uncertainty. They represent for example external disturbances like unknown forces is mechanical system. Additionally, unstructured uncertainties include effects which require a very complex description which may result in a very unhandy model which makes the theoretical analysis more complicated than necessary. A very common use-case for this is friction in a mechanical systems, that may depend on the position or the velocity. However, friction models can achieve any kind of complexity[1], such that it is a common way to use a simplified model and describe the friction as structured uncertainty. Another way in dealing with unstructured uncertainty is to obtain some kind of bound that helps characterizing its properties. The bounds can e.g. limit the amplitude, the derivative or the frequency of the uncertainty.

From the separation of the possible uncertainties in a dynamical model, we derive directly the motivation of this thesis. On the one hand we have an adaptive controller that is very good in handling structured uncertainties that depend on the system states. However, the performance of the adaptive controller may be degraded as soon as an unknown input acts on the system [61].

Sliding-mode control on the other hands is very capable in dealing with unstructured, but bounded uncertainties. Yet, many SMC approach are not suited to handle state-

[1]For an overview of different friction models, consider [10]

depended uncertainties as they require *a priori* knowledge about the upper bound. If this bound depends on a system state, the stability proof would require the state to be bounded in order show that it actually is, resulting in a circular argumentation.

With the combination of sliding-mode and adaptive control, we tackle the following issues: First of all we extend the class of admissible uncertainties compared to the individual approaches. The resulting controller should be able to handle structured as well as unstructured uncertainties. The main focus lies on uncertainties that are state dependent since their boundedness cannot be guaranteed *a priori*. Hence, we exploit the available information about the system as much as possible in order to find an appropriate controller. In the conventional sliding mode design, the dynamics of the system are often neglected or interpreted as "uncertainties". As a result, many sliding mode approaches would require unnecessarily high gains. We know from Chapter 4, in particular Section 4.3, that a high control gain will inevitably lead to a larger chattering amplitude. While it is only a secondary objective, the chattering behavior of the proposed combination should be better than a conventional sliding-mode design or at least on a similar level.

Another crucial point is the stability proof for the whole adaptive system. As we have discussed in Chapter 4, the early stability proofs for sliding-mode controller, especially HOSM, relies on geometric considerations. Just recently, it became more popular in the sliding-mode community to use Lyapunov theory in order to verify stability. The aim of this work is to provide an adaptive sliding-mode approach where the stability is ensured via Lyapunov theory.

5.2. Problem Statement

Before we discuss the proposed adaptive controllers, we have to specify the system class we are dealing with. This includes the assumptions that are required for the ideas in this chapter.

Let us start with a general nonlinear system given by the following differential equation

$$\dot{\eta}(t) = \phi(t, \eta(t)) + \gamma(t, x)\, v(t) + d(t, \eta(t))) \tag{5.6}$$

with the initial value $\eta(0) = \eta_0$. In (5.6), $\eta \in \mathrm{R}^n$ denotes the state vector of dimension n and $v(t) \in \mathrm{R}$ the scalar control input. The vector fields $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and

$\gamma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are known. Furthermore, the system (5.6) is affected by an uncertain term $d : \mathrm{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which can be time and state depended. This uncertainty shall be subject to the following assumption:

Assumption 5.1 (Matched Uncertainties)**.** *We assume that the uncertainty $d(t, \eta(t))$ fulfills the matching condition given in Definition 4.1.*

In addition, we require that the system (5.6) is exact input-state linearizable, see the definition in Section A.1 and proof in [149]. Based on this statements, according to [133, 150], we can write (5.6) as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f(t, x) + g(t, x)\upsilon + \Delta(t, x) \end{aligned} \tag{5.7}$$

with the smooth state transformation

$$x = \varXi(\eta)$$

where $\varXi : \mathbb{R}^n \to \mathbb{R}^n$ being a smooth state transformation. Note that the time argument for x is dropped for readability.

With the assumption

Assumption 5.2 (Uncertainty Separation)**.** *We can split the uncertainty $\Delta(t, x)$ in the following manner $\Delta(t, x) = \Theta^\top \omega(t, x) + \delta(t)$ such that we end up with a structured or parametric uncertainty $\Theta^\top \omega(t, x)$ and an unstructured term $\delta(t)$. Furthermore, we assume that the parameter vector $\Theta \in \mathbb{R}^p$ is unknown but constant, whereas the regressor function $\omega : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$ is assumed to be known.*

and a feedback-linearizing control law

$$\upsilon = g(t, x)^{-1} \left(u - f(t, x)\right) \tag{5.8}$$

we can rearrange (5.7) to

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_n &= u + \Theta^\top \omega(t, x) + \delta(t)
\end{aligned}
\tag{5.9}
$$

with the new control input $u \in \mathbb{R}$.

In this thesis we consider state feedback control only. Therefore we demand:

Assumption 5.3 (State Information)**.** *The state x from* (5.9) *is available for control.*

5.3. Main Idea

In order to solve the task given in the previous section, we propose a concept similar to the conventional adaptive control with the major difference that the nominal controller is replaced by a sliding-mode approach. The resulting controller should be able to handle structured and unstructured uncertainties. A consequence is that the structure of the uncertainty has to be somehow included in the control law. Due to the system structure given in (5.9), we consider a concept based on the certainty equivalence (CE) principle. The procedure mainly considers the following design steps:

Uncertainty Characterization Identify the uncertain terms in the original system description (5.6) in such a way that the separation into a structured and unstructured uncertainty is possible. Loosely speaking, the goal of this separation is that most of the amplitude of $\Delta(t, x)$ in (5.7) can be expressed by the structured part. Additionally, one has to find an appropriate upper bound on the unstructured part $\delta(t)$ in (5.9).

Nominal Controller Design A suitable sliding-mode controller should be selected in such a way that it suppresses the unstructured uncertainty completely. In this step, it shall be assumed that the parameter Θ of the structured uncertainty in (5.9) is perfectly known and therefore, can be compensated with an additional term in the control law. Hence, the nominal closed loop does not contain the structured uncertainty anymore.

Stability Proof for the nominal Case Now, one has to provide a stability proof for the nominal closed loop with a Lyapunov function. While this is a common step in conventional adaptive control, proofs via Lyapunov function for sliding-mode controllers became common just is the last decade. As a result, the requirements on these Lyapunov functions are not yet fully explored.

Design of an appropriate Adaptation Law The Lyapunov function from the previous step is used to obtain the adaptation law for the unknown parameters of the structured uncertainty. The assumption of perfectly known parameters is dropped in favor of an estimate $\hat{\Theta}$ of the parameters. Now, the nominal Lyapunov function is extended by a quadratic term of the estimation error. Finally, the adaptation law is selected in such a way that the derivative of the extended Lyapunov function is rendered independent of the estimation error.

While this procedure is common in the conventional controller design, the application towards HOSM approaches is not common and should be investigated in the following sections.

5.4. Application to the Super-Twisting Algorithm

The first sliding-mode controller that we consider is the well-known super-twisting algorithm (STA). As we have seen in Subsection 4.2.2, it is a SOSM implementation that provides a continuous control signal. The idea is to force the system to a given manifold that describes the desired dynamics. For the given system (5.9), we design a sliding variable $s : \mathbb{R}^n \to \mathbb{R}$ such that $s(x)$ is sufficiency smooth and the zero dynamics $s(x) = 0$ is stable. In the case of STA, we require that the relative-degree between $s(x)$ and the control input u is equal to one.

Note that the system dynamics can also be expressed by

$$\dot{x} = A\,x + B\,\left(u + \Theta^\top\,\omega(t,x) + \delta(t)\right)$$

with $A \in \mathbb{R}^{n\times n}$ a Frobenius matrix where all eigenvalues are located at zero and $B \in \mathbb{R}^n$ the unit vector of dimension n.

The time derivative of $s(x)$ yields

$$\dot{s} = \frac{\partial s}{\partial x}\,\dot{x} = \frac{\partial s}{\partial x} A\,x + \frac{\partial s}{\partial x} B\,\left(u + \Theta^\top \omega(t,x) + \delta(t)\right) \tag{5.10}$$

which is subject to the STA. The proposed controller is given by

$$\begin{aligned} u &= \left(\frac{\partial s}{\partial x} B\right)^{-1} \left(-\frac{\partial s}{\partial x} A\,x - k_1 \lceil s \rfloor^{\frac{1}{2}} + v - \hat{\Theta}^\top \omega(t,x)\right) \\ \dot{v} &= -k_2 \,\mathrm{sign}\,(s) \end{aligned} \tag{5.11}$$

with the gains $k_1, k_2 > 0$ and the initial value $v(0) = v_0$. Note that the control law is almost similar to the one given in Subsection 4.2.2 with the major difference that the term $\hat{\Theta}^\top \omega(t,x)$ is added in order to compensate for the structured uncertainty. If we apply the control law (5.11) to the dynamics of the sliding variable in (5.10), we obtain

$$\begin{aligned} \dot{s} &= -k_1 \lceil s \rfloor^{\frac{1}{2}} + v + \tilde{\Theta}^\top \omega(t,x) + \delta(t) \\ \dot{v} &= -k_2 \mathrm{sign}\,(s) \end{aligned} \tag{5.12}$$

where the estimation error is defined as

$$\tilde{\Theta} := \Theta - \hat{\Theta}. \tag{5.13}$$

Now we have to find a suitable adaptation law that provides an estimation $\hat{\Theta}$ of the plant parameters that can be used in (5.11). In the following we discuss three different Lyapunov functions for the nominal system which is a requirement according to the CE principle. We will refer to this combination in the following as certainty-equivalence based super-twisting algorithm (CESTA).

5.4.1. Weak Quadratic Lyapunov Function

The following approach was firstly presented in [14] and combines the STA and CE principle. As we have discussed in the Section 5.3, we now have to find an adaptation law that can be used to adjust the estimated parameters $\hat{\Theta}$ continuously. Hence we provide the following theorem:

Theorem 5.4. *Let the uncertainty $\delta(t)$ be bounded by*

$$|\delta(t)| \leq \Omega_1 \, |s(x(t))|^{\frac{1}{2}} \tag{5.14}$$

for all $t \in \mathbb{R}$ and let $k_1 > \Omega_1$. Then, the adaptation law

$$\dot{\hat{\Theta}} = \Gamma \,\mathrm{sign}\,(s(x))\; \omega(t,x) \tag{5.15}$$

with $\Gamma > 0$ ensures that the trajectories of the system (5.9) with the control law (5.11) will converge to the sliding manifold $s(x) = 0$ despite the presence of the structured uncertainty.

Proof. According to the procedure given in Section 5.3, we have to show that the closed loop is stable under the assumption of perfectly known parameters, i.e. $\hat{\Theta} = \Theta$. While the first stability proofs for the STA were carried out geometrically, see [76], in [105] the Lyapunov function

$$V_1(s, v) = k_2 |s| + \frac{1}{2} v^2 . \tag{5.16}$$

is proposed.

Now we can calculate the time derivative of (5.16) along the trajectories of (5.12) and obtain

$$\begin{aligned} \dot{V}_1 &= k_2 \operatorname{sign}(s) \left(-k_1 \lceil s \rfloor^{\frac{1}{2}} + v + \tilde{\Theta}^\top \omega(t, x) + \delta(t) \right) - k_2 \operatorname{sign}(s)\ v \\ &= k_2 \operatorname{sign}(s) \left(-k_1 \lceil s \rfloor^{\frac{1}{2}} + \tilde{\Theta}^\top \alpha(t, x) + \delta(t) \right) . \end{aligned} \tag{5.17}$$

As mentioned before, we assume in this first step, that the parameters are perfectly known. The outcome of this consideration is, that the estimation error $\tilde{\Theta}$ is zero. If we furthermore include the assumption on the unstructured uncertainty given in (5.14), we can provide a bound for $\dot{V}_1$ given by

$$\dot{V}_1 \leq k_2 \left(-k_1 + \Omega_1 \right) |s|^{\frac{1}{2}} . \tag{5.18}$$

We may conclude that (5.18) is negative semi-definite in s and v if we select the gain k_1 larger than the upper bound Ω_1 and $k_2 > 0$. By using Theorem 2.19, we can ensure that the s will converge asymptotically to zero.

Now, according to the procedure in Section 5.3, we may extend the Lyapunov function (5.16) by a quadratic term of the estimation error $\tilde{\Theta}$. We obtain

$$V_{\mathrm{ex},1}(s, v, \tilde{\Theta}) = k_2 |s| + \frac{1}{2} v^2 + \frac{k_2}{2\,\Gamma} \tilde{\Theta}^\top \tilde{\Theta}. \tag{5.19}$$

with $\Gamma > 0$ being a constant that can be used to adjust the rate of the adaptation law. The time derivative of (5.19) yields

$$\dot{V}_{\mathrm{ex},1} = k_2 \operatorname{sign}(s) \left(-k_1 \lceil s \rfloor^{\frac{1}{2}} + \tilde{\Theta}^\top \omega(t, x) + \delta(t) \right) + \frac{k_2}{\Gamma} \tilde{\Theta}^\top \dot{\tilde{\Theta}}$$

which can be rewritten as

$$\dot{V}_{\mathrm{ex},1} = k_2 \operatorname{sign}(s) \left(-k_1 \lceil s \rfloor^{\frac{1}{2}} + \tilde{\Theta}^\top \omega(t, x) + \delta(t) \right) - \frac{k_2}{\Gamma} \tilde{\Theta}^\top \dot{\hat{\Theta}}. \tag{5.20}$$

due to Assumption 5.2, where we supposed that the plant parameters are constant, i.e. $\dot{\tilde{\Theta}} = -\dot{\hat{\Theta}}$. The adaptation law given in the theorem renders the derivative of the extended Lyapunov function (5.20) independent of the estimation error:

$$\dot{V}_{\text{ex},1} \leq k_2 \left(-k_1 + \Omega_1\right) |s|^{\frac{1}{2}} \leq 0 \tag{5.21}$$

which is similar to the nominal case (5.18). Again, by choosing $k_1 > \Omega_1$ we ensure that (5.21) is negative semi-definite. Similar to the nominal case, we recall Theorem 2.19 to show that the sliding variable s converges to zero. □

At this point we shall have a closer look at the derivative of the Lyapunov function (5.21), which contains neither the estimation error $\tilde{\Theta}$ nor the controller state v is appearing. Hence, we can only guarantee the convergence of the sliding variable s but not for the controller state v or the estimation error $\tilde{\Theta}$.

As we have discussed in Chapter 3, the main goal of an adaptive controller is to ensure stability of the state. Hence, many approaches lack the convergence of the parameters unless some special requirements on $\omega(t, x(t))$ are fulfilled, see Section 3.5. So it is not astonishing that the proposed combination of adaptive and sliding-mode control does not deliver this property.

Furthermore, (5.18) does also not include the controller state v. On the first glace, this might not be an issue as the main objective is to ensure the convergence of s. However, as we have seen in Subsection 4.2.2, if we allow unstructured uncertainties $\delta(t)$ where only the derivative of the uncertainty is bounded, we need a Lyapunov function that is strictly negative definite. This shall be the issue of the subsequent section.

5.4.2. Strict Quadratic Lyapunov Function

The method proposed in this section was firstly presented in [16] and is motivated by the observations from the previous section: Since V_1 from (5.16) is only a weak Lyapunov function it does not prove convergence of the controller state. Hence, it is also not useful in showing that the STA can handle disturbances where the derivative is bounded.

However, as we have discussed in Subsection 4.2.2, the quadratic Lyapunov function proposed in [97] eliminates these restrictions caused by the weak Lyapunov function V_1.

The quadratic Lyapunov function is given by

$$V_2(z) = z^\top P\, z \tag{5.22}$$

with a constant, symmetric and positive matrix $P = P^\top > 0$ and the state $z = (z_1, z_2)^\top$ defining in the following manner

$$z_1 = \lceil s \rfloor^{\frac{1}{2}} \tag{5.23a}$$

$$z_2 = v + \delta(t). \tag{5.23b}$$

Thanks to monotony of the transformation it is ensured that if z_1 converge to zero, s will converge to zero as well. On the other hand, if z_2 converges to zero, the closed loop in (5.12) reveals that the disturbance $\delta(t)$ will be compensated exactly.

Now, the z-dynamics can be expressed by

$$\begin{aligned} \dot{z} &= \frac{1}{|z_1|} \begin{pmatrix} \frac{1}{2}\big(-k_1 z_1 + z_2\big) \\ -k_2\, z_1 \end{pmatrix} + \frac{1}{2\,|z_1|} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{\Theta}^\top \omega(t,x) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{\delta}(t) \\ &= \frac{1}{|z_1|} \bar{A}\, z + \frac{1}{2\,|z_1|} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{\Theta}^\top \omega(t,x) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{\delta}(t) \end{aligned} \tag{5.24}$$

with $\bar{A}$ defined as

$$\bar{A} := \begin{pmatrix} \frac{k_1}{2} & \frac{1}{2} \\ -k_2 & 0 \end{pmatrix}.$$

Similar to the previous section, we have to show that V_2 is a Lyapunov function for the closed loop in the nominal case, i.e. $\tilde{\Theta} = 0$. In Subsection 4.2.2 we have seen that in the undisturbed case, i.e. $\delta \equiv 0$, V_2 is a Lyapunov function for (5.12). Additionally, in [97] it is shown that V_2 is also a Lyapunov function for (5.12) if

$$\left| \dot{\delta}(t) \right| \le \Omega_2 \tag{5.25}$$

and a proper selection of the gains k_1 and k_2. Furthermore, according to [97] it is possible to find a symmetric, positive definite matrix Q_r such that

$$\dot{V}_2 \le -\frac{1}{|z_1|} z^\top Q_\mathrm{r} z \tag{5.26}$$

holds. Hence, we can easily verify the following inequality

$$\dot{V}_2(z) \leq -\kappa V_2^{\frac{1}{2}}(z) \tag{5.27}$$

with κ defined as

$$\kappa := \sqrt{\lambda_{\min}\{P\}}\,\frac{\lambda_{\min}\{Q_{\mathrm{r}}\}}{\lambda_{\max}\{P\}}$$

holds, where $\lambda_{\min,\max}$ denotes the minimum or maximum eigenvalue, respectively. Similar to Proposition 4.4, we use (5.27) to conclude that the closed loop converges to the origin in finite time in the nominal case.

Adaptive Extension

Now, according to the proposed design procedure, we will discard the assumption that the plant parameters are known. In order to obtain the adaptation law, we have to extend the Lyapunov function (5.22) by a term including the estimation error. Therefore, we propose the following Lyapunov function candidate

$$V(z, \tilde{\Theta})_{\mathrm{ex},2} = z^\top P\, z + \frac{1}{2\,\Gamma}\tilde{\Theta}^\top \tilde{\Theta} \tag{5.28}$$

with a constant $\Gamma > 0$. The derivative of (5.28) reads as

$$\begin{aligned}\dot{V}_{\mathrm{ex},2} =& \frac{1}{|z_1|} z^\top (\bar{A}^\top P + P\,\bar{A})\, z + z^\top P \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{\delta}(t) + \\ & \frac{1}{|z_1|} z^\top P \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{\Theta}^\top \omega(t,x) + \frac{1}{\Gamma}\tilde{\Theta}^\top \dot{\tilde{\Theta}}.\end{aligned} \tag{5.29}$$

By using (5.26), we derive the following estimation

$$\dot{V}_{\mathrm{ex},2} \leq -\frac{1}{|z_1|} z^\top Q_{\mathrm{r}} z + \frac{1}{|z_1|} z^\top P \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tilde{\Theta}^\top \omega(t,x) - \frac{1}{\Gamma}\tilde{\Theta}^\top \dot{\hat{\Theta}} \tag{5.30}$$

with $\dot{\tilde{\Theta}} = -\dot{\hat{\Theta}}$. To ensure $\dot{V}_{\mathrm{ex},2} \leq 0$, we demand

$$\frac{1}{|z_1|} z^\top P \begin{pmatrix} 1 \\ 0 \end{pmatrix} \omega(t,x) - \frac{1}{\Gamma}\dot{\hat{\Theta}} = 0 \tag{5.31}$$

By selecting the adaptation law to

$$\dot{\hat{\Theta}} = \frac{\Gamma}{|z_1|}\omega(t,x)\begin{pmatrix}1 & 0\end{pmatrix} P\,z, \tag{5.32}$$

we make sure that the derivative of the Lyapunov function (5.30) independent of $\tilde{\Theta}$. Since $\dot{V}_{\mathrm{ex},2}$ is negative semi-definite as $\tilde{\Theta}$ is not appearing, we apply Theorem 2.19 to show that z converges to zero. As mentioned before, the convergence of s to the origin and v to the negative of $\delta(t)$ follows from this observation. Since the adaptation law (5.32) removes the influence of the estimation error on $\dot{V}_{\mathrm{ex},2}$, we cannot prove the convergence of the estimated parameters to their actual values. It is only ensured that $\tilde{\Theta}$ is bounded. As mentioned already before, this is a common characteristic of adaptive controllers that have been designed based on the CE principle.

A far more significant issue can be observed in (5.32): While the term $\frac{1}{|z_1|}$ is not a problem for the proof for the nominal system since we can show that $\dot{V}_2$ along the trajectories is AC, it becomes an issue in derivation of the adaptation law based on this Lyapunov function. As soon as z_1 approaches zero, the actual goal of the controller, the adaptation law becomes unbounded. Hence, the solution of the differential equation (5.32) cannot be ensured. While the solution concept given in Subsection 2.2.1 deals with a discontinuous right-hand side, it explicitly demands that the right hand side is bounded along the discontinuous surface. This requirement cannot be fulfilled in case of an unbounded adaptation law.

In the following we discuss two possible circumstances under which the adaptation law (5.32) can yet be used: The first solution to the presence of the singularity in (5.32) is to require a bound on the regressor $\omega(t,w)$ of the structured uncertainty. Assuming that there exist the following constants β_1, $\beta_2, \Pi \in \mathbb{R}$ with $\Pi > \frac{1}{2}, \beta_1, \beta_2 > 0$ such that

$$\beta_1\,|s(x)|^{\frac{1}{2}} \leq |\omega_i(t,x)| \leq \beta_2\,|s(x)|^{\Pi}$$

holds component-wise for every $i = \{1, ..., p\}$, the right-hand side of the adaptation law (5.32) is bounded. However, in many practical applications it would be difficult to ensure this requirement. The assumption basically implies that the structured uncertainty has to vanish as soon as the system reaches the sliding surface $s = 0$.

On the other hand, for a large z_1 the rate of adaptation may be slow if the uncertainty $\omega(t,x)$ does not grow fast enough. While stability is still ensured, the performance of the adaptation becomes very poor which is actually not intended by the proposed method.

In the following section we discuss another possible way to avoid the singularity in the adaptation law.

Diagonal Quadratic Lyapunov Function

Another way to overcome the singularity in the adaptation law (5.32) is the selection of a particular P in (5.22). Due to brevity, we consider the case $\delta \equiv 0$ for this analysis. Let P be given as a diagonal matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & p_2 \end{pmatrix} \tag{5.33}$$

with $p_2 > 0$ such that P is symmetric and positive definite according to the requirements of a Lyapunov function. By using this special choice we can apply the following simplification

$$\begin{pmatrix} 1 & 0 \end{pmatrix} P\, z = \begin{pmatrix} 1 & 0 \end{pmatrix} z = z_1,$$

to the adaptation law, which reads as

$$\begin{aligned} \dot{\hat{\Theta}} &= \frac{1}{|z_1|} \Gamma \omega(t, x)\, z_1 \\ &= \Gamma\, \omega(t, x)\, \mathrm{sign}\,(z_1) \\ &= \Gamma\, \omega(t, x)\, \mathrm{sign}\,(s)\,. \end{aligned} \tag{5.34}$$

The result is an adaptation law that does not contain a singularity in z_1.

It seems that this special selection of the matrix P in (5.22) allows us to avoid the issue of the singular point. However, if we calculate the derivative of the Lyapunov function according to (5.29), we obtain

$$Q = -\bar{A}^{\top} P - P\, \bar{A} = \begin{pmatrix} k_1 & k_2\, p_2 - \frac{1}{2} \\ k_2\, p_2 - \frac{1}{2} & 0 \end{pmatrix}. \tag{5.35}$$

If we analyze the definiteness, we can see that for $k_1 > 0$ the matrix Q is positive semi-definite if

$$\left(k_2\,p_2 - \frac{1}{2}\right)^2 = 0$$

and indefinite otherwise. This is actually the same statement yielded by the weak Lyapunov function (5.16) from Subsection 5.4.1. In fact, selecting the free parameter to $p_2 = \frac{1}{2\,k_2}$ leads us to a Lyapunov function

$$V = z^\top P z = z_1^2 + \frac{1}{2\,k_2} z_2^2 = |s| + \frac{1}{2\,k_2} v^2$$

which is, apart from the scaling with $\frac{1}{k_2}$, equivalent to (5.16). Therefore this selection of P is not sufficient to prove robustness against disturbances $\delta(t)$ with a bounded derivative.

5.4.3. Differentiable strict Lyapunov Function

So far we have seen that the choice of the Lyapunov function has significant influence on the adaptation law. In both investigated case we have seen different issues that comes with the particular selection: While the weak Lyapunov function provides a bounded adaptation law, we are not able to show that both, the sliding variable and the controller state v converge to zero. It is also not possible for us to show robustness for the unstructured uncertainty if its derivative is bounded, which is a common use-case for the STA. On the other hand, the approach with the quadratic (5.16) Lyapunov function suffers from an adaptation law that contains a singularity. Since the singular point is unfortunately located at the sliding surface $s = 0$, we cannot implement this adaptation law unless we require some unusual conditions for the regressor $\omega(t, x)$ of the structured uncertainty.

These issues provide the motivation for the results proposed in [15] where the main idea is to use a strict and differentiable Lyapunov function for the design of the adaptation law. It comes in handy that in [98, 119] an interesting approach was published. In the following we use the proposed Lyapunov function to design an adaptation law that avoids the issues from the previous approaches. We will start with a brief recapitulation of the idea of the Lyapunov function given in [119]. For brevity we shall focus only on the case with no disturbance.

Stability Proof for the Nominal System

Let us start with the nominal case, i.e. $\tilde{\Theta} = 0$ and $\delta \equiv 0$. We consider the following Lyapunov function

$$V_3(s,v) = \frac{2}{3}\left(k_1^2 + 2\,k_2\right)\,|s|^{\frac{3}{2}} - k_1\,s\,v + \frac{1}{3\,k_1}\,|v|^3 \tag{5.36}$$

in order to prove stability of the closed loop given in (5.12). As mentioned before, this Lyapunov function [2] is proposed in [119], including a detailed proof of its positive definiteness.

In order to demonstrate that (5.36) is a Lyapunov function for (5.12) in the nominal case we calculate the time derivative of (5.36) and receive

$$\dot{V}_3 = -k_1\,\left(k_1^2 + k_2\right)|s| + 2\,\left(k_1^2 + k_2\right)\lceil s\rfloor^{\frac{1}{2}}\,v - k_1\,|v|^2 - \frac{k_2}{k_1}\,\lceil s\,v\rfloor^0\,|v|^2 \tag{5.37}$$

where $k_1, k_2 > 0$ denote the controller parameters given in (5.11). By defining the following constants

$$\begin{aligned} \eta_1 &:= k_1^2 + k_2, & \eta_2 &:= 2\,\left(k_1^2 + k_2\right), \\ \eta_3 &:= k_1 - \frac{k_2}{k_1}, & \eta_4 &:= k_1 + \frac{k_2}{k_1} \end{aligned} \tag{5.38}$$

and using the state transformation given in (5.23) with $\dot{\delta}(t) = 0$, we can express (5.37) with

$$\dot{V}_3 \leq \begin{cases} -\eta_1\,z_1^2 + \eta_2\,|z_1|\,|z_2| - \eta_3\,|z_2|^2, & z_1 z_2 \geq 0 \\ -\eta_1\,z_1^2 - \eta_2\,|z_1|\,|z_2| - \eta_4\,|z_2|^2, & z_1 z_2 < 0 \end{cases} \tag{5.39}$$

which can be simplified to

$$\dot{V}_3 \leq \begin{cases} -z^\top W_1\,z, & z_1 z_2 \geq 0 \\ -z^\top W_2\,z, & z_1 z_2 < 0 \end{cases} \tag{5.40}$$

[2] The actual proposed Lyapunov function is $V = \frac{2}{3}\gamma_1\,|s|^{\frac{3}{2}} - \gamma_{12} s\,v + \frac{1}{3}\gamma_2\,|v|^3$ and we select $\gamma_1, \gamma_{12}, \gamma_2$ to $\gamma_1 = k_1^2 + 2\,k_2$, $\gamma_{12} = k_1$, $\gamma_2 = \frac{1}{k_1}$.

with W_1 and W_2 defined as

$$W_1 =: \begin{pmatrix} \eta_1 & -\frac{\eta_2}{2} \\ -\frac{\eta_2}{2} & \eta_3 \end{pmatrix} \quad \text{and} \quad W_2 := \begin{pmatrix} \eta_1 & \frac{\eta_2}{2} \\ \frac{\eta_2}{2} & \eta_4 \end{pmatrix}. \tag{5.41}$$

Now, to show that $\dot{V}_3$ is negative definite, we have to validate the positive definiteness of the matrices W_1 and W_2. As shown in [119], if the parameters satisfy the inequality

$$k_1^2 > k_2 \tag{5.42}$$

then both matrices W_1 and W_2 are positive definite and V_3 as well. At this point we shall notice that $\dot{V}_3$, in contrast to $\dot{V}_2$, is free of singularities. This comes at the price of an additional requirement on the controller gains given by (5.42).

Furthermore, this Lyapunov function can also be used to prove robustness against the disturbance $\delta(t)$ in (5.10). Let us assume that the time derivative of $\delta(t)$ is bounded as stated in (5.25). By the change of coordinates

$$\tilde{v} = v + \delta(t) \tag{5.43}$$

we obtain the following structure for the system dynamics

$$\begin{aligned} \dot{s} &= -k_1 \lceil s \rfloor^{\frac{1}{2}} + \tilde{v} \\ \dot{\tilde{v}} &= -k_2 \text{sign}(s) + \dot{\delta}(t) \end{aligned} \tag{5.44}$$

which is required to show that the closed loop system is invariant to $\delta(t)$. By replacing v with $\tilde{v}$ in (5.36), we can conclude that if

$$k_2 > \Omega_2 \tag{5.45a}$$

$$k_1^2 > \frac{(k_2 + \Omega_2)^2}{k_2 - \Omega_2}, \tag{5.45b}$$

is satisfied, s and $\tilde{v}$ will tend to zero. Therefore, v will take the negative value of $\delta(t)$ which eliminates the influence of the unstructured uncertainty $\delta(t)$ on the sliding surface $s = 0$ completely.

Adaptive Extension

As already stated in the beginning of this section, using the Lyapunov function (5.36) is motivated by the drawbacks of the other two approaches.

Theorem 5.5. *Let the gains k_1 and k_2 fulfill the inequalities* (5.45) *and* $\Gamma \in \mathbb{R} : \Gamma > 0$. *Then the control law* (5.11) *will stabilize the system* (5.10) *in the presence of the structured uncertainty* $\Theta^\top \omega(t, x)$ *with the adaptation law*

$$\dot{\hat{\Theta}} = \Gamma\, \omega(t, x) \left(\left(k_1^2 + 2\, k_2 \right)\, |s|^{\frac{1}{2}} - k_1 \tilde{v} \right) \tag{5.46}$$

for an unknown parameter vector Θ.

Proof. Similar to the other design examples, we will use the proposed design procedure. We start by extending the nominal Lyapunov function V_3 by a quadratic term of the estimation error defined in (5.13). In order to include the unstructured uncertainty, we have to replace v with z_2 from (5.23b) so as to achieve the structure required in [119]. We end up with

$$V_{\text{ex},3}(s, v, \tilde{\Theta}) = \frac{2}{3} \left(k_1^2 + 2\, k_2 \right)\, |s|^{\frac{3}{2}} - k_1\, s\, \tilde{v} + \frac{1}{3\, k_1}\, |\tilde{v}|^3 + \frac{1}{2\, \Gamma} \tilde{\Theta}^\top\, \tilde{\Theta}\,. \tag{5.47}$$

Calculating the time derivative yields

$$\dot{V}_{\text{ex},3} = \dot{V}_3 + \left(\left(k_1^2 + 2\, k_2 \right)\, |s|^{\frac{1}{2}} - k_1 \tilde{v} \right)\, \tilde{\Theta}^\top\, \omega(t, x) + \frac{1}{\Gamma} \tilde{\Theta}^\top\, \dot{\tilde{\Theta}}$$

which can be rewritten to

$$\dot{V}_{\text{ex},3} = \dot{V}_3 + \tilde{\Theta}^\top \left(\omega(t, x) \left(\left(k_1^2 + 2\, k_2 \right)\, |s|^{\frac{1}{2}} - k_1 \tilde{v} \right) - \frac{1}{\Gamma} \dot{\hat{\Theta}} \right) \tag{5.48}$$

by using $\dot{\tilde{\Theta}} = -\dot{\hat{\Theta}}$. If we replace $\dot{\hat{\Theta}}$ in (5.48) with the proposed adaptation law (5.46), we render $\dot{V}_{\text{ex},3}$ independent of the estimation error $\tilde{\Theta}$. We known from the nominal case that $\dot{V}_3$ is negative definite which allows us to conclude that (5.48) is negative semi-definite in s, $\tilde{v}$ and $\tilde{\Theta}$. Theorem 2.19 allows us to conclude that s and $\tilde{v}$ will converge to zero despite the presence of the structured uncertainty. □

Similar to the two other approaches, the derivative of (5.47) does not contain the estimation error. Hence, we cannot be sure that the parameter estimation $\hat{\Theta}$ converges to the actual values.

The significant difference to the design with the quadratic Lyapunov function is the fact, that the latest adaptation law (5.46) is continuous in s and $\tilde{v}$. Therefore we do not face any issues while implementing this adaptation law.

5.4.4. Discussion

In this section, we have discussed several approaches to combine the certainty equivalence (CE) with sliding-mode control (SMC). The idea is to use the well known super-twisting algorithm (STA) and extend it by an adaptive part that compensates for structural uncertainties. In contrast to the adaptive-gain super-twisting algorithm (AGSTA), we do not propose a direct adaptation scheme. Instead we favor an indirect approach that adapts the unknown parameters of the plant. This procedure brings several advantages: First of all, the proposed approach allows to include state-depended uncertainties, which is a significant improvement compared to the AGSTA and the conventional STA as well. Since the STA requires an upper bound on the derivative of the disturbance, it cannot handle state dependent uncertainties because the state-dependency would include the derivative of the system states into the upper bound estimation. While the Lyapunov function requires the upper bound on the derivative in order to prove that the state variables are bounded, we cannot assume that the states are finite in order to find the upper bound. This would result in a circular argument.

Another important advantage of the proposed method are the gains of the sliding-mode part. This comes especially into play if the overall amplitude of the structured uncertainty is significantly larger than the one oft he unstructured uncertainty, i.e. loosely speaking

$$\|\Theta^\top \omega(t, x)\| >> \|\delta(t)\|.$$

In the conventional design, the sliding-mode controller would have to compensate for both uncertainties and needs an arbitrary high controller gain. This can be avoided by the proposed approach because the sliding-mode part of the controller (5.11), is only responsible for the unstructured part of the uncertainty. While this is not the main intention of this proposed combination, reducing the gains of the sliding-mode part will also help reducing the chattering amplitude as we have already discussed in Section 4.3.

We will investigate in Chapter 6 whether we can underline this statement in a practical setup.

Compared to conventional adaptive controllers, like MRAC, we are to steer the system exactly to zero, even in the presence of unstructured uncertainty. In contrast to those approaches, we do not require any modifications on the update law that would inhibit the convergence to the origin.

At this point we should draw the attention to some drawbacks of the certainty-equivalence based super-twisting algorithm (CESTA) approach in this section. As we have seen, the first adaptation law can easily be implemented but the Lyapunov function for the nominal closed loop is not suitable to ensure robustness against the typical disturbances for the STA. The seconds approach introduces a singularity in the adaptation law which can be avoided. While the third law seems to solve the problems from the two approaches before, it has a drawback that should not be uncommented: The adaptation rule (5.46) contains the variable $\tilde{v}$ from the nominal Lyapunov function. From (5.43) we can obtain the rule to compute $\tilde{v}$ which contains the unstructured uncertainty $\delta(t)$. To use this adaptation rule in a practical application, it would be necessary to estimate this disturbance for example with an unknown input observer.

Hence in the following section, we investigate whether we can find an adaptive higher-order sliding-mode with the same robustness against structured uncertainty that does not possess the mentioned disadvantages.

5.5. Adaptive Arbitrary-Order Sliding Mode

Up to this point, we have discussed adaptive sliding-mode approaches that stabilize a manifold in state-space. This manifold is designed such that the zero-dynamics are stable and the system will converge to zero on this manifold. Furthermore, the manifold is described by a sliding-variable, which is zero as soon the system reaches the manifold. However, the SMC approaches we have discussed so far, require a relative degree of one between the sliding variable and the control input.

Now, we address the system (5.9) directly. In other words, the proposed design should not require a sliding-surface which separates this procedure from the STA design in the previous section.

Let us now consider the disturbed integrator chain given in (5.9). Similar to the previous adaptive controller designs, we require the same assumptions: The uncertainty should be matched, see Assumption 5.1, and it can be separated into a structured and unstructured part, according to Assumption 5.2. We assume that the regressor $\omega(t, x)$ of the structured uncertainty is known. It is important to notice that $\omega(t, x)$ can be state-depended as well. Furthermore, we assume that

$$|\delta(t)| < \Omega \tag{5.49}$$

holds $\forall t > 0$.

The design of the adaptive control will now follow the same procedure described in Section 5.3: We will use a higher-order sliding-mode controller to stabilize the nominal system, i.e. the parameters Θ of the structured uncertainty are perfectly known. The stability of the nominal controller is validated with a Lyapunov function, which will be used to obtain the adaptation law.

We propose the following control law

$$u = u_{\text{smc}}(x) - \hat{\Theta}^\top \omega(t, x) \tag{5.50}$$

where u_{smc} denotes the discontinuous control law which will be presented in the following section. The remaining term $\hat{\Theta}^\top \omega(t, x)$ compensates the structured uncertainty.

5.5.1. Nominal System Stabilization

We use a higher-order sliding-mode controller for the stabilization of the system (5.9) in the nominal case, i.e. $\hat{\Theta} = \Theta$. It is easy to see that the control law (5.50) will eliminate the structured uncertainty in this case completely. Furthermore, the controller should be designed such that we can apply the approach to an arbitrary system order. Discontinuous controllers that solve this task are presented in [80]. Compared to FOSM or STA design, the most significant difference is that the design of sliding surface with relative degree of one is not needed. The controllers from [80] can be used to an arbitrary system order. Unfortunately, the stability proofs in [80] rely on geometric considerations which prevents us to obtain a suitable adaptation law according to the proposed design procedure. However, just recently, a constructive method for a Lyapunov function for these HOSM controllers was presented in [34, 94]. This circumstance enables us to extend the controllers from [80] by an adaptive part.

It shall be noted that due to brevity we focus on the so-called Relay Polynomial Controllers which is only one of the possible HOSM controllers. The stability proof for Nested or Quasi-Continuous Controllers from [80] can be obtain in a similar manner, see [34, 94].

In general, the discontinuous controller shall be given by

$$u_{\text{smc}} = \varphi_n(x) \tag{5.51}$$

with n the system order of (5.9) and

$$\varphi_n(x) = -k_n \text{sign}\left(\sigma_n(x)\right). \tag{5.52}$$

with k_n selected such that $k_n > \Omega$ and Ω from (5.49). Here, σ_i for $i = 2, \ldots, n$ is defined as

$$\begin{aligned} \sigma_1(x_1) &= \lceil x_1 \rfloor^{\frac{n+\alpha}{n}}, \\ \sigma_i(\bar{x}_i) &= \lceil x_i \rfloor^{\frac{n+\alpha}{n-i+1}} + k_{i-1}^{\frac{n+\alpha}{n-i+1}} \sigma_{i-1}(\bar{x}_{i-1}). \end{aligned} \tag{5.53}$$

for some $k_i > 0, i = 1, ..., n$. For a system order of 2 or 3, the control law reads as

$$\varphi_2 = -k_2 \operatorname{sign}\left(\lceil x_2 \rfloor^{2+a} + k_1^{\frac{2+\alpha}{2}} \lceil x_1 \rfloor^{\frac{2+\alpha}{2}} \right) \tag{5.54a}$$

$$\varphi_3 = -k_3 \operatorname{sign}\left(\lceil x_3 \rfloor^{3+a} + k_2^{3+\alpha} \lceil x_2 \rfloor^{\frac{3+\alpha}{2}} + k_2^{3+\alpha} k_1^{\frac{3+\alpha}{2}} \lceil x_1 \rfloor^{\frac{3+\alpha}{3}} \right) \tag{5.54b}$$

with constant $\alpha \geq 0$, see [94]. We can easily verify that the control law (5.52) is r-homogeneous of degree 0 according to Definition 2.21 with

$$r = (n, n-1, \cdots, 1) \ . \tag{5.55}$$

If we apply the proposed controller (5.50) to the system (5.9), we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &\in \varphi_n(x) + \delta(t) \end{aligned} \tag{5.56}$$

as the closed loop dynamics if the parameters of the structured uncertainty are perfectly known.

Since the geometric stability proofs for HOSM controllers provided in [80] are not suitable for the intended adaptive controller design, we will rely on the Lyapunov functions presented in [94]. At this point, we only provide a sketch of the proof for the sake of brevity. A complete verification of the stability can be found in [34] and [94], respectively.

The special feature of the proposed Lyapunov function for (5.56) is the recursive construction. We start with

$$V_1(x_1) = \frac{n}{m} |x_1|^{\frac{m}{n}} \tag{5.57}$$

in the first step and m given by $m = n + \alpha + 1$ with α from the control law (5.52). The subsequent V_i are obtained by evaluating

$$V_i(\bar{x}_i) = V_{i-1}(\bar{x}_{i-1}) + W_i(\bar{x}_i) \tag{5.58}$$

with W_i given as

$$W_i(\bar{x}_i) = \frac{r_i}{m} |x_i|^{\frac{m}{r_i}} - \lceil \nu_{i-1}(\bar{x}_{i-1}) \rfloor^{\frac{m-r_i}{r_i}} x_i + \left(1 - \frac{r_i}{m}\right) |\nu_{i-1}(\bar{x}_{i-1})|^{\frac{m}{r_i}} \tag{5.59}$$

while r_i is taken from (5.55) and

$$\begin{aligned} \nu_1(x_1) &= -\, k_1 \lceil \sigma_1 \rfloor^{\frac{r_2}{n+\alpha}} \;, \\ \nu_i(\bar{x}_i) &= -\, k_i \lceil \sigma_i(\bar{x}_i) \rfloor^{\frac{r_{i+1}}{n+\alpha}} \;. \end{aligned} \tag{5.60}$$

Note that the vector $\bar{x}_i \in \mathbb{R}^i$ contains the i first elements of x, i.e. $\bar{x}_i = (x_1, ..., x_i)$.

The first step in proving the stability with the help of a Lyapunov function is to validate whether $V_n(x)$ is a valid candidate. Therefore we have to check if V_n is positive definite. We can obtain directly from (5.57) that V_1 is positive as long as $x_1 \neq 0$. In order to show that the subsequent V_i up to V_n are positive definite, we use a modified version of Young's Lemma from [48], given by

$$a \lceil b \rfloor^{\beta} \leq \frac{1}{\gamma} |a|^{\gamma} + \left(1 - \frac{1}{\gamma}\right) |b|^{\beta \frac{\gamma}{\gamma-1}} \;, \tag{5.61}$$

and select $a = x_i$, $b = \nu_{i-1}(\bar{x}_{i-1})$, $\beta = \frac{m-r_i}{r_i} > 1$ and $\gamma = \frac{m}{r_i} > 1$. Now, we can verify that W_i is positive semi-definite and $W_i = 0$ if and only if $x_i = \nu_{i-1}$. As a result, V_n is a valid Lyapunov function candidate.

For the stability proof, we have to show that the derivative of the Lyapunov function is negative definite. Calculating the time derivative of V_n along the system trajectories of (5.56) yields

$$\dot{V}_n = \sum_{i=1}^{n-1} \frac{\partial V_n}{\partial x_j} x_{j+1} + \frac{\partial V_n}{\partial x_n} \left(\delta(t) + \varphi_n(x)\right). \tag{5.62}$$

Since we assumed that k_n is selected such that $k_n > \Omega$, we can rewrite the equation (5.62) as

$$\dot{V}_n \leq \sum_{i=1}^{n-1} \frac{\partial V_n}{\partial x_j} x_{j+1} + \frac{\partial V_n}{\partial x_n} \left(-\left(k_n - \Omega\right) \lceil \sigma_n(x) \rfloor^{0}\right) \;. \tag{5.63}$$

with Ω given in (5.49). Furthermore, we apply the following simplification

$$\begin{aligned} \frac{\partial V_n(x)}{\partial x_n} = \frac{\partial W_n(x)}{\partial x_n} &= \lceil x_n \rfloor^{n+\alpha} + k_{n-1}^{n+\alpha} \lceil \sigma_{i-1}(\bar{x}_{n-1}) \rfloor \\ &= \sigma_n(x) \end{aligned} \tag{5.64}$$

based on the fact that $r_n = 1$ as stated in (5.55). Now we can use (5.64) in order to rewrite (5.63) as

$$\dot{V}_n \leq \sum_{i=1}^{n-1} \frac{\partial V_n}{\partial x_j} x_{j+1} - (k_n - \Omega) \left| \sigma_n(x) \right| . \tag{5.65}$$

It is now left to us to show that (5.65) is negative definite.

In the first step, we assume that

$$\frac{\partial V_n}{\partial x_j} x_{j+1} < 0 \tag{5.66}$$

holds for $\sigma_n(x) = 0$. In that case, for a sufficiently large k_n, we can conclude that $\dot{V}_n < 0$. Using (5.65), we can see that k_n has to fulfill

$$k_n > \frac{\sum_{i=1}^{n-1} \frac{\partial V_n}{\partial x_j} x_{j+1}}{|\sigma_n(x)|} + \Omega \tag{5.67}$$

for all $x \in \mathbb{R}^n$. At this point the homogeneity properties come in to play: We known that $V_n(x)$ is an r-homogeneous function of degree m according to the Definition 2.21. Furthermore, $\frac{\partial V_n}{\partial x_j} x_{j+1}$ is a r-homogeneous function of degree $m - 1 = n + \alpha$ which is equal to the homogeneous degree of $\sigma_n(x)$. Hence, the fraction in (5.67) is a function of homogeneous degree of zero. Due to the homogeneity, we only have to verify this condition on the homogeneous unit sphere, see Definition 2.23. In fact it has to be shown that the fraction is a lower semi-continuous function. Indeed, by the usage of Lemma 4 in [94], we can verify the existence of an upper bound of the fraction and be sure that we are able to find a k_n such that (5.67) holds.

The next step is to verify that (5.66) is satisfied when $\sigma_n = 0$. At this point we will refer to [94] for a detailed proof due to brevity. However, the idea is that $\nu_{i-1}(\bar{x}_{i-i})$ given in (5.60) acts like a virtual controller for x_i. The well-disposed reader might recognize a similarity to the classical backstepping. In fact, as it is shown in [34, 94], we can apply a similar argumentation to conclude that (5.66) holds.

Finally, we can summarize that $\dot{V}_n$ is negative definite as long the gains $k_1, \dots, k_n$ are selected properly. There exists several methods to obtain a suitable selection of the gains $k_1, \dots, k_n$, some possible solutions are given in [94]. Furthermore, it is demonstrated in [34, 94] that the origin of the nominal system is finite time stable.

5.5.2. Adaptive Extension

In the previous section we assumed in the so-called nominal case that the parametric uncertainty Θ is perfectly known. Now it is time to relax this condition in order to derive an adaptation law that is capable of dealing with the structured uncertainty. The following approach is presented in [13].

Theorem 5.6. *Let the disturbance $\delta(t)$ in* (5.9) *be bounded by* (5.49) *and $\omega(t, x)$ be a known function. Then, the control law* (5.50) *in combination with the adaptation law*

$$\dot{\hat{\Theta}} = \Gamma \, \sigma_n(x) \, \omega(t, x) \tag{5.68}$$

where $\sigma_n(x)$ from (5.53) *with the initial value $\hat{\Theta}(0) = \hat{\Theta}_0$ and $\Gamma \in \mathbb{R}$ with $\Gamma > 0$, will stabilize the system* (5.9) *despite the presence of the (possibly state-dependent) structured uncertainty $\Theta^\top \omega(t, x)$.*

Proof. Applying the control law (5.50) to (5.9) yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &\in \varphi_n(x) + \tilde{\Theta}^\top \omega(t, x) + \delta(t) \end{aligned} \tag{5.69}$$

Since we do not assume any bounds on the structured uncertainty $\Theta^\top \omega(t, x)$, we cannot use the proof for the nominal higher-order closed-loop system.

We will start this proof by extending the Lyapunov function (5.58) by a term that contains the parameter estimation error $\tilde{\Theta} := \Theta - \hat{\Theta}$. This Lyapunov function candidate reads as

$$V_{\text{ext}}(x, \tilde{\Theta}) = V_n(x) + \frac{1}{2\,\Gamma} \tilde{\Theta}^\top \tilde{\Theta} \tag{5.70}$$

with $\Gamma > 0$.

Calculating the time derivative yields

$$\dot{V}_{\text{ext}} = \sum_{i=1}^{n-1} \frac{\partial V_n}{\partial x_j} x_{j+1} + \frac{\partial V_n}{\partial x_n} \Big(\varphi_n(x) + \delta(t) + \tilde{\Theta}^\top \omega(t,x) \Big) + \tilde{\Theta}^\top \, \Gamma^{-1} \, \dot{\tilde{\Theta}}$$

which can be rearranged to

$$\dot{V}_{\text{ext}} = \sum_{i=1}^{n-1} \frac{\partial V_n}{\partial x_j} x_{j+1} + \frac{\partial V_n}{\partial x_n} \left(\varphi_n(x) + \delta(t) \right) + \frac{\partial V_n}{\partial x_n} \tilde{\Theta}^\top \omega(t,x) - \frac{1}{\Gamma} \tilde{\Theta}^\top \dot{\hat{\Theta}} \tag{5.71}$$

by using the fact that the plant parameters are constant, i.e. $\dot{\tilde{\Theta}} = -\dot{\hat{\Theta}}$. Due to (5.64) and , we can simplify (5.71) to

$$\dot{V}_{\text{ext}} \leq \sum_{i=1}^{n-1} \frac{\partial V_n}{\partial x_j} x_{j+1} - (k_n - \Omega) \left| \sigma_n(x) \right| + \tilde{\Theta}^\top \left(\sigma_n(x) \omega(t,x) - \frac{1}{\Gamma} \dot{\hat{\Theta}} \right) \tag{5.72}$$

Now, by inserting the adaptation law (5.68) into (5.72), we render $\dot{V}_{\text{ext}}$ independent of the estimation error. Since the remaining part is equivalent to (5.65), we can conclude that $\dot{V}_{\text{ext}}$ negative. However, due to absence of $\tilde{\Theta}$, $\dot{V}_{\text{ext}}$ is only negative semi-definite. Therefore, we apply Theorem 2.19 and conclude that x will converge to zero and the estimation error $\tilde{\Theta}$ is bounded. □

5.5.3. Discussion

We shall notice that the given control law does not contain any singular points due to the continuously differentiable Lyapunov function. Furthermore, the adaptation law uses only the system state which are assumed to be known, see Assumption 5.3. In contrast to some of the adaptation laws for the STA, no disturbance estimation is required while we are still able to demonstrate the robustness against unstructured uncertainties in the nominal case.

As it is shown in [94], the nominal system converges in finite time to the origin. Unfortunately we cannot apply the same strategy used in the contribution to show the finite convergence time in the case of the presented adaptive controller. In the context of sliding-mode, we can often establish the following inequality

$$\dot{V} \leq -\kappa V^{\beta} \tag{5.73}$$

with the real scalar constants satisfying $\kappa > 0$ and $0 < \beta < 1$, which cannot be applied in this case. The extended Lyapunov function V_{ext} depends on the system state x and the parameter estimation error. On the other hand, the adaptation law renders $\dot{V}_{\text{ext}}$ independent of $\tilde{\Theta}$ which implies that we cannot find a κ and β such that (5.73) holds. This can be easily seen at the point $x = 0$, where the $\dot{V}_{\text{ext}}$ is zero but V_{ext} remains possibly non-zero since we cannot ensure the convergence of the parameter estimation.

The convergence of the parameters draws the attention to another worth mentioning point: It is a well known fact in classic adaptive control that persistent excitation (PE) ensures parameter convergence, see [99]. As we known from Section 3.5, this is done by the analysis of a linear time-variant system given by (3.32). Unfortunately, we cannot apply the same transformation in order to obtain a linear time-variant system on which the whole PE argumentation is based on.

But this is not the only difference to the approach presented in this section: In the classical adaptive control, the control law is a linear combination of the states which makes the transformation to the linear time-variant system possible in the first place. Hence, whenever the actual system state (or the tracking error $e(t)$ in (3.32)) is zero, the control output of the nominal controller will be zero. Therefore it is necessary that the unstructured uncertainty disappears, which happens either when the estimation error is zero or the scalar product of the regressor and the estimation error is zero.

However, the proposed control approach introduce a third possible solution. Let us have a look at the last line in (5.69): We shall assume that the control error is zero, i.e. $x = 0$ and

$$k_n > \left| \delta(t) + \tilde{\Theta}^{\top} \omega(t, x) \right|$$

holds for (a possible small) estimation error $\tilde{\Theta} \neq 0$. Therefore, $\dot{x}_n = 0$ lies in the convex combination given by the differential inclusion of the last line in (5.69). According to the solution concept defined in Subsection 2.2.1, the system will stay at the origin even if $\tilde{\Theta}^{\top} \omega(t, x) \neq 0$.

Loosely speaking, for small estimation errors, the structured uncertainty will be compensated by the sliding-mode part of the controller and makes the parameter convergence, in contrast to the classic adaptive control, not necessary.

5.6. Conclusion

The previous chapter introduces a different approach in combining adaptive and higher-order sliding mode control. In contrast to the existing methods, we incorporated the structure of an uncertainty into the controller design, whereas conventional approaches assume only an upper bound on the disturbance or its derivative.

The certainty equivalence principle helps us to find an adaptive sliding-mode controller where the stability is ensured by a Lyapunov function. We therefore require that we can start with a Lyapunov function for the nominal case, i.e. it is assumed that the linear parameters of the structured uncertainty are known. As it turned out in Section 5.4, the Lyapunov function for the nominal case has a significant influence on the adaptation law. We have seen that a continuous differentiable Lyapunov function is required in order to obtain a continuous adaptation law, otherwise it may contain a singularity.

It must be pointed out that the STA controller has a drawback that should be mentioned: Whenever a strict Lyapunov function for the STA is used, the adaptation law will contain the difference between the controller state v and the unstructured uncertainty. This is caused by the mixing term of the sliding variable and the controller state in the Lyapunov function which is needed to show asymptotic convergence in the nominal case. However, this will inevitably induce a term containing the difference between the controller state and the unstructured uncertainty into the adaptation law.

Hence, we moved forward to other sliding-mode controllers that can deal with an arbitrary system order with the expectation of avoiding the afore-mentioned issues. Indeed, using a memoryless sliding-mode approach in combination with a continuously differentiable Lyapunov function helped us to create an adaptation law which can be directly implemented.

All in all we were able to extend the class of admissible disturbances in comparison to the conventional approach as we include state depended disturbances as well. We evaluate the advantages that come with this fact in the following chapter in simulation and on a laboratory test-bench.

6. Experiments

Beside the theoretical analysis in the previous chapter, we demonstrate the advantages in some selected experiments.

6.1. Simulation Examples

We will start with two simulation examples that should underline the effect of including the structure of the uncertainty into the controller design. In the first example, we drive an unstable system to the origin in the presence of unstructured uncertainty.

The next example involves the analysis of the performance of the proposed combination of adaptive and higher-order sliding-mode control. We demonstrate that the usage of available information helps to improve the performance while tracking a reference signal.

6.1.1. Unstable System

In the first simulation example, we consider a double-integrator similar to the examples shown in Chapter 3 and 4. The dynamics of the system are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \Delta(t, x_1, x_2) \end{aligned} \tag{6.1}$$

with the initial value $x_0 = (x_1(0), x_2(0))$. The disturbance $\Delta(t, x_1, x_2)$ shall be defined as

$$\Delta(t, x) = x_1 + 2\,\frac{\arctan(x_1)}{\pi}\,\cos(5\,t). \tag{6.2}$$

The objective of this example is to show that the conventional STA and the AGSTA are not robust against state-dependent uncertainties. A closer look at (6.2) reveals that the system (6.1) with this particular uncertainty is unstable. The uncertain term x_1 is simple but yet effective since it will grow with the system state as well.

We compare the proposed adaptive method using the classic STA with the adaptive-gain STA and the fixed gain variant. All three controllers shall use the same sliding-variable defined by

$$s = x_1 + x_2 \tag{6.3}$$

and the control law given in (5.11). Note that the AGSTA uses a slightly modified version where k_1 and k_2 are replaced by their time-variant equivalents $\alpha(t), \beta(t)$ and are adapted using (5.2). Table 6.1 contains the parameters of the three controllers in this

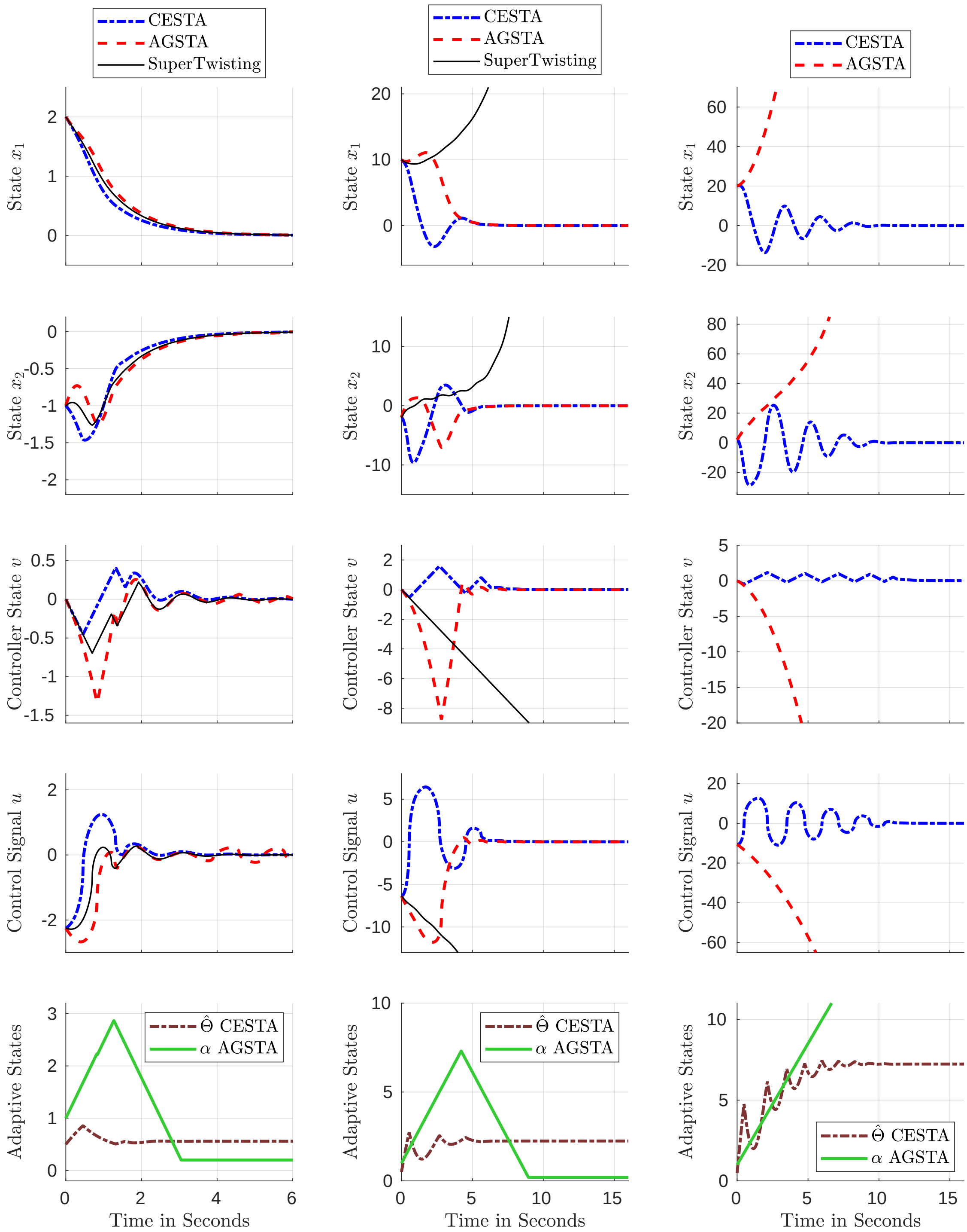
Controller	Parameter	Value
STA	k_1	2.25
	k_2	1
CESTA	k_1	2.25
	k_2	1
	Γ	0.2
AGSTA	ϵ	0.5
	μ	0.01
	ω_1	1.5
	γ_1	1
	α_{m}	0.2
	η	0.1

Table 6.1.: Controller Parameters in the Setpoint Stabilization Example

simulation experiment. Furthermore, for the proposed CESTA we use the control law given by (5.11) and the adaptation rule (5.15). The structured uncertainty is identified with $\omega(t, x) = x_1$ and unknown parameter $\Theta = 1$.

The results of the simulation are shown in Figure 6.1 including three different initial values for each control approach. From the first simulation shown in Figure 6.1a with the initial value $x_0 = (2, -1)^\top$, we obtain that the performance of all three controllers is almost the same close to the origin. Despite the state-dependency of the uncertainty, all three controllers are able to reach the sliding-manifold. So one could assume that even the conventional STA is able to deal with the state-dependent uncertainty which is a misconception that can be found in some publications. It is assumed that if the state of system is bounded, the upper bound of the uncertainty can be found. However, if we change the initial point x_0 to a different value, the upper bound on the uncertainty might not be valid anymore, which is exactly the case in the second simulation example displayed in Figure 6.1b. In that case, the conventional STA is not able to steer the system trajectories to the origin, while the AGSTA and the proposed CESTA are still able to solve the given task. The ingenuous user might assume that any adaptation algorithm is able to fulfill the task of stabilizing the given system due to the observations made in this second example. However, the simulation with the third initial point in Figure 6.1c shows that this assumption is not valid. In this example, all but the proposed CESTA fail to reach the origin. From the lowest plot in Figure 6.1c we obtain that the the gains of the AGSTA increase far beyond the values of the sliding-mode part of the CESTA. Nevertheless, the gains of the controller do not grow fast enough to suppress the uncertainty which increases with the state as well.

At this point we may conclude that the proposed CESTA is able to demonstrate the advantages over a conventional STA and the adaptive-gain version. This could be mainly achieved by including the structure of the uncertainty into the design of the controller. Hence, we can ensure the stability for any initial value.



(a) $x_0 = (2, -1)^\top$ **(b)** $x_0 = (10, -2)^\top$ **(c)** $x_0 = (20, 2)^\top$

Figure 6.1.: Simulation Results for an unstable System

6.1.2. Tracking Control

For the next simulation example, we consider the same double-integrator system (6.1) as before, but with a different disturbance. In contrast to the previous simulation, the uncertainty is given by

$$\Delta(t, x) = \Delta_{\mathrm{s}}(x) + \Delta_{\mathrm{u}}(t) \tag{6.4}$$

where $\Delta_{\mathrm{u}}(t) = 0.5\,\sin(10\,\pi\,t)$ denotes the unstructured uncertainty. The remaining

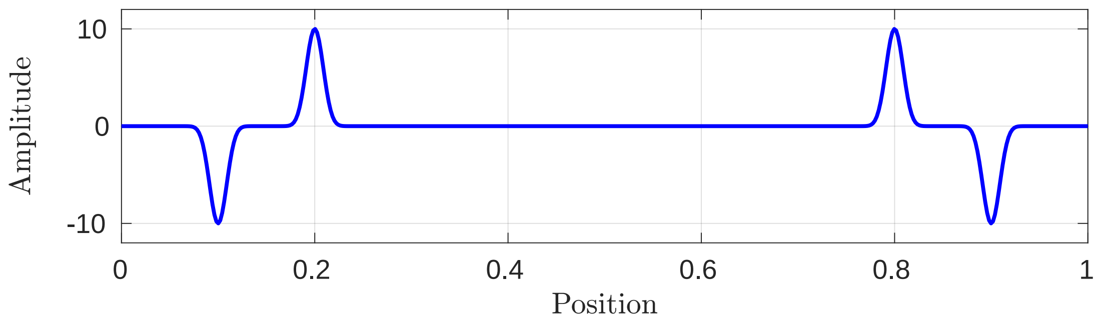


Figure 6.2.: Structured Disturbance for the Tracking System

term $\Delta_{\mathrm{s}}(x)$ is represented by the function shown in Figure 6.2. Based on the reasonable assumption that the position of the spikes in Figure 6.2 is known but the amplitude is not, we can apply an adaptive HOSM controller as proposed in Section 5.5 to the system.

The objective of the controller is to track a reference trajectory given by

$$x_1^\star(t) = \cos(2\,\pi\,\psi\,t) \tag{6.5a}$$

$$x_2^\star(t) = -2\pi\psi\,\sin(2\,\pi\,\psi\,t) \tag{6.5b}$$

$$u^\star(t) = -4\pi^2\psi^2\,\cos(2\,\pi\,\psi\,t) \tag{6.5c}$$

with $\psi = 0.5$. The control law reads

$$u(t) = u^\star(t) + u_{\mathrm{FB}}(t, x)$$

where $u_{\mathrm{FB}}(t, x)$ denotes the corresponding feedback control signal.

In this example we compare the arbitrary HOSM with the adaptive version given by (5.50), where x is replaced by the tracking error $e := x - x^\star$. Furthermore we include a conventional MRAC design, according to Section 3.3, in the comparison. The parameters of the adaptive HOSM are identical to the non-adaptive version and have been selected as $k_1 = 1$ and $k_2 = 3$ and $\alpha = 0$. For the adaptive HOSM and the MRAC, we select the adaptation gain as $\Gamma = 10^5$ and the gain of the feedback part in the MRAC controller to $k^\top = (-30, -60)$.

The results of all three controller implementations are given in Figure 6.3 and show the progress of the system state over a time horizon of 20 seconds. Note that in all three graphs in Figure 6.3, the MRAC is printed in blue, the conventional HOSM controller in black and the proposed adaptive HOSM approach in red.

In the upper graph, we locate the position tracking error e_1. Here, we can clearly see the advantages of the proposed adaptive HOSM approach compared to the two other designs. Indeed, the tracking error of the proposed method is significantly lower than the conventional HOSM controller. We observe that the spikes of the structured uncertainty lead to a relatively large deviation in the position error e_1. While the MRAC approach is able to compensate this effect partially, it suffers from the influence of the unstructured uncertainty Δ_u. Due to the linear feedback, the controller is not able to compensate the overall uncertainty completely which results in the mentioned tracking error.

Compared to that, the proposed adaptive HOSM shows a significantly better performance. After a short adaptation period, the tracking error is negligible in comparison to the two other approaches.

Similar to the first simulation example, the usage of the available information helps to improve the performance of the sliding-mode controller significantly. On the other hand, the discontinuous control law renders the closed-loop system invariant towards unstructured uncertainties which is a clear advantage compared to the MRAC design.

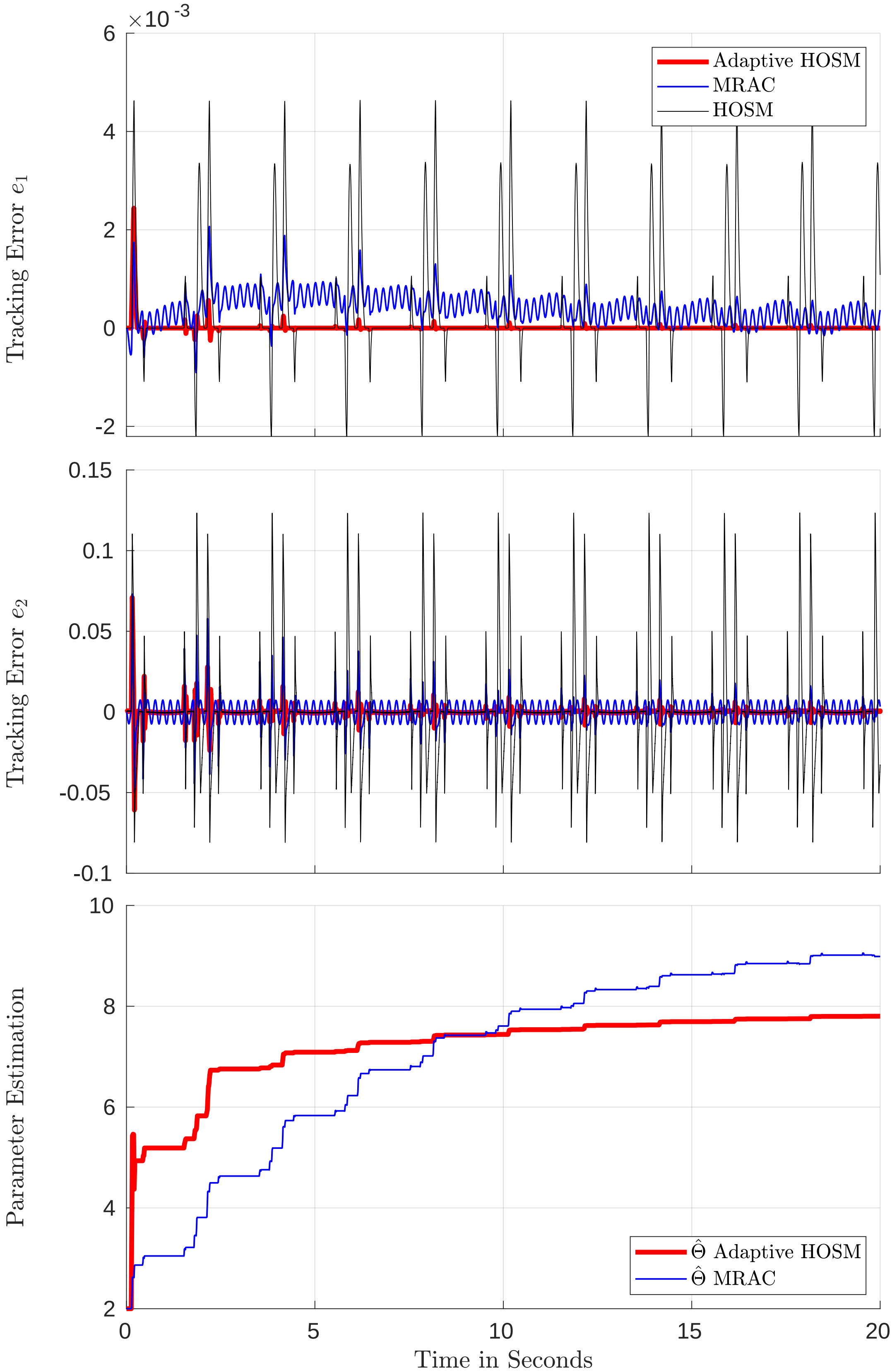


Figure 6.3.: Tracking Performance Comparison

6.2. Laboratory Experiment

Next to the theoretical validation and simulation analysis, a major survey of every control algorithm is the practical implementation. In this section we validate the result from the previous chapters and sections and review some of the statements we posed before.

For the practical test, we use a simple but effective test-bench that is convenient to underline the advantages of the proposed control approach. A plain mechanical setup satisfies those requirement. In Figure 6.4, a picture of the test-bench is displayed. The

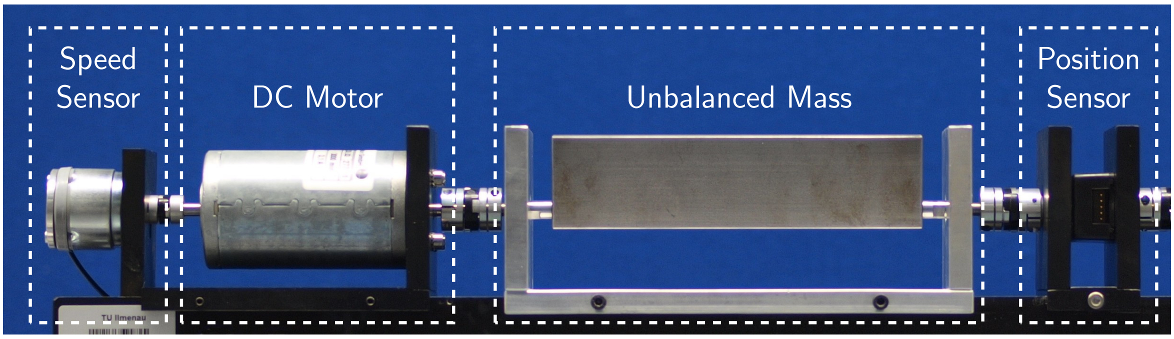


Figure 6.4.: Picture of the Testbench

setup contains just two major elements: An unbalanced mass is coupled to a DC motor. The unbalanced mass is a metal cuboid where the center of mass is located outside the rotary axis. One could compare this to a washing machine whose loaded drum rotates at a certain angular velocity. Next to the mass, we can find a sensor that tracks the angular position. Furthermore, a speed sensor is directly mounted to the DC motor.

The system can be described by the following differential equation

$$J\,\dot{\psi}(t) + d\,\sin(\varphi(t)) + \delta(t) = \tau(t) \tag{6.6}$$

where ψ denotes the angular velocity, φ the angle of the imbalance and τ is the torque generated by the motor. We should keep in mind that the angular position is computed by

$$\varphi(t) = \int_0^t \psi(\tau)\ \mathrm{d}\tau.$$

In favor of brevity we drop the time arguments in the following.

The system dynamics are described by two parameters, namely the moment of inertia J and the parameter d of the imbalance. We expect that d is not available at the moment when the controller is designed. Since we assume that the additional term $\delta(t)$ contains mainly friction effects, thus we abstain from modeling it.

6.2.1. Controller Design

For the given task, we implement a controller based on the CESTA presented in Section 5.4. The controller should maintain a specified angular velocity ψ_d. Since we have a system of order one, we specify the sliding variable simply as the error between the actual velocity and the reference, i.e.

$$s := \psi - \psi_d \,. \tag{6.7}$$

According to the design procedure given in Section 5.3, we have to identify the structured and unstructured uncertainty by analyzing the system (6.6). In this example, the unbalanced load is the structured uncertainty since it can be rewritten in the form $\Theta^\top \omega(t, x)$ from Assumption 5.2 with

$$\Theta = d \tag{6.8a}$$

$$\omega(t, \psi) = \sin(\varphi(t)) \tag{6.8b}$$

where φ can be measured directly. The term $\delta(t)$ in (6.6) should be covered by the sliding-mode part of the controller since it is interpreted as unstructured uncertainty.

To solve the given task, we implemented a CESTA according to the methodology presented in Subsection 5.4.1 since we do not intend to implement a disturbance observer. Hence, the adaptation law is given by (5.15) with an adaptation rate chosen to $\gamma = 2$. We select the gains of the sliding-mode part of the controller as

$$k_1 = 0.15, \quad k_2 = 1.2 \,. \tag{6.9}$$

The given set of parameters is a result of experimental tuning. In general one would stick to the design rule given in [76]. However, we obtained better results regarding chattering with the one proposed in (6.9).

Next to the proposed CESTA, a conventional STA controller was set up in order to compare it with a commonly used approach.

6.2.2. Implementation

All controllers have been implemented on a dSpace real-time platform. The sample time was set to 200μs. The torques were commanded to an underlying current controller.

6.2.3. Results

With this experiment we address the following questions: Can the proposed CESTA approach improve the performance in this particular experiment and confirm the theoretical results? Does the novel design help reduce the chattering?

In Figure 6.5 we can locate the results of the different controller which should track a constant reference signal $\psi_\mathrm{d} = 360\,{}^{deg}\!/\mathrm{s}$. The first graph in Figure 6.5 displays the evolution of the angular velocities, where the reference is given by a thick red line, the angular velocity using the CESTA controller is printed in black and the conventional STA approach in blue. Furthermore, we plot the controller states in the third graph.

We can obtain from the first plot in Figure 6.5 that the proposed controller is able to track the reference signal almost perfectly. As slight overshot in the beginning is the result of a wrong initial value of the estimated parameter $\hat{\Theta}$. However, the adaptation law is able to improve the estimation significantly. Hence, after a short adaptation phase, the CESTA tracks the reference velocity with a minimal deviation.

On the other hand, the conventional STA design is not able to achieve a similar performance and fails to solve the given task. We can see in the first graph in Figure 6.5 that the velocity for this controller shows a significant oscillation with peaks up to $900\,{}^{deg}\!/\mathrm{s}$. This is almost 3 times the actual reference value and would not be acceptable in a practical application.

Now, one could expect that the better performance of the CESTA comes at the price of an increase amplitude of the control signal produced by the adaptive part. However, a closer look at the second graph in Figure 6.5 reveals, that this expectation cannot be confirmed. The control signal of the CESTA approach is displayed as a black line, the conventional design is shown in red. Indeed, the difference between the maximum

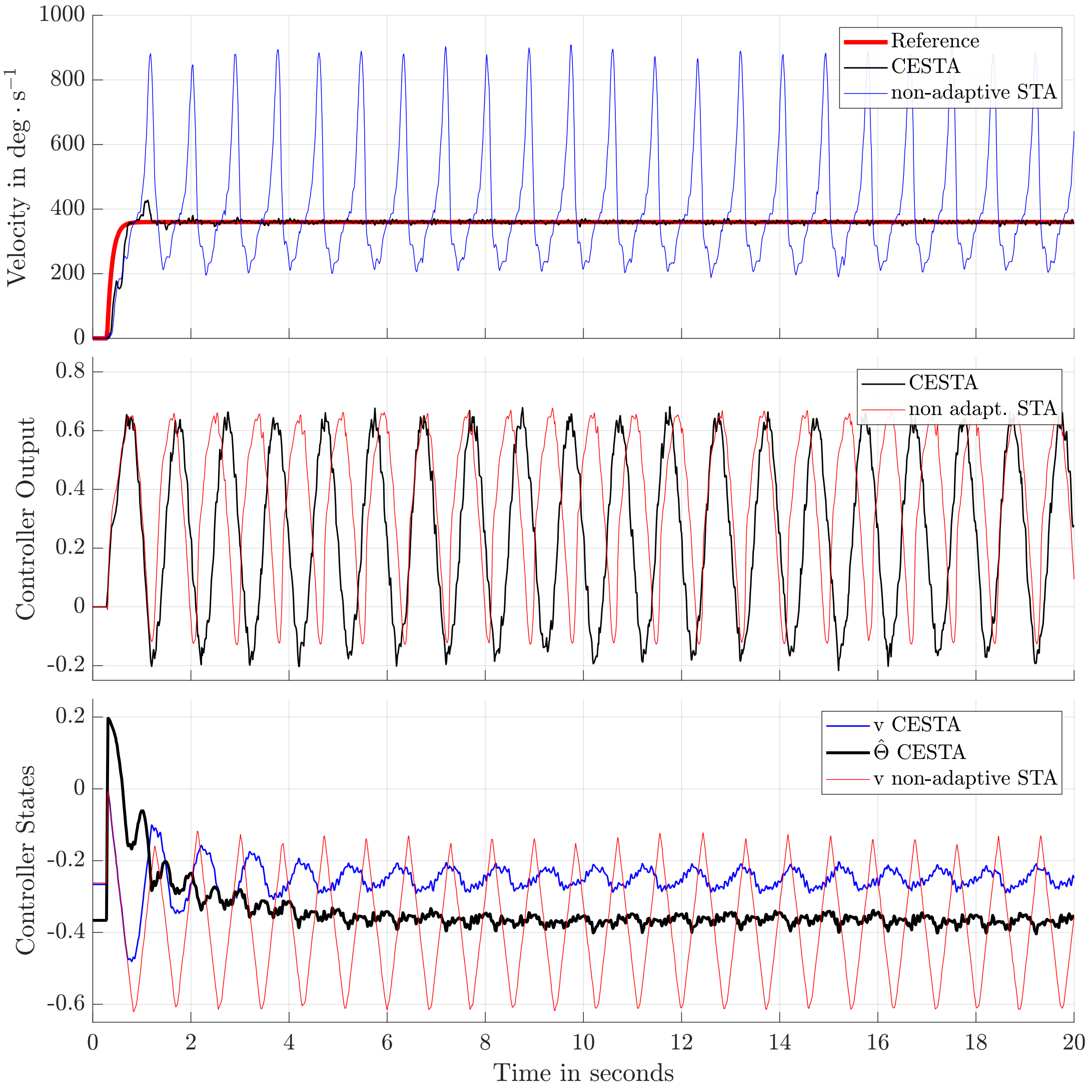


Figure 6.5.: Tracking performance comparison between conventional STA and CESTA

control amplitude is rather small. By using the available information of the structured uncertainty, the CESTA generates a control signal that fits almost perfectly to the uncertainty and therefore keeps the velocity at the desired level.

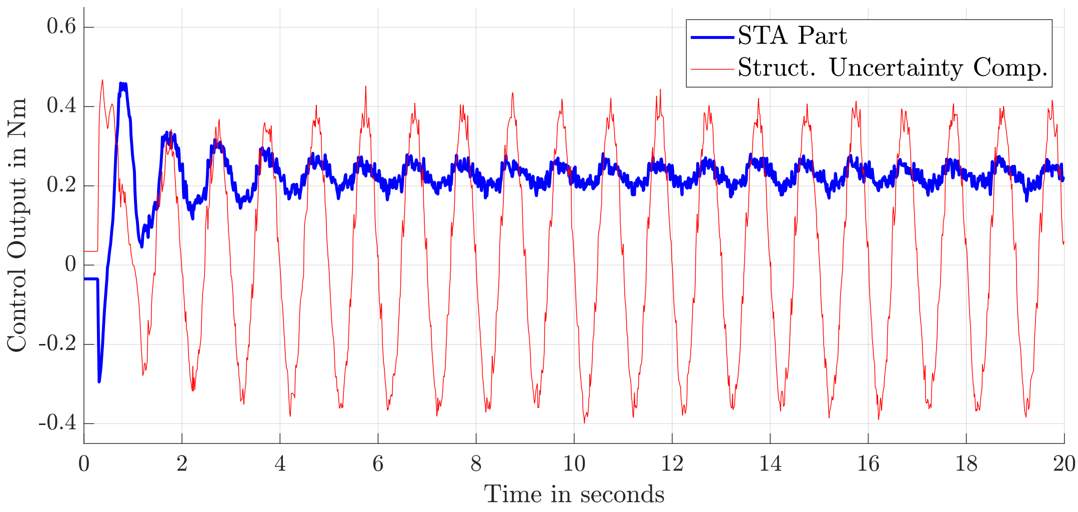


Figure 6.6.: Different components of the CESTA control signal

To underline this statement, Figure 6.6 displays each component of the control signal of the CESTA separately: The sliding-mode control signal is plotted in blue, while the adaptive part that shall compensate the structured uncertainty is marked in red. We known from (5.11) that the control signal which is applied to the plant is the sum of those two parts.

Indeed, it is clear to see in Figure 6.6 that the amplitude of the sliding-mode control signal is significantly lower than in the case of the conventional design as shown in the middle graph of Figure 6.5. Since the adaptive part of the controller compensates the significant part of the unbalanced load, the super-twisting part has to cope only with the remaining friction effects. In a separated experiment, the unknown parameter d is identified to be ≈ 0.4. This is achieved by holding the unbalanced load at an angle of $\pi/2$ such that the regressor (6.8b) takes its maximum value. We can see in Figure 6.5 converges to a vicinity around the identified value which indicates that the adaptive part solves its intended purpose. The remaining uncertainty is no match of the STA controller such that an almost perfect tracking performance.

Now, one could argue that the design of the conventional STA is not accomplished correctly since the gains given in (6.9) are too small. Indeed, if we assume that $d = 0.4$

from the experiment described before, the value L for the upper bound of the derivative of the disturbance can be calculated to $L = 2.51$, assuming a desired speed of 360 $^{\mathrm{deg}}/\mathrm{s}$. By using the design rule given in (4.65), we end up with the following gains

$$k_1 = 2.38, \quad k_2 = 2.76 \,. \tag{6.10}$$

Unfortunately, we observe strong chattering effects when using this set of controller parameters.

The best-tracking performance was achieved with

$$k_1 = 0.4, \quad k_2 = 2.4 \tag{6.11}$$

as control gains for the conventional STA. With this new set of parameters, we repeated

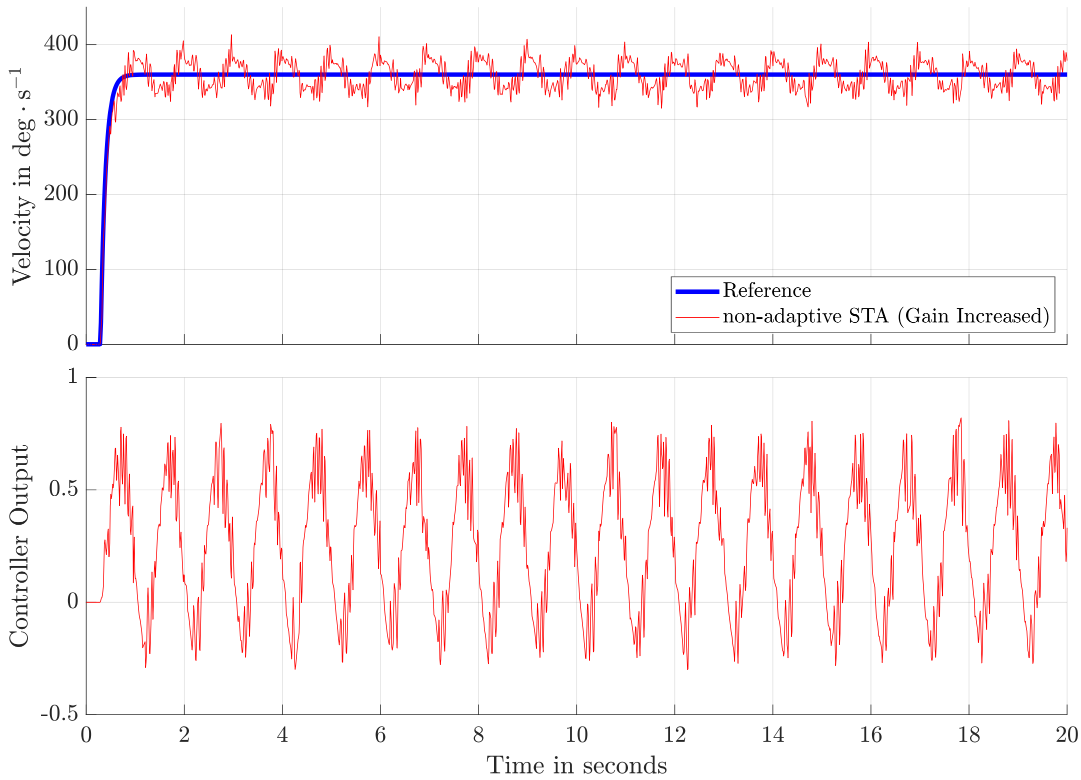


Figure 6.7.: Tracking of the STA with increased gains

the same experiment from above at the same reference speed. This improves the tracking performance significantly. However, in this constellation, we can examine high-frequency noise signals from the test-bench, indicating the presence of a significant chattering which may be caused by the unmodeled current dynamics and the discretization of the controller. In Figure 6.7 the first impression of the sounds emitted by the test-bench can be validated by the control signal which contains much more high-frequency oscillations compared to the first experiment. As we know from Section 4.3, this is a direct consequence of the larger gains in this constellation since the chattering parameters depend strongly on the controller gains, see Table 4.1.

These observations lead to another point that should be considered. As mentioned before, there exist approaches that modify the sliding-mode gains during run-time, for example the AGSTA presented in [125] which is a direct adaptive control method. Another algorithm is given [46], also known as variable-gain super-twisting algorithm (VGSTA), which uses time-variant gains that could also include some structural information of the plant. However, both methods will have to increase the gains to approximately the values given in (6.10) in order to be able to suppress the uncertainties in the system which will lead inevitably to the same chattering effects shown in Figure 6.7.

Finally we can conclude that the proposed adaptive sliding mode controller is a useful extension to the STA approach. Instead of using high-frequency switching to compensate the disturbance, we propose to include as much information about the plant as possible. Hence, the performance of the proposed approach is significantly better than the conventional design. An indirect advantage of the CESTA method is the reduction of the chattering amplitude since the gains of the sliding-mode part can be lowered substantially.

7. Conclusion and Outlook

7.1. Conclusion

In this thesis we have discussed a possible way to combine adaptive control with higher-order sliding-mode (HOSM) approaches. Therefore we analyzed each method separately and collect their individual advantages and deficits.

As it turned out, conventional adaptive control based on the certainty equivalence principle handles structured uncertainties very well. However, the main drawback is the reduced performance in the presence of disturbances that are not covered by the assumed structure.

This gap is covered by sliding-mode approaches that have only little requirements on the uncertainty since they only demand an upper bound on the disturbance. Unfortunately, in the presence of state-depended uncertainties, the existence of this bound cannot be ensured in general during the controller design. Another drawback is the chattering that occurs due to a discrete time implementation or a unmodeled input-dynamics. We have seen that this effect is directly proportional to the gains used in the discontinuous control law.

From this observations we proposed a state-feedback control law that combines an adaptive and a HOSM controller. Before analyzing the actual control approach, some theoretical preparation was necessary since we face Lyapunov functions whose time derivatives is negative semi-definite. Unfortunately, to the best knowledge to the author, the literature does not provide a lemma which could be used to demonstrate convergence to a domain where the derivative is zero in the case of systems with a discontinuous right-hand side. Hence, a theorem based on a modified version of a known result is provided in Subsection 2.3.3, which allows to apply the certainty equivalence (CE) principle to sliding-mode controllers.

In a next step we combine the STA with an adaptive part in order to improve the class of admissible disturbances and the reduction of the gains of the sliding-mode part of the controller. It turned out that the Lyapunov function of the nominal controller has a significant influence on the adaptation law. Indeed, it is advisable to use a continuously differentiable Lyapunov function for the nominal sliding-mode controller design since this will lead to a continuous control law as well.

However, the design with the STA has the disadvantage of requiring an estimation of the unstructured uncertainty in some cases. Hence, we combine the adaptive part with higher-order sliding-mode controllers that use memory-less feedback only. These approaches do not require any further estimation beside the original states of the system, thanks to the nominal Lyapunov function that is continuously differentiable.

The advantages of the proposed method are demonstrated in two simulation examples and on a laboratory test-bench. It is demonstrated that the adaptive controller is able to stabilize the system even in the presence of state-dependent disturbances where a conventional sliding-mode approach fails. Furthermore, we have seen in the laboratory experiment, that the gains of the sliding-mode part of the controller can be reduced significantly compared to a sliding-mode only design. In this particular case, an adaptive-gain sliding-mode approach would not help improve the result since a higher-gain of the sliding-mode part leads to a larger chattering amplitude. The proposed approach is able to reduce the chattering without the use of a special discretization method since the variable structure part has to deal with a much smaller uncertainty than before. Hence, the gains can be selected smaller witch results directly in a lower chattering amplitude.

7.2. Future Work

In view of the observations made in this thesis, several open questions arise that may be considered for future analysis.

It should be analyzed under which conditions a finite convergence time can be achieved. Due to the negative semi-definiteness of the Lypunov function, the finite convergence time cannot be reasoned in the common way. It is assumed that there are some conditions to the regressor of the structured uncertainty that have to be fulfilled such that the closed-loop system will converge in finite time. One possible way to show this property is that by some contraction argument that the system converts to a positive-invariant set in finite time where the sliding-mode part is able to steer it in finite time to the origin.

Additionally it could be analyzed if it is possible to use some condition similar to persistent excitation (PE) and show that the parameter estimation error will converge to a small domain that depends possibly on the gain of the sliding-mode part of the controller. It is expected that in the case of a memory-less sliding-mode design it is not possible for the estimated parameters to converge to their actual values. Indeed, the sliding-mode part does not distinguish between structured and unstructured part and at a certain point, will compensate the structured uncertainty as well. Once the control objective is reached, no further adaptation takes place and the parameter estimation error will remain at a (possibly very small) deviation.

Another field is the adaptive output feedback that should definitely be considered as a future research topic. Since a conventional higher-order sliding-mode observer cannot handle state-dependent disturbances as well, it may be interesting which properties have to be met such that the stability in the presence of uncertainties can be ensured if the estimation of the plant parameters is used in the controller and observer as well.

Finally, the discrete time implementation should be investigated as well, since the latest implicit discretized sliding-mode controllers show a significant improvement in terms of chattering reduction. However, these approaches may converge only to a small vicinity around the origin as soon as a disturbance is present. It should be analyzed whether this domain can be reduced when the proposed controller is implemented using an implicit discretization.

Appendix

A. Miscellaneous

A.1. Feedback Linearization

The following collection of definitions and results can be found in [63, 68] and are given here without any proofs.

Definition A.1 (Feedback Linearizable [68]). *The nonlinear system*

$$\dot{x} = f(x) + g(x)\,u \tag{A.1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and sufficiently smooth vector fields f, g on a domain $\mathcal{D} \subset \mathbb{R}^n$, is said to be feedback linearizable at $\bar{x} \in \mathcal{D}$ if there exists a diffeomorphism $T : \mathcal{D} \to \mathbb{R}^n$ such that $\mathcal{D}_z = T(\mathcal{D})$ contains the origin and the change of variables $z = T(x)$ transforms the system (A.1) *into the form*

$$\dot{z} = A\,z + B\,\gamma(x)\,(u - \alpha(x)) \tag{A.2}$$

with the pair (A, B) controllable and $\gamma(x)$ nonsingular $\forall x \in \mathcal{D}$.

Definition A.2 (Lie-Derivative). *Let f, h be sufficiently smooth vector fields defined on the domain $\mathcal{D} \subset \mathbb{R}^n$. The expression*

$$\mathcal{L}_f h(x) := \frac{\partial h(x)}{\partial x} f(x)$$

with $x \in \mathcal{D}$ is called the Lie Derivative of h along f.

Definition A.3 (Lie-Bracket). *Let f, g be sufficiently smooth vector fields defined on the domain $\mathcal{D} \subset \mathbb{R}^n$. The Lie Bracket is a vector field defined by*

$$[f, g]\,(x) := \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x)$$

with $x \in \mathcal{D}$.

We define the following notation to express the repeated application of the Lie brackets:

$$\begin{aligned}
\mathrm{ad}_f^0 g(x) &:= g(x) \\
\mathrm{ad}_f g(x) &:= [f, g](x) \\
\mathrm{ad}_f^i g(x) &:= [f, \mathrm{ad}_f^{i-1} g(x)](x), \text{ with } i \geq 2
\end{aligned}$$

Definition A.4 (Smooth Distribution)**.** *Let* $f_1(x), ..., f_m(x)$ *be a set of sufficiently smooth vector field. The distribution* $\Delta(x)$ *is defined as*

$$\Delta(x) = \mathrm{span}\{f_1(x), f_2(x), ..., f_(x)\}, \forall x \in \mathcal{D} \subset \mathbb{R}^n.$$

Definition A.5 (Involutive Distribution)**.** *A distribution* Δ *is involutive if for any two vector fields* $\varphi_1 \in \Delta(x)$ *and* $\varphi_2 \in \Delta(x)$ *the following statement holds*

$$[\varphi_1, \varphi_2] \in \Delta.$$

Theorem A.6 (Feedback Linearization [68])**.** *The system* (A.1) *is said to be feedback linearizable if and only if there is a domain* $\mathcal{D}_0 \subset \mathcal{D}$ *such that*

1. *the matrix* $G(x) = [g(x), \mathrm{ad}_f g(x), ..., \mathrm{ad}_f^{n-1} g(x)]$ *has* rank n *for all* $x \in \mathcal{D}_0$
2. *the distribution* $\Delta = \mathrm{span}\{g(x), \mathrm{ad}_f g(x), ..., \mathrm{ad}_f^{n-2} g(x)\}$ *is involutive in* $\mathcal{D}_0$.

A.2. Comparison Functions

Definition A.7 (Class $\mathcal{K}/\mathcal{K}_\infty$ Function [68])**.** *A scalar continuous function* $V(x)$*, defined for* $x \in [0, a)$ *is said to belong to class* $\mathcal{K}$ *if it is strictly increasing and* $V(0) = 0$*. It is said to belong to class* $\mathcal{K}_\infty$ *if it is defined* $\forall x \geq 0$ *and* $V(x) \to \infty$ *as* $x \to \infty$.

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