



Technische Universität Ilmenau
Institut für Mathematik
Fachgebiet Kombinatorik/Graphentheorie

DISSERTATION

zur Erlangung des akademischen Grades Dr. rer. nat.

COLORINGS OF GRAPHS, DIGRAPHS, AND HYPERGRAPHS

vorgelegt der Fakultät für Mathematik und Naturwissenschaften
der Technischen Universität Ilmenau von

THOMAS SCHWESER

1. Gutachter: Prof. Dr. Michael Stiebitz
2. Gutachter: Prof. Dr. Maria Axenovich
3. Gutachter: Prof. Dr. Eckhard Steffen

Tag der Einreichung: 10. Juli 2020

Tag der wissenschaftlichen Aussprache: 13. November 2020

ABSTRACT

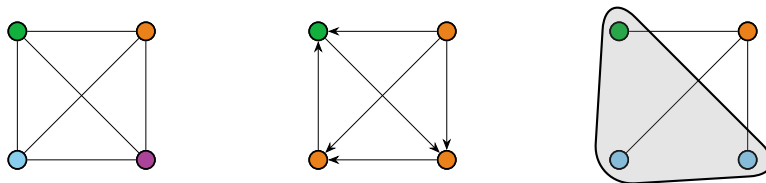
In 1940, a certain William T. TUTTE presented the *Cambridge Philosophical Society* a paper called “Coloring of abstract graphs”; authored by Rowland L. BROOKS. In this paper, he proved one of the most known results in graph coloring theory:

Let G be a graph with maximum degree $\Delta > 2$, all of whose components are distinct from the complete graph $K_{\Delta+1}$. Then, there is a coloring of the vertices of G with Δ colors such that no two vertices that are joined by an edge get the same color.

In the following years, based on BROOKS’ result, a large variety of new research topics in graph coloring arose. Among other things, different proof techniques were developed that can be used in order to verify the above theorem. Furthermore, it became evident that BROOKS’ Theorem could be transferred to many other graph- respectively coloring-concepts. The present thesis puts its focus especially on two of those concepts: hypergraphs and digraphs. A proper coloring of a hypergraph H is a coloring of its vertices such that each edge contains at least two vertices of distinct color. BROOKS’ Theorem for hypergraphs was obtained in 1975 by Rhys P. JONES; in particular, he showed that the above statement equally holds for hypergraphs. In the first part of this thesis, we present several possible ways how to further extend JONES’ theorem. The key element is a partition result, to which the entire second chapter is devoted. Given a hypergraph H and a sequence $f = (f_1, f_2, \dots, f_p)$ of functions mapping from the vertex set of H to the set of non-negative integers, we examine if there is a partition of H into induced subhypergraphs H_1, H_2, \dots, H_p such that each of the hypergraphs H_i is strictly f_i -degenerate. This means that in each non-empty subhypergraph H'_i of H_i there is a vertex v having degree $d_{H'_i}(v) < f_i(v)$. We prove that the condition $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ for all vertices v of H is almost always sufficient for the existence of such a partition and we further show that the exceptional cases can be fully characterized. By choosing an appropriate function f , many well known coloring results can

be derived from the partition result, as shown in the third chapter. In the fourth and fifth chapter we prove two further generalizations of JONES' theorem. Here, one main result is a theorem on DP-colorings of hypergraphs, a recent concept that is based on list-colorings but significantly more general. The second main result links the chromatic number of a hypergraph with its maximum local edge connectivity, *i.e.*, the maximum number of edge disjoint paths between any two of the hypergraph's vertices.

The second part of the thesis is devoted to colorings of digraphs. An acyclic coloring of a digraph is a coloring of its vertex set such that the same-colored vertices induce subdigraphs that do not contain a directed cycle; so called acyclic subdigraphs. This coloring concept is practical for many reasons: on one hand, every proper coloring of an undirected graph G is also an acyclic coloring of the digraph $D(G)$ that results from G by replacing each edge of G with a pair of opposite arcs (and vice versa). On the other hand, many classic coloring results can be transferred to this coloring concept. This also includes BROOKS' Theorem, as proved by MOHAR in 2010. In Chapter 7, similar to Chapter 4, we examine DP-colorings; this time, however, for digraphs. The chapter's main result is the transfer of MOHAR's theorem to DP-colorings. The following chapter deals with critical digraphs. In particular, we describe construction methods for critical digraphs and prove the digraph version of HAJÓS' Theorem. The thesis concludes with a collection of open problems regarding colorings of digraphs.



Colorings of a graph, a digraph, and a hypergraph.

ZUSAMMENFASSUNG

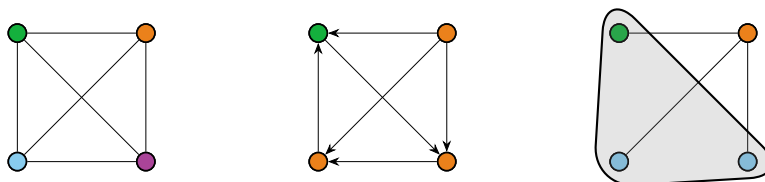
Im Jahre 1940 stellte ein gewisser William T. TUTTE der *Cambridge Philosophical Society* eine Abhandlung mit dem Namen “Coloring of abstract graphs” vor; Verfasser der Arbeit war Rowland L. BROOKS. In dieser Arbeit bewies er eines der bis heute bekanntesten Resultate der Färbungstheorie von Graphen:

Es sei G ein Graph mit Maximalgrad $\Delta > 2$, dessen sämtliche Komponenten verschieden vom vollständigen Graphen $K_{\Delta+1}$ sind. Dann lassen sich die Ecken von G so mit Δ Farben färben, dass keine zwei durch eine Kante verbundenen Ecken dieselbe Farbe erhalten.

In den darauffolgenden Jahrzehnten entwickelten sich, ausgehend von BROOKS Resultat, eine Vielzahl neuer Forschungsthematiken in Bezug auf Färbungen von Graphen. Es wurden unter anderem verschiedene Beweistechniken entdeckt, mit denen sich obiges Theorem verifizieren lässt, zudem zeigte sich, dass sich BROOKS Resultat auch auf viele andere Graphenbeziehungsweise Färbungskonzepte übertragen lässt. Besonderer Fokus liegt in der vorliegenden Arbeit auf zweien dieser Konzepte: Hypergraphen und gerichtete Graphen. Eine zulässige Färbung eines Hypergraphen H ist eine Färbung der Ecken von H so, dass jede Kante wenigstens zwei Ecken unterschiedlicher Farbe enthält. Für Hypergraphen wurde der Satz von BROOKS im Jahre 1975 von Rhys P. JONES publiziert, welcher zeigen konnte, dass obige Aussage ebenso auf Hypergraphen zutrifft. Im ersten Teil dieser Dissertation werden mehrere Möglichkeiten aufgezeigt, das Resultat von JONES weiter zu verallgemeinern. Kernstück ist dabei ein Zerlegungsresultat, dem das zweite Kapitel vollständig gewidmet ist. Zu einem Hypergraphen H und einer Folge $f = (f_1, f_2, \dots, f_p)$ von Funktionen, welche von der Eckenmenge von H in die nicht-negativen ganzen Zahlen abbilden, wird untersucht, ob es eine Zerlegung des Hypergraphen in induzierte Unterhypergraphen H_1, H_2, \dots, H_p derart gibt, dass jeder der Hypergraphen H_i strikt f_i -degeneriert ist. Dies bedeutet, dass jeder der Unterhypergraphen H'_i von H_i eine Ecke v enthält, deren Grad in H'_i kleiner als

$f_i(v)$ ist. Es wird bewiesen, dass die Bedingung $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ für alle Ecken v fast immer ausreichend für die Existenz einer solchen Zerlegung ist und gezeigt, dass sich die Ausnahmefälle vollständig charakterisieren lassen. Durch geeignete Wahl der Funktion f lassen sich schließlich viele bekannte Färbungsergebnisse ableiten, was im dritten Kapitel erörtert wird. Im vierten und fünften Kapitel werden zwei weitere Verallgemeinerungen des Satzes von JONES bewiesen. Ein Hauptresultat ist hierbei das Theorem zu DP-Färbungen von Hypergraphen, einem erst 2016 entwickelten Färbungskonzept, welches auf Listenfärbungen beruht, obgleich aber wesentlich allgemeiner ist. Das zweite Hauptresultat verbindet die chromatische Zahl eines Hypergraphen mit dessen maximalem lokalem Kantenzusammenhang, d.h., der maximalen Anzahl kantendisjunkter Wege zwischen zwei Ecken des Hypergraphen.

Der zweite Teil der Dissertation handelt von Färbungen gerichteter Graphen. Eine azyklische Färbung eines gerichteten Graphen ist eine Färbung der Eckenmenge des gerichteten Graphen, sodass die von gleichgefärbten Ecken induzierten Untergraphen keinen gerichteten Kreis enthalten. Dieses Färbungskonzept ist aus mehreren Gründen praktikabel: Zum Einen ist dadurch jede zulässige Färbung eines ungerichteten Graphen G auch immer eine azyklische Färbung des gerichteten Graphen $D(G)$, welcher entsteht, indem jede Kante von G durch zwei gegensätzlich orientierte Kanten ersetzt wird (und andersherum). Zum Anderen lassen sich auf dieses Konzept viele klassische Färbungsergebnisse übertragen. Dazu zählt auch BROOKS Theorem, wie 2010 von MOHAR bewiesen wurde. Im siebten Kapitel werden, wie bereits in Kapitel 4, DP-Färbungen untersucht, diesmal jedoch für gerichtete Graphen. Das Hauptresultat des Kapitels stellt den Transfer von MOHARS Theorem auf DP-Färbungen dar. Das darauffolgende Kapitel befasst sich mit kritischen gerichteten Graphen. Insbesondere werden Konstruktionen für kritische gerichtete Graphen angegeben und die gerichtete Version des Satzes von HAJÓS bewiesen. Den Abschluss der Dissertation bildet eine Sammlung offener Probleme in Bezug auf Färbungen gerichteter Graphen.



Färbungen eines Graphen, gerichteten Graphen, und eines Hypergraphen.

CONTENTS

Introduction: The Theorems of Brooks and Gallai	1
I. Partitions and Colorings of Hypergraphs	5
1. Preliminaries: Hypergraphs	6
1.1. Basic Terminology	6
1.2. Partitions and Colorings of Hypergraphs	12
1.3. Hypergraph Properties	14
1.4. Criticality	16
1.5. Structure of the Following Chapters	16
2. Partitions of Hypergraphs into Degenerate Subhypergraphs	19
2.1. Introduction and Main Result	19
2.2. Proof of Theorem 2.3	23
3. Generalized Colorings of Hypergraphs	42
3.1. Brooks' Theorem for List-Colorings of Hypergraphs	42
3.2. Additional Degree Constraints	43
3.3. Hypergraph Partitions and Maximum Degenerate Subhypergraphs	46
3.4. Point-partition Number	50
3.5. \mathcal{P} -coloring Problem	53
3.5.1. Colorings with Respect to Hypergraph Properties	53
3.5.2. Proof of Theorem 3.13	59
3.5.3. A Gallai-type Bound for the Degree Sum of Critical Hypergraphs	61
4. DP-coloring of Hypergraphs	67
4.1. Introduction	67

4.2. DP-degree Colorable Hypergraphs	70
4.3. Proof of Theorem 4.2	75
4.3.1. Necessary Tools	75
4.3.2. The Main Proof	78
4.4. A Brooks-type Theorem for χ_{DP}	85
5. Coloring Hypergraphs of Low Connectivity	87
5.1. Connectivity of Critical Hypergraphs	89
5.2. Proof of Theorem 5.2	97
5.3. Splitting Operation	100
5.4. Concluding Remarks	104
II. Colorings of Digraphs	105
6. Preliminaries: Digraphs	106
6.1. Basic Digraph Terminology	106
6.2. Colorings of Digraphs	108
6.3. Focus of our Research	112
7. DP-coloring of Digraphs	114
7.1. The DP-dichromatic Number	114
7.2. DP-degree Colorable Digraphs	117
7.3. Proof of Theorem 7.5	125
7.4. A Brooks-type Theorem for DP-colorings of Digraphs	135
7.5. Ohba's Conjecture for DP-colorings of Digraphs	136
8. Critical Digraphs	138
8.1. Introduction	138
8.2. A Gallai-type Theorem for Critical Digraphs	141
8.3. Construction of Critical Digraphs	147
8.4. The Ore Construction	153
9. Some Nice Conjectures on Digraph Coloring	161
9.1. Emerging Questions Regarding DP-colorings	161
9.2. Digraphs and Variable (Weak) Degeneracy	165
9.3. The Hajós Construction	168

9.4. Critical Digraphs with Few Arcs and Few Vertices	168
References	173
List of Symbols	182

Introduction

THE THEOREMS OF BROOKS AND GALLAI

The chromatic number is probably the most examined parameter in coloring theory of graphs. Given a (simple) graph G , the chromatic number $\chi(G)$ of G is defined as the least integer k for which there is a coloring of the vertices of G with k colors such that no same-colored vertices are adjacent. Such a coloring is also referred to as a proper coloring of G and a proper k -coloring of G . Until the 1940s, papers on graph coloring solely investigated map-colorings; the common goal was to prove the *Four-Color-Conjecture*, which had been postulated by Francis GUTHRIE in 1852. The first paper on coloring of abstract graphs, however, is due to Rowland Leonard BROOKS. In November 1940, he asked his fellow student William T. TUTTE to communicate his work “On colouring the nodes of a network” [26] to the Cambridge Philosophical Society. In this paper, he proved the following remarkable result, nowadays known as BROOKS’ Theorem.

Theorem 1 (BROOKS, 1941). *Let G be a connected graph of maximum degree Δ . Then, $\chi(G) \leq \Delta + 1$ and equality holds if and only if G is a complete graph or a cycle of odd length.* ◊

Although BROOKS initially confirmed his result for disconnected graphs with $\Delta > 2$, he noticed that it is sufficient to prove the theorem only for connected graphs and that $\chi(G) \leq \Delta + 1$ also holds true for $\Delta \in \{0, 1, 2\}$; the theorem as stated above is the common way how BROOKS’ Theorem is referred to. After finishing his studies at Trinity College in Cambridge, BROOKS became an income-tax inspector in London. Throughout his life, he published only five papers [26, 27, 28, 29, 30]; three of those being joint work with Cedric

A.B. SMITH, Arthur H. STONE, and William T. TUTTE. Particularly famous is their paper “The dissection of rectangles into squares”, which they wrote together at Trinity College.

In the 1970s, Paul ERDŐS, Arthur L. RUBIN, and Herbert TAYLOR [44] and, independently, Vadim G. VIZING [120] developed a more general coloring concept, so called list-coloring. Given a graph G , we assign each vertex v a set (list) $L(v)$ of admissible colors. Then, a proper L -coloring of G is a proper coloring of G where each vertices color is taken from the vertices color list. This concept extends the usual coloring concept as a proper k -coloring of a graph resembles a proper L -coloring with $L(v) = \{1, 2, \dots, k\}$ for all vertices v . ERDŐS, RUBIN, and TAYLOR [44] and, independently, Oleg V. BORODIN [19, 20] proved the following degree version of BROOKS’ Theorem. Note that a block of a graph G is a maximal connected subgraph without separating vertices.

Theorem 2 (ERDŐS, RUBIN, and TAYLOR, 1979). *Let G be a connected graph and, for each vertex v of G , let $L(v)$ be a list of at least degree of v many colors. Then, G admits a proper L -coloring, unless each block of G is a complete graph, or an odd cycle.* \diamond

Critical Graphs

The critical graph method is a useful concept for proving results related to graph coloring. As defined by Gabriel A. DIRAC in his PhD thesis “On the Colouring of Graphs” [35] in 1951, a graph G is critical and k -critical if $\chi(G) = k$ but $\chi(G') < k$ for every proper subgraph G' of G . The usefulness of critical graphs comes from two important observations. First of all,

every graph contains a critical subgraph having the same chromatic number.

In order to verify this, let G be a graph with $\chi(G) = k$ and let G' be a minimal subgraph of G with $\chi(G') = \chi(G)$. Then, $\chi(\tilde{G}) < k$ for all $\tilde{G} \subset G'$, but $\chi(G') = k$ and so G' is k -critical. The second important observation is the following. Let \mathcal{P} be a monotone graph property (*i.e.* $G \in \mathcal{P}$ implies $G' \in \mathcal{P}$ for all $G' \subseteq G$) and let ρ be a monotone graph parameter for \mathcal{P} , *i.e.*, a mapping that assigns each graph $G \in \mathcal{P}$ a real number $\rho(G)$ such that $\rho(G') \leq \rho(G)$ whenever G' is a subgraph of G . Then, the following holds true.

If $\chi(G') \leq \rho(G')$ whenever G' is a critical graph from \mathcal{P} , then every graph $G \in \mathcal{P}$ satisfies $\chi(G) \leq \rho(G)$.

This is due to the fact that every graph G contains a critical graph G' having the same chromatic number and so $\chi(G) = \chi(G') \leq \rho(G') \leq \rho(G)$, as claimed. Those two observations are commonly referred to as the *Critical Graph Method*. Thus, if we want to prove upper bounds for the chromatic number it usually suffices to prove the bounds for critical graphs. Moreover, for every fixed $k \geq 1$, every graph G satisfies

$\chi(G) \leq k - 1$ if and only if no k -critical graph is a subgraph of G .

Thus, extensive knowledge about the class $\text{CRI}(k)$ of all k -critical graphs would be helpful in order to solve coloring problems. It is obvious that the complete graph K_k is the only k -critical graph with at most k vertices and that for $k = 1, 2$, there are no others. Furthermore, KÖNIG's characterization of bipartite graphs [68] is equivalent to the fact that the class $\text{CRI}(3)$ coincides with the class of cycles having odd length. However, it is very unlikely that there is a good characterization of the class of $\text{CRI}(k)$ for any fixed $k \geq 4$. This is related to the fact that the decision problem whether a given graph G satisfies $\chi(G) \leq 3$ is already NP-complete (see [112]).

The critical graph method was initially observed by DIRAC. As pointed out in his thesis,

“Every general feature of k -chromatic graphs is possessed also by critical k -chromatic graphs, on the other hand a critical graph is more sharply defined and less arbitrary than a non-critical graph.”

G. A. DIRAC, 1951 [35]

Suprisingly, even 4-critical graphs with fixed order are ten a penny, as proved by Vojtěch RÖDL in the 1970s (see [118]). Apparently, DIRAC told Bjarne TOFT that he did not know about BROOKS' paper when he was writing his thesis, but instead had been inspired to consider graph colorings by Peter UNGER; he was informed about BROOKS' Theorem by his external examiner, who was no other than C.A.B. SMITH. DIRAC later published his results on critical graphs within various papers (see, for instance, [36, 37, 38, 39, 40]). In [39], he proved that every k -critical graph is $(k - 1)$ -edge-connected. As a trivial consequence, every k -critical graph has minimum degree at least $k - 1$. This motivated Tibor GALLAI [48], who continued DIRAC's studies in the well-known papers [48, 49], to classify the vertices of a k -critical graph into two parts. A low vertex of a k -critical graph G is a vertex of degree $k - 1$, vertices of higher degree are called high vertices. Moreover, the low vertex subgraph of G is the subgraph of G that is induced by the set of low vertices; we denote it by G_L . GALLAI [48] proved the following result, thereby extending BROOKS' Theorem.

Theorem 3 (GALLAI, 1963). *If G is a critical graph, then each block of G_L is a complete graph or an odd cycle.* ◊

Note that the above theorem can be deduced from Theorem 2 by using a trick that will occur frequently in this thesis: Let G be a k -critical graph and let B be a block of G_L . Then, as G is critical, $G - V(B)$ admits a proper $(k - 1)$ -coloring. By renaming the colors

if necessary we may assume that all colors are chosen from the set $\{1, 2, \dots, k-1\}$. Now we define the lists for the vertices of B as follows: for $v \in V(B)$ let $L(v)$ be the list of colors from $\{1, 2, \dots, k-1\}$ that do not occur in the neighborhood of v . Then, as all vertices from B are low vertices and, therefore, have degree $k-1$ in G , we obtain that the cardinality of $L(v)$ is at least the degree of v in B for all vertices $v \in V(B)$. As consequence, the premises of Theorem 2 are fulfilled and, hence, B admits a proper L -coloring, unless B is a complete graph or an odd cycle. Since a proper L -coloring of B would lead to a proper $(k-1)$ -coloring of G , which is impossible, this implies Theorem 3's statement.

In this thesis, we aim to generalize the theorems of BROOKS, ERDŐS, RUBIN, and TAYLOR, and GALLAI in various different settings. The partition result of Chapter 2 will lead to a hypergraph version of all those results, as well as to a version of those results for generalized hypergraph coloring. Moreover, we will prove a hypergraph as well as a digraph version of them for DP-coloring. That it is possible to obtain these results in the described and even in many more versions emphasizes their—up to this day—outstanding importance in graph coloring theory.

The author took inspiration for the introduction from the very readable survey “Brooks’s theorem” by Michael STIEBITZ and Bjarne TOFT [110]. For the standard reference work on results and open problems in graph coloring theory we refer the reader to the book “Graph Coloring Problems” by Tommy R. JENSEN and Bjarne TOFT [62].

Part I

PARTITIONS AND COLORINGS OF HYPERGRAPHS

Chapter 1

PRELIMINARIES: HYPERGRAPHS

1.1. Basic Terminology

As usual, \mathbb{N} denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of non-negative integers. For $k, \ell \in \mathbb{N}_0$ let $[k, \ell] = \{h \in \mathbb{N}_0 \mid k \leq h \leq \ell\}$. Given a set V , we denote the **cardinality** of V by $|V|$ and the **power set** of V by 2^V . The empty set is denoted by \emptyset .

Hypergraphs

A **hypergraph** $H = (V, E, i)$ is a triple consisting of two finite sets, V and E , and a function i from E to the power set 2^V with $|i(e)| \geq 2$ for $e \in E$ (*i.e.*, no loops are allowed). The set $V = V(H)$ is called **vertex set** of H ; its elements are the **vertices** of H . The cardinality of V is the **order** of H , we denote it by $|H|$. We call $E = E(H)$ **edge set** of H and its elements **edges**. The function $i = i_H$ is the **incidence function** of H : For an edge $e \in E$, $i(e)$ is the set of vertices that are **incident** to e . A **hyperedge** is an edge e with $|i(e)| \geq 3$, an **ordinary edge** is one with $|i(e)| = 2$. Two distinct edges e and e' are **parallel** if $i(e) = i(e')$. In this thesis, we usually allow for parallel edges unless explicitly stated otherwise. A hypergraph H is **simple** if none of its edges is contained in another edge, *i.e.*, $i_H(e) \not\subseteq i_H(e')$ for all distinct edges e, e' of H . A **q -uniform** hypergraph is a hypergraph H with $|i(e)| = q$ for all $e \in E$. Thus, a **graph** is just a 2-uniform hypergraph; in particular, all definitions relevant to us regarding graphs can be obtained from their hypergraph counterparts by considering 2-uniform hypergraphs. Throughout this thesis, we will consistently use the notation H, H', \tilde{H} etc. for hypergraphs; G, G', \tilde{G} etc. will always refer to a graph unless

explicitly stated otherwise. As usual, K_n denotes the **complete graph** on n vertices with $n \geq 0$, and C_n denotes an ordinary cycle as a 2-uniform simple hypergraph of order n with $n \geq 3$. A cycle is called **odd** or **even** depending on whether its order is odd or even. An **odd wheel** is a graph that is obtained from an odd cycle by adding one vertex and joining it to all others. A **hyperwheel** is a hypergraph that results from an edge by adding one vertex and joining it to all others by ordinary edges.

If H is a hypergraph such that there exists an edge $e \in E(H)$ with $V(H) = i_H(e)$ and $E(H) = \{e\}$, we will briefly write $H = \langle e \rangle$. The hypergraph H is the **empty hypergraph** if $V(H) = E(H) = \emptyset$, in this case we also write $H = \emptyset$. A hypergraph H' is a **subhypergraph** of H , written $H' \subseteq H$, if $V(H') \subseteq V(H)$, $E(H') \subseteq E(H)$, and $i_{H'} = i_H|_{E(H')}$. If $H' \subseteq H$ and $H' \neq H$, then H' is said to be a **proper subhypergraph** of H and we briefly write $H' \subset H$.

Hypergraph Operations

There are various well known operations in order to create new hypergraphs from given ones: The **union** of two subhypergraphs H_1 and H_2 of a hypergraph H is the hypergraph $H' = H_1 \cup H_2$ with $V(H') = V(H_1) \cup V(H_2)$, $E(H') = E(H_1) \cup E(H_2)$ and $i_{H'} = i_H|_{E(H')}$; the **intersection** of H_1 and H_2 is the hypergraph $H' = H_1 \cap H_2$ with $V(H') = V(H_1) \cap V(H_2)$, $E(H') = E(H_1) \cap E(H_2)$, and $i_{H'} = i_H|_{E(H')}$.

Now let H^1 and H^2 be two **disjoint hypergraphs**, that is, $V(H^1) \cap V(H^2) = \emptyset$ and $E(H^1) \cap E(H^2) = \emptyset$. Furthermore, for $i \in \{1, 2\}$, let $v^i \in V(H^i)$ and let $v^* \notin (V(H^1) \cup V(H^2)) \setminus \{v^1, v^2\}$. We obtain a new hypergraph H with $V(H) = ((V(H^1) \cup V(H^2)) \setminus \{v^1, v^2\}) \cup \{v^*\}$, $E(H) = E(H^1) \cup E(H^2)$, and

$$i_H(e) = \begin{cases} i_{H^j}(e) & \text{if } e \in E(H^j), v^j \notin i_{H^j}(e) \text{ (} j \in \{1, 2\}\text{),} \\ (i_{H^j}(e) \setminus \{v^j\}) \cup \{v^*\} & \text{if } e \in E(H^j), v^j \in i_{H^j}(e) \text{ (} j \in \{1, 2\}\text{).} \end{cases}$$

In this case we say that H is obtained from H^1 and H^2 by **merging** v^1 and v^2 to v^* . For the purpose of improved clarity, we will usually identify the vertices v^1, v^2 and v^* .

Given a hypergraph H and a vertex set $X \subseteq V(H)$ we define two new hypergraphs, both having X as its vertex set. Firstly, $H[X]$ is the subhypergraph of H satisfying

$$V(H[X]) = X, E(H[X]) = \{e \in E \mid i_H(e) \subseteq X\}, \text{ and } i_{H[X]} = i_H|_{E(H[X])}.$$

We call $H[X]$ the **subhypergraph of H induced by X** . Note that a vertex set $X \subseteq V(H)$ is called an **independent set** of H if $H[X]$ has no edge. A hypergraph H' is called **induced subhypergraph** of H if $V(H') \subseteq V(H)$ and $H' = H[V(H')]$.

Secondly, $H(X)$ is the hypergraph that is obtained by **shrinking H to X**, *i.e.*, the hypergraph satisfying

$$V(H(X)) = X, E(H(X)) = \{e \in E(H) \mid |i_H(e) \cap X| \geq 2\},$$

and

$$i_{H(X)}(e) = i_H(e) \cap X \text{ for all } e \in E(H(X)).$$

Those two hypergraphs naturally introduce the following ones: $H - X = H[V(H) \setminus X]$ and $H \div X = H(V(H) \setminus X)$. We say that $H \div X$ is obtained by **shrinking H at X** (not to be confused with shrinking H to X). If $X = \{v\}$ is a singleton, we rather write $H - v$ and $H \div v$ than $H - X$ and $H \div X$. An example of shrinking a hypergraph at a vertex is displayed in Figure 1.1. Note that for a graph G shrinking at a vertex set X coincides with deleting the vertex X , *i.e.* $G \div X = G - X$ for $X \subseteq V(G)$. Given an edge set $F \subseteq E(H)$, let $H - F = (V(H), E(H) \setminus F, i_{H-F})$ where $i_{H-F} = i_H|_{E \setminus F}$. Again, if $F = \{e\}$ is a singleton we prefer writing $H - e$ rather than $H - F$. To obtain the reverse operations, let H' be a proper subhypergraph of H and let $v \in V(H) \setminus V(H')$, respectively $e \in E(H) \setminus E(H')$ with $i(e) \subseteq V(H')$. Then, $H' + v = H[V(H') \cup \{v\}]$ and $H' + e = (V(H'), E(H') \cup \{e\}, i_{H'+e})$ where $i_{H'+e} = i_H|_{E' \cup \{e\}}$. Note that for distinct vertices u and v of H it clearly holds true that

$$(H \div u) \div v = (H \div v) \div u. \tag{1.1}$$

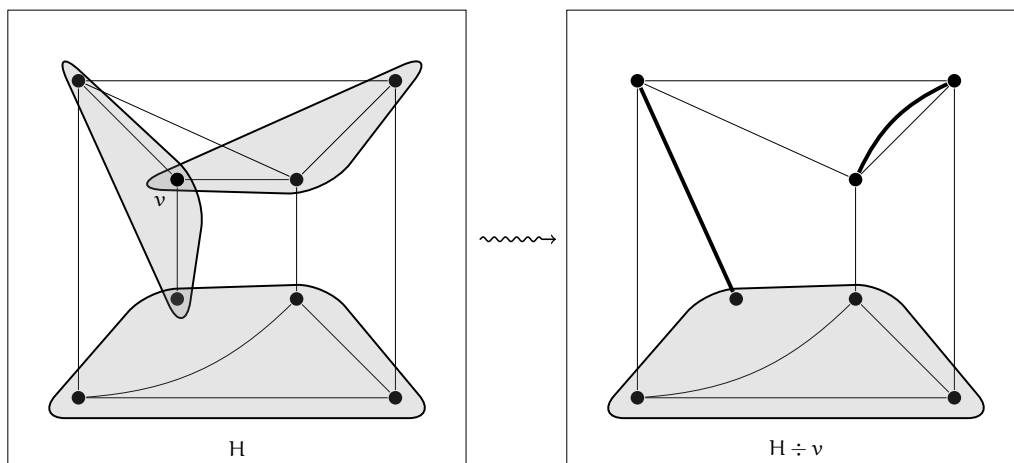
Lastly, for a simple hypergraph H and an integer $t \geq 1$, let $H' = tH$ be the hypergraph obtained from H by replacing each edge of H by t parallel edges; we call H' the **t-uniform inflation of H**.

Matchings and Connectivity

A **matching** of a hypergraph H is a subset M of $E(H)$ such that each vertex $v \in V(H)$ is incident with at most one edge from M . A hypermatching M of H is **perfect** if for each $v \in V(H)$ there is an edge $e \in M$ that **contains** v (*i.e.* $v \in i_H(e)$).

Similar to graphs, there are various equivalent ways how to define connectivity. We say that a hypergraph H is **connected** if H is non-empty and for every non-empty vertex set $X \subseteq V(H)$ there is at least one edge $e \in E(H)$ such that $i_H(e)$ contains a vertex of X as well as a vertex of $V(H) \setminus X$.

A **(hyper)-path of length ℓ** in H is a sequence $P = (v_0, e_0, v_1, e_1, \dots, v_{\ell-1}, e_{\ell-1}, v_\ell)$ of distinct vertices v_0, v_1, \dots, v_ℓ and distinct edges $e_0, e_1, \dots, e_{\ell-1}$ such that $\{v_i, v_{i+1}\} \subseteq i_H(e_i)$


 FIG. 1.1. Shrinking a hypergraph H at the vertex v .

for $i \in [0, \ell - 1]$. In this case we also say that P is a (v_0, v_ℓ) -**hyperpath** and that P is a hyperpath **between** v_0 **and** v_ℓ . The vertices $v_1, v_2, \dots, v_{\ell-1}$ are called **inner vertices** of P . If P is a (u, v) -hyperpath for some vertices u, v and if w is an inner vertex of P , we denote the **subhyperpath** of P between u and w by uPw and the one between w and v by wPv . Moreover, we denote the length of a shortest (u, v) -hyperpath in H by $\text{dist}_H(u, v)$. If there is no hyperpath in H between the vertices u and v , we set $\text{dist}_H(u, v) = \infty$. It is easy to see that a non-empty hypergraph H is connected if and only if there is a hyperpath in H between any two of its vertices (which is equivalent to $\text{dist}_H(u, v) \in \mathbb{N}$ for all distinct vertices u, v from H). A (connected) **component** of a hypergraph H is a maximal connected subhypergraph of H .

A **separating vertex set** of a hypergraph H is a set $S \subseteq V(H)$ such that H is the union of two induced subhypergraphs H_1 and H_2 satisfying $V(H_1) \cap V(H_2) = S$ and $|H_i| > |S|$ for $i \in \{1, 2\}$. If $S = \{v\}$ is a singleton, we say that v is a **separating vertex** of H . Clearly, if G is a connected graph, then a vertex v of G is non-separating if and only if $G - v$ is empty or connected. This is not true in general for hypergraphs: take a hypergraph $H = \langle e \rangle$ with $|i(e)| \geq 3$. Then, every vertex v of H is non-separating but $H - v$ is not connected. Instead, it is easy to check that a vertex v of a connected hypergraph H is non-separating if and only if the hypergraph $H \div v$ is empty or connected. Similar, a vertex set $S \subseteq V(H)$ is non-separating if and only if $H \div S$ is empty or connected. This indicates that for hypergraphs, the shrinking operation might be the correct generalization of vertex deletion in the graph case. Regarding edges, an edge e is a **bridge** of a hypergraph H if $H - e$ has $|i_H(e)| - 1$

more components than H . Thus, an edge e is a bridge if and only if every vertex from $i_H(e)$ belongs to a different component of $H - e$.

Similar to the graph case, however, is the definition of blocks: A **block** of a hypergraph H is a maximal connected subhypergraph of H not containing a separating vertex. By $\mathcal{B}(H)$ we denote the set of all blocks of H and, given a vertex $v \in V(H)$, $\mathcal{B}_v(H)$ is the set of all blocks B from $\mathcal{B}(H)$ with $v \in V(B)$. Clearly, $\mathcal{B}(\emptyset) = \emptyset$ and every block of a non-empty hypergraph H is a connected induced subhypergraph of H . If H contains only one block, we also say that H is a **block**. As for graphs it is not difficult to show that any two distinct blocks of a hypergraph H have at most one vertex in common, and a vertex of H is a separating vertex of H if and only if it belongs to more than one block. An **end-block** of H is a block that contains at most one separating vertex of H . It is important to note that H has at least two end-blocks if H contains a separating vertex.

Degrees and Ordinary Neighbors

Let H be a hypergraph. A vertex $v \in V(H)$ is **incident** with an edge $e \in E(H)$ if $v \in i_H(e)$. Moreover, two distinct vertices u, v of H are **adjacent** if there is an edge $e \in E(H)$ such that $\{u, v\} \subseteq i_H(e)$. In this case we say that u is a **neighbor** of v and vice versa. The **ordinary neighborhood** of a vertex v in a hypergraph H is the set of all vertices $u \in V(H)$ such that there is an edge e with $i_H(e) = \{u, v\}$; we denote it by $N_H(v)$. For a vertex set $X \subseteq V(H)$, let

$$E_H(X) = \{e \in E(H) \mid i_H(e) \cap X \neq \emptyset \text{ and } i_H(e) \cap (V(H) \setminus X) \neq \emptyset\}.$$

If $X = \{v\}$ is a singleton, we rather write $E_H(v)$ than $E_H(\{v\})$. In particular, we have

$$E_H(v) = \{e \in E(H) \mid v \in i_H(e)\}.$$

The **degree** of v in H is $d_H(v) = |E_H(v)|$. A **regular** and **r-regular** hypergraph is one whose vertices all have degree r . As usual, $\delta(H) = \min_{v \in V(H)} d_H(v)$ is the **minimum degree** of H and $\Delta(H) = \max_{v \in V(H)} d_H(v)$ is the **maximum degree** of H . If $H = \emptyset$, then we define $\delta(H) = \Delta(H) = 0$. For an ordinary edge e of H with $i_H(e) = \{u, v\}$, we also write $e = uv$ and $e = vu$ and say that e **joins** u and v . Given two distinct edges u and v of H , note that $E(H[\{u, v\}])$ is the set of ordinary edges joining u and v . We define the **multiplicity** of (u, v) in H as $\mu_H(u, v) = |E(H[\{u, v\}])|$. Be aware that if $v \in V(H)$, then every vertex $u \in V(H) \setminus \{v\}$ satisfies

$$d_{H \div v}(u) = d_H(u) - \mu_H(u, v). \tag{1.2}$$

Degeneracy and Coloring Number

The concept of degeneracy is closely related to graph colorings. Following the definition of LICK and WHITE [78], we say that a hypergraph H is **k -degenerate**, where k is a non-negative integer, if every non-empty subgraph H' of H contains a vertex v with $d_{H'}(v) \leq k$. Moreover, H is **strictly k -degenerate** if the previous inequality is strict, *i.e.*, if every non-empty subgraph H' of H contains a vertex v with $d_{H'}(v) < k$. Obviously, H is strictly k -degenerate if and only if H is $(k - 1)$ -degenerate, but our results can be expressed more naturally by using strict degeneracy; so we prefer this notation over the original one. Note that H is strictly 0-degenerate if and only if $H = \emptyset$ and H is strictly 1-degenerate if and only if H is edgeless. The strictly 2-degenerate graphs are precisely the forests. If we are looking for the minimum integer k for which a hypergraph H is strictly k -degenerate, then we call this integer **coloring number** $\text{col}(H)$ of H . Thus,

$$\text{col}(H) = \max_{H' \subseteq H} \delta(H') + 1.$$

Often, the coloring number of a hypergraph H is defined in a different, yet equivalent way: as the least integer k for which there is an ordering (v_1, v_2, \dots, v_n) of the vertices of H such that $d_{H[\{v_1, v_2, \dots, v_i\}]}(v_i) + 1 \leq k$ for all $i \in [1, n]$. For graphs, this definition goes back to ERDŐS and HAJNAL [42]; SZEKERES and WILF [113] were the first to examine the coloring number of a graph as we define it and established it as an upper bound for the chromatic number (see also (1.3)). The equivalence of both definitions can easily be checked as we shall demonstrate here.

Let (v_1, v_2, \dots, v_n) be any ordering of the vertices of a hypergraph H . Moreover, let H' be a subhypergraph of H with maximum minimum degree, and let i be the largest index from $[1, n]$ such that v_i is contained in H' . Then, $H' \subseteq H[\{v_1, v_2, \dots, v_i\}]$ and so

$$\text{col}(H) = \delta(H') + 1 \leq d_{H'}(v_i) + 1 \leq d_{H[\{v_1, v_2, \dots, v_i\}]}(v_i) + 1.$$

To obtain the converse, let (v_1, v_2, \dots, v_n) be an ordering of the vertices of H such that v_i has minimum degree in the subhypergraph $H_i = H - \{v_{i+1}, v_{i+2}, \dots, v_n\}$ for $i = n, n-1, \dots, 1$ where $H_n = H$ (a so called **smallest last order** of H). Then, for all $i \in [1, n]$, we have

$$\text{col}(H) = \max_{\emptyset \neq H' \subseteq H} \delta(H') + 1 \geq \delta(H_i) + 1 = d_{H[\{v_1, v_2, \dots, v_i\}]}(v_i) + 1.$$

Summarizing, we conclude the following proposition.

Proposition 1.1. *Let H be a non-empty hypergraph and let (v_1, v_2, \dots, v_n) be a smallest last order of H . Then,*

$$\text{col}(H) = \max_{1 \leq i \leq n} d_{H[\{v_1, v_2, \dots, v_i\}]}(v_i) + 1. \quad \diamond$$

For graphs, this way of computing the coloring number leads to a polynomial time algorithm as pointed out by FINCK and SACHS [45] and, independently, by MATULA [85].

A natural generalization of strict k -degeneracy is the following variable version. Given a hypergraph H and a function h from $V(H)$ to the set \mathbb{N}_0 , we say that H is **strictly h -degenerate** if every non-empty subhypergraph H' of H contains a vertex v with $d_{H'}(v) < h(v)$. Clearly, this includes the definition of being strictly k -degenerate by setting $h(v) = k$ for all vertices v of H . The concept of variable degeneracy was introduced by STIEBITZ [109] as a tool to prove a conjecture of THOMASSEN [114] regarding graph partitions under minimum degree constraints. Variable degeneracy was further examined by BORODIN, KOSTOCHKA, and TOFT in 2000 [23].

1.2. Partitions and Colorings of Hypergraphs

Based on our experience, (hyper-)graph colorists are divided into two groups: for some, a coloring of a (hyper-)graph H is a partition of H into parts that satisfy certain conditions; for the others, a coloring of H is a function that assigns each vertex of the hypergraph a color according to some given rules. Of course, both approaches are completely right and technically equivalent. However, since we like to work flexibly, we want to introduce both ways and use the best fitting one depending on the situation.

Let H be a hypergraph. A **partition** or **p-partition** of H is a sequence (H_1, H_2, \dots, H_p) of $p \geq 1$ pairwise vertex disjoint induced subhypergraphs with

$$V(H) = V(H_1) \cup V(H_2) \cup \dots \cup V(H_p);$$

we call the subhypergraphs H_1, H_2, \dots, H_p **parts** of the partition. Note that parts may also be empty by definition; this is due to technical reasons.

A **coloring** of H with a finite **color set** Γ is a function $\varphi : V(H) \rightarrow \Gamma$. If the color set's cardinality equals k , we also say that the coloring φ is a **k -coloring** of H . For each **color** $\alpha \in \Gamma$, the preimage

$$\varphi^{-1}(\alpha) = \{v \in V(H) \mid \varphi(v) = \alpha\}$$

is called **color class** of H with respect to φ . A subhypergraph of H is **monochromatic** with respect to the coloring φ if all of its vertices belong to the same color class.

In many coloring problems like scheduling or the channel assignment problem not every color is available for each vertex. Instead, each vertex v gets assigned a set (list) $L(v)$ of colors from which the vertices color must be chosen. To formalize this approach, a **list-assignment** of a hypergraph H with color set Γ is a function L from $V(H)$ to the power set 2^Γ . Given a list-assignment L of H with color set Γ , we say that a coloring φ of H is an **L-coloring** of H if $\varphi(v) \in L(v)$ for all $v \in V(H)$.

As emphasized at the beginning of this section, the concepts of hypergraph partitions and hypergraph coloring are two sides of the same coin. Let H be a hypergraph and let $\Gamma = [1, p]$. If φ is a coloring of H with color set Γ , then (H_1, H_2, \dots, H_p) with $H_\alpha = H[\varphi^{-1}(\alpha)]$ for $\alpha \in \Gamma$ is a partition of H . Conversely, if (H_1, H_2, \dots, H_p) is a partition of H , then the function φ with $\varphi(v) = \alpha$ if $v \in V(H_\alpha)$ is a coloring of H with color set Γ .

Colorings and partitions of hypergraphs become a subject of interest only when some restrictions to the color classes, respectively to the parts of the partition, are imposed. The probably most investigated type of colorings are proper colorings: A coloring or L-coloring of a hypergraph H with color set Γ is called a **proper coloring**, respectively a **proper L-coloring** of H if each color class is an independent set of H . Equivalently, a coloring (respectively L-coloring) φ of H is a proper coloring (respectively proper L-coloring) if $|\varphi(i_H(e))| \geq 2$ for all $e \in E(H)$, *i.e.* every edge contains at least two vertices of distinct colors. This coloring concept is also related to degeneracy; in a proper coloring, each color class induces an edgeless subhypergraph and, therefore, a strictly 1-degenerate subhypergraph. Consequently, a hypergraph H admits a proper p -coloring if and only if H has a p -partition all of which parts are strictly 1-degenerate.

The **chromatic number** of a hypergraph H , denoted by $\chi(H)$, is the least integer k such that H admits a proper k -coloring. Similar, the **list-chromatic number** of H (also commonly known as **choice number**), denoted by $\chi_\ell(H)$, is the least integer k such that H admits a proper L-coloring for each list-assignment L satisfying $|L(v)| \geq k$ for all $v \in V(H)$. Since $\chi_\ell(H) = k$ implies that H has a proper L-coloring for the constant list-assignment L with $L(v) = [1, k]$ for all $v \in V(H)$, we obtain $\chi(H) \leq \chi_\ell(H)$. Furthermore, a simple sequential coloring argument shows that

$$\chi(H) \leq \chi_\ell(H) \leq \text{col}(H) \leq \Delta(H) + 1. \quad (1.3)$$

Note that the chromatic number and the list-chromatic number of a hypergraph H is equal

to the chromatic number, respectively list-chromatic number of its **underlying simple hypergraph**, that is, the hypergraph obtained from H by deleting all edges $e \in E(H)$ for which there exists an edge $e' \in E(H)$ with $i_H(e') \subset i_H(e)$ and then replacing all parallel edges by a single edge.

1.3. Hypergraph Properties

We have seen above that in a proper coloring of a hypergraph, each color class induces a strictly 1-degenerate subhypergraph. So wouldn't it be natural also to examine colorings in which each color class induces a strictly k -degenerate subhypergraph for some $k \geq 2$? In fact, we can get even more general and just require each color class to induce a subhypergraph belonging to some prescribed hypergraph property. In Chapter 3, we will examine these kind of colorings and show that one can prove generalized versions of famous theorems like the ones of BROOKS and GALLAI if we only demand the hypergraph property to satisfy two reasonable conditions. Let's get to the definitions. By \mathcal{H} we denote the class of all hypergraphs. A **hypergraph property** \mathcal{P} is a subclass of \mathcal{H} that is closed under isomorphisms. The hypergraph property \mathcal{P} is **smooth** if the following two conditions hold.

(P1) \mathcal{P} is **hereditary**, i.e., \mathcal{P} is closed under induced subhypergraphs, and

(P2) \mathcal{P} is **non-trivial**, i.e., \mathcal{P} contains a non-empty hypergraph but is different from \mathcal{H} .

Hereditary properties for graphs have been studied extensively, an interesting overview can be found in [24]. Some important hereditary hypergraph properties that are smooth, in particular, are the following:

$$\mathcal{O} = \{H \in \mathcal{H} \mid H \text{ is edgeless}\},$$

$$\mathcal{S}_k = \{H \in \mathcal{H} \mid \Delta(H) \leq k\}, \text{ and}$$

$$\mathcal{D}_k = \{H \in \mathcal{H} \mid H \text{ is strictly } (k+1)\text{-degenerate}\}$$

with $k \geq 0$. Consequently, a proper coloring of a hypergraph H is just a coloring such that each color class induces a subhypergraph of H that belongs to \mathcal{O} . Note that the two classes \mathcal{O} and \mathcal{D}_0 coincide.

For a smooth hypergraph property \mathcal{P} , let

$$\mathcal{F}(\mathcal{P}) = \{H \in \mathcal{H} \mid H \notin \mathcal{P}, \text{ but } H - v \in \mathcal{P} \text{ for all } v \in V(H)\},$$

and let

$$d(\mathcal{P}) = \min\{\delta(H) \mid H \in \mathcal{F}(\mathcal{P})\}.$$

For instance, $\mathcal{F}(\mathcal{C}) = \{H \in \mathcal{H} \mid H = \langle e \rangle \text{ for some edge } e\}$ and $d(\mathcal{C}) = 1$. Moreover, $\mathcal{F}(\mathcal{S}_k)$ contains the star on $k+2$ vertices and so $d(\mathcal{S}_k) = 1$. Obviously, the class of $(k+1)$ -regular connected hypergraphs is contained in $\mathcal{F}(\mathcal{D}_k)$, but a full characterization of $\mathcal{F}(\mathcal{D}_k)$ for $k \geq 1$ has not yet been obtained. Still, it is an easy observation that $d(\mathcal{D}_k) = k+1$.

The next proposition states some basic facts on hypergraph properties; the statements are well-known for graphs and easily extend to hypergraphs.

Proposition 1.2. *Let \mathcal{P} be a smooth hypergraph property. Then, the following statements hold:*

- (a) \mathcal{P} contains K_0 and K_1 .
- (b) A hypergraph H belongs to $\mathcal{F}(\mathcal{P})$ if and only if each proper induced subhypergraph of H belongs to \mathcal{P} , but H does not.
- (c) A hypergraph H does not belong to \mathcal{P} if and only if H contains an induced subhypergraph from $\mathcal{F}(\mathcal{P})$.
- (d) The class $\mathcal{F}(\mathcal{P})$ is non-empty and $d(\mathcal{P})$ is from \mathbb{N}_0 .
- (e) If a hypergraph H does not belong to \mathcal{P} , but $H - v \in \mathcal{P}$ for some $v \in V(H)$, then $d_H(v) \geq d(\mathcal{P})$. ◊

Proof. Since \mathcal{P} is non-trivial, \mathcal{P} contains a non-empty hypergraph H . As \mathcal{P} is hereditary, it contains all induced subhypergraphs of H and, therefore, K_0 and K_1 . Statement (b) follows from (P1) and the definition of $\mathcal{F}(\mathcal{P})$ since $H - v$ is a proper induced subhypergraph of H for all $v \in V(H)$. In order to prove (c), let H be a hypergraph. If H contains an induced subhypergraph H' from $\mathcal{F}(\mathcal{P})$, then clearly $H \notin \mathcal{P}$ (by (P1)). Conversely, if H does not belong to \mathcal{P} , there is an induced subhypergraph H' of H such that $H' \notin \mathcal{P}$ and $|H'|$ is minimum. Then, $H' - v \in \mathcal{P}$ for all $v \in V(H')$ and H' belongs to $\mathcal{F}(\mathcal{P})$. Since \mathcal{P} is different from \mathcal{H} (by (P2)), statement (d) is an immediate consequence of (c).

It remains to prove statement (e). To this end, let $H \notin \mathcal{P}$ be a hypergraph such that $H - v \in \mathcal{P}$ for some $v \in V(H)$. By (c), H contains an induced subhypergraph H' from $\mathcal{F}(\mathcal{P})$. Then, H' contains v , since otherwise H' would be an induced subhypergraph of $H - v$ and

would therefore belong to \mathcal{P} (by (P1)). Thus,

$$d(\mathcal{P}) \leq \delta(H') \leq d_{H'}(v) \leq d_H(v),$$

which proves (e). ■

1.4. Criticality

Criticality plays a vital role in the investigation of various hypergraph properties. Given a hypergraph property \mathcal{P} , a hypergraph H is called **\mathcal{P} -vertex-critical** (respectively, **\mathcal{P} -critical**) if every proper induced subhypergraph (respectively, every proper subhypergraph) belongs to \mathcal{P} , but H itself does not. Clearly, every \mathcal{P} -critical hypergraph is also \mathcal{P} -vertex-critical; yet the converse implication does not necessarily hold true. Note that if \mathcal{P} is a smooth hypergraph property, then a hypergraph H is \mathcal{P} -vertex-critical if and only if H does not belong to \mathcal{P} and $H - v \in \mathcal{P}$ for all $v \in V(H)$. Consequently, $\mathcal{F}(\mathcal{P})$ is the set of \mathcal{P} -vertex-critical hypergraphs. Moreover, a hypergraph H does not belong to \mathcal{P} if and only if H has a \mathcal{P} -vertex-critical induced subhypergraph.

As mentioned in the introduction, the concept of criticality was first introduced and investigated by DIRAC in the 1950s for the class of simple graphs and with respect to the graph property consisting of all graphs G with $\chi(G) \leq k$ for fixed k ; see e.g. [35, 36, 37, 38, 39, 40].

1.5. Structure of the Following Chapters

In Chapter 2 we consider the following partition problem: given a hypergraph H and a sequence $f = (f_1, f_2, \dots, f_p)$ of functions from $V(H)$ to \mathbb{N}_0 , we want to find a p -partition (H_1, H_2, \dots, H_p) of H such that H_i is strictly f_i -degenerate for all $i \in [1, p]$ (a so called **f -partition** of H). Such kind of partition is especially interesting as we can model many of the common coloring problems by choosing f appropriately. For example, if $f_i \equiv 1$ for all $i \in [1, p]$, then an f -partition of H corresponds to a proper p -coloring of the hypergraph (and vice versa). Similarly, proper list-colorings and even generalized colorings regarding smooth hypergraph properties can be designed by a suitable function f . Thus, by analyzing which requirements in regard to f are sufficient for the existence of an f -partition, we can solve a lot of questions related to hypergraph coloring. We shall prove that the condition $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ for all $v \in V(H)$ is nearly sufficient and give a full characterization of the hypergraphs H that are not f -partitionable under this requirement (see Theorem 2.3). This

main result of Chapter 2 generalizes a theorem of BORODIN, KOSTOCHKA, and TOFT [23] on f -partitions of simple graphs. Although the proof of our result is both technical as well as quite exhausting, the theorem itself is a really nice meta result with multiple interesting applications. The results of Chapter 2 as well as those in the first part of Chapter 3 are joint work with MICHAEL STIEBITZ and have been published as **Partitions of hypergraphs under variable degeneracy constraints** in *Journal of Graph Theory* [104].

In Chapter 3 we demonstrate how to obtain various well-known coloring results for graphs and hypergraphs from Theorem 2.3 by choosing the correct function f . This includes the list version of BROOKS' Theorem for hypergraphs [72] (see Theorem 3.2), a BROOKS-type result regarding the coloring number that was proved for graphs by BORODIN [20], respectively BOLLOBÁS and MANVEL [16] (see Theorem 3.4'), and even a generalization of this result that was obtained for graphs by MATAMALA [84] (see Theorem 3.8). Moreover, we obtain a result on the (list-)point-partition number, *i.e.*, on colorings of which every color class induces a strictly s -degenerate subhypergraph (see Corollary 3.10). Sections 3.1 and 3.2 are the hypergraph-counterparts to the respective sections in the paper [23]; the proofs are easy adaptations. The MATAMALA-related result in Section 3.3 is from the paper **Vertex partition of hypergraphs and maximum denegenerate subhypergraphs** [103], which is also joint work with MICHAEL STIEBITZ and has been submitted to *Electronic Journal of Graph Theory and Applications*.

Afterwards, we examine generalized hypergraph coloring as motivated in the previous section: given a hypergraph H , a list-assignment L of H , and a smooth hypergraph property \mathcal{P} , a (\mathcal{P}, L) -coloring of H is an L -coloring of H such that each color class induces a subhypergraph belonging to \mathcal{P} . We then use Theorem 2.3 in order to deduce a GALLAI-type result regarding (\mathcal{P}, L) -vertex-critical hypergraphs, *i.e.*, vertex-critical hypergraphs with respect to the property of having a (\mathcal{P}, L) -coloring (see Theorem 3.13). This again leads to a BROOKS-type theorem for generalized hypergraph coloring (see Theorem 3.16) and a GALLAI-type bound for the number of edges in critical hypergraphs (see Section 3.5.3). These results have been published as **Generalized hypergraph coloring** in *Discussiones Mathematicae Graph Theory* [100].

In Chapter 4 we analyze DP-colorings of hypergraphs. DP-coloring is a new coloring concept that has been introduced recently by DVOŘÁK and POSTLE [41] and, since then, has aroused curiosity of many graph theorists. Their main idea was to generalize the list-coloring concept in order to make advantage of classical proof techniques like vertex identification that usually do not work for list-colorings. To this end, given a list-assignment L of a graph

G , they construct an auxiliary graph \mathcal{G} as follows: $V(\mathcal{G}) = \{(v, \alpha) \mid v \in V(G), \alpha \in L(v)\}$ and $E(\mathcal{G}) = \{(v, \alpha)(v', \alpha') \mid vv' \in E(G) \text{ and } \alpha = \alpha'\}$. Then, it is an easy observation that G admits a proper L -coloring if and only if \mathcal{G} has an independent transversal T , *i.e.*, for each vertex $v \in V(G)$ exactly one of the vertices (v, α) , $\alpha \in L(v)$, belongs to T and $\mathcal{G}[T]$ is edgeless. Note that for distinct vertices $u, v \in V(G)$ the edges between the corresponding vertices in \mathcal{G} form a specific matching in this setting. By allowing any kind of matching between the corresponding vertices, we get to the concept of DP-coloring. In Chapter 4, we transfer the definition of DP-coloring to hypergraphs. Moreover, we obtain a BROOKS-type result and also its degree version (see Corollary 4.9 and Theorem 4.10), thereby transferring results by BERNSHTEYN, KOSTOCHKA, and PRON [14] as well as KIM and OZEKI [66] to hypergraphs. Chapter 4 is based on the paper **DP-degree colorable hypergraphs** [101] that has been published in *Theoretical Computer Science*.

In Chapter 5 we examine the connection between the chromatic number of a hypergraph and its maximum local edge connectivity. Given two distinct vertices u, v of a hypergraph H , the local edge connectivity $\lambda_H(u, v)$ of u and v is the maximum number of edge-disjoint (u, v) -hyperpaths; the maximum local edge connectivity $\lambda(H)$ of H is the maximum local edge connectivity over all possible pairs (u, v) . As proved by TOFT in [117], every hypergraph H satisfies $\chi(H) \leq \lambda(H) + 1$. Following up on that, we prove that, for $\lambda(H) \geq 3$, $\chi(H) = \lambda(H) + 1$ if and only if some block of H belongs to a predescribed hypergraph property $\mathcal{H}_{\lambda(H)}$. Here, \mathcal{H}_3 is the smallest class of hypergraphs that contains all odd wheels and is closed under so called HAJÓS joins and, for $k \geq 4$, \mathcal{H}_k is the smallest class of hypergraphs containing all complete graphs of order $k + 1$ that is closed under taking HAJÓS joins. Introduced in [54] for graphs, the concept of HAJÓS joins is a well known method for creating infinite families of critical graphs and has been transferred to hypergraphs by TOFT [117]. The results displayed in Chapter 5 are from the paper **Coloring hypergraphs of low connectivity** [105], which is joint work with MICHAEL STIEBITZ and BJARNE TOFT and has been submitted to *Journal of Combinatorics*.

Large parts of the following chapters (and also of this chapter) are similar to those from the corresponding papers or have been changed only slightly.

Chapter 2

PARTITIONS OF HYPERGRAPHS INTO DEGENERATE SUBHYPERGRAPHS

2.1. Introduction and Main Result

Let H be an arbitrary hypergraph. A function $f : V(H) \rightarrow \mathbb{N}_0^p$ is called **vector function** of H . By f_i we name the i th **coordinate** of f , i.e., $f = (f_1, f_2, \dots, f_p)$. The set of all vector functions of H with p coordinates is denoted by $\mathcal{V}_p(H)$. For $f \in \mathcal{V}_p(H)$, an **f -partition** of H is a p -partition (H_1, H_2, \dots, H_p) of H such that H_i is strictly f_i -degenerate for all $i \in [1, p]$. If the hypergraph H admits an f -partition, then H is said to be **f -partitionable**. Since a vector function $f \in \mathcal{V}_p(H)$ can naturally serve as vector function for any subhypergraph H' of H , we also denote the restriction of f to H' by f . Note that if H is f -partitionable, then each of its subhypergraphs is f -partitionable, too.

Although the above definition might seem quite unimpressive at first glance, it is astonishing how many coloring problems can be stated in terms of f -partitions. As a first example, let us examine how to reduce the question of finding a proper list-coloring to finding an f -partition. To this end, let H be a simple hypergraph and let L be a list-assignment for H with color set Γ . By renaming the colors if necessary we may assume $\Gamma = [1, p]$. For $v \in V(H)$ and for $i \in [1, p]$ let $f_i(v) = 1$ if $i \in L(v)$ and $f_i(v) = 0$, otherwise (see also Figure 2.1). Then, for any proper L -coloring φ of H , setting $H_i = [\varphi^{-1}(i)]$ leads to an f -partition (H_1, H_2, \dots, H_p) of H . This is due to the fact that each H_i is strictly 1-degenerate and that $f_i(v) = 1$ for all $v \in V(H_i)$. Conversely, given an f -partition (H_1, H_2, \dots, H_p) of H ,

we assign vertex v color i if $v \in V(H_i)$ and thereby obtain a proper L -coloring of H . Thus, finding an L -coloring of a hypergraph H is equivalent to finding an f -partition of H with f defined as above.

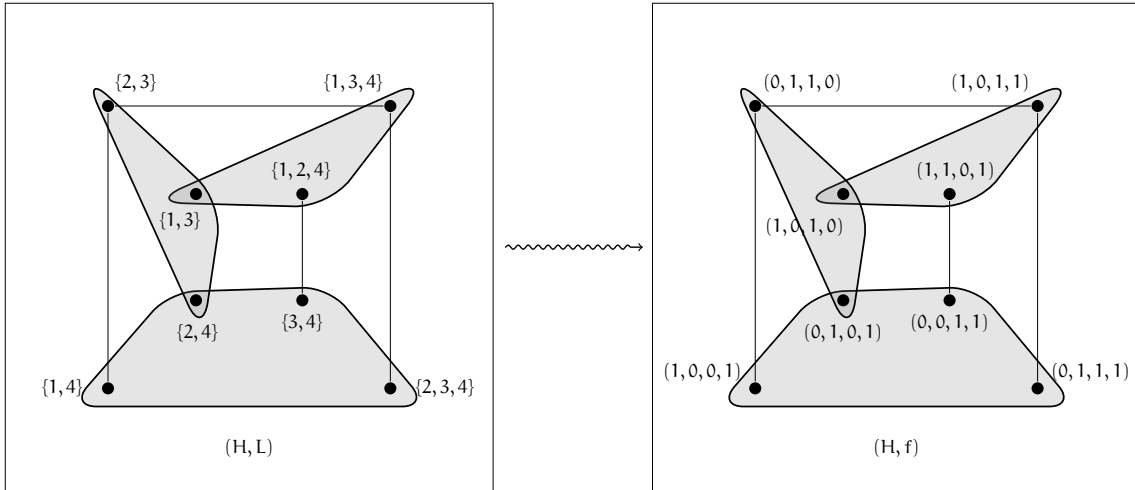


FIG. 2.1. Transforming a list-assignment L with color set $[1, 4]$ into a function f .

As a consequence, results on the existence of f -partitions can be used in order to obtain results in classic coloring theory, as well. But how difficult is it, to decide, whether a hypergraph H is f -partitionable for a given vector function f ? Since the determination of the chromatic number of a graph is already NP-hard, yet just a special case of the f -partition problem, this problem is NP-complete. Nevertheless, maybe some restrictions to the function f could make the problem polynomial time solvable but still be non-trivial. In order to find a suitable condition, we need to go back to an old result of LOVÁSZ [80], who was one of the first to regard graph partitions under degree constraints. In 1966, he proved that, for integers $d_1, d_2, \dots, d_p \geq 0$ and $p \geq 2$, each simple graph G with $d_G(v) < d_1 + d_2 + \dots + d_p$ for all $v \in V(G)$ admits a p -partition (G_1, G_2, \dots, G_p) such that $\Delta(G_i) < d_i$ for all $i \in [1, p]$. A version for variable functions was published eleven years later by BORODIN and KOSTOCHKA [22]. Although they formulated their result for graphs, the proof works for hypergraphs, too. Since the proof is quite simple yet elegant, we include it in this thesis.

Theorem 2.1 (BORODIN and KOSTOCHKA, 1977). *Let H be a hypergraph, and, for an integer $p \geq 2$, let f_1, f_2, \dots, f_p be functions from $V(G)$ to the set of positive integers. If*

$$d_H(v) < f_1(v) + f_2(v) + \dots + f_p(v)$$

for all $v \in V(H)$, then there is a partition (H_1, H_2, \dots, H_p) such that $d_{H_i}(v) < f_i(v)$ for all $i \in [1, p]$ and all $v \in V(H_i)$. \diamond

Proof. As the result for $p > 2$ follows from an easy induction, we may assume that $p = 2$. We claim that a 2-partition (H_1, H_2) of H that minimizes

$$w(H_1, H_2) = |E(H_1)| + |E(H_2)| + \sum_{v \in V(H_1)} f_2(v) + \sum_{v \in V(H_2)} f_1(v)$$

is the one that we are looking for. For otherwise, by symmetry, there is a vertex $v \in V(H_1)$ such that $d_{H_1}(v) \geq f_1(v)$. Since $d_H(v) < f_1(v) + f_2(v)$, this implies that $d_{H_2}(v) < f_2(v)$ and, thus,

$$w(H_1 - v, H_2 + v) - w(H_1, H_2) = d_{H_2}(v) - d_{H_1}(v) + f_1(v) - f_2(v) < 0,$$

which is impossible. Thus, (H_1, H_2) is the desired partition. \blacksquare

As a consequence of the above theorem, if we require the hypergraph H to satisfy $d_H(v) < f_1(v) + f_2(v) + \dots + f_p(v)$ for all $v \in V(H)$, then H does not only admit an f -partition, but a partition (H_1, H_2, \dots, H_p) such that for all $i \in [1, p]$, every vertex of H_i has degree less than its f_i -value in H_i . But what happens if we do not require the inequality to be strict, *i.e.*, if we just request that $d_H(v) \leq f_1(v) + f_2(v) + \dots + f_p(v)$ for all vertices v ? Unfortunately, it is not hard to find infinitely many examples of pairs (H, f) that meet this requirement such that H is not f -partitionable. To see this, we regard BROOKS' Theorem for hypergraphs, which was obtained by JONES [64] in 1975.

Theorem 2.2 (JONES, 1975). *Let H be a connected hypergraph. Then, $\chi(H) \leq \Delta(H) + 1$ and equality holds if and only if H is a complete graph, an odd cycle, or consists of just one hyperedge.* \diamond

Since we have already demonstrated how to transform the list-coloring problem (and therefore also the problem of finding a proper coloring) into the one of finding an f -partition, Jones' theorem immediately implies that complete graphs, odd cycles, and hypergraphs of the form $\langle e \rangle$ for some hyperedge e together with appropriate functions f fulfill the above degree condition, but are not f -partitionable. The good news, however, is that all counterexamples can actually be created from those three basic types by the merging operation and, therefore, may be characterized nicely. To this end, we introduce the following, recursively defined class of configurations.

Let H be a connected hypergraph and let $f \in \mathcal{V}_p(H)$ be a vector-function for some $p \geq 1$.

We say that H is **f-hard**, or, equivalently, that (H, f) is a **hard pair**, if one of the following four conditions hold:

- (1) H is a block and there exists an index $j \in [1, p]$ such that

$$f_i(v) = \begin{cases} d_H(v) & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in [1, p]$ and for each $v \in V(H)$. In this case, we say that H is a **monoblock** or a block of type **(M)**.

- (2) $H = tK_n$ for some $t \geq 1, n \geq 3$ and there are integers $n_1, n_2, \dots, n_p \geq 0$ with at least two n_i different from zero such that $n_1 + n_2 + \dots + n_p = n - 1$ and that

$$f(v) = (tn_1, tn_2, \dots, tn_p)$$

for all $v \in V(H)$. In this case, we say that H is a block of type **(K)**.

- (3) $H = tC_n$ with $t \geq 1$ and $n \geq 5$ odd and there are two indices $k \neq \ell$ from the set $[1, p]$ such that

$$f_i(v) = \begin{cases} t & \text{if } i \in \{k, \ell\}, \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in [1, p]$ and for each $v \in V(H)$. In this case, we say that H is a block of type **(C)**.

- (4) There are two hard pairs (H^1, f^1) and (H^2, f^2) with $f^1 \in \mathcal{V}_p(H^1)$ and $f^2 \in \mathcal{V}_p(H^2)$ such that H is obtained from H^1 and H^2 by merging two vertices $v^1 \in V(H_1)$ and $v^2 \in V(H_2)$ to a new vertex v^* . Furthermore, it holds

$$f(v) = \begin{cases} f^1(v) & \text{if } v \in V(H_1) \setminus \{v^1\}, \\ f^2(v) & \text{if } v \in V(H_2) \setminus \{v^2\}, \\ f^1(v^1) + f^2(v^2) & \text{if } v = v^* \end{cases}$$

for all $v \in V(H)$. In this case we say that (H, f) is obtained from (H^1, f^1) and (H^2, f^2) by merging v^1 and v^2 to v^* .

In order to develop a better feeling of how hard pairs may look like we refer the reader to

Figure 2.2. An example of merging twice is displayed in Figure 2.3.

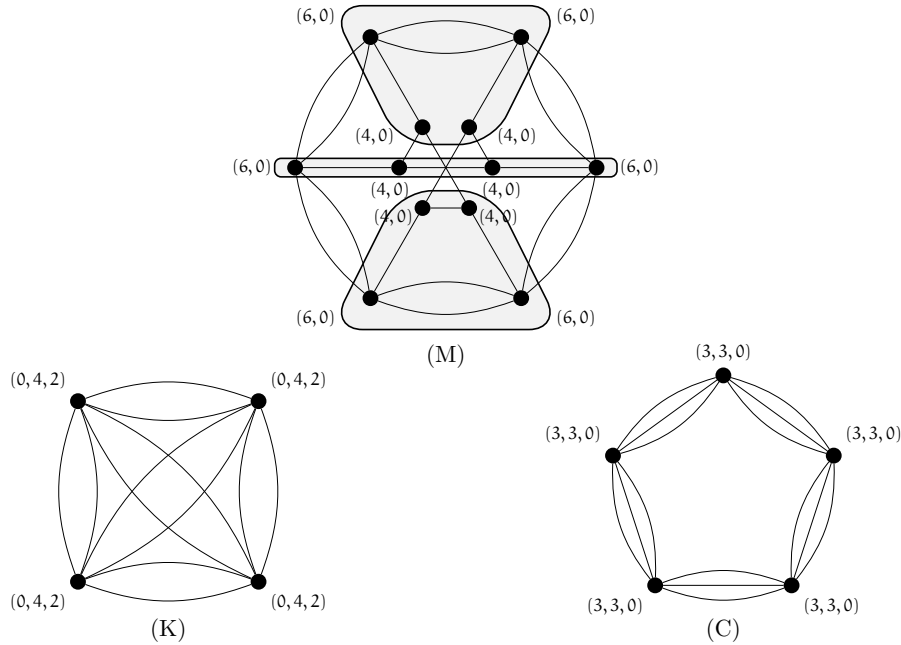


FIG. 2.2. A block of type (M), (K), and (C).

In the following section, we will show that if H is a hypergraph and $f \in \mathcal{V}_p(H)$ is a function ($p \geq 1$) the condition $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ for all $v \in V(H)$ is not sufficient for the existence of an f -partition of H if and only if at least one component of H is f -hard. Note that H is f -partitionable if and only if each component of H is f -partitionable. Thus, it is satisfactory to consider only connected hypergraphs. The next result was proved by BORODIN, KOSTOCHKA and TOFT [23] for the class of simple graphs. In the next section, we will show how to extend it to hypergraphs.

Theorem 2.3. *Let H be a connected hypergraph and let $f \in \mathcal{V}_p(H)$ be a vector function with $p \geq 1$ such that $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ for all $v \in V(H)$. Then H is not f -partitionable if and only if (H, f) is a hard pair. \diamond*

2.2. Proof of Theorem 2.3

We begin this section with an apologize to the reader: within the following, it will be unavoidable to use also the letter G for hypergraphs in order to ensure readability. Nevertheless, this is the only section of this thesis in which G refers to a hypergraph, so we hope

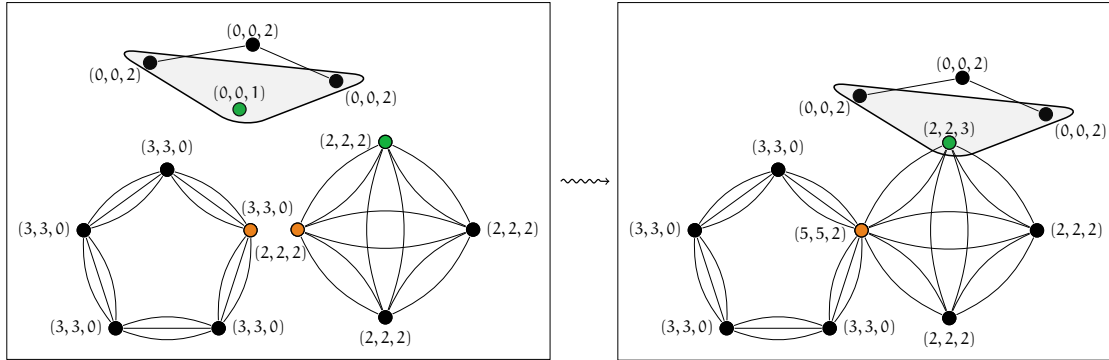


FIG. 2.3. Merging hard pairs.

that the reader will forgive us the stylistic weakness. The next proposition—although fairly trivial—will be used frequently in the following.

Proposition 2.4. *Let H be a connected hypergraph, and let $h \in \mathcal{V}_1(H)$. If $h(v) = d_H(v)$ for all $v \in V(H)$, then each proper subhypergraph of H is strictly h -degenerate. \diamond*

The proof of Theorem 2.3 is divided into two parts. In the first part, we prove some properties of hard pairs and show that any hard pair is not f -partitionable. The proof of the next proposition can be done by induction on the number of blocks of H and is straightforward.

Proposition 2.5. *Let H be a connected hypergraph and let $f \in \mathcal{V}_p(H)$ be a vector function with $p \geq 1$ such that H is f -hard. Then, for each $B \in \mathcal{B}(H)$ there is a uniquely determined function $f_B \in \mathcal{V}_p(B)$ such that the following statements hold:*

- (a) (B, f_B) is a hard pair of type (M), (K), or (C).
- (b) $f(v) = \sum_{B \in \mathcal{B}_v(H)} f_B(v)$ for all $v \in V(H)$.
- (c) $f_B(v) = f(v)$ for all non-separating vertices v of H belonging to B . \diamond

Note that the above proposition clearly implies that $f_B(v) \leq f(v)$ holds coordinatewise. The next proposition shows that f -hard hypergraphs are not f -partitionable.

Proposition 2.6. *Let H be a connected hypergraph, and let $f \in \mathcal{V}_p(H)$ be a vector function with $p \geq 1$. If H is f -hard, then the following statements hold:*

- (a) $f_1(v) + f_2(v) + \dots + f_p(v) = d_H(v)$ for all $v \in V(H)$.

(b) If $u \neq u'$ are two non-separating vertices contained in the same block of H , then either $f(u) = f(u')$ or $f_i(u) = f_i(u') = 0$ for all but one index $i \in [1, p]$.

(c) H is not f -partitionable. ◊

Proof. Statements (a) and (b) are simple consequences of Proposition 2.5. The proof of (c) is by reductio ad absurdum. To this end, choose (H, f) such that

- (1) H is f -hard,
- (2) there is an f -partition (H_1, H_2, \dots, H_p) of H , and
- (3) $|H|$ is minimum with respect to (1) and (2).

Note that the empty hypergraph is the only hypergraph that is strictly 0-degenerate; thus, if $f_i \equiv 0$ for some i , then $H_i = \emptyset$ must hold. As a consequence, if (H, f) is of type (M), there is an index j such that $H_i = \emptyset$ for all $i \in [1, p] \setminus \{j\}$ and $f_j(v) = d_H(v)$ for all $v \in V(H)$. Therefore, H_j is not strictly f_j -degenerate, contradicting (2).

If (H, f) is of type (K), then $H = tK_n$ for some $t \geq 1, n \geq 3$ and there are integers n_1, n_2, \dots, n_p such that $n_1 + n_2 + \dots + n_p = n - 1$ and $f(v) = (tn_1, tn_2, \dots, tn_p)$ for all $v \in V(H)$. Thus, H_i is a tK_{m_i} for some $m_i \geq 0$ for all $i \in [1, p]$. Since H_i is strictly f_i -degenerate, it holds $|H_i| = m_i \leq n_i$ for all $i \in [1, p]$. Consequently, we obtain

$$|H| = |H_1| + |H_2| + \dots + |H_p| \leq n - 1,$$

which is impossible.

If (H, f) is of type (C), then $H = tC_n$ for some $t \geq 1$ and $n \geq 5$ odd, and there are two indices $k \neq \ell$ from the set $[1, p]$ such that $f_i(v) = t$ for $i \in \{k, \ell\}$ and $f_i(v) = 0$, otherwise. Then, (H_k, H_ℓ) is a 2-partition of H and $f_k(v) = f_\ell(v) = t$ for all $v \in V(H)$. Since n is odd, one of the parts, say H_k , contains two adjacent vertices that are joined by t parallel edges. Therefore, H_k is not strictly f_k -degenerate, a contradiction.

It remains to consider the case that (H, f) is obtained from two hard pairs (H^1, f^1) and (H^2, f^2) by merging v^1 and v^2 to v^* . In order to simplify the proof, we assume $v^1 = v^2 = v^*$. By (3), H^j is not f^j -partitionable for $j \in \{1, 2\}$. Let $H_i^j = H^j \cap H_i$ for $i \in [1, p]$ and $j \in \{1, 2\}$. By symmetry, we may assume $v^* \in V(H_1)$. Since (H_1, H_2, \dots, H_p) is an f -partition of H , it follows that H_i^1 is strictly f_i^1 -degenerate and H_i^2 is strictly f_i^2 -degenerate for all $i \in [2, p]$. As a consequence, for $j \in \{1, 2\}$, the hypergraph H_1^j is not strictly f_1^j -degenerate and, thus, there is a non-empty subhypergraph $G^j \subseteq H_1^j$ such that $d_{G^j}(v) \geq f_1^j(v)$ for all $v \in V(G^j)$.

Nevertheless, this implies that $G = G^1 \cup G^2$ is a non-empty subhypergraph of H_1 such that $d_G(v) \geq f_1(v)$ for all $v \in V(G)$, a contradiction. This completes the proof. \blacksquare

Thus, the “if”-direction is proved. For the remaining part, we will need the following notation. We say that (H, f) is a **non-partitionable pair of dimension p** if H is a connected hypergraph, $f \in \mathcal{V}_p(H)$ is a vector function satisfying

$$f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$$

for all $v \in V(H)$, and H is not f -partitionable. The next two propositions describe characteristics of non-partitionable pairs.

Proposition 2.7. *Let (H, f) be a non-partitionable pair of dimension p , let z be a non-separating vertex of H , and let $j \in [1, p]$ such that $f_j(z) \neq 0$. For the hypergraph $H' = H \div z$, define $f' \in \mathcal{V}_p(H')$ to be the vector function satisfying*

$$f'_i(v) = \begin{cases} \max\{0, f_j(v) - \mu_H(z, v)\} & \text{if } i = j, \\ f_i(v) & \text{otherwise} \end{cases}$$

for all $v \in V(H')$ and $i \in [1, p]$. Then, (H', f') is a non-partitionable pair of dimension p , and in what follows, we write $(H', f') = (H, f)/(z, j)$. \diamond

Proof. By symmetry, we may assume $j = 1$. Then, $f_1(z) \geq 1$ and H is not f -partitionable. Thus, $|H| \geq 2$ holds and $H' = H \div z$ is connected. Assume that H' admits an f' -partition (H_1, H_2, \dots, H_p) . To arrive at a contradiction, let $H_1^* = H[V(H_1) \cup \{z\}]$. We show that H_1^* is strictly f_1 -degenerate. To this end, choose a non-empty subhypergraph $G^* \subseteq H_1^*$. Then, $H_1 = H_1^* \div z$ and $G = G^* \div z$ is a subhypergraph of H_1 . As H_1 is strictly f'_1 -degenerate, G is strictly f'_1 -degenerate, too. If G is non-empty, this implies that there is a vertex v satisfying $d_G(v) < f'_1(v)$. But then, $f'_1(v) > 0$ and, by using (1.2), we obtain

$$d_{G^*}(v) = d_G(v) + \mu_H(v, z) < f'_1(v) + \mu_H(v, z) = f_1(v),$$

and we are done. If G is empty, then $V(G^*) = \{z\}$ and $d_{G^*}(z) = 0 < f_1(z)$. Hence, H_1^* is strictly f_1 -degenerate. As $H_j^* = H[V(H_j)]$ is a subhypergraph of H_j for $j \in [2, p]$, H_j^* is strictly- f_j -degenerate and, thus, the sequence $(H_1^*, H_2^*, \dots, H_p^*)$ is an f -partition of H , which is impossible. \blacksquare

By applying the above introduced reduction method, we obtain the following statements.

Proposition 2.8. *Let (H, f) be a non-partitionable pair of dimension $p \geq 1$. Then, the following statements hold:*

- (a) $f_1(v) + f_2(v) + \dots + f_p(v) = d_H(v)$ for all $v \in V(H)$.
- (b) *If z is a non-separating vertex of H satisfying $f_j(z) \neq 0$ for some $j \in [1, p]$, then $f_j(v) \geq \mu_H(z, v)$ holds for all $v \in V(H) \setminus \{z\}$.*
- (c) *If $|H| \geq 2$ and if u is an arbitrary vertex of H , then $H - u$ admits an f -partition. Furthermore, for any such f -partition (H_1, H_2, \dots, H_p) it holds $f_i(u) = d_{H_i+u}(u)$ for all $i \in [1, p]$ and $E_H(u) = E_{H_1+u}(u) \cup E_{H_2+u}(u) \cup \dots \cup E_{H_p+u}(u)$. \diamond*

Proof. The proof of statement (a) is by induction on the order n of H . For $n = 1$, the statement is evident. Let $n \geq 2$ and let v be an arbitrary vertex. Since H is connected, there is a non-separating vertex $z \neq v$ in H . Since

$$f_1(z) + f_2(z) + \dots + f_p(z) \geq d_H(z) \geq 1,$$

it holds $f_j(z) \geq 1$ for some $j \in [1, p]$. By Proposition 2.7, the pair $(H', f') = (H, f)/(z, j)$ is non-partitionable, $f'_i(v) = f_i(v)$ for all $i \neq j$ from the set $[1, p]$ and, moreover, $f'_j(v) = \max\{0, f_j(v) - \mu_H(v, z)\}$. From the induction hypothesis it follows that

$$f'_1(v) + f'_2(v) + \dots + f'_p(v) = d_{H'}(v).$$

Since $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$, this leads to

$$\begin{aligned} d_H(v) &\leq f_1(v) + f_2(v) + \dots + f_p(v) \\ &\leq f'_1(v) + f'_2(v) + \dots + f'_p(v) + \mu_H(v, z) \\ &= d_{H'}(v) + \mu_H(v, z) = d_H(v) \end{aligned}$$

(see (1.2)), and the proof of (a) is complete.

The proof of (b) is by contradiction. Assume that there exist a non-separating vertex z of H and a vertex $v \neq z$ such that $f_j(z) \neq 0$ and $f_j(v) < \mu_H(v, z)$ for some $j \in [1, p]$. By symmetry, we may assume $j = 1$. Then, $(H', f') = (H, f)/(z, 1)$ is a non-partitionable pair such that

$$f_1(v) - \mu_H(z, v) < 0 = f'_1(v)$$

and $f_i(v) = f'_i(v)$ for all $i \in [2, p]$. Using (1.2) and applying (a) to (H', f') as well as (H, f)

leads to

$$\begin{aligned} d_H(v) - \mu(z, v) &= d_{H'}(v) = f'_1(v) + f'_2(v) + \dots + f'_p(v) \\ &> f_1(v) - \mu_H(z, v) + f_2(v) + \dots + f_p(v) \\ &= d_H(v) - \mu_H(z, v), \end{aligned}$$

which is impossible.

In order to prove (c), let u be an arbitrary vertex of H and let $H' = H - u$. Since H is connected, each component G of H' contains a vertex u' , which is a neighbor of u in H and, so, $f_1(u') + f_2(u') + \dots + f_p(u') \geq d_H(u') > d_G(u')$. Applying (a) to (G, f) , this implies that G is f -partitionable and, thus, H' is f -partitionable. Hence, there is an f -partition (H_1, H_2, \dots, H_p) of H' . Since H is not f -partitionable, we conclude that $H'_i = H_i + u$ is not strictly f_i -degenerate for each $i \in [1, p]$. Hence, there is a non-empty subhypergraph G_i of H'_i such that $d_{G_i}(v) \geq f_i(v)$ for all $v \in V(G_i)$. Since H_i is strictly f_i -degenerate, $u \in V(G_i)$ for all $i \in [1, p]$. Due to the fact that $f_1(u) + f_2(u) + \dots + f_p(u) = d_H(u)$ (by (a)) and

$$d_H(u) \geq d_{H'_1}(u) + d_{H'_2}(u) + \dots + d_{H'_p}(u),$$

it follows that

$$\begin{aligned} f_1(u) + f_2(u) + \dots + f_p(u) &= d_H(u) \\ &\geq d_{H'_1}(u) + d_{H'_2}(u) + \dots + d_{H'_p}(u) \\ &\geq d_{G_1}(u) + d_{G_2}(u) + \dots + d_{G_p}(u) \\ &\geq f_1(u) + f_2(u) + \dots + f_p(u), \end{aligned}$$

which leads to $f_i(u) = d_{H'_i}(u)$ for all $i \in [1, p]$. Furthermore, it follows

$$d_H(u) = d_{H'_1}(u) + d_{H'_2}(u) + \dots + d_{H'_p}(u),$$

which clearly implies the last part of the statement. ■

Now we are able to prove the remaining part of Theorem 2.3.

Theorem 2.9. *If (H, f) is a non-partitionable pair of dimension $p \geq 1$, then H is f -hard.*

Proof. The proof is by reductio ad absurdum. So let (H, f) be a smallest counterexample, that is,

- (1) (H, f) is a non-partitionable pair of dimension $p \geq 1$,
- (2) (H, f) is not a hard pair, and
- (3) $|H|$ is minimum subject to (1) and (2).

From Proposition 2.8(a) it then follows that

$$f_1(v) + f_2(v) + \dots + f_p(v) = d_H(v) \quad (2.1)$$

for all $v \in V(H)$. Furthermore, $|H| \geq 2$, for otherwise, (H, f) would be a hard pair of type (M), contradicting (2). To arrive at a contradiction, we shall establish eight claims analyzing the structure of the pair (H, f) .

Claim 2.9.1. *H is a block, that is, H has no separating vertex.* ◊

Proof. Suppose, to the contrary, that H has a separating vertex v^* . Then, H is the union of two connected induced subhypergraphs H^1 and H^2 with $V(H^1) \cap V(H^2) = \{v^*\}$ and $|H^j| < |H|$ for $j \in \{1, 2\}$. By Proposition 2.8(c), $H - v^*$ admits an f -partition (H_1, H_2, \dots, H_p) satisfying $f_i(v^*) = d_{H_i+v^*}(v^*)$ for all $i \in [1, p]$ and

$$E_H(v^*) = E_{H_1+v^*}(v^*) \cup E_{H_2+v^*}(v^*) \cup \dots \cup E_{H_p+v^*}(v^*).$$

For $i \in [1, p]$, we define $H_i^1 = H_i \cap H^1$ and $H_i^2 = H_i \cap H^2$. Then, $H_i = H_i^1 \cup H_i^2$ and

$$f_i(v^*) = d_{H_i+v^*}(v^*) = d_{H_i^1+v^*}(v^*) + d_{H_i^2+v^*}(v^*) \quad (2.2)$$

for all $i \in [1, p]$. For $j \in \{1, 2\}$ let $f^j \in \mathcal{V}_p(H^j)$ be the function satisfying

$$f_i^j(v) = \begin{cases} f_i(v) & \text{if } v \in V(H^j - v^*), \\ d_{H_i^j+v^*}(v^*) & \text{if } v = v^* \end{cases}$$

for all $v \in V(H^j)$ and all $i \in [1, p]$. By (2.1) and (2.2) together with Proposition 2.8(c), we conclude that $f_1^j(v) + f_2^j(v) + \dots + f_p^j(v) = d_{H^j}(v)$ for each $j \in \{1, 2\}$ and $v \in V(H^j)$. If H^j is not f^j -partitionable for $j \in \{1, 2\}$, then, as (H^j, f^j) satisfies (1) and since $|H^j| < |H|$, it follows from (3) that H^j is f^j -hard. Therefore, (H, f) is obtained from two hard pairs by merging two

vertices, and so H is f -hard. Otherwise, by symmetry, we may assume that H^1 admits an f^1 -partition $(H'_1, H'_2, \dots, H'_p)$ and that $v^* \in V(H'_1)$. Consider the p -partition (G_1, G_2, \dots, G_p) of H , where $G_1 = H'_1 \cup (H_1^2 + v^*)$ and $G_i = H'_i \cup H_i^2$ for $i \in [2, p]$. By construction, G_i is strictly f_i -degenerate for $i \in [2, p]$. We claim that G_1 is strictly f_1 -degenerate. In order to prove this, let G be a non-empty subhypergraph of G_1 . If $G \subseteq H_1^2$, then $d_G(v) < f(v)$ for some vertex $v \in V(G)$ since H_1^2 is strictly f_1 -degenerate. Otherwise, $G' = G \cap H'_1$ is a non-empty subhypergraph of H'_1 and, since H'_1 is strictly f_1^1 -degenerate, there is a vertex $v \in V(G')$ such that $d_{G'}(v) < f_1^1(v)$. If $v \neq v^*$, then $d_G(v) = d_{G'}(v) < f_1^1(v) = f_1(v)$ and we are done. Else, $v = v^*$ and it follows from (2.2) and from the definition of f_1^j that

$$d_G(v^*) \leq d_{G'}(v^*) + d_{H_1^2+v^*}(v^*) < f_1^1(v^*) + f_1^2(v^*) = f_1(v^*).$$

This shows that G_1 is strictly f_1 -degenerate and, hence, H is f -partitionable, contradicting the premise. Thus, the proof of the first claim is complete. \square

Claim 2.9.2. *If there exists a vertex $z \in V(H)$ and an index $j \in [1, p]$ such that $f_j(z) \neq 0$, then $(H', f') = (H, f)/(z, j)$ is a non-partitionable pair and the following statements hold:*

(a) (H', f') is a hard pair.

(b) $f_j(v) \geq \mu_H(v, z)$ for all $v \in V(H) \setminus \{z\}$. \diamond

Proof. Since H is a block (by Claim 2.9.1) and $|H| \geq 2$, z is a non-separating vertex of H and $H' = H \div z \neq \emptyset$. Since (H, f) is a non-partitionable pair (by (1)), (H', f') is a non-partitionable pair, too (by Proposition 2.7). From (3) it then follows that (H', f') is a hard pair. Statement (b) is a consequence of Proposition 2.8(b). \square

Now let $z \in V(H)$ be an arbitrary vertex. Since $|H| \geq 2$ and since H is connected, there is an index $j \in [1, p]$ with $f_j(z) \neq 0$ (by (2.1)). By symmetry, we may assume $j = 1$. Then, $(H', f') = (H, f)/(z, 1)$ is a hard pair (by Claim 2.9.2(a)). Furthermore, for all $v \in V(H')$, $f_1(v) \geq \mu_H(v, z)$ (by Claim 2.9.2(b)), and so

$$f'(v) = (f_1(v) - \mu_H(z, v), f_2(v), \dots, f_p(v)). \quad (2.3)$$

Claim 2.9.3. *The hard pair (H', f') is not of type (M). \diamond*

Proof. Assume, to the contrary, that (H', f') is of type (M). If $n = 2$ this implies that (H, f) is of type (M) since otherwise (H, f) would clearly admit an f -partition. But then (H, f) is a

hard pair, contradicting (2). Now let $n \geq 3$. Since (H', f') is of type (M), there is an index $k \in [1, p]$ such that $f'_k(v) = d_{H'}(v)$ and $f'_i(v) = 0$ for all $i \in [1, p] \setminus \{k\}$ and for all $v \in V(H')$.

Case A: *Vertex z is contained in a hyperedge.* Then, $H - z$ is a proper subhypergraph of $H' = H \div z$. Since each vertex v of H' satisfies $d_{H'}(v) = f'_k(v)$, Proposition 2.4 implies that $H - z$ is strictly f'_k -degenerate and therefore strictly f_k -degenerate. If $k \neq 1$, then setting $H_1 = H[\{z\}]$, $H_k = H - z$, and $H_i = \emptyset$ for $i \in [2, p] \setminus \{k\}$ gives us an f -partition of H , which is impossible. Thus, $k = 1$. Moreover, by a similar argumentation it must hold $f(z) = (d_H(z), 0, \dots, 0)$ and, thus, (H, f) is a hard pair of type (M), contradicting (2).

Case B: *Vertex z is contained only in ordinary edges.* Since H is a block, this implies that the ordinary neighborhood $N = N_H(z)$ of z is non-empty. If $k = 1$, then (2.3) leads to $f_2(v) = f_3(v) = \dots = f_p(v) = 0$ for all $v \in V(H')$. By Claim 2.9.2(b), it then follows $f_2(z) = f_3(z) = \dots = f_p(z) = 0$ and, thus, (H, f) is of type (M), a contradiction to (2).

It remains to consider the case that $k \neq 1$, say $k = 2$ (by symmetry). Then, $f_3(v) = f_4(v) = \dots = f_p(v) = 0$ and $f_2(v) = f'_2(v) = d_{H'}(v) > 0$ for all $v \in V(H')$. Since N is non-empty, Claim 2.9.2(b) (applied on a vertex from N) implies that $f_2(z) > 0$. Then, $(H', f'') = (H, f)/(z, 2)$ is a hard pair (by Claim 2.9.2(a)), too, and it holds

$$f''(v) = (f_1(v), d_{H'}(v) - \mu_H(v, z), 0, 0, \dots, 0)$$

for all $v \in V(H')$. Assume that there is a vertex $u \in V(H') \setminus N$. Then, $\mu_H(u, z) = 0$ and $f''(u) = f'(u) = (0, d_{H'}(u), 0, 0, \dots, 0)$. Since (H', f'') is a hard pair of type (M) (and hence H' is a block), we conclude that (H', f'') is a hard pair of type (M) with $f''(v) = (0, d_{H'}(v), 0, 0, \dots, 0)$ for all $v \in V(H')$. However, since $f_1(z) > 0$, Claim 2.9.2(b) leads to $f_1(v) > 0$ for all $v \in N$ and, thus, $f''_1(v) = f_1(v) > 0$ for all $v \in N$, a contradiction. As a consequence, $N = V(H')$. Then, $f_1(z) > 0$ and $f_2(v) = f'_2(v) = d_{H'}(v) > 0$ for all $v \in V(H')$. By Claim 2.9.2(b), this leads to f_1 and f_2 being nowhere-zero in $V(H)$. Let $v \in V(H')$ be an arbitrary vertex. Then, $H_2 = H - z - v$ is a proper subhypergraph of H' and, therefore, strictly f'_2 -degenerate (by Proposition 2.4). If $H_1 = H[\{v, z\}]$ is strictly f_1 -degenerate, then $(H_1, H_2, \emptyset, \emptyset, \dots, \emptyset)$ is an f -partition of H , which is impossible. Thus, $H_1 = H[\{v, z\}]$ is not strictly f_1 -degenerate and, since $\min\{f_1(z), f_1(v)\} \geq \mu_H(v, z) \geq 1$ (by Claim 2.9.2(b)), this leads to $f_1(z) = f_1(v) = \mu_H(v, z)$. Since v was chosen arbitrarily, this implies that there is an integer $m \geq 1$ such that $m = \mu_H(v, z) = f_1(v) = f_1(z)$ for all $v \in V(H') = N$. Since $n \geq 3$, N contains at least two vertices. We choose two different vertices from N , say u and v and

show that $\mu_H(\mathbf{u}, \mathbf{v}) = m$. Let $H_2 = H - \mathbf{u} - \mathbf{v}$. We claim that H_2 is strictly f_2 -degenerate. To this end, let G be a non-empty subhypergraph of H_2 . If z is contained in G , then

$$d_G(z) \leq d_{H_2}(z) = d_H(z) - 2m < d_H(z) - m = f_2(z),$$

(as $f_1(z) = m$ and as $f_3(z) = f_4(z) = \dots = f_p(z) = 0$) and we are done. If z is not contained in G , then G is a proper subhypergraph of $H' = H \div z = H - z$. As $d_{H'}(\mathbf{v}) = f'_2(\mathbf{v})$ for all $\mathbf{v} \in V(H')$, it follows from Proposition 2.4 that G is strictly f'_2 -degenerate and, therefore, strictly f_2 -degenerate. Consequently, H_2 is strictly f_2 -degenerate. If $H_1 = H[\{\mathbf{u}, \mathbf{v}\}]$ is strictly f_1 -degenerate, then $(H_1, H_2, \emptyset, \emptyset, \dots, \emptyset)$ is an f -partition of H , which is impossible. Thus, H_1 is not strictly f_1 -degenerate and, as $m = f_1(\mathbf{u}) = f_1(\mathbf{v}) \geq \mu_H(\mathbf{u}, \mathbf{v})$ (by Claim 2.9.2(b)), it follows that $\mu_H(\mathbf{u}, \mathbf{v}) = m$.

Since \mathbf{u} and \mathbf{v} were chosen arbitrarily from N , we conclude that $\mu_H(\mathbf{u}, \mathbf{v}) = m$ for all $\mathbf{u} \neq \mathbf{v}$ from $V(H)$. As a consequence, we obtain $f_2(\mathbf{v}) \geq m(n-2)$ for all $\mathbf{v} \in V(H')$. Assume that H contains a hyperedge e . Then, since z is contained only in ordinary edges, e belongs to $E(H')$. Moreover, regarding $(H', f'') = (H, f)/(z, 2)$, it follows that (H', f'') is a hard pair of type (M) (by Claim 2.9.2(a) and as H' contains e). Furthermore, as $f_1(\mathbf{v}) = f''_1(\mathbf{v}) > 0$ for all $\mathbf{v} \in V(H')$, it holds $f''_2(\mathbf{v}) = 0$ for all $\mathbf{v} \in V(H')$ and, therefore, $f_2(\mathbf{v}) = m(n-2)$ for all $\mathbf{v} \in V(H')$ and $n = 3$. However, this leads to $|H'| = n - 1 = 2$ and, thus, H' cannot contain any hyperedges, a contradiction. Hence, H does not contain any hyperedges and, therefore, $H = mK_n$ and $f_2(\mathbf{v}) = m(n-2)$ for all $\mathbf{v} \in V(H')$. Since we have $d_H(z) = m(n-1) = f_1(z) + f_2(z) = m + f_2(z)$ (by (2.1)), we conclude that $f_2(z) = m(n-2)$ and (H, f) is a hard pair of type (K), contradicting (2). This completes the proof. \square

Claim 2.9.4. *The hard pair (H', f') is not of type (K).* \diamond

Proof. Assume, to the contrary, that $(H', f') = (H, f)/(z, 1)$ is of type (K). Then, it holds $H' = tK_{n-1}$ for some $t \geq 1$ and $n \geq 4$ and there are integers n_1, n_2, \dots, n_p with at least two n_i different from zero such that $n_1 + n_2 + \dots + n_p = n - 2$ and that $f'(\mathbf{v}) = (tn_1, tn_2, \dots, tn_p)$ for all $\mathbf{v} \in V(H')$. By symmetry, we may assume $n_2 > 0$ and, thus, $f'_2(\mathbf{v}) > 0$ for all $\mathbf{v} \in V(H')$. We distinguish between two cases.

Case A: $E_H(z)$ contains an ordinary edge. Then, the ordinary neighborhood $N = N_H(z)$ of z is non-empty. Since $f_2(\mathbf{v}) = f'_2(\mathbf{v}) > 0$ for all $\mathbf{v} \in N$, it follows from Claim 2.9.2(b) that $f_2(z) > 0$. Let $\mathbf{v} \in N$ and let $m = \mu_H(\mathbf{v}, z)$. Then, by (2.3), we have $f(\mathbf{v}) = (tn_1 + m, tn_2, tn_3, \dots, tn_p)$. Since $f_2(z) > 0$, Claim 2.9.2(a) implies that $(H', f'') = (H, f)/(z, 2)$ is also a hard pair, which can be only of type (M) or (K). Since H' is regular, we conclude

that f'' is constant. Furthermore,

$$f''(v) = (tn_1 + m, tn_2 - m, tn_3, \dots, tn_p) \neq f'(v)$$

and

$$f''(w) = (tn_1 + \mu_H(w, z), tn_2 - \mu_H(w, z), tn_3, \dots, tn_p)$$

for all $w \in V(H') \setminus \{v\}$ (by Claim 2.9.2(b)). Consequently, $\mu_H(z, u) = m \geq 1$ for all $u \in V(H')$. Since (H', f'') is of type (M) or (K) and since all vertices of H' have degree $t(n-2)$ in H' , we furthermore conclude that $m \equiv 0 \pmod{t}$ and so $m \geq t$. Finally, we obtain that f_1 as well as f_2 are nowhere-zero in $V(H)$.

Next we claim that $H \div v$ is a block for all $v \in V(H)$. Otherwise, there would exist a vertex $v \in V(H)$ different from z such that $H \div v$ is not a block. Since z is joined to all other vertices by ordinary edges, z would be the only possible separating vertex of $H \div v$. However, as $(H \div z) \div v = (H \div v) \div z$, the hypergraph $(H \div v) \div z$ is complete and therefore connected, a contradiction.

Assume that there is a hyperedge $e \in E(H)$. Then, $z \in i_H(e)$ and $|i_H(e)| = 3$ since H' does not contain any hyperedges. Since $n \geq 4$, there is a vertex $x \in V(H) \setminus i_H(e)$. As $H \div x$ is a block containing the hyperedge e , the hard pair $(H, f)/(x, j)$ must be of type (M) for $j \in \{1, 2\}$. Since $\mu_H(x, z) = m$ and since $f_1(z), f_2(z) \geq m$ (by Claim 2.9.2(b)), this implies that $f_1(z) = f_2(z) = m$ and $f_3(z) = f_4(z) = \dots = f_p(z) = 0$. As a consequence, $f_1(z) + f_2(z) + \dots + f_p(z) = 2m < 3m \leq d_H(z)$, a contradiction.

Hence, there are no hyperedges in H . Then, $H' = H \div z = H - z$ and so $\mu_H(u, v) = t$ for all $u \neq v$ from $V(H) \setminus \{z\}$. Moreover, as $\mu_H(v, z) = m$ for all $v \in V(H) \setminus \{z\}$ and since f_1 and f_2 are nowhere-zero, it holds $\min\{f_1(v), f_2(v)\} \geq m$ (by Claim 2.9.2(b)). We show that $t = m$ and so $H = tK_n$. Otherwise, $t < m$. As $n \geq 4$, $(H^*, f^*) = (H, f)/(x, 1)$ must be of type (M) for any $x \in V(H) \setminus \{z\}$. However, since $t < m$ it holds $f_1^*(v) = f_1(v) - t > 0$ and $f_2^*(v) = f_2(v) > 0$ for all $v \in V(H^*) \setminus \{z\} \neq \emptyset$, a contradiction. Thus, $m = t$ and so $H = tK_n$.

To conclude the case, we show that (H, f) is of type (K), giving a contradiction to statement (2). To this end, we choose two distinct vertices u and v in H . By Proposition 2.8(c), $H-u$ admits an f -partition (H_1, H_2, \dots, H_p) and $f_i(u) = d_{H_i+u}(u) = t|H_i|$ for every $i \in [1, p]$. By symmetry, we may assume $v \in V(H_1)$. Due to the fact that H_1 is strictly f_1 -degenerate and since $f_1(u) = d_{H_1+u}(u) > d_{(H_1-v)+u}(u)$, the hypergraph $H'_1 = (H_1-v)+u$ is also strictly f_1 -degenerate. Thus, (H'_1, H_2, \dots, H_p) is an f -partition of $H-v$ satisfying $|H'_1| = |H_1|$. As a consequence, $f_i(v) = t|H_i|$ for every $i \in [1, p]$ (by Proposition 2.8(c)). In conclusion,

$f_i(u) = f_i(v) = t|H_i|$ for each $i \in [1, p]$ and, since f_1 and f_2 are nowhere-zero, at least two $|H_i|$ are non-empty. Therefore, (H, f) is of type (K), contradicting (2).

Case B: $E_H(z)$ contains only hyperedges. This implies $f(v) = f'(v) = (tn_1, tn_2, \dots, tn_p)$ for all $v \in V(H')$. First assume that there is a vertex $v \in V(H')$ such that $H \div v$ has a separating vertex. Since $(H \div v) \div z = (H \div z) \div v$ is complete, z is a non-separating vertex of $\tilde{H} = H \div v$. Let B be the block of \tilde{H} containing z . Due to the fact that any two distinct vertices of $V(H) \setminus \{v, z\}$ are either contained in an ordinary edge of H or in a hyperedge of H together with z , they all are contained in the same block B' of \tilde{H} and B' is a tK_{n-2} . Since \tilde{H} has at least two blocks, this implies that B and B' are the only blocks of \tilde{H} and that there is exactly one separating vertex u in \tilde{H} . Moreover, we conclude that there is a hyperedge e in H with $i_H(e) = \{u, v, z\}$. Let x be a non-separating vertex of B' . Then, $(H'', f'') = (H, f)/(x, 2)$ is a hard pair (since $f_2(x) = f'_2(x) > 0$ and by Claim 2.9.2(a)). As B' is a tK_{n-2} and as v is joined to all vertices from $V(B') \setminus \{u\}$ by ordinary edges (since $H \div z$ is a tK_{n-1}), we conclude that H'' is a block, which contains the hyperedge e . Thus, (H'', f'') is of type (M) and there is an index $i \in [1, p]$ such that $f''_i(w) = d_{H''}(w) > 0$ and $f''_k(w) = 0$ for all $k \in [1, p] \setminus \{i\}$ and for all $w \in V(H'')$. In particular, since $f_1(z) = f''_1(z) > 0$, it holds $i = 1$. Thus, $f_1(w) = f''_1(w) = d_{H''}(w) > 0$ and $f''_k(w) = 0$ for all $k \in [2, p]$ and for all $w \in V(H'')$. However, this also implies that $f_1(x) > 0$ (since $f_1 = f'_1$ is constant in $V(H')$). By Claim 2.9.2(a), $(H'', f^*) = (H, f)/(x, 1)$ again is a hard pair with $f_1(w) = d_{H''}(w)$ and $f_k(w) = 0$ for all $k \in [2, p]$ and for all $w \in V(H')$ (as $f_1(z) = f''_1(z) > 0$). However, in this case we obtain $f^*_2(v) = f_2(v) > 0$, a contradiction.

It remains to consider the case that $H \div v$ is a block for all $v \in V(H)$. Suppose first that $f(z) = (d_H(z), 0, 0, \dots, 0)$ and let $v \in V(H')$. Then, since $f_2(v) = f'_2(v) > 0$ and by Claim 2.9.2(a), the pair $(H'', f'') = (H, f)/(v, 2)$ is a hard pair of type (M) with $f''_1(u) = d_{H''}(u) > 0$ for any $u \in V(H'')$. Since $f''_1(u) = f_1(u) = f_1(v)$ for any $u \in V(H'') \setminus \{z\}$ (as f_1 is constant in $V(H) \setminus \{z\}$), this implies that $f_1(v) > 0$ and $(H'', f^*) = (H, f)/(v, 1)$ is a hard pair of type (M). However, it holds $f^*_1(z) = f_1(z) > 0$ and $f^*_2(u) = f_2(u) > 0$ for all $u \in V(H'') \setminus \{z\}$, which is impossible.

Now suppose that $f(z) \neq (d_H(z), 0, 0, \dots, 0)$. Then there is an index $j \neq 1$ such that $f_j(z) > 0$. Since $E_H(z)$ contains only hyperedges, this implies that $(H, f)/(v, k)$ is not of type (M) for any $v \in V(H')$ and for any $k \in [1, p]$ with $f_k(v) > 0$. Thus, after shrinking H at any vertex, no hyperedge remains. Nevertheless, since $n \geq 4$, this is impossible. This completes the proof. \square

Claim 2.9.5. *The hard pair (H', f') is not of type (C).* \diamond

Proof. Assume, to the contrary, that $(H', f') = (H, f)/(z, 1)$ is of type (C) and, thus, $H = tC_{n-1}$ for some $t \geq 1, n \geq 6$ even. Moreover, there are two indices $k \neq \ell$ from $[1, p]$ such that $f'_k(v) = f'_\ell(v) = t$ and $f'_j(v) = 0$ for all $j \in [1, p] \setminus \{k, \ell\}$ and for all $v \in V(H')$. By symmetry, we may assume $k = 2$ and $\ell \in \{1, 3\}$. If $\ell = 1$, then we obtain

$$\textcircled{*} \quad f'(v) = (t, t, 0, 0, \dots, 0) \text{ and } f(v) = (t + \mu_H(v, z), t, 0, 0, \dots, 0)$$

for all $v \in V(H')$. If $\ell = 3$, then

$$\textcircled{\odot} \quad f'(v) = (0, t, t, 0, 0, \dots, 0) \text{ and } f(v) = (\mu_H(v, z), t, t, 0, 0, \dots, 0)$$

for all $v \in V(H')$. Similar to the proof of Claim 2.9.4, we distinguish between two cases.

Case A: $E_H(z)$ contains an ordinary edge. Then, the ordinary neighborhood $N = N_H(z)$ of z is non-empty. Since $f_2(v) = f'_2(v) = t > 0$ for all $v \in V(H')$ (by (2.3)), this implies that $f_2(z) > 0$ (by Claim 2.9.2(b)) and so $(H', f'') = (H, f)/(z, 2)$ is a hard pair of type (M) or (C). If $\textcircled{*}$ holds, then

$$f''(v) = (t + \mu_H(v, z), t - \mu_H(v, z), 0, 0, \dots, 0)$$

for all $v \in V(H')$ and since $\mu_H(v, z) \geq 1$ for all $v \in N$ this implies that (H', f'') is a bad pair of type (M). Then we conclude that $t - \mu_H(v, z) = 0$ for all $v \in V(H')$ and so $\mu_H(v, z) = t$ for all $v \in V(H')$ and $N = V(H')$. If $\textcircled{\odot}$ holds, we have

$$f''(v) = (\mu_H(v, z), t - \mu_H(v, z), t, 0, 0, \dots, 0)$$

for all $v \in V(H')$ and again, since $\mu_H(v, z) \geq 1$ for all $v \in N$, this implies that (H', f'') is a bad pair of type (C). Hence, in both cases we have $\mu_H(v, z) = t$ for all $v \in V(H')$ and $N = V(H')$. Thus, we obtain

$$\textcircled{\tilde{*}} \quad f(v) = (2t, t, 0, 0, \dots, 0) \text{ for all } v \in V(H') \text{ (if } \ell = 1), \text{ or}$$

$$\textcircled{\tilde{\odot}} \quad f(v) = (t, t, t, 0, 0, \dots, 0) \text{ for all } v \in V(H') \text{ (if } \ell = 3).$$

Since z is joined in H to all other vertices by ordinary edges and since $H \div v \div z = H \div z \div v$ is a path (with multiple edges) and therefore connected for any $v \in V(H')$, $H \div v$ is a block for all $v \in V(H)$. As a consequence, for any vertex $v \in V(H')$, the hard pair $(H, f)/(v, 1)$ must be of type (M). However, since for H either $\textcircled{\tilde{*}}$ or $\textcircled{\tilde{\odot}}$ holds, it is easy to check that this is impossible.

Case B: $E_H(z)$ contains only hyperedges. As a consequence, $f(v) = f'(v)$ for all $v \in V(H')$. We claim that H admits an f -partition. To this end, let $e \in E(H)$ be an arbitrary hyperedge of H , and let $i_H(e) = \{x, y, z\}$ (e must contain z since $H \div z$ is a tC_{n-1}). Then, since $H' = H \div z$, the vertices x and y are adjacent in H' . Let v_1, v_2, \dots, v_{n-1} be a cyclic order of the vertices of $H' = tC_{n-1}$ with $x = v_1$ and $y = v_{n-1}$. Then, we define $H_2 = H[\{v_i \in V(H') \mid i \geq 1 \text{ odd}\}]$, $H_\ell = H[\{v_i \in V(H') \mid i \geq 2 \text{ even}\}]$, and $H_j = \emptyset$ for all $j \in [1, p] \setminus \{2, \ell\}$. Since $d_{H_2}(v_1) = d_{H_2}(v_{n-1}) < t = f_2(v_1) = f_2(v_{n-1})$, $E(H_\ell) = \emptyset$, and $f_\ell(v) = t$ for all $v \in V(H')$ (see \circledast , \circledcirc), the sequence (H_1, H_2, \dots, H_p) is an f -partition of $H - z$. As H_1 is an edgeless induced subhypergraph of $H \div z$, it follows that $d_{H_1+z}(z) = 0 < f_1(z)$, contradicting Proposition 2.8(c). This proves the claim. \square

Claim 2.9.6. For every vertex $z \in V(H)$, $H \div z$ is not a block. \diamond

Proof. Suppose, to the contrary, that there exists a vertex z such that $H \div z$ is a block. Let $j \in [1, p]$ such that $f_j(z) > 0$. Then, by Claim 2.9.2(a), $(H', f') = (H, f)/(z, j)$ is a hard pair and, since $H' = H \div z$ is a block, (H', f') must be of type (M), (K), or (C). However, the three above claims imply that this is not possible. \square

Claim 2.9.7. For every vertex $z \in V(H)$, $H \div z$ has exactly two end-blocks. \diamond

Proof. Assume, to the contrary, that there is a vertex $z \in V(H)$ such that $H' = H \div z$ does not have exactly two end-blocks. By Claim 2.9.6 this implies that H' has at least three end-blocks.

Let T denote the block graph of H' , that is, the simple graph having vertex set $V(T) = \mathcal{B}(H') \cup S$, where S is the set of all separating vertices of H' , and edge set $E(T) = \{vB \mid v \in S, B \in \mathcal{B}(H') \text{ and } v \in V(B)\}$. Note that T is a tree with bipartition $(\mathcal{B}(H'), S)$ and the end-blocks of H' coincide with the leaves of T . Since H' has at least three end-blocks, $\Delta(T) \geq 3$. Let B be an arbitrary end-block of H' . Since B is a leaf of T and $\Delta(T) \geq 3$, there is a unique vertex $x_B \in V(T)$ such that x_B is the only vertex of degree at least 3 in T belonging to the subpath P_B of T between x_B and B . Moreover, there exists a unique subtree T_B of T such that $T = T_B \cup P_B$ and $V(T_B) \cap V(P_B) = \{x_B\}$. Finally, there is a unique vertex $v_B \in S$ such that $v_B = x_B$ or $x_B v_B$ is an edge of P_B .

Let B_1, B_2, B_3 be three distinct end-blocks of H' . For $i \in [1, 3]$, let $v_i \in V(B_i)$ be the only separating vertex of H' contained in B_i , let $u_i \in V(B_i) \setminus \{v_i\}$, let V_i be the set of all vertices contained in a block of H' belonging to T_{B_i} , and let $\tilde{B}^i = H[V_i \cup \{z\}]$. Since H is a block, for each end-block B of H' there is an edge $e \in E_H(z)$ such that the vertex set $i_H(e) - \{z\}$ belongs to B and contains a non-separating vertex of H' . Consequently,

\tilde{B}^i is a block contained in $H \div \mathbf{u}_i$ as an induced subhypergraph and, therefore, there is a unique block B^i of $H \div \mathbf{u}_i$ containing \tilde{B}^i . Furthermore, let $(H^i, f^i) = (H, f)/(\mathbf{u}_i, j_i)$ for some $j_i \in [1, p]$ satisfying $f_{j_i}(\mathbf{u}_i) > 0$. Since (H^i, f^i) is a hard pair (by Claim 2.9.2(a)), it follows from Proposition 2.5 that (H^i, f^i) resulted from hard pairs $(B', f_{B'}^i)$ (with $B' \in \mathcal{B}(H^i)$) of type (M), (K), and (C) by merging them appropriately (H^i has at least two blocks by Claim 2.9.6). Note that for $i \neq j$ from $[1, 3]$, the vertices \mathbf{u}_i and \mathbf{u}_j are not adjacent in $H \div z$ and, therefore, not adjacent in H . As a consequence, the hard pair $(B^i, f_{B^i}^i)$ cannot be of type (K) for $i \in [1, 3]$. Furthermore, for $i \in [1, 3]$ and for $j \in [1, 3] \setminus \{i\}$ vertex \mathbf{u}_i is not a separating vertex of H^j contained in B^j and, together with Proposition 2.5, we conclude

$$\otimes \quad f_{B^j}^j(\mathbf{u}_i) = f^j(\mathbf{u}_i) = f(\mathbf{u}_i).$$

In the following, we consider the hard pairs $(B^1, f_{B^1}^1)$, $(B^2, f_{B^2}^2)$, and $(B^3, f_{B^3}^3)$.

Case A: *One of the three hard pairs, say $(B^1, f_{B^1}^1)$ is of type (C).* Then, $B^1 = tC_m$ for some $t \geq 1, m \geq 5$ odd and by symmetry we may assume $f_{B^1}^1(v) = (t, t, 0, \dots, 0)$ for all $v \in V(B^1)$. Since \tilde{B}^1 contains no separating vertex and as B^1 is a tC_m , $\tilde{B}^1 = B^1$. Furthermore, this implies that z is joined to \mathbf{u}_2 and \mathbf{u}_3 by ordinary edges in H (since \mathbf{u}_1 and \mathbf{u}_i are not adjacent in $H \div z$ for $i \in \{2, 3\}$), that $V(B_i) = \{\mathbf{u}_i, v_i\}$ for $i \in \{2, 3\}$ (as B_i is an end-block of H' contained in B^1), and that $P^1 = T_{B^1}$ is a path and each block on P^1 is a tK_2 (as each block on P^1 is contained in $B^1 = tC_m$). Since $f_{B^3}^3(\mathbf{u}_2) = f(\mathbf{u}_2) = f_{B^1}^1(\mathbf{u}_2) = (t, t, 0, 0, \dots, 0)$ (by \otimes) and since $(B^3, f_{B^3}^3)$ is a hard pair (not of type (K)) with $\mathbf{u}_2 \in V(B^3)$ not being a separating vertex of H^3 , we obtain that $(B^3, f_{B^3}^3)$ is a hard pair of type (C), too, and that $P^3 = T_{B^3}$ is a path and each block on P^3 is a tK_2 . Analogously we can show that $(B^2, f_{B^2}^2)$ is a hard pair of type (C) and that each block of the path $P^2 = T_{B^2}$ is a tK_2 . Since $\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 are not pairwise adjacent in H , this implies that $B^1 = tC_m$ contains exactly one separating vertex v_B of H^1 and that H is the union of the three (multi-)cycles B^1, B^2, B^3 with $V(B^1) \cap V(B^2) \cap V(B^3) = \{z, v_B\}$ and $\mathbf{u}_i \notin V(B^i)$. Let ℓ_i be the length of the (z, v_B) -(multi-)path in H containing the ordinary edge $z\mathbf{u}_i$. Then,

$$\begin{aligned} |B^1| &= \ell_2 + \ell_3, \\ |B^2| &= \ell_1 + \ell_3, \text{ and} \\ |B^3| &= \ell_1 + \ell_2. \end{aligned}$$

However, since $|\mathcal{B}^i|$ is odd for $i \in [1, 3]$, we obtain

$$\ell_1 + \ell_2 \equiv \ell_1 + \ell_3 \equiv \ell_2 + \ell_3 \equiv 1 \pmod{p},$$

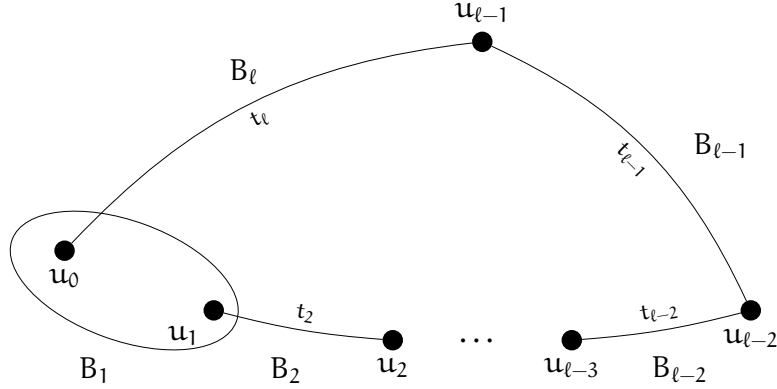
which is impossible.

Case B: All three hard-pairs $(\mathcal{B}^i, f_{\mathcal{B}^i}^i)$ ($i \in [1, 3]$) are of type (M). By symmetry, we may assume that $f_{\mathcal{B}^1}^1(v) = (d_{\mathcal{B}^1}(v), 0, 0, \dots, 0)$ for all $v \in V(\mathcal{B}^1)$. Since the vertex u_2 is contained in \mathcal{B}^1 and \mathcal{B}^3 , and since $f_{\mathcal{B}^1}^1(u_2) = f^1(u_2) = f(u_2) = f_{\mathcal{B}^3}^3(u_2)$ (by \otimes), we conclude that $f_{\mathcal{B}^3}^3(v) = (d_{\mathcal{B}^3}(v), 0, 0, \dots, 0)$ for all $v \in V(\mathcal{B}^3)$. Analogously, regarding $u_3 \in V(\mathcal{B}^2)$, we conclude that $f_{\mathcal{B}^2}^2(v) = (d_{\mathcal{B}^2}(v), 0, 0, \dots, 0)$ for all $v \in V(\mathcal{B}^2)$. As $(H^i, f^i) = (H, f)/(u_i, j_i)$, this implies that $j_i = 1$ for $i \in [1, 3]$. Furthermore, since z is a non-separating vertex of H^i contained in \mathcal{B}^i for all $i \in [1, 3]$ (as $H^i \div z = H \div u_i \div z = H \div z \div u_i$ is connected) and as $j_i = 1$ for $i \in [1, 3]$, it follows that $f(z) = (d_H(z), 0, 0, \dots, 0)$. Let $v \in V(H) \setminus \{z\}$ be an arbitrary vertex. We claim that $f(v) = (d_H(v), 0, 0, \dots, 0)$. Assume this is false. Then, by symmetry, $f_2(v) > 0$. Clearly, v belongs to $\tilde{\mathcal{B}}^i$ for some $i \in [1, 3]$, say v belongs to $\tilde{\mathcal{B}}^1$. Note that $\tilde{\mathcal{B}}^1$ is an induced subhypergraph of $\mathcal{B}^1 \in \mathcal{B}(H^1)$. If v is a non-separating vertex of H^1 , then it holds $f(v) = (d_H(v), 0, 0, \dots, 0)$ (as $j_1 = 1$, as $f^1(v) = f_{\mathcal{B}^1}^1(v) = (d_{\mathcal{B}^1}(v), 0, 0, \dots, 0)$ and by (2.1)), contradicting our assumption. Otherwise, $v = v_{\mathcal{B}^1}$ is a separating vertex of H^1 and so $\tilde{\mathcal{B}}^1 = \mathcal{B}^1$. As (H^1, f^1) results from merging hard pairs (by Proposition 2.5) and as $f_2(v) > 0$, this implies that v is contained in a block \mathcal{B}' of H^1 with $V(\mathcal{B}') \cap V(\mathcal{B}^1) = \{v\}$ and $f_2(w) > 0$ for all $w \in V(\mathcal{B}')$. Let $v' \in V(\mathcal{B}') \setminus \{v\}$. Then, v' is contained in a block belonging to the subpath of the block graph Γ between $v_{\mathcal{B}^1}$ and \mathcal{B}_1 . But then, v' is a non-separating vertex of H^j contained in $\tilde{\mathcal{B}}^j$ for $j \in \{2, 3\}$. This however implies that $f_{\mathcal{B}^2}^2(v') = (d_{\mathcal{B}^2}(v'), 0, 0, \dots, 0)$, a contradiction. Hence, the claim is proved and, thus, (H, f) is a hard pair of type (M), contradicting (2). This settles case B. Thus, the proof of the claim is complete. \square

Claim 2.9.8. *There exists a sequence $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_\ell$ of induced subhypergraphs of H and a sequence $u_0, u_1, \dots, u_{\ell-1}$ of distinct vertices of H with $\ell \geq 4$ such that the following statements hold:*

- (a) $\mathcal{B}_i = t_i K_2$ for $i \in [2, \ell]$ and some $t_i \geq 1$, \mathcal{B}_1 has no separating vertex, and $|\mathcal{B}_1| \geq 2$.
- (b) $H = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_\ell$, for $i \in [1, \ell - 1]$ we have $V(\mathcal{B}_i) \cap V(\mathcal{B}_{i+1}) = \{u_i\}$, and $V(\mathcal{B}_1) \cap V(\mathcal{B}_\ell) = \{u_0\}$. \diamond

Proof. Let z be an arbitrary vertex of H . Then, $H \div z$ has exactly two end-blocks (by Claim 2.9.7) and, therefore, there is a uniquely determined sequence $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ of $k \geq 2$


 FIG. 2.4. The structure of H .

blocks of $H \div z$ and a sequence u_1, u_2, \dots, u_{k-1} of distinct vertices such that $V(B_i) \cap V(B_{i+1}) = \{u_i\}$ for all $i \in [1, k-1]$ and $H \div z = B_1 \cup B_2 \cup \dots \cup B_k$. In particular, B_1 and B_k are the end-blocks of $H \div z$. Let $b_z = \max\{|B_i| \mid i \in [1, k]\}$. Among all vertices z of H we may choose one for which b_z is maximum. Let B_j be a block of $H \div z$ with $|B_j| = b_z$. Since H is a block, there are vertices $u_0 \in V(B_1)$ and $u_k \in V(B_k)$ which are non-separating vertices of $H \div z$ and adjacent to z . Assume that there is an index $i \neq j$ from the set $[1, k]$ such that B_i contains a non-separating vertex v of $H \div z$ different from u_0 and u_k . Then, it follows that $H \div v$ has a block B containing B_j as well as z and, thus, $|B| > |B_j| = b_z$, a contradiction. As a consequence, for each index i from the non-empty set $[1, k] \setminus \{j\}$, there exists an integer $t_i \geq 1$ such that $B_i = t_i K_2$. To complete the proof, all we need to show is that z is not adjacent to any vertex besides u_0 and u_k . By symmetry, we may assume that $j \neq 1$ and, thus, B_1 is a $t_1 K_2$ for some $t_1 \geq 1$ and $V(B_1) = \{u_0, u_1\}$. If there was a hyperedge e with $i_H(e) = \{z, u_0, u_1\}$, then clearly $H \div u_0$ would still be a block, which is impossible. Thus, u_0 is adjacent only to z and u_1 and not contained in any hyperedge. As a consequence, if z is adjacent to any vertex from $(V(B_1) \cup V(B_2) \cup \dots \cup V(B_j)) \setminus \{u_j\}$ other than u_0 and u_k , then $H \div u_0$ has a block B that contains B_j as well as z , giving a contradiction to the maximality of $|B_j|$. Similarly, if z is adjacent to any vertex from $(V(B_j) \cup V(B_{j+1}) \cup \dots \cup V(B_k)) \setminus \{u_{j-1}\}$ except u_k (implying $j \neq k$), by a similar argumentation we conclude that $H \div u_k$ has a block B containing B_j as well as z , again contradicting the maximality of $|B_j|$. Consequently, z is only adjacent to u_0 and u_k . By setting $B_{k+1} = H[\{z, u_k\}]$, $B_{k+2} = H[\{z, u_0\}]$, $u_{k+1} = z$, and by shifting the block-sequence and the vertex-sequence we obtain the statement. \square

To conclude the proof of Theorem 2.9, we show that (H, f) is a hard pair, giving a contradiction to statement (2). Consider the sequences B_1, B_2, \dots, B_ℓ and $u_0, u_1, \dots, u_{\ell-1}$

as described in Claim 2.9.8 (see also Figure 2.4). For technical reasons, let $u_\ell = u_0$. Let $j \in [1, p]$ such that $f_j(u_{\ell-1}) > 0$, and consider the hard pair $(H', f') = (H, f)/(u_{\ell-1}, j)$. Then, B_1 is a block of $H' = H \div u_{\ell-1}$. Let f'_{B_1} be defined as in Proposition 2.5.

First we claim that (B_1, f'_{B_1}) is of type (M). If (B_1, f'_{B_1}) is of type (K), then $|B_1| \geq 3$ and there is a vertex $v \in V(B_1)$ such that $H \div v$ is a block, which is impossible. If (B_1, f'_{B_1}) is of type (C), say $B_1 = tC_m$ with $m \geq 5$ odd, by symmetry we may assume that $f'_{B_1}(v) = (t, t, 0, 0, \dots, 0)$ for all $v \in V(B_1)$. Let P_1 and P_2 be the two disjoint (u_0, u_1) -(multi-)paths in B_1 . As $m \geq 5$, we may suppose that P_2 has length at least three and so there is a vertex v on P_2 that is adjacent to u_0 and not adjacent to u_1 . Then, as v is not adjacent to $u_{\ell-1}$, we have $f(v) = f'_{B_1}(v) = (t, t, 0, 0, \dots, 0)$. Let $(H^*, f^1) = (H, f)/(v, 1)$ and $(H^*, f^2) = (H, f)/(v, 2)$. Due to the fact that $B_1 = tC_m$, that $B_i = t_iK_2$ for $i \in [2, \ell]$, and that $P^* = P_2 - u_0 - v$ is a path of length at least one, we conclude that the union $P_1 \cup B_2 \cup B_3 \cup \dots \cup B_\ell$ itself is a block B^* of H^* , and therefore, $(B^*, f^1_{B^*})$ and $(B^*, f^2_{B^*})$ are both hard pairs (of type (M) or (C)). Note that $H^* = B^* \cup P^*$ and u_1 is the only separating vertex of H^* contained in B^* . Hence, $f^i(u_{\ell-1}) = f^i_{B^*}(u_{\ell-1})$ and $f^i(u_0) = f^i_{B^*}(u_0)$ for $i \in \{1, 2\}$ (by Proposition 2.5(c)). However, $f^1(u_0) \neq f^2(u_0)$ but $f^1(u_{\ell-1}) = f^2(u_{\ell-1})$, contradicting Proposition 2.8(b). This proves the claim that (B_1, f'_{B_1}) is of type (M).

As a consequence, there is an index $j' \in [1, p]$ such that $f_{j'}(v) = d_{B_1}(v) = d_H(v)$ and $f_k(v) = 0$ for $k \in [1, p] \setminus \{j'\}$ and for all $v \in V(B_1) \setminus \{u_0, u_1\}$. Moreover, since (B_1, f'_{B_1}) is of type (M), for $v \in \{u_0, u_1\}$ the j th component of $f_{B_1}(v)$ equals the degree of v in B_1 and so it follows $f_{j'}(v) \geq d_{B_1}(v)$ (as $f(v) \geq f^1_{B_1}(v)$ coordinatewise).

Recall that for $i \in [2, \ell]$, we have $B_i = t_iK_2$ and $V(B_i) = \{u_{i-1}, u_i\}$. Further recall that there is an index $j \in [1, p]$ such that $f_j(u_{\ell-1}) > 0$. By symmetry, we may assume that $j = 1$. We claim that either

$$\otimes f(u_i) = (t_i + t_{i+1}, 0, 0, \dots, 0) \text{ for all } i \in [2, \ell - 1], \text{ or}$$

$$\odot f(u_i) = (t, t, 0, 0, \dots, 0) \text{ for all } i \in [2, \ell - 1] \text{ (except for symmetry) and } B_i = tK_2 \text{ for all } i \in [2, \ell].$$

As $f_1(u_{\ell-1}) > 0$, by repeated application of Claim 2.9.2(b) we conclude that $f_1(u_i) \geq \max\{t_i, t_{i+1}\}$ for $i \in [2, \ell - 1]$. If there exists an index $k \neq 1$, say $k = 2$ (by symmetry) such that $f_2(u_i) > 0$ for some $i \in [2, \ell - 1]$, then, similarly to above, we get $f_2(u_i) \geq \max\{t_i, t_{i+1}\}$ for $i \in [2, \ell - 1]$. By (2.1), this implies $t_i = t_{i+1} = t$ as well as $f(u_i) = (t, t, 0, 0, \dots, 0)$ for some $t \geq 1$ and for all $i \in [2, \ell - 1]$, and so \odot holds. If $f_k(u_i) = 0$ for all $k \in [2, p]$ and all $i \in [2, \ell - 1]$, equation (2.1) implies that $f(u_i) = (t_i + t_{i+1}, 0, 0, \dots, 0)$ for all $i \in [2, \ell - 1]$ and \otimes holds.

If \otimes is satisfied, then by Claim 2.9.2(b) it holds $f(v) = (d_H(v), 0, 0, \dots, 0)$ for $v \in \{u_0, u_1\}$ and hence $j' = 1$ and (H, f) is a hard pair of type (M), contradicting (2).

Thus, it remains to consider the case that \odot holds. If $|B_1| = 2$, then $B_1 = t_1 K_2$ for some $t_1 \geq 1$. Then, again we conclude $f(u_0) = f(u_1) = (t, t, 0, 0, \dots, 0)$ and so $H = tC_n$ and $f(v) = (t, t, 0, 0, \dots, 0)$ for all $v \in V(H)$. Furthermore, n must be odd since otherwise H would clearly admit an f -partition. Consequently, (H, f) is of type (C), which contradicts (2).

Finally, assume that $|B_1| \geq 3$. Then, there is a vertex $z \in V(B_1)$ different from u_0 and u_1 and $f_{j'}(z) = f'_{j'}(z) = d_{B_1}(z) = d_H(z)$ and $f_k(z) = 0$ for $k \in [1, p] \setminus \{j'\}$. As $f(u_{\ell-1}) = (t, t, 0, 0, \dots, 0)$, it follows from Claim 2.9.2(b) that $f_1(u_0) \geq t > 0$ and $f_2(u_0) \geq t > 0$. Since $(H', f') = (H, f)/(u_{\ell-1}, 1)$ and since (B_1, f'_{B_1}) is a hard pair of type (M), it must hold $f_1(u_0) = t$ and $j' = 2$. Therefore, we have $f(z) = f'(z) = (0, d_H(z), 0, 0, \dots, 0)$. Moreover, as $f_2(u_{\ell-1}) = t > 0$, $(H', f'') = (H, f)/(u_{\ell-1}, 2)$ is a hard pair, too, and (B_1, f''_{B_1}) is a hard pair of type (M). Consequently, it must hold $f_2(u_0) = t$ and $f''_1(v) > 0$ for all $v \in V(B_1)$. However, as $f''_1(z) = f_1(z) = f'_1(z) = 0$, this is impossible. This completes the proof of Theorem 2.9 and, therefore, also Theorem 2.3 is proved. \blacksquare

Chapter 3

GENERALIZED COLORINGS OF HYPERGRAPHS

3.1. Brooks' Theorem for List-Colorings of Hypergraphs

Recall from the first chapter that the chromatic number, respectively list-chromatic number of a hypergraph H is always less than or equal to the coloring number of H . In particular, in (1.3), we obtained that

$$\chi(H) \leq \chi_\ell(H) \leq \text{col}(H) \leq \Delta(H) + 1.$$

The inequality $\chi(H) \leq \Delta(H) + 1$ naturally raises the question in which cases equality holds. For simple graphs, the answer was given by BROOKS [26] in 1941. His famous Theorem 1 states that complete graphs and odd cycles are the only connected graphs, for which the chromatic number is equal to the maximum degree plus one. For list-colorings, the solution was found by ERDŐS, RUBIN, and TAYLOR [44] (see also Theorem 2) and, independently, by VIZING [120]. A (slightly more general) degree-version was proved by ERDŐS, RUBIN, and TAYLOR and, independently, by Borodin [19, 20].

Theorem 3.1 (ERDŐS, RUBIN, and TAYLOR, 1979). *Let G be a connected simple graph and let L be a list-assignment satisfying $|L(v)| \geq d_G(v)$ for all $v \in V(G)$. If G does not admit a proper L -coloring, then $|L(v)| = d_G(v)$ for all $v \in V(H)$ and each block of G is either a complete graph or an odd cycle. As a consequence, $\chi_\ell(G) \leq \Delta(G) + 1$ and equality holds if and only if G is a complete graph or an odd cycle. \diamond*

It turns out that those theorems can be extended to hypergraphs, as well. An analogue to BROOKS' Theorem was given by JONES [64] in 1975 (see Theorem 2.2). BROOKS' Theorem for list-colorings of hypergraphs was obtained by KOSTOCHKA, STIEBITZ, and WIRTH [72].

Theorem 3.2 (KOSTOCHKA, STIEBITZ, and WIRTH, 1996). *Let H be a connected simple hypergraph and let L be a list-assignment satisfying $|L(v)| \geq d_H(v)$ for all $v \in V(H)$. If H does not admit a proper L -coloring, then it holds $|L(v)| = d_H(v)$ for all $v \in V(H)$ and each block B of H is either a complete graph, an odd cycle, or B has just one edge. As a consequence, $\chi_\ell(H) \leq \Delta(H) + 1$ and equality holds if and only if H is either a complete graph, an odd cycle, or if H contains just one edge.* \diamond

How can we conclude the above theorem from Theorem 2.3? Recall from Section 2.1 that it is possible to transform the list-coloring problem to the one of finding an f -partition: given a simple hypergraph H and a list-assignment L of H with color set $\Gamma = [1, p]$, set $f_i(v) = 1$ if $i \in L(v)$ and $f_i(v) = 0$, otherwise. Then, H has a proper L -coloring if and only if H is f -partitionable. Moreover, if we have $|L(v)| \geq d_H(v)$ for all $v \in V(H)$, then the definition of f implies $f_1(v) + f_2(v) + \dots + f_p(v) = |L(v)| \geq d_H(v)$ for all $v \in V(H)$. Thus, if H does not admit a proper L -coloring, then (H, f) is a hard pair by Theorem 2.3 and so each block of H is of type (M), (K) or (C), which proves the first part of Theorem 3.2. Moreover, from Proposition 2.6(a) it follows that $|L(v)| = f_1(v) + f_2(v) + \dots + f_p(v) = d_H(v)$ for all $v \in V(H)$. In order to deduce the second part of the above theorem, we argue as follows. If $\chi_\ell(H) = \Delta(H) + 1$, then there is a list-assignment L satisfying $|L(v)| = \Delta(H)$ for all $v \in V(H)$ such that H does not admit a proper L -coloring. Consequently, it must hold that $d_H(v) = |L(v)| = \Delta(H)$ for all $v \in V(H)$, and so H is $\Delta(H)$ -regular. Moreover, we already know that each block of H is a complete graph, an odd cycle, or contains just one edge. But then, as H is $\Delta(H)$ -regular, H can only consist of exactly one block, and we are done. On the other hand, if H is a complete graph, an odd cycle, or if $|E(H)| = 1$, then it is easy to see that $\chi_\ell(H) = \Delta + 1$.

3.2. Additional Degree Constraints

BORODIN [18] and, independently, BOLLOBÁS and MANVEL [16], proved another extension of BROOKS' Theorem for the class of simple graphs.

Theorem 3.3 (BORODIN, 1976/BOLLOBÁS and MANVEL, 1979). *Let G be a connected simple graph with maximum degree $\Delta \geq 3$ different from $K_{\Delta+1}$. Let also k_1, k_2, \dots, k_p be positive*

integers, $p \geq 2$, such that $k_1 + k_2 + \dots + k_p \geq \Delta$. Then, there is a p -partition (G_1, G_2, \dots, G_p) of G such that $\text{col}(G) \leq k_i$ whenever $1 \leq i \leq p$. \diamond

Clearly, by setting $k_1 = k_2 = \dots = k_p = 1$ one can immediately deduce BROOKS' Theorem from the above theorem. However, BORODIN [20] generalized Theorem 3.3 even further with the help of a simple argument. BOLLOBÁS and MANVEL [16] proved the same extension independently.

Theorem 3.4 (BORODIN, 1979/BOLLOBÁS and MANVEL, 1979). *Let G be a connected simple graph with maximum degree $\Delta \geq 3$ different from $K_{\Delta+1}$. Let also k_1, k_2, \dots, k_p be positive integers, $p \geq 2$, such that $k_1 + k_2 + \dots + k_p \geq \Delta$. Then, there is a p -partition (G_1, G_2, \dots, G_p) of G satisfying $\text{col}(G) \leq k_i$ and $\Delta(G_i) \leq k_i$ whenever $1 \leq i \leq p$.* \diamond

It turns out that it is possible to prove a similar result for arbitrary hypergraphs. For an example of how such a partition may look like in the hypergraph case consider Figure 3.1.

Theorem 3.4'. *Let H be a connected hypergraph having maximum degree $\Delta(H) = \Delta \geq 1$ that is not a tK_n for some $t, n \geq 1$ and not a tC_n for $t \geq 1, n \geq 5$ odd. Let also k_1, k_2, \dots, k_p be positive integers, $p \geq 2$, such that $k_1 + k_2 + \dots + k_p \geq \Delta$. Then, there is a p -partition (H_1, H_2, \dots, H_p) of H such that $\text{col}(H_i) \leq k_i$ and $\Delta(H_i) \leq k_i$ whenever $1 \leq i \leq p$.* \diamond

The condition $\Delta(G) \geq 3$ in the simple case ensures that G is not an odd cycle. However, since the hypergraphs of type (C) may have arbitrarily large maximum degree, we have to exclude this case manually. Before we prove Theorem 3.4', it is necessary to obtain the following statement.

Proposition 3.5. *If a hypergraph H is f -partitionable for some $f \in \mathcal{V}_p(H)$ with*

$$f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$$

for all $v \in V(H)$, then there is an f -partition (H_1, H_2, \dots, H_p) of H such that $d_{H_i}(v) \leq f_i(v)$ for all $v \in V(H_i)$ and for all $i \in [1, p]$. \diamond

Proof. Given an arbitrary p -partition (H_1, H_2, \dots, H_p) of H , define its weight by

$$W_{(H_1, H_2, \dots, H_p)} = \sum_{i=1}^p (|E(H_i)| - \sum_{v \in V(H_i)} f_i(v)).$$

If there is a $v \in V(H)$ and two indices $i \neq j$ from $[1, p]$ such that $v \in V(H_i)$, $d_{H_i}(v) \geq f_i(v)$ and $d_{H_j}(v) < f_j(v)$, then shifting v from H_i to H_j decreases $W_{(H_1, H_2, \dots, H_p)}$. In order to

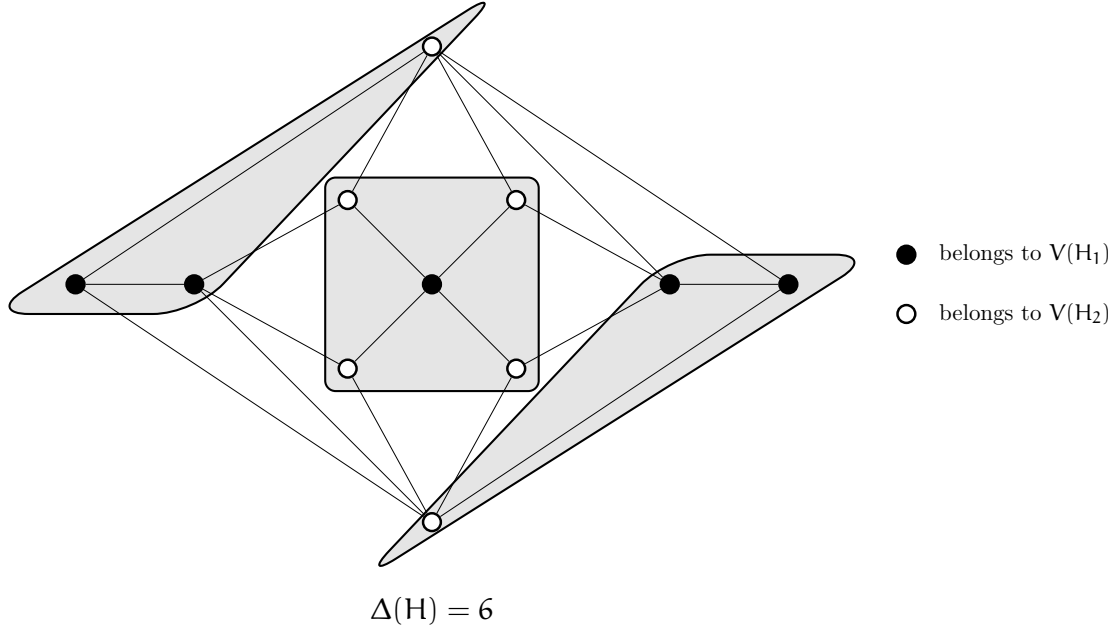


FIG. 3.1. A partition (H_1, H_2) of H such that $\text{col}(H_i) \leq 3$ and $\Delta(H_i) \leq 3$ for $i = 1, 2$.

prove this, let $H'_i = H_i - v$, $H'_j = H_j + v$, and let $H'_k = H_k$ for all $k \in [1, p] \setminus \{i, j\}$. Then, for $W = W_{(H_1, H_2, \dots, H_p)}$ and for $W' = W_{(H'_1, H'_2, \dots, H'_p)}$ it holds

$$W' - W = -d_{H_i}(v) + f_i(v) + d_{H'_j}(v) - f_j(v) < 0.$$

Let (H_1, H_2, \dots, H_p) be an f -partition of H that minimizes $W_{(H_1, H_2, \dots, H_p)}$. We claim that (H_1, H_2, \dots, H_p) has the desired property. Otherwise there is an index $i \in [1, p]$ and a vertex $v \in V(H_i)$ such that $v \in V(H_i)$ and $d_{H_i}(v) > f_i(v)$. As $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$, there is an index $j \in [1, p]$ such that $d_{H_j+v}(v) < f_j(v)$. Thus, $H'_j = H_j + v$ is still strictly f_j -degenerate. Furthermore, $H'_i = H_i - v$ is strictly f_i -degenerate as well, and by the above observation, we obtain a new p -partition $(H'_1, H'_2, \dots, H'_p)$ with $W_{(H'_1, H'_2, \dots, H'_p)} < W_{(H_1, H_2, \dots, H_p)}$, a contradiction. \blacksquare

It is notable that the above proposition leads to a stronger version of Theorem 2.3.

Theorem 2.3'. *Let H be a connected hypergraph, and let $f \in \mathcal{V}_p(H)$ be a vector function with $p \geq 1$ such that $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ for all $v \in V(H)$. Then, there is an f -partition (H_1, H_2, \dots, H_p) of H such that $d_{H_i}(v) \leq f_i(v)$ for all $v \in V(H_i)$ and for all $i \in [1, p]$ if and only if (H, f) is not a hard pair. \diamond*

Now we are able to prove Theorem 3.4'.

Proof of Theorem 3.4'. Let $f_i(v) = k_i$ for all $v \in V(H)$ and for each $i \in [1, p]$. Then, $f_1(v) + f_2(v) + \dots + f_p(v) \geq \Delta(H) \geq d_H(v)$ and $f_i(v) \geq 1$ for all $i \in [1, p]$ and for all $v \in V(H_i)$. Since $p \geq 2$, this implies that (H, f) cannot be of type (M). Moreover, since H is not a tK_n for some $t, n \geq 1$ nor a tC_n for $t \geq 1$ and $n \geq 5$ odd, it is easy to see that (H, f) is not a hard pair (see Proposition 2.5). Thus, by Theorem 2.3', H admits an f -partition (H_1, H_2, \dots, H_p) such that $d_{H_i}(v) \leq f_i(v) = k_i$ for all $v \in V(H_i)$ and each $i \in [1, p]$. In particular, H_i is strictly k_i -degenerate and, thus, $\text{col}(H_i) \leq k_i$ for all $i \in [1, p]$. ■

3.3. Hypergraph Partitions and Maximum Degenerate Subhypergraphs

When examining proper colorings of hypergraphs, it is a natural question to ask whether it is possible to find an optimal coloring such that one color class is a maximum independent set of the hypergraph. Regarding this question in the graph case, CATLIN [31] and, independently, GERENCSEÉR [51] proved the following.

Theorem 3.6 (CATLIN, 1979/GERENCSEÉR, 1965). *Let G be a connected simple graph with maximum degree $\Delta \geq 3$ different from $K_{\Delta+1}$. Then, there is a Δ -coloring of G in which one color class is a maximum independent set.* ◊

Hence, given a graph G as in the above theorem, there is a partition (G_1, G_2) of G such that G_1 is a maximum independent set and G_2 does not contain a K_Δ . Consequently, we can destroy all K_Δ 's in G by deleting an appropriate maximum independent set from G . The cases $\Delta(G) = 3, 4, 5$ were further examined by CATLIN and LAI [32]. In 2007, MATAMALA [84] obtained a result that not only connects Theorem 3.3 of BORODIN/BOLLOBÁS and MANVEL with Theorem 3.6, but strengthens both of them (as well as the results of CATLIN and LAI [32]).

Theorem 3.7 (MATAMALA, 2007). *Let G be a connected simple graph with maximum degree $\Delta \geq 3$ different from $K_{\Delta+1}$ and let d_1, d_2 be positive integers with $d_1 + d_2 \geq \Delta$. Then, there is a partition (G_1, G_2) of G such that G_1 is a maximum order induced subgraph with $\text{col}(G_1) \leq d_1$ and $\text{col}(G_2) \leq d_2$.* ◊

Note that CATLIN's theorem follows from the above theorem by setting $d_1 = 1$ and $d_2 = \Delta - 1$. It shows that with the help of Theorem 2.3, we can obtain a generalization of MATAMALA's result.

Theorem 3.8. *Let H be a hypergraph and let $f \in \mathcal{V}_p(H)$ be a vector function of H with $p \geq 2$ such that $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ for all $v \in V(H)$. Furthermore, assume that if H' is a component of H , then (H', f) is not a hard pair. Then, there is a partition (H_1, H_2, \dots, H_p) of H such that H_1 is a maximum order strictly f_1 -degenerate subhypergraph of H , and for $i \in [2, p-1]$, the hypergraph H_i is a maximum order strictly f_i -degenerate subhypergraph of $H - (V(H_1) \cup V(H_2) \cup \dots \cup V(H_{i-1}))$. \diamond*

For the proof of the above theorem we will use the following key lemma.

Lemma 3.9. *Let H be a hypergraph and let $f \in \mathcal{V}_p(H)$ be a vector function of H with $p \geq 2$ such that $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ for all $v \in V(H)$. If H is f -partitionable, then there is an f -partition (H_1, H_2, \dots, H_p) of H such that H_1 is a maximum order strictly f_1 -degenerate subhypergraph of H .*

Proof. The lemma's proof is by reductio ad absurdum. Let \mathcal{F} denote the set of tuples $(H_1, H_2, \dots, H_p, H_1^*, H_2^*)$ such that

- (1) (H_1, H_2, \dots, H_p) is an f -partition of H ,
- (2) H_1^* is a maximum order strictly f_1 -degenerate subhypergraph of H , and
- (3) $H_2^* = H - V(H_1^*)$.

Furthermore, let $f' = (f_2, f_3, \dots, f_p)$ and let $h = f_2 + f_3 + \dots + f_p$. By assumption, H admits an f -partition and, obviously, contains a maximum order strictly f_1 -degenerate subhypergraph. Hence, \mathcal{F} is non-empty.

Claim 3.9.1. *Let $(H_1, H_2, \dots, H_p, H_1^*, H_2^*) \in \mathcal{F}$ be an arbitrary tuple. Then, the following statements hold:*

- (a) *Let $v \in V(H_2^*)$ be an arbitrary vertex. Then, there is a hypergraph $H' \subseteq H_1^* + v$ with $d_{H'}(w) \geq f_1(w)$ for all $w \in V(H')$ and each such hypergraph contains the vertex v . As a consequence, $d_{H_2^*}(v) \leq f_2(v) + f_3(v) + \dots + f_p(v) = h(v)$ for all $v \in V(H_2^*)$.*
- (b) *The hypergraph H_2^* is not f' -partitionable and any non- f' -partitionable component K of H_2^* is h -regular (i.e. $d_K(w) = h(w)$ for all $w \in V(K)$) and contains a vertex v^* from H_1 .*
- (c) *Let K be a non- f' -partitionable component of H_2^* and let $v^* \in V(K) \cap V(H_1)$. Moreover, let $H' \subseteq H_1^* + v^*$ be a hypergraph with $d_{H'}(w) \geq f_1(w)$ for all $w \in V(H')$. Then, H' contains a vertex w^* from $V(H) \setminus V(H_1)$.*

- (d) Let K be a non- f' -partitionable component of H_2^* and let $v^* \in V(K) \cap V(H_1)$. Moreover, let $H' \subseteq H_1^* + v^*$ be a hypergraph with $d_{H'}(w) \geq f_1(w)$ for all $w \in V(H')$ and let u^* be a vertex that is adjacent to v^* in H' . Then, $\tilde{H}_1 = H_1^* + v^* - u^*$ is a maximum order strictly f_1 -degenerate subhypergraph of H and with $\tilde{H}_2 = H_2^* + u^* - v^*$ we have $(H_1, H_2, \dots, H_p, \tilde{H}_1, \tilde{H}_2) \in \mathcal{F}$. Furthermore, \tilde{H}_2 has at most as many non- f' -partitionable components as H_2^* and if equality holds, then u^* is contained in a non- f' -partitionable component of \tilde{H}_2 . \diamond

Proof. For the proof of (a) let $v \in V(H_2^*)$ be an arbitrary vertex. Since H_1^* is a maximum order strictly f_1 -degenerate subhypergraph of H , the hypergraph $H_1^* + v$ is not strictly f_1 -degenerate and, thus, there is a subhypergraph H' of $H_1^* + v$ such that $d_{H'}(w) \geq f_1(w)$ for all $w \in V(H')$. As H_1 is strictly f_1 -degenerate, H' contains the vertex v and so $d_{H_1^*}(v) \geq d_{H'}(v) \geq f_1(v)$. As $d_{H_1^*}(v) + d_{H_2^*}(v) \leq d_H(v) \leq f_1(v) + f_2(v) + \dots + f_p(v)$, this implies that $d_{H_2^*}(v) \leq f_2(v) + f_3(v) + \dots + f_p(v)$, which proves statement (a).

For the proof of (b) assume that H_2^* admits an f' -partition $(H_2', H_3', \dots, H_p')$. Then, the tuple $(H_1^*, H_2', H_3', \dots, H_p')$ is an f -partition of H such that H_1^* is a maximum order strictly f_1 -degenerate subhypergraph of H , contradicting the assumption that the lemma is wrong. Hence, H_2^* is not f' -partitionable, *i.e.*, H_2^* has at least one non- f' -partitionable component. Now let K be a component of H_2^* that is not f' -partitionable. Then, by (a) and by Proposition 2.8(a), $d_K(v) = d_{H_2^*}(v) = f_2(v) + f_3(v) + \dots + f_p(v)$ for all $v \in V(K)$, *i.e.*, K is h -regular. As $H - V(H_1)$ is f' -partitionable, K clearly contains a vertex v^* from H_1 . This proves (b).

For the proof of (c) and (d), let $H' \subseteq H_1^* + v^*$ be a hypergraph with $d_{H'}(w) \geq f_1(w)$ for all $w \in V(H')$ (which exists by (a)). By (a), H' contains the vertex v^* . As H_1 is strictly f_1 -degenerate, H' contains a vertex w^* from $V(H) \setminus V(H_1)$, which proves (c). Now let u^* be a vertex that is adjacent to v^* in H' . Then, $d_{H_2^*}(v^*) = d_K(v^*) = f_2(v^*) + f_3(v^*) + \dots + f_p(v^*)$ (by (b)), $d_{H_1^*}(v^*) \geq d_{H'}(v^*) \geq f_1(v^*)$, and $d_{H_1^*}(v^*) + d_{H_2^*}(v^*) \leq d_H(v^*) \leq f_1(v^*) + f_2(v^*) + \dots + f_p(v^*)$. As a consequence, we have $d_{H_1^*}(v^*) = f_1(v^*)$ and so $d_{H_1^*}(v^*) = d_{H'}(v^*)$. Hence, $d_{H_1^* - u^*}(v^*) < f_1(v^*)$. As $H_1^* - u^* \subseteq H_1^*$ and H_1^* is strictly f_1 -degenerate, this implies that $H_1^* + v^* - u^*$ is strictly f_1 -degenerate as well and so $\tilde{H}_1 = H_1^* + v^* - u^*$ is a maximum order strictly f_1 -degenerate subhypergraph of H . Note that $K - v^*$ is f' -partitionable (as K is h -regular by (b) and by Proposition 2.4) and so $H_2^* - v^*$ has one non- f' -partitionable component less than H_2^* . Clearly, $\tilde{H}_2 = H_2^* - v^* + u^*$ may have only one more non- f' -partitionable component than $H_2^* - v^*$ and if so, u^* must be contained in this component. Since \tilde{H}_1 is a maximum order strictly f_1 -degenerate subhypergraph of H , $(H_1, H_2, \dots, H_p, \tilde{H}_1, \tilde{H}_2) \in \mathcal{F}$ and the proof is complete. \square

Let $(H_1, H_2, \dots, H_p, H_1^*, H_2^*) \in \mathcal{F}$ be an arbitrary tuple. Since we assume that the lemma is false, $|H_1| < |H_1^*|$. By Claim 3.9.1(b), H_2^* is not f' -partitionable and so there is a non- f' -partitionable component of H_2^* . Let $\mathcal{K}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}$ denote the set of non- f' -partitionable components of H_2^* . Then, by Claim 3.9.1(c), for any $K \in \mathcal{K}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}$ we have $V(K) \cap V(H_1) \neq \emptyset$. Let

$$V_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)} = \bigcup_{K \in \mathcal{K}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}} (V(K) \cap V(H_1)).$$

Moreover, let $\mathcal{T}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}$ denote the set of all tuples (v^*, H', w^*) such that $v^* \in V_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}$, H' is a subhypergraph of $H_1^* + v^*$ with $d_{H'}(w) \geq f_1(w)$ for all $w \in V(H')$ and $w^* \in V(H') \setminus V(H_1)$. By Claim 3.9.1(a),(c), each vertex $v^* \in V_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}$ is contained in some tuple from $\mathcal{T}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}$.

Now we choose $(H_1, H_2, \dots, H_p, H_1^*, H_2^*) \in \mathcal{F}$ such that

- (1) $|H_1 \cap H_1^*|$ is maximum.
- (2) $|\mathcal{K}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}|$ is minimum subject to (1).
- (3) $m = \min\{\text{dist}_{H'}(v^*, w^*) \mid (v^*, H', w^*) \in \mathcal{T}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}\}$ is minimum subject to (1),(2).

Let $(v^*, H', w^*) \in \mathcal{T}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}$ such that $\text{dist}_{H'}(v^*, w^*) = m$. If $m = 1$, then w^* is in H' adjacent to v^* and it follows from Claim 3.9.1(d) that $\tilde{H}_1 = H_1^* + v^* - w^*$ is a maximum order strictly f_1 -degenerate subgraph of H . Moreover, $|V(H_1) \cap V(\tilde{H}_1)| > |V(H_1) \cap V(H_1^*)|$, contradicting (1). Hence, $m \geq 2$. Let u^* be a vertex that is adjacent to v^* in H' and is contained in a shortest (v^*, w^*) -hyperpath of H' . As $m \geq 2$ and by (3), $u^* \in V(H_1)$. By Claim 3.9.1(d), $\tilde{H}_1 = H_1 + v^* - u^*$ is a maximum order strictly f_1 -degenerate subhypergraph of H and $\tilde{H}_2 = H_2 + u^* - v^*$ has at most $|\mathcal{K}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}|$ non- f' -partitionable components. By (2), \tilde{H}_2 has exactly $|\mathcal{K}_{(H_1, H_2, \dots, H_p, H_1^*, H_2^*)}|$ non- f' -partitionable components implying (by Claim 3.9.1(d)) that u^* is contained in a non- f' -partitionable component K of \tilde{H}_2 . Then, $(H_1, H_2, \dots, H_p, \tilde{H}_1, \tilde{H}_2) \in \mathcal{F}$ is a tuple satisfying (1) and (2) and $(u^*, H', w^*) \in \mathcal{T}_{(H_1, H_2, \dots, H_p, \tilde{H}_1, \tilde{H}_2)}$ with $\text{dist}_{H'}(u^*, w^*) < \text{dist}_{H'}(v^*, w^*) = m$, contradicting (3). This proves the lemma. \blacksquare

Proof of Theorem 3.8. Let (H, f) be as described in the theorem. Then, since for any component H' of H the hypergraph H' is not f -hard, it follows from Theorem 2.3 that H is

f -partitionable. Then, by Lemma 3.9, H admits an f -partition (H_1, H_2, \dots, H_p) such that H_1 is a maximum order strictly f_1 -degenerate subhypergraph. Now let $H' = H - V(H_1)$. We claim that $f_2(v) + f_3(v) + \dots + f_p(v) \geq d_{H'}(v)$ for all $v \in V(H')$. Otherwise, $f_2(v) + f_3(v) + \dots + f_p(v) < d_{H'}(v)$ for some $v \in V(H')$ and, as $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$, we conclude $d_{H_1}(v) < f_1(v)$. As a consequence, $H_1 + v$ is a strictly f_1 -degenerate subhypergraph of H with $|H_1 + v| > |H_1|$, contradicting the maximality of H_1 . Hence, $f_2(v) + f_3(v) + \dots + f_p(v) \geq d_{H'}(v)$ for all $v \in V(H')$. Let $f' = (f_2, f_3, \dots, f_p)$. Since (H_2, H_3, \dots, H_p) is an f' -partition of H' , we may once again apply Lemma 3.9, which leads to an f' -partition $(H'_2, H'_3, \dots, H'_p)$ of H' such that H'_2 is a maximum order strictly f_2 -degenerate subhypergraph. By repeated application of the above arguments we eventually obtain the demanded f -partition. ■

The next Theorem is the counter-part to MATAMALA's Theorem 3.7 and can easily be deduced from Theorem 2.3 and Theorem 3.8.

Theorem 3.7'. *Let H be a connected hypergraph with maximum degree $\Delta \geq 1$. Moreover, let d_1, d_2, \dots, d_p be positive integers, $p \geq 2$, such that $d_1 + d_2 + \dots + d_p \geq \Delta$. Then, there is a partition (H_1, H_2, \dots, H_p) of H such that H_1 is a maximum order subhypergraph of H with $\text{col}(H_1) \leq d_1$, and for $i \in [2, p-1]$, the hypergraph H_i is a maximum order subhypergraph of $H - ((V(H_1)) \cup V(H_2) \cup \dots \cup V(H_{i-1}))$ with $\text{col}(H_i) \leq d_i$, unless H is a tK_n for some $t, n \geq 1$, $d_i = tn_i$ for some $n_i \geq 1$, $i \in [1, p]$, and $d_1 + d_2 + \dots + d_p = t(n-1) = \Delta$, or $H = tC_n$ for $t \geq 1$ and $n \geq 3$ odd, $p = 2$, and $d_i = t$ for $i \in \{1, 2\}$. ◇*

3.4. Point-partition Number

The **point-partition number** $\chi^s(H)$ of a hypergraph H (with $s \geq 0$) is the minimum number k such that H admits a k -coloring in which each color class induces an s -degenerate subhypergraph of H . Thus, $\chi^0(H)$ corresponds to the chromatic number of H . Furthermore, the **list-point partition number** $\chi_\ell^s(H)$ of a hypergraph H is the least integer k such that for any list-assignment L fulfilling $|L(v)| \geq k$ for all $v \in V(H)$, there is an L -coloring of H such that each color class induces an s -degenerate subhypergraph. The point-partition number was originally introduced by LICK and WHITE [79] for simple graphs; BOLLOBÁS and MANVEL [16] later used the term **s -chromatic number** for χ^s . It is notable that for a graph G , the **point arboricity** of G is defined as the least number k of forests forming a k -partition of G and, thus, corresponds to $\chi^1(G)$.

If we consider Theorem 3.4', by setting $k_1 = k_2 = \dots = k_p = s + 1$, we obtain that the point-partition number $\chi^s(H)$ is at most p if H is a connected hypergraph with maximum

degree $\Delta \geq 1$ different from tK_n with $t, n \geq 1$ and tC_n for $t \geq 1$ and $n \geq 3$ odd such that $p(s+1) = k_1 + k_2 + \dots + k_p \geq \Delta$. For simple graphs, these cases were originally solved by KRONK and MITCHEM [76] and MITCHEM [87].

Let H be a hypergraph and let L be an arbitrary list-assignment of H . We say that H is (L, s) -colorable if there is an L -coloring of H such that each color class induces a strictly s -degenerate subhypergraph. As a simple consequence of Theorem 2.3, we shall prove a list-version of the above result:

Theorem 3.10. *Let s be a positive integer, let H be a connected hypergraph, and let L be a list-assignment satisfying $|L(v)| \geq \Delta(H)/s$ for each $v \in V(H)$. Then, H is not (L, s) -colorable if and only if the following two conditions are fulfilled:*

- (a) $H = tK_n$ with $1 \leq t \leq s$ and $t(n-1) \equiv 0 \pmod{s}$, or $H = sC_n$ with $n \geq 5$ odd, or H is an s -regular hypergraph.
- (b) There is a color set Γ such that $L(v) = \Gamma$ for all $v \in V(H)$ and $|\Gamma| = \Delta(H)/s$. ◊

Proof. If $H = K_1$ and, hence, $\Delta(H) = 0$, the statement is evident. Thus, we may assume $\Delta(H) \geq 1$. Let $\Gamma = \bigcup_{v \in V(H)} L(v)$. By renaming the colors if necessary, we get $\Gamma = [1, p]$ with $p \geq 1$. Let $f \in \mathcal{V}_p(H)$ be the function with

$$f_i(v) = \begin{cases} s & \text{if } i \in L(v), \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in [1, p]$. Then,

$$\sum_{i=1}^p f_i(v) = |L(v)|s \geq \Delta(H) \geq d_H(v) \tag{3.1}$$

for all $v \in V(H)$. Clearly, H is not (L, s) -colorable if and only if H does not admit an f -partition. Thus, by (3.1) and Theorem 2.3, H is not (L, s) -colorable if and only if (H, f) is a hard pair.

First assume that H and L satisfy (a) and (b). Then it is easy to check that (H, f) is a hard pair and so H is not (L, s) -colorable. Conversely, assume that H is not (L, s) -colorable. Then (H, f) is a hard pair, which implies, in particular, that $|L(v)|s = \Delta(H) = d_H(v)$ for all $v \in V(H)$ (by (3.1) and Proposition 2.6(a)), and so H is Δ -regular. If $\Delta(H) = s$, then H is an s -regular connected hypergraph and, hence, each proper subhypergraph of H is strictly

s -degenerate (by Proposition 2.4). Since H is not (L, s) -colorable, this clearly implies that there is a color α such that $L(v) = \{\alpha\}$ for all $v \in V(H)$, and we are done. Otherwise, $\Delta(H)/s \geq 2$ and so for each vertex $v \in V(H)$ there are two indices $i \neq j$ from $[1, p]$ such that $f_i(v) = s$ and $f_j(v) = s$. Consequently, no end-block of H is a mono-block and, since H is Δ -regular, this implies that H itself is a block and so (H, f) is either of type (K) or (C). In the first case, $H = tK_n$ for some $t \geq 1, n \geq 3$ and there are integers n_1, n_2, \dots, n_p with at least two n_i different from zero such that $n_1 + n_2 + \dots + n_p = n - 1$ and $f(v) = (tn_1, n_2, \dots, tn_p)$ for all $v \in V(H)$. Since every coordinate of $f(v)$ is either s or zero, this is only possible if $t(n - 1) \equiv 0 \pmod{s}$ and if $1 \leq t \leq s$. In the second case, $H = sC_n$ with $n \geq 5$ odd and $f(v) = (s, s, 0, \dots, 0)$ (except for symmetry). As the vector function f is constant in both cases, we easily deduce that the list-assignment L is constant, too, and $|L(v)| = \Delta/s$ for all $v \in V(H)$. ■

The final question that we want to address in this section is if it is possible to obtain a degree version of the above result. For $s = 1$, this corresponds to Theorem 3.2; the characterization of the ‘‘uncolorable’’ list-assignments was given by BORODIN [19, 20] for graphs and by KOSTOCHKA and STIEBITZ [70] for hypergraphs. Unfortunately, we did not succeed in extending the result for $s \geq 2$, except for $s = 2$ and the class of graphs.

Theorem 3.11. *Let $s \in \{1, 2\}$, let G be a connected graph with $|G| \geq 2$, and let L be a list-assignment satisfying $|L(v)| \geq d_G(v)/s$ for each $v \in V(G)$. Then, G is not (L, s) -colorable if and only if the following two conditions are fulfilled:*

- (a) *If $B \in \mathcal{B}(G)$, then $B = tK_n$ with $1 \leq t \leq s$ and $t(n - 1) \equiv 0 \pmod{s}$, or $B = sC_n$ with n odd, or B is s -regular.*
- (b) *For each $B \in \mathcal{B}(G)$, there is a set Γ_B of $\Delta(B)/s$ colors such that for every $v \in V(G)$, the sets Γ_B with $B \in \mathcal{B}_v(G)$ are pairwise disjoint and $L(v) = \bigcup_{B \in \mathcal{B}_v(G)} \Gamma_B$. ◊*

Proof. As in the previous proof let Γ be the set of colors used in the union of all lists $L(v)$ and assume $\Gamma = [1, p]$. Moreover, define $f \in \mathcal{V}_p(G)$ with

$$f_i(v) = \begin{cases} s & \text{if } i \in L(v), \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in [1, p]$. Then, $f_1(v) + f_2(v) + \dots + f_p(v) \geq |L(v)|s \geq d_G(v)$ for all $v \in V(G)$. Again, G is not (L, s) -colorable if and only if (G, f) is a hard pair.

If G and L satisfy (a) and (b), it is easy to check that (G, f) is a hard pair and so G is not (L, s) -colorable. Now assume that G is not (L, s) -colorable. Then (G, f) is a hard pair and it follows from Proposition 2.6(a) that $|L(v)|s = d_G(v)$ for all $v \in V(G)$. Moreover, by Proposition 2.5(a)(b), for each $B \in \mathcal{B}(G)$ there is a uniquely determined function $f_B \in \mathcal{V}_p(B)$ such that (B, f_B) is of type (M),(K), or (C) and that $f(v) = \sum_{B \in \mathcal{B}_v(G)} f_B(v)$ for all $v \in V(G)$. We claim that $f_B(v) \in \{0, 2\}^p$ whenever $B \in \mathcal{B}(G)$ and $v \in V(B)$. The prove is by induction on the number m of blocks of G . For $m = 1$, the claim trivially holds. So assume $m \geq 2$. Let B be an end-block of G and let v be the only separating vertex of G contained in B . Furthermore, let $G' = G - (V(B) \setminus \{v\})$ and let $f'(w) = f(w)$ for all $w \in V(G') \setminus \{v\}$ and $f'(v) = f(v) - f_B(v)$. Then, (G', f') is a hard pair with $f'(w) \in \{0, 2\}^p$ for all $w \in V(G') \setminus \{v\}$ and $f'(v) \in \{0, 1, 2\}^p$. If $f'(v) \in \{0, 2\}^p$, then also $f_B(v) \in \{0, 2\}^p$ and we are done by induction. So assume that one coordinate of $f'(v)$ equals 1. Then, the same coordinate of $f_B(v)$ is also 1. Consequently, (B, f_B) is a mono-block (block of type (M)). Thus, $d_B(v) = 1$ and, as G is a graph and B is a block, $B = K_2$, which is impossible. This proves the claim that $f_B(v) \in \{0, 2\}^p$ whenever $B \in \mathcal{B}(G)$ and $v \in V(B)$. With the help of similar arguments as in the proof of Theorem 3.10, it is now easy to see that G and L satisfy (a) and (b). ■

The reason why the above theorem does not hold for larger s , respectively for hypergraphs with $s \geq 2$, is that the mono-blocks do not need to be regular in this case. Figure 3.2 displays s -regular counter-examples for graphs with $s = 3$ and $s = 4$ and for hypergraphs with $s = 2$. If $L \equiv \{1\}$ is the constant list-assignment, the three pictured (hyper-)graphs H satisfy $|L(v)| \geq d_H(v)/s = 1$ for all $v \in V(H)$, but are not (L, s) -colorable, yet there are blocks not satisfying statement (a) of the theorem. Note that it is possible to prove the “if”-part of the above theorem if we forbid $B = tK_n$ and $B = tC_n$ for all $n \geq 1$ and $1 \leq t \leq s$ and replace the phrase “ B is s -regular” with “ $\Delta(B) \leq s$ ”.

3.5. \mathcal{P} -coloring Problem

3.5.1. Colorings with Respect to Hypergraph Properties

As mentioned in Section 1.3, we can use hypergraph properties in order to generalize the ordinary coloring concept for hypergraphs. For instance, a proper coloring of a hypergraph H is a coloring of H in which each color class induces a subhypergraph belonging to the property \mathcal{C} of edgeless hypergraphs. Of course, we can replace \mathcal{C} with any other property. In this section, we aim to generalize this approach. To this end, let \mathcal{P} be an arbitrary hypergraph property. Moreover, let H be a hypergraph, and let Γ be a color set. A coloring

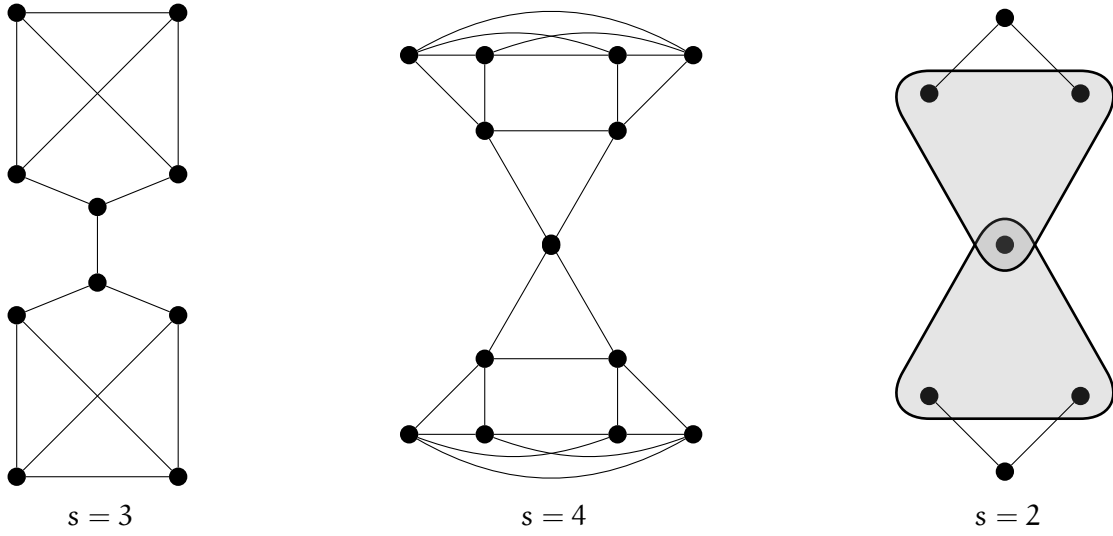


FIG. 3.2. Counter-examples to Theorem 3.11 for $s = 3, 4$ and for hypergraphs with $s = 2$, where $L \equiv \{1\}$ is the constant list-assignment.

$\varphi : V(H) \rightarrow \Gamma$ is a \mathcal{P} -coloring of H if each color class induces a subhypergraph of H belonging to \mathcal{P} . Furthermore, the \mathcal{P} -chromatic number $\chi(H : \mathcal{P})$ of H is the least integer k such that H admits a \mathcal{P} -coloring with color set $[1, k]$. Similar, given a list-assignment $L : V(H) \rightarrow 2^\Gamma$, a (\mathcal{P}, L) -coloring of H is an L -coloring φ of H such that $H[\varphi^{-1}(\alpha)] \in \mathcal{P}$ for all $\alpha \in \Gamma$. If H admits a (\mathcal{P}, L) -coloring, we also say that H is (\mathcal{P}, L) -colorable. Finally, we define the \mathcal{P} -list-chromatic number $\chi_\ell(H : \mathcal{P})$ of H as the least integer k such that H is (\mathcal{P}, L) -colorable for all list-assignments L with $|L(v)| \geq k$ for all $v \in V(H)$. From the definition it immediately follows that

$$\chi(H) = \chi(H : \mathcal{C}), \quad \chi_\ell(H) = \chi_\ell(H : \mathcal{C}), \quad \chi^s(H) = \chi(H : \mathcal{D}_s), \quad \text{and} \quad \chi_\ell^s(H) = \chi_\ell(H : \mathcal{D}_s).$$

As the reader will recall from Section 1.3, a smooth hypergraph property is a hypergraph property, which is hereditary and non-trivial. If \mathcal{P} is a smooth hypergraph property, then $K_0, K_1 \in \mathcal{P}$ (by Proposition 1.2), which implies that

$$\chi(H : \mathcal{P}) \leq \chi_\ell(H : \mathcal{P}) \leq |H|$$

for all hypergraphs H . Moreover, it holds

$$\chi_\ell(H : \mathcal{P}) - 1 \leq \chi_\ell(H - v : \mathcal{P}) \leq \chi_\ell(H : \mathcal{P})$$

for all hypergraphs H and for each vertex $v \in V(H)$. The second inequality is obvious. In order to obtain the first inequality, assume that $\chi_\ell(H, \mathcal{P}) = k$, but $\chi_\ell(H - v : \mathcal{P}) \leq k - 2$ for some vertex $v \in V(H)$, that is, $H - v$ is (\mathcal{P}, L') -colorable for each list-assignment L' such that $|L'(u)| \geq k - 2$ for all $u \in V(H - v)$. Now let L be an arbitrary list-assignment of H with $|L(u)| \geq k - 1$ for all $u \in V(H)$. Then, we may assign v an arbitrary color α from $L(v)$ and set $L'(u) = L(u) \setminus \{\alpha\}$ for all $u \in V(H) \setminus \{v\}$. As a consequence, L' is a list-assignment of $V(H - v)$ such that $|L'(u)| \geq k - 2$ for all $u \in V(H - v)$ and, thus, $H - v$ admits an L' -coloring, which leads to an L -coloring of H . Since L was chosen arbitrarily, this implies that $\chi_\ell(H : \mathcal{P}) \leq k - 1$, a contradiction.

Let L be a list-assignment of a hypergraph H . We say that H is (\mathcal{P}, L) -**vertex-critical** if $H - v$ is (\mathcal{P}, L) -colorable for all $v \in V(H)$, but H itself is not. Recall from Section 1.3 that, for a smooth hypergraph property \mathcal{P} ,

$$\mathcal{F}(\mathcal{P}) = \{H \mid H \notin \mathcal{P}, \text{ but } H - v \in \mathcal{P} \text{ for all } v \in V(H)\},$$

and

$$d(\mathcal{P}) = \min\{\delta(H) \mid H \in \mathcal{F}(\mathcal{P})\}.$$

Proposition 3.12. *Let \mathcal{P} be a smooth graph property with $d(\mathcal{P}) = r$, let H be a non-empty hypergraph, and let L be a list-assignment of H . If H is (\mathcal{P}, L) -critical, then the following conditions hold:*

- (a) $d_H(v) \geq r|L(v)|$ for all $v \in V(H)$.
- (b) Let v be a vertex of H with $d_H(v) = r|L(v)|$, and let φ be a (\mathcal{P}, L) -coloring of $H - v$ with color set Γ . Moreover, for $\alpha \in L(v)$, let

$$H_{\alpha,v} = H[\varphi^{-1}(\alpha) \cup \{v\}] \text{ and } d_\alpha = d_{H_{\alpha,v}}(v)$$

Then, $d_\alpha = r$ for all $\alpha \in L(v)$ and $E_H(v) = \bigcup_{\alpha \in L(v)} E_{H_{\alpha,v}}(v)$. ◊

Proof. Let v be an arbitrary vertex of H . Since H is (\mathcal{P}, L) -critical, there is a (\mathcal{P}, L) -coloring φ of $H - v$. As H is not (\mathcal{P}, L) -colorable, $H_{\alpha,v} = H[\varphi^{-1}(\alpha) \cup \{v\}]$ is not in \mathcal{P} for all $\alpha \in L(v)$, and thus, by Proposition 1.2(e),

$$r = d(\mathcal{P}) \leq d_{H_{\alpha,v}}(v) = d_\alpha$$

for each $\alpha \in L(v)$. Consequently, we obtain

$$d_H(v) \geq \sum_{\alpha \in L(v)} d_\alpha \geq r|L(v)|.$$

This proves (a). If v is a vertex of H with $d_H(v) = r|L(v)|$, then the above inequalities immediately imply that $r = d_\alpha$ for all $\alpha \in L(v)$ and that $E_H(v) = \bigcup_{\alpha \in L(v)} E_{H_\alpha, v}(v)$, which proves (b). \blacksquare

Let \mathcal{P} be a smooth hypergraph property with $d(\mathcal{P}) = r$, let H be a hypergraph, and let L be a list-assignment of H such that H is (\mathcal{P}, L) -critical. Then, we know from Proposition 3.12 that $d_H(v) \geq r|L(v)|$ for all $v \in V(H)$. This gives us a natural way to divide the vertices of H into two classes: high vertices and low vertices. A vertex v is a **low vertex** of H if $d_H(v) = r|L(v)|$ and a **high vertex**, otherwise. By $V(H, \mathcal{P}, L)$, we denote the set of low vertices of H . Moreover, we call $H(V(H, \mathcal{P}, L))$ the **low-vertex hypergraph** with respect to (H, \mathcal{P}, L) . Note that $H(V(H, \mathcal{P}, L))$, contrary to the case for graphs, is not necessarily a subhypergraph of H . The main result of this section is a GALLAI-type theorem that characterizes the structure of the low-vertex hypergraph. For simple graphs, it was obtained in 1995 by BOROWIECKI, DRGAS-BURCHARDT and MIHÓK [25]. We say that a hypergraph H is a **brick** if $H = tC_n$ for some $t \geq 1$ and $n \geq 3$ odd or $H = tK_n$ for some $t, n \geq 1$.

Theorem 3.13. *Let \mathcal{P} be a smooth hypergraph property with $d(\mathcal{P}) = r$, let H be a non-empty hypergraph, and let L be a list-assignment of H such that H is (\mathcal{P}, L) -critical and $F = H(V(H, \mathcal{P}, L))$ is non-empty. If B is a block of F , then B is a brick, or $B \in \mathcal{F}(\mathcal{P})$ and B is r -regular, or $B \in \mathcal{P}$ and $\Delta(B) \leq r$. \diamond*

The proof of Theorem 3.13 is presented in the next subsection. Before that, we want to discuss some applications of the above theorem; the first one deals with proper colorings. To this end, let us introduce an important class of hypergraphs: a connected hypergraph is a **GALLAI tree** if each of its block is a complete graph, an odd cycle, or consists of just one hyperedge. Moreover, a **GALLAI forest** is a hypergraph all of whose components are GALLAI trees. We want to stress the fact that if H is a GALLAI forest, then H is a simple hypergraph and, for all $v \in V(H)$, the hypergraph $H \div v$ is a GALLAI forest, too.

Theorem 3.14. *Let H be a hypergraph, and let L be a list-assignment of H such that H is (\mathcal{G}, L) -critical. Then, the low vertex hypergraph $F = H(V(H, \mathcal{G}, L))$ is a GALLAI forest (possibly empty). \diamond*

Proof. Recall from Section 1.3 that $d(\mathcal{C}) = 1$ and that $\mathcal{F}(\mathcal{C})$ is the class of connected hypergraphs having just one edge. If $v \in V(F)$ is a low vertex, then it follows from Proposition 3.12(b) that any pair of distinct edges $e, e' \in E_H(v)$ satisfy $i_H(e) \cap i_H(e') = \{v\}$. As a consequence, F is a simple hypergraph and Theorem 3.13 implies that F is a GALLAI forest. ■

Note that the above theorem implies GALLAI's Theorem 3 by setting $L(v) = [1, k - 1]$ for all $v \in V(H)$. Hypergraphs that are (\mathcal{C}, L) -critical for a given list-assignment L are also called **list-(vertex)-critical** hypergraphs. THOMASSEN [115] proved Theorem 3.14 for list-critical simple graphs; for list-critical simple hypergraphs it was proved by KOSTOCHKA, STIEBITZ, and WIRTH [72].

Now, we shall demonstrate how to use Theorem 3.13 in order to obtain a BROOKS-type result for the \mathcal{P} -chromatic number as well as for the \mathcal{P} -list-chromatic number. To this end, let \mathcal{P} be a smooth hypergraph property. A hypergraph H is (χ_ℓ, \mathcal{P}) -**(vertex)-critical** if $\chi_\ell(H' : \mathcal{P}) < \chi_\ell(H : \mathcal{P})$ for each proper induced subhypergraph H' of H . Note that H is (χ_ℓ, \mathcal{P}) -critical if and only if $\chi_\ell(H - v : \mathcal{P}) = \chi_\ell(H : \mathcal{P}) - 1$ for each vertex $v \in V(H)$.

Lemma 3.15. *If \mathcal{P} is a smooth hypergraph property with $d(\mathcal{P}) = r \geq 1$, then the following statements hold:*

- (a) *For each hypergraph H there is a (χ_ℓ, \mathcal{P}) -critical induced subhypergraph H' such that $\chi_\ell(H' : \mathcal{P}) = \chi_\ell(H : \mathcal{P})$.*
- (b) *If H is a (χ_ℓ, \mathcal{P}) -critical hypergraph with $\chi_\ell(H : \mathcal{P}) = k$, then $\delta(H) \geq r(k - 1)$. Moreover, if $U = \{v \in V(H) \mid d_H(v) = r(k - 1)\}$ is non-empty, then each block B of $H(U)$ is a brick, or $B \in \mathcal{F}(\mathcal{P})$ and B is r -regular, or $B \in \mathcal{P}$ and $\Delta(B) \leq r$.*
- (c) *For each hypergraph H it holds $\chi_\ell(H : \mathcal{P}) \leq \frac{\Delta(H)}{r} + 1$. ◊*

Proof. Let H' be an induced subhypergraph of H with $\chi_\ell(H' : \mathcal{P}) = \chi_\ell(H : \mathcal{P})$ whose order is minimum; this hypergraph clearly fulfills statement (a). To prove (b), let H be a (χ_ℓ, \mathcal{P}) -critical hypergraph with $\chi_\ell(H : \mathcal{P}) = k$ and let $U = \{v \in V(H) \mid d_H(v) = r(k - 1)\}$. Then, there exists a list-assignment L of H with $|L(v)| = k - 1$ for all $v \in V(H)$ such that H is not (\mathcal{P}, L) -colorable, but $H - v$ is (\mathcal{P}, L) -colorable for each $v \in V(H)$. As a consequence, H is (\mathcal{P}, L) -critical and, by Proposition 3.12(a), it holds $\delta(H) \geq r(k - 1)$ and $U = V(H, \mathcal{P}, L)$. Applying Theorem 3.13 then leads to each block B of $H(U)$ having the structure that is required in (b).

For the proof of (c), let H be an arbitrary hypergraph with $\chi_\ell(H : \mathcal{P}) = k$. By (a), H contains a (χ_ℓ, \mathcal{P}) -critical induced subhypergraph H' such that $\chi_\ell(H' : \mathcal{P}) = \chi_\ell(H : \mathcal{P})$. By

(b), H' has minimum degree at least $r(k-1)$ and we conclude $\Delta(H) \geq \Delta(H') \geq \delta(H') \geq r(k-1)$ and, hence, $\chi_\ell(H : \mathcal{P}) \leq \frac{\Delta(H)}{r} + 1$. \blacksquare

We say that a hypergraph property \mathcal{P} is **additive** if \mathcal{P} is closed under vertex disjoint unions. This means that a non-empty hypergraph H is in \mathcal{P} if and only if each component of H is in \mathcal{P} . If we also require \mathcal{P} to be smooth, then each hypergraph H from $\mathcal{F}(\mathcal{P})$ is connected and it holds $d(\mathcal{P}) \geq 1$ (since $K_0, K_1 \in \mathcal{P}$ by Proposition 1.2(a)).

Recall that \mathcal{C} is the class of edgeless hypergraphs. The property \mathcal{C} obviously is non-trivial, hereditary and additive, and \mathcal{C} is contained in each property \mathcal{P} that is smooth and additive (by Proposition 1.2(a)). As a consequence, each hypergraph H satisfies

$$\chi_\ell(H : \mathcal{P}) \leq \chi_\ell(H : \mathcal{C}) = \chi_\ell(H)$$

for any smooth and additive hypergraph property \mathcal{P} . With the help of Lemma 3.15 we are able to deduce a BROOKS-type result for smooth and additive hypergraph properties. This theorem was proved for simple graphs in [25].

Theorem 3.16. *Let \mathcal{P} be a non-trivial, hereditary and additive hypergraph property with $d(\mathcal{P}) = r$ and let H be a connected hypergraph. Then,*

$$\chi_\ell(H : \mathcal{P}) \leq \left\lceil \frac{\Delta(H)}{r} \right\rceil + 1,$$

and if equality holds, then $H = tK_{(kr+t)/t}$ for some integers $t \geq 1, k \geq 0$, or H is a tC_n for $t = r, n \geq 3$ odd, or H is r -regular and $H \in \mathcal{F}(\mathcal{P})$. \diamond

Proof. Let H be an arbitrary connected hypergraph. If $\Delta(H)$ is not divisible by r , then the statement follows directly from Lemma 3.15(c) (in particular, equality cannot hold). Thus, we may assume $\Delta(H) = kr$ for some integer $k \geq 0$ and so $\chi_\ell(H : \mathcal{P}) \leq k + 1$ (by Lemma 3.15(c)). If $\chi_\ell(H : \mathcal{P}) \leq k$, there is nothing left to show. Otherwise, we have $\chi_\ell(H : \mathcal{P}) = k + 1$. Then, by Lemma 3.15(a),(b), H contains a (χ_ℓ, \mathcal{P}) -critical subhypergraph H' satisfying $\chi_\ell(H' : \mathcal{P}) = k + 1$ and $\delta(H') \geq kr$. As H is connected and as $\Delta(H') \leq \Delta(H) = kr$, this implies that $H = H'$ and, hence, H is kr -regular and (χ_ℓ, \mathcal{P}) -critical. Thus, $H = H(\mathbf{U})$, where $\mathbf{U} = \{v \in V(H) \mid d_H(v) = rk\}$ and, by Lemma 3.15(b), each block B of H is a brick, or $B \in \mathcal{F}(\mathcal{P})$ and B is r -regular, or $B \in \mathcal{P}$ and $\Delta(B) \leq r$. As H itself is kr -regular, this clearly implies that H is a block.

If $H = tK_n$ with $t, n \geq 1$, then $d_H(v) = t(n-1) = kr$ and thus $n = \frac{kr+t}{t}$. Hence, we are done. If $H = tC_n$ for some $t \geq 1$ and $n \geq 3$ odd, we have $kr = 2t \geq 2$. In the case $k = 1$,

it follows that $\chi_\ell(H : \mathcal{P}) = 2$ and $r = 2t$. As H is (χ_ℓ, \mathcal{P}) -critical, this implies that H is in $\mathcal{F}(\mathcal{P})$ and H is r -regular. For $k \geq 2$, we argue as follows. Since $\chi_\ell(H : \mathcal{P}) \leq \chi_\ell(H) \leq 3$ and as $\chi_\ell(H : \mathcal{P}) = k + 1$, it follows that $\chi_\ell(H : \mathcal{P}) = 3$, $k = 2$ and, thus, $r = t$. Hence, we are done. If $H \in \mathcal{F}(\mathcal{P})$ and H is r -regular, then $k = 1$ (as H is kr -regular), and we are done, too. Finally, if $H \in \mathcal{P}$ and $\Delta(H) \leq r$, then $\chi_\ell(H : \mathcal{P}) = 1$, but $k = 1$, contradicting the premise that $\chi_\ell(H : \mathcal{P}) = k + 1$. This completes the proof. ■

To conclude this subsection, we present a generalized version of ERDŐS, RUBIN, and TAYLOR's Theorem 3.1, respectively the hypergraph version by KOSTOCHKA, STIEBITZ and WIRTH (see Theorem 3.2).

Theorem 3.17. *Let \mathcal{P} be a non-trivial, hereditary and additive hypergraph property with $d(\mathcal{P}) = r$, and let H be a connected hypergraph. Moreover, let L be a list-assignment of H such that $r|L(v)| \geq d_H(v)$ for all $v \in V(H)$. Then, H is (\mathcal{P}, L) -colorable, unless $d_H(v) = |L(v)|$ for all $v \in V(H)$ and each block B of H is a brick, or $B \in \mathcal{F}(\mathcal{P})$ is r -regular, or $B \in \mathcal{P}$ and $\Delta(B) \leq r$. ◊*

Proof. If H is (\mathcal{P}, L) -colorable, there is nothing left to show. Suppose that H is not (\mathcal{P}, L) -colorable. Then, there is a (\mathcal{P}, L) -critical subhypergraph H' of H . By Proposition 3.12(a), $d_{H'}(v) \geq r|L(v)|$ for all $v \in V(H')$ and, thus, $d_{H'}(v) = d_H(v) = r|L(v)|$ for all $v \in V(H')$. As H is connected, this implies that $H' = H$, i.e., H is (\mathcal{P}, L) -critical. Moreover, it follows that $d_H(v) = r|L(v)|$ for all $v \in V(H)$ and so $V(H) = V(H, \mathcal{P}, L)$. Applying Theorem 3.13 then completes the proof. ■

If we set $\mathcal{P} = \mathcal{C}$ in the above theorem, we have $d(\mathcal{P}) = 1$ and so the condition $r|L(v)| \geq d_H(v)$ for all $v \in V(H)$ is the same as the requirement $|L(v)| \geq d_H(v)$ in Theorem 3.2. Moreover, $\mathcal{F}(\mathcal{C})$ corresponds to the class of hypergraphs H with $H = \langle e \rangle$ for some edge e . Since the only connected non-empty hypergraph H from \mathcal{C} with $\Delta(H) \leq 1$ is the complete graph K_1 , the statement of Theorem 3.2 immediately follows from Theorem 3.17 if we restrict ourselves to simple hypergraphs.

3.5.2. Proof of Theorem 3.13

For the reader's convenience, let us recall Theorem 3.13.

Theorem 3.13. *Let \mathcal{P} be a smooth hypergraph property with $d(\mathcal{P}) = r$, let H be a non-empty hypergraph, and let L be a list-assignment of H such that H is (\mathcal{P}, L) -critical and*

$F = H(V(H, \mathcal{P}, L))$ is non-empty. If B is a block of F , then B is a brick, or $B \in \mathcal{F}(\mathcal{P})$ and B is r -regular, or $B \in \mathcal{P}$ and $\Delta(B) \leq r$. \diamond

Proof. Once again, the proof is based on Theorem 2.3. The main idea is, given a block B of $H(V(H, \mathcal{P}, L))$ and a (\mathcal{P}, L) -coloring φ of $H - V(B)$, to define a function f such that φ cannot be extended to a (\mathcal{P}, L) -coloring of H if and only if B is not f -partitionable. Then we apply Theorem 2.3 to (B, f) and analyze the three types of blocks (M), (K), and (C) that may occur. These will be exactly the blocks as described in Theorem 3.13. Let's get into the proof.

Let B be an arbitrary block of $F = H(V(H, \mathcal{P}, L))$. Since H is (\mathcal{P}, L) -critical, there is a (\mathcal{P}, L) -coloring φ of $H - V(B)$ with a set Γ of p colors. By renaming the colors we may assume $\Gamma = [1, p]$. Let $H_i = H[\varphi^{-1}(i)]$ for each $i \in [1, p]$. Then, for $v \in V(B)$, we define the vector function $f : V(B) \rightarrow \mathbb{N}_0^p$ as follows. For each $v \in V(B)$, let $f_i(v) = \max\{0, r - d_{H_i+v}(v)\}$ if $i \in L(v)$, and $f_i(v) = 0$, otherwise.

We claim that B is not f -partitionable. Assume, to the contrary, that B admits an f -partition $(H'_1, H'_2, \dots, H'_p)$. Then, for $i \in [1, p]$ let $\tilde{H}_i = H[V(H_i) \cup V(H'_i)]$. Obviously, $(\tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_p)$ is a partition of H . Note that $v \in V(\tilde{H}_i)$ implies that $i \in L(v)$ (since $f_i(v) \geq 1$ for $v \in V(H'_i)$). If $\tilde{H}_i \in \mathcal{P}$ for all $i \in [1, p]$, it follows that H is (\mathcal{P}, L) -colorable, a contradiction. As a consequence, there is an $i \in [1, p]$ such that $\tilde{H}_i \notin \mathcal{P}$. By Proposition 1.2(c), there exists an induced subhypergraph H' of \tilde{H}_i such that $H' \in \mathcal{F}(\mathcal{P})$ and, thus, $\delta(H') \geq d(\mathcal{P}) = r$. Since H_i is in \mathcal{P} but H' is not, H' contains a vertex of H'_i . Thus, the hypergraph $H'' = H'_i[V(H') \cap V(H'_i)]$ is non-empty. However, since H'_i is strictly f_i -degenerate, there is a vertex v in H'' such that $d_{H''}(v) < f_i(v) = r - d_{H_i+v}(v)$ and thus $d_{H'}(v) \leq d_{H''}(v) + d_{H_i+v}(v) < r$, a contradiction. Hence, B is not f -partitionable.

Since $d_H(v) = r|L(v)|$ for all $v \in V(B)$, we obtain that

$$\begin{aligned} \sum_{i=1}^p f_i(v) &= \sum_{i \in L(v)} f_i(v) \geq \sum_{i \in L(v)} (r - d_{H_i+v}(v)) \\ &= d_H(v) - \sum_{i \in L(v)} d_{H_i+v}(v) \geq d_B(v) \end{aligned}$$

for all $v \in V(B)$. Thus, by Theorem 2.3 and as B is a block, (B, f) is of type (M), (K) or (C). If (B, f) is not of type (M), then B is a brick and we are done. Thus assume that (B, f) is of type (M). Then, there is exactly one index i such that $f_i(v) = d_B(v)$ for all $v \in V(B)$ and $f_j(v) = 0$ for $j \neq i$ from the set $[1, p]$. As a consequence, $d_{H_j+v}(v) \geq r$ for all $j \in L(v) \setminus \{i\}$ and thus, $d_B(v) \leq r$ for all $v \in V(B)$. If $B \in \mathcal{P}$, we have $\Delta(B) \leq r$ and there is nothing left

to show. If $B \notin \mathcal{P}$, then by Proposition 1.2(c), B contains an induced subhypergraph B' from $\mathcal{F}(\mathcal{P})$. Since $d_B(v) \leq r$ for all $v \in V(B)$ and since $\delta(B') \geq d(\mathcal{P}) = r$, we conclude that $B = B'$ and $d_B(v) = r$ for all $v \in V(B)$. Consequently, $B \in \mathcal{F}(\mathcal{P})$ and B is r -regular. This completes the proof. ■

3.5.3. A Gallai-type Bound for the Degree Sum of Critical Hypergraphs

The topic of finding lower bounds for the number of edges, respectively the degree sum of critical graphs and hypergraphs with respect to some coloring concept has already been examined extensively in the past. Given a hypergraph H , let $d(H) = \sum_{v \in V(H)} d_H(v)$ denote the degree-sum over all vertices of H . Note that, contrary to the graph case, $d(H)$ coincides with $\sum_{e \in E(H)} |i_H(e)|$ but usually not with $2|E(H)|$. Regarding proper colorings of simple graphs, GALLAI [49] proved that for a $(k+1)$ -critical graph $G \neq K_{k+1}$, that is, a graph, which has chromatic number $k+1$, but each proper subgraph has chromatic number at most k (see also the introduction), it holds

$$d(G) \geq k|G| + \frac{k-2}{k^2+2k-2}|G|$$

if $k \geq 3$. For simple hypergraphs, an even stronger bound was proved by KOSTOCHKA and STIEBITZ [70]. MIHÓK and ŠKREKOVSKI [86] proved a GALLAI-type bound for the case of (\mathcal{P}, L) -critical simple graphs. In this subsection, with the help of STIEBITZ and KOSTOCHKA's approach, we aim to show a similar inequality for (\mathcal{P}, L) -critical hypergraphs, provided that all lists have same size and the low vertex hypergraph is simple. To this end, for $\delta, n \in \mathbb{N}$, we define

$$a(\delta, n) = \delta n + \frac{\delta-2}{\delta^2+2\delta-2}n.$$

We shall prove the following theorem.

Theorem 3.18. *Let \mathcal{P} be a smooth additive hypergraph property with $d(\mathcal{P}) = r \geq 1$, let $k \geq 2$, and let $\delta = kr \geq 3$. Furthermore, let H be a (\mathcal{P}, L) -critical hypergraph, where L is a list-assignment of H with $|L(v)| = k$ for all $v \in V(H)$. If the low vertex hypergraph $H(V(H), \mathcal{P}, L)$ is simple and each of its components is distinct from $K_{\delta+1}$, then $d(H) \geq a(\delta, |H|)$. ◊*

At this point, the reader might wonder if we could make life easier by just assuming that the hypergraph itself is simple. Unfortunately, however, the shrinking operation may still lead to parallel edges in this case. Thus, since it will be crucial for us that the low

vertex hypergraph is simple, we really do need this technical assumption. At least in the case of $\mathcal{P} = \mathcal{O}$, the low vertex hypergraph of any (\mathcal{P}, L) -critical hypergraph is simple (see [70]). Before proving the above theorem, let us deduce a simple corollary. For that purpose, let H be a (χ_ℓ, \mathcal{P}) -critical hypergraph with $\chi_\ell(H : \mathcal{P}) = k + 1$. Then, $\delta(H) \geq rk$ (by Lemma 3.15(b)) and we set $V(H, \mathcal{P}, \chi_\ell) = \{v \in V(H) \mid d_H(v) = rk\}$. As before, we call $H(V(H, \mathcal{P}, \chi_\ell))$ the **low-vertex hypergraph** of H . As H is (χ_ℓ, \mathcal{P}) -critical with $\chi_\ell(H : \mathcal{P}) = k + 1$, there is a list-assignment L of H with $|L(v)| = k$ for all $v \in V(H)$ such that H is (\mathcal{P}, L) -critical. Furthermore, we have $V(H, \mathcal{P}, \chi_\ell) = V(H, \mathcal{P}, L)$ and so the next result easily follows from the above theorem.

Corollary 3.19. *Let \mathcal{P} be a smooth additive hypergraph property with $d(\mathcal{P}) = r \geq 1$, let $k \geq 2$, and let $\delta = kr \geq 3$. Furthermore, let H be a (χ_ℓ, \mathcal{P}) -critical hypergraph with $\chi_\ell(H : \mathcal{P}) = k + 1$. If the low vertex hypergraph $H(V(H, \mathcal{P}, \chi_\ell))$ is simple and each of its components is distinct from $K_{\delta+1}$, then $d(H) \geq \alpha(\delta, |H|)$. \diamond*

Surely by now, the question has arisen why the restrictions for k , δ , and H are imposed. To answer this, let \mathcal{P} be a smooth additive hypergraph property with $d(\mathcal{P}) = r \geq 1$, let $k \geq 1$ and let $\delta = kr$. First, we assume $k = 1$. Then, $\delta = r$ and $\alpha(\delta, n) > rn$ provided that $r \geq 3$. Moreover, the class of (χ_ℓ, \mathcal{P}) -critical hypergraphs H with $\chi_\ell(H : \mathcal{P}) = k + 1 = 2$ coincides with the class $\mathcal{F}(\mathcal{P})$ (by Proposition 1.2(b)). Here, however, the inequality $d(H) \geq \alpha(\delta, |H|)$ is not true for infinitely many hypergraphs H and certain properties \mathcal{P} . For instance, let $\mathcal{P} = \mathcal{D}_{r-1}$ be the class of strictly r -degenerate hypergraphs. Then, it is easy to check that $d(\mathcal{D}_{r-1}) = r$ and that $\mathcal{F}(\mathcal{D}_{r-1})$ contains all r -regular connected hypergraphs and each such simple hypergraph H coincides with its low vertex hypergraph, but $d(H) = r|H| < \alpha(\delta, |H|)$ for $r \geq 3$. Therefore, in the following we demand that $k \geq 2$, and so $\delta \geq 2$. If $\delta = 2$, then $\alpha(\delta, n) = rn$ and the inequality $d(H) \geq \alpha(\delta, |H|)$ holds for every (χ_ℓ, \mathcal{P}) -critical hypergraph with $\chi_\ell(H : \mathcal{P}) = k + 1$. Hence, as of now we may suppose $\delta \geq 3$. Lastly, it is important to note that if $H = K_{\delta+1}$, then H is its own low-vertex hypergraph and, clearly, $d(H) < \alpha(\delta, |H|)$ for $\delta \geq 3$; thus the bound is not true in this case.

The remaining part of this section is dedicated to the proof of Theorem 3.18, which is done via three lemmas. At first, we show that the bound always holds if a specific condition is fulfilled (see Lemma 3.20). Of course, this is only useful if the condition is true and so we subsequently prove that this is always the case. Most parts of the next three lemmas are similar to those in the paper by KOSTOCHKA and STIEBITZ [70]. Since all considered low vertex hypergraphs are supposed to be simple, the structures described in Theorem 3.13 can be simplified. Therefore, we say that a connected simple hypergraph H is a **GALLAI-**

\mathcal{P} -tree if each block B of H is a complete graph, or B is a cycle of odd length, or $B \in \mathcal{F}(\mathcal{P})$ and B is r -regular, or $B \in \mathcal{P}$ and $\Delta(B) \leq r$.

Lemma 3.20. *Let \mathcal{P} be a smooth additive hypergraph property with $d(\mathcal{P}) = r \geq 1$, let $k \geq 2$, and let $\delta = kr \geq 3$. Furthermore, let H be a connected hypergraph with $\delta(H) \geq \delta$. We define*

$$\mathbf{U} = \{v \in V(H) \mid d_H(v) = \delta\}, \quad r_\delta = \delta - 1 + \frac{2}{\delta},$$

and

$$\sigma = |\mathbf{U}|r_\delta - d(H(\mathbf{U})).$$

If each component of $H(\mathbf{U})$ is a GALLAI- \mathcal{P} -tree distinct from $K_{\delta+1}$ and $\sigma \geq 0$, then

$$d(H) \geq a(\delta, n). \quad \diamond$$

Proof. First we claim $\mathbf{U} \neq V(H)$. Otherwise, $H = H(\mathbf{U})$ would be a δ -regular GALLAI- \mathcal{P} -tree (by assumption), and this is only possible if $H = K_{\delta+1}$ (as $\delta > r$ and $\delta \geq 3$). Hence, $\mathbf{U} \neq V(H)$. If $\mathbf{U} = \emptyset$, we obtain $d(H) \geq (\delta + 1)n \geq a(\delta, n)$ and there is nothing left to prove. Thus, we may assume $\mathbf{U} \neq \emptyset$. Then,

$$\begin{aligned} d(H) &= \delta|\mathbf{U}| + \sum_{v \in V(H) \setminus \mathbf{U}} d_H(v) \\ &\geq d(H - \mathbf{U}) + 2\delta|\mathbf{U}| - d(H(\mathbf{U})) \\ &= d(H - \mathbf{U}) + \sigma + (2\delta - r_\delta)|\mathbf{U}| \\ &= d(H - \mathbf{U}) + \sigma + (\delta + 1 - \frac{2}{\delta})|\mathbf{U}| \\ &\geq (\delta + 1 - \frac{2}{\delta})|\mathbf{U}|. \end{aligned}$$

On the other hand, it is obvious that $d(H) \geq (\delta + 1)n - |\mathbf{U}|$, and so we obtain

$$\begin{aligned} d(H) + d(H)(\delta + 1 - \frac{2}{\delta}) &\geq (\delta + 1 - \frac{2}{\delta})|\mathbf{U}| + (\delta + 1)(\delta + 1 - \frac{2}{\delta})n \\ &\quad - |\mathbf{U}|(\delta + 1 - \frac{2}{\delta}) \\ &= (\delta + 1)(\delta + 1 - \frac{2}{\delta})n. \end{aligned}$$

By solving the inequation for $d(H)$, we easily deduce the required result. \blacksquare

Thus, the only remaining question is if $\sigma \geq 0$ is always fulfilled. That this is indeed the case is proved in the next two lemmas. As in the above lemma, let $r_\delta = \delta - 1 + \frac{2}{\delta}$ and, for an arbitrary hypergraph H , let

$$\sigma(H) = |H|r_\delta - d(H).$$

Furthermore, let \mathcal{T}_δ denote the class of GALLAI- \mathcal{P} -trees distinct from $K_{\delta+1}$ whose maximum degree is at most δ . Lastly, for $T \in \mathcal{T}_\delta$ and for an end-block B of T , we define

$$T_B = T - (V(B) \setminus \{x\}),$$

where x denotes the only separating vertex of T in B (if T has only one block choose an arbitrary vertex x of $V(T)$).

Lemma 3.21. *Let $T \in \mathcal{T}_\delta$ and let $\delta \geq 3$. Then, the following statements hold:*

(a) *If $B \in \mathcal{B}(T)$, then $\sigma(B) = 2$ if $B = K_\delta$ and $\sigma(B) \geq r_\delta$ otherwise.*

(b) *If B is an end-block of T , then $\sigma(T) = \sigma(T_B) + \sigma(B) - r_\delta$.* ◊

Proof. If B is a K_b for some $b \in \{1, 2, \dots, \delta\}$, then

$$\sigma(B) = b(r_\delta - b + 1) \begin{cases} \geq r_\delta, & \text{if } 1 \leq b \leq \delta - 1, \text{ and} \\ = 2, & \text{if } b = \delta. \end{cases}$$

Otherwise, if B is a cycle of odd length with at least 5 vertices, then it is easy to check that

$$\sigma(B) = |B|(r_\delta - 2) \geq 5(r_\delta - 2) \geq r_\delta.$$

If $B = \langle e \rangle$ for some edge e , then $\sigma(B) = |i_T(e)|(r_\delta - 1) \geq r_\delta$ (as $r_\delta \geq 2$).

It remains to consider the case that B is a block with $\Delta(B) \leq r$ that is not of the above mentioned types. This implies, in particular, that $|B| \geq 3$. If $k \geq 3$, then $rk \geq 2r + 1$ and

we conclude

$$\begin{aligned}
\sigma(B) &= |B|(rk - 1 + \frac{2}{rk}) - \sum_{v \in V(B)} d_B(v) \\
&\geq |B|(rk - 1 + \frac{2}{rk}) - |B|r \\
&= |B|(r(k-1) - 1 + \frac{2}{rk}) \\
&\geq 2rk - 2r - 2 + \frac{4}{rk} \\
&= r_\delta + rk - 2r - 1 + \frac{2}{rk} \geq r_\delta.
\end{aligned}$$

Otherwise, $k = 2$ and, since $\delta \geq 3$, we have $r \geq 2$. Then, since $|B| \geq 3$, we get

$$\begin{aligned}
\sigma(B) &\geq |B|(r(k-1) - 1 + \frac{2}{rk}) \\
&\geq 3rk - 3r - 3 + \frac{6}{rk} \\
&= r_\delta + 2rk - 3r - 2 + \frac{4}{rk} \geq r_\delta,
\end{aligned}$$

as $2rk = 4r \geq 3r + 2$. Due to the fact that T_B and B share exactly one vertex, statement (b) is evident. \blacksquare

Following GALLAI [49], we say that a hypergraph is an ε_δ -hypergraph if each separating vertex belongs to exactly two blocks, one being a K_δ and the other one being of the form $\langle e \rangle$ for some edge e , and if each non-separating vertex is contained in a block, which is a K_δ . An example of an ε_δ -hypergraph with $\delta = 4$ is given in Figure 3.3.

Lemma 3.22. *Let $T \in \mathcal{T}_\delta$ and let $\delta \geq 4$. Then, $\sigma(T) \geq 2$ if T is an ε_δ -hypergraph and $\sigma(T) \geq r_\delta$, otherwise. \diamond*

Proof. The proof is by induction on the number m of blocks of T . If $m = 1$, the statement follows immediately from Lemma 3.21. Assume $m \geq 2$. If T is an ε_δ -hypergraph, then T_B is not an ε_δ -hypergraph for any end-block B of T and, by Lemma 3.15 we have $\sigma(T) \geq \sigma(T_B) + \sigma(B) - r_\delta \geq 2$ (as $\sigma(T_B) \geq r_\delta$ by the induction hypothesis).

If T is not an ε_δ -hypergraph, assume that T has a block B of the form $B = \langle e \rangle$. Then, clearly e is a bridge of T . For $x \in i_T(e)$, let T_x denote the component of $T - i_T(e)$ containing x . As T is not an ε_δ -hypergraph, T_x is not an ε_δ -hypergraph for at least one $x \in i_T(e)$.

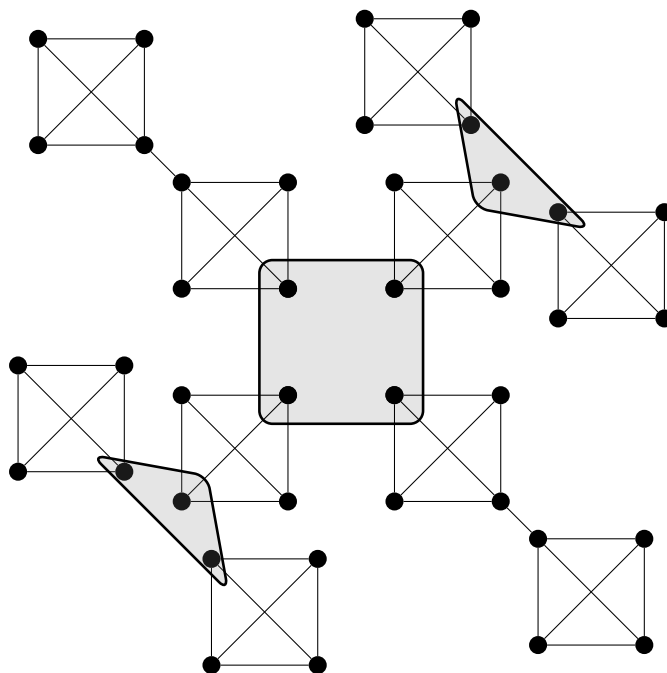


FIG. 3.3. An ϵ_4 -hypergraph.

Moreover, $r_\delta \geq \delta - 2 \geq 2$. By applying the induction hypothesis, we conclude

$$\sigma(T) = \sum_{x \in i_T(e)} \sigma(T_x) - |i_T(e)| \geq 2(|i_T(e)| - 1) + r_\delta - |i_T(e)| \geq r_\delta.$$

If T has no block of the form $\langle e \rangle$, then no block of T is a K_δ . Let B be an end-block of T . Then, T_B is not a ϵ_δ -hypergraph and, by the induction hypothesis and Lemma 3.21, $\sigma(T) = \sigma(T_B) + \sigma(B) - r_k \geq r_k$. ■

Now we shall finally prove Theorem 3.18.

Proof of Theorem 3.18. Let $\mathcal{P}, r, k, \delta$ and H and L be defined as in Theorem 3.18. Furthermore, let $U = \{v \in V(H) \mid d_H(v) = \delta\}$. Since H is (\mathcal{P}, L) -critical and $|L(v)| = k$ for all $v \in V(H)$, the set U coincides with $V(H, \mathcal{P}, L)$ and so $H(U)$ is the low vertex hypergraph. Then, each component of $H(U)$ is a GALLAI tree (by Theorem 5.2) and no component of $H(U)$ is a $K_{\delta+1}$ (by assumption). Thus, each component H' of $H(U)$ belongs to \mathcal{T}_δ and, hence, $\sigma(H') \geq 2$ by Lemma 3.22. As a consequence, $\sigma(H(U)) \geq 0$ and, by Lemma 3.20, we conclude $d(H) \geq \alpha(\delta, |H|)$. ■

Chapter 4

DP-COLORING OF HYPERGRAPHS

4.1. Introduction

DP-coloring is a very recent concept by DVOŘÁK and POSTLE [41] that, nevertheless, has found plenty of attention within the last four years (see, for instance, [10, 11, 12, 14, 15, 17, 66]). Searching for the term “DP-coloring” on arXiv already provides over 50 related papers (status as of June 2020). So what makes this concept such worthwhile examining? As DVOŘÁK and POSTLE point out in their initial paper [41], a difficulty in proving results on list-coloring is that the common technique of vertex identification usually does not work. By way of example, they recall the proof that every planar graph G admits a proper 5-coloring: let G be a counter-example of minimum order and choose a vertex $v \in V(G)$ of degree at most five. If $d_G(v) = 4$, then any proper coloring of $G - v$ extends to a proper coloring of G . Hence suppose $d_G(v) = 5$. Then, as G is planar, there are two non-adjacent neighbors u and w of v . Let G' be the graph that results from $G - v$ by identifying u and w to a vertex u^* . Then, G' is planar with $|G'| < |G|$ and, therefore, admits a proper 5-coloring φ . By assigning the vertices u and w the color $\varphi(u^*)$, we get a proper 5-coloring of $G - v$ such that at most four distinct colors appear in the neighborhood of v , and so G admits a proper 5-coloring, which is impossible.

Nevertheless, regarding list-colorings, the lists might differ for each vertex and so vertex identification is not possible. In order to overcome this difficulty, DVOŘÁK and POSTLE transform the problem of finding a list-coloring to the one of finding a large independent set in an auxiliary graph. Since the definition of the auxiliary graph is contained in our

definition of the auxiliary hypergraph, we only mention the hypergraph case.

Let H be a hypergraph. A **cover** of H is a pair (X, \mathcal{H}) consisting of a map X and a hypergraph \mathcal{H} such that the following conditions are fulfilled:

- (C1) $X : V(H) \rightarrow 2^{V(\mathcal{H})}$ is a function that assigns each vertex $v \in V(H)$ a vertex set $X_v = X(v) \subseteq V(\mathcal{H})$ such that the sets X_v with $v \in V(H)$ are pairwise disjoint.
- (C2) \mathcal{H} is a hypergraph with $V(\mathcal{H}) = \bigcup_{v \in V(H)} X_v$ such that X_v is an independent set of \mathcal{H} , and for each edge $e \in E(H)$ there is a possibly empty (hyper-)matching M_e in $\mathcal{H}[\bigcup_{v \in i_H(e)} X_v]$ with $|i_{\mathcal{H}}(\tilde{e}) \cap X_v| = 1$ for all $v \in i_H(e)$ and for all $\tilde{e} \in M_e$. Moreover, $E(\mathcal{H}) = \bigcup_{e \in E(H)} M_e$.

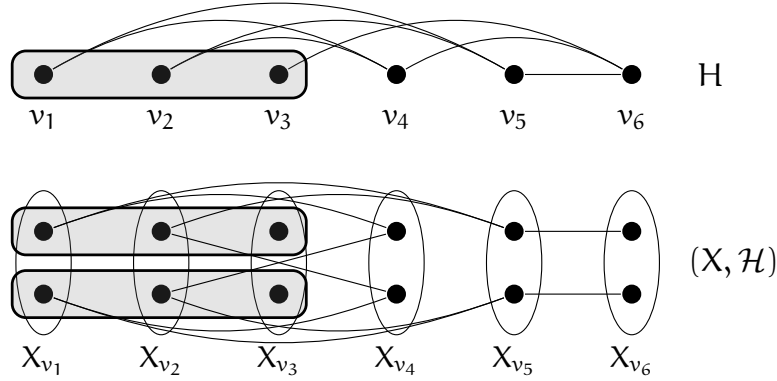


FIG. 4.1. A cover (X, \mathcal{H}) of a hypergraph H .

An example of how a cover may look like is given in Figure 4.1. Now let (X, \mathcal{H}) be a cover of H . A vertex set $T \subseteq V(\mathcal{H})$ is a **transversal** of (X, \mathcal{H}) if $|T \cap X_v| = 1$ for each vertex $v \in V(H)$. An **independent transversal** of (X, \mathcal{H}) is a transversal of (X, \mathcal{H}) , which is an independent set of \mathcal{H} . An independent transversal of (X, \mathcal{H}) is also called an (X, \mathcal{H}) -**coloring** of H ; the vertices of \mathcal{H} are called **colors**. We say that H is (X, \mathcal{H}) -**colorable** if H admits an (X, \mathcal{H}) -coloring. Let $f : V(H) \rightarrow \mathbb{N}_0$ be a function. Then, H is said to be **DP-f-colorable** if H is (X, \mathcal{H}) -colorable for any cover (X, \mathcal{H}) of H satisfying $|X_v| \geq f(v)$ for all $v \in V(H)$. When $f(v) = k$ for all $v \in V(H)$, the term becomes **DP-k-colorable**. The **DP-chromatic number** $\chi_{\text{DP}}(H)$ is the least integer $k \geq 0$ such that H is DP- k -colorable. Recently, BERNSHTEYN and KOSTOCHKA [13] also introduced the DP-chromatic number of a hypergraph in an equivalent but slightly different way. Initially, DVOŘÁK and POSTLE named their concept *correspondence coloring*, the name DP-coloring is due

to BERNSHTEYN, KOSTOCHKA, and PRON [14] as an appreciation for the inventors. Most papers that have appeared since then seem to use the term DP-coloring (maybe due to its catchyness). A similar concept to DP-coloring of graphs was obtained independently by FRAIGNAUD, HEINRICH, and KOSOWSKI [46].

So how can we transform the problem of finding a proper list-coloring to DP-colorings? To answer this, let H be a hypergraph and let L be a list-assignment for H . Let (X, \mathcal{H}) be a cover of H as follows:

- For $v \in V(H)$, let $X_v = \{(v, x) \mid x \in L(v)\}$ and let $V(\mathcal{H}) = \bigcup_{v \in V(H)} X_v$.
- For any set $S = \{(v_1, x_1), (v_2, x_2), \dots, (v_\ell, x_\ell)\}$ of vertices from \mathcal{H} , there is an edge $e' \in E(\mathcal{H})$ with $i_{\mathcal{H}}(e') = S$ if and only if in H there is an edge e with $i_H(e) = \{v_1, v_2, \dots, v_\ell\}$ and if $x_1 = x_2 = \dots = x_\ell$.

It is easy to check that (X, \mathcal{H}) is indeed a cover of H . Furthermore, if φ is a proper L -coloring of H , then $T = \{(v, \varphi(v)) \mid v \in V(H)\}$ clearly is an independent transversal of (X, \mathcal{H}) and so H is (X, \mathcal{H}) -colorable. If conversely T is an independent transversal of (X, \mathcal{H}) , then $T = \{(v, x_v) \mid v \in V(H)\}$ and it is easy to see that the mapping $v \mapsto x_v$ is a proper L -coloring of H . Thus, H admits a proper L -coloring if and only if H is (X, \mathcal{H}) -colorable. Furthermore, we clearly have $|X_v| = |L(v)|$ for all $v \in V(H)$. Consequently, if $k \geq 0$ is an integer, then H is k -list-colorable if H is DP- k -colorable and, in particular, $\chi_\ell(H) \leq \chi_{\text{DP}}(H)$.

Thus, proper list-coloring is just a special case of DP-coloring. However, the definition of a cover is significantly more general than the setting for proper list-colorings and the list-chromatic number and the DP-chromatic number may differ. For instance, it is not difficult to check that the hypergraph H in Figure 4.1 satisfies $\chi_\ell(H) = 2$, but admits no independent transversal for the given cover and, hence, $\chi_{\text{DP}}(H) \geq 3$. So is it at all possible to obtain non-trivial results for DP-coloring and, in particular, to use the advantages of DP-coloring in order to prove new results for the list-chromatic number? The answer is yes, for example, DVORÁK and POSTLE [41] proved that every planar graph without cycles of lengths four to eight has list-chromatic number at most three, thereby answering a question by BORODIN [21]. Also, it shows that the coloring number is an upper bound for the DP-chromatic number. To obtain this, we use a sequential coloring method as described in Algorithm 1.

Clearly, if $|X_{v_i}| \geq d_{H[\{v_1, v_2, \dots, v_i\}]}(v_i) + 1$ for all $i \in [1, n]$, in step 5 there is always a possible choice for x_i and, thus, the algorithm terminates with an (X, \mathcal{H}) -coloring of H . This is due to the fact that for each edge $e \in E(H)$ with $v_i \in i_H(e) \subseteq \{v_1, v_2, \dots, v_i\}$ and for any set of

Algorithm 1 Sequential coloring algorithm

- 1: Input: hypergraph H and cover (X, \mathcal{H}) of H .
 - 2: Choose an arbitrary vertex order (v_1, v_2, \dots, v_n) of H .
 - 3: Let $T = \emptyset$.
 - 4: **for all** $i = 1, 2, \dots, n$ **do**
 - 5: Choose a vertex (color) x_i from X_{v_i} such that $E(\mathcal{H}[T \cup \{x_i\}]) = \emptyset$.
 - 6: Let $T = T \cup \{x_i\}$.
 - 7: **end for**
 - 8: Return: Independent transversal T .
-

fixed colors

$$\{x_k \mid x_k \in X_{v_k}, v_k \in i_H(e), k \in [1, i-1]\},$$

at most one color from X_{v_i} is prohibited. Hence, in X_{v_i} , at most $d_{H[v_1, v_2, \dots, v_i]}(v_i)$ vertices are forbidden. As a consequence, if $f(v) \geq d_H(v) + 1$ for all $v \in V(H)$, then H is DP- f -colorable. If we apply Algorithm 1 to a smallest last order (v_1, v_2, \dots, v_n) of H , the algorithm returns an independent transversal provided that $|X_{v_i}| \geq \text{col}(H)$ for all $i \in [1, n]$ (by Proposition 1.1 and the above argumentation). Therefore, $\chi_{\text{DP}}(H) \leq \text{col}(H)$ and, summarizing, we obtain

$$\chi(H) \leq \chi_\ell(H) \leq \chi_{\text{DP}}(H) \leq \text{col}(H) \leq \Delta(H) + 1. \quad (4.1)$$

Our aim is to obtain a BROOKS-type result for DP-colorings of hypergraphs, *i.e.*, to characterize the hypergraphs H for which $\chi_{\text{DP}}(H) = \Delta(H) + 1$ holds. Clearly, if H is an odd cycle, we have $\chi(H) = \Delta(H) + 1 = 3$ and, thus, equality holds. To see that $\chi_{\text{DP}}(H) = 3$ holds for even cycles as well, we construct an appropriate cover of H , following [14]. Assume that $V(H) = [1, n]$ with $n \geq 2$ even and $E(H) = \{uv \mid u, v \in V(H) \text{ and } u - v \equiv 1 \pmod{n}\}$. Let (X, \mathcal{H}) be the cover of H with $X_v = \{v\} \times \{1, 2\}$ for all $v \in V(H)$ and $E(\mathcal{H}) = \{(u, i)(v, j) \mid |u - v| = 1 \text{ and } i = j; \text{ or } \{u, v\} = \{1, n\} \text{ and } i - j \equiv 1 \pmod{2}\}$. Then, (X, \mathcal{H}) is a cover of H with $|X_v| = 2$ for all $v \in V(H)$. Moreover, $\mathcal{H} = C_{2n}$ and (X, \mathcal{H}) has no independent transversal. The cover for $n = 4$ is displayed in Figure 4.2. As emphasized in [14], the fact that $\chi_{\text{DP}}(C_n) = 3$ for all $n \geq 2$ and not only for odd $n \geq 3$ marks an important difference between the DP-chromatic number and the list-chromatic number.

4.2. DP-degree Colorable Hypergraphs

We say that a hypergraph H is **DP-degree colorable** if H is (X, \mathcal{H}) -colorable whenever (X, \mathcal{H}) is a cover of H such that $|X_v| \geq d_H(v)$ for all $v \in V(H)$. Regarding graphs, BERN-

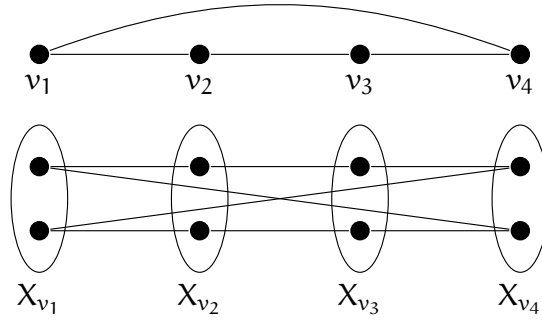


FIG. 4.2. The cover admits no independent transversal and so $\chi_{DP}(C_4) = 3$.

SHTEYN, KOSTOCHKA, and PRON [14] proved that a connected graph G is not DP-degree colorable if and only if each block of G is a tK_n or a tC_n for some integers $t, n \geq 1$. Of course, when dealing with DP-coloring, it is not only of interest to characterize the non DP-degree colorable graphs, but also the corresponding “bad” covers. This was done by KIM and OZEKI [66] (see Theorem 4.7). The aim of this section is to give the corresponding characterizations for DP-degree-colorable hypergraphs.

A **feasible configuration** is a triple (H, X, \mathcal{H}) consisting of a connected hypergraph H and a cover (X, \mathcal{H}) of H . A feasible configuration is said to be **degree-feasible** if $|X_v| \geq d_H(v)$ for each vertex $v \in V(H)$. Furthermore, (H, X, \mathcal{H}) is **colorable** if H is (X, \mathcal{H}) -colorable, otherwise it is called **uncolorable**.

The next proposition lists some basic properties of feasible configurations; the proofs are straightforward and left to the reader. Recall that for distinct vertices u, v of a hypergraph H , we denote by $\mu_H(u, v)$ the number of ordinary edges of H that are incident with both u and v .

Proposition 4.1. *Let (H, X, \mathcal{H}) be a feasible configuration. Then, the following statements hold:*

- (a) *For distinct vertices u, v of H , the hypergraph $\mathcal{H}[X_u \cup X_v]$ is a bipartite graph with parts X_u and X_v whose maximum degree is at most $\mu_H(u, v)$. Furthermore, for every vertex $v \in V(H)$ and every vertex $x \in X_v$, we have $d_{\mathcal{H}}(x) \leq d_H(v)$.*
- (b) *Let \mathcal{H}' be a spanning subhypergraph of \mathcal{H} . Then, (H, X, \mathcal{H}') is a feasible configuration. If (H, X, \mathcal{H}) is colorable, then (H, X, \mathcal{H}') is colorable, too. Furthermore, (H, X, \mathcal{H}) is degree-feasible if and only if (H, X, \mathcal{H}') is degree feasible. \diamond*

The above proposition leads to the following concept. We say that a feasible configuration (H, X, \mathcal{H}) is **minimal uncolorable** if (H, X, \mathcal{H}) is uncolorable, but $(H, X, \mathcal{H} - e)$ is colorable for each $e \in E(\mathcal{H})$. Clearly, if $|H| \geq 2$ and if $\tilde{\mathcal{H}}$ is the edgeless spanning hypergraph of H , then $(G, X, \tilde{\mathcal{H}})$ is colorable. Thus, it follows from the above Proposition that if (H, X, \mathcal{H}) is an uncolorable feasible configuration, then there is a spanning subhypergraph \mathcal{H}' of \mathcal{H} such that (H, X, \mathcal{H}') is a minimal uncolorable feasible configuration. Furthermore, if (H, X, \mathcal{H}) is a minimal uncolorable feasible configuration, then \mathcal{H} clearly is a simple hypergraph.

In order to characterize the class of minimal uncolorable degree-feasible configurations, we firstly need to introduce three basic types of degree-feasible configurations. To this end, we need some more definitions. As usual, for $n, t \geq 1$, by $K_{(n,t)}$ we denote the **complete n-partite graph** all of whose partite sets have t vertices. In particular, $K_{(2,t)}$ is the complete bipartite graph $K_{t,t}$. Given a hypergraph H , by $\mathcal{A}(H)$ we denote the set of all two-subsets $\{u, v\} \subseteq V(H)$ such that $\mu_H(u, v) > 0$, *i.e.*, the set of all two-subsets $\{u, v\}$ that are joined by at least one ordinary edge in H .

We say that (H, X, \mathcal{H}) is a **K-configuration** if $H = tK_n$ for some integers $t, n \geq 1$ and if (X, \mathcal{H}) is a cover of H such that for every vertex $v \in V(H)$, there is a partition $(X_v^1, X_v^2, \dots, X_v^{n-1})$ of X_v satisfying the following conditions:

- For every $i \in [1, n - 1]$, the graph $\mathcal{H}^i = \mathcal{H}[\bigcup_{v \in V(H)} X_v^i]$ is a $K_{(n,t)}$ whose partite sets are the sets X_v^i with $v \in V(H)$, and
- $\mathcal{H} = \mathcal{H}^1 \cup \mathcal{H}^2 \cup \dots \cup \mathcal{H}^{n-1}$.

It is an easy exercise to check that each K-configuration is a minimal uncolorable degree-feasible configuration. Note that for $n = 1$, we have $H = K_1$, $X = \emptyset$, and $\mathcal{H} = \emptyset$.

Next we define the so called C-configurations. We say that (H, X, \mathcal{H}) is an **odd C-configuration** if $H = tC_n$ for some integers $t \geq 1$ and $n \geq 5$ odd and if (X, \mathcal{H}) is a cover of H such that for every vertex $v \in V(H)$, there is a partition (X_v^1, X_v^2) of X_v satisfying the following conditions:

- For every $i \in \{1, 2\}$ and for every set $\{u, v\} \in \mathcal{A}(H)$, the graph $\mathcal{H}_{\{u,v\}}^i = \mathcal{H}[X_u^i \cup X_v^i]$ is a $K_{t,t}$ whose partite sets are X_u^i and X_v^i , and
- \mathcal{H} is the union of all graphs $\mathcal{H}_{\{u,v\}}^i$ with $i \in \{1, 2\}$ and $\{u, v\} \in \mathcal{A}(H)$.

It is easy to verify that any odd C-configuration is a minimal uncolorable degree-feasible configuration.

We call (H, X, \mathcal{H}) an **even C-configuration** if $H = tC_n$ for some integers $t \geq 1, n \geq 4$ even and if (X, \mathcal{H}) is a cover of H such that for every vertex $v \in V(H)$, there is a partition (X_v^1, X_v^2) of X_v and a set $\{w, w'\} \in \mathcal{A}(H)$ satisfying the following conditions:

- For every $i \in \{1, 2\}$ and for every set $\{u, v\} \in \mathcal{A}(H)$ different from $\{w, w'\}$, the graph $\mathcal{H}_{\{u,v\}}^i = \mathcal{H}[X_u^i \cup X_v^i]$ is a $K_{t,t}$ whose partite sets are X_u^i and X_v^i ,
- $\mathcal{H}_{\{w,w'\}}^1 = \mathcal{H}[X_w^1 \cup X_{w'}^2]$ is a $K_{t,t}$ whose partite sets are X_w^1 and $X_{w'}^2$,
- $\mathcal{H}_{\{w,w'\}}^2 = \mathcal{H}[X_w^2 \cup X_{w'}^1]$ is a $K_{t,t}$ whose partite sets are X_w^2 and $X_{w'}^1$, and
- \mathcal{H} is the union of all graphs $\mathcal{H}_{\{u,v\}}^i$ with $i \in \{1, 2\}$ and $\{u, v\} \in \mathcal{A}(H)$.

Again, it is easy to check that any even C-configuration is a minimal uncolorable degree-feasible configuration. By a **C-configuration** we mean either an even or an odd C-configuration. An example of a K- and a C-configuration is given in Figure 4.3.

Finally, we say that (H, X, \mathcal{H}) is an **E-configuration** if $H = \langle e \rangle$ for some hyperedge e , if $|X_v| = 1$ for each $v \in V(H)$ and if $\mathcal{H} \cong H$. Clearly, each E-configuration is a minimal uncolorable degree-feasible configuration.

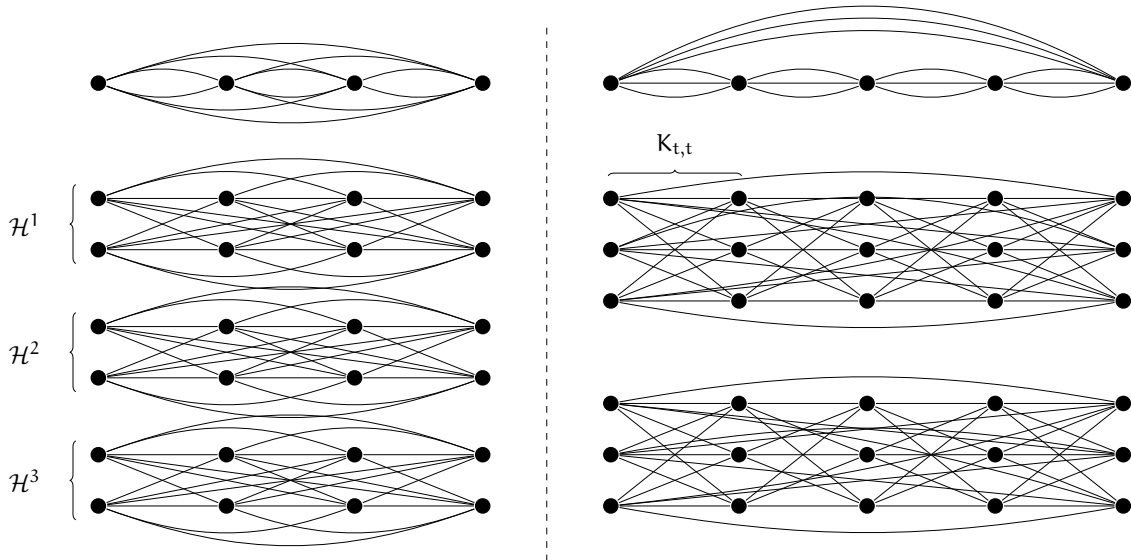


FIG. 4.3. A K- and an odd C-configuration.

We will show that we can construct any minimal uncolorable degree-feasible configuration from these three basic configurations using the following operation.

Let $(H^1, X^1, \mathcal{H}^1)$ and $(H^2, X^2, \mathcal{H}^2)$ be two feasible configurations, which are **disjoint**, that is, $V(H^1) \cap V(H^2) = \emptyset$ and $V(\mathcal{H}^1) \cap V(\mathcal{H}^2) = \emptyset$. Furthermore, let H be the hypergraph obtained from H^1 and H^2 by merging two vertices $v^1 \in V(H^1)$ and $v^2 \in V(H^2)$ to a new vertex v^* . Finally, let $H = \mathcal{H}^1 \cup \mathcal{H}^2$ and let $X : V(H) \rightarrow 2^{V(\mathcal{H})}$ be the mapping such that

$$X_v = \begin{cases} X_{v^1}^1 \cup X_{v^2}^2 & \text{if } v = v^*, \\ X_v^i & \text{if } v \in V(H^i) \setminus \{v^i\} \text{ and } i \in \{1, 2\} \end{cases}$$

for $v \in V(H)$. Then, (H, X, \mathcal{H}) is a feasible configuration and we say that (H, X, \mathcal{H}) is obtained from $(H^1, X^1, \mathcal{H}^1)$ and $(H^2, X^2, \mathcal{H}^2)$ by **merging** v^1 and v^2 to v^* .

Since $d_H(v^*) = d_{H^1}(v^1) + d_{H^2}(v^2)$, it follows that (H, X, \mathcal{H}) is degree-feasible if both $(H^1, X^1, \mathcal{H}^1)$ and $(H^2, X^2, \mathcal{H}^2)$ are degree-feasible. Figure 4.4 displays a configuration that is obtained from an even C-configuration and an E-configuration by merging two vertices.

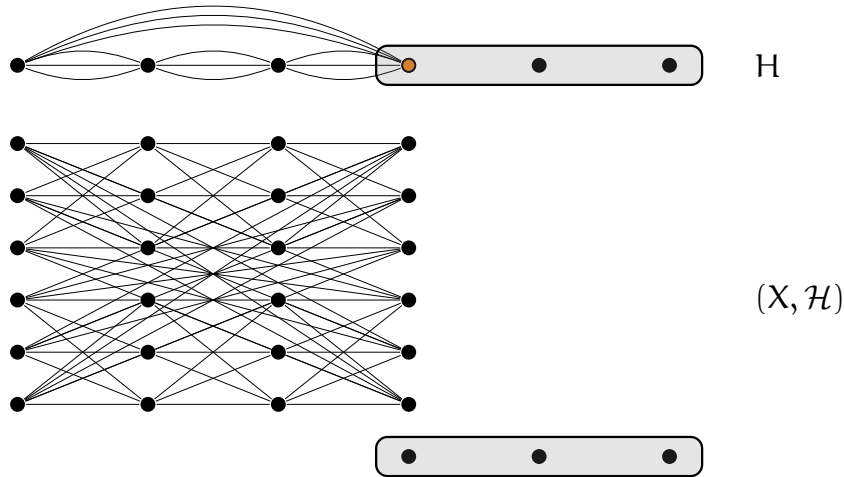


FIG. 4.4. The configuration results from merging a C- with an E- configuration at the colored vertex.

Now we define the class of **constructible configurations** as the smallest class of feasible configurations that contains each K-configuration, each C-configuration and each E-configuration and that is closed under the merging operation. Thus, if (H, X, \mathcal{H}) is a constructible configuration, then each block of H is a tK_n for $t \geq 1, n \geq 1$, a tC_n for $t \geq 1, n \geq 3$, or of the form $\langle e \rangle$ for some edge e . A block B of H is called a **DP-brick** if $B = tK_n$ for some $t \geq 1, n \geq 1$ or if $B = tC_n$ for some $t \geq 1, n \geq 3$. Moreover, we say that B is a **DP-hyperbrick**, if B is either a DP-brick or $B = \langle e \rangle$ for some edge e . The main result of this chapter is as follows.

Theorem 4.2. *Let (H, X, \mathcal{H}) be a degree-feasible configuration. Then, (H, X, \mathcal{H}) is minimal uncolorable if and only if (H, X, \mathcal{H}) is constructible.* \diamond

4.3. Proof of Theorem 4.2

4.3.1. Necessary Tools

The structure of the proof strongly resembles the one of the proof of Theorem 2.3. In particular, we again use the method of reducible configurations. Fortunately, it is significantly easier to deal with DP-colorings than with variable degeneracy and so it won't take 18 pages this time. Still, we first need various propositions. The next one describes the block-configurations of constructible configurations; the proof can be done by induction on the number of blocks and is left to the reader.

Proposition 4.3. *Let (H, X, \mathcal{H}) be a constructible configuration. Then, for each block $B \in \mathcal{B}(G)$ there is a uniquely determined cover (X^B, \mathcal{H}^B) of B such that the following statements hold:*

- (a) *For each block $B \in \mathcal{B}(H)$, the triple (B, X^B, \mathcal{H}^B) is a K-configuration, C-configuration, or E-configuration.*
- (b) *The hypergraphs \mathcal{H}^B with $B \in \mathcal{B}(H)$ are pairwise disjoint and $\mathcal{H} = \cup_{B \in \mathcal{B}(H)} \mathcal{H}^B$.*
- (c) *For every vertex $v \in V(H)$ we have $X_v = \cup_{B \in \mathcal{B}_v(H)} X_v^B$.* \diamond

The next proposition is key in order to obtain our main result, it describes the reduction method that allows us to use induction on the number of vertices of H . Similar propositions to the next two propositions were proved by BERNSHTEYN, KOSTOCHKA, and PRON for graphs in [14]. Recall that, given a hypergraph H and a vertex $v \in V(H)$, the set $N_H(v)$ denotes the ordinary neighborhood of v , *i.e.*, the set of vertices u with $\mu_H(u, v) > 0$.

Proposition 4.4. *Let (H, X, \mathcal{H}) be a feasible configuration with $|H| \geq 2$, let v be a non-separating vertex of H , and let $x \in X_v$ be a color. We define a cover of the hypergraph $H' = H \div v$ as follows. For $u \in V(H')$ let*

$$X'_u = X_u \setminus N_{\mathcal{H}}(x)$$

and let \mathcal{H}' be the hypergraph with $V(\mathcal{H}') = \cup_{u \in V(H')} X'_u$,

$$E(\mathcal{H}') = \{e \mid e \in E(\mathcal{H}), |i_{\mathcal{H}}(e) \setminus \{x\}| \geq 2, \text{ and } (i_{\mathcal{H}}(e) \setminus \{x\}) \subseteq V(\mathcal{H}')\},$$

and

$$i_{\mathcal{H}'}(e) = i_{\mathcal{H}}(e) \setminus \{x\}$$

for all $e \in E(\mathcal{H}')$. Then, (H', X', \mathcal{H}') is a feasible configuration, and in what follows we write $(H', X', \mathcal{H}') = (H, X, \mathcal{H}) / (v, x)$. Moreover, the following statements hold:

(a) If (H, X, \mathcal{H}) is degree-feasible, then (H', X', \mathcal{H}') is degree-feasible, too.

(b) If (H, X, \mathcal{H}) is uncolorable, then (H', X', \mathcal{H}') is uncolorable, too. \diamond

Proof. Clearly, (X', \mathcal{H}') is a cover of H' and, hence, (H', X', \mathcal{H}') is a feasible configuration. Moreover, for $u \in V(H')$ it holds $d_{H'}(u) = d_H(u) - \mu_H(u, v)$ and $|N_{\mathcal{H}}(x) \cap X_u| \leq \mu_H(u, v)$ (see (1.2) and Proposition 4.1). Thus, we obtain

$$|X'_u| = |X_u| - |N_{\mathcal{H}}(x) \cap X_u| \geq |X_u| - \mu_H(u, v).$$

As $|X_u| \geq d_H(u)$, this leads to $|X'_u| \geq d_{H'}(u)$ and (H', X', \mathcal{H}') is degree-feasible. Furthermore, if T' is an independent transversal of (X', \mathcal{H}') , then $T = T' \cup \{x\}$ is an independent transversal of (X, \mathcal{H}) . This proves (b). \blacksquare

Using the above introduced reduction method, we obtain the following.

Proposition 4.5. *Let (H, X, \mathcal{H}) be an uncolorable degree-feasible configuration. Then, the following statements hold:*

(a) $|X_v| = d_H(v)$ for all $v \in V(H)$.

(b) For each non-separating vertex z of H and each vertex $v \neq z$ of H , $|N_{\mathcal{H}}(x) \cap X_v| = \mu_H(v, z)$ for all $x \in X_z$.

(c) Every hyperedge e of H is a bridge of H and, therefore, $\langle e \rangle$ is a block of H . As a consequence, there are no parallel hyperedges in H .

(d) If H is a block, then H is regular, and for distinct vertices u, v of H , the hypergraph $\mathcal{H}[X_u \cup X_v]$ is a $\mu_H(u, v)$ -regular bipartite graph whose partite sets are X_u and X_v .

(e) For each vertex $v \in V(H)$ there is an independent set T in \mathcal{H} satisfying $|T \cap X_u| = 1$ for all $u \in V(H) \setminus \{v\}$.

Proof. We prove (a) by induction on the order of H . If H consists of only one vertex v , then $X_v = \emptyset$ and $\mathcal{H} = \emptyset$. Thus, (a) is fulfilled. Now assume $|H| \geq 2$ and choose an arbitrary vertex v of H . As H is connected, there is a non-separating vertex $z \neq v$ in H and $X_z \neq \emptyset$. Let $x \in X_z$. Then, $(H', X', \mathcal{H}') = (H, X, \mathcal{H}) / (z, x)$ is an uncolorable degree-feasible configuration (by Proposition 4.4). Applying the induction hypothesis then leads to $|X'_v| = d_{H'}(v)$ and we conclude

$$\begin{aligned} d_{H'}(v) &= |X'_v| = |X_v| - |N_{\mathcal{H}}(x) \cap X_v| \\ &\geq |X_v| - \mu_H(v, z) \geq d_H(v) - \mu_H(v, z) = d_{H'}(v). \end{aligned}$$

This implies $|X_v| = d_H(v)$ and $|N_{\mathcal{H}}(x) \cap X_v| = \mu_H(v, z)$; thus, (a) is proved. The same argument can be applied in order to prove (b).

For the proof of (c) assume that some hyperedge $e \in E(H)$ is not a bridge of H . Then, for some vertex $v \in i_H(e)$, the hypergraph $H' = (V(H), E(H), i_{H'})$ with $i_{H'}(e) = i_H(e) \setminus \{v\}$ and $i_{H'}(e') = i_H(e')$ for $e' \in E(H) \setminus \{e\}$ is connected. Let $X' = X$ and let \mathcal{H}' be the hypergraph with vertex set $V(\mathcal{H})$ and edge set $(E(\mathcal{H}) \setminus M_e) \cup M'_e$, whereas M'_e denotes the restriction of M_e to the vertices of $\bigcup_{u \in i_H(e) \setminus \{v\}} X_u$. Clearly, (H', X', \mathcal{H}') is a degree-feasible configuration. However, (a) implies that $|X'_v| = |X_v| = d_H(v) > d_{H'}(v)$ and so, again by (a), (H', X', \mathcal{H}') is colorable. Hence, there is an independent transversal T' of (H', X', \mathcal{H}') . We claim that T' is also an independent transversal of (H, X, \mathcal{H}) . Otherwise, by construction of \mathcal{H}' there would be an edge $\tilde{e} \in E(\mathcal{H})$ with $v \in i_{\mathcal{H}}(\tilde{e}) \subseteq T'$. But then, $i_{\mathcal{H}'}(\tilde{e} - v) \subseteq T'$ and so T' is not an independent transversal of \mathcal{H}' , a contradiction. Hence, T' is an independent transversal of (H, X, \mathcal{H}) and so (H, X, \mathcal{H}) is colorable, which is impossible. This settles the case (c).

In order to prove (d), assume that H is a block. If $H = \langle e \rangle$ for some hyperedge e , then H is regular and the statement clearly holds. Thus, by (c), we may assume that H does not contain any hyperedge. Let u, v be distinct vertices of H . Then, $\mathcal{H}[X_u \cup X_v]$ is a $\mu_H(u, v)$ -regular bipartite graph with parts X_u and X_v (by (b)). This is only possible if $|X_u| = |X_v|$. By (a), this leads to $d_H(u) = d_H(v)$ and (d) is proved.

Finally, for the proof of (e), let v be an arbitrary vertex of H . Let U be the vertex set of a component of $H - v$, let X' be the restriction of X to U , and let $\mathcal{H}' = \mathcal{H}[\bigcup_{u \in U} X_u]$. Then, $(H[U], X', \mathcal{H}')$ is a degree-feasible configuration and, as H is connected, it holds $|X_u| = d_H(u) > d_{H[U]}(u)$ for at least one vertex $u \in U$. Hence, there is an independent transversal T_U of (X', \mathcal{H}') (by (a)). Let T be the union of the independent transversals T_U over all components $H[U]$ of $H - v$. Clearly, T is an independent set of \mathcal{H} such that $|T \cap X_w| = 1$ for

all $w \in V(H) \setminus \{v\}$. This proves (e). ■

Finally, we connect the concept of being minimal uncolorable with the merging operation.

Proposition 4.6. *Let (H, X, \mathcal{H}) be obtained from two disjoint degree-feasible configurations $(H^1, X^1, \mathcal{H}^1)$ and $(H^2, X^2, \mathcal{H}^2)$ by merging $v^1 \in V(H^1)$ and $v^2 \in V(H^2)$ to a new vertex v^* . Then, (H, X, \mathcal{H}) is a degree-feasible configuration and the following conditions are equivalent:*

(a) *Both $(H^1, X^1, \mathcal{H}^1)$ and $(H^2, X^2, \mathcal{H}^2)$ are minimal uncolorable.*

(b) *(H, X, \mathcal{H}) is minimal uncolorable.* ◇

Proof. First we show that (a) implies (b). Assume that (H, X, \mathcal{H}) is colorable. Then, there is an independent transversal T of (X, \mathcal{H}) , that is, an independent set of H such that $|T \cap X_u| = 1$ for all $u \in V(H)$. As $X_{v^*} = X_{v^1} \cup X_{v^2}$, this implies (by symmetry) that $|T \cap X_{v^1}| = 1$. As a consequence, $T^1 = T \cap V(\mathcal{H}^1)$ is an independent transversal of (X^1, \mathcal{H}^1) and so $(H^1, X^1, \mathcal{H}^1)$ is colorable, a contradiction to (a). Thus, (H, X, \mathcal{H}) is uncolorable. Let $e \in E(\mathcal{H})$ be an arbitrary edge. By the structure of $\mathcal{H} = \mathcal{H}^1 \cup \mathcal{H}^2$, we may assume that $e \in E(\mathcal{H}^1)$. Due to the fact that $(H^1, X^1, \mathcal{H}^1)$ is minimal uncolorable, there is an independent transversal T^1 of the cover $(X^1, \mathcal{H}^1 - e)$. Since $(H^2, X^2, \mathcal{H}^2)$ is also minimal uncolorable and as H^2 is connected, it follows from Proposition 4.5(e) that there is an independent set T^2 in \mathcal{H}^2 satisfying $|T^2 \cap X_u^2| = 1$ for all $u \in V(H^2) \setminus \{v^2\}$. However, as $\mathcal{H} = \mathcal{H}^1 \cup \mathcal{H}^2$ and $\mathcal{H}^1 \cap \mathcal{H}^2 = \emptyset$, the set $T = T^1 \cup T^2$ is an independent transversal of $(X, \mathcal{H} - e)$ and so $(H, X, \mathcal{H} - e)$ is colorable. Thus, (b) holds.

In order to prove that (a) can be deduced from (b), we only need to show that $(H^1, X^1, \mathcal{H}^1)$ is minimal uncolorable (by symmetry). First assume that $(H^1, X^1, \mathcal{H}^1)$ is colorable, that is, (X^1, \mathcal{H}^1) has an independent transversal T^1 . Since (H, X, \mathcal{H}) is minimal uncolorable and connected and as $\mathcal{H}^2 - X_{v^2}$ is a subhypergraph of \mathcal{H} , Proposition 4.5(e) implies that there is an independent set T^2 in $\mathcal{H}^2 - X_{v^2}$ such that $|T^2 \cap X_u^2| = 1$ for all $u \in V(H^2) \setminus \{v^2\}$. Then again, $T = T^1 \cup T^2$ is an independent transversal of (X, \mathcal{H}) , contradicting (b). Thus, $(H^1, X^1, \mathcal{H}^1)$ is uncolorable. Now let $e \in E(\mathcal{H}^1)$ be an arbitrary edge. Then, as (H, X, \mathcal{H}) is minimal uncolorable, there is an independent transversal T of $(X, \mathcal{H} - e)$ and $T^1 = T \cap V(\mathcal{H}^1)$ clearly is an independent transversal of (X^1, \mathcal{H}^1) . Consequently, $(H^1, X^1, \mathcal{H}^1 - e)$ is colorable and the proof is complete. ■

4.3.2. The Main Proof

Before proving it, let us again state the main result of this chapter.

Theorem 4.2. *Let (H, X, \mathcal{H}) be a degree-feasible configuration. Then, (H, X, \mathcal{H}) is minimal uncolorable if and only if (H, X, \mathcal{H}) is constructible. \diamond*

Proof. If (H, X, \mathcal{H}) is constructible, then (H, X, \mathcal{H}) is minimal uncolorable (by Proposition 4.6 and as each K,C and E-configuration is a minimal uncolorable degree-feasible configuration). Let (H, X, \mathcal{H}) be a minimal uncolorable degree-feasible configuration. We prove that (H, X, \mathcal{H}) is constructible by induction on the order of H . Clearly, if $|H| = 1$, then $X = \emptyset$, $\mathcal{H} = \emptyset$ and (H, X, \mathcal{H}) is a K-configuration. So assume that $|H| \geq 2$. By Proposition 4.5(a), it holds

$$|X_v| = d_H(v) \quad (4.2)$$

for each vertex $v \in V(H)$. We distinguish between two cases.

Case 1: H contains a separating vertex v^* . Then, H is the union of two connected induced subhypergraphs H^1 and H^2 with $V(H^1) \cap V(H^2) = \{v^*\}$ and $|H^j| < |H|$ for $j \in \{1, 2\}$. For $j \in \{1, 2\}$, by \mathcal{T}^j we denote the set of all independent sets T of \mathcal{H} such that $|T \cap X_v| = 1$ for all $v \in V(H^j)$. By Proposition 4.5(e), both \mathcal{T}^1 and \mathcal{T}^2 are non-empty. For $j \in \{1, 2\}$, let X_j be the set of all vertices of X_{v^*} that do not occur in any independent set from \mathcal{T}^j . We claim that $X_{v^*} = X_1 \cup X_2$. For otherwise, there is a vertex $u \in X_{v^*} \setminus (X_1 \cup X_2)$. Then, u is contained in two independent sets $T^j \in \mathcal{T}^j$ with $(j \in \{1, 2\})$ and $T = T^1 \cup T^2$ is an independent transversal of (X, \mathcal{H}) . This is due to the fact that each hyperedge of H is contained in H^j for some $j \in \{1, 2\}$ and that for $u \in V(H^1) \setminus \{v^*\}$ and $v \in V(H^2) \setminus \{v^*\}$ we have $\mu_H(u, v) = 0$ and so $\mathcal{H}[X_u \cup X_v]$ is edgeless (by Proposition 4.1(a)). Thus, (H, X, \mathcal{H}) is colorable, a contradiction. Consequently, $X_{v^*} = X_1 \cup X_2$, as claimed. For $j \in \{1, 2\}$, let (X^j, \mathcal{H}^j) be a cover of H^j as follows. For $v \in V(H^j)$, let

$$X_v^j = \begin{cases} X_v & \text{if } v \neq v^* \\ X_j & \text{if } v = v^*, \end{cases}$$

and let $\mathcal{H}^j = \mathcal{H}[\bigcup_{v \in V(H^j)} X_v^j]$. Then, $(H^j, X^j, \mathcal{H}^j)$ is a feasible configuration and, by definition of $X_j = X_{v^*}^j$, $(H^j, X^j, \mathcal{H}^j)$ is uncolorable. Moreover, for each vertex $v \in V(H^j) \setminus \{v^*\}$ it holds $|X_v| = d_H(v) = d_{H^j}(v)$ (by (4.2)). As $(H^j, X^j, \mathcal{H}^j)$ is uncolorable, it follows from Proposition 4.5(a) that $|X_{v^*}^j| \leq d_{H^j}(v^*)$ for $j \in \{1, 2\}$. Since $X_{v^*} = X_1 \cup X_2 = X_{v^*}^1 \cup X_{v^*}^2$ we conclude from (4.2) that

$$|X_{v^*}^1| + |X_{v^*}^2| \geq |X_{v^*}^1 \cup X_{v^*}^2| = |X_{v^*}| = d_H(v^*) = d_{H^1}(v^*) + d_{H^2}(v^*),$$

and, thus, $|X_{v^*}^j| = d_{H^j}(v^*)$ and $X_{v^*}^1 \cap X_{v^*}^2 = \emptyset$. Hence, $(H^j, X^j, \mathcal{H}^j)$ is a degree-feasible configuration. Moreover, $\mathcal{H}' = \mathcal{H}^1 \cup \mathcal{H}^2$ is a spanning subhypergraph of \mathcal{H} and $V(\mathcal{H}^1) \cap V(\mathcal{H}^2) = \emptyset$. So, (H, X, \mathcal{H}') is a degree-feasible configuration (by Proposition 4.1(b)) and (H, X, \mathcal{H}') is obtained from two isomorphic copies of $(H^1, X^1, \mathcal{H}^1)$ and $(H^2, X^2, \mathcal{H}^2)$ by the merging operation. Clearly, (H, X, \mathcal{H}') is uncolorable. Otherwise, there would exist an independent transversal T of (X, \mathcal{H}') and, by symmetry, T would contain a vertex of $X_{v^*}^1$. But then, $T^1 = T \cap V(\mathcal{H}^1)$ would be an independent transversal of (X^1, \mathcal{H}^1) , which is impossible. As (H, X, \mathcal{H}) is minimal uncolorable and as \mathcal{H}' is a spanning subhypergraph of \mathcal{H} , this implies that $\mathcal{H} = \mathcal{H}'$ and (H, X, \mathcal{H}) is obtained from two isomorphic copies of $(H^1, X^1, \mathcal{H}^1)$ and $(H^2, X^2, \mathcal{H}^2)$ by the merging operation. By Proposition 4.6, both $(H^1, X^1, \mathcal{H}^1)$ and $(H^2, X^2, \mathcal{H}^2)$ are minimal uncolorable (and also degree-feasible). Applying the induction hypotheses leads to $(H^j, X^j, \mathcal{H}^j)$ being constructible for $j \in \{1, 2\}$, and so (H, X, \mathcal{H}) is constructible. Thus, the first case is complete.

Case 2: H is a block. If H contains any hyperedge e , then it follows from Proposition 4.5(c) that $H = \langle e \rangle$ and (H, X, \mathcal{H}) is not colorable if and only if (H, X, \mathcal{H}) is an E-configuration. Thus, in the following we may assume that H does not contain any hyperedges. We prove that (H, X, \mathcal{H}) is either a K-configuration or a C-configuration. This is done via a sequence of four claims.

Claim 4.6.1. *Let v be an arbitrary vertex of H , let $x \in X_v$ be an arbitrary color, and let $(H', X', \mathcal{H}') = (H, X, \mathcal{H}) / (v, x)$. Then, there is a spanning subhypergraph $\tilde{\mathcal{H}}$ of \mathcal{H}' such that $(H', X', \tilde{\mathcal{H}})$ is minimal uncolorable. Moreover, $(H', X', \tilde{\mathcal{H}})$ is constructible and so each block of $H' = H - v$ is a DP-brick.* \diamond

Proof. Since $|H| \geq 2$ and H is connected, $X_v \neq \emptyset$ (by (4.2)). Thus, $(H', X', \mathcal{H}') = (H, X, \mathcal{H}) / (v, x)$ is an uncolorable degree-feasible configuration (by Proposition 4.4) and, therefore, there is a spanning subhypergraph $\tilde{\mathcal{H}}$ of \mathcal{H}' such that $(H', X', \tilde{\mathcal{H}})$ is minimal uncolorable. Then, the induction hypothesis implies that $(H', X', \tilde{\mathcal{H}})$ is constructible, and, as $H' = H - v$, this particularly implies that each block of H' is a DP-brick (since H does not contain any hyperedge). \square

By a **multicycle** or **multipath** we mean a graph that can be obtained from a cycle, respectively a path, by replacing each edge e of the cycle or path by a set of t_e parallel edges, where $t_e \geq 1$. Given integers $s, t \geq 1$, we say that a graph G is an **(s, t)-multicycle** if G can be obtained from an even cycle C by replacing each edge of a perfect matching of C by a set of s parallel edges and each other edge of C by a set of t parallel edges. Clearly,

each (s, t) -multicycle is r -regular with $r = s + t$. Moreover, if G is a regular multicycle, then either $G = tC_n$ for some integers $t \geq 1$ and $n \geq 3$, or G is an (s, t) -multicycle for some integers $s, t \geq 1$.

Claim 4.6.2. *The graph H is a DP-brick.* ◊

Proof. Since H is a block, Proposition 4.5(d) implies that H is r -regular for some integer $r \geq 1$. For every vertex v of H , each block of $H - v$ is a DP-brick (by Claim 4.6.1). Let S denote the set of all vertices v of H such that $H - v$ is a block. Then, for every vertex $v \in S$, $H - v$ is a DP-brick and, therefore, regular. As H is regular, too, for $v \in S$ there must be an integer $t_v \geq 1$ such that $\mu_H(u, v) = t_v$ for all $u \in V(H) \setminus \{v\}$. As a consequence, if S is non-empty, then $S = V(H)$ and it clearly holds $t_v = t$ for all $v \in V(H)$. Thus, $H = tK_n$ with $n = |H|$. It remains to consider the case that $S = \emptyset$. Let v be an arbitrary vertex of H . Then, $H - v$ has at least two end-blocks and each block of $H - v$ is a DP-brick and therefore regular. Let B be an arbitrary end-block of $H - v$. Then, B is t_B -regular for some $t_B \geq 1$ and B contains exactly one separating vertex v_B of $H - v$. As H is r -regular, there is an integer s_B such that $\mu_H(u, v) = s_B$ for all vertices $u \in V(B) \setminus \{v_B\}$. As a consequence, $|B| = 2$, since otherwise every vertex of $B - v_B$ belongs to S and so $S \neq \emptyset$, which is impossible. Hence, $B = t_B K_2$, $r = t_B + s_B$, $V(B) = \{v', v_B\}$, and $N_H(v') = \{v, v_B\}$. Repeating the above argumentation with v' instead of v proves that H is a multicycle. Since H is regular, this implies that either $H = tC_n$ with $t \geq 1$ and $n \geq 3$, or H is an (s, t) -multicycle with $s \neq t$. If $H = tC_n$, we are done. We prove that H cannot be an (s, t) -multicycle by reductio ad absurdum. By symmetry, we may assume $1 \leq s < t$. By (4.2), for each vertex v we have $|X_v| = s + t$. Let $v \in V(H)$. Then, $H - v$ is a multipath and one end-block of $H - v$, say B , is a tK_2 . Then, B consists of two vertices u and w with $d_{H-v}(u) = t$ and $d_{H-v}(w) = s + t$. Let $x \in X_v$ be an arbitrary color and set $(H', X', \mathcal{H}') = (H, X, \mathcal{H}) / (v, x)$. Then, there is a spanning subgraph $\tilde{\mathcal{H}}$ of \mathcal{H}' such that $(H', X', \tilde{\mathcal{H}})$ is constructible (by Claim 4.6.1). Moreover, (4.2) together with Proposition 4.3 implies that $|X'_u| = t, |X'_w| = s + t$ and that there is a subset X_w^1 of X'_w such that $|X_w^1| = t$ and $\mathcal{H}^1 = \mathcal{H}[X'_u \cup X_w^1]$ is a $K_{t,t}$ with parts X'_u and X_w^1 . The graph \mathcal{H}^1 is a subgraph of $\mathcal{H}^2 = \mathcal{H}[X_u \cup X_w]$, and \mathcal{H}^2 is a t -regular bipartite graph with parts X_u and X_w (by Proposition 4.5(d)). Since $|X_u| = |X_w| = s + t$ and $1 \leq s < t$, this is impossible and the claim is proved. □

By Claim 4.6.2, H is either a tK_n with $t \geq 1$ and $n \geq 2$, or $H = tC_n$ with $t \geq 1$ and $n \geq 4$. In order to complete the proof we show that in the first case, (H, X, \mathcal{H}) is a K -configuration, and, in the second case, (H, X, \mathcal{H}) is a C -configuration.

Claim 4.6.3. *If $H = tK_n$ for integers $t \geq 1, n \geq 2$, then (H, X, \mathcal{H}) is a K-configuration. \diamond*

Proof. Since (H, X, \mathcal{H}) is minimal uncolorable, for each vertex v of H and each pair u, w of distinct vertices of H , it holds

(a) $|X_v| = t(n-1)$ and $\mathcal{H}[X_u \cup X_w]$ is a t -regular bipartite graph with parts X_u and X_w

(by (4.2) and by Proposition 4.5(d)). If $n = 2$, then H has exactly two vertices, say u and w , and $\mathcal{H}[X_u \cup X_w]$ is a $K_{t,t}$ (by (a)), and so (H, X, \mathcal{H}) is a K-configuration as claimed.

Now assume that $n \geq 3$. Let v be an arbitrary vertex of H , and let $x \in X_v$ be an arbitrary color. Moreover, let $(H', X', \mathcal{H}') = (H, X, \mathcal{H}) / (v, x)$. Then, there is a spanning subgraph $\tilde{\mathcal{H}}$ of $\mathcal{H}' = \mathcal{H} - (X_v \cup N_{\mathcal{H}}(x))$ such that $(H', X', \tilde{\mathcal{H}})$ is a constructible configuration (by Claim 4.6.1). As $H' = H - v = tK_{n-1}$, $(H', X', \tilde{\mathcal{H}})$ is a K-configuration. Consequently, for every vertex $u \in V(H)$, there is a partition $(X_u^1, X_u^2, \dots, X_u^{n-2})$ of $X'_u = X_u \setminus N_{\mathcal{H}}(x)$ such that, for $i \in [1, n-2]$,

(b) the graph $\mathcal{H}^i = \tilde{\mathcal{H}}[\bigcup_{u \in V(H')} X_u^i]$ is a $K_{(n-1), t}$ whose partite sets are the sets X_u^i with $u \in V(H')$, and $\tilde{\mathcal{H}} = \mathcal{H}^1 \cup \mathcal{H}^2 \cup \dots \cup \mathcal{H}^{n-2}$.

For $u \in V(H')$ let $X_u^{n-1} = X_u \setminus X'_u$. Then, for every vertex $u \in V(H')$, $|X_u^{n-1}| = t$ and $(X_u^1, X_u^2, \dots, X_u^{n-1})$ is a partition of X_u . Since $\tilde{\mathcal{H}}$ is a spanning subgraph of \mathcal{H}' , it follows from (a) and (b) that \mathcal{H}^i is an induced subgraph of \mathcal{H} (for $i \in [1, n-2]$), and the graph

$$\mathcal{H}^{n-1} = \mathcal{H}[\bigcup_{u \in V(H')} X_u^{n-1}]$$

is a $K_{(n-1), t}$ whose partite sets are the sets X_u^{n-1} with $u \in V(H')$. Moreover,

$$\mathcal{H} - X_v = \mathcal{H}^1 \cup \mathcal{H}^2 \cup \dots \cup \mathcal{H}^{n-1} \text{ and } N_{\mathcal{H}}(x) = V(\mathcal{H}^{n-1}).$$

Since the color $x \in X_v$ was chosen arbitrarily, this implies that for each $x \in X_v$ there is an index $i \in [1, n-1]$ such that $N_{\mathcal{H}}(x) = V(\mathcal{H}^i)$, and, by (a) and (b), for each index $i \in [1, n-1]$ there are exactly t colors x from X_v such that $N_{\mathcal{H}}(x) = V(\mathcal{H}^i)$. As a consequence, there is a partition $(X_v^1, X_v^2, \dots, X_v^{n-1})$ of X_v such that $|X_v^i| = t$ and $N_{\mathcal{H}}(x) = V(\mathcal{H}^i)$ for $x \in X_v^i$ and for $i \in [1, n-1]$. Hence, for $i \in [1, n]$, the graph

$$\mathcal{H}_i = \mathcal{H}[\bigcup_{u \in V(H)} X_u^i]$$

is a $K_{(n,t)}$ whose partite sets are the sets X_u^i with $u \in V(H)$, and, moreover, $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \dots \cup \mathcal{H}_n$. Thus, (H, X, \mathcal{H}) is a K -configuration. \square

Claim 4.6.4. *If $H = tC_n$ for integers $t \geq 1, n \geq 4$, then (H, X, \mathcal{H}) is a C -configuration. \diamond*

Proof. Since (H, X, \mathcal{H}) is minimal uncolorable, for each vertex $v \in V(H)$ and each two-set $\{u, w\} \in \mathcal{A}(H)$, it holds

(a) $|X_v| = 2t$ and $\mathcal{H}[X_u \cup X_w]$ is a t -regular bipartite graph with parts X_u and X_w

(by (4.2) and by Proposition 4.5(d)). Let v be an arbitrary vertex of H , and let $x \in X_v$ be an arbitrary color. Moreover, let $(H', X', \mathcal{H}') = (H, X, \mathcal{H}) / (v, x)$. Then, there is a spanning subgraph $\tilde{\mathcal{H}}$ of $\mathcal{H}' = \mathcal{H} - (X_v \cup N_{\mathcal{H}}(x))$ such that $(H', X', \tilde{\mathcal{H}})$ is a constructible configuration (by Claim 4.6.1). Since $H' = H - v = tP_{n-1}$, the vertices of H' can be arranged in a sequence, say v_1, v_2, \dots, v_{n-1} , such that two vertices are adjacent in H' if and only if they are consecutive in the sequence. Note that $N_H(v) = \{v_1, v_{n-1}\}$ and each block of H' is a tK_2 . We claim that for each vertex u of H' there is a partition (X_u^1, X_u^2) of X_u such that the following conditions hold:

(b) For every $i \in \{1, 2\}$ and every $k \in [1, n-2]$, the graph $\mathcal{H}_k^i = \mathcal{H}[X_{v_k}^i \cup X_{v_{k+1}}^i]$ is a $K_{t,t}$ whose partite sets are $X_{v_k}^i$ and $X_{v_{k+1}}^i$.

(c) The graph $\mathcal{H} - X_v$ is the union of all graphs \mathcal{H}_k^i with $i \in \{1, 2\}$ and $k \in [1, n-2]$.

(d) If n is even, then $N_{\mathcal{H}}(x) = X_{v_1}^1 \cup X_{v_{n-1}}^2$, or $N_{\mathcal{H}}(x) = X_{v_1}^2 \cup X_{v_{n-1}}^1$.

(e) If n is odd, then $N_{\mathcal{H}}(x) = X_{v_1}^1 \cup X_{v_{n-1}}^1$, or $N_{\mathcal{H}}(x) = X_{v_1}^2 \cup X_{v_{n-1}}^2$.

For $k \in [1, n-2]$, the graph $B^k = H[\{v_k, v_{k+1}\}]$ is a block of H' . Clearly, $\mathcal{B}(H') = \{B^1, B^2, \dots, B^{n-2}\}$ and the only end-blocks of H' are B^1 and B^{n-2} . Since $(H', X', \tilde{\mathcal{H}})$ is a constructible configuration and since each block of H' is a tK_2 , it follows from Proposition 4.3 that for each $k \in [1, n-2]$ there is a uniquely determined cover $(\tilde{X}^k, \tilde{\mathcal{H}}^k)$ of B^k such that

- $\tilde{\mathcal{H}}^k$ is a $K_{t,t}$ with parts $\tilde{X}_{v_k}^k$ and $\tilde{X}_{v_{k+1}}^k$,
- $\tilde{\mathcal{H}}$ is the disjoint union of the graphs $\tilde{\mathcal{H}}^1, \tilde{\mathcal{H}}^2, \dots, \tilde{\mathcal{H}}^{n-2}$,
- $X'_{v_1} = \tilde{X}_{v_1}^1$, $X'_{v_k} = \tilde{X}_{v_k}^{k-1} \cup \tilde{X}_{v_k}^k$ for $k \in [2, n-2]$, and $X'_{v_{n-1}} = \tilde{X}_{v_{n-1}}^{n-2}$.

Since $\{v_k, v_{k+1}\} \in \mathcal{A}(H)$ for $k \in [1, n-2]$, it follows from (a) that $\tilde{\mathcal{H}}^k$ is an induced subgraph of \mathcal{H} . Let $\tilde{X}_{v_1}^0 = X_{v_1} \setminus X'_{v_1}$ and $\tilde{X}_{v_{n-1}}^{n-1} = X_{v_{n-1}} \setminus X'_{v_{n-1}}$. Then, both sets $\tilde{X}_{v_1}^0$ and $\tilde{X}_{v_{n-1}}^{n-1}$ have exactly t elements, and $N_{\mathcal{H}}(x) = \tilde{X}_{v_1}^0 \cup \tilde{X}_{v_{n-1}}^{n-1}$. Furthermore, we conclude from (a) that, for $k \in [1, n-2]$,

- the graph $\mathcal{H}[\tilde{X}_{v_k}^{k-1} \cup \tilde{X}_{v_{k+1}}^{k+1}]$ is a $K_{t,t}$ with parts $\tilde{X}_{v_k}^{k-1}$ and $\tilde{X}_{v_{k+1}}^{k+1}$.

If n is even, we set

$$(X_{v_1}^1, X_{v_2}^1, \dots, X_{v_{n-1}}^1) = (\tilde{X}_{v_1}^1, \tilde{X}_{v_2}^1, \tilde{X}_{v_3}^3, \tilde{X}_{v_4}^3, \dots, \tilde{X}_{v_{n-3}}^{n-3}, \tilde{X}_{v_{n-2}}^{n-3}, \tilde{X}_{v_{n-1}}^{n-1}),$$

and

$$(X_{v_1}^2, X_{v_2}^2, \dots, X_{v_{n-1}}^2) = (\tilde{X}_{v_1}^0, \tilde{X}_{v_2}^2, \tilde{X}_{v_3}^2, \tilde{X}_{v_4}^4, \tilde{X}_{v_5}^4, \dots, \tilde{X}_{v_{n-2}}^{n-2}, \tilde{X}_{v_{n-1}}^{n-2}).$$

If n is odd, let

$$(X_{v_1}^1, X_{v_2}^1, \dots, X_{v_{n-1}}^1) = (\tilde{X}_{v_1}^1, \tilde{X}_{v_2}^1, \tilde{X}_{v_3}^3, \tilde{X}_{v_4}^3, \dots, \tilde{X}_{v_{n-2}}^{n-2}, \tilde{X}_{v_{n-1}}^{n-2}),$$

and

$$(X_{v_1}^2, X_{v_2}^2, \dots, X_{v_{n-1}}^2) = (\tilde{X}_{v_1}^0, \tilde{X}_{v_2}^2, \tilde{X}_{v_3}^2, \dots, \tilde{X}_{v_{n-3}}^{n-3}, \tilde{X}_{v_{n-2}}^{n-3}, \tilde{X}_{v_{n-1}}^{n-1}).$$

By using (a) and Proposition 4.5(b), it is easy to check that, for every vertex u of H' , (X_u^1, X_u^2) is a partition of X_u such that the conditions (b), (c), (d), and (e) are satisfied. Since the color $x \in X_v$ was chosen arbitrarily, it follows from (a) and Proposition 4.5(b) that there is a partition (X_v^1, X_v^2) of X_v such that $|X_v^1| = |X_v^2| = t$ and the following conditions hold:

- If n is even, then $N_{\mathcal{H}}(x) = X_{v_1}^1 \cup X_{v_{n-1}}^2$ for all $x \in X_v^1$ and $N_{\mathcal{H}}(x) = X_{v_1}^2 \cup X_{v_{n-1}}^1$ for all $x \in X_v^2$.
- If n is odd, then $N_{\mathcal{H}}(x) = X_{v_1}^1 \cup X_{v_{n-1}}^1$ for all $x \in X_v^1$ and $N_{\mathcal{H}}(x) = X_{v_1}^2 \cup X_{v_{n-1}}^2$ for all $x \in X_v^2$.

Clearly, this implies that (H, X, \mathcal{H}) is a C-configuration, and the claim is proved. \square

This settles Case 2. Hence, in both cases we showed that (H, X, \mathcal{H}) is a constructible configuration and the proof of the theorem is complete. \blacksquare

As mentioned earlier, KIM and OZEKI [66] characterized the “bad” covers for non-DP-degree colorable graphs; many ideas of their proof are similar to ours. In our terminology, they proved the following:

Theorem 4.7 (KIM and OZEKI, 2019). *Let G be a graph and let (G, X, \mathcal{G}) be a degree-feasible configuration. Then, G is not (X, \mathcal{G}) -colorable if and only if for each block $B \in \mathcal{B}(G)$ there is a cover (X^B, \mathcal{G}^B) of B such that the following statements hold:*

- (a) *For every block $B \in \mathcal{B}(G)$, the triple (B, X^B, \mathcal{G}^B) is a K-configuration, or a C-configuration.*
- (b) *The graphs \mathcal{G}^B with $B \in \mathcal{B}(G)$ are pairwise disjoint and $\mathcal{G} \supseteq \bigcup_{B \in \mathcal{B}(G)} \mathcal{G}^B$.*
- (c) *For each vertex $v \in V(G)$ it holds $X_v = \bigcup_{B \in \mathcal{B}(G), v \in V(B)} X_v^B$.* ◇

In particular, if G itself is a block it follows from their theorem that (G, X, \mathcal{G}) is either a K-, or a C-configuration. Thus, by using their result, we could omit Claim 4.6.3 and 4.6.4. However, for the reader's convenience, the entire proof is displayed so that the reader receives a complete presentation how both the characterization of the "bad" blocks as well as the corresponding "bad" covers in the hypergraph and the graph case can be done simultaneously.

4.4. A Brooks-type Theorem for χ_{DP}

The next two corollaries are direct consequences of Theorem 4.2 and Proposition 4.3. The second corollary is the degree version of BROOKS' Theorem for DP-coloring of hypergraphs and thereby extends ERDŐS, RUBIN and TAYLOR'S Theorem 3.1 as well as KOSTOCHKA, STIEBITZ and WIRTH'S Theorem 3.2.

Corollary 4.8. *Let (H, X, \mathcal{H}) be a degree-feasible configuration. If (H, X, \mathcal{H}) is minimal uncolorable, then for each block $B \in \mathcal{B}(H)$ there is a uniquely determined cover (X^B, \mathcal{H}^B) of B such that the following statements hold:*

- (a) *For every block $B \in \mathcal{B}(H)$, (B, X^B, \mathcal{H}^B) is a K-configuration, C-configuration, or E-configuration.*
- (b) *The hypergraphs \mathcal{H}^B with $B \in \mathcal{B}(H)$ are pairwise disjoint and $\mathcal{H} = \bigcup_{B \in \mathcal{B}(H)} \mathcal{H}^B$.*
- (c) *For each vertex $v \in V(H)$ it holds $X_v = \bigcup_{B \in \mathcal{B}(H), v \in V(B)} X_v^B$.* ◇

Corollary 4.9. *A connected hypergraph H is not DP-degree-colorable if and only if each block of H is a DP-hyperbrick.* ◇

To conclude this chapter, we prove a BROOKS-type theorem for DP-colorings of hypergraphs. For graphs, the theorem was proved by BERNSHTEYN, KOSTOCHKA, and PRON [14].

Theorem 4.10. *Let H be a connected hypergraph. Then, $\chi_{\text{DP}}(H) \leq \Delta(H) + 1$ and equality holds if and only if H is a DP-hyperbrick. \diamond*

Proof. It follows from (4.1) that $\chi_{\text{DP}}(H) \leq \Delta(H) + 1$. Moreover, it is obvious that every DP-hyperbrick H satisfies $\chi_{\text{DP}}(H) = \Delta(H) + 1$, just take a K-, C-, or E-configuration. Now assume that $\chi_{\text{DP}}(H) = \Delta(H) + 1$. Then, there is a cover (X, \mathcal{H}) of H such that $|X_v| \geq \Delta(H)$ for all $v \in V(H)$ and H is not (X, \mathcal{H}) -colorable. Hence, (H, X, \mathcal{H}) is an uncolorable degree-feasible configuration and there is a spanning subhypergraph \mathcal{H}' of \mathcal{H} such that (H, X, \mathcal{H}') is minimal uncolorable. Then, H is regular (by Proposition 4.5(a)) and each block of H is a DP-hyperbrick (by Theorem 4.2). As every DP-hyperbrick is regular, this implies that H has only one block and, therefore, is a DP-hyperbrick. This completes the proof. \blacksquare

Chapter 5

COLORING HYPERGRAPHS OF LOW CONNECTIVITY

In this chapter we only examine proper colorings of hypergraphs. As the chromatic number of a hypergraph is not affected by parallel edges, it is sufficient to examine only hypergraphs in which parallel edges are absent. This offers several benefits: we may now regard a hypergraph as a pair $H = (V, E)$ where E is a subset of the power set 2^V and $i(e) = e$ for all edges $e \in E(H)$. Hence, for a vertex $v \in V(H)$, we have $E_H(v) = \{e \in E(H) \mid v \in e\}$ and, for a vertex set $X \subseteq V(H)$, we obtain

$$E_H(X) = \{e \in E(H) \mid e \cap X \neq \emptyset \text{ and } e \cap (V(H) \setminus X) \neq \emptyset\}.$$

Note that $E_H(X) = E_H(V(H) \setminus X)$. In addition, many other concepts can be simplified, too. For instance, the subhypergraph $H[X]$ of H induced by X is the hypergraph with vertex set $V(H[X]) = X$ and edge set $E(H[X]) = \{e \in E(H) \mid e \subseteq X\}$. Nonetheless, we need to make a minor adjustment to the shrinking operation. Now, shrinking a hypergraph H to a vertex set $X \subseteq V(H)$ results in a hypergraph $H(X)$ with vertex set $V(H(X)) = X$ and edge set

$$E(H(X)) = \{e \cap X \mid e \in E(H) \text{ and } |e \cap X| \geq 2\}.$$

Thus, unlike in the previous chapters, all parallel edges that would occur while shrinking get replaced by a single edge. Again, let $H \div X = H(V(H) \setminus X)$. We would like to point out

that it is still possible that an edge is contained in another; a hypergraph H is simple if and only if this is forbidden. Since graphs are 2-uniform hypergraphs, all graphs considered in this chapter are assumed to be simple.

The aim of this chapter is to examine the relation between the chromatic number of a hypergraph and its local edge connectivity. Let H be a hypergraph with at least two vertices. The **local edge connectivity** $\lambda_H(\mathbf{u}, \mathbf{v})$ of distinct vertices \mathbf{u}, \mathbf{v} in the hypergraph H is the maximum number of edge-disjoint (\mathbf{u}, \mathbf{v}) -hyperpaths of H . The **maximum local edge connectivity** of a hypergraph H is

$$\lambda(H) = \max\{\lambda_H(\mathbf{u}, \mathbf{v}) \mid \mathbf{u}, \mathbf{v} \in V(H), \mathbf{u} \neq \mathbf{v}\}.$$

If H has at most one vertex, we set $\lambda(H) = 0$. The local edge connectivity is an important parameter for hypergraphs. For instance, it is well known that MENGER'S Theorem also holds for hypergraphs (see [47, Theorem 2.5.28] and [67]). It states the following.

Theorem 5.1. *If H is a hypergraph and \mathbf{u}, \mathbf{v} are distinct vertices of H , then*

$$\lambda_H(\mathbf{u}, \mathbf{v}) = \min\{|E_H(X)| \mid \mathbf{u} \in X \subseteq V(H) \setminus \{\mathbf{v}\}\}.$$

But what is the connection between the local edge connectivity and an optimal proper coloring of a hypergraph? By a result of TOFT [117], each hypergraph H satisfies $\chi(H) \leq \lambda(H) + 1$. As the maximum degree $\Delta(H)$ of H is a trivial upper bound for $\lambda(H)$, this immediately raises the question if there is a related BROOKS-type result, *i.e.*, if it is possible to find a nice characterization of the class of hypergraphs for which equality hold. In this chapter, we show that for all values of $\lambda(H)$ except for $\lambda(H) = 2$ there is indeed such a characterization (the case $\lambda(H) = 2$ is still open). To this end, we use a famous construction by HAJÓS [54], which was extended to hypergraphs by TOFT [116].

Let H_1 and H_2 be two vertex disjoint hypergraphs and, for $i \in \{1, 2\}$, let $e_i \in E(H_i)$ and $v_i \in e_i$. Then, we create a new hypergraph H by deleting e_1 and e_2 , identifying the vertices v_1 and v_2 to a new vertex v^* , and adding a new edge $e^* \in E(H)$ either with $e^* = (e_1 \cup e_2) \setminus \{v_1, v_2\}$ or with $e^* = (e_1 \cup e_2 \cup v^*) \setminus \{v_1, v_2\}$. Then, H is a **HAJÓS join** of H_1 and H_2 and we write $H = (H_1, v_1, e_1) \nabla (H_2, v_2, e_2)$ or, briefly, $H = H_1 \nabla H_2$. Figure 5.1 shows the two possible HAJÓS joins of two K_4 .

For an integer $k \geq 3$ we define a class \mathcal{H}_k of hypergraphs as follows. Let \mathcal{H}_3 be the smallest class of hypergraphs that contains all odd wheels and is closed under taking HAJÓS joins. Moreover, for $k \geq 4$, let \mathcal{H}_k be the smallest class of hypergraphs that contains all

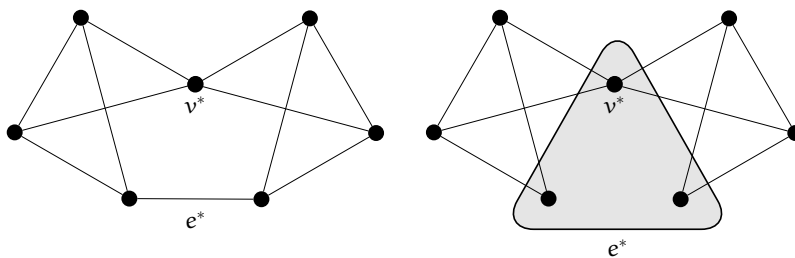


FIG. 5.1. The two possible HAJÓS joins of two K_4 .

complete graphs of order $k + 1$ and is closed under taking HAJÓS joins.

Recall that a **block** of a hypergraph H is a maximal connected subhypergraph of H that does not contain a separating vertex. It is well known that any two blocks of H have at most one vertex in common. In particular,

$$\chi(H) = \max\{\chi(B) \mid B \text{ is a block of } H\}. \tag{5.1}$$

This is due to the fact that if we have optimal proper colorings of the blocks of H , then, by permuting the colors in the blocks, we can create an optimal proper coloring of H .

The next theorem is the main result of this chapter and generalizes BROOKS' Theorem for hypergraphs (see Theorem 2.2). The graph-counterpart was proved by ABOULKER, BRETTELL, HAVET, MARX, and TROTIGNON [1] for graphs G with $\lambda(G) = 3$ and by STIEBITZ and TOFT [111] for $\lambda(G) \geq 4$.

Theorem 5.2. *Let H be a hypergraph with $\lambda(H) \geq 3$. Then, $\chi(H) \leq \lambda(H) + 1$ and equality holds if and only if H has a block belonging to the class $\mathcal{H}_{\lambda(H)}$.* \diamond

Note that for $\lambda(H) \in \{0, 1\}$, it is obvious that a connected hypergraph H satisfies $\chi(H) = \lambda(H) + 1$ if and only if $\lambda(H) = 0$ and $H = K_1$, or $\lambda(H) = 1$ and each block of H consists of just one edge. The case $\lambda(H) = 2$ has not yet been solved in a satisfactory way, that is, we do not know with certainty what \mathcal{H}_2 is. In the graph case, however, \mathcal{H}_2 just consists of all cycles having odd length.

5.1. Connectivity of Critical Hypergraphs

In order to prove Theorem 5.2, we use the concept of **critical hypergraphs**. Critical graphs were introduced by DIRAC in his Ph.D. thesis and the resulting papers [38] and [39]. His concept was extended to hypergraphs by LOVÁSZ [81]. We say that a hypergraph H is

$(k + 1)$ -critical or, briefly, **critical** if $\chi(H) = k + 1$, but $\chi(H') \leq k$ for any proper subhypergraph H' of H . Note that, unlike in Section 3.5 where we only regarded vertex-critical hypergraphs, H' does not have to be an induced subhypergraph of H . Critical hypergraphs are a useful concept in chromatic number theory as many problems can be reduced to critical hypergraphs. In particular, each hypergraph H contains a critical hypergraph H' with $\chi(H') = \chi(H)$. The next two propositions state some well known facts about critical hypergraphs.

Proposition 5.3. *Let H be a hypergraph with $\delta(H) \geq 1$, and let $k \geq 1$ be an integer. Then, H is $(k + 1)$ -critical if and only if $\chi(H - e) \leq k < \chi(H)$ for each edge $e \in E(H)$. \diamond*

It is easy to see that K_1 is the only 1-critical hypergraph and that the only 2-critical hypergraphs are the connected hypergraphs that contain only one edge. Regarding graphs, it is also easy to obtain that the only 3-critical graphs are the odd cycles. However, it seems unlikely that there is a good characterization of 3-critical hypergraphs as even the decision whether a given hypergraph H satisfies $\chi(H) \leq 2$ is NP-complete (see [82]).

Proposition 5.4. *Let H be a $(k + 1)$ -critical hypergraph for some integer $k \geq 1$. Then, the following statements hold:*

- (a) $\delta(H) \geq k$, in fact each vertex v is contained in k edges having pairwise only v in common.
- (b) $\lambda_H(u, v) \geq k$ for distinct vertices $u, v \in V(H)$.
- (c) H is a block.
- (d) H is a simple hypergraph. \diamond

Statement (a) follows from the fact that there is a proper coloring of $H - v$ with color set $\Gamma = [1, k]$. This coloring, however, cannot be extended to a proper k -coloring of H , and therefore for each color $\alpha \in \Gamma$ there is an edge in $E_H(v)$ where all vertices have color α , except v . This proves (a). Statement (b) was proved by TOFT in [116]; we also give a proof in Theorem 5.10. Statement (c) is a direct consequence of (5.1), and (d) is obvious.

Similar to Section 3.5, where we examined (\mathcal{P}, L) -vertex-critical hypergraphs, Proposition 5.4(a) leads to a classification of the vertices of critical hypergraphs. Let H be a $(k + 1)$ -critical hypergraph. Then, a vertex is said to be a **low vertex** of H if it has degree k in H , and a **high vertex**, otherwise. Thus, each high vertex of H has degree at least $k + 1$ in H . Clearly, every $(k + 1)$ -critical hypergraph H satisfies $\chi(H - v) < \chi(H)$ for all

$v \in V(H)$. As a consequence, H is also (\mathcal{C}, L) -vertex-critical, where L is the list-assignment with $L(v) = [1, k]$ for all $v \in V(H)$ and so the set V_L of low vertices of H coincides with $V(H, \mathcal{C}, L)$. Thus, it follows from Theorem 3.14 that the **low vertex hypergraph** $H(V_L)$ of H , *i.e.*, the subhypergraph of H that results from shrinking H to the set V_L , is a GALLAI forest. Recall that a connected hypergraph is a **GALLAI tree** if each of its blocks is a complete graph, an odd cycle, or consists of just one hyperedge, and that a **GALLAI forest** is a hypergraph whose components are all GALLAI trees.

GALLAI [48] characterized the critical graphs having exactly one high vertex. A similar characterization holds for hypergraphs; however, we only need the following easy observation. Recall that a **hyperwheel** is a hypergraph obtained from a (hyper-)edge by adding a new vertex and joining it to all others by ordinary edges.

Lemma 5.5. *Let H be a $(k+1)$ -critical hypergraph for some integer $k \geq 2$. If H has exactly one high vertex, then either H has a separating vertex set of size 2, or $k = 2$ and H is a hyperwheel, or $k = 3$ and H is an odd wheel. \diamond*

Proof. Let v be the only high vertex of H . Then, $V_L = V(H) \setminus \{v\}$ is the set of low vertices of H and $H(V_L) = H \div v$. By Theorem 3.14, $H(V_L)$ is a GALLAI forest. As H is a block (by Proposition 5.4(c)), $H(V_L)$ is connected and therefore a GALLAI tree. Let B be an end-block of $H(V_L)$. If B is not the only block of $H(V_L)$, then $H(V_L)$ has a separating vertex u and $\{v, u\}$ is a separating vertex set of H , so we are done. Otherwise, $H(V_L) = B$, where B is a complete graph, an odd cycle, or consists of just one hyperedge. In particular, B is regular. We claim that $E_H(v)$ contains only ordinary edges. Assume, to the contrary, that $E_H(v)$ contains a hyperedge e . Then, as H is simple (by Proposition 5.4(d)), we have $d_B(w) = d_H(w) = k$ for all $w \in e \setminus \{v\}$, and so B is k -regular. As each low vertex of H has degree k in H , it follows that $E_H(v)$ contains only hyperedges. Since v is a high vertex of H , this implies that $|B| \geq 3$ and B is a complete graph or an odd cycle and, so, $|e'| = 3$ for all $e' \in E_H(v)$. Let H' be the hypergraph that results from H by replacing e with the ordinary edge vw for one vertex $w \in e \setminus \{v\}$. Clearly, $\chi(H) \leq \chi(H')$. Moreover, H' is connected (as H is a block and $|B| \geq 3$) and we have $d_{H'}(u) = k - 1$ for the vertex $u \in e \setminus \{v, w\}$, and $d_{H'}(u') = k$ for all $u' \in V_L \setminus \{u\}$. This implies that H' is strictly k -degenerate and so the coloring number of H' is at most k . Then, by (1.3),

$$\chi(H) \leq \chi(H') \leq \text{col}(H') \leq k,$$

which is impossible. This proves the claim that $E_H(v)$ contains only ordinary edges.

As a consequence, $H(V_L) = H[V_L]$. Since $H(V_L)$ is a GALLAI tree consisting only of the block B , this block B is regular of degree $k - 1$ and v is joined to each vertex of B by an ordinary edge. Then, $|B| \geq d_H(v) \geq k + 1$ and so $k = 2$ and B consists of just one edge, or $k = 3$ and B is an odd cycle. Thus, $k = 2$ and H is a hyperwheel, or $k = 3$ and H is an odd wheel, as claimed. \blacksquare

As was previously noted, a critical graph is connected and contains no separating vertex. DIRAC [38] as well as GALLAI [48] characterized critical graphs having a separating vertex set of size 2. The next theorem is the hypergraph counterpart. For a hypergraph H , by $\mathcal{C}\mathcal{O}_k(H)$ we denote the set of all proper k -colorings of H , *i.e.*, all proper colorings of H with color set $[1, k]$.

Theorem 5.6. *Let H be a $(k + 1)$ -critical hypergraph for an integer $k \geq 2$, and let $S \subseteq V(H)$ be a separating vertex set of H satisfying $|S| \leq 2$. Then S is an independent set of H consisting of two vertices, say v and w , and $H \div S$ has exactly two components H_1 and H_2 . Moreover, if $H_i = H[V(H_i) \cup S]$ for $i \in \{1, 2\}$, we can adjust the notation so that for a coloring $\varphi_1 \in \mathcal{C}\mathcal{O}_k(H_1)$ we have $\varphi_1(v) = \varphi_1(w)$. Then, the following statements hold:*

- (a) *Each coloring $\varphi \in \mathcal{C}\mathcal{O}_k(H_1)$ satisfies $\varphi(v) = \varphi(w)$ and each coloring $\varphi \in \mathcal{C}\mathcal{O}_k(H_2)$ satisfies $\varphi(v) \neq \varphi(w)$.*
- (b) *The hypergraph $H'_1 = H_1 + vw$ obtained from H by adding the edge vw is $(k + 1)$ -critical.*
- (c) *The hypergraph H'_2 obtained from H_2 by identifying v and w is $(k + 1)$ -critical. \diamond*

Proof. Since H is $(k + 1)$ -critical with $k \geq 2$, the separating set S consists of exactly two elements, say $S = \{v, w\}$. Then, H is the union of two induced subhypergraphs H_1 and H_2 with $V(H_1) \cap V(H_2) = \{v, w\}$ and $|H_i| > 2$ for $i \in \{1, 2\}$. Since H_i is a proper subhypergraph of H , there is a coloring $\varphi_i \in \mathcal{C}\mathcal{O}_k(H_i)$ ($i \in \{1, 2\}$). Then, for one coloring, say φ_1 , we have $\varphi_1(v) = \varphi_1(w)$ and for φ_2 , we have $\varphi_2(v) \neq \varphi_2(w)$. For otherwise, we could permute the colors in one coloring such that $\varphi_1(v) = \varphi_2(v)$ and $\varphi_1(w) = \varphi_2(w)$ so that $\varphi_1 \cup \varphi_2$ would be a proper k -coloring of H , which is impossible. Consequently, S is an independent set of H . Furthermore it follows that each coloring $\varphi \in \mathcal{C}\mathcal{O}_k(H_1)$ satisfies $\varphi(v) = \varphi(w)$ and each coloring $\varphi \in \mathcal{C}\mathcal{O}_k(H_2)$ satisfies $\varphi(v) \neq \varphi(w)$. Hence, (a) is proved.

For the proof of (b), let $H'_1 = H_1 + vw$. Then, it follows from (a) that $\chi(H'_1) \geq k + 1$. Let e be an arbitrary edge of H'_1 . We show that $H'_1 - e$ admits a proper k -coloring. If $e = vw$, this is evident. Otherwise, $e \in E(H_1)$ and there is a proper k -coloring φ of $H - e$. By (a),

it follows that $\varphi(v) \neq \varphi(w)$ and so φ induces a proper k -coloring of $H'_1 - e$. Hence, H'_1 is $(k+1)$ -critical (see Proposition 5.3).

In order to prove (c), let H'_2 be the hypergraph obtained from H_2 by identifying v and w to a new vertex v^* . Then, by (a), $\chi(H'_2) \geq k+1$. Let e be an arbitrary edge of H'_2 and let e' be a corresponding edge of H_2 . Then, $H - e'$ admits a proper k -coloring φ and, by (a), $\varphi(v) = \varphi(w)$ and so φ induces a proper k -coloring of $H'_2 - e$. Hence, H'_2 is $(k+1)$ -critical.

Finally, we obtain that

$$H \div S = (H_1 \div S) \cup (H_2 \div S) = (H'_1 \div S) \cup (H'_2 \div v^*).$$

Since S is not an independent set of H'_1 and since H'_1 is critical, $H'_1 \div S$ is connected. Moreover, since H'_2 is critical, $H'_2 \div v^*$ is connected. This proves that $H \div S$ has exactly two components H_1 and H_2 as claimed and the proof is complete. ■

Theorem 5.7. *Let $H = (H_1, v_1, e_1) \nabla (H_2, v_2, e_2)$ be a Hajós join of two hypergraphs H_1 and H_2 , and let $k \geq 2$ be an integer. Then, the following statements hold:*

- (a) *If both H_1 and H_2 are $(k+1)$ -critical, then H is $(k+1)$ -critical.*
- (b) *If H is $(k+1)$ -critical and $k \geq 3$, then both H_1 and H_2 are $(k+1)$ -critical.* ◊

Proof. For the proof of (a), assume that both H_1 and H_2 are $(k+1)$ -critical. First we claim that $\chi(H) \geq k+1$. For otherwise, there is a coloring $\varphi \in \mathcal{C}\mathcal{O}_k(H)$. Then, there are vertices $x \neq y$ from e^* such that $\varphi(x) \neq \varphi(y)$ and at least one vertex, say x , satisfies $\varphi(x) \neq \varphi(v^*)$. By symmetry, we may assume $x \in V(H_1)$. However, then the mapping φ_1 with $\varphi_1(u) = \varphi(u)$ for all $u \in V(H_1) \setminus \{v_1\}$ and $\varphi_1(v_1) = \varphi(v^*)$ is a proper k -coloring of H_1 and, thus, $\chi(H_1) \leq k$, a contradiction. Hence, $\chi(H) \geq k+1$. In order to see that H is $(k+1)$ -critical, let $H' = H - e$ for some edge $e \in E(H)$. If $e = e^*$, then, as H_1 and H_2 are critical, we can create a proper k -coloring φ of H' by choosing proper k -colorings φ_1 of $H_1 - e_1$ and φ_2 of $H_2 - e_2$, permuting the colors such that $\varphi_1(v_1) = \varphi_2(v_2)$, and setting $\varphi(u) = \varphi_i(u)$ if $u \in V(H_i)$. If $e \neq e^*$, then $e \in E(H_i)$ for some $i \in \{1, 2\}$, say $e \in E(H_1)$. Then, $H_1 - e$ admits a proper k -coloring φ_1 and there is a vertex $u \in e_1$ with $\varphi_1(u) \neq \varphi_1(v_1)$. Moreover, $H_2 - e_2$ admits a proper k -coloring φ_2 and all vertices from e_2 have the same color. Again by permuting the colors it is easy to see that one can create a proper k -coloring of H . Thus H is $(k+1)$ -critical, and (a) is proved.

In order to prove (b) assume that H is $(k+1)$ -critical with $k \geq 3$. By symmetry, it suffices to show that H_1 is $(k+1)$ -critical, as well. Clearly, if $\chi(H_1) \leq k$, then there is a proper

k -coloring φ_1 of H_1 with $\varphi_1(u) = \alpha \neq \beta = \varphi_1(v_1)$ for at least one $u \in e_1$. Moreover, as H is $(k+1)$ -critical and since $k \geq 3$, there is a proper k -coloring of $H - e^*$ and hence a proper k -coloring φ_2 of $H_2 - e_2$ such that $\varphi_2(v_2) = \beta$ and $\varphi_2(u') \neq \alpha$ for at least one $u' \in e_2 \setminus \{v_2\}$. Then, the union of the colorings φ_1 and φ_2 is a proper k -coloring of H , a contradiction. Thus, $\chi(H_1) \geq k+1$. Similarly, one can show that $\chi(H_2) \geq k+1$. Now let $H'_1 = H_1 - e$ for some $e \in E(H_1)$. If $e = e_1$, then the restriction of any proper k -coloring φ of $H - e^*$ to $V(H_1)$ is a proper k -coloring of H'_1 and we are done. If $e \neq e_1$, then there is a proper k -coloring φ of $H - e$. If $\varphi(u) \neq \varphi(v^*)$ for at least one $u \in e^* \cap V(H_1)$, we are done. Otherwise, there is a vertex $u \in e \cap V(H_2)$ with $\varphi(u) \neq \varphi(v^*)$ and the restriction of φ to $V(H_2)$ is a proper k -coloring of H_2 , a contradiction to $\chi(H_2) \geq k+1$. This proves (b). ■

Note that (b) does not hold for $k = 2$, not even in the graph case as demonstrated for example by a cycle C_7 being obtained as HAJÓS join of two cycles C_4 .

Let H be a connected hypergraph, $v \in V(H)$, and $e \in E(H)$. Then, $\{v, e\}$ is a **separating set** (consisting of one edge and one vertex) if v is a separating vertex of $H - e$ (no matter whether $v \in e$ or not).

Theorem 5.8. *Let H be a $(k+1)$ -critical hypergraph with $k \geq 3$. If H has a separating set consisting of one edge and one vertex, then H is a HAJÓS join of two hypergraphs.* ◊

Proof. There is a vertex $v^* \in V(H)$ and an edge $e^* \in E(H)$ such that $H - e^* = H_1 \cup H_2$ with $V(H_1) \cap V(H_2) = \{v^*\}$ and $|H_i| \geq 2$ for $i \in \{1, 2\}$. As H is a block (by Proposition 5.4(c)), $e^* \cap V(H_i) \neq \emptyset$ for $i \in \{1, 2\}$. For $i \in \{1, 2\}$, let $e_i = (e^* \cap V(H_i)) \cup \{v^*\}$. If we can show that $e_i \notin E(H)$, then H is the HAJÓS join of $H_1 + e_1$ and $H_2 + e_2$, and we are done. By symmetry, assume that $e_1 \in E(H)$. As H is $(k+1)$ -critical, there is a proper k -coloring φ of $H - e^*$ and all vertices from e^* have the same color α . Moreover, as $e_1 \in E(H)$, v^* has a color $\beta \neq \alpha$. Since $k \geq 3$, there is a color $\gamma \notin \{\alpha, \beta\}$. By coloring all vertices from H_2 having color α with γ and vice versa, we obtain a proper k -coloring of H , a contradiction. This completes the proof. ■

The next theorem examines decompositions of $(k+1)$ -critical hypergraphs having a separating edge set of size k . A **separating edge set** of a hypergraph H is a set $F \subseteq E(H)$ such that $H - F$ has more components than H . If F is a separating edge set and there is no proper subset of F that is also a separating edge set, then F is said to be a **minimal separating edge set**. It is well known that if F is a minimal separating edge set of a connected hypergraph H , then $F = E_H(X)$ for some non-empty proper subset X of $V(H)$. A hypergraph H is **k -edge-connected** for an integer $k \geq 1$ if $|H| \geq 2$ and $H - F$ is connected

for any set $F \subseteq E(H)$ with $|F| \leq k-1$. Consequently, MENGER's Theorem 5.1 implies that H is k -edge-connected if and only if the minimum local edge connectivity over all pairs (u, v) of distinct vertices of H is at least k . Thus, it follows from Proposition 5.4(b) that every $(k+1)$ -critical hypergraph is k -edge-connected.

Now let H be an arbitrary hypergraph. An **edge cut** of H is a triple (X, Y, F) such that X is a non-empty proper subset of $V(H)$, $Y = V(H) \setminus X$, and $F = E_H(X) = E_H(Y)$. If (X, Y, F) is an edge cut of H , by X_F (respectively Y_F) we denote the set of vertices of X (respectively Y) that are incident to some edge of F . An edge cut (X, Y, F) of H is non-trivial if $|X_F| \geq 2$ and $|Y_F| \geq 2$.

That a $(k+1)$ -critical graph is k -edge-connected was proved by Dirac [39]. A characterization of $(k+1)$ -critical graphs having a separating edge set of size k was given by TOFT [117] and, independently, by GALLAI (oral communication to Bjarne TOFT). GALLAI used the following lemma about complements of bipartite graphs. The **clique number** $\omega(G)$ of a graph G is the maximum integer n such that K_n is a subgraph of G . A graph G is **perfect** if each induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. It is well known that complements of bipartite graphs are perfect. For the reader's convenience we repeat the proof of the following lemma from [111].

Lemma 5.9. *Let G be a graph and let $k \geq 3$ be an integer. Suppose that (A, B, F') is an edge cut of G such that $|F'| \leq k$ and A as well as B are cliques of G with $|A| = |B| = k$. If $\chi(G) \geq k+1$, then $|F'| = k$ and $F' \subseteq E_G(v)$ for some vertex v of G .*

Proof. The graph G is perfect and so $\omega(G) = \chi(G) \geq k+1$. Consequently, G contains a clique X with $|X| = k+1$. Let $s = |A \cap X|$ and hence $k+1-s = |B \cap X|$. Since $|A| = |B| = k$, this implies that $s \geq 1$ and $k+1-s \geq 1$. Since X is a clique of H , the set E' of edges of G joining a vertex of $A \cap X$ with a vertex of $B \cap X$ satisfies $E' \subseteq F'$ and $|E'| = s(k+1-s)$. Clearly, the function $g(s) = s(k+1-s)$ is strictly concave on the real interval $[1, k]$ as $g''(s) = -2$. Since $g(1) = g(k) = k$, we conclude that $g(s) > k$ for all $s \in (1, k)$. Since $g(s) = |E'| \leq |F'| \leq k$, this implies that $s = 1$ or $s = k$. In both cases we obtain that $E' = F' \subseteq E_G(v)$ for some vertex v of G and $|E'| = |F'| = k$. ■

Theorem 5.10. *Let H be a $(k+1)$ -critical hypergraph with $k \geq 2$, and let $F \subseteq E(H)$ be a separating edge set of H with $|F| \leq k$. Then, $|F| = k$ and there is an edge cut (X, Y, F) of H satisfying the following properties:*

- (a) *Every proper k -coloring φ of $H[X]$ satisfies $|\varphi(X_F)| = 1$, every proper k -coloring φ of $H[Y]$ satisfies $|\varphi(Y_F)| = k$ and for every color $i \in [1, k]$ there is an edge $e \in F$ such that*

$$\varphi(e \cap Y) = \{i\}.$$

- (b) Each vertex of Y_F is incident to exactly one edge of F .
- (c) If $|X_F| \geq 2$, then the hypergraph H_1 obtained from $H[X]$ by adding the hyperedge with vertex set X_F is $(k+1)$ -critical.
- (d) The hypergraph H_2 obtained from $H[Y]$ by adding a new vertex v and adding for each edge $e \in F$ the new edge $(e \setminus X) \cup \{v\}$ is $(k+1)$ -critical. \diamond

Proof. We may assume that F is a minimal separating edge set of H and, hence, there exists an edge cut (X, Y, F) of H . Since H is $(k+1)$ -critical, for every set $Z \in \{X, Y\}$ there is a coloring $\varphi_Z \in \mathcal{C}\mathcal{O}_k(H[Z])$. Now we construct an auxiliary graph G as follows. The vertex set of G consists of two disjoint cliques A and B with $|A| = |B| = k$, say $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$. The edge set of G consists of the edges of the cliques A and B and an additional edge set $F' \subseteq E_G(A) = E_G(B)$. An edge $a_i b_j$ belongs to F' if and only if there is an edge $e \in F$ such that $\varphi_X(e \cap X) = \{i\}$ and $\varphi_Y(e \cap Y) = \{j\}$. Then $|F'| \leq k$ and we claim that $\chi(G) \geq k+1$. For otherwise, there exists a coloring $\varphi' \in \mathcal{C}\mathcal{O}_k(G)$ and we may assume that $\varphi'(a_i) = i$ and $\varphi'(b_j) = \pi(j)$ for a permutation $\pi \in \mathcal{S}_k$. Then $\varphi'_Y = \pi \circ \varphi_Y$ belongs to $\mathcal{C}\mathcal{O}_k(H[Y])$ and the function $\varphi_X \cup \varphi'_Y$ belongs to $\mathcal{C}\mathcal{O}_k(H)$, which is impossible. This proves the claim that $\chi(G) \geq k+1$. From Lemma 5.9 it then follows that $|F'| = k$ and $F' \subseteq E_G(v)$ for some vertex $v \in V(G) = A \cup B$. By symmetry, we may assume that $v \in A$. Then, $|F'| = |F|$ and we conclude that $|\varphi_X(X_F)| = 1$ (since otherwise $|F'| < |F|$ by definition of F' and as $F' \subseteq E_G(v)$) and so X_F is an independent set of H . Moreover, it follows that $|\varphi_Y(Y_F)| = k$ and for every color $i \in [1, k]$ there is an edge $e \in F$ such that $\varphi_Y(e \cap Y) = \{i\}$. If $\varphi \in \mathcal{C}\mathcal{O}_k(H[X])$ we can apply the same argument to the colorings φ and φ_Y , which leads to $|\varphi(X_F)| = 1$. Similar, if $\varphi \in \mathcal{C}\mathcal{O}_k(H[Y])$, we apply the same argument to the colorings φ_X and φ , and obtain $|\varphi(Y_F)| = k$. This proves (a) and (b).

For the proof of (c) assume that $|X_F| \geq 2$ and let H_1 be the hypergraph obtained from $H[X]$ by adding the hyperedge with vertex set X_F . By (a), $\chi(H_1) \geq k+1$. Let e be an arbitrary edge from H_1 . We show that $H_1 - e$ has a proper k -coloring. If $e = X_F$, this is evident. Otherwise, e belongs to $H[X]$ and since H is $(k+1)$ -critical, there is a proper k -coloring φ of $H - e$. Clearly, φ induces a proper k -coloring of $H[Y]$ and we conclude from (a) that $|\varphi(X_F)| \geq 2$. Hence, φ induces a proper k -coloring of $H_1 - e$. Consequently, H_1 is $(k+1)$ -critical (see Proposition 5.3).

In order to prove statement (d) let H_2 be the hypergraph obtained from $H[Y]$ by adding a new vertex v and adding for each edge $e \in F$ the new edge $(e \setminus X) \cup \{v\}$. By (a), $\chi(H_2) \geq k+1$.

Let e be an arbitrary edge of H_2 . We show that $H_2 - e$ admits a proper k -coloring. Let e' be the corresponding edge of H . Then, $e' \in F \cup E(H[Y])$. As H is $(k+1)$ -critical, there is a proper k -coloring φ of $H - e'$ and, by (a), $|\varphi(X_F)| = 1$. Hence, φ induces a proper k -coloring of $H_2 - e$ and we are done. ■

5.2. Proof of Theorem 5.2

Let H be a hypergraph with $\lambda(H) \geq 3$. Then, H contains a critical hypergraph H' with $\chi(H) = \chi(H')$. Furthermore, $\chi(H') \leq \lambda(H') + 1$ (by Proposition 5.4(b), respectively by Theorem 5.10 and Theorem 5.1). As λ is a monotone hypergraph parameter, *i.e.*, $\lambda(\tilde{H}) \leq \lambda(H)$ for any subhypergraph $\tilde{H} \subseteq H$, it follows $\chi(H) \leq \lambda(H) + 1$ and the first part of the main result is proved.

It remains to be shown that $\chi(H) = \lambda(H) + 1$ if and only if some block of H belongs to $\mathcal{H}_{\lambda(H)}$. We will show that the critical subhypergraph H' is a block of H which belongs to $\mathcal{H}_{\lambda(H)}$. For an integer $k \geq 2$, let \mathcal{C}_k denote the class of hypergraphs H such that H is a critical hypergraph with chromatic number $k+1$ and with $\lambda(H) \leq k$. We first prove that $\mathcal{C}_k = \mathcal{H}_k$.

Theorem 5.11. *Let $k \geq 3$ be an integer. Then, the two classes \mathcal{C}_k and \mathcal{H}_k coincide.* ◊

Proof. The proof of Theorem 5.11 is divided into five claims. Proving the following claim is straightforward and therefore left to the reader.

Claim 5.11.1. *The odd wheels belong to the class \mathcal{C}_3 and the complete graphs of order $k+1$ belong to the class \mathcal{C}_k .* ◊

Claim 5.11.2. *Let $k \geq 3$ be an integer, and let $H = H_1 \nabla H_2$ be a HAJÓS join of two hypergraphs H_1 and H_2 . Then, H belongs to the class \mathcal{C}_k if and only if both H_1 and H_2 belong to the class \mathcal{C}_k .* ◊

Proof. Let $H = (H_1, v_1, e_1) \nabla (H_2, v_2, e_2)$, let v^* be the new vertex, and let e^* be the new edge of H . First suppose that H_1 and H_2 are from \mathcal{C}_k . Then, by Theorem 5.7, H is $(k+1)$ -critical. In order to show that $\lambda(H) \leq k$ let u and u' be distinct vertices of H and let $p = \lambda_H(u, u')$. Then, there is a system \mathcal{P} of p edge disjoint (u, u') -hyperpaths in H . If u and u' are both from H_1 , then only one hyperpath P of \mathcal{P} may contain vertices from H_2 (distinct from v^*). In this case, P contains the vertex v^* as well as the edge e^* . Let $u^* \in V(H_1)$ be the vertex from P such that u^* and e^* are consecutive in P . Then, replacing the subhyperpath u^*Pv^*

of P by the hyperpath $P' = (u^*, e_1, v_1)$ leads to a system of p edge disjoint (u, u') -paths in H_1 , and, thus, $p \leq \lambda_{H_1}(u, u') \leq k$. The same argument can be used if $u, u' \in V(H_2)$. It remains to consider the case that one vertex, say u , belongs to H_1 and the other vertex u' belongs to H_2 . By symmetry we may assume that $u \neq v^*$. Again at most one hyperpath P of \mathcal{P} uses the edge e^* and all other hyperpaths of \mathcal{P} contain the vertex $v^* (= v_1 = v_2)$. As before, let u^* be the vertex from $V(H_1)$ such that u^* and e^* are consecutive in P and let $P' = (u^*, e_1, v_1)$. If we replace P by the hyperpath $uPu^* + P'$, then we obtain p edge disjoint (u, v_1) -hyperpaths in H_1 , and thus, $p \leq \lambda_{H_1}(u, v_1) \leq k$. Hence, $\lambda(H) \leq k$ and so $H \in \mathcal{C}_k$.

Now suppose that $H \in \mathcal{C}_k$. As $k \geq 3$, it follows from Theorem 5.7(b) that both H_1 and H_2 are $(k + 1)$ -critical hypergraphs. It remains to be shown that $\lambda(H_i) \leq k$ for $i \in \{1, 2\}$. By symmetry, it is sufficient to prove that $\lambda(H_1) \leq k$. Let u and u' be distinct vertices of H_1 and let $p = \lambda_{H_1}(u, u')$. Then, there is a system \mathcal{P} of p edge disjoint (u, u') -hyperpaths in H_1 . At most one hyperpath P of \mathcal{P} may contain the edge e_1 . If v_1 and e_1 are not consecutive in P , replacing e_1 by e^* leads to a system of p edge-disjoint (u, u') -hyperpaths of H and so $p \leq \lambda_H(u, u') \leq k$ and we are done. So assume that v_1 and e_1 are consecutive in P . Let u'' be a vertex from $e_2 \setminus \{v_2\}$. As H_2 is critical, Proposition 5.4(b) implies that there is a (u'', v_2) -hyperpath P' , which does not contain the edge e_2 . So, replacing the edge e_1 in P by the sequence e^*P' , we get p edge-disjoint (u, u') -hyperpaths of H , and hence, $p \leq \lambda_H(u, u') \leq k$. Thus, $\lambda(H_1) \leq k$ and the claim is proved. \square

The next claim is a direct consequence of Claims 5.11.1 and 5.11.2.

Claim 5.11.3. *Let $k \geq 3$ be an integer. Then, \mathcal{H}_k is a subclass of \mathcal{C}_k .* \diamond

Claim 5.11.4. *Let $k \geq 3$ be an integer, and let H be a hypergraph from \mathcal{C}_k . If H does not admit a separating vertex set of size at most 2, then either $k = 3$ and H is an odd wheel, or $k \geq 4$ and H is a complete graph of order $k + 1$.* \diamond

Proof. The proof is by contradiction; we consider a counter-example H with minimum order $|H|$. Then, $H \in \mathcal{C}_k$ having no separating set of size at most 2 and either $k = 3$ and H is not an odd wheel, or $k \geq 4$ and H is not a complete graph of order $k + 1$. First we show that the set V_H of high vertices of H contains at least two vertices. If $V_H = \emptyset$, then, as H is a block and as $k \geq 3$, it follows from Theorem 3.14 that H is a complete graph of order $k + 1$, a contradiction. If $|V_H| = 1$, then Lemma 5.5 implies that $k = 3$ and H is an odd wheel, a contradiction. Thus, $|V_H| \geq 2$. Let u and v be distinct high vertices of H . As $H \in \mathcal{C}_k$, it follows from Proposition 5.4(b) that $\lambda(H) = k$ and, therefore, H contains a separating edge set F with $|F| = k$, which separates u and v . From Theorem 5.10 it follows that there is an

edge cut (X, Y, F) satisfying the four properties of that theorem. Since F separates u and v , we may assume that $u \in X$ and $v \in Y$. As u is a high vertex and H has no separating vertex set of size at most two, it follows that $|X_F| \geq 3$. Now we consider the hypergraph H_1 obtained from $H[X]$ by adding the hyperedge e with vertex set X_F . By Theorem 5.10(c), H_1 is $(k + 1)$ -critical. As H has no separating vertex set of size at most 2 and since $|X_F| \geq 3$, H_1 neither has.

Now we claim that $\lambda(H_1) \leq k$. To this end, let x and y be distinct vertices of H_1 and let \mathcal{P} be a set of $p = \lambda_{H_1}(x, y)$ edge disjoint (x, y) -hyperpaths of H_1 . Then, at most one hyperpath P contains the edge e . The hyperpath P contains a subhyperpath $P' = (z, e, z')$. Then, there is a (z, z') -hyperpath P^* in H containing only edges of F and $H[Y]$. This follows from Theorem 5.10(d). By replacing the hyperpath P' by P^* we obtain a system of p edge-disjoint (x, y) -hyperpaths in H and so $p \leq \lambda_H(x, y) \leq k$. Hence, $\lambda(H_1) \leq k$ and so $H_1 \in \mathcal{C}_k$. Clearly, $|H_1| < |H|$ and either $k = 3$ and H_1 is not an odd wheel, or $k \geq 4$ and H_1 is not a complete graph of order $k + 1$. This gives a contradiction to the choice of H . Thus, the claim is proved. \square

Claim 5.11.5. *Let $k \geq 3$ be an integer, and let H be a hypergraph from \mathcal{C}_k . If H has a separating vertex set of size 2, then $H = H_1 \nabla H_2$ is the HAJÓS join of two hypergraphs H_1 and H_2 , which both belong to \mathcal{C}_k .* \diamond

Proof. If H has a separating set consisting of one edge and one vertex, then Theorem 5.8 implies that H is the HAJÓS join of two hypergraphs H_1 and H_2 . By Claim 5.11.2 it then follows that both H_1 and H_2 belong to \mathcal{C}_k and we are done. It remains to consider the case that H does not contain a separating set consisting of one edge and one vertex. By assumption, there is a separating vertex set of size 2, say $S = \{v, w\}$. Then, Theorem 5.6 implies that $H \div S$ has exactly two components H_1 and H_2 such that the hypergraphs $H_i = H[V(H_i) \cup S]$ with $i \in \{1, 2\}$ satisfy the three properties of this theorem. In particular, we get that $H'_1 = H_1 + vw$ is a $(k + 1)$ -critical hypergraph. By Proposition 5.4(b) it then follows that $\lambda_{H'_1}(v, w) \geq k$ implying that $\lambda_{H_1}(v, w) \geq k - 1$. As $H \in \mathcal{C}_k$, $\lambda_H(v, w) \leq k$, which implies that $\lambda_{H_2}(v, w) \leq 1$. Since H_2 is connected, this implies that H_2 has a separating edge e . But then, $\{v, e\}$ or $\{w, e\}$ is a separating set of H consisting of one edge and one vertex, a contradiction. \square

As a consequence of Claim 5.11.4 and Claim 5.11.5, we conclude that the class \mathcal{C}_k is contained in the class \mathcal{H}_k and so $\mathcal{C}_k = \mathcal{H}_k$, as claimed. \blacksquare

Proof of Theorem 5.2. In order to complete the proof of Theorem 5.2, let H be a hypergraph with $\lambda(H) = k$ and $k \geq 3$. As shown at the beginning of the section, we have $\chi(H) \leq k+1$. If one block B of H belongs to \mathcal{H}_k , then $B \in \mathcal{C}_k$ (by Theorem 5.11) and hence $\chi(H) = k+1$ (by (5.1)).

Assume conversely that $\chi(H) = k+1$. Then, H contains a critical subhypergraph H' such that $\chi(H') = k+1$. Since $\lambda(H') \leq \lambda(H) \leq k$, $H' \in \mathcal{C}_k$. By Proposition 5.4(c), H' contains no separating vertex. We claim that H' is a block of H . Otherwise, H' would be a proper subhypergraph of a block B of H . This implies that there are distinct vertices v and w in H' which are joined by a hyperpath P of H satisfying $E(P) \cap E(H') = \emptyset$. Since $\lambda_{H'}(v, w) \geq k$ (by Proposition 5.4(c)), this implies that $\lambda_H(v, w) \geq k+1$ and thus $\lambda(H) \geq k+1$, a contradiction. This proves the claim that H' is a block of H . As $\mathcal{C}_k = \mathcal{H}_k$ by Theorem 5.11, it follows that $H' \in \mathcal{H}_k$. This completes the proof of the theorem.

5.3. Splitting Operation

In this section, we want to characterize the $(k+1)$ -critical hypergraphs having a separating edge set of size k . As we have already demonstrated in Theorem 5.10, these hypergraphs can be decomposed into smaller critical hypergraphs. We now want to introduce a reverse operation, called **splitting**. Let H_1 and H_2 be two disjoint hypergraphs, let $\tilde{e} \in E(H_1)$ and $\tilde{v} \in V(H_2)$. Furthermore, let $s : E_{H_2}(\tilde{v}) \rightarrow 2^{\tilde{e}}$ be a mapping such that $s(e) \neq \emptyset$ for all $e \in E_{H_2}(\tilde{v})$ and

$$\bigcup_{e \in E_{H_2}(\tilde{v})} s(e) = \tilde{e}.$$

Now let H be the hypergraph with vertex set $V(H) = V(H_1) \cup (V(H_2) \setminus \{\tilde{v}\})$ and edge set

$$E(H) = (E(H_1) \setminus \{\tilde{e}\}) \cup (E(H_2) \setminus E_{H_2}(\tilde{v})) \cup \{(e - \{\tilde{v}\}) \cup s(e) \mid e \in E_{H_2}(\tilde{v})\}.$$

We then say that H is obtained from H_1 and H_2 by **splitting** the vertex \tilde{v} into the edge \tilde{e} , and we briefly write $H = S(H_1, \tilde{e}, H_2, \tilde{v}, s)$. If $|s(e)| = 1$ for all $e \in E_{H_2}(\tilde{v})$, we call the splitting s a **simple splitting**. An example of splitting is displayed in Figure 5.2.

Theorem 5.12. *Let H_1 and H_2 be two disjoint $(k+1)$ -critical hypergraphs with $k \geq 2$, let $\tilde{e} \in E(H_1)$, and let $\tilde{v} \in V(H_2)$ be a low vertex of H_2 . Then the hypergraph $H = S(H_1, \tilde{e}, H_2, \tilde{v}, s)$ is $(k+1)$ -critical, too, and $F = E_H(V(H_1))$ is a separating edge set of size k . \diamond*

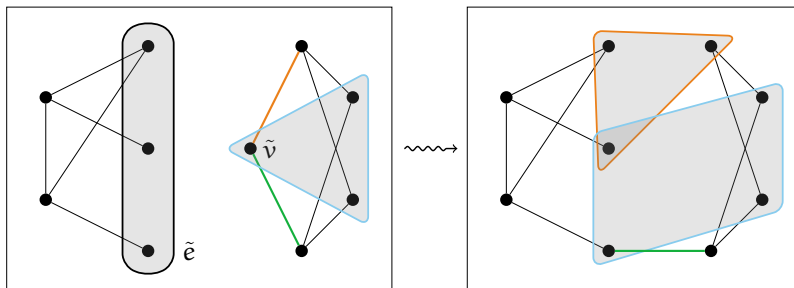


FIG. 5.2. The hypergraph on the right is obtained from splitting the vertex \tilde{v} into the edge \tilde{e} .

Proof. Since \tilde{v} is a low vertex of H_2 and so $E_{H_2}(\tilde{v})$ is a separating edge set of size k , for each coloring $\varphi \in \mathcal{C}\mathcal{O}_k(H_2 - \tilde{v})$ and for each color $i \in [1, k]$ there is an edge $e \in E_{H_2}(\tilde{v})$ with $\varphi(e \setminus \{\tilde{v}\}) = \{i\}$ (by Theorem 5.10). Furthermore, in each proper k -coloring φ of $H_1 - \tilde{e}$, the edge \tilde{e} is monochromatic with respect to φ . Consequently, $\chi(H) \geq k + 1$. It remains to show that $\chi(H - e) \leq k$ for all edges $e \in E(H)$. If $e \in E(H_1)$, then $H_1 - e$ admits a proper k -coloring φ_1 in which the edge \tilde{e} is not monochromatic. Hence, we can choose any proper k -coloring φ_2 of $H_2 - \tilde{v}$ and permute the colors such that $\varphi_1 \cup \varphi_2$ is a proper k -coloring of $H - e$ (see Lemma 5.9). If $e \notin E(H_1)$, we choose the corresponding edge $e' \in E(H_2)$. Then, there is a coloring $\varphi_2 \in \mathcal{C}\mathcal{O}_k(H_2 - e')$. Combining φ_2 with a coloring $\varphi_1 \in \mathcal{C}\mathcal{O}_k(H_1 - \tilde{e})$ results in a proper k -coloring of $H - e$. Thus, H is $(k + 1)$ -critical (see Proposition 5.3). By construction, F is a separating edge set with $|F| = d_{H_2}(\tilde{v}) = k$. This completes the proof. ■

Combining Theorem 5.6 with the next results provides a characterization of $(k + 1)$ -critical hypergraphs having a separating vertex set of size 2.

Theorem 5.13. *Let H_1 and H_2 be two disjoint $(k + 1)$ -critical hypergraphs with $k \geq 2$, let $\tilde{e} \in E(H_1)$ be an ordinary edge of H_1 , and let $\tilde{v} \in V(H_2)$ be an arbitrary vertex. Let $H = S(H_1, \tilde{e}, H_2, \tilde{v}, s)$ and let $H'_2 = H[(V(H_2) \setminus \{\tilde{v}\}) \cup \tilde{e}]$. If $\chi(H'_2) \leq k$, then H is a $(k + 1)$ -critical hypergraph and \tilde{e} is a separating vertex set of H of size 2. ◊*

Proof. Let $\tilde{e} = uw$ and $H'_1 = H_1 - \tilde{e}$. Then, H is the union of the two induced subhypergraphs H'_1 and H'_2 with $V(H'_1) \cap V(H'_2) = \{u, w\}$ and $|H'_1| > 2$ as $|H_1| \geq k + 1 \geq 3$. So $S = \{u, w\}$ is a separating set of H . Furthermore, H_1 is obtained from H'_1 by adding the edge uw , and H'_2 is obtained from H_2 by identifying u and v to the new vertex \tilde{v} . Since $\chi(H_2) = k + 1$ and $\chi(H'_2) \leq k$, each coloring $\varphi_2 \in \mathcal{C}\mathcal{O}_k(H'_2)$ satisfies $\varphi_2(u) \neq \varphi_2(w)$. Since H_1 is $(k + 1)$ -critical and $H'_1 = H_1 - uw$, each coloring $\varphi_1 \in \mathcal{C}\mathcal{O}_k(H'_1)$ satisfies $\varphi_1(u) = \varphi_1(w)$. Consequently, $\chi(H) \geq k + 1$. Now let e be an arbitrary edge of H . It remains to show that $\chi(H - e) \leq k$.

First assume that e belongs to H'_1 and hence to H_1 . As H_1 is $(k + 1)$ -critical, there is a coloring $\varphi_1 \in \mathcal{CO}_k(H_1 - e)$ and so $\varphi_1(u) \neq \varphi_1(w)$. There is a coloring $\varphi_2 \in \mathcal{CO}_k(H'_2)$ and $\varphi_2(u) \neq \varphi_2(w)$. By permuting colors if necessary, $\varphi_1 \cup \varphi_2$ is a proper k -coloring of $H - e$. Now assume that e belongs to H'_2 and let e' be the corresponding edge of H_2 . As H_2 is $(k + 1)$ -critical, there is a coloring $\varphi_2 \in \mathcal{CO}_k(H_2 - e')$ which leads to a coloring $\varphi'_2 \in \mathcal{CO}_k(H'_2 - e)$ such that $\varphi'_2(u) = \varphi'_2(w) = \varphi_2(\tilde{v})$. As H_1 is $(k + 1)$ -critical, there is a coloring $\varphi_1 \in \mathcal{CO}_k(H_1 - \tilde{e})$ and so $\varphi_1(u) = \varphi_1(w)$. By permuting colors if necessary, $\varphi_1 \cup \varphi'_2$ yields a proper k -coloring of $H - e$. Hence H is $(k + 1)$ -critical (by Proposition 5.3). ■

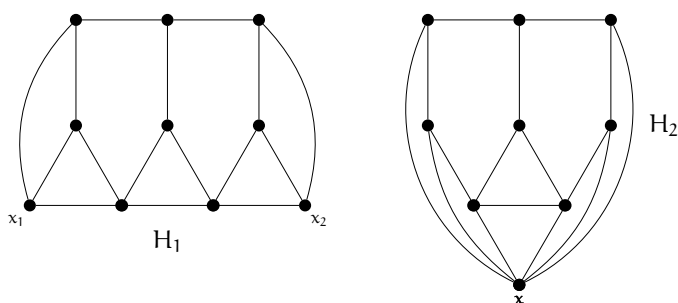


FIG. 5.3. Two 4-critical graphs.

There are $(k + 1)$ -critical graphs H_2 and vertices v of H_2 such that the resulting graph H'_2 obtained from H_2 by splitting v into an independent set of size at least 2 satisfies $\chi(H'_2) \geq k + 1$; in this case H'_2 is $(k + 1)$ -critical, too. An example with $k = 3$ is shown in Figure 5.3; both graphs H_1 and H_2 are 4-critical and H_1 is obtained from H_2 by splitting x into the vertex set $\{x_1, x_2\}$. The graph H_1 is a Hajos join of the form $H_1 = (K_4 \nabla K_4) \nabla K_4$ and hence 4-critical. That H_2 is 4-critical can also easily be checked by hand using Proposition 5.3.

Both Theorems 5.12 and 5.13 are special cases of a more general theorem about the splitting operation for critical hypergraphs. The proof of the next result is almost the same as the proof of the former theorem.

Theorem 5.14. *Let H_1 and H_2 be two disjoint $(k + 1)$ -critical hypergraphs with $k \geq 2$, let $\tilde{e} \in E(H_1)$ be an arbitrary edge of H_1 , and let $\tilde{v} \in V(H_2)$ be an arbitrary vertex. Let $H = S(H_1, \tilde{e}, H_2, \tilde{v}, s)$ and let $H'_2 = H[(V(H_2) \setminus \{\tilde{v}\}) \cup \tilde{e}]$. Assume that for every coloring $\varphi \in \mathcal{CO}_k(H[\tilde{e}])$ with $|\varphi(\tilde{e})| \geq 2$ there is a coloring $\varphi' \in \mathcal{CO}_k(H'_2)$ such that $\varphi'|_{\tilde{e}} = \varphi$. Then, H is a $(k + 1)$ -critical hypergraph. ◊*

A slightly weaker version of the above theorem has already been proved by TOFT [116]; he only considered the case when H_2 is a critical graph and s is a simple splitting. Then,

the resulting critical hypergraph H has one hyperedge less. By repeated application of the splitting operation one can finally obtain a critical graph.

Let $H_1, H_2, \tilde{e}, \tilde{v}, H$ and H'_2 as in Theorem 5.14. As H_1 is critical, H_1 is a simple hypergraph (by Proposition 5.4(d)). Hence, \tilde{e} is an independent set of H as well as of H'_2 and $H[\tilde{e}] = H'_2[\tilde{e}]$. We then say that H'_2 is obtained from H_2 by splitting \tilde{v} into the independent set \tilde{e} , and write $H'_2 = S(H_2, \tilde{v}, \tilde{e}, s)$.

Let H be a $(k + 1)$ -critical hypergraph with $k \geq 2$, and let v be a vertex of H . We say that v is a **universal vertex** of H , if for every hypergraph $H' = S(H, v, X, s)$, where X is a set, and every coloring $\varphi' \in \mathcal{CO}_k(H'[X])$ with $|\varphi'(X)| \geq 2$ there is a coloring $\varphi \in \mathcal{CO}_k(H)$ with $\varphi|_X = \varphi'$.

Theorem 5.14 then implies that if H_1 and H_2 are disjoint $(k + 1)$ -critical hypergraphs, and \tilde{v} is a universal vertex of H_2 , then any hypergraph H obtained from H_1 and H_2 by splitting \tilde{v} into an edge \tilde{e} of H_2 is a $(k + 1)$ -critical hypergraph, too. However, a good characterization of universal vertices in critical hypergraphs or graphs seems not available. From the proof of Theorem 5.12 it follows that any low vertex of a $(k + 1)$ -critical hypergraph with $k \geq 2$ is universal. Further cases were given by TOFT in [117] and [116].

Next to the HAJÓS construction there is another construction for critical hypergraphs, first used by DIRAC for critical graphs (see GALLAI [48, (2.1)]). Let H_1 and H_2 be two disjoint hypergraphs, and let H be the hypergraph obtained from the union $H_1 \cup H_2$ by adding all ordinary edges between H_1 and H_2 , that is, $V(H) = V(H_1) \cup V(H_2)$ and $E(H) = E(H_1) \cup E(H_2) \cup \{uv \mid u \in V(H_1), v \in V(H_2)\}$. We call H the **DIRAC sum**, or the **join** of H_1 and H_2 and write $H = H_1 \boxtimes H_2$. Then, it is straightforward to show that $\chi(H) = \chi(H_1) + \chi(H_2)$, and, moreover, H is critical if and only if both H_1 and H_2 are critical. For example, $KC_{n,p} = K_n \boxtimes C_{2p+1}$ is a $(n + 3)$ -critical graph and, as proved by TOFT [116], each high vertex of $KC_{n,p}$ is universal. These graphs enable us to construct from any $(k + 1)$ -critical hypergraph with $k \geq 3$ and copies of $KC_{k-2,p}$ a $(k + 1)$ -critical graph. Note that if $H = S(H_1, \tilde{e}, H_2, \tilde{v}, s)$ and s is a simple splitting, then $d_{H_2}(\tilde{v}) \geq |\tilde{e}|$. One popular example of a critical graph obtained from a critical hypergraph was presented by TOFT [117]. For $i \in \{1, 2\}$, let H_i be a connected hypergraph with one edge e_i of size $2p + 1$, so H_i is a 2-critical hypergraph. Then the DIRAC sum $H' = H_1 \boxtimes H_2$ is a 4-critical hypergraph. If we now apply the splitting operation with two copies of the odd wheels $KC_{1,p}$ and the high vertex v , that is, we first construct $\tilde{H} = S(H', e_1, KC_{1,p}, v, s)$ with a simple splitting s and then $H = S(\tilde{H}, e_2, KC_{1,p}, v, s')$ with a simple splitting s' , then the resulting graph H is a 4-critical graph of order $n = 8p + 4$ and with $m = (2p + 1)^2 + 8p + 4 = \frac{1}{16}n^2 + n$ edges, *i.e.*, H has many edges. The constant $\frac{1}{16}$ has not been improved.

5.4. Concluding Remarks

Surprisingly, we are not able to characterize the hypergraphs with $\lambda = 2$ and $\chi = 3$. If \mathcal{H}_2 denotes the smallest class of hypergraphs that contains all hyperwheels and is closed under taking HAJÓS joins, then it is easy to show that \mathcal{H}_2 is contained in the class \mathcal{C}_2 of 3-critical hypergraphs with $\lambda \leq 2$. As proved in Claim 5.11.4 if H belongs to \mathcal{C}_k with $k \geq 3$ and H has no separating vertex set of size at most 2, then H is a base graph of \mathcal{H}_k , that is, either $k = 3$ and H is an odd wheel or $k \geq 4$ and H is a K_{k+1} . However, there are hypergraphs in \mathcal{C}_2 that do not have a separating vertex set of size at most 2, but that are different from hyperwheels. Examples of such 3-critical hypergraphs can be obtained as follows. Let T be an arbitrary rooted tree such that the root has degree at least 2 and the distance between the leafs of T and the root all have the same parity. If H is the hypergraph obtained from T by adding the hyperedge consisting of the leafs of T , then it is easy to check that $H \in \mathcal{C}_2$. If the non-leaf vertices of T have degree at least 3, then H has no separating vertex set of size at most 2; one such hypergraph is shown in Figure 5.4. On the other hand, H belongs to \mathcal{H}_2 , and we do not know any hypergraph belonging to \mathcal{C}_2 , but not to \mathcal{H}_2 . If $H \in \mathcal{C}_2$ then H has a separating edge set of size 2, and according to Theorem 5.10 the hypergraph H can be decomposed into two 3-critical hypergraphs H_1 and H_2 . It can easily be shown that $\lambda(H_i) \leq 2$ for $i \in \{1, 2\}$ implying that both H_1 and H_2 belong to \mathcal{C}_2 . The problem is the converse splitting operation.

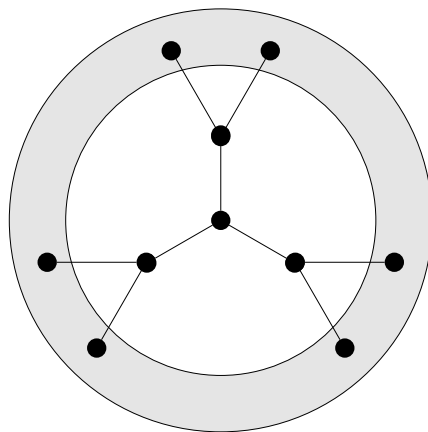


FIG. 5.4. A member in \mathcal{C}_2 without a separating vertex set of size 2.

It seems likely that one can obtain a polynomial time algorithm from the proof of Theorem 5.2, which, given a hypergraph H with $\lambda(H) \leq k$ and $k \geq 3$, either finds a proper k -coloring of H or a block belonging to \mathcal{H}_k . We did not explore this question.

Part II

COLORINGS OF DIGRAPHS

Chapter 6

PRELIMINARIES: DIGRAPHS

The topics and results presented within the next three chapters are joint work with JØRGEN BANG-JENSEN, THOMAS BELLITTO, and MICHAEL STIEBITZ. The results of Chapter 7 have been published under the title *On DP-coloring of digraphs* in *Journal of Graph Theory* (see [6]); Chapter 8 reflects the outcomes of the paper *Hajós and Ore constructions for digraphs*, which has been published in *The Electronic Journal of Combinatorics* (see [7]). Most parts of the following three chapters strongly resemble their counterparts in the corresponding papers or have been taken over one-to-one.

6.1. Basic Digraph Terminology

The digraph terminology used in this thesis is mostly based on the book of BANG-JENSEN and GUTIN [8]. A digraph $D = (V(D), A(D))$ consists of a finite set $V(D)$ of so called **vertices** and a finite set $A(D)$ of ordered pairs of distinct vertices of D , so called **arcs** of the digraph D . Accordingly, $V(D)$ is the **vertex set** of D and $A(D)$ is the **arc set** of D . The size of the vertex set of D is called the **order** of D ; we denote it by $|D|$. If $\alpha = (u, v)$ is an arc of D with **end-vertices** u and v , we say that u is the **initial vertex** of α and v is the **terminal vertex** of α . For the sake of readability, we write $\alpha = uv$ instead of $\alpha = (u, v)$. Note that our definition neither allows for loops (*i.e.*, arcs of which initial and terminal vertex coincide) nor parallel arcs (*i.e.*, two or more arcs with the same initial vertex and the same terminal vertex). However, it may happen that there are two arcs going in opposite directions between two vertices; in this case, we say that the two arcs are **opposite**. Two

vertices u and v are **adjacent** if at least one of uv and vu belongs to $A(D)$, we then also say that u is a **neighbor** of v and vice versa. A vertex and an arc are **incident** if the vertex is either initial vertex or terminal vertex of the arc. If $uv \in A(D)$, then v is an **out-neighbor** of u and u is an **in-neighbor** of v . Let $N_D^+(v)$ (respectively $N_D^-(v)$) denote the **set of out-neighbors** (respectively **set of in-neighbors**) of v in D . If $X, Y \subseteq V(D)$, then $A_D(X, Y)$ denotes the set of arcs from $A(D)$ having initial vertex in X and terminal vertex in Y . Hence, $|N_D^+(v)| = |A_D(\{v\}, V(D) \setminus \{v\})|$ and $|N_D^-(v)| = |A_D(V(D) \setminus \{v\}, \{v\})|$.

Degree Concepts for Digraphs

Due to the directedness of arcs, there are a few degree concepts for digraphs that naturally arise. First of all, the **out-degree** $d_D^+(v)$ of v in a digraph D is the number of arcs whose initial vertex is v and so $d_D^+(v) = |N_D^+(v)|$. Similarly, the **in-degree** $d_D^-(v) = |N_D^-(v)|$ of v in D is the number of arcs whose terminal vertex is v . Moreover, the **total degree** $d_D(v)$ of v in D is the sum of in-degree and out-degree of v , *i.e.* $d_D(v) = d_D^+(v) + d_D^-(v)$. A vertex v of D is **EULERian** if $d_D^+(v) = d_D^-(v)$. We say that the digraph D is **EULERian** if every vertex of D is EULERian. As usual, $\Delta^+(D) = \max_{v \in V(D)} d_D^+(v)$ denotes the **maximum out-degree** of D and $\Delta^-(D) = \max_{v \in V(D)} d_D^-(v)$ denotes the **maximum in-degree** of D . Unsurprisingly, $\Delta(D) = \max_{v \in V(D)} d_D(v)$ is the **maximum total degree** of D . By substituting *max* with *min* in the previous three sentences we obtain the definitions of **minimum out-degree** $\delta^+(D)$, **minimum in-degree** $\delta^-(D)$, and **minimum total degree** $\delta(D)$ of D .

Subdigraphs and Induced Subdigraphs

Let D be a digraph. A **subdigraph** D' of D is a digraph fulfilling $V(D') \subseteq V(D)$ and $A(D') \subseteq A(D)$; we then write $D' \subseteq D$. The subdigraph D' is a **proper subdigraph** of D (written $D' \subset D$) if $V(D') \subset V(D)$ or $A(D') \subset A(D)$. If $X \subseteq V(D)$, then $D[X]$ denotes the **subdigraph of D induced by X** , *i.e.*, $V(D[X]) = X$ and $A(D[X]) = A_D(X, X)$. A subdigraph D' of D is **induced** if there exists a vertex set $X \subseteq V(D)$ such that $D' = D[X]$. Conversely, $D - X = D[V(D) \setminus X]$ denotes the subdigraph of D that results from D by deleting all vertices of X as well as all incident arcs from D . If $X = \{v\}$ is a singleton, we prefer writing $D - v$ instead of $D - \{v\}$.

Paths, Cycles, and Connectivity

A **directed path** of length $\ell \geq 0$ is a digraph P with vertex set $V(P) = \{v_0, v_1, \dots, v_\ell\}$ and arc set $A(P) = \{v_0v_1, v_1v_2, \dots, v_{\ell-1}v_\ell\}$ where all of the v_i are pairwise distinct; we also say that P is a directed path **from v_0 to v_ℓ** . A **directed cycle** of length $\ell + 1 \geq 2$ is a digraph C that results from a directed path of length ℓ from v_0 to v_ℓ by adding the arc $v_\ell v_0$. A directed

cycle of length 2 is called **digon**. In digraph theory, it is often interesting to examine **digon-free** digraphs and, in particular, tournaments: a **tournament** is a digraph that results from a complete graph by orienting each edge, that is, replacing the edge between any pair of vertices u, v by either the arc uv or vu .

The **underlying graph** $G(D)$ of a digraph D is the simple graph with $V(G(D)) = V(D)$ and $\{u, v\} \in E(G(D))$ if and only if at least one of uv and vu belongs to $A(D)$. The digraph D is **(weakly) connected** if $G(D)$ is connected. A **separating vertex** of a connected digraph D is a vertex $v \in V(D)$ such that $D - v$ is not connected. Moreover, a **block** of D is a maximal connected subdigraph B of D such that B contains no separating vertex. As previously, $\mathcal{B}(D)$ denotes the **set of blocks** of D and, for $v \in V(D)$, $\mathcal{B}_v(D)$ denotes the set of blocks of D containing v . Note that the vertex sets of the blocks of D are exactly the vertex sets of the blocks of $G(D)$. The **complement** of D is the digraph \bar{D} with $V(\bar{D}) = V(D)$ and $A(\bar{D}) = \{uv \mid u, v \in V(D) \text{ and } uv \notin A(D)\}$. Clearly, \bar{D} is connected if D is not connected, but the converse does not hold true in general.

By taking into account the orientation of arcs, we obtain a second connectivity concept. We say that the digraph D is **strongly connected** if there exists a directed path in D between any pair of distinct vertices of D . Obviously, every strongly connected digraph is also connected but the converse does not always hold true.

If D is a digraph and C is a cycle in the underlying graph $G(D)$, we denote by D_C the maximal subdigraph of D satisfying $G(D_C) = C$. A **bidirected** graph is a digraph that can be obtained from a simple graph G by replacing each edge by two opposite arcs; we denote it by $D(G)$. A bidirected complete graph is also called a **complete digraph**. Note that if $D = D(G)$ is a bidirected graph, then $G(D) = G$.

6.2. Colorings of Digraphs

In this thesis' first part, we have exhaustively examined colorings of hypergraphs. Those were mappings such that each hyperedge of the respective hypergraph contains at least two vertices of distinct colors. In particular, we have seen that this coloring approach generalizes the usual one for graphs as it forces vertices to have distinct colors if they are joined by an edge. Now, is there a reasonable coloring concept for digraphs that generalizes the usual graph coloring method, too? Certainly, given a digraph D , we could just define a coloring of D to be a coloring of the underlying graph $G(D)$. In this instance, however, there is no point in studying digraph colorings, as everything would be settled by investigating colorings of undirected graphs. Instead, in 1982, NEUMANN-LARA [94] came up with another concept

that takes into account the orientation of edges. According to him, an **(acyclic) coloring** of a digraph D is a mapping from the vertex set $V(D)$ to a color set Γ such that each color class induces an **acyclic subdigraph** of D , *i.e.*, a subdigraph that does not contain any directed cycles. If $|\Gamma| = k$, we call such a coloring an **(acyclic) k -coloring** of D and say that D is **k -colorable** if D admits such a k -coloring. The **dichromatic number** $\vec{\chi}(D)$ of a digraph D is the smallest integer k such that D is k -colorable. Here again, this concept generalizes the usual concept for graphs. This is due to the fact that every acyclic set in a bidirected graph D is also an independent set of D . Hence, any acyclic coloring of a bidirected graph D induces a proper coloring of its underlying graph and vice versa, and so the dichromatic number of a bidirected graph and the chromatic number of its underlying graph coincide, that is, if G is a simple graph, then

$$\vec{\chi}(D(G)) = \chi(G). \quad (6.1)$$

Since the introduction of this digraph coloring concept, various well known results for graph coloring have been transferred to the digraph setting. As in the previous chapters, we will especially focus on BROOKS' Theorem and related research.

A first step towards BROOKS' Theorem was already made in the initial paper by NEUMANN-LARA [94]. He proved the following, simple theorem (see also Section 8.1).

Theorem 6.1 (NEUMANN-LARA, 1982). *Let D be a digraph. Then,*

$$\vec{\chi}(D) \leq \min\{\Delta^-(D), \Delta^+(D)\} + 1. \quad \diamond$$

For many years, NEUMANN-LARA's digraph coloring concept did not receive much attention. Only after it was rediscovered by MOHAR [88], more people started to investigate the dichromatic number (see, for instance, [2, 5, 56, 57, 58, 89, 90, 106]). In 2010, MOHAR finally published BROOKS' Theorem for digraphs. Although he proved a slightly different version, it is equivalent to the following theorem (see also Section 8.1).

Theorem 6.2 (MOHAR, 2010). *Let D be a connected digraph. Then, D satisfies $\vec{\chi}(D) \leq \max\{\Delta^-(D), \Delta^+(D)\} + 1$ and equality holds if and only if D is*

- (a) *a directed cycle of length ≥ 2 , or*
- (b) *a bidirected cycle of odd length ≥ 3 , or*
- (c) *a bidirected complete graph.* \diamond

Following up on this, HARUTYUNYAN and MOHAR [58] examined list-colorings of digraphs. Given a digraph D , some color set Γ , and a function $L : V(D) \rightarrow 2^\Gamma$ (which we still call **list-assignment**), an **L-coloring** of D is a function $\varphi : V(D) \rightarrow \Gamma$ such that $\varphi(v) \in L(v)$ for all $v \in V(D)$ and $D[\varphi^{-1}(\{\alpha\})]$ contains no directed cycle for each $\alpha \in \Gamma$ (if such a coloring exists, we say that D is **L-colorable**). HARUTYUNYAN and MOHAR [58] extended ERDŐS, RUBIN and TAYLOR's Theorem 3.1 to digraphs, thereby obtaining a generalization of Theorem 6.2.

Theorem 6.3 (HARUTYUNYAN and MOHAR, 2011). *Let D be a connected digraph, and let L be a list-assignment such that $|L(v)| \geq \max\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$. Suppose that D is not L-colorable. Then, the following statements hold:*

- (a) D is EULERIAN and $|L(v)| = \max\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$.
- (b) If $B \in \mathcal{B}(D)$, then B is a directed cycle of length ≥ 2 , or B is a bidirected complete graph, or B is a bidirected cycle of odd length ≥ 5 .
- (c) For each $B \in \mathcal{B}(D)$ there is a set Γ_B of $\Delta^+(B)$ colors such that for every $v \in V(D)$, the sets Γ_B with $B \in \mathcal{B}_v(D)$ are pairwise disjoint and $L(v) = \bigcup_{B \in \mathcal{B}_v(D)} \Gamma_B$. \diamond

While HARUTYUNYAN and MOHAR originally only proved statements (a) and (b) of the above theorem, it is possible to deduce the theorem in the form presented here from Theorem 3.11, as we shall demonstrate. For the reader's convenience, let us recall Theorem 3.11. As defined in Chapter 3, given a graph G , a list-assignment L of G , and a non-negative integer s , the graph G is (L, s) -colorable if there is an L-coloring of G such that every color class induces a strictly s -degenerate subgraph of G .

Theorem 3.11. *Let $s \in \{1, 2\}$, let G be a connected graph with $|G| \geq 2$, and let L be a list-assignment satisfying $|L(v)| \geq d_G(v)/s$ for each $v \in V(G)$. Then, G is not (L, s) -colorable if and only if the following two conditions are fulfilled:*

- (a) If $B \in \mathcal{B}(G)$, then $B = tK_n$ with $1 \leq t \leq s$ and $t(n-1) \equiv 0 \pmod{s}$, or $B = sC_n$ with n odd, or B is s -regular.
- (b) For each $B \in \mathcal{B}(G)$, there is a set Γ_B of $\Delta(B)/s$ colors such that for every $v \in V(G)$, the sets Γ_B with $B \in \mathcal{B}_v(G)$ are pairwise disjoint and $L(v) = \bigcup_{B \in \mathcal{B}_v(G)} \Gamma_B$. \diamond

Proof of Theorem 6.3. For $v \in V(D)$, let $\ell(v) = \max\{d_D^+(v), d_D^-(v)\}$ and let L be a list-assignment as described in the theorem. Note that $|L(v)| \geq \ell(v)$ for all $v \in V(D)$. Now, we

create an auxiliary graph G as follows: let $V(G) = V(D)$ and, for each arc $uv \in A(D)$ add an edge between the vertices u and v in G . Then, between any pair of distinct vertices of G there are at most two parallel edges, and $d_G(v) = d_D(v) \leq 2\ell(v)$ for all $v \in V(G)$. Clearly, if there is an $(L, 2)$ -coloring of G , then every color class induces a forest in G and, therefore, an acyclic subdigraph of D , contradicting the premise. Hence, G is not $(L, 2)$ -colorable and it follows from Theorem 3.11 that if $B \in \mathcal{B}(G)$, then $B = tK_n$ with $t \in \{1, 2\}$ and $t(n-1) \equiv 0 \pmod{2}$, $B = 2C_n$ with n odd, or B is 2-regular. Moreover, Theorem 3.11(b) implies that $d_G(v) = d_D(v) = 2\ell(v) = 2|L(v)|$ for all $v \in V(G)$ and so statements (a) and (c) of Theorem 6.3 hold true. As D is EULERIAN, an easy induction on the number of blocks of D shows that each block of D is EULERIAN, too (as every block of G is regular and as it is not possible that all but one vertices of a block are EULERIAN). Consequently, if $B \in \mathcal{B}(G)$ is 2-regular, then the corresponding block of $\mathcal{B}(D)$ is a directed cycle. Similarly, if $B = 2C_n$ with n odd, then B corresponds to a bidirected cycle of odd length n . Finally, if $B = tK_n$, we claim that $t = 2$ and, hence, B corresponds to a bidirected complete graph, or that $t = 1, n = 3$ and B corresponds to a directed cycle in D . Clearly, this claim is equivalent to the statement that no block of D is a tournament of odd order at least five.

So assume, to the contrary, that there is a block B of D , which is a tournament of order n with $n \geq 5$ odd. Let Γ_B be as described in statement (c) of Theorem 6.3. Then, $|\Gamma_B| = (n-1)/2$ and $\Gamma_B \subseteq L(v)$ for all $v \in V(B)$. We claim that B admits an acyclic coloring with color set Γ_B . To this end, we choose a vertex $v \in V(B)$ and two distinct out-neighbors $u, w \in V(B)$ of v (which exist since $n \geq 5$ and $d_D^+(v) = (n-1)/2 \geq 2$). Clearly, $B[\{u, v, w\}]$ is acyclic and so we assign the three vertices u, v and w the same color α from Γ_B . Afterwards, we group the remaining vertices into $(n-1)/2 - 1$ pairs and assign each pair a unique color from $\Gamma_B \setminus \{\alpha\}$. As B contains no digon, this leads to an acyclic coloring φ_B of B with color set Γ_B , as claimed. In order to derive a contradiction, we prove that we can extend this coloring φ_B to an L -coloring of D . For that purpose, let $v \in V(B)$ be an arbitrary vertex, and let D_v be the component of $D - (V(B) \setminus \{v\})$ containing v . Similarly, let G_v be the component of $G - (V(B) \setminus \{v\})$ containing v . Note that $V(D_v) = V(G_v)$. Moreover, let L_v be the list-assignment of D_v (and, therefore, of G_v) with $L_v(v) = L(v) \setminus \Gamma_B$ and $L_v(w) = L(w)$ for $w \in V(D_v) \setminus \{v\}$. If v is not a separating vertex of D , then $G_v = K_1$ and $L_v(v) = \emptyset$ and so G_v is not $(L_v, 2)$ -colorable. If, however, v is a separating vertex of D , then $|G_v| \geq 2$, $|L_v(w)| \geq d_{G_v}(w)/s$ for all $w \in V(G_v)$ and G_v and L_v satisfy statements (a) and (b) of Theorem 3.11. As a consequence, G_v is not $(L_v, 2)$ -colorable in both cases. Now let L'_v be the list assignment of D_v (and G_v) with $L'_v(w) = L(w) = L_v(w)$ for all $w \in V(D_v) \setminus \{v\}$ and $L'_v(v) = L(v) \setminus (\Gamma_B \setminus \varphi_B(v)) = L_v(v) \cup \{\varphi_B(v)\}$. If v is not a separating vertex of D ,

then $G_v = K_1$, $L'_v(v) = \{\varphi_B(v)\}$, and φ_v with $\varphi_v(v) = \varphi_B(v)$ clearly is an $(L'_v, 2)$ -coloring of G_v and, therefore, an L'_v -coloring of D_v . Now assume that v is a separating vertex of D . Then $|G_v| \geq 2$, $|L'_v(v)| > d_{G_v}(v)/s$, and $|L'_v(w)| \geq d_{G_v}(w)/s$ for all $w \in V(G_v) \setminus \{v\}$. By Theorem 3.11, G_v admits an $(L'_v, 2)$ -coloring φ_v and, as G_v is not $(L_v, 2)$ -colorable, we have $\varphi_v(v) = \varphi_B(v)$. Consequently, φ_v is an L'_v -coloring of D_v . As $D = B \cup \bigcup_{v \in V(B)} D_v$ and since the digraphs D_v are pairwise disjoint, it follows that $\varphi = \bigcup_{v \in V(B)} \varphi_v$ is an L -coloring of D , which is impossible. This proves the claim that no block of D is a tournament of odd order at least five and so the theorem's proof is complete. ■

It is natural to wonder if the requirement $|L(v)| \geq \min\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$ is already sufficient for implying the above statement. That this is not the case was shown by HARUTYUNYAN and MOHAR [58] in the same paper; they designed an easy counter-example on four vertices, which is depicted in Figure 6.1. Note that, for better readability, we will always display opposite arcs as one arc with arrowheads in both directions. HARUTYUNYAN and MOHAR [58] further proved that it is even NP-complete to decide whether a planar digraph satisfying this condition is L -colorable, or not.

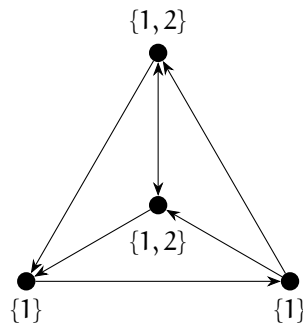


FIG. 6.1. ([58]) A non- L -colorable digraph with $|L(v)| \geq \min\{d^+(v), d^-(v)\}$ that is not EULERIAN.

6.3. Focus of our Research

When we became attentive to the paper of HARUTYUNYAN and MOHAR [58], we immediately thought about how it would be possible to transfer DP-colorings to digraphs. Obviously, directed cycles needed to play a key role, which led us to the following, quite intuitive approach. Given a digraph D , a cover (X, \mathcal{D}) consists of a mapping X and a digraph \mathcal{D} nearly as before, but for each arc uv we add a directed matching from the corresponding vertex set X_u to X_v (i.e. a matching in which all initial vertices are in X_u and all terminal

vertices are in X_v), instead of an undirected matching as we did in Chapter 4. Then, an (X, \mathcal{D}) -coloring of \mathcal{D} is an acyclic transversal of (X, \mathcal{D}) , *i.e.*, a transversal that does not induce any directed cycles in \mathcal{D} . The exact definition of covers and DP-coloring of digraphs is given in Chapter 7. Moreover, there it is proved that DP-colorings of bidirected graphs and DP-colorings of its underlying graphs coincide. Finally, we obtain a BROOKS-type theorem for the DP-chromatic number.

This served us as motivation to investigate digraph coloring in more detail. To date, many questions and subjects in digraph coloring are still wide open and we decided to take a deeper look at critical digraphs. A digraph \mathcal{D} is **critical** and **k-critical** if $\vec{\chi}(\mathcal{D}) = k$ but $\vec{\chi}(\mathcal{D}') < k$ for each proper subdigraph \mathcal{D}' of \mathcal{D} . NEUMANN-LARA already considered critical digraphs in his initial paper [94] (he called them **minimal k-chromatic**) and proved that critical digraphs are strongly connected and have no separating vertices. In Chapter 8, we prove a GALLAI-type result for critical digraphs. Afterwards, we regard a classical construction for critical graphs, the HAJÓS join, and transfer his most famous theorem, stating that each graph of chromatic number at least k contains a HAJÓS-constructible graph, *i.e.*, a simple graph that can be obtained from copies of K_k by applying the HAJÓS join and identifying non-adjacent vertices, to digraphs. Following up on that, we introduce the ORE join and prove the counterpart of a well-known theorem of URQUHART [119], which states that every simple graph of chromatic number at least k not only contains a HAJÓS-constructible digraph but itself is HAJÓS-constructible.

Nonetheless, there is still a lot to be done. A collection of interesting open problems and possible ways how to approach those is presented in Chapter 9.

Chapter 7

DP-COLORING OF DIGRAPHS

Although the reader might still have good memories from the first part of this thesis, let us briefly recall the concept of DP-coloring. The main idea was to generalize the list-coloring concept by transforming the problem of finding an L-coloring of a graph G to that of finding an independent transversal in an auxiliary graph \mathcal{G} with vertex set $\{(v, \alpha) \mid v \in V(G), \alpha \in L(v)\}$ and edge set $\{(v, \alpha)(v', \alpha') \mid vv' \in E(G) \text{ and } \alpha = \alpha'\}$. Then, each edge uv of G corresponds to a particular matching in \mathcal{G} . By allowing for arbitrary matchings instead, we end up with so called **covers** of G . In the next section, we will transfer this definition to digraphs and examine the relation between DP-colorings of bidirected graphs and those of their underlying graphs. Afterwards, we will prove the digraph version of Theorem 4.2 (see Theorem 7.5), which characterizes DP-degree colorable digraphs. This will also lead to a generalization of Theorem 6.3 (see Theorem 7.13). Before going on to the next section, note that a **matching** in a digraph D is a set M of arcs of D with no common end-vertices. A matching in D is **perfect** if it contains $\frac{|D|}{2}$ arcs.

7.1. The DP-dichromatic Number

Let D be a digraph. A **cover** of D is a pair (X, \mathcal{D}) satisfying the following conditions:

- (C1) \mathcal{D} is a digraph and $X : V(D) \rightarrow 2^{V(\mathcal{D})}$ is a function that assigns to each vertex $v \in V(D)$ a vertex set $X_v = X(v) \subseteq V(\mathcal{D})$ such that the sets X_v with $v \in V(D)$ are pairwise disjoint.

(C2) We have $V(\mathcal{D}) = \bigcup_{v \in V(D)} X_v$ and each X_v is an independent set of \mathcal{D} . For each arc $a = uv \in A(D)$, the arcs from $A_{\mathcal{D}}(X_u, X_v)$ form a possibly empty matching M_a in $\mathcal{D}[X_u \cup X_v]$. Furthermore, the arcs of \mathcal{D} are $A(\mathcal{D}) = \bigcup_{a \in A(D)} M_a$.

Now let (X, \mathcal{D}) be a cover of D . A vertex set $T \subseteq V(\mathcal{D})$ is a **transversal** of (X, \mathcal{D}) if $|T \cap X_v| = 1$ for each vertex $v \in V(D)$. An **acyclic transversal** of (X, \mathcal{D}) is a transversal T of (X, \mathcal{D}) such that $\mathcal{D}[T]$ contains no directed cycle. An acyclic transversal of (X, \mathcal{D}) is also called an (X, \mathcal{D}) -**coloring** of D ; the vertices of \mathcal{D} are called **colors**. We say that D is (X, \mathcal{D}) -**colorable** if D admits an (X, \mathcal{D}) -coloring. Let $f : V(D) \rightarrow \mathbb{N}_0$ be a function. Then, D is said to be **DP-f-colorable** if D is (X, \mathcal{D}) -colorable for every cover (X, \mathcal{D}) of D satisfying $|X_v| \geq f(v)$ for all $v \in V(D)$ (we will call such a cover an **f-cover**). If D is DP-f-colorable for a function f such that $f(v) = k$ for all $v \in V(D)$, then we say that D is **DP-k-colorable**. The **DP-dichromatic number** $\vec{\chi}_{DP}(D)$ is the smallest integer $k \geq 0$ such that D is DP-k-colorable.

In the following, we want to examine the relation between DP-colorings of bidirected graphs and those of their underlying graphs. For the reader's convenience, let us recall the essential definitions regarding simple graphs from Section 4.1.

Let G be a simple graph. A **cover** of G is a pair (X, \mathcal{G}) satisfying (C1) and (C2) where we suppress the orientations, *i.e.*, in particular, the matching M_e associated to an edge $e = uv \in E(G)$ is an (undirected) matching of \mathcal{G} between X_u and X_v (and \mathcal{G} is therefore an undirected graph). An (X, \mathcal{G}) -**coloring** of G is an **independent transversal** T of (X, \mathcal{G}) , *i.e.*, T is a transversal of (X, \mathcal{G}) such that $\mathcal{G}[T]$ is edgeless. The definitions of DP-f-colorable, DP-k-colorable and the DP-chromatic number are analogous.

Theorem 7.1. *A bidirected graph D is DP-f-colorable if and only if its underlying graph $G(D)$ is DP-f-colorable.* ◊

Proof. We prove the two implications separately. First assume that D is DP-f-colorable. In order to show that $G = G(D)$ is DP-f-colorable, let (X, \mathcal{G}) be an f-cover of G and let $\mathcal{D} = D(\mathcal{G})$ be the bidirected graph associated to \mathcal{G} . Then, (X, \mathcal{D}) is an f-cover of D . By assumption, there is an acyclic transversal T of (X, \mathcal{D}) . As \mathcal{D} is bidirected, T is an independent transversal of (X, \mathcal{G}) and so G is DP-f-colorable.

The converse implication is less obvious since even if D is bidirected, its covers do not necessarily have to be. We prove the implication's contraposition. To this end, let f be such that D is not DP-f-colorable. We show that $G(D)$ is also not DP-f-colorable. Let (X, \mathcal{D}) be an f-cover of D for which D is not (X, \mathcal{D}) -colorable such that the number of opposite

arcs in \mathcal{D} is maximum. If \mathcal{D} is a bidirected graph, then it is not difficult to check that $(X, G(\mathcal{D}))$ is an f -cover of $G(\mathcal{D})$ such that $G(\mathcal{D})$ is not $(X, G(\mathcal{D}))$ -colorable, and we are done. Otherwise, \mathcal{D} is not a bidirected graph, and hence there are distinct vertices u and v from D with $x_u x_v \in A(\mathcal{D})$ but $x_v x_u \notin A(\mathcal{D})$ for some vertices $x_u \in X_u, x_v \in X_v$. Let (X, \mathcal{D}') be the f -cover of D obtained from (X, \mathcal{D}) by replacing M_{wu} by the opposite of M_{uw} for every vertex w adjacent to u . Clearly, the number of opposite arcs in \mathcal{D}' is larger than in \mathcal{D} and so there must exist an acyclic transversal T of (X, \mathcal{D}') (by the choice of (X, \mathcal{D})). Then, T is also a transversal of (X, \mathcal{D}) , and, since D is not (X, \mathcal{D}) -colorable, $\mathcal{D}[T]$ contains a directed cycle C . As $\mathcal{D} - X_u$ is isomorphic to $\mathcal{D}' - X_u$, it follows from the choice of T that C must contain a vertex $x \in X_u$. Hence, there exists a vertex w adjacent to u in D and a vertex $x' \in X_w$ such that $xx' \in M_{uw}$ and $x' \in T$. Since the digraph \mathcal{D}' contains both the arcs xx' and $x'x$, $\mathcal{D}'[\{x, x'\}]$ is a digon and, hence, $\mathcal{D}'[T]$ also contains a directed cycle, contradicting the choice of T . This completes the proof. ■

What makes the dichromatic number especially reasonable is that the dichromatic number of a bidirected graph coincides with the chromatic number of its underlying graph. Theorem 7.1 implies that this also holds true for DP-colorings:

Corollary 7.2. *The DP-dichromatic number of a bidirected graph is equal to the DP-chromatic number of its underlying graph.* ◊

As we have already examined for hypergraphs, DP-colorings are of special interest because they constitute a generalization of list-colorings: Let D be a digraph, Γ be a color set, and let $L : V(D) \rightarrow 2^\Gamma$ be a list-assignment. We define a cover (X, \mathcal{D}) of D as follows: let $X_v = \{v\} \times L(v)$ for all $v \in V(D)$, $V(\mathcal{D}) = \bigcup_{v \in V(D)} X_v$, and $A(\mathcal{D}) = \{(v, \alpha)(v', \alpha') \mid vv' \in A(D) \text{ and } \alpha = \alpha'\}$. It is obvious that (X, \mathcal{D}) indeed is a cover of D . Moreover, if φ is an L -coloring of D , then $T = \{(v, \varphi(v)) \mid v \in V(D)\}$ is an acyclic transversal of (X, \mathcal{D}) . On the other hand, given an acyclic transversal $T = \{(v_1, \alpha_1), \dots, (v_n, \alpha_n)\}$ of \mathcal{D} , we obtain an L -coloring of D by coloring the vertex v_i with α_i for $i \in [1, n]$. Thus, finding an L -coloring of D is equivalent to finding an acyclic transversal of (X, \mathcal{D}) . Hence, the **list-dichromatic number** $\vec{\chi}_\ell(D)$ of D , which is the smallest integer k such that D admits an L -coloring for every list-assignment L satisfying $|L(v)| \geq k$ for all $v \in V(D)$, is always at most the DP-dichromatic number $\vec{\chi}_{DP}(D)$. Moreover, by using a sequential coloring algorithm similar to Algorithm 1, it is easy to verify that $\vec{\chi}_{DP}(D) \leq \min\{\Delta^+(D), \Delta^-(D)\} + 1$. Hence, we obtain

the following sequence of inequalities:

$$\vec{\chi}(\mathbb{D}) \leq \vec{\chi}_\ell(\mathbb{D}) \leq \vec{\chi}_{\text{DP}}(\mathbb{D}) \leq \min\{\Delta^+(\mathbb{D}), \Delta^-(\mathbb{D})\} + 1. \quad (7.1)$$

7.2. DP-degree Colorable Digraphs

In Chapter 4, we have characterized DP-degree colorable hypergraphs (Theorem 4.2) and thereby obtained a generalization of KIM and OZEKI's Theorem [66] regarding DP-degree colorable graphs. In the following, we will similarly examine DP-degree colorable digraphs.

We say that a digraph \mathbb{D} is **DP-degree colorable** if \mathbb{D} is (X, \mathcal{D}) -colorable whenever (X, \mathcal{D}) is a cover of \mathbb{D} such that $|X_v| \geq \max\{d_{\mathbb{D}}^+(v), d_{\mathbb{D}}^-(v)\}$ for all $v \in V(\mathbb{D})$. In the following, we will give a characterization of the non-DP-degree-colorable digraphs as well as a characterization of the edge-minimal corresponding “bad” covers (see Theorem 7.5). Clearly, it suffices to do this only for connected digraphs.

A **feasible configuration** is a triple $(\mathbb{D}, X, \mathcal{D})$ consisting of a connected digraph \mathbb{D} and a cover (X, \mathcal{D}) of \mathbb{D} . A feasible configuration $(\mathbb{D}, X, \mathcal{D})$ is said to be **degree-feasible** if $|X_v| \geq \max\{d_{\mathbb{D}}^+(v), d_{\mathbb{D}}^-(v)\}$ for each vertex $v \in V(\mathbb{D})$. Furthermore, $(\mathbb{D}, X, \mathcal{D})$ is **colorable** if \mathbb{D} is (X, \mathcal{D}) -colorable, otherwise it is called **uncolorable**. The next proposition lists some basic properties of feasible configurations; the proofs are straightforward and left to the reader.

Proposition 7.3. *Let $(\mathbb{D}, X, \mathcal{D})$ be a feasible configuration. Then, the following statements hold:*

- (a) *For every vertex $v \in V(\mathbb{D})$ and every vertex $x \in X_v$, we have $d_{\mathcal{D}}^+(x) \leq d_{\mathbb{D}}^+(v)$ and $d_{\mathcal{D}}^-(x) \leq d_{\mathbb{D}}^-(v)$.*
- (b) *Let \mathcal{D}' be a spanning subdigraph of \mathcal{D} . Then, $(\mathbb{D}, X, \mathcal{D}')$ is a feasible configuration. If $(\mathbb{D}, X, \mathcal{D})$ is colorable, then $(\mathbb{D}, X, \mathcal{D}')$ is colorable, too. Furthermore, $(\mathbb{D}, X, \mathcal{D})$ is degree-feasible if and only if $(\mathbb{D}, X, \mathcal{D}')$ is degree-feasible. \diamond*

The above proposition leads to the following concept. We say that a feasible configuration $(\mathbb{D}, X, \mathcal{D})$ is **minimal uncolorable** if $(\mathbb{D}, X, \mathcal{D})$ is uncolorable, but $(\mathbb{D}, X, \mathcal{D} - \mathbf{a})$ is colorable for each arc $\mathbf{a} \in A(\mathcal{D})$. As usual, $\mathcal{D} - \mathbf{a}$ denotes the digraph obtained from \mathcal{D} by deleting the arc \mathbf{a} . Clearly, it follows from the above Proposition that if $(\mathbb{D}, X, \mathcal{D})$ is an uncolorable feasible configuration, then there is a spanning subdigraph \mathcal{D}' of \mathcal{D} such that $(\mathbb{D}, X, \mathcal{D}')$ is a minimal uncolorable feasible configuration.

In order to characterize the class of minimal uncolorable degree-feasible configurations, we first need to introduce three basic types of degree-feasible configurations.

We say that (D, X, \mathcal{D}) is a **K-configuration** if D is a complete digraph of order n for some $n \geq 1$, and (X, \mathcal{D}) is a cover of D such that the following conditions hold:

- $|X_v| = n - 1$ for all $v \in V(D)$,
- for each $v \in V(D)$ there is a labeling $x_v^1, x_v^2, \dots, x_v^{n-1}$ of the vertices of X_v such that $\mathcal{D}^i = \mathcal{D}[\{x_v^i \mid v \in V(D)\}]$ is a complete digraph for $i \in [1, n - 1]$, and
- $\mathcal{D} = \mathcal{D}^1 \cup \mathcal{D}^2 \cup \dots \cup \mathcal{D}^{n-1}$.

An example of a K-configuration with $n = 4$ is given in Figure 7.1. It is an easy exercise to check that each K-configuration is a minimal uncolorable degree-feasible configuration. Note that for $|D| = 1$, we have $X_v = \emptyset$ for the only vertex $v \in V(D)$ and $\mathcal{D} = \emptyset$ (and so there is no transversal of (X, \mathcal{D})).

We say that (D, X, \mathcal{D}) is a **DC-configuration** if D is a directed cycle of length $n \geq 2$ and (X, \mathcal{D}) is a cover such that $X_v = \{x_v\}$ for all $v \in V(D)$ and $A(\mathcal{D}) = \{x_v x_u \mid vu \in A(D)\}$. Note that in this case, \mathcal{D} is a copy of D . Clearly, each DC-configuration is a minimal uncolorable degree-feasible configuration.

We say that (D, X, \mathcal{D}) is an **odd C-configuration** if D is a bidirected cycle of odd length ≥ 5 and (X, \mathcal{D}) is a cover of D such that the following conditions are fulfilled:

- $|X_v| = 2$ for all $v \in V(D)$,
- for each $v \in V(D)$ there is a labeling x_v^1, x_v^2 of the vertices of X_v such that $A(\mathcal{D}) = \{x_v^i x_w^i \mid vw \in A(D) \text{ and } i \in \{1, 2\}\}$.

Note that $\mathcal{D}^i = \mathcal{D}[\{x_v^i \mid v \in V(D)\}]$ is a bidirected cycle in \mathcal{D} and $\mathcal{D} = \mathcal{D}^1 \cup \mathcal{D}^2$. It is easy to verify that every odd C-configuration is a minimal uncolorable degree-feasible configuration.

We call (D, X, \mathcal{D}) an **even C-configuration** if D is a bidirected cycle of even length ≥ 4 , (X, \mathcal{D}) is a cover of D , and there is an arc $uu' \in A(D)$ such that:

- $|X_v| = 2$ for all $v \in V(D)$,
- for each $v \in V(D)$ there is a labeling x_v^1, x_v^2 of the vertices of X_v such that $A(\mathcal{D}) = \{x_v^i x_w^i \mid \{v, w\} \neq \{u, u'\}, vw \in A(D), \text{ and } i \in \{1, 2\}\} \cup \{x_u^1 x_{u'}^2, x_u^2 x_{u'}^1, x_u^1 x_{u'}^1, x_u^2 x_{u'}^2\}$

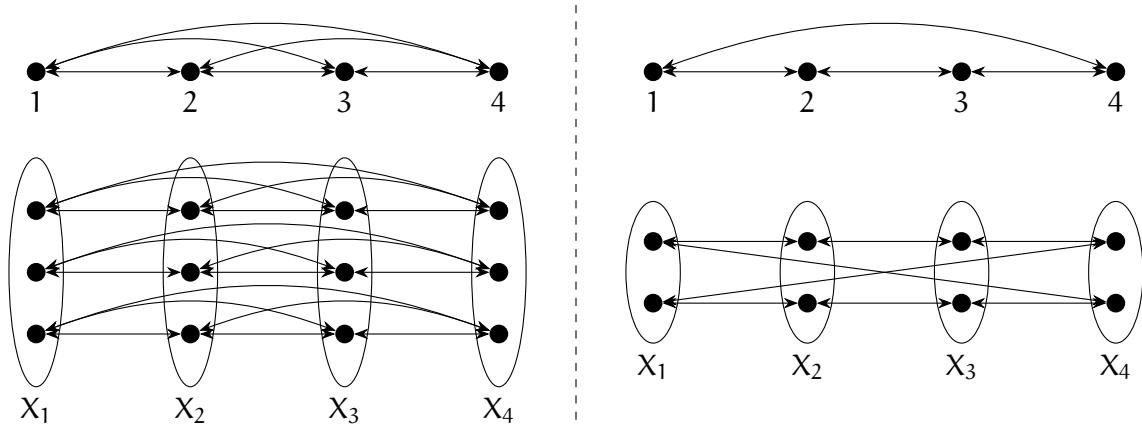


FIG. 7.1. A K -configuration and an even C -configuration for digraphs.

Again, it is easy to check that every even C -configuration is a minimal uncolorable degree-feasible configuration. By a **C-configuration** we either mean an even or an odd C -configuration.

Our aim is to show that we can construct every minimal uncolorable degree-feasible configuration from the three basic configurations by using the following operation. Let $(D^1, X^1, \mathcal{D}^1)$ and $(D^2, X^2, \mathcal{D}^2)$ be two feasible configurations, which are **disjoint**, that is, $V(D^1) \cap V(D^2) = \emptyset$ and $V(\mathcal{D}^1) \cap V(\mathcal{D}^2) = \emptyset$. Furthermore, let D be the digraph obtained from D^1 and D^2 by identifying two vertices $v^1 \in V(D^1)$ and $v^2 \in V(D^2)$ to a new vertex v^* . Finally, let $\mathcal{D} = \mathcal{D}^1 \cup \mathcal{D}^2$ and let $X : V(D) \rightarrow 2^{V(\mathcal{D})}$ be the mapping such that

$$X_v = \begin{cases} X_{v^1}^1 \cup X_{v^2}^2 & \text{if } v = v^*, \\ X_v^i & \text{if } v \in V(D^i) \setminus \{v^i\} \text{ and } i \in \{1, 2\} \end{cases}$$

for $v \in V(D)$. Then, (D, X, \mathcal{D}) is a feasible configuration and we say that (D, X, \mathcal{D}) is obtained from $(D^1, X^1, \mathcal{D}^1)$ and $(D^2, X^2, \mathcal{D}^2)$ by **merging** v^1 and v^2 to v^* .

Now we define the class of **constructible configurations** as the smallest class of feasible configurations that contains each K -configuration, each DC -configuration and each C -configuration and that is closed under the merging operation. We say that a digraph is a **DP-brick** if it is either a complete digraph, a directed cycle, or a bidirected cycle. Thus, if (D, X, \mathcal{D}) is a constructible configuration, then each block of D is a DP -brick. The next proposition is straightforward and left to the reader.

Proposition 7.4. *Let (D, X, \mathcal{D}) be a constructible configuration. Then, for each block $B \in \mathcal{B}(D)$ there is a uniquely determined cover (X^B, \mathcal{D}^B) of B such that the following statements hold:*

- (a) *For each block $B \in \mathcal{B}(D)$, the triple (B, X^B, \mathcal{D}^B) is a K-configuration, a DC-configuration, or a C-configuration.*
- (b) *The digraphs \mathcal{D}^B with $B \in \mathcal{B}(D)$ are pairwise disjoint and $\mathcal{D} = \bigcup_{B \in \mathcal{B}(D)} \mathcal{D}^B$.*
- (c) *For every vertex v from $V(D)$ we have $X_v = \bigcup_{B \in \mathcal{B}_v(D)} X_v^B$.* ◊

Our aim is to prove that the class of constructible configurations and the class of minimal uncolorable degree-feasible configurations coincide. This leads to the following theorem.

Theorem 7.5. *Suppose that (D, X, \mathcal{D}) is a degree-feasible configuration. Then, (D, X, \mathcal{D}) is minimal uncolorable if and only if (D, X, \mathcal{D}) is constructible.* ◊

When we take another look at the definitions of K-configuration and C-configuration in Section 4.2 of Chapter 4, it is not difficult to see a connection to the current definitions. In fact, we obtain the definition of K-configuration and C-configuration in the simple undirected case (*i.e.* for $t = 1$) by taking a (directed) K-, respectively C-configuration (D, X, \mathcal{D}) and regarding the underlying graphs, *i.e.* $(G(D), X, G(\mathcal{D}))$, instead (see also Figure 7.2 as a quick reminder). On this basis, we will use Theorem 4.2 as a tool in order to prove Theorem 7.5.

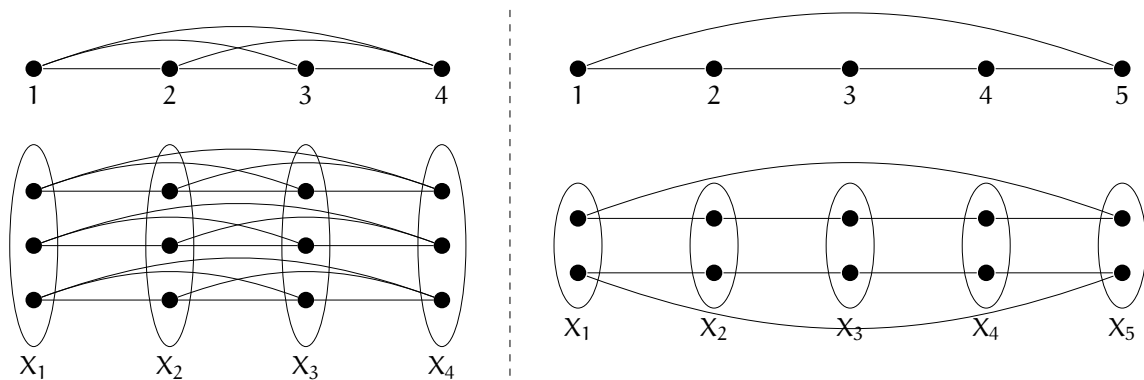


FIG. 7.2. A K-configuration and a C-configuration for simple graphs.

In the following, given a feasible configuration (D, X, \mathcal{D}) , we will often fix a vertex $v \in V(D)$ and regard the feasible configuration (D', X', \mathcal{D}') , where $D' = D - v$, X' is

the restriction of X to $V(D) \setminus \{v\}$ and $\mathcal{D}' = \mathcal{D} - X_v$. For the sake of readability, we will write $(X', \mathcal{D}') = (X, \mathcal{D})/v$.

First we state some important facts about minimal uncolorable degree-feasible configurations. Recall that the digraph D of a degree-feasible configuration (D, X, \mathcal{D}) is connected by definition.

Proposition 7.6. *Let (D, X, \mathcal{D}) be a degree-feasible configuration. If (D, X, \mathcal{D}) is uncolorable, then the following statements hold:*

- (a) $|X_v| = d_D^+(v) = d_D^-(v)$ for all $v \in V(D)$. As a consequence, D is EULERIAN.
- (b) Let $v \in V(D)$ and let $(X', \mathcal{D}') = (X, \mathcal{D})/v$. Then, there is an acyclic transversal of (X', \mathcal{D}') .
- (c) Let $v \in V(D)$ and let T be an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v$. Moreover, let $T^+ = \bigcup_{u \in N_D^+(v)} (X_u \cap T)$ and let $T^- = \bigcup_{u \in N_D^-(v)} (X_u \cap T)$. Then, the arcs from $E_{\mathcal{D}}(X_v, T^+)$ form a perfect matching in $\mathcal{D}[X_v \cup T^+]$ and the arcs from $E_{\mathcal{D}}(T^-, X_v)$ form a perfect matching in $\mathcal{D}[X_v \cup T^-]$. \diamond

Proof. (a) The proof is by induction on the order of D . The statement is clear if $|D| = 1$ as in this case $X_v = \emptyset$ for the only vertex v of D . Now assume that $|D| \geq 2$. By assumption, $|X_v| \geq \max\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$. Hence, it suffices to show $|X_v| \leq \min\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$. Suppose, to the contrary, that there is a vertex $v \in V(D)$ with $|X_v| > \min\{d_D^+(v), d_D^-(v)\}$, say $|X_v| > d_D^-(v)$ (by symmetry). Let $D' = D - v$ and let $(X', \mathcal{D}') = (X, \mathcal{D})/v$. We claim that D' is not (X', \mathcal{D}') -colorable. Otherwise, there would be an acyclic transversal T of (X', \mathcal{D}') . As $|X_v| > d_D^-(v)$ it follows from (C2) that there is a vertex $x \in X_v$ such that $x'x \notin A(\mathcal{D})$ for all $x' \in T$. Consequently, $T \cup \{x\}$ is an acyclic transversal of (X, \mathcal{D}) as x has no in-neighbor in $\mathcal{D}[T \cup \{x\}]$, that is, (D, X, \mathcal{D}) is colorable, a contradiction. Thus, D' is not (X', \mathcal{D}') -colorable, as claimed. Hence, D' contains a connected component D'' such that $(D'', X'', \mathcal{D}'')$ is uncolorable, where X'' is the restriction of X' to $V(D'')$ and $\mathcal{D}'' = \mathcal{D}'[\bigcup_{v \in V(D'')} X_v]$. By applying the induction hypothesis to $(D'', X'', \mathcal{D}'')$ we conclude that $|X_w| = d_{D''}^+(w) = d_{D''}^-(w)$ for all $w \in D''$. As D is connected, there is a vertex $w \in D''$ that is adjacent to v in D . By symmetry, we may assume $wv \in A(D)$. But then,

$$d_{D''}^+(w) = |X_w| \geq \max\{d_D^+(w), d_D^-(w)\} \geq d_{D''}^+(w) + 1,$$

which is impossible. This proves (a).

(b) For this proof, let $D' = D - v$ and let $(X', \mathcal{D}') = (X, \mathcal{D})/v$. Let D'' be an arbitrary component of D' , let X'' be the restriction of X' to $V(D'')$, and let $\mathcal{D}'' = \mathcal{D}[\bigcup_{u \in V(D'')} X_u]$. Then, $(D'', X'', \mathcal{D}'')$ is a degree-feasible configuration. As D is connected, there is at least one vertex $u \in V(D'')$ that is in D adjacent to v , say $uv \in A(D)$. By (a), this implies $|X_u| = d_D^+(u) > d_{D''}^+(u)$. Again by (a), we conclude that $(D'', X'', \mathcal{D}'')$ is colorable, *i.e.*, (X'', \mathcal{D}'') admits an acyclic transversal $T_{D''}$. Let T be the union of the sets $T_{D''}$ over all components D'' of $D - v$. Then, T is an acyclic transversal of (X', \mathcal{D}') .

(c) For the proof, we first assume that there is a vertex $x \in X_v$ such that no vertex of T is an out-neighbor of x in \mathcal{D} . Then, similarly to the proof of (a), we conclude that $T \cup \{x\}$ is an acyclic transversal of (X, \mathcal{D}) , a contradiction. Hence, each vertex $x \in X_v$ has in \mathcal{D} at least one out-neighbor belonging to T . Moreover, for each vertex $u \in N_D^+(v)$ and for the unique vertex $x' \in T \cap X_u$ there may be at most one vertex $x \in X_v$ with $xx' \in A(\mathcal{D})$ (by (C2)). As $|X_v| = d_D^+(v) = |N_D^+(v)|$, this implies that for each vertex $x \in X_v$ there is exactly one vertex $x' \in T$ with $xx' \in A(\mathcal{D})$. Thus, the arcs from X_v to $T^+ = \bigcup_{u \in N_D^+(v)} (X_u \cap T)$ are a perfect matching in $\mathcal{D}[X_v \cup T^+]$ as claimed. Using a similar argument, it follows that $E_{\mathcal{D}}(T^-, X_v)$ is a perfect matching in $\mathcal{D}[X_v \cup T^-]$. ■

The next proposition shows the usefulness of the merging operation.

Proposition 7.7. *Let $(D^1, X^1, \mathcal{D}^1)$ and $(D^2, X^2, \mathcal{D}^2)$ be two disjoint feasible configurations, and let (D, X, \mathcal{D}) be the configuration that is obtained from $(D^1, X^1, \mathcal{D}^1)$ and $(D^2, X^2, \mathcal{D}^2)$ by merging two vertices $v^1 \in V(D^1)$ and $v^2 \in V(D^2)$ to a new vertex v^* . Then, (D, X, \mathcal{D}) is a feasible configuration and the following statements are equivalent:*

- (a) *Both $(D^1, X^1, \mathcal{D}^1)$ and $(D^2, X^2, \mathcal{D}^2)$ are minimal uncolorable degree-feasible configurations.*
- (b) *(D, X, \mathcal{D}) is a minimal uncolorable degree-feasible configuration.* ◊

Proof. First we show that (a) implies (b). Clearly, (D, X, \mathcal{D}) is degree-feasible. Assume that (D, X, \mathcal{D}) is colorable. Then, there is an acyclic transversal T of (X, \mathcal{D}) . As $X_{v^*} = X_{v^1} \cup X_{v^2}$, this implies that at least one of v^1 and v^2 (by symmetry, we can assume it is v^1) satisfies $|T \cap X_{v^1}| = 1$. Thus, $T^1 = T \cap V(\mathcal{D}^1)$ is an acyclic transversal of (X^1, \mathcal{D}^1) and so $(D^1, X^1, \mathcal{D}^1)$ is colorable, a contradiction to (a). This proves that (D, X, \mathcal{D}) is uncolorable. Now let $a \in A(\mathcal{D})$ be an arbitrary arc. By symmetry, we may assume $a \in A(\mathcal{D}^1)$. Since $(D^1, X^1, \mathcal{D}^1)$ is minimal uncolorable, there is an acyclic transversal T^1 of $(X^1, \mathcal{D}^1 - a)$. Since $(D^2, X^2, \mathcal{D}^2)$ is also uncolorable and degree-feasible, there is an acyclic transversal T^2 of $(X^2, \mathcal{D}^2)/v^2$ (by

Proposition 7.6(b)). However, as $\mathcal{D} = \mathcal{D}^1 \cup \mathcal{D}^2$ and $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, the set $\mathsf{T} = \mathsf{T}^1 \cup \mathsf{T}^2$ is an acyclic transversal of $(X, \mathcal{D} - \mathbf{a})$ and so $(\mathsf{D}, X, \mathcal{D} - \mathbf{a})$ is colorable. Thus, (b) holds.

To prove that (b) implies (a), we first show that $(\mathsf{D}^1, X^1, \mathcal{D}^1)$ is minimal uncolorable. Assume that $(\mathsf{D}^1, X^1, \mathcal{D}^1)$ is colorable, that is, (X^1, \mathcal{D}^1) has an acyclic transversal T^1 . Since $(\mathsf{D}, X, \mathcal{D})$ is a minimal uncolorable degree-feasible configuration and as $\mathcal{D}^2 - X_{v^2}$ is a proper subdigraph of $\mathcal{D} - X_{v^*}$, there is an acyclic transversal T^2 of $(X^2, \mathcal{D}^2)/v^2$ (by Proposition 7.6(b)). Then again, $\mathsf{T} = \mathsf{T}^1 \cup \mathsf{T}^2$ is an acyclic transversal of (X, \mathcal{D}) , contradicting (b). Thus, $(\mathsf{D}^1, X^1, \mathcal{D}^1)$ is uncolorable. Now let $\mathbf{a} \in \mathsf{A}(\mathsf{D}^1)$ be an arbitrary arc. Then, as $(\mathsf{D}, X, \mathcal{D})$ is minimal uncolorable and $\mathbf{a} \in \mathsf{A}(\mathcal{D})$, there is an acyclic transversal T of $(X, \mathcal{D} - \mathbf{a})$ and $\mathsf{T}^1 = \mathsf{T} \cap \mathsf{V}(\mathcal{D}^1)$ clearly is an acyclic transversal of $(X^1, \mathcal{D}^1 - \mathbf{a})$. Consequently, $(\mathsf{D}^1, X^1, \mathcal{D}^1 - \mathbf{a})$ is colorable. This shows that $(\mathsf{D}^1, X^1, \mathcal{D}^1)$ is minimal uncolorable. By symmetry $(\mathsf{D}^2, X^2, \mathcal{D}^2)$ is minimal uncolorable, too.

It remains to show that $(\mathsf{D}^j, X^j, \mathcal{D}^j)$ is degree-feasible for $j \in \{1, 2\}$. As $(\mathsf{D}, X, \mathcal{D})$ is an uncolorable degree-feasible configuration, Proposition 7.6(a) implies that

$$|X_v| = d_{\mathsf{D}}^+(v) = d_{\mathsf{D}}^-(v) \text{ for all } v \in \mathsf{V}(\mathsf{D}). \tag{7.2}$$

Consequently, each vertex from $\mathsf{D}^j - v^j$ is EULERian in D^j . Since

$$\sum_{u \in \mathsf{V}(\mathsf{D}^j)} d_{\mathsf{D}^j}^+(u) = \sum_{u \in \mathsf{V}(\mathsf{D}^j)} d_{\mathsf{D}^j}^-(u) = |\mathsf{A}(\mathsf{D}^j)|$$

is the number of arcs of D^j , it follows that $d_{\mathsf{D}^j}^+(v^j) = d_{\mathsf{D}^j}^-(v^j)$, and so D^j is EULERian for $j \in \{1, 2\}$. Moreover, it follows from (7.2) that $|X_v| = d_{\mathsf{D}}^+(v) = d_{\mathsf{D}^j}^+(v) = d_{\mathsf{D}^j}^-(v)$ for all $v \in \mathsf{V}(\mathsf{D}^j) \setminus \{v^j\}$ and $j \in \{1, 2\}$. If $|X_{v^j}| < d_{\mathsf{D}}^+(v^j)$ for some $j \in \{1, 2\}$, then $|X_{v^{3-j}}| > d_{\mathsf{D}}^+(v^{3-j})$ and so $(\mathsf{D}^{3-j}, X^{3-j}, \mathcal{D}^{3-j})$ would be colorable by Proposition 7.6(a), a contradiction. Hence, $(\mathsf{D}^j, X^j, \mathcal{D}^j)$ is degree-feasible for $j \in \{1, 2\}$. ■

In order to prove Theorem 7.5, we need some more tools. The first one, which will be frequently used in the following, is the so-called **shifting operation**. Let $(\mathsf{D}, X, \mathcal{D})$ be a minimal uncolorable degree-feasible configuration, let $\mathsf{D}' = \mathsf{D} - v$ for some $v \in \mathsf{V}(\mathsf{D})$, and let T be an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v$ (which exists by Proposition 7.6(b)). Then it follows from Proposition 7.6(c) that for each vertex $x \in X_v$ there is exactly one vertex $x' \in \mathsf{T}$ with $xx' \in \mathsf{A}(\mathcal{D})$ and exactly one vertex $x'' \in \mathsf{T}$ with $x''x \in \mathsf{A}(\mathcal{D})$. Let v' and v'' be the vertices from $\mathsf{V}(\mathsf{D})$ such that $x' \in X_{v'}$ and $x'' \in X_{v''}$. Then, $\mathsf{T}' = \mathsf{T} \setminus \{x'\} \cup \{x\}$ and $\mathsf{T}'' = \mathsf{T} \setminus \{x''\} \cup \{x\}$ are acyclic transversals of $(X, \mathcal{D})/v'$ and $(X, \mathcal{D})/v''$, respectively, since in

$\mathcal{D}[T']$ (respectively $\mathcal{D}[T'']$) the vertex x has no out-neighbor (respectively no in-neighbor) and, hence, x cannot be contained in a directed cycle. We say that T' (respectively T'') evolves from T by **shifting** the color x' (respectively x'') to x . Of course, the shifting operation may be applied repeatedly. The next proposition can be easily deduced from Proposition 7.6 by applying the shifting operation. The statements of the proposition are illustrated in Figure 7.3.

Proposition 7.8. *Let (D, X, \mathcal{D}) be a minimal uncolorable degree-feasible configuration, let $v \in V(D)$, and let T be an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v$. Then, the following statements hold:*

- (a) *For every vertex $x \in X_v$ we have $|N_{\mathcal{D}}^+(x) \cap T| = 1$ and $|N_{\mathcal{D}}^-(x) \cap T| = 1$.*
- (b) *Let $u \in N_{\mathcal{D}}^+(v)$ and let $X_u \cap T = \{x_u\}$. Then, there is a vertex $x \in X_v$ such that $xx_u \in A(\mathcal{D})$ and $N_{\mathcal{D}}^-(x_u) \cap T = \emptyset$.*
- (c) *Let $w \in N_{\mathcal{D}}^-(v)$ and let $X_w \cap T = \{x_w\}$. Then, there is a vertex $x \in X_v$ such that $x_w x \in A(\mathcal{D})$ and $N_{\mathcal{D}}^+(x_w) \cap T = \emptyset$.* ◊

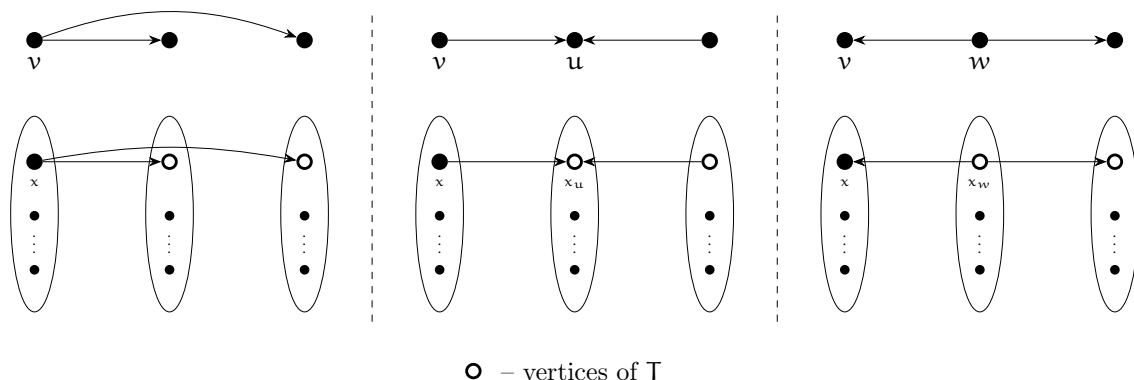


FIG. 7.3. Forbidden configurations for (D, X, \mathcal{D}) .

Proof. Statement (a) is a direct consequence of Proposition 7.6(c). In order to prove (b) let $u \in N_{\mathcal{D}}^+(v)$ and let $X_u \cap T = \{x_u\}$. Again from Proposition 7.6(c) it follows that there is a vertex $x \in X_v$ with $xx_u \in A(\mathcal{D})$. Now assume that there is a vertex $x' \in N_{\mathcal{D}}^-(x_u) \cap T$. Let T' be the transversal of $(X, \mathcal{D})/u$ that evolves from T by shifting x_u to x . Then, both x' and x are in-neighbors of x_u in \mathcal{D} and so $|N_{\mathcal{D}}^-(x_u) \cap T'| \geq 2$, a contradiction to (a). This proves (b). By symmetry, (c) follows. ■

Proposition 7.9. *Let (D, X, \mathcal{D}) be a minimal uncolorable degree-feasible configuration and let $u, v \in V(D)$ such that there are opposite arcs between u and v . Then, $\mathcal{D}[X_u \cup X_v]$ is a bidirected graph. \diamond*

Proof. Suppose the statement is false. Then there are vertices $x_u \in X_u$ and $x_v \in X_v$ with $x_u x_v \in A(\mathcal{D})$ and $x_v x_u \notin A(\mathcal{D})$. Since (D, X, \mathcal{D}) is minimal uncolorable, there is an acyclic transversal T of $(X, \mathcal{D} - x_u x_v)$. Furthermore, T must contain both x_u and x_v as otherwise T would be an acyclic transversal of (X, \mathcal{D}) , a contradiction. Then, $T' = T \setminus \{x_v\}$ is an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v$. As $u \in N_D^+(v)$, it follows from Proposition 7.8(b) that there is a vertex $x \in X_v$ with $x x_u \in A(\mathcal{D})$. Since $x_v x_u \notin A(\mathcal{D})$, $x \neq x_v$. Let T^* be the transversal that evolves from T' by shifting x_u to x_v . Then, x_u has an in-neighbor x^* from T^* in \mathcal{D} (by Proposition 7.8(a)) and $x^* \notin X_v$ (since $x_v x_u \notin A(\mathcal{D})$). Moreover, x^* is contained in the transversal \tilde{T} that evolves from T' by shifting x_u to x and so $\{x, x^*\} \subseteq N_{\mathcal{D}}^-(x_u) \cap \tilde{T}$. Consequently, $|N_{\mathcal{D}}^-(x_u) \cap \tilde{T}| > 1$, which contradicts Proposition 7.8(a). Hence $x = x_v$, and so $x_v x_u \in A(\mathcal{D})$, a contradiction. \blacksquare

In particular, the above proposition implies the following, concerning the shifting operation. Let (D, X, \mathcal{D}) be a minimal uncolorable degree-feasible configuration, let $v \in V(D)$ and let T be an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v$ (which exists by Proposition 7.6(b)). Then the above proposition together with Proposition 7.8(b)(c) implies that for each vertex u that is in D adjacent to v and for the unique vertex $x_u \in X_u \cap T$ there is exactly one vertex $x_v \in X_v$ that is in \mathcal{D} adjacent to x_u . Hence, x_v is the unique vertex from X_v to which we can shift the color x_u . Thus, in the following we may regard the shifting operation as an operation in the digraph D rather than in \mathcal{D} and write $u \rightarrow v$ in order to express that we shift the color from the corresponding vertex x_u to x_v .

As another consequence of Proposition 7.9 we easily obtain the following corollary.

Corollary 7.10. *Let (D, X, \mathcal{D}) be a degree-feasible minimal uncolorable configuration such that D is a bidirected graph. Then \mathcal{D} is a bidirected graph, too. \diamond*

Having all those tools available, we are finally ready to prove our main theorem.

7.3. Proof of Theorem 7.5

This subsection is devoted to the proof of Theorem 7.5, which we recall for convenience.

Theorem 7.5. *Suppose that (D, X, \mathcal{D}) is a degree-feasible configuration. Then, (D, X, \mathcal{D}) is minimal uncolorable if and only if (D, X, \mathcal{D}) is constructible. \diamond*

Proof. If (D, X, \mathcal{D}) is constructible, then (D, X, \mathcal{D}) is minimal uncolorable (by Proposition 7.7 and as each K-, DC-, and C-configuration is a minimal uncolorable degree-feasible configuration).

Now let (D, X, \mathcal{D}) be a minimal uncolorable degree-feasible configuration. We prove that (D, X, \mathcal{D}) is constructible by induction on the order of D . If $|D| = 1$, then $V(D) = \{v\}$, $X_v = \emptyset$ and $\mathcal{D} = \emptyset$ and so (D, X, \mathcal{D}) is a K-configuration. If $|D| = 2$, then $|\mathcal{D}| = 2$ and (D, X, \mathcal{D}) is uncolorable if and only if both D and \mathcal{D} are copies of $D(K_2)$. Thus, we may assume that $|D| \geq 3$. By Proposition 7.6(a),

$$|X_v| = d_D^+(v) = d_D^-(v) \quad \text{for all } v \in V(D). \tag{7.3}$$

We distinguish between two cases.

Case 1: D contains a separating vertex v^* . Then, D is the union of two connected induced subdigraphs D^1 and D^2 with $V(D^1) \cap V(D^2) = \{v^*\}$ and $|D^j| < |D|$ for $j \in \{1, 2\}$. By equation (7.3), all vertices from D^j except from v^* are EULERian in D^j (for $j \in \{1, 2\}$). However, since

$$\sum_{u \in V(D^j)} d_{D^j}^+(u) = \sum_{u \in V(D^j)} d_{D^j}^-(u) = |A(D^j)|$$

is the number of arcs of D^j , it follows that $d_{D^j}^+(v^*) = d_{D^j}^-(v^*)$ and so D^j is EULERian for $j \in \{1, 2\}$. For $j \in \{1, 2\}$, by \mathcal{T}^j we denote the set of all subsets T of \mathcal{D} with $|T \cap X_v| = 1$ for all $v \in V(D^j)$ and $|T \cap X_u| = 0$ for all $u \in V(D^{3-j}) \setminus \{v^*\}$ such that $\mathcal{D}[T]$ is acyclic. As (D, X, \mathcal{D}) is uncolorable and degree-feasible, both \mathcal{T}^1 and \mathcal{T}^2 are non-empty (by Proposition 7.6(b)). Moreover, for $j \in \{1, 2\}$, let X_j be the set of all vertices of X_{v^*} that do not occur in any set from \mathcal{T}^j . We claim that $X_{v^*} = X_1 \cup X_2$. For otherwise, there is a vertex $x \in X_{v^*} \setminus (X_1 \cup X_2)$. Then, x is contained in two sets $T^1 \in \mathcal{T}^1$ and $T^2 \in \mathcal{T}^2$, and so $T = T^1 \cup T^2$ is an acyclic transversal of (X, \mathcal{D}) . Thus, (D, X, \mathcal{D}) is colorable, a contradiction. Consequently, $X_{v^*} = X_1 \cup X_2$. For $j \in \{1, 2\}$, we define a cover (X^j, \mathcal{D}^j) of D^j as follows. For $v \in V(D^j)$, let

$$X_v^j = \begin{cases} X_v & \text{if } v \neq v^* \\ X_j & \text{if } v = v^*, \end{cases}$$

and let $\mathcal{D}^j = \mathcal{D}[\bigcup_{v \in V(D^j)} X_v^j]$. Then, $(D^j, X^j, \mathcal{D}^j)$ is an uncolorable feasible configuration for $j \in \{1, 2\}$: Suppose w.l.o.g. that $(D^1, X^1, \mathcal{D}^1)$ has an acyclic transversal T . Then T is in \mathcal{T}^1 , but T contains a vertex $x \in X_{v^*}^1 = X_1$, which is impossible. Furthermore, for each vertex $v \in V(D^j) \setminus \{v^*\}$, equation (7.3) implies that $|X_v^j| = d_{D^j}^+(v) = d_{D^j}^-(v)$. As $(D^j, X^j, \mathcal{D}^j)$ is

uncolorable and D^j is connected, it follows from Proposition 7.6(a) that $|X_{v^*}^j| \leq d_{D^j}^+(v^*)$ for $j \in \{1, 2\}$. Since $X_{v^*} = X_1 \cup X_2 = X_{v^*}^1 \cup X_{v^*}^2$, we conclude from (7.3) that

$$|X_{v^*}^1| + |X_{v^*}^2| \geq |X_{v^*}^1 \cup X_{v^*}^2| = |X_{v^*}| = d_D^+(v^*) = d_{D^1}^+(v^*) + d_{D^2}^+(v^*),$$

and, thus, $|X_{v^*}^j| = d_{D^j}^+(v^*) (= d_{D^j}^-(v^*))$ and $X_{v^*}^1 \cap X_{v^*}^2 = \emptyset$. Consequently, (D^j, X^j, D^j) is a degree-feasible configuration. Moreover, $\mathcal{D}' = \mathcal{D}^1 \cup \mathcal{D}^2$ is a spanning subdigraph of \mathcal{D} and $V(\mathcal{D}^1) \cap V(\mathcal{D}^2) = \emptyset$. So, (D, X, \mathcal{D}') is a degree-feasible configuration that is obtained from two isomorphic copies of $(D^1, X^1, \mathcal{D}^1)$ and $(D^2, X^2, \mathcal{D}^2)$ by the merging operation. Clearly, (D, X, \mathcal{D}') is uncolorable. Otherwise, there would exist an acyclic transversal T of (X, \mathcal{D}') and by symmetry we may assume that T would contain a vertex of $X_{v^*}^1$. But then, $T^1 = T \cap V(\mathcal{D}^1)$ would be an acyclic transversal of (X^1, \mathcal{D}^1) , contradicting that $(D^1, X^1, \mathcal{D}^1)$ is uncolorable. As (D, X, \mathcal{D}) is minimal uncolorable and as \mathcal{D}' is a spanning subhypergraph of \mathcal{D} , this implies that $\mathcal{D} = \mathcal{D}'$ and (D, X, \mathcal{D}) is obtained from two isomorphic copies of $(D^1, X^1, \mathcal{D}^1)$ and $(D^2, X^2, \mathcal{D}^2)$ by the merging operation. Then, by Proposition 7.7, both $(D^1, X^1, \mathcal{D}^1)$ and $(D^2, X^2, \mathcal{D}^2)$ are minimal uncolorable. Applying the induction hypothesis leads to $(D^j, X^j, \mathcal{D}^j)$ being constructible for $j \in \{1, 2\}$, and so (D, X, \mathcal{D}) is constructible. Thus, the proof of the first case is complete.

Case 2: D is a block. Then, since $|D| \geq 3$, each vertex of D is contained in a cycle of the underlying graph $G(D)$. We prove that (D, X, \mathcal{D}) is a K-, DC- or C-configuration by examining the cycles that may occur in $G(D)$ and showing that those always force the structure of (D, X, \mathcal{D}) to be as claimed. This is done via a sequence of claims. In the first three claims we analyze the case where D contains a digon and show that in this case, both D and \mathcal{D} are bidirected. Then, we can apply Theorem 4.2 to the undirected configuration $(G(D), X, G(\mathcal{D}))$ in order to deduce that (D, X, \mathcal{D}) is a K- or C-configuration. Afterwards, we analyze the case that D does not contain any digons and prove that this implies that (D, X, \mathcal{D}) is a DC-configuration. Recall that if C is a cycle in the underlying graph $G(D)$, then D_C is the maximal subdigraph of D such that $G(D_C) = C$.

Claim 7.5.1. *Let C be a cycle of length 3 in the underlying graph $G(D)$. If D_C is not a directed cycle, then $V(C)$ induces a complete digraph in D .* \diamond

Proof. Let v_1, v_2, v_3 be the vertices of C . By symmetry, assume that $\{v_3v_1, v_1v_2, v_3v_2\} \subseteq A(D)$. We prove that $v_1v_3 \in A(D)$. Let T be an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v_1$, let x_j be the unique vertex from $X_{v_j} \cap T$ (for $j \in \{2, 3\}$) and let $x_1 \in X_{v_1}$ such that $x_3x_1 \in A(\mathcal{D})$ (such a vertex exists by Proposition 7.8(c)). Then, by Proposition 7.8(c), $x_3x_2 \notin A(\mathcal{D})$.

Furthermore, by Proposition 7.8(a), x_1 must have an out-neighbor x in T . Assume that $x \in T \setminus \{x_2, x_3\}$. Then we can shift $v_3 \rightarrow v_1$, $v_2 \rightarrow v_3$ and $v_1 \rightarrow v_2$ and get a new acyclic transversal T' of (X', \mathcal{D}') . Moreover, if x'_2 is the vertex from $X_{v_2} \cap T'$, due to the shifting we have $x_1 x'_2 \in A(\mathcal{D})$. Since $T \setminus (X_{v_2} \cup X_{v_3}) = T' \setminus (X_{v_2} \cup X_{v_3})$ we conclude $N_{\mathcal{D}}^+(x_1) \cap T' \supseteq \{x'_2, x\}$ and so $|N_{\mathcal{D}}^+(x_1) \cap T'| \geq 2$, contradicting Proposition 7.8(a) (see Figure 7.4). Hence, $x \in \{x_2, x_3\}$. If $x = x_2$ (and so $x'_2 = x_2$), then starting from T and then shifting $v_3 \rightarrow v_1$ and $v_2 \rightarrow v_3$ leads to an acyclic transversal T^* of $(X, \mathcal{D})/v_2$ such that $|N_{\mathcal{D}}^-(x_2) \cap T^*| \geq 2$, in contradiction to Proposition 7.8(a). Thus, $x = x_3$ and so $x_1 x_3 \in A(\mathcal{D})$. However, this implies $v_1 v_3 \in A(\mathcal{D})$ (by (C2)), as claimed. By symmetry we conclude that $D[V(C)]$ is a complete digraph and the proof is complete. \square

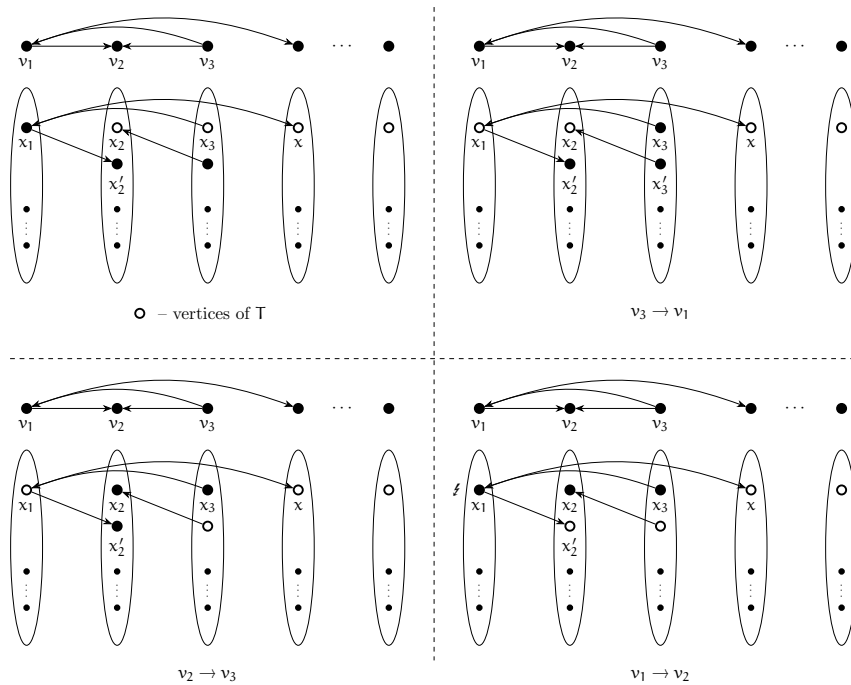


FIG. 7.4. (D, X, \mathcal{D}) before and after shifting $v_3 \rightarrow v_1, v_2 \rightarrow v_3$ and $v_1 \rightarrow v_2$.

Claim 7.5.2. *Let C be an induced cycle in the underlying graph $G(\mathcal{D})$. If D_C contains a digon, then D_C is a bidirected cycle.* \diamond

Proof. Assume, to the contrary, that D_C is not bidirected. Then (by symmetry) we can choose a cyclic ordering v_1, v_2, \dots, v_p of the vertices of C such that $v_1 v_2, v_2 v_1$ and $v_1 v_p$ are arcs of \mathcal{D} and that $v_p v_1 \notin A(\mathcal{D})$. Let T be an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v_1$. For

$i \in [2, p]$ let x_i be the vertex from $X_{v_i} \cap T$. By Proposition 7.8(b) and Proposition 7.9, there is a vertex $x \in X_{v_1}$ that is joined to x_2 by opposite arcs and a vertex $x' \in X_{v_1}$ with $x'x_p \in A(\mathcal{D})$. Moreover, by Proposition 7.8(a), $x \neq x'$. By shifting the vertices $v_2 \rightarrow v_1, v_3 \rightarrow v_2, \dots, v_p \rightarrow v_{p-1}$ counterclockwise on the cycle C we obtain from Proposition 7.8(c) that x has an out-neighbor x'_p in X_p . If we further shift $v_1 \rightarrow v_p$, we get a new acyclic transversal T' of (X', \mathcal{D}') such that $x'_p \in T'$. By Proposition 7.8(a), there must exist a vertex $y \in T'$ with $yx \in A(\mathcal{D})$. As x_2 is the unique in-neighbor of x from T , since v_1 has no neighbors besides v_2 and v_p from $V(C)$, and as the shifting only affected vertices from C , we conclude that $y \in X_{v_2} \cup X_{v_p}$. However, since $xx'_p \in A(\mathcal{D})$, it follows from Proposition 7.8(a) that $x_2 \notin T'$. Hence, $y \in X_{v_p}$ and so $v_p v_1 \in A(\mathcal{D})$, a contradiction. \square

Claim 7.5.3. *Suppose that D contains a digon. Then, D is bidirected.* \diamond

Proof. Assume, to the contrary, that D is not bidirected. As D is a block this implies that in the underlying graph $G(D)$ there is a cycle C of minimum length such that D_C contains a digon but is not bidirected. Since C has minimum length, we conclude that C is an induced cycle of $G(D)$, but then it follows from Claim 7.5.2 that D_C is bidirected, a contradiction. This proves the claim. \square

Suppose that D contains at least one digon. Then, D is bidirected (by Claim 7.5.3) and it follows from Corollary 7.10 that \mathcal{D} is bidirected, too. Consequently, $(G(D), X, G(\mathcal{D}))$ is a degree-feasible configuration. Furthermore, an acyclic transversal of (X, \mathcal{D}) is an independent transversal of $(X, G(\mathcal{D}))$ and vice versa, and it easy to check that $(G(D), X, G(\mathcal{D}))$ is minimal uncolorable (as (D, X, \mathcal{D}) is minimal uncolorable). Then, as $G(D)$ is a block, it follows from Theorem 4.2 that $(G(D), X, G(\mathcal{D}))$ is a K - or a C -configuration. As a consequence, (D, X, \mathcal{D}) is a K - or a C -configuration and there is nothing left to show. Hence, from now on we may assume the following:

$$D \text{ does not contain a digon.} \tag{7.4}$$

In the remaining part of the proof we will show that under the assumption (7.4), the configuration (D, X, \mathcal{D}) is a DC-configuration.

Claim 7.5.4. *The underlying graph $G(D)$ does not contain K_4 .* \diamond

Proof. Otherwise, $G(D)$ contains a cycle C such that D_C is not a directed cycle. Hence, by Claim 7.5.1, D would contain a complete digraph on three vertices, which contradicts (7.4). \square

Recall that K_4^- denotes the (undirected) graph that results from K_4 by deleting any edge.

Claim 7.5.5. *The underlying graph $G(\mathcal{D})$ does not contain any induced K_4^- .* \diamond

Proof. Assume that $G(\mathcal{D})$ contains an induced K_4^- , say $\tilde{G} = G(\tilde{\mathcal{D}})$. Then, by (7.4) and Claim 7.5.1, $V(\tilde{\mathcal{D}}) = \{v_1, v_2, v_3, v_4\}$ and $A(\tilde{\mathcal{D}}) = \{v_1v_2, v_1v_3, v_2v_4, v_3v_4, v_4v_1\}$. Let T be an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v_1$, and for $i \in [2, 4]$ let $x_i \in X_{v_i} \cap T$. Then it follows from Proposition 7.8(b),(c) that there are vertices $x, x' \in X_{v_1}$ with $x'x_2 \in A(\mathcal{D})$ and $xx_3 \in A(\mathcal{D})$. By Proposition 7.8(a), $x \neq x'$. By shifting $v_3 \rightarrow v_1$, we obtain that x_4 has an in-neighbor $x'_3 \in X_{v_3}$ (by Proposition 7.8(c)). We claim that $x'x'_3 \in A(\mathcal{D})$. To see this, starting from T , we can shift $v_3 \rightarrow v_1, v_4 \rightarrow v_3, v_2 \rightarrow v_4$ and then $v_1 \rightarrow v_2$ and obtain another acyclic transversal T' of (X', \mathcal{D}') with $x'_3 \in T'$. Then, x' must have an out-neighbor y in T' (by Proposition 7.8(a)). However, as $x \neq x'$, we deduce that $y \notin X_{v_2}$. As we only shifted along vertices of $\tilde{\mathcal{D}}$, we conclude that $y \notin T' \setminus (X_2 \cup X_3 \cup X_4)$ (since otherwise $\{y, x_2\} \subseteq |N_{\tilde{\mathcal{D}}}^+(x') \cap T|$, which leads to a contradiction to Proposition 7.8(a)). Moreover, as $v_1v_4 \notin A(\mathcal{D})$, this implies that $y \in X_{v_3}$ and so $y = x'_3$. Hence, $x'x'_3 \in A(\mathcal{D})$, as claimed. But now, starting from T we can shift $v_3 \rightarrow v_1, v_4 \rightarrow v_3$ and $v_1 \rightarrow v_4$ and obtain an acyclic transversal T^* of (X', \mathcal{D}') that contains both x_2 and x'_3 . As a consequence, $|N_{\tilde{\mathcal{D}}}^+(x') \cap T^*| \geq 2$, which contradicts Proposition 7.8(a). This proves the claim. \square

Claim 7.5.6. *Let C be an induced cycle of the underlying graph $G(\mathcal{D})$. Then, \mathcal{D}_C is a directed cycle.* \diamond

Proof. The proof is by reductio ad absurdum. Then, we can choose a cyclic ordering of the vertices of C , say v_1, v_2, \dots, v_p , such that $\{v_1v_2, v_1v_p\} \subseteq A(\mathcal{D})$. Furthermore, let T be an acyclic transversal of $(X', \mathcal{D}') = (X, \mathcal{D})/v_1$ and, for $i \in [2, p]$ let $x_i \in X_{v_i} \cap T$. Then, by Proposition 7.8(a),(b), there are vertices $x \neq x'$ from X_{v_1} with $xx_2 \in A(\mathcal{D})$ and $x'x_p \in A(\mathcal{D})$. Moreover, by shifting $v_p \rightarrow v_1, v_{p-1} \rightarrow v_p, \dots, v_2 \rightarrow v_3$ clockwise around C , we obtain that x' has an out-neighbor $x'_2 \in X_{v_2}$ (by Proposition 7.8(c)). We claim that $x_3x'_2 \in A(\mathcal{D})$. Assume, to the contrary, that $x_3x'_2 \notin A(\mathcal{D})$ and let T' be the transversal that results from T by shifting $v_2 \rightarrow v_1$. Then, x'_2 must have an in-neighbor y in T' (by Proposition 7.8(a)) and $y \notin X_{v_i}$ for $i \in [1, p]$ (as $x_3x'_2 \notin A(\mathcal{D})$, as $x' \notin T'$ and as C is an induced cycle). If instead, starting from T , we shift the vertices $v_p \rightarrow v_1, v_{p-1}v_p, \dots, v_2 \rightarrow v_3$, we obtain an acyclic transversal T^* of $(X, \mathcal{D})/v_2$ that contains both x' as well as y , contradicting Proposition 7.8(a) (as x'_2 has the two in-neighbors x', y in T^*). Thus, $x_3x'_2 \in A(\mathcal{D})$ and hence $v_3v_2 \in A(\mathcal{D})$. As a consequence, there is also a vertex $x'_3 \neq x_3$ from X_{v_3} such that $x'_3x_2 \in A(\mathcal{D})$. Now we can shift $v_2 \rightarrow v_1$ and obtain an acyclic transversal of $(X, \mathcal{D})/v_2$. By

repeating the same argumentation as above we conclude that $x'_3x_4 \in A(\mathcal{D})$. Now, we can iterate this procedure for the remaining vertices of C and obtain the following:

D_C is **alternating**, *i.e.* the vertices from D_C alternatively have two in-neighbours and two out-neighbours in D_C . (7.5)

Note that this implies, in particular, that C is even. Moreover, we conclude that for $i \in [2, p]$ there are vertices $x_i \neq x'_i$ from X_{v_i} such that the following holds:

- There is an acyclic transversal T of $(X', \mathcal{D}') = (X, \mathcal{D})/v_1$ that contains the vertices x_2, x_3, \dots, x_p , and
- $\{xx_2, x'_2x'_2, xx'_p, x'_px'_p\} \subseteq A(\mathcal{D})$ and for $i \in [2, p-2]$ we have $x_{i+1}x'_i, x'_{i+1}x_i \in A(\mathcal{D})$.

Note that (beginning from T) by shifting $v_2 \rightarrow v_1, v_3 \rightarrow v_2, \dots, v_p \rightarrow v_{p-1}$ counterclockwise around C and then shifting $v_1 \rightarrow v_p$ we obtain an acyclic transversal T' of (X', \mathcal{D}') that contains the vertices x'_2, x'_3, \dots, x'_p .

Since (D, X, \mathcal{D}) is minimal uncolorable, $\mathcal{D}[T \cup \{x\}]$ contains a directed cycle that must contain x , say C_x . Moreover, by Proposition 7.8(a) and since $xx_2 \in A(\mathcal{D})$, x and x_2 are consecutive on C_x . Let z denote the vertex different from x_2 such that x and z are consecutive on C_x . Then, $z \notin \{x_3, x_4, \dots, x_p\}$. This is due to the fact that C is an induced cycle in $G(D)$ (and so $v_1v_i \notin A(D)$ for $i \in [3, p-1]$) and that $xx'_p \in A(\mathcal{D})$ and, therefore, $xx_p \notin A(\mathcal{D})$. Moreover, we obtain the following:

C_x is an induced directed cycle of $\mathcal{D}[T \cup \{x\}]$ and
no vertex from C_x is adjacent to any vertex from $T \setminus V(C_x)$. (7.6)

Otherwise, starting from T we could shift the vertices around C_x and would obtain vertices $v^* \in V(D)$, $x^* \in X_{v^*} \cap V(C_x)$ and an acyclic transversal T^* of $(X, \mathcal{D})/v^*$ such that the neighbors of x^* on C_x are in T^* and such that x^* has another in- or out-neighbor in T^* , contradicting Proposition 7.8(a). Finally, we conclude that

no vertex from $\{x_3, x_4, \dots, x_p\}$ is in $V(C_x)$. (7.7)

Assume, to the contrary, that there is an index $i \neq 2$ with $x_i \in V(C_x)$. Then, as C is induced and since x_ix_{i+1} as well as $x_{i-1}x_i$ are not arcs of \mathcal{D} , both neighbors of x_i in C_x must be from $V(D) \setminus \{x_2, x_3, \dots, x_p\}$. But then, starting from T we can shift $x_2 \rightarrow x, x_3 \rightarrow x_2, \dots, x_i \rightarrow x_{i-1}$

and obtain an acyclic transversal \tilde{T} of $(X, \mathcal{D})/v_i$ such that x_i either has two in- or out-neighbors from \tilde{T} , contradicting Proposition 7.8(a).

By analogous arguments we conclude that $\mathcal{D}[T' \cup \{x\}]$ contains a directed cycle C'_x and x and x'_p are consecutive on C'_x . Furthermore, if z' denotes the vertex different from x'_p such that x and z' are consecutive on C'_x , we have $z \notin \{x'_2, x'_3, \dots, x'_{p-1}\}$. Moreover, the following holds:

$$\begin{aligned} C'_x \text{ is an induced directed cycle of } \mathcal{D}[T' \cup \{x\}] \text{ and} \\ \text{no vertex from } C'_x \text{ is adjacent to any vertex from } T' \setminus V(C'_x) \end{aligned} \tag{7.8}$$

and

$$\text{no vertex from } \{x'_2, x'_3, \dots, x'_{p-1}\} \text{ is in } V(C'_x). \tag{7.9}$$

Since $T \setminus \{x_2, x_3, \dots, x_p\} = T' \setminus \{x'_2, x'_3, \dots, x'_p\}$, it follows from Proposition 7.8(a) that $z = z'$. Let y denote the vertex from C_x different from x such that x_2 and y are consecutive on C_x and let y' denote the vertex from C'_x different from x such that x'_p and y' are consecutive on C'_x . From (7.7) and (7.9) we obtain that y and y' are from $T \setminus \{x_2, x_3, \dots, x_p\} = T' \setminus \{x'_2, x'_3, \dots, x'_p\}$. Combining (7.6) and (7.8) with the fact that z is contained in both C_x as well as $C_{x'}$ then leads to $y = y'$ and to $\mathcal{D}[V(C_x) \setminus \{x_2\}] = \mathcal{D}[V(C'_x) \setminus \{x'_p\}]$ being an induced directed path of \mathcal{D} . Let $v \in V(D)$ denote the vertex such that $y \in X_v$. Then we have $v_2v \in A(D)$ and $v_pv \in A(D)$ and so $\{v_1, v_2, v_p, v\}$ either induces a K_4^- in $G(D)$ (which is impossible by Claim 7.5.5) or a cycle C' of length 4 in $G(D)$ such that $D_{C'}$ is non-alternating in D , contradicting (7.5). This proves the claim. \square

Claim 7.5.7. *All cycles in $G(D)$ are induced, i.e., no cycle has a chord.* \diamond

Proof. Let C be a cycle in $G(D)$. We prove that C cannot contain a chord by induction on the length p of C . If $p = 4$, then C has no chord as otherwise, the vertices of C would either induce a K_4 or a K_4^- in $G(D)$, contradicting Claim 7.5.4 or Claim 7.5.5. Now assume $p \geq 5$. If C has a chord, say $uv \in E(G)$, then the edge uv divides the cycle C into two smaller cycles C_1 and C_2 . Then it follows from the induction hypothesis that neither C_1 nor C_2 has a chord. Hence, C_1 and C_2 are induced cycles of $G(D)$, and Claim 7.5.6 implies that D_{C_1} and D_{C_2} are directed cycles. Furthermore, uv is the only chord of C , since otherwise $G[V(C)]$ would contain a smaller cycle than C whose edges would have no cyclic orientation in D , contradicting Claim 7.5.6. By symmetry, suppose that $uv \in A(D)$. Then, in D_C the vertex u has two in-neighbors, and the vertex v has two out-neighbors, say w and w' . Moreover, by

symmetry, C_1 contains the vertices u, v , and w and C_2 contains the vertices u, v , and w' . Let T be an acyclic transversal of $(X, \mathcal{D})/v$ and let $u_1 \in X_u \cap T$, $w_1 \in X_w \cap T$, and $w'_1 \in X_{w'} \cap T$. Furthermore we choose a cyclic ordering of the vertices of C such that w is the left neighbor of v and w' is the right neighbor. Then, there are vertices $v_1, v_2, v_3 \in X_v$ with $v_1 w_1, v_2 w'_1$ and $u_1 v_3 \in A(\mathcal{D})$ (by Proposition 7.8(b),(c)). Furthermore, by Proposition 7.8(a), $v_1 \neq v_2$. By shifting $w \rightarrow v$ and the remaining vertices of C (except v) counterclockwise around C , we get an acyclic transversal T' of $(X, \mathcal{D})/w'$ with $v_1 \in T'$. Thus, by Proposition 7.8(c), there is a vertex $w'_2 \in X_{w'}$ with $v_1 w'_2 \in A(\mathcal{D})$. In particular, $w'_2 \neq w'_1$ (as $v_1 \neq v_2$). By similar argumentation, v_2 has an out-neighbor $w_2 \neq w_1$ from X_w (see Figure 7.5).

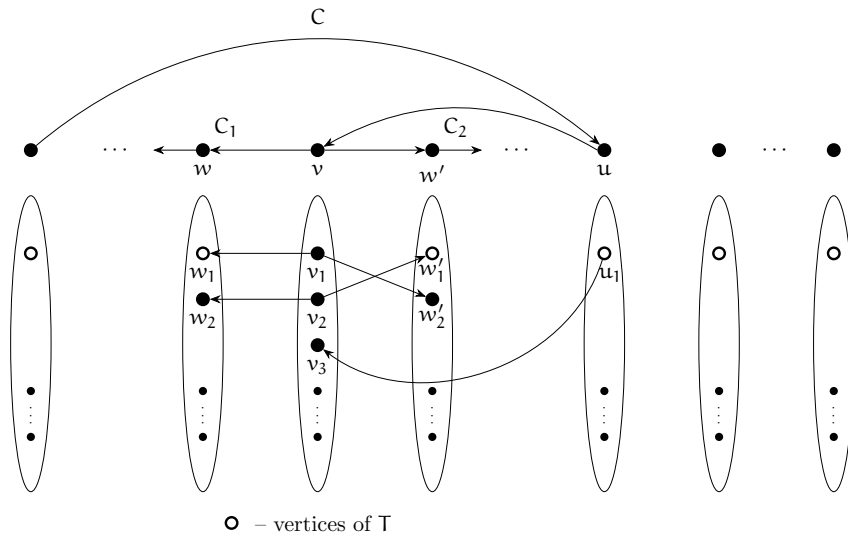


FIG. 7.5. Setting up (D, X, \mathcal{D}) .

Now we claim that $v_3 \notin \{v_1, v_2\}$. Assume that $v_3 = v_1$. Then, starting from T , we can shift each vertex from C_2 counterclockwise (beginning with $u \rightarrow v$) around C_2 (which gives us a transversal of $(X, \mathcal{D})/w'$ containing v_1) and, afterwards shift $v \rightarrow w'$. Then we get an acyclic transversal T^* of $(X, \mathcal{D})/v$ that contains w_1 as well as w'_2 and so $|N_{\mathcal{D}}^+(v_1) \cap T^*| \geq 2$, a contradiction to Proposition 7.8(a). Hence, $v_3 \neq v_1$. By repeating the argumentation with C_1 instead of C_2 we conclude that $v_3 \neq v_2$. Clearly, v_3 has an out-neighbor $w'_3 \in X_{w'}$ and an out-neighbor $w_3 \in X_w$ (shift clockwise around C_2 , respectively C_1). This is also illustrated in Figure 7.6. By (C2) and since $v_3 \notin \{v_1, v_2\}$, the vertex w'_3 is neither w'_1 nor w'_2 . Now finally, starting from T , we shift each vertex (beginning with $u \rightarrow v$, *i.e.* $u_1 \rightarrow v_3$) counterclockwise around C_2 such that we get an acyclic transversal of $(X, \mathcal{D})/w'$ and, afterwards, we shift $v \rightarrow w'$ (*i.e.* $v_3 \rightarrow w'_3$). This gives us an acyclic transversal \tilde{T} of $(X, \mathcal{D})/v$ with $w'_3 \in \tilde{T}$. We

claim that v_2 has no out-neighbor in \tilde{T} (which would contradict Proposition 7.8(a)). As uv is the unique chord of C , we conclude that $w \notin V(C_2)$ and so $w_1 \in \tilde{T}$. Since $v_1w_1 \in A(\mathcal{D})$, (C2) implies that $v_2w_1 \notin A(\mathcal{D})$. Furthermore, the out-neighbor of v_2 from \tilde{T} must be contained in $\bigcup_{v' \in V(C_2)} X_{v'}$ as w'_1 is the out-neighbor of v_2 from T and since we only shifted around C_2 . But since C_2 has no chords and since $vu \notin A(\mathcal{D})$, the out-neighbor of v_2 from \tilde{T} can only be the vertex from $X_w \cap \tilde{T}$, that is, w'_3 . However, $v_3w'_3 \in A(\mathcal{D})$ and so $v_2w'_3 \notin A(\mathcal{D})$. Thus, v_2 has no out-neighbor from \tilde{T} , a contradiction. This proves the claim. \square

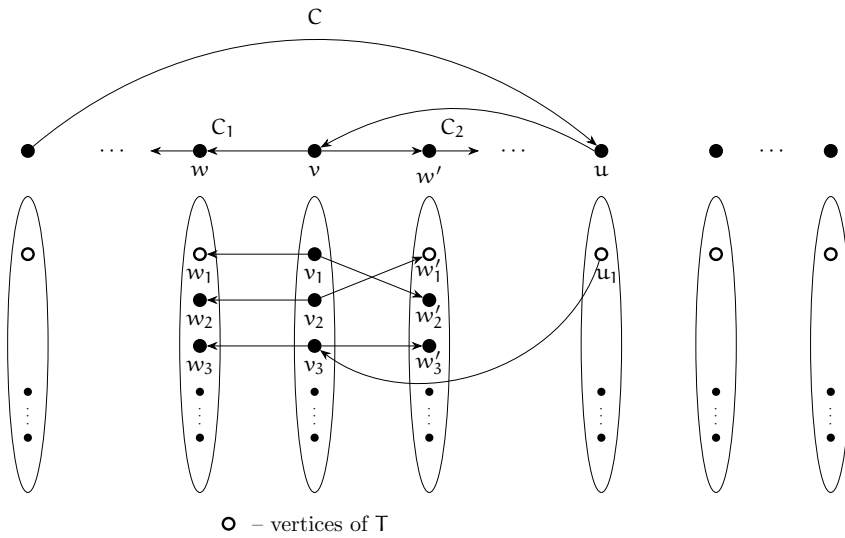


FIG. 7.6. Including the neighbors of v_3 .

The remaining part of the proof is straightforward: As D is a block, $G(D)$ contains an induced cycle C . Then, D_C is a directed cycle by Claim 7.5.6. We claim that $D = D_C$. Otherwise, there would be a vertex $v \in V(D) \setminus V(C)$. Moreover, since D and therefore $G(D)$ is a block, there are two internally disjoint paths P and P' in $G(D)$ from v to vertices $w \neq w'$ such that $V(P) \cap V(C) = \{w\}$ and $V(P') \cap V(C) = \{w'\}$. Since all cycles of $G(D)$ are induced (by Claim 7.5.7), w and w' are not consecutive in C . Let P_C and P'_C denote the two internally disjoint paths between w and w' contained in C . Then, P, P' together with P_C , respectively P, P' together with P'_C form induced cycles C_1 and C_2 of $G(D)$. Since D_C is a directed cycle, either D_{C_1} or D_{C_2} is not a directed cycle, contradicting Claim 7.5.6. Hence, $D = D_C$, *i.e.*, D is a directed cycle. As (D, X, \mathcal{D}) is a minimal uncolorable degree-feasible configuration, we easily conclude that (D, X, \mathcal{D}) is a DC-configuration. This completes the proof of the theorem. \blacksquare

7.4. A Brooks-type Theorem for DP-colorings of Digraphs

The next two statements are direct consequences of Theorem 7.5 and Proposition 7.4. In particular, Theorem 7.13 is a generalization of Theorem 6.3.

Corollary 7.12. *Let (D, X, \mathcal{D}) be a degree-feasible configuration. If (D, X, \mathcal{D}) is minimal uncolorable, then for each block $B \in \mathcal{B}(D)$ there is a uniquely determined cover (X^B, \mathcal{D}^B) of B such that the following statements hold:*

- (a) *For every block $B \in \mathcal{B}(D)$, the triple (B, X^B, \mathcal{D}^B) is a K-configuration, a DC-configuration, or a C-configuration.*
- (b) *The digraphs \mathcal{D}^B with $B \in \mathcal{B}(D)$ are pairwise disjoint and $\mathcal{D} = \bigcup_{B \in \mathcal{B}(D)} \mathcal{D}^B$.*
- (c) *For each vertex $v \in V(D)$ it holds $X_v = \bigcup_{B \in \mathcal{B}_v(D)} X_v^B$.* ◊

Theorem 7.13. *A connected digraph D is not DP-degree-colorable if and only if for every block B of D one of the following cases occurs:*

- (a) *B is a directed cycle of length ≥ 2 .*
- (b) *B is a bidirected cycle of length ≥ 3 .*
- (c) *B is a bidirected complete graph.* ◊

To conclude this chapter adequately, we deduce a BROOKS-type theorem for DP-colorings of digraphs. For undirected graphs, the theorem was proved by BERNSHTEYN, KOSTOCHKA, and PRON [14]. Note that the upcoming theorem generalizes the BROOKS-type Theorem 6.2 by Mohar as $\vec{\chi}(D) \leq \vec{\chi}_{DP}(D)$ for every digraph D by (7.1).

Theorem 7.14. *Let D be a connected digraph. Then, $\vec{\chi}_{DP}(D) \leq \max\{\Delta^+(D), \Delta^-(D)\} + 1$ and equality holds if and only if D is*

- (a) *a directed cycle of length ≥ 2 , or*
- (b) *a bidirected cycle of length ≥ 3 , or*
- (c) *a bidirected complete graph.* ◊

Proof. As mentioned earlier, $\vec{\chi}_{\text{DP}}(\mathbf{D}) \leq \max\{\Delta^+(\mathbf{D}), \Delta^-(\mathbf{D})\} + 1$ is always true. Moreover, if \mathbf{D} satisfies (a),(b), or (c), then $\vec{\chi}_{\text{DP}}(\mathbf{D}) = \max\{\Delta(\mathbf{D})^+, \Delta^-(\mathbf{D})\} + 1$, take a C-, DC-, or K-configuration. Now assume $\vec{\chi}_{\text{DP}}(\mathbf{D}) = \max\{\Delta^+(\mathbf{D}), \Delta^-(\mathbf{D})\} + 1$. Then, there is a cover $(\mathbf{X}, \mathcal{D})$ of \mathbf{D} such that $|X_v| \geq \max\{\Delta^+(\mathbf{D}), \Delta^-(\mathbf{D})\}$ for all $v \in V(\mathbf{D})$ and \mathbf{D} is not $(\mathbf{X}, \mathcal{D})$ -colorable. Hence, $(\mathbf{D}, \mathbf{X}, \mathcal{D})$ is an uncolorable degree-feasible configuration and there is a spanning subdigraph \mathcal{D}' of \mathcal{D} such that $(\mathbf{D}, \mathbf{X}, \mathcal{D}')$ is minimal uncolorable. Then, $|X_v| = d_{\mathbf{D}}^+(v) = d_{\mathbf{D}}^-(v)$ for all $v \in V(\mathbf{D})$ (by Proposition 7.6(a)) and each block of \mathbf{D} satisfies (a),(b) or (c) (by Theorem 7.13). Thus, $|X_v| = \max\{\Delta^+(\mathbf{D}), \Delta^-(\mathbf{D})\}$ for all $v \in V(\mathbf{D})$ and we conclude that \mathbf{D} has only one block and, therefore, satisfies (a), (b) or (c). This completes the proof. ■

7.5. Ohba's Conjecture for DP-colorings of Digraphs

In [97], OHBA conjectured that for graphs with few vertices compared to their chromatic number, the chromatic number and the list-chromatic number coincide. This conjecture was recently proved by NOEL, REED, and WU [96].

Theorem 7.15 (OHBA's Conjecture). *For every graph G satisfying $\chi(G) \geq (|G| - 1)/2$, we have $\chi(G) = \chi_\ell(G)$.* ◇

Regarding digraphs, BENSMAIL, HARUTYUNYAN and LE [9] came up with a simple transformation in order to obtain the directed version of OHBA's Conjecture from the undirected case.

Theorem 7.16 (BENSMAIL, HARUTYUNYAN and LE, 2018). *For every digraph \mathbf{D} satisfying $\vec{\chi}(\mathbf{D}) \geq (|\mathbf{D}| - 1)/2$, we have $\vec{\chi}(\mathbf{D}) = \vec{\chi}_\ell(\mathbf{D})$.* ◇

It is easy to see that OHBA's Conjecture does not hold if we take DP-colorings instead of list-colorings neither in the undirected nor in the directed case (just take C_4 , or the bidirected C_4 , respectively). However, BERNSHTEYN, KOSTOCHKA, and ZHU [15] proved the following, sharp, bound.

Theorem 7.17 (BERNSHTEYN, KOSTOCHKA, and ZHU, 2017). *For $n \in \mathbb{N}$, let $r(n)$ denote the minimum $r \in \mathbb{N}$ such that every graph G with $|G| = n$ and $\chi(G) \geq r$ satisfies $\chi_{\text{DP}}(G) = \chi(G)$. Then,*

$$n - r(n) = \Theta(\sqrt{n}). \quad \diamond$$

By combining the ideas from [9] with a similar technique to the one used in the proof of Theorem 7.1, we obtain the following theorem.

Theorem 7.18. *Let D be a digraph and let (V_1, V_2, \dots, V_k) be a partition of $V(D)$ such that $D[V_i]$ contains no directed cycle for $i \in [1, k]$. Furthermore, let G be the complete multipartite graph with classes V_1, V_2, \dots, V_k . Then, $\vec{\chi}_{\text{DP}}(D) \leq \chi_{\text{DP}}(G)$. \diamond*

Proof. Let $\chi_{\text{DP}}(G) = \ell$. Suppose that there is an ℓ -cover (X, \mathcal{D}) of D such that (X, \mathcal{D}) contains no acyclic transversal. We define an ℓ -cover (X_G, \mathcal{G}) of G as follows. Let $X_G = X$, and let $E(\mathcal{G})$ be the set of all edges $x_v x_w$ such that there are indices $i, j \in [1, k]$ with $i < j$ and vertices $v \in V_i, w \in V_j$ with $x_v \in X_v, x_w \in X_w$, and $x_v x_w \in A(\mathcal{D})$. As $\chi_{\text{DP}}(G) = \ell$, there is an independent transversal T_G of (X_G, \mathcal{G}) . As (X, \mathcal{D}) contains no acyclic transversal, $\mathcal{D}[T_G]$ contains a directed cycle C . Let $V' = \{v \in V(D) \mid X_v \cap C \neq \emptyset\}$. Then, $D[V']$ contains a directed cycle, as well. Since V_i is acyclic for all $i \in [1, k]$, this implies that there are indices $i < j$ from $[1, k]$ and vertices $v \in V_i, w \in V_j$ such that $vw \in A(D[V']), X_v \cap T_G = \{x_v\} \in V(C), X_w \cap T_G = \{x_w\} \in V(C)$, and $x_v x_w \in A(\mathcal{D})$. Consequently, $x_v x_w \in E(\mathcal{G}[T_G])$ and so T_G is not an independent transversal of (X_G, \mathcal{G}) , a contradiction. This completes the proof. \blacksquare

Corollary 7.19. *For $n \in \mathbb{N}$, let $r(n)$ denote the minimum $r \in \mathbb{N}$ such that every digraph D with $|D| = n$ and $\vec{\chi}(D) \geq r$ satisfies $\vec{\chi}_{\text{DP}}(D) = \vec{\chi}(D)$. Then,*

$$n - r(n) = \Theta(\sqrt{n}). \quad \diamond$$

Proof. That $n - r(n) = \mathcal{O}(\sqrt{n})$ follows from the fact that for each bidirected digraph D we have $\vec{\chi}_{\text{DP}}(D) = \chi_{\text{DP}}(G(D))$ (by Corollary 7.2) and from Theorem 7.17. The fact that $n - r(n) = \Omega(\sqrt{n})$ can easily be deduced by combining Theorems 7.17 and 7.18. \blacksquare

Chapter 8

CRITICAL DIGRAPHS

8.1. Introduction

As we have already examined in Chapter 3, critical graphs are one of the main tools for proving coloring results. Introduced by DIRAC in his doctoral thesis [35], who intended to use them for proving the *Four-Color-Conjecture*, critical graphs could not really serve their original purpose, but, nevertheless, have countless applications. However, not much is known about critical digraphs. Recall that a digraph D is **critical** and **k -critical** if $\vec{\chi}(D) = k$, but $\vec{\chi}(D') < k$ for each proper subdigraph D' of D . Since removing an arc or a vertex from a digraph decreases the dichromatic number at most by one, it is easy to see that every digraph D contains a $\vec{\chi}(D)$ -critical subdigraph. The next proposition goes back to NEUMANN-LARA [94] (the exact formulation is from [71, Prop. 1]) and states some basic facts on critical digraphs. The proofs are straightforward and left to the reader.

Proposition 8.1. *Let $k \geq 1$ be an integer and let D be a k -critical digraph. Then, the following statements hold:*

- (a) *If $v \in V(D)$ and if φ is a $(k-1)$ -coloring of $D - v$, then for each $\alpha \in [1, k-1]$ the color class $\varphi^{-1}(\alpha)$ contains an out-neighbor of v and an in-neighbor of v . As a consequence, $|\varphi(N_D^+(v))| = |\varphi(N_D^-(v))| = k-1$.*
- (b) *Each vertex $v \in V(D)$ satisfies $\min\{d_D^-(v), d_D^+(v)\} \geq k-1$.*

- (c) For each arc $a = uv \in A(D)$ and for each $(k-1)$ -coloring φ of $D - a$, there is a monochromatic directed path from v to u with respect to φ .
- (d) $|D| \geq k$ and equality holds if and only if $D = D(K_k)$.
- (e) If $k = 1$ then $|D| = 1$, and if $k = 2$ then D is a directed cycle. \diamond

In order to emphasize the usefulness of the concept of criticality, let us give an astonishingly short proof of Theorem 6.1, which we recall for convenience.

Theorem 6.1. *Let D be a digraph. Then,*

$$\vec{\chi}(D) \leq \min\{\Delta^-(D), \Delta^+(D)\} + 1. \quad \diamond$$

Proof. Let $k = \vec{\chi}(D)$. Then, D contains a k -critical digraph D' and, by Proposition 8.1(b), we have $\min\{d_{D'}^+(v), d_{D'}^-(v)\} \geq k-1$ for all $v \in V(D')$. As a consequence,

$$\min\{\Delta^+(D), \Delta^-(D)\} \geq \min\{\Delta^+(D'), \Delta^-(D')\} \geq k-1 = \vec{\chi}(D) - 1,$$

which settles the proof. \blacksquare

Moreover, MOHAR [90] formulated his BROOKS-type Theorem (see Theorem 6.2) using critical digraph terminology:

Theorem 8.2 (MOHAR, 2010). *Let D be a k -critical digraph with $k \geq 2$ in which each vertex satisfies $d_D^+(v) = d_D^-(v) = k-1$. Then, one of the following cases occurs:*

- (a) $k = 2$ and D is a directed cycle, or
- (b) $k = 3$ and D is a bidirected cycle of odd length ≥ 3 , or
- (c) $k \geq 4$ and D is a bidirected complete graph of order k . \diamond

Even if it does not appear so at first sight, Theorem 6.2 is equivalent to the above theorem:

Theorem 6.2. *Let D be a connected digraph. Then, $\vec{\chi}(D) \leq \max\{\Delta^-(D), \Delta^+(D)\} + 1$ and equality holds if and only if D is*

- (a) a directed cycle of length ≥ 2 , or
- (b) a bidirected cycle of odd length ≥ 3 , or

(c) a bidirected complete graph. ◊

Proof of the equivalence. As the digraph D from Theorem 8.2 satisfies $\chi(D) = k = \max\{\Delta^-(D), \Delta^+(D)\} + 1$, Theorem 8.2 immediately follows from Theorem 6.2.

The converse is less obvious: from Theorem 6.1 we obtain $\vec{\chi}(D) \leq \max\{\Delta^-(D), \Delta^+(D)\} + 1$. Moreover, it is easy to check that if D is a directed cycle, a bidirected cycle of odd length, or a bidirected complete graph, then $\vec{\chi}(D) = \max\{\Delta^-(D), \Delta^+(D)\} + 1$. It remains to be proved that if $\vec{\chi}(D) = \max\{\Delta^-(D), \Delta^+(D)\} + 1$, then D satisfies (a), (b), or (c) of Theorem 6.2. To this end, let D be a minimal counter-example, that is:

- (1) $\vec{\chi}(D) = \max\{\Delta^-(D), \Delta^+(D)\} + 1$,
- (2) D neither satisfies (a), (b), nor (c), and
- (3) $|A(D)|$ is minimum with respect to (1) and (2).

Let $k = \vec{\chi}(D)$. Then, by (3), $\vec{\chi}(D - \alpha) = k - 1$ for any $\alpha \in A(D)$ and so D is k -critical. Note that $k \geq 2$ as $D(K_1)$ is the only 1-critical digraph. From Proposition 8.1(b) it follows that

$$\min\{d_D^-(v), d_D^+(v)\} \geq k - 1 = \max\{\Delta^-(D), \Delta^+(D)\}$$

for all $v \in V(D)$ and so we have $d_D^+(v) = d_D^-(v)$ for each vertex $v \in V(D)$. Then, by Theorem 8.2, D satisfies (a), (b), or (c), which is impossible. This completes the proof. ■

Now let D be a k -critical digraph. As we have already utilized, $\min\{d_D^-(v), d_D^+(v)\} \geq k - 1$ for every vertex v of D . This again gives us a way to classify the vertices of D as we did in Chapters 3 and 5. A vertex $v \in V(D)$ is a **low-vertex** of D if $d_D^+(v) = d_D^-(v) = k - 1$ and a **high vertex** of D , otherwise. The **low vertex subdigraph** D_L of D is the digraph that is induced by the set of low vertices of D . In the next section, we will transfer GALLAI's well-known Theorem 3, which describes the structure of the low vertex subgraph of a k -critical graph, to digraphs (see Theorem 8.3').

Later in this chapter, we will try to analyze the structure of a k -critical digraph as a whole. Let $\vec{CRI}(k)$ denote the class of k -critical digraphs and $CRI(k)$ denote the class of k -critical graphs, respectively. Then, it is easy to see that $\vec{CRI}(0) = \{\emptyset\}$, $\vec{CRI}(1) = \{K_1\}$, and $\vec{CRI}(2)$ consists of all directed cycles. Nevertheless, it is not even known which digraphs $\vec{CRI}(3)$ consists of; unlike in the undirected case, where it follows from KÖNIG's characterization of bipartite graphs [68] that $CRI(3)$ coincides with the class of all odd cycles. So, how can we

possibly try to describe the structure of k -critical digraphs for $k \geq 3$? In Chapter 5, we have examined a method for creating an infinite class of critical hypergraphs (see Theorem 5.7), the so called HAJÓS join. Initially, HAJÓS developed the HAJÓS join only for graphs [54]; he proved that a simple graph G has chromatic number at least k if and only if it contains a subgraph that can be obtained from K_k 's by applying HAJÓS joins and identifying non-adjacent vertices. As these two operations preserve the chromatic number, it immediately follows that every k -critical graph can be constructed from copies of K_k by those operations. In Section 8.3, we introduce the digraph version of HAJÓS' operation, the directed and the bidirected HAJÓS join. Moreover, we transfer his result to digraphs (see Theorem 8.8'). In Section 8.4 eventually, we prove the digraph counterpart to a strengthening of HAJÓS' theorem due to URQUHART [119] (see Theorem 8.17).

8.2. A Gallai-type Theorem for Critical Digraphs

Recall from the introduction, respectively from Chapter 5 that we classified the vertices of a k -critical graph G into two groups: high vertices and low vertices. The subgraph G_L of G induced by the low vertices of G , *i.e.*, the vertices of G having degree $k - 1$ in G , is called **low-vertex subgraph** of G . GALLAI [48] (see also Theorem 3) proved that the blocks of the low vertex subgraph have a specific structure:

Theorem 8.3 (GALLAI, 1963). *Let G_L be the low vertex subgraph of a k -critical graph G . Then, each block of G_L is a complete graph or an odd cycle.* \diamond

Looking back at Theorems 6.2 and 6.3, we recall that we have always had three kinds of 'bad' blocks in the digraph setting; directed cycles, bidirected cycles of odd length, and bidirected complete graphs. Thus, it comes without surprise that we will see the same kind of blocks in the low vertex subdigraph. Somewhat surprising, however, is the fourth kind of block that may occur:

Theorem 8.3'. *Let D_L be the low vertex subdigraph of a k -critical digraph D . Then, each block B of D_L satisfies at least one of the following statements:*

- (a) B consists of just one single arc.
- (b) B is a directed cycle of length ≥ 2 .
- (c) B is a bidirected cycle of odd length ≥ 3 .
- (d) B is a bidirected complete graph. \diamond

For the proof of Theorem 8.3', we once again need Theorem 6.3, which we recall for convenience.

Theorem 6.3 (HARUTYUNYAN and MOHAR, 2011). *Let D be a connected digraph, and let L be a list-assignment such that $|L(v)| \geq \max\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$. Suppose that D is not L -colorable. Then, the following statements hold:*

- (a) D is EULERian and $|L(v)| = \max\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$.
- (b) If $B \in \mathcal{B}(D)$, then B is a directed cycle of length ≥ 2 , or B is a bidirected complete graph, or B is a bidirected cycle of odd length ≥ 5 .
- (c) For each $B \in \mathcal{B}(D)$ there is a set Γ_B of $\Delta^+(B)$ colors such that for every $v \in V(D)$, the sets Γ_B with $B \in \mathcal{B}_v(D)$ are pairwise disjoint and $L(v) = \bigcup_{B \in \mathcal{B}_v(D)} \Gamma_B$. \diamond

The next proposition states some important facts that will be needed for the proof of Theorem 8.3'.

Proposition 8.4. *Let D_L be the low vertex subdigraph of a k -critical digraph D . Moreover, given a vertex $v \in V(D_L)$, let φ be a $(k-1)$ -coloring of $D - v$ with color set $\Gamma = [1, k-1]$. Then the following statements hold:*

- (a) Each color from Γ appears exactly once in $N_D^+(v)$ and in $N_D^-(v)$.
- (b) If $u \in V(D_L)$ is adjacent to v , then uncoloring u and coloring v with the color of u leads to a $(k-1)$ -coloring of $D - u$. \diamond

Proof. Suppose (by symmetry) that there is a color $\alpha \in \Gamma$ such that α does not appear in $N_D^+(v)$. Then, coloring v with α cannot create a monochromatic cycle in D (as v has no out-neighbor with color α) and, thus, D would be $(k-1)$ -colorable, a contradiction. As $d_D^+(v) = k-1 = |\Gamma|$, this proves (a).

For the proof of (b), assume (by symmetry) that $uv \in A(D)$. Then it follows from (a) that after uncoloring u , vertex v has no in-neighbor with color $\varphi(u)$ and so coloring v with color $\varphi(u)$ cannot create a monochromatic cycle. \blacksquare

If the reader still remembers details of the previous chapter, he or she might become suspicious of statement (b) of the above proposition. Indeed, the procedure described there strongly resembles the shifting operation that we used in order to prove Theorem 7.5. The only difference makes the absence of a cover digraph and so we have to shift in the original

digraph, instead. Thus, in the following, we will call the procedure that is depicted in Proposition 8.4(b) **shifting** the color from u to v and briefly write $u \rightarrow v$.

Now let D be a k -critical digraph, let C be a (not necessarily directed) cycle in D_L and let $v \in V(C)$. Moreover, let φ be a $(k-1)$ -coloring of $D-v$ and let u and w be the vertices such that u, v and w are consecutive in C . Then, beginning with $u \rightarrow v$, we can shift each vertex of C , one after another, clockwise and obtain a new $(k-1)$ -coloring of $D-v$ (see Figure 8.1). Similar, beginning with $w \rightarrow v$, we can shift each vertex of C counter-clockwise and obtain a third $(k-1)$ -coloring of $D-v$. The main idea for this goes back to GALLAI [48]; we will use this observation frequently in the following.

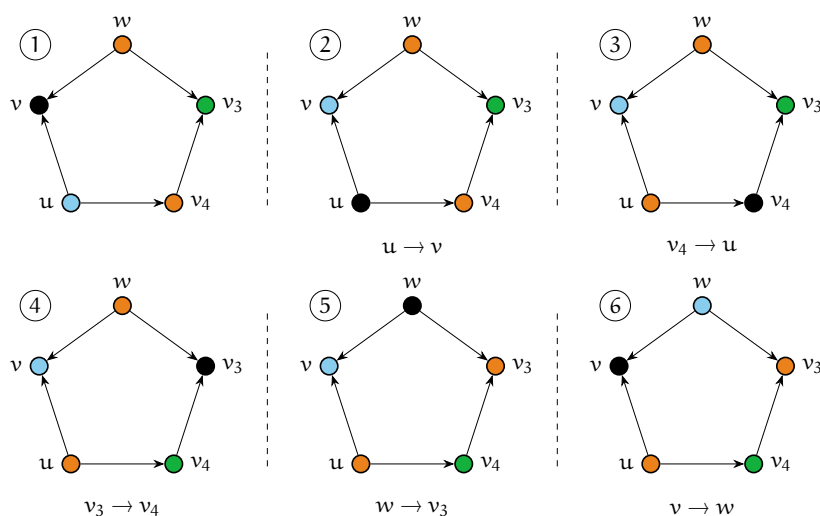


FIG. 8.1. The black (uncolored) vertex denotes the clockwise shifting around a cycle.

Proof of Theorem 8.3'. Let D_L be the low vertex subdigraph of a k -critical digraph D and let B be an arbitrary block of D_L . If $|B| = 1$, then $B = D(K_1)$ and we are done. If $|B| = 2$, then either B consists of just one arc or B is a bidirected complete graph and so there is nothing to show. Thus, we may assume $|B| \geq 3$.

Claim 8.4.1. For all vertices $v \in V(B)$ we have $d_B^+(v) = d_B^-(v)$, i.e., B is EULERIAN. \diamond

Proof. For otherwise, we may assume that $d_B^+(v) < d_B^-(v)$ for some $v \in V(B)$. Let φ be a $(k-1)$ -coloring of $D-v$. Since $d_D^+(v) = d_D^-(v) = k-1$, it follows from Proposition 8.4(a) that there is a color α that appears in $N_B^-(v)$ but not in $N_B^+(v)$. Let u be the vertex from $N_B^-(v)$ with $\varphi(u) = \alpha$. Note that Proposition 8.4(a) furthermore implies that there is a vertex in

$v' \in N_D^+(v) \cap (V(D) \setminus V(B))$ that has color α . First we show that $d_B^+(v) = 0$. Suppose, to the contrary, that $d_B^+(v) > 0$ and let w be an out-neighbor of v in D_L . Then, in B there is a (not-necessarily directed) induced cycle C such that u, v and w are consecutive on C . Beginning with $u \rightarrow v$, we shift all vertices of C clockwise and obtain a new $(k-1)$ -coloring φ' of $D-v$ with $\varphi'(w) = \alpha$. Since no vertex from $V(D) \setminus V(C)$ took part in the shifting, we have $\varphi'(v') = \varphi(v') = \alpha$ and so α appears twice in $N_D^+(v)$, contradicting Proposition 8.4(a). This proves that $d_B^+(v) = 0$.

Let again C be an (undirected) induced cycle in B such that u and v are consecutive on C and let w be the other neighbor of v in C . Then, w is also an in-neighbor of v (as $d_B^+(v) = 0$). Thus, it follows from Proposition 8.4(a) that $\varphi(w) \neq \varphi(u)$, say $\varphi(w) = \beta$. Moreover, we obtain that the vertices of C (except from v) are colored alternately with β and α . Otherwise, there are two consecutive vertices x, x' on C such that $\{\varphi(x), \varphi(x')\} \neq \{\alpha, \beta\}$. Then we can shift the colors around the vertices of C such that u gets color $\varphi(x)$ and w gets color $\varphi(x')$ and obtain a $(k-1)$ -coloring φ' of $D-v$ with $\{\varphi'(u), \varphi'(w)\} \neq \{\alpha, \beta\}$, which contradicts Proposition 8.4(a) as C is induced and so no neighbors of v besides u and w take part in the shifting.

As a consequence, C has odd length. Now let $v = v_1, w = v_2, v_3, \dots, u = v_r, v_1$ be a cyclic ordering of the vertices of C . We claim that $v_3v_2 \notin A(D)$. Assume, to the contrary, $v_3v_2 \in A(D)$. Then, we can shift $w \rightarrow v$ and obtain a coloring φ' of $D-w$ with $\varphi'(v) = \beta$ and $\varphi'(v_3) = \alpha$. In particular, v_3 is the only in-neighbor of w that has color α with respect to φ' . On the other hand, beginning from φ with $u \rightarrow v$, we can shift every vertex besides v clockwise around C (the last shift is $w \rightarrow v_3$) and get a $(k-1)$ -coloring φ^* of $D-w$ with $\varphi^*(v) = \alpha$ and $\varphi^*(v_3) = \beta$. As $vw \notin A(D)$ and as C is induced, it follows that w has no in-neighbor that has color α with respect to φ^* , a contradiction. Hence, $v_3v_2 \notin A(D)$ and so $v_2v_3 \in A(D)$. By repeating this argumentation, we obtain that $v_{i+1}v_i \notin A(D)$ but $v_iv_{i+1} \in A(D)$ for $i \geq 2$ even and that $v_iv_{i+1} \notin A(D)$ but $v_{i+1}v_i \in A(D)$ for $i \geq 3$ odd. In particular, this leads to $v_rv \notin A(D)$, a contradiction. This proves the claim. \square

Now let φ be a $(k-1)$ -coloring of $D-B$ with color set $\Gamma = [1, k-1]$. For $v \in V(B)$, let

$$L(v) = \Gamma \setminus \varphi(N_D^+(v) \setminus V(B)).$$

Then, as $d_D^+(v) = d_D^-(v) = k-1 = |\Gamma|$ and since $d_B^+(v) = d_B^-(v)$ by Claim 8.4.1, we have $|L(v)| \geq \max\{d_B^+(v), d_B^-(v)\}$ for all $v \in V(B)$. Moreover, B is not L -colorable, as the union of any L -coloring of B with φ would clearly lead to a $(k-1)$ -coloring of D . Hence, we can apply Theorem 6.3 and so B is a directed cycle, or an odd bidirected cycle, or a bidirected

complete graph, as claimed. ■

In the undirected case, GALLAI [48] showed that the only blocks of the low vertex graph are complete graphs or odd cycles. Although for digraphs the directed cycles arise naturally, it was quite unexpected that there may also be blocks that consist of just one arc. That this indeed may happen is illustrated in Figure 8.2, where a 4-critical digraph is displayed; here the low vertex subdigraph consists of every vertex except the vertex v . The reader might wonder how we came up with this example as it is not fully trivial to see that the digraph is 4-critical: In fact, the digraph is the so called HAJÓS join of two bidirected K_4 , which is the main topic of the next section. Note that it is even possible to create infinite families of digraphs D such that there are blocks of D_L consisting of just a single arc just by performing a HAJÓS join between two copies of the bidirected K_k . If the reader already wants to know how the HAJÓS construction works, we recommend having a look at Theorem 8.7.

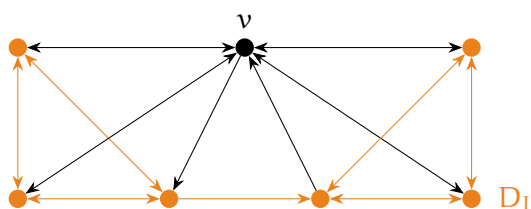


FIG. 8.2. Here, one block of D_L consists of just one arc.

GALLAI used the characterization of the low vertex subgraph of critical graphs that he obtained in [48] to establish a lower bound for the number of edges of critical graphs. We can apply the same approach to achieve a similar bound for the number of arcs in critical digraphs.

Theorem 8.5. *Let D be a $(k + 1)$ -critical digraph with $k \geq 3$ and without digons. Then*

$$2|A(D)| \geq \left(2k + \frac{k}{3k + 1}\right) |D|.$$

Proof. Let $V = V(D)$ and let $n = |V|$. For a set $X \subseteq V$, let $a(X)$ denote the number of arcs of $D[X]$. Furthermore, let

$$R = \left(2k + \frac{k}{3k + 1}\right).$$

Our aim is to show that $2a(V) \geq Rn$. If $|D_L| = 0$, then every vertex v of D satisfies $d_D^+(v) + d_D^-(v) \geq 2k + 1$, which leads to $2a(V) \geq (2k + 1)n \geq Rn$, and we are done. So

assume that $|D_L| \geq 1$. Since D has no digons, it follows from Theorem 8.3' that each block of D_L consists of an isolated vertex, or exactly one arc, or is a directed cycle of length at least three.

Now we claim that $3|D_L| \geq 2|A(D_L)|$. It suffices to prove this claim for each component of D_L separately. Thus, we may assume that D_L is connected. The proof of the inequality is by induction on the number of blocks of D_L . If D_L itself is a block, the statement clearly holds. If D_L consists of more than one block, let B be an end-block of D_L , *i.e.*, B is a block of D_L containing exactly one separating vertex v_B of D_L . Now let $D'_L = D_L - (V(B) \setminus \{v_B\})$. Then, by the induction hypothesis, we have $3|D'_L| \geq 2|A(D'_L)|$. As B either consists of exactly one arc or is a directed cycle of length $\ell \geq 3$, we have $3(|B| - 1) - 2|A(B)| \geq 0$. This leads to

$$3|D_L| = 3|D'_L| + 3(|B| - 1) \geq 2|A(D'_L)| - 2|A(B)| + 3(|B| - 1) \geq 2|A(D_L)|,$$

which proves the claim.

Since every vertex of $U = V(D_L)$ has total degree $2k$ in D (*i.e.*, $d_D^+(v) + d_D^-(v) = 2k$ for all $v \in U$) and since $k \geq 3$, we obtain that

$$2\alpha(V) = 2\alpha(W) + 4k|U| - 2\alpha(U) \geq 4k|U| - 2\alpha(U) \geq (4k - 3)|U| \geq 3k|U|.$$

On the other hand, since every vertex in W has total degree at least $2k + 1$ and since $n = |U| + |W|$, we have

$$2\alpha(V) \geq 2kn + |W| \geq (2k + 1)n - |U|.$$

Adding the first inequality to the second inequality multiplied with $3k$ yields

$$2\alpha(V)(3k + 1) \geq 3k(2k + 1)n,$$

and, as $3k(2k + 1) = 2k(3k + 1) + k$, we conclude

$$2\alpha(V) \geq \left(2k + \frac{k}{3k + 1}\right)n = Rn.$$

Thus, the proof is complete. ■

8.3. Construction of Critical Digraphs

In Figure 8.2 we have already seen that it might be useful to know some ways how to create critical digraphs. In this section, we will present two methods on how to do so, the DIRAC join and the directed and bidirected HAJÓS join. Note that, given a simple graph G , it follows from (6.1) that

$$G \text{ is } k\text{-critical (with respect to } \chi) \text{ if and only if } D(G) \text{ is } k\text{-critical.} \quad (8.1)$$

Now let D_1 and D_2 be two disjoint digraphs. Let D be the digraph obtained from the union $D_1 \cup D_2$ by adding all possible arcs in both directions between D_1 and D_2 , *i.e.*, $V(D) = V(D_1) \cup V(D_2)$ and $A(D) = A(D_1) \cup A(D_2) \cup \{uv, vu \mid u \in V(D_1) \text{ and } v \in V(D_2)\}$. We say that D is the **DIRAC join** of D_1 and D_2 and denote it by $D = D_1 \boxtimes D_2$. The proof of the next theorem is quite simple and therefore left to the reader.

Theorem 8.6 (DIRAC Construction). *Let $D = D_1 \boxtimes D_2$ be the DIRAC join of two disjoint non-empty digraphs D_1 and D_2 . Then, $\vec{\chi}(D) = \vec{\chi}(D_1) + \vec{\chi}(D_2)$ and D is critical if and only if both D_1 and D_2 are critical.* \diamond

Unlike in Chapter 5, the HAJÓS join is usually a tool for undirected graphs that can be used to create infinite families of k -critical graphs, see *e. g.* [54]. For digraphs, an equivalent construction was defined by HOSHINO and KAWARABAYASHI in [60]. Let D_1 and D_2 be two disjoint digraphs and select an arc u_1v_1 of D_1 as well as an arc v_2u_2 of D_2 . Let D be the digraph obtained from the union $D_1 \cup D_2$ by deleting both arcs u_1v_1 and v_2u_2 , identifying the vertices v_1 and v_2 to a new vertex v , and adding the arc u_1u_2 . We say that D is the **(directed) HAJÓS join** of D_1 and D_2 and write $D = (D_1, v_1, u_1) \nabla (D_2, v_2, u_2)$ or, briefly, $D = D_1 \nabla D_2$. Recall that for the **undirected HAJÓS join** of two undirected graphs G_1 and G_2 , we just choose two edges $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$ and perform exactly the same procedure as described above (except for the orientations). Statement (c) of the following theorem has already been mentioned in [60, Proposition 2].

Theorem 8.7 (HAJÓS Construction). *Let $D = D_1 \nabla D_2$ be the HAJÓS join of two disjoint non-empty digraphs D_1 and D_2 . Then, the following statements hold:*

- (a) $\vec{\chi}(D) \geq \min\{\vec{\chi}(D_1), \vec{\chi}(D_2)\}$.
- (b) *If $\vec{\chi}(D_1) = \vec{\chi}(D_2) = k$ and $k \geq 2$, then $\vec{\chi}(D) = k$.*
- (c) *If both D_1 and D_2 are k -critical and $k \geq 2$, then D is k -critical.*

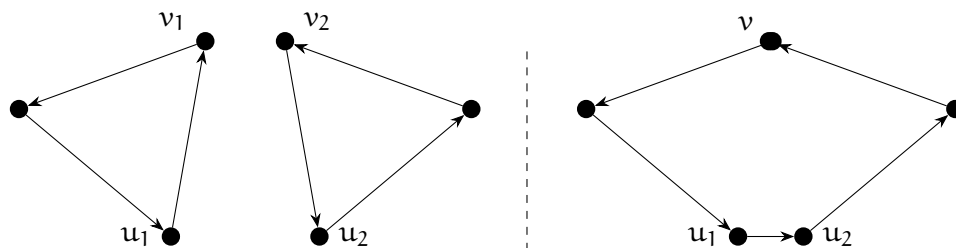


FIG. 8.3. The Hajós join of two directed cycles of length 3.

(d) If D is k -critical and $k \geq 2$, then both D_1 and D_2 are k -critical. ◊

Proof. Suppose that $D = (D_1, v_1, u_1) \nabla (D_2, v_2, u_2)$ and let v denote the vertex that is obtained from identifying v_1 and v_2 . To simplify the proof, we assume that $v = v_1 = v_2$. For the proof of (a) let $\vec{\chi}(D) = k$ and let φ be a k -coloring of D . For $i \in \{1, 2\}$, let φ_i denote the restriction of φ to D_i , where $\varphi_i(v_i) = \varphi(v)$. We claim that either φ_1 is a k -coloring of D_1 or φ_2 is a k -coloring of D_2 . Otherwise, in D_1 there is a monochromatic directed cycle C_1 that contains the arc u_1v_1 (as $D_1 - u_1v_1$ is a subdigraph of D and therefore k -colorable). Similar, in D_2 there exists a monochromatic cycle C_2 that contains the arc v_2u_2 . But then, $C_1 \cup C_2 - u_1v_1 - v_2u_2 + u_1u_2$ is a monochromatic directed cycle in D , a contradiction. This proves (a).

In order to prove (b), let $\vec{\chi}(D_1) = \vec{\chi}(D_2) = k$. By (a), $\vec{\chi}(D) \geq k$. Thus, it suffices to show that $\vec{\chi}(D) \leq k$. For $i \in \{1, 2\}$, let φ_i be a k -coloring of D_i . By permuting the colors if necessary we obtain $\varphi_1(v_1) = \varphi_2(v_2)$. For $w \in V(D)$ let

$$\varphi(w) = \begin{cases} \varphi_1(w) & \text{if } w \in V(D_1), \\ \varphi_2(w) & \text{if } w \in V(D_2), \text{ and} \\ \varphi_1(v_1) & \text{if } w = v. \end{cases}$$

We claim that φ is a k -coloring of D . For otherwise, D would contain a monochromatic directed cycle C with $\{u_1, u_2, v\} \subseteq V(C)$ and $u_1u_2 \in A(C)$. But then, $(C \cap D_1) + u_1v_1$ is a monochromatic directed cycle in D_1 , which is impossible.

For the proof of (c) it suffices to show that $\vec{\chi}(D - a) < k$ for all $a \in A(D)$ (by (b)). If $a = u_1u_2$, then choosing $(k - 1)$ -colorings of $D_1 - u_1v_1$ and $D_2 - v_2u_2$ that assign the same color to v_1 and v_2 and taking the union of those colorings clearly leads to a $(k - 1)$ -coloring of $D - a$. Let $a \in A(D) \setminus \{u_1u_2\}$. By symmetry, we may assume that $a \in A(D_1)$. Then,

there is a $(k-1)$ -coloring φ_1 of $D_1 - \mathbf{a}$ and a $(k-1)$ -coloring φ_2 of $D_2 - v_2u_2$ such that $\varphi_1(v_1) = \varphi_2(v_2)$. We claim that taking the union of those colorings gives us a $(k-1)$ -coloring of $D - \mathbf{a}$. Suppose, to the contrary, that there exists a monochromatic cycle C in $D - \mathbf{a}$. Then C contains the arc u_1u_2 and $(C \cap D_1) + u_1v_1$ is a monochromatic cycle in $D_1 - \mathbf{a}$, which is impossible. Hence, D is k -critical, as claimed.

To prove statement (d) first assume that $\vec{\chi}(D_1) \leq k-1$. Then there is a $(k-1)$ -coloring φ_1 of D_1 . Since D is k -critical, there furthermore exists a $(k-1)$ -coloring φ_2 of $D_2 - v_2u_2$ with $\varphi_2(v_2) = \varphi_1(v_1)$ and the union of φ_1 and φ_2 leads to a $(k-1)$ -coloring of D (by the same arguments as in (c)). Hence, $\vec{\chi}(D_1) \geq k$ and, by symmetry, we obtain $\vec{\chi}(D_2) \geq k$. In order to complete the proof, it suffices to show that $\vec{\chi}(D_i - \mathbf{a}) < k$ for $i \in \{1, 2\}$ and for $\mathbf{a} \in A(D_i)$. By symmetry, we may assume that $i = 1$. If $\mathbf{a} = u_1v_1$, then $\vec{\chi}(D_1 - \mathbf{a}) < k$ as $D_1 - \mathbf{a}$ is a proper subdigraph of D and therefore $(k-1)$ -colorable. Let $\mathbf{a} \in A(D_1) \setminus \{u_1v_1\}$. Then, there is a $(k-1)$ -coloring φ of $D - \mathbf{a}$. We claim that the restriction of φ to $V(D_1)$ is a $(k-1)$ -coloring of $D_1 - \mathbf{a}$. For otherwise, in $D_1 - \mathbf{a}$ there would exist a monochromatic directed cycle C_1 that contains the arc u_1v_1 . Since $\vec{\chi}(D_2) \geq k$, the restriction of φ to $V(D_2)$ creates a monochromatic directed cycle C_2 in D_2 that contains the arc u_2v_2 . However, $C_1 \cup C_2 - u_1v_1 - v_2u_2 + u_1u_2$ is a monochromatic directed cycle in $D - \mathbf{a}$ with respect to φ , a contradiction. This completes the proof. ■

Another common operation for graphs and digraphs is the identification of independent sets. Let D be a digraph and let I be a non-empty independent set of D , *i.e.*, $D[I]$ has no arcs. Then, we can create a new digraph H from $D - I$ by adding a new vertex v and adding all arcs from v to $N_D^+(I) = \bigcup_{u \in I} N_D^+(u)$ and all arcs from $N_D^-(I) = \bigcup_{u \in I} N_D^-(u)$ to v . We say that H is obtained from D by **identifying I with v** , or briefly by **identifying independent vertices** and write $H = D/(I \rightarrow v)$ (briefly $H = D/I$). It is obvious that any k -coloring of D/I can be extended to a k -coloring of D by coloring each vertex of I with the color of v . Thus, $\vec{\chi}(D/I) \geq \vec{\chi}(D)$.

We define the class of **HAJÓS- k -constructible digraphs** as the smallest family of digraphs that contains all bidirected complete graphs of order k and is closed under HAJÓS joins and identifying independent vertices. The class of **HAJÓS- k -constructible graphs** is defined accordingly. In 1961, HAJÓS [54] proved the following remarkable result.

Theorem 8.8 (HAJÓS). *Let $k \geq 3$ be an integer. A graph has chromatic number at least k if and only if it contains a HAJÓS- k -constructible subgraph.* ◊

For digraphs, we obtain a similar result. While our proof also uses some of the original

ideas from HAJÓS, we need some new tricks related to perfect digraphs, which are examined below.

Theorem 8.8'. *Let $k \geq 3$ be an integer. A digraph has dichromatic number at least k if and only if it contains a HAJÓS- k -constructible subdigraph.* \diamond

The **clique number** $\omega(D)$ of a digraph D is the order of the largest bidirected complete subdigraph of D . As $\vec{\chi}(D(K_n)) = n$, every digraph D satisfies $\omega(D) \leq \vec{\chi}(D)$. A **perfect digraph** is a digraph D satisfying that for each induced subdigraph D' of D it holds $\vec{\chi}(D') = \omega(D')$. An **odd hole** is an (undirected) cycle of odd length at least 5 and an **odd antihole** is the complement of an odd hole. Moreover, a **filled odd hole/filled odd antihole** is a digraph D so that $S(D)$ is an odd hole/antihole, where $S(D)$ is the **symmetric part** of D , that is, the graph with vertex set $V(D)$ and edge set

$$E(S(D)) = \{uv \mid uv \in A(D) \text{ and } vu \in A(D)\}.$$

ANDRES and HOCHSTÄTTLER [4, Corollary 5] proved the following result on perfect digraphs.

Theorem 8.9 (ANDRES and HOCHSTÄTTLER). *A digraph D is perfect if and only if it contains none of the following as an induced subdigraph: a filled odd hole, a filled odd antihole, and a directed cycle of length at least 3.* \diamond

This theorem is a really nice and powerful tool in many ways. For the class of bidirected graphs, the theorem is equivalent to the Strong Perfect Graph Theorem (SPGT) by CHUDNOVSKY, ROBERTSON, SEYMOUR, and THOMAS [33], and hence, the SPGT follows from ANDRES and HOCHSTÄTTLER's result. Nevertheless, their proof heavily relies on the SPGT. We will use their result for the following corollary.

Corollary 8.10. *Let D be a digraph and for $u, v \in V(D)$ let $u \sim v$ denote the relation that $uv \notin A(D)$. If \sim is transitive, then D is perfect.* \diamond

Proof. By Theorem 8.9, we only need to prove that D does neither contain a filled odd hole, nor a filled odd antihole, nor an induced directed cycle of length at least 3 as an induced subdigraph.

First assume that D contains a filled odd hole C as an induced subdigraph and choose a cyclic ordering $v_1, v_2, \dots, v_r, v_1$ of the vertices of the filled odd hole. Then r is odd and $r \geq 5$. By symmetry, we may assume that $v_3 \sim v_1$. As \sim is transitive, this implies that $v_1 v_4 \in A(D)$ (as otherwise $v_3 \sim v_1, v_1 \sim v_4$, but $v_3 \not\sim v_4$) and so $v_4 \sim v_1$. As a consequence,

$v_1v_3 \in A(D)$ (since $v_4 \sim v_1$ and $v_4v_3 \in A(D)$). By continuing this argumentation we obtain that $v_1v_i \in A(D)$ for all $i \in [2, r]$. Moreover, regarding v_2 , it follows that $v_2v_4 \in A(D)$ (as otherwise $v_2 \sim v_4, v_4 \sim v_1$, but $v_2 \not\sim v_1$, a contradiction). As a consequence, $v_2v_i \in A(D)$ for all $i \in [4, r]$. Finally, $v_3v_r \in A(D)$ (as otherwise $v_3 \sim v_r, v_r \sim v_2$, but $v_3 \not\sim v_2$). However, since C is a filled odd hole, this gives us $v_r \sim v_3$ and so $v_r \sim v_3, v_3 \sim v_1$, but $v_1 \not\sim v_r$, a contradiction. Thus, D cannot contain a filled odd hole as an induced subdigraph.

Next assume that D contains a filled odd antihole C as an induced subdigraph. Let again $v_1, v_2, \dots, v_r, v_1$ be a cyclic ordering of the vertices. Then r is odd and $r \geq 5$. By symmetry, we may assume that $v_1 \sim v_2$. Then, $v_2v_3 \in A(D)$ as otherwise \sim would not be transitive. Continuing this argument, we obtain that $v_i \sim v_{i+1}$ for i odd and $v_iv_{i+1} \in A(D)$ for even i . As r is odd this implies $v_r \sim v_1$. As a consequence, $v_r \sim v_1, v_1 \sim v_2$, but $v_rv_2 \in A(D)$, a contradiction. Thus, D contains no filled antiholes as induced subdigraphs.

Finally, assume that D contains an directed cycle C of length at least 3 as an induced subdigraph. Again, let $v_1, v_2, \dots, v_r, v_1$ be a cyclic ordering of the vertices of C . Then, $v_1 \sim v_r, v_r \sim v_2$, but $v_1v_2 \in A(D)$, a contradiction. As a consequence, D is perfect by Theorem 8.9, and we are done. \blacksquare

Proof of Theorem 8.8'. Let $k \geq 3$ be an integer. Clearly, every HAJÓS- k -constructible digraph has dichromatic number at least k (by Theorem 8.7 and since $\vec{\chi}(D/I) \geq \vec{\chi}(D)$ for each independent set I of a digraph D). This proves the “if”-implication. The proof of the “only if”-implication is by reductio ad absurdum. Let D be a maximal counter-example in the sense that $\vec{\chi}(D) \geq k$ and D does not contain a HAJÓS- k -constructible subdigraph, but adding a new arc $a \in A(\overline{D})$ to D implies the existence of a HAJÓS- k -constructible subdigraph D_a of $D + a$ with $a \in A(D_a)$. For two vertices $u, v \in V(D)$, let $u \sim v$ denote the relation that $uv \notin A(D)$. We distinguish between two cases and show that both of them lead to a contradiction.

Case 1: \sim is transitive. Then, D is perfect by Corollary 8.10 and so D contains a bidirected complete graph of order at least k as a subdigraph and, therefore, a HAJÓS- k -constructible subdigraph, which is impossible.

Case 2: \sim is not transitive. Then there are vertices $u, v, w \in V(D)$ such that $uv \notin A(D)$, $vw \notin A(D)$, but $uw \in A(D)$. Hence, both arcs uv and vw belong to $A(\overline{D})$. By the maximality of D , there exist HAJÓS- k -constructible subdigraphs $D_{uv} \subseteq D + uv$ and $D_{vw} \subseteq D + vw$ with $uv \in A(D_{uv})$ and $vw \in A(D_{vw})$. Let D' be the graph obtained from the union $(D_{uv} - uv) \cup (D_{vw} - vw)$ by adding the arc uw . Then, D' is a subdigraph of D that can be obtained from disjoint copies of D_{uv} and D_{vw} as follows. First we apply the HAJÓS join

by removing the copies of the arcs uv and vw , identifying the two copies of v , and adding the arc from $u \in V(D_{uv})$ to $w \in V(D_{vw})$. Afterwards, for each vertex x that belongs to both D_{uv} and D_{vw} , we identify the two copies of x . Hence, D' is a HAJÓS- k -constructible subdigraph of D , a contradiction. This completes the proof. ■

While reading our submission of the paper [7], one of the referees suggested another idea for proving Corollary 6 using DILWORTH's Theorem. To this end, note that a **preorder** $P = (X, \prec)$ consists of a set X and a binary relation \prec , which is reflexive and transitive. Two elements $x, y \in X$ are **comparable** (with respect to P) if $x \prec y$ or $y \prec x$ and **incomparable**, otherwise. A **chain** in P is a subset $Y \subseteq X$ of pairwise comparable elements, an **antichain** on P is a subset $Z \subseteq X$ of pairwise incomparable elements. The well known theorem of DILWORTH [34] (see also [8, Theorem 13.5.8]) states the following.

Theorem 8.11 (DILWORTH). *Let $P = (X, \prec)$ be a preorder. Then the minimum number of chains needed to cover X equals the maximum number of elements in an antichain.* ◊

Alternate proof of Corollary 8.10. Let $D' \subseteq D$ be an induced subdigraph of D and let $\vec{\chi}(D') = k$. We claim that $\omega(D') = k$. Recall that for vertices $u, v \in V(D)$, $u \sim v$ denotes the relation that $uv \notin A(D)$. Since \sim is transitive on $A(D)$, the relation \sim is transitive on $A(D')$, and so $P = (V(D'), \sim)$ is a preorder. Then, an antichain on P induces a bidirected complete graph in D' (as $u, v \in V(D')$ are incomparable if and only if $uv \in A(D')$ and $vu \in A(D')$). Furthermore, it is easy to see that Y is a chain in P if and only if $D'[Y]$ is an acyclic subdigraph of D' . Hence, a cover of $V(D')$ with ℓ chains corresponds to an ℓ -coloring of D' . As $\chi(D') = k$, we need k chains in order to cover $V(D')$ and so there is an antichain $Z \subseteq V(D')$ of order k , i.e. $D'[Z] = D(K_k)$. Thus, $\omega(D') \geq k$ and, as $\vec{\chi}(D') = k$, we have $\omega(D') = k$. Consequently, D is perfect. ■

A third short proof of Corollary 8.10 can be obtained by applying the GALLAI-MILGRAM Theorem [50] (see also [8, Theorem 13.5.2]) to the complement \bar{D} of D .

In the last two decades HAJÓS' theorem (Theorem 8.8) became very popular among graph theorists. HAJÓS-like theorems were established for the list-chromatic number by GRAVIER [53] and KRÁL [74], for the circular chromatic number by ZHU [121], for the signed chromatic number by KANG [65], for the chromatic number of edge weighted graphs by MOHAR [89], for graph homomorphisms by NEŠETRIL [93], and for GRASSMANN homomorphisms (a homomorphism concept that provides a common generalization of graph colorings, hypergraph colorings and nowhere-zero flows) by JENSEN [61].

8.4. The Ore Construction

Regarding undirected graphs, URQUHART [119] proved that each graph with chromatic number at least k does not only contain a HAJÓS- k -constructible subgraph but itself is HAJÓS- k -constructible. The aim of this section is to point out that the same result does not hold for digraphs and to prove that, however, a slight modification of the HAJÓS join does the trick.

Theorem 8.12. *Let $k \geq 3$ be an integer and let D be a HAJÓS- k -constructible digraph. Then, D is strongly connected.* \diamond

Proof. Clearly, if D is a strongly connected digraph, then identifying non-adjacent vertices still leads to a strongly connected digraph. Moreover, if D_1 and D_2 are strongly connected, then the directed HAJÓS-join of D_1 and D_2 is strongly connected, too, as vertices on directed cycles are still on directed cycles after the HAJÓS join (consider Figure 8.3 for a visualization). \blacksquare

As a consequence of the above theorem, every digraph with dichromatic number at least k that is not strongly connected is not HAJÓS- k -constructible and so URQUHART's Theorem cannot be directly transferred to digraphs. Nevertheless, it turns out that we get an URQUHART-type theorem by further allowing the following join. Let D_1 and D_2 be two digraphs and let $u_1, v_1 \in V(D_1)$ and $u_2, v_2 \in V(D_2)$ such that $D_i[\{u_i, v_i\}]$ is a digon for $i \in \{1, 2\}$. Now let D be the digraph obtained from the union $D_1 \cup D_2$ by deleting both arcs between u_1 and v_1 as well as both arcs between u_2 and v_2 , identifying the vertices v_1 and v_2 to a new vertex v , and adding both arcs $u_1 u_2$ and $u_2 u_1$. We say that D is the **bidirected HAJÓS join** of D_1 and D_2 and write $D = (D_1, v_1, u_1) \overleftrightarrow{\nabla} (D_2, v_2, u_2)$ or, briefly, $D = D_1 \overleftrightarrow{\nabla} D_2$. Note that the bidirected HAJÓS join is the exact analogue of the undirected HAJÓS join. By a slight modification of the proof of Theorem 8.7(a)-(c) one can easily show that the following holds.

Theorem 8.13 (Bidirected HAJÓS Construction). *Let $D = D_1 \overleftrightarrow{\nabla} D_2$ result from the bidirected HAJÓS join of two disjoint non-empty digraphs D_1 and D_2 . Then, the following statements hold:*

- (a) $\vec{\chi}(D) \geq \min\{\vec{\chi}(D_1), \vec{\chi}(D_2)\}$.
- (b) If $\vec{\chi}(D_1) = \vec{\chi}(D_2) = k$ and $k \geq 3$, then $\vec{\chi}(D) = k$.
- (c) If both D_1 and D_2 are k -critical and $k \geq 3$, then D is k -critical. \diamond

Note that for the proof of statement (b), we use the fact that $k \geq 3$ and so we can choose φ_1 and φ_2 such that $\varphi_1(v_1) = \varphi_2(v_2)$ and $\varphi_1(u_1) \neq \varphi_2(u_2)$. For $k = 2$, the statement is not true: for example, $D(C_4) \overset{\leftrightarrow}{\nabla} D(C_4) = D(C_7)$, whereas $\vec{\chi}(D(C_4)) = 2 \neq 3 = \vec{\chi}(D(C_7))$. The same trick works for statement (c).

For the proof of his theorem, URQUHART even used a more restricted class of constructible (undirected) graphs than the class of HAJÓS- k -constructible graphs, which originally was introduced by ORE [98, Chapter 11]. Transferred to digraphs, we get the following. Let D_1 and D_2 be two vertex-disjoint digraphs, let u_1v_1 be an arc of D_1 , and let v_2u_2 be an arc of D_2 . Furthermore, let $\iota : S_1 \rightarrow S_2$ be a bijection with $S_i \subseteq V(G_i - v_i)$ for $i \in \{1, 2\}$ and $\iota(u_1) \neq u_2$. Let D be the digraph obtained from $(D_1, v_1, u_1) \nabla (D_2, v_2, u_2)$ by identifying w with $\iota(w)$ for each $w \in S_1$. Then, D is a **directed ORE join** of D_1 and D_2 and we write $D = (D_1, v_1, u_1) \nabla_{\iota}^o (D_2, v_2, u_2)$. Note that the **undirected ORE join** of two undirected graphs G_1 and G_2 is performed via an undirected HAJÓS join and identification afterwards. However, for digraphs we need a second type of ORE join: If $u_1, v_1 \in V(D_1)$ and $u_2, v_2 \in V(D_2)$ are vertices such that $D_i[[u_i, v_i]]$ is a digon for $i \in \{1, 2\}$ and if ι is the bijection from above, then the digraph D obtained from $(D_1, v_1, u_1) \overset{\leftrightarrow}{\nabla} (D_2, v_2, u_2)$ by identifying w with $\iota(w)$ for each $w \in S_1$ is a **bidirected ORE join** of D_1 and D_2 and we write $D = (D_1, v_1, u_1) \overset{\leftrightarrow}{\nabla}_{\iota}^o (D_2, v_2, u_2)$. Recall that if the identification would lead to more than one arc in the same direction between two vertices, all but one of those arcs get deleted.

We define the class of **ORE- k -constructible** digraphs as the smallest family of digraphs that contains all bidirected complete graphs of order k and is closed under both directed and bidirected ORE joins. The proof of Theorem 8.8' immediately implies the following theorem (see [98] for the undirected analogue). In particular, here we do not need any bidirected ORE joins.

Theorem 8.14. *Let $k \geq 3$ be an integer. A digraph has dichromatic number at least k if and only if it contains an ORE- k -constructible subdigraph. \diamond*

URQUHART [119] proved the following result, thereby answering a conjecture by HANSON, ROBINSON, and TOFT [55] (the conjecture was also proposed by JENSEN and TOFT in their book on graph coloring problems [62, Problem 11.5]).

Theorem 8.15 (URQUHART). *Let $k \geq 3$ be an integer. For a graph G the following conditions are equivalent:*

- (a) G satisfies $\chi(G) \geq k$.

(b) G is HAJÓS- k -constructible.

(c) G is ORE- k -constructible. \diamond

Note that if G is the HAJÓS join of two graphs G_1 and G_2 , then $D(G)$ is the bidirected HAJÓS join of $D(G_1)$ and $D(G_2)$. Furthermore, $\vec{\chi}(D(G)) = \chi(G)$ (by (6.1)) and so the above theorem immediately implies the following.

Observation 8.16. *Each bidirected graph with dichromatic number at least $k \geq 3$ is ORE- k -constructible.*

Now we have all the tools that we need in order to prove our URQUHART-type theorem.

Theorem 8.17. *Let $k \geq 3$ be an integer. A digraph has dichromatic number at least k if and only if it is ORE- k -constructible. \diamond*

Proof. It immediately follows from Theorem 8.7(a) and Theorem 8.13(a) that each ORE- k -constructible digraph has dichromatic number at least k .

Thus, it suffices to show that each digraph with dichromatic number at least k is ORE- k -constructible. We will do this via a sequence of claims. In the following, we will denote by $D(K_k) \vec{+} v$ (respectively $D(K_k) \overleftarrow{+} v$) the digraph that results from $D(K_k)$ by adding a new vertex v and the arc uv (respectively vu) for some vertex u of $D(K_k)$. Moreover, let $D(K_k) + a$ be the digraph that results from $D(K_k)$ by adding two new vertices u, v and the arc $a = uv$. Finally, \mathcal{O}_k denotes the class of ORE- k -constructible digraphs and \mathcal{O}_k^* denotes the class of ORE- k -constructible digraphs containing a bidirected complete graph of order k . It follows from Observation 8.16 that

Claim 8.17.1. *The digraph obtained from $D(K_k)$ by adding an isolated vertex belongs to \mathcal{O}_k^* . \diamond*

Claim 8.17.2. *The digraph $D(K_k) + a$ belongs to \mathcal{O}_k^* . \diamond*

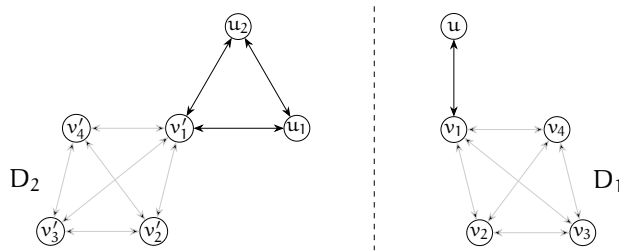
Proof. It is clear that $D(K_k) + a$ still contains a copy of $D(K_k)$. We claim that $D(K_k) + a$ is ORE-constructible. To this end, let D_1 (respectively D_2) be the bidirected graph obtained by identifying a vertex of $D(K_k)$ with a vertex of a disjoint copy of $D(K_2)$ (respectively

$D(K_3)$). More formally,

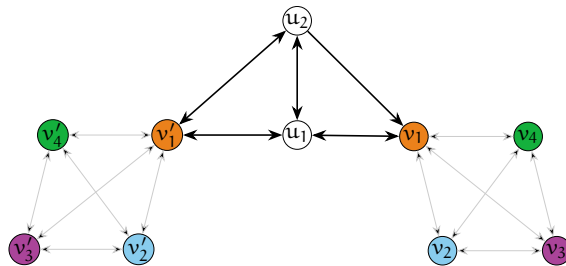
$$\begin{aligned} V(D_1) &= \{v_1, v_2, \dots, v_k, u\}, \\ A(D_1) &= \{v_i v_j \mid i \neq j\} \cup \{v_1 u, u v_1\}, \\ V(D_2) &= \{v'_1, v'_2, \dots, v'_k, u_1, u_2\}, \text{ and} \\ A(D_2) &= \{v'_i v'_j \mid i \neq j\} \cup \{v'_1 u_1, v'_1 u_2, u_1 v'_1, u_2 v'_1, u_1 u_2, u_2 u_1\} \end{aligned}$$

(see Figure 8.4a). Let ι be the bijection with $\iota(v_i) = v'_i$ for all $i \in [1, k]$ and let $D'_2 = (D_1, u, v_1) \nabla_{\iota}^o (D_2, u_2, u_1)$ (see Figure 8.4(a) and (b)). This ORE-join leads to the digraph $D'_2 = D_2 - u_2 u_1$ (see Figure 8.4(c)). By v_i^* we denote the vertex that results from identifying v_i with $\iota(v_i) = v'_i$. Now we take a new copy of D_1 , define ι' to be the bijection with $\iota'(v_i^*) = v_{i+1}$ for all $i \in [1, k]$ (where $v_{k+1} = v_1$), and set $D''_2 = (D'_2, u_1, v_1^*) \overset{\leftarrow}{\nabla}_{\iota'}^o (D_1, u, v_1)$ (see Figure 8.5b). Still, let v_i^* denote the vertex that results from identifying v_i^* with $\iota'(v_i^*)$.

Finally, we take another copy of D_1 , set $\iota''(v_i^*) = v_{i+1}$ for $i \in [1, k]$ (where $v_{k+1} = v_1$) and perform the ORE join $(D''_2, u_2, v_1^*) \overset{\leftarrow}{\nabla}_{\iota''}^o (D_1, u, v_1)$ (see Figure 8.6(a)(b)). This gives us the digraph $D(K_k) + u_1 u_2$ as required. \square

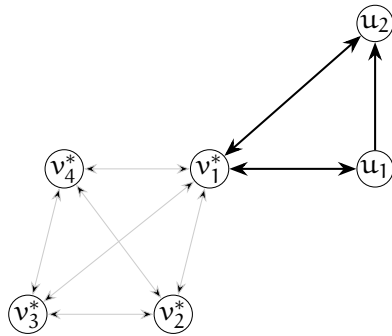


(a) The digraphs D_2 and D_1 . Firstly, we perform the directed HAJÓS join $(D_1, u, v_1) \Delta (D_2, u_2, u_1)$.



(b) The digraph we obtain after the HAJÓS join. Now, we identify the vertices of the same color.

Claim 8.17.3. The digraphs $D(K_k) \overset{\rightarrow}{+} v$ and $D(K_k) \overset{\leftarrow}{+} v$ are in \mathcal{O}_k^* . \diamond



(c) This gives us the digraph D'_2 .

FIG. 8.4. The first step of the construction of Claim 8.17.2.

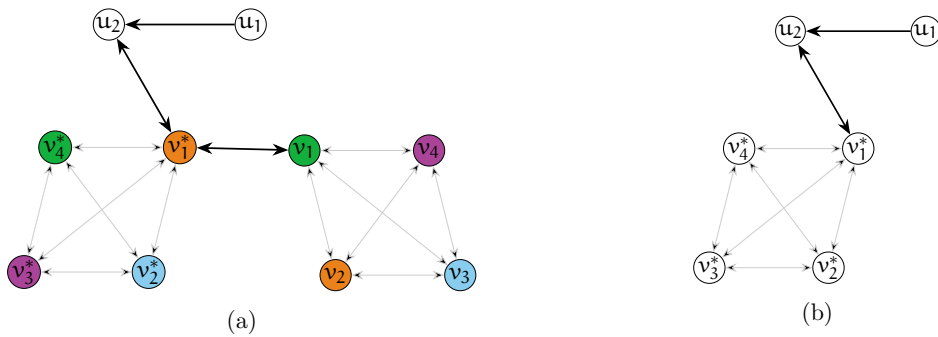


FIG. 8.5. How to build the digraph D''_2 .

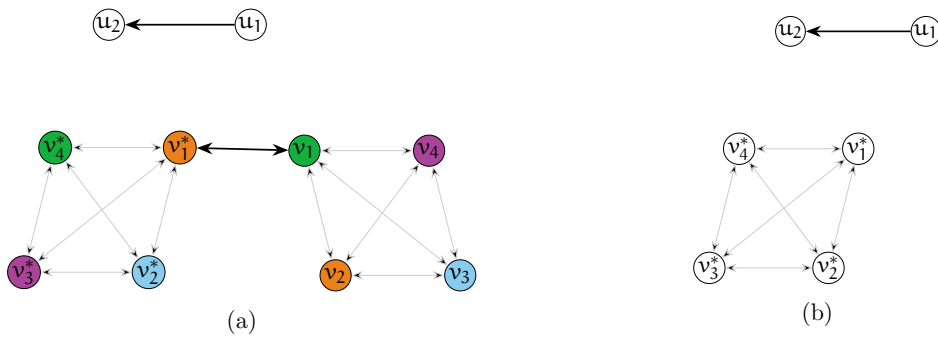


FIG. 8.6. The final step of the construction.

Proof. For the exact construction of $D(K_k) \overset{\rightarrow}{+} v$ see Figure 8.7; we start with two digraphs that result from $D(K_k)$ by adding a vertex and joining it to either one or two vertices of $D(K_k)$ by arcs in both directions. The construction of $D(K_k) \overset{\leftarrow}{+} v$ can be obtained (by symmetry) by changing the order of the digraphs in the directed HAJÓS join in Figure 8.7(a). \square

From now on, we may argue similar to the original proof of URQUHART.

Claim 8.17.4. *Let D be a digraph belonging to \mathcal{O}_k^* . Then, the digraph D' obtained from D by adding an isolated vertex belongs to \mathcal{O}_k^* , too.*

Proof. It suffices to show that $D' \in \mathcal{O}_k$. Let D_2 be a copy of $D(K_k)$ plus an isolated vertex (which belongs to \mathcal{O}_k^* by Claim 1). As $D \in \mathcal{O}_k^*$, there is a vertex set $X_1 \subseteq V(D)$ such that $D[X_1]$ is isomorphic to $D(K_k)$. Let X_2 be the vertex set of the bidirected complete graph of order k contained in D_2 and, for $i \in \{1, 2\}$, let v_i, w_i, u_i be three vertices of X_i . Furthermore, let $\iota : X_1 \setminus \{v_1\} \rightarrow X_2 \setminus \{v_2\}$ be a bijection such that $\iota(u_1) = w_2$ and $\iota(w_1) = u_2$. Then $(D, v_1, u_1) \overset{\leftrightarrow}{\nabla}_\iota^0(D_2, v_2, u_2) \in \mathcal{O}_k$ is a copy of D' and we are done. \square

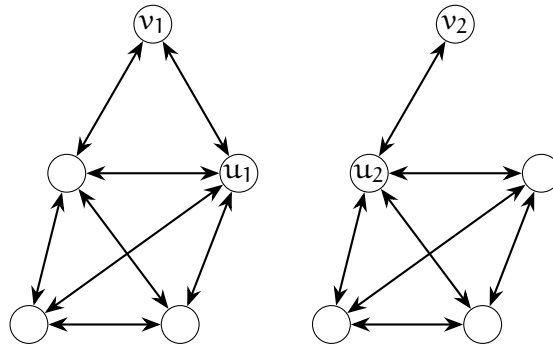
Claim 8.17.5. *Let D be a digraph belonging to \mathcal{O}_k^* and let $a \in A(\overline{D})$. Then, the digraph $D + a$ belongs to \mathcal{O}_k^* , too.*

Proof. Since $D \in \mathcal{O}_k^*$, there is a vertex set $X \subseteq V(D)$ such that $D[X]$ is a copy of $D(K_k)$. We distinguish between two cases.

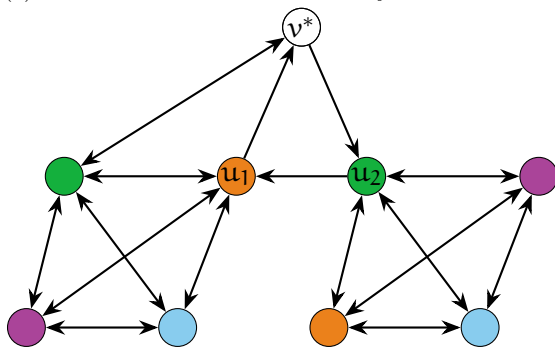
Case 1: *One end-vertex of the arc a belongs to X .* Then, we may assume $a = uv$ with $u \in X$ and $v \in V \setminus X$ (the case $a = vu$ can be done analogously). Moreover, let D' be a copy of $D(K_k) + \vec{v}'$, let $X' = V(D') \setminus \{v'\}$, and let u' be the vertex adjacent to v' in D' . Finally, let $w, z \in X \setminus \{u\}$ and let $w', z' \in X' \setminus \{u'\}$. By Claim 8.17.3, $D' \in \mathcal{O}_k$. Now let ι be a bijection from $(X \setminus \{u\}) \cup \{v\}$ to $(X' \setminus \{u'\}) \cup \{v'\}$ with $\iota(v) = v'$, $\iota(w) = z'$, and $\iota(z) = w'$. Then, $(D, u, w) \overset{\leftrightarrow}{\nabla}_\iota^0(D', u', w') \in \mathcal{O}_k$ is a copy of $D + a$, and we are done.

Case 2: *No end-vertex of a belongs to X .* Then, let $a = uv$, and D' be a copy of $D(K_k) + u'v'$. By Claim 8.17.2, D' belongs to \mathcal{O}_k . Now let x, y, z be three vertices from X and let $\{x', y', z'\} \subseteq D' \setminus \{u, v\}$. Finally, let ι be a bijection from $X \setminus \{x\} \cup \{u, v\}$ to $D' \setminus \{x'\}$ with $\iota(u) = u'$, $\iota(v) = v'$, $\iota(y) = z'$, and $\iota(z) = y'$. Then, $(D, x, y) \overset{\leftrightarrow}{\nabla}_\iota^0(D', x', y') \in \mathcal{O}_k$ is a copy of $D + a$ and the proof of the claim is complete. \square

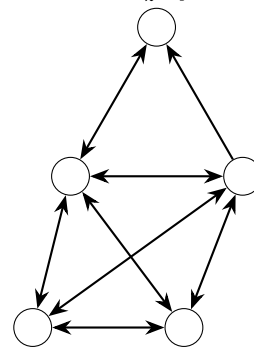
It follows from Claims 8.17.4 and 8.17.5 that each digraph containing $D(K_k)$ belongs to \mathcal{O}_k^* . The remaining part of the proof is by reductio ad absurdum. Let D be a maximal counterexample on a fixed number of vertices in the sense that $\vec{\chi}(D) \geq k$, D is not ORE- k -constructible, and D has maximum number of arcs with respect to this property. Then, D does not contain $D(K_k)$ and if $a \in A(\overline{D})$, $D + a$ belongs to \mathcal{O}_k . Now we argue as in the proof of Theorem 8.8'. For two vertices $u, v \in V(D)$, let $u \sim v$ denote the relation that $uv \notin A(D)$. If \sim is transitive we again conclude from Corollary 8.10 that D is perfect and,



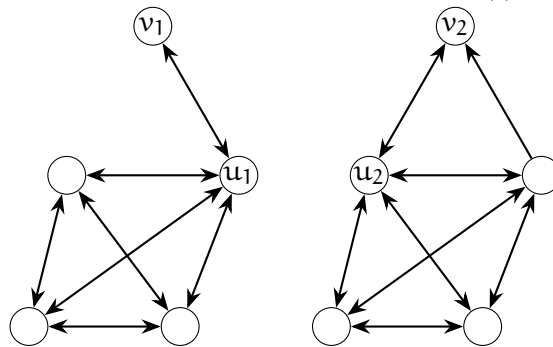
(a) We start with a directed HAJÓS join between the two digraphs, which are in \mathcal{G}_k^* by Theorem 8.15.



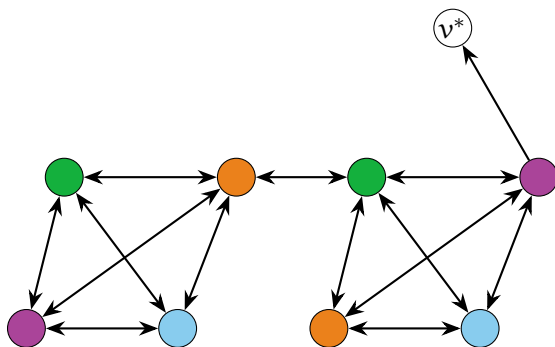
(b) Afterwards, we identify the vertices of the same color.



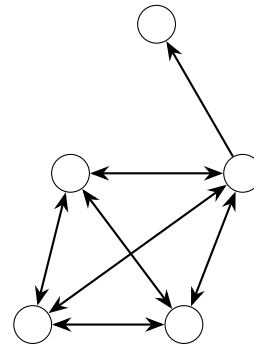
(c) End of the first step.



(d) Now we perform a bidirected HAJÓS join.



(e) Again we identify vertices of the same color.



(f) We get the digraph $D(K_k) \vec{+} v$.

FIG. 8.7. The construction of Claim 8.17.3.

hence, contains $D(K_k)$, a contradiction. Hence, \sim is not transitive and so there are vertices $u, v, w \in V(D)$ with $uv \notin A(D)$, $vw \notin A(D)$, but $uw \in A(D)$. Then, both digraphs $D + uv$ as well as $D + vw$ belong to \mathcal{C}_k and D is the ORE join of two disjoint copies of these two digraphs. Thus, D belongs to \mathcal{C}_k , a contradiction. ■

Chapter 9

SOME NICE CONJECTURES ON DIGRAPH COLORING

9.1. Emerging Questions Regarding DP-colorings

In Chapter 7, we have transferred the concept of DP-coloring to digraphs and obtained the BROOKS-type Theorem 7.14 as well as the solution to OHBA's Conjecture in this setting (see Corollary 7.19). Nevertheless, as we are the first to examine DP-colorings of digraphs, there are still various interesting problems left to be investigated.

The Multi-Case

A major benefit of DP-coloring is that the definition of a cover naturally works for multi-graphs, respectively multidigraphs, as well: if there are ℓ arcs from u to v in a digraph D , just add ℓ matchings from X_u to X_v in the auxiliary digraph \mathcal{D} . Thus, in contrast to (list-)coloring of digraphs, forbidding parallel arcs between vertices makes a big difference. But how would the “bad” configurations look like if we allow parallel arcs in the digraph? Obviously, we get the K- and the C-configurations from the (hyper-)graph case in Chapter 4 by replacing each edge with a digon (see Figure 9.1). Moreover, if D is a directed multicycle in which two consecutive vertices are joined by ℓ arcs, then we have $|X_v| = \ell$ for all $v \in V(D)$ and for two consecutive vertices u, v on the cycle, $A(\mathcal{D}[X_u \cup X_v])$ consists of all possible arcs $x_u x_v$ with $x_u \in X_u, x_v \in X_v$ and is the union of ℓ matchings from X_u to X_v (see also Figure 9.1). It is easy to check that this, indeed, gives us a minimal uncolorable configu-

ration. Nevertheless, by considering multidigraphs we get in trouble regarding shifting in the auxiliary digraph. Since we can have more than one matching between two sets X_u and X_v , there does not exist a unique vertex anymore to which we can shift the color and so many arguments would have to be reassessed significantly. Still, we strongly believe that the following holds true. Recall that a (multi-)digraph D together with a cover (X, \mathcal{D}) is a degree-feasible configuration if $|X_v| \geq \max\{d_D^+(v), d_D^-(v)\}$ for all $v \in V(D)$. A configuration (D, X, \mathcal{D}) is constructible if it can be obtained from K-, C- and DC-configurations (here in the multi-version) via the merging operation.

Conjecture 9.1. *Suppose that (D, X, \mathcal{D}) is a degree-feasible configuration, where D is a multidigraph. Then, (D, X, \mathcal{D}) is minimal uncolorable if and only if (D, X, \mathcal{D}) is constructible.◦*

DP-chromatic Number of a Tournament

We have proved in Corollary 7.2 that the DP-chromatic number of a bidirected graph D always coincides with the DP-chromatic number of its underlying graph $G(D)$. Consequently, many bounds for the DP-chromatic number of a graph also hold for digraphs and, often, the sharp digraphs are just bidirected graphs (like for example the K- and the C-configuration in our result). However, by forbidding digons, *i.e.*, by examining orientations of graphs, instead, many new and interesting results can be obtained. Here, the first step is usually to regard tournaments. For instance, NEUMANN-LARA [95] proved that there are exactly four non-isomorphic tournaments on seven vertices that have dichromatic number 3 and that there is a unique tournament on eleven vertices with dichromatic number 4. The 3-chromatic tournaments of order 7 are displayed in Figure 9.2. In particular, the first two tournaments are also critical; for the other two the removal of the colored arc in each case leads (up to isomorphisms) to the same critical digraph (see also [60]). We will go more into detail on this topic in Section 9.4.

But there are also extremal results on coloring tournaments: HARUTYUNYAN proved in his PhD thesis [56] that the random tournament T of order n satisfies $\vec{\chi}(T) \geq \frac{n}{2 \log_2 n + 2}$ a.a.s., and that every tournament T of order n fulfills $\vec{\chi}(T) \leq \frac{n}{\log_2 n} (1 + o(1))$. A **random tournament** of order n is a tournament obtained from the complete graph K_n by replacing each edge between two vertices u and v either by the arc uv or vu with equal probability $1/2$. As a consequence, the random tournament T of order n satisfies a.a.s. $\vec{\chi}(T) \sim \frac{n}{2 \log_2 n}$. Later, BENSMAIL, HARUTYUNYAN, and LE [9] extended the argumentation to list-colorings of digraphs, they obtained the following result.

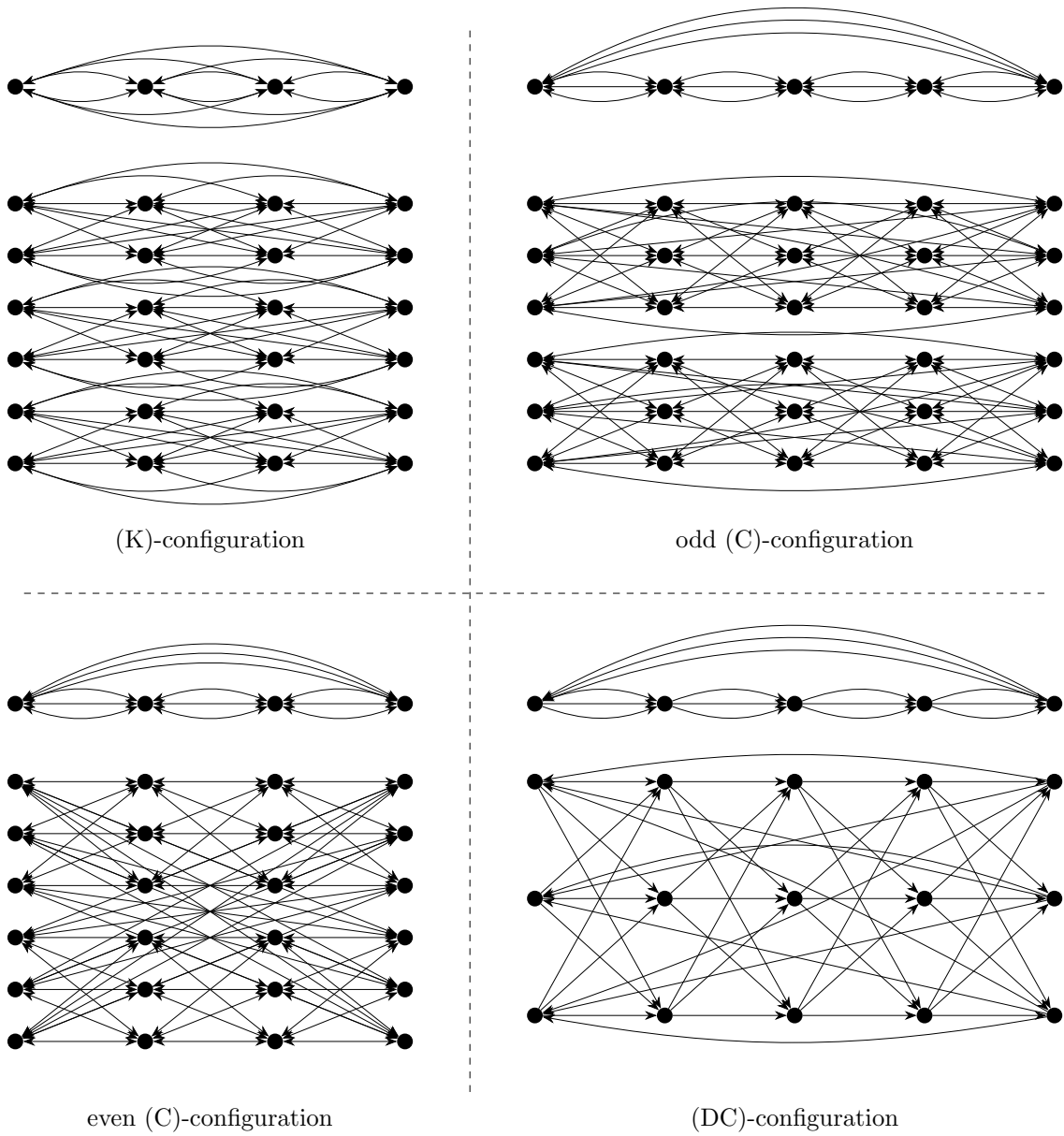


FIG. 9.1. The “bad” configurations for multidigraphs.

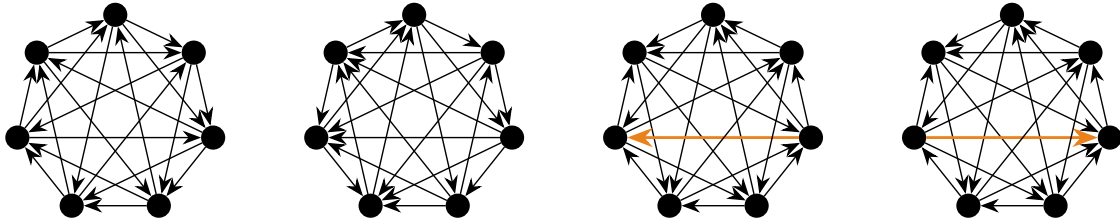


FIG. 9.2. The 3-chromatic tournaments of order 7.

Theorem 9.2 (BENSMAIL, HARUTYUNYAN, and LE, 2018). *Let T be the random tournament of order n . Then a.a.s.*

$$\vec{\chi}(T) \sim \vec{\chi}_\ell(T) \sim \frac{n}{2 \log_2 n}. \quad \diamond$$

Obviously, the question arises whether this also true for DP-coloring.

Question 9.3. *Let T be the random tournament of order n . Does it hold a.a.s.*

$$\vec{\chi}_{\text{DP}}(T) = \mathcal{O}\left(\frac{n}{\log_2 n}\right)? \quad \diamond$$

In order to tackle this question, we should have a deeper look into BENSMAIL et al.'s approach: they rely heavily on the following Lemma due to ERDŐS and MOSER [43].

Lemma 9.4 (ERDŐS and MOSER, 1964). *Every Tournament T with order n has an acyclic set of size at least $\log_2 n + 1$.* \diamond

In the paper [9], it is first proved that every tournament T of order n satisfies $\vec{\chi}_\ell(T) \leq \frac{n}{\log_2 n}(1 + o(1))$. Here, the key idea is basically to choose a large set of vertices whose lists share a common color α . Then, the subdigraph of T induced by those vertices is again a tournament and, by the above lemma, contains a large acyclic set S . Thus, we can color the vertices from S with α and, thereby, do not create a monochromatic cycle. Afterwards, we remove those vertices as well as the color α from T , respectively the lists, and repeat this procedure until there is no large set of vertices left whose lists contain a common color. For the remaining vertices, we create a bipartite auxiliary graph in which the remaining vertices form one and the remaining colors form the other class. A vertex v is joined with a color if the color is contained in its (reduced) list $L(v)$. Then, applying HALL's Theorem leads to a matching that covers all the vertices and, hence, to the required list-coloring. The proof that the random tournament T of order n even satisfies $\vec{\chi}_\ell(T) \leq \frac{n}{2 \log_2 n}(1 + o(1))$ is based on

the same ideas but uses probabilistic tools (in particular, the Extended JANSON Inequality, cf. [3]).

The problem with DP-coloring, however, is that we do not have the concept of *coloring vertices with the same color* anymore. Clearly, if we have an acyclic set in a digraph D , we can choose any vertices in the auxiliary digraph \mathcal{D} and do not create a directed cycle there. But how is it possible to choose those vertices from \mathcal{D} and to reduce the configuration (D, X, \mathcal{D}) such that only few vertices are forbidden? This question is definitely interesting and should be examined in detail. Since the author has already spent (maybe too) much time on trying to find a solution for this problem, it would be highly appreciated if the reader tries his luck.

9.2. Digraphs and Variable (Weak) Degeneracy

In the paper [52], GOLOWICH introduced the concept of degeneracy for digraphs: given a positive integer k , a digraph D is **weakly k -degenerate** if every non-empty subdigraph D' of D contains a vertex v with $\min\{d_{D'}^+(v), d_{D'}^-(v)\} < k$. Consequently, a digraph is acyclic if and only if it is weakly 1-degenerate. Moreover, if D is a bidirected graph, D is weakly k -degenerate if and only if $G(D)$ is strictly k -degenerate. Due to this relation, the question arises if we might obtain a decomposition result for digraphs similar to the one in Chapter 2: given a digraph D and a function $h : V(D) \rightarrow \mathbb{N}_0$, a digraph D is **weakly h -degenerate** if in each non-empty subdigraph D' of D there is a vertex v with $\min\{d_{D'}^-(v), d_{D'}^+(v)\} < h(v)$. Moreover, if $f : V(D) \rightarrow \mathbb{N}_0^p$ is a vector function, then an **f -partition** of D is a partition (D_1, D_2, \dots, D_p) of D such that D_i is weakly f_i -degenerate for $i \in [1, p]$. We want to determine under which degree condition D does admit an f partition. Theorems 6.2 and 6.3 suggest that the requirement

$$f_1(v) + f_2(v) + \dots + f_p(v) \geq \max\{d_D^+(v), d_D^-(v)\}$$

for all $v \in V(D)$ is the correct condition as (list-)coloring can be modeled by choosing f appropriately analogous to what we did in Chapter 3. Then, the definition of blocks of type (K) and (C) can be transferred directly to digraphs by taking $D(K_n)$, respectively $D(C_n)$ instead of K_n , respectively C_n . Obviously, there has to be a "bad"-type regarding directed cycles, in this case there needs to exist an index j such that $f_j(v) = 1$ for all v and $f_i(v) = 0$ for $i \neq j$. Thus, the directed cycles would fall under the definition of monoblocks. The digraph D is a monoblock if D is a block, D is EULERian, and there is an index $j \in [1, p]$

with $f_i(v) = \max\{d_H^+(v), d_H^-(v)\}$ if $i = j$ and $f_i(v) = 0$, otherwise. An example of a block of type (M), (K), and (C) is given in Figure 9.3. By defining the types (M), (K), and (C) in this way, the merging operation still works. We say that (D, f) is a **hard pair** if D is a block of type (M), (K), or (C), or if (D, f) is obtained from two hard pairs by the merging operation.

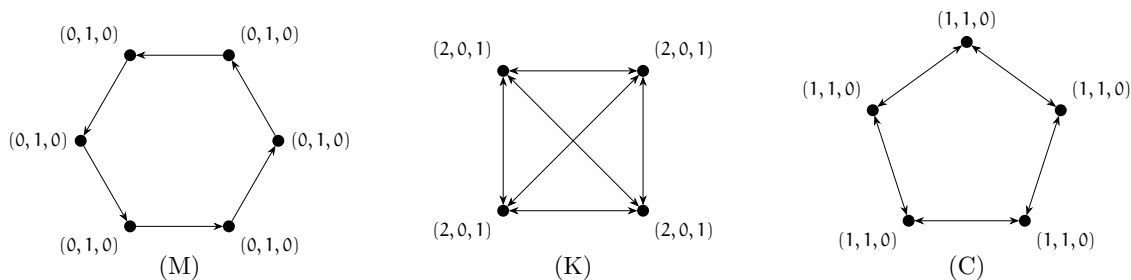


FIG. 9.3. A block of type (M), (K), and (C).

Conjecture 9.5. Let D be a digraph, and let $f \in \mathcal{V}_p(D)$ be a vector function with $p \geq 1$ such that $f_1(v) + f_2(v) + \dots + f_p(v) \geq d_H(v)$ for all $v \in V(H)$. Then, D is not f -partitionable if and only if (D, f) is a hard pair. \diamond

By examining digraph degeneracy, GOLOWICH [52] aimed to tackle a particularly challenging conjecture due to ERDŐS [44] from 1979 that was, according to [59], stated independently by MCDIARMID and MOHAR in 2002.

Conjecture 9.6. Let D be a digon-free digraph. Then, $\vec{\chi}(D) = \mathcal{O}\left(\frac{\Delta(D)}{\log_2 \Delta(D)}\right)$. \diamond

This conjecture is the digraph-counterpart to JOHANSSON’s [63] celebrated theorem that there exists a positive constant C such that every triangle-free graph G with maximum degree Δ satisfies $\chi(G) \leq (C + o(1))\frac{\Delta}{\ln \Delta}$. JOHANSSON even proved his result for the list-chromatic number with $C = 9$; MOLLOY [92] recently improved the constant to $C = 1$. Surprisingly, it is possible to use MOLLOY’s probabilistic approach to obtain the result also for the DP-chromatic number of G , as demonstrated in the highly recommended paper [10] by BERNSHTEYN. If ERDŐS’ conjecture holds true, it would be asymptotically best possible as the dichromatic number of a random tournament of order n is approximately $\frac{n}{2 \log_2 n}$ as pointed out above. So far, the conjecture is still wide open. A first step was made by HARUTYUNYAN and MOHAR [57], who proved the following by using the probabilistic method. Note that, given a digraph D , $\tilde{d}_D(v)$ denotes the geometric mean of $d_D^+(v)$ and

$d_{\vec{D}}^-(v)$ and $\tilde{\Delta}(\mathcal{D})$ is the maximum geometric mean over all vertices of \mathcal{D} .

Theorem 9.7 (HARUTYUNYAN and MOHAR, 2011). *There is an absolute constant Δ_1 such that every digon-free digraph \mathcal{D} with $\tilde{\Delta}(\mathcal{D}) \geq \Delta_1$ satisfies $\vec{\chi}(\mathcal{D}) \leq (1 - e^{-13})\tilde{\Delta}(\mathcal{D})$.* \diamond

A similar theorem regarding the list-dichromatic number was obtained later by BENSMAIL, HARUTYUNYAN, and LE [9]. GOLOWICH [52] obtained a strengthening of the above theorem by regarding the **m-degenerate dichromatic number** $\vec{\chi}_m(\mathcal{D})$ of a digraph \mathcal{D} , *i.e.*, the smallest integer k such that \mathcal{D} admits a partition (D_1, D_2, \dots, D_k) of which each part D_i is weakly m -degenerate.

Theorem 9.8 (GOLOWICH, 2016). *Let m be a positive integer. For any digon-free digraph \mathcal{D} , we have*

$$\vec{\chi}_m(\mathcal{D}) \leq \left\lfloor \frac{\Delta(\mathcal{D}) - \lfloor \frac{\Delta(\mathcal{D})+1}{4m+1} \rfloor}{2m} \right\rfloor + 1. \quad \diamond$$

Note that $m = 1$ corresponds to the usual acyclic coloring concept for digraphs and so the above theorem implies $\vec{\chi}(\mathcal{D}) \leq \lfloor 2/5 \cdot (\Delta(\mathcal{D}) + 1) \rfloor + 1$ for all digon-free digraphs \mathcal{D} . GOLOWICH further utilized his theorem to substantially improve HARUTYUNYAN and MOHAR's theorem in terms of $\tilde{\Delta}(\mathcal{D})$:

Theorem 9.9 (GOLOWICH, 2016). *Every digon-free digraph \mathcal{D} satisfies*

$$\vec{\chi}(\mathcal{D}) \leq \left\lfloor \sqrt{\frac{2}{3}} \cdot \tilde{\Delta}(\mathcal{D}) + \frac{7}{5} \right\rfloor. \quad \diamond$$

Obviously, GOLOWICH's theorem is much weaker than ERDŐS' conjecture, but it gives a precise bound for all digon-free digraphs. In this regard, HARUTYUNYAN and MOHAR [57] conjectured that the following holds true.

Conjecture 9.10 (HARUTYUNYAN and MOHAR, 2011). *Every digon-free digraph \mathcal{D} satisfies*

$$\vec{\chi}(\mathcal{D}) \leq \left\lceil \frac{\tilde{\Delta}(\mathcal{D})}{2} \right\rceil + 1. \quad \diamond$$

9.3. The Hajós Construction

Regarding critical digraphs, a lot of questions immediately come to mind. It follows from Theorem 8.8' that each k -critical digraph is HAJÓS- k -constructible. However, the proof of Theorem 8.8' is not constructive at all and it has proved quite challenging to construct specific critical digraphs by the HAJÓS construction. The first barrier to take should be the following.

Question 9.11. *How can a bidirected C_5 be constructed from copies of $D(K_3)$ by only using directed HAJÓS joins and identifying non-adjacent vertices?* \diamond

Building upon this question, it is of particular interest to study the connection of the HAJÓS construction to computational complexity. In the undirected case, MANSFIELD and WELSH [83] stated the problem of determining the complexity of the HAJÓS construction. They noted that if for any $k \geq 3$ there would exist a polynomial P such that every graph of order n with chromatic number k contains a HAJÓS- k -constructible subgraph that can be obtained by at most $P(n)$ uses of the HAJÓS-join and identification of non-adjacent vertices, then $NP = coNP$. Hence, it is very likely that the HAJÓS construction is not polynomially bounded, but not much progress has been made on this problem yet. PITASSI and URQUHART [99] found a linkage to another important open problem in logic; they proved that a restricted version of the HAJÓS construction is polynomially bounded if and only if extended FREGE systems are polynomially bounded.

Question 9.12. *For $k \geq 3$, is there a polynomial P such that every digraph of order n contains a HAJÓS- k -constructible subdigraph that can be obtained from bidirected complete graphs of order k by at most $P(n)$ uses of the directed HAJÓS-join and identification of non-adjacent vertices?* \diamond

9.4. Critical Digraphs with Few Arcs and Few Vertices

A beautiful theorem of GALLAI [49] states that any k -critical graph with order at most $2k - 2$ and $k \geq 2$ is the DIRAC join of two disjoint non-empty critical graphs. Within the last decades, various different proofs of this theorem have been published (see e.g. [91] and [107]). Clearly, a graph G is the DIRAC join of two disjoint non-empty graphs if and only if \overline{G} is disconnected and so most of the proofs use matching theory for the complement graph \overline{G} . Recently, STEHLÍK [108] transferred GALLAI's theorem to digraphs, thereby answering a question that we raised in an early version of the paper [7].

Theorem 9.13 (STEHLÍK, 2019). *Let $k \geq 3$ be an integer and let D be a k -critical digraph on at most $2k - 2$ vertices. Then, \overline{D} is disconnected and so D is the DIRAC join of two proper subdigraphs D_1 and D_2 .* \diamond

GALLAI [49] used his Theorem in order to determine the minimum number of edges in k -critical graphs of order n with $k + 2 \leq n \leq 2k - 2$. It seems natural to apply his approach to the digraph setting. To this end, let $\vec{\text{CRI}}(k, n)$ denote the class of k -critical digraphs of order n . Our aim is to analyze the function

$$\vec{\text{ext}}(k, n) = \min\{|A(D)| \mid D \in \vec{\text{CRI}}(k, n)\}$$

as well as the corresponding class

$$\vec{\text{Ext}}(k, n) = \{D \in \vec{\text{CRI}}(k, n) \mid |A(D)| = \vec{\text{ext}}(k, n)\}.$$

The definition of $\text{ext}(k, n)$ and $\text{Ext}(k, n)$ for graphs are accordingly.

That it is worthwhile studying the function $\text{ext}(k, n)$ was already recognized by DIRAC in his PhD thesis. As every k -critical graph has minimum degree at least $k - 1$, we have the trivial bound $2\text{ext}(k, n) \geq (k - 1)n$. Note that BROOKS' Theorem is equivalent to the fact that equality holds if and only if $n = k$ or $k = 3$ and n is odd. In 1957, DIRAC [40] improved this bound by showing that

$$2\text{ext}(k, n) \geq (k - 1)n + k - 3 \text{ for } k \geq 4 \text{ and } n \geq k + 2.$$

Note that there is no k -critical graph on $k + 1$ vertices since no two vertices in a critical graph can have the same neighborhood. GALLAI [49] further improved DIRAC's bound. Utilizing his Theorem 3 on the low-vertex subgraph of a k -critical graph, he deduced that

$$2\text{ext}(k, n) \geq \left(k - 1 + \frac{k - 3}{k^2 - 3}\right)n \text{ for } n \geq k + 2.$$

As already indicated above, GALLAI also used his characterization of k -critical graphs with order at most $2k - 2$ to determine the exact values of $\text{ext}(k, n)$ for $k \geq 4$ and $k + 2 \leq n \leq 2k - 1$: he proved that

$$\text{ext}(k, n) = \binom{n}{2} - ((n - k)^2 + 1).$$

After a series of additional enhancements (see [70, 75] and also the survey by KOSTOCHKA [69]), KOSTOCHKA and YANCEY [73] proved the following remarkable result that establishes the

asymptotics of $\text{ext}(k, n)$.

Theorem 9.14 (KOSTOCHKA and YANCEY, 2014). *If $k \geq 4$ and $n \geq k$ with $n \neq k + 1$, then*

$$\text{ext}(k, n) \geq \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil$$

and equality holds if $n \equiv 1 \pmod{k-1}$, or $k = 4$, or $k = 5$ and $n \equiv 2 \pmod{4}$. \diamond

In particular, we get $\text{ext}(4, n) = \lceil \frac{5n-2}{3} \rceil$ for $n \geq 4$ and $n \neq 5$. Moreover, it follows that

$$\lim_{n \rightarrow \infty} \frac{2\text{ext}(k, n)}{n} = k - \frac{2}{k-1},$$

which confirms a conjecture by ORE [98].

However, not much is known about the function $\vec{\text{ext}}(k, n)$. As every k -critical digraph D satisfies $\min\{\delta^+(D), \delta^-(D)\} \geq k-1$ (by Proposition 8.1(b)), it trivially holds $\vec{\text{ext}}(k, n) \geq (k-1)n$ and it follows from Theorem 8.2 that equality holds if and only if $k = 2$ and $n \geq 3$, or $k = 3$ and $n \geq 3$ odd, or $n = k \geq 4$. Moreover, $\vec{\text{Ext}}(2, n)$ consists only of the directed cycle of order n ($n \geq 2$), $\vec{\text{Ext}}(3, n)$ only contains the bidirected C_n (for $n \geq 3$ odd), and $\vec{\text{Ext}}(k, k) = \{D(K_k)\}$ for $k \geq 4$.

A natural first approach for the determination of $\vec{\text{ext}}(k, n)$ is to analyze the relation between $\vec{\text{ext}}(k, n)$ and $\text{ext}(k, n)$. In (8.1) we have already obtained that a bidirected graph D is k -critical if and only if its underlying graph $D(G)$ is k -critical (with respect to χ). As a consequence, $\vec{\text{ext}}(k, n) \leq 2\text{ext}(k, n)$ for $n \geq k \geq 4$ and $n \neq k + 1$. Note that this inequality does not hold for $n = k + 1$ as $\text{CRI}(k, k + 1)$ is empty, but $\vec{\text{CRI}}(k, k + 1)$ is not. In order to verify that $\vec{\text{CRI}}(k, k + 1)$ indeed is non-empty, let C be a directed cycle of length 3 and consider the DIRAC join $D = D(K_{k-2}) \boxtimes C$ with $k \geq 2$. Then, D is k -critical by Theorem 8.6 and $|D| = k + 1$. In fact, it is possible to show that this is the only digraph contained in $\vec{\text{CRI}}(k, k + 1)$ [102]. KOSTOCHKA and STIEBITZ [71] conjectured the following:

Conjecture 9.15 (KOSTOCHKA and STIEBITZ, 2020). *Let $k, n \in \mathbb{N}$ with $n \geq k \geq 4$ and $n \neq k + 1$. Then $\vec{\text{ext}}(k, n) = 2\text{ext}(k, n)$ and hence*

$$\lim_{n \rightarrow \infty} \frac{\vec{\text{ext}}(k, n)}{n} = k - \frac{2}{k-1}.$$

Furthermore, $\vec{\text{Ext}}(k, n) = \{D(G) \mid G \in \text{Ext}(k, n)\}$. \diamond

KOSTOCHKA and STIEBITZ [71] also made a first step to confirming their conjecture; they proved that for $n \geq 4$,

$$\vec{\text{ext}}(4, n) \geq \frac{10n - 4}{3}.$$

Note that this corresponds with the above mentioned fact that $\text{ext}(4, n) = \lceil \frac{5n-2}{3} \rceil$. Thus, if $n \geq 4$ and $n \neq 5$, then

$$2\text{ext}(4, n) - 1 \leq \vec{\text{ext}}(4, n) \leq 2\text{ext}(4, n)$$

and $\vec{\text{ext}}(4, n) = 2\text{ext}(4, n)$ if $n \equiv 1 \pmod{3}$ or $n \equiv 2 \pmod{3}$. We are optimistic that we can use STEHLÍK's result together with GALLAI's approach in order to prove the following conjecture.

Conjecture 9.16. *Let k and n be integers satisfying $k \geq 4$ and $k + 2 \leq n \leq 2k - 1$. Then, $\vec{\text{ext}}(k, n) = 2\left(\binom{n}{2} - ((n - k)^2 + 1)\right)$. \diamond*

Note that the above conjecture would imply that $\vec{\text{ext}}(k, n) = 2\text{ext}(k, n)$ if $k \geq 4$ and $k + 2 \leq n \leq 2k - 1$.

If KOSTOCHKA and STIEBITZ's conjecture can be confirmed true, there is no longer any reason to consider $\vec{\text{ext}}(k, n)$ as (nearly) everything could be settled within the graph case. What would still be interesting, though, is to examine how the number of arcs in critical digraphs behaves if we forbid digons. Related to this question, HOSHINO and KAWARABAYASHI [60] proved that there is an infinite family of sparse k -critical digon-free digraphs D with $|A(D)| < (\frac{k^2-k+1}{2})|D|$ and an infinite family of dense k -critical digon-free digraphs D satisfying $|A(D)| > (\frac{1}{2} - \frac{1}{2^{k-1}})|D|^2$. However, they point out that $\frac{k^2-k+1}{2}$ is presumably not optimal, for instance, there is an infinite family of 3-critical digon-free digraphs with D with $|A(D)| < \frac{20}{7}|D|$, where the above bound would only give $\frac{7}{2}$. This motivates the following problem.

Problem 9.17 ([60]). *For each $k \geq 3$, determine a function $y(k) < \frac{k^2-k+1}{2}$ for which there exist infinitely many k -critical digon-free digraphs D satisfying $|A(D)| < y(k)|D|$. What is the minimum of all such functions $y(k)$? \diamond*

HOSHINO and KAWARABAYASHI further conjectured that $y(k) = \frac{k^2}{2} - O(k)$, to the author's knowledge this is still open. Regarding the upper bound, they guessed that $\frac{1}{2} - \frac{1}{2^{k-1}}$ is at least close-to-optimal. They still proposed the following problem.

Problem 9.18 ([60]). *For each $k \geq 3$, determine a function $\chi(k) > \frac{1}{2} - \frac{1}{2^{k-1}}$ for which there exist infinitely many k -critical digon-free digraphs D satisfying $|A(D)| > \chi(k)|D|^2$. What is the supremum of all such functions $\chi(k)$?* \diamond

Instead of regarding the relation between arcs and vertices, one could also examine what are the "smallest" critical digraphs with respect only to their order.

Question 9.19. *For fixed $k \geq 3$, what is the minimum integer $N(k)$ such that there is a k -critical digon-free digraph on $N(k)$ vertices?* \diamond

As $k-1 \leq \min\{d_D^+(v), d_D^-(v)\}$ for all vertices v of a k -critical digraph D , we trivially have $N(k) \geq 2k-1$. In fact, BROOKS' Theorem 8.2 for digraphs implies that $N(k) \geq 2k$ for $k \geq 3$. Moreover, some small values are already known: the directed triangle shows that $N(2) = 3$, and NEUMANN-LARA [95] proved that $N(3) = 7$, $N(4) = 11$, and $17 \leq N(5) \leq 19$; he conjectured that $N(5) = 17$. In fact, there are exactly three non-isomorphic 3-critical digraphs of order seven, two of them having 21 arcs and the third having 20 arcs. Those three digraphs are displayed in Figure 9.2 by regarding only the black arcs. The third and the fourth digraph are isomorphic.

Finally, let us return to the problem of determining the minimum number of arcs in critical digon-free digraphs: from Theorem 8.5 it follows that each 3-critical digon-free digraph D satisfies $|A(D)| \geq (2 + \frac{1}{7})|D|$. We believe that this bound can be improved to $|A(D)| \geq (2 + \frac{1}{2})|D|$. As a consequence of our bound, we obtain that if D is a 3-critical digon-free planar digraph, then $G(D)$ contains a triangle. However, LI and MOHAR [77] in fact proved that in this case D contains not only a triangle but a directed cycle of length three. This implies that every planar digraph of digirth at least 4 admits a 2-coloring. This result is a first step in proving the following, famous conjecture proposed by ERDŐS and NEUMANN-LARA, and, independently, by ŠKREKOVSKI (for a reference see [77]).

Conjecture 9.20 (THE TWO COLOR CONJECTURE). *Every digon-free planar digraph D satisfies $\vec{\chi}(D) \leq 2$.* \diamond

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LIST OF SYMBOLS

The number at the end of each line refers to the page where the term is defined.
The items are sorted according to content as far as possible.

Basic Terminology

<p>\mathbb{N} set of positive integers, 6</p> <p>\mathbb{N}_0 set of non-negative integers, 6</p> <p>$[k, \ell]$ all $h \in \mathbb{N}_0$ with $k \leq h \leq \ell$, 6</p> <p>\emptyset empty set, 6</p> <p>2^V power set of V, 6</p> <p>V cardinality of the set V, 6</p> <p>G, G', \tilde{G} graphs, 6</p> <p>$\omega(G)$ clique number of G, 95</p> <p>K_n complete graph on n vertices, 7</p> <p>C_n cycle on n vertices, 7</p> <p>Γ color set, 12</p>	<p>L list-assignment, 13</p> <p>α color, 13</p> <p>φ coloring of graph/hypergraph/digraph, 12</p> <p>$\chi(G)$ chromatic number of a graph G, 1</p> <p>$\mathbf{CRI}(k)$ class of k-critical graphs, 3</p> <p>$\mathbf{ext}(k, n)$ minimum number of edges in a k-critical graph of order n, 169</p> <p>$\mathbf{Ext}(k, n)$ class of k-critical graphs of order n with minimum number of edges, 169</p> <p>G_L low vertex subgraph of G, 3</p>
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Hypergraph Terminology

<p>H, H', \tilde{H} hypergraphs, 6</p> <p>$V(H)$ vertex set of a hypergraph H, 6</p> <p>$E(H)$ edge set of a hypergraph H, 6</p> <p>i_H incidence function of H, 6</p> <p>H order of the hypergraph H, 6</p> <p>$\langle e \rangle$ hypergraph consisting of the edge e, 7</p> <p>tH t-uniform inflation of H, 8</p> <p>$H' \subseteq H$ H' subhypergraph of H, 7</p>	<p>$H' \subset H$ H' proper subhypergraph of H, 7</p> <p>$H_1 \cup H_2$ union of H_1 and H_2, 7</p> <p>$H_1 \cap H_2$ intersection of H_1 and H_2, 7</p> <p>$H[X]$ subhypergraph induced by X, 7</p> <p>$H(X)$ H shrunk to X, 8</p> <p>$H \div X$ H shrunk at X, 8</p> <p>$H - X$ subhypergraph of H induced by $V(H) \setminus X$, 8</p>
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- $\mathbf{H} - \mathbf{F}$ subhypergraph with vertex set $V(\mathbf{H})$ and edge set $E(\mathbf{H}) \setminus F$, 8
 $\mathbf{H}' + \mathbf{v}$ subhypergraph induced by $V(\mathbf{H}') \cup \{\mathbf{v}\}$, 8
 $\mathbf{H}' + \mathbf{e}$ subhypergraph with vertex set $V(\mathbf{H}')$ and edge set $E(\mathbf{H}') \cup \{\mathbf{e}\}$, 8
 \mathbf{M} (hyper-)matching, 8
 \mathbf{P} (hyper-)path, 9
 \mathbf{uPw} subhyperpath between \mathbf{u} and \mathbf{w} , 9
 $\text{dist}_{\mathbf{H}}(\mathbf{u}, \mathbf{v})$ distance between \mathbf{u} and \mathbf{v} , 9
 $\lambda_{\mathbf{H}}(\mathbf{u}, \mathbf{v})$ local edge connectivity of \mathbf{u}, \mathbf{v} , 88
 $\lambda(\mathbf{H})$ maximum local edge connectivity, 88
 \mathbf{B} block, 10
 $\mathcal{B}(\mathbf{H})$ set of all blocks of \mathbf{H} , 10
 $\mathcal{B}_{\mathbf{v}}(\mathbf{H})$ set of blocks of \mathbf{H} containing \mathbf{v} , 10
 $(\mathbf{X}, \mathbf{Y}, \mathbf{F})$ edge cut of \mathbf{H} , 95
 $\mathbf{X}_{\mathbf{F}}$ set of vertices of \mathbf{X} incident to \mathbf{F} with respect to $(\mathbf{X}, \mathbf{Y}, \mathbf{F})$, 95
 $\mathbf{N}_{\mathbf{H}}(\mathbf{v})$ ordinary neighborhood of \mathbf{v} , 10
 $\mathbf{E}_{\mathbf{H}}(\mathbf{X})$ set of edges containing at least one vertex from \mathbf{X} as well as $V(\mathbf{H}) \setminus \mathbf{X}$, 10
 $\mathbf{E}_{\mathbf{H}}(\mathbf{v})$ set of edges incident with \mathbf{v} , 10
 $\mathbf{d}_{\mathbf{H}}(\mathbf{v})$ degree of \mathbf{v} in \mathbf{H} , 10
 $\delta(\mathbf{H})$ minimum degree of \mathbf{H} , 10
 $\Delta(\mathbf{H})$ maximum degree of \mathbf{H} , 10
 $\mathbf{d}(\mathbf{H})$ degree sum of \mathbf{H} , 61
 $\mu_{\mathbf{H}}(\mathbf{u}, \mathbf{v})$ multiplicity of (\mathbf{u}, \mathbf{v}) in \mathbf{H} , 10
 $\mathcal{A}(\mathbf{H})$ set of all two-subsets $\{\mathbf{u}, \mathbf{v}\}$ of $V(\mathbf{H})$ with $\mu_{\mathbf{H}}(\mathbf{u}, \mathbf{v}) > 0$, 72
 $\text{col}(\mathbf{H})$ coloring number of \mathbf{H} , 11
 $(\mathbf{H}_1, \mathbf{H}_2, \dots, \mathbf{H}_p)$ p -partition of \mathbf{H} , 12
 $\chi(\mathbf{H})$ chromatic number of \mathbf{H} , 13
 $\chi_{\ell}(\mathbf{H})$ list-chromatic number of \mathbf{H} , 13
 $\chi_{\text{DP}}(\mathbf{H})$ DP-chromatic number of \mathbf{H} , 68
 $\chi^s(\mathbf{H})$ point-partition number of \mathbf{H} , 50
 $\chi_{\ell}^s(\mathbf{H})$ list point-partition number of \mathbf{H} , 50
 $\chi(\mathbf{H} : \mathcal{P})$ \mathcal{P} -chromatic number of \mathbf{H} , 54
 $\chi_{\ell}(\mathbf{H} : \mathcal{P})$ \mathcal{P} -list-chromatic number of \mathbf{H} , 54
 $\mathcal{CO}_k(\mathbf{H})$ set of proper k -colorings of \mathbf{H} , 92
 \mathcal{H} class of all hypergraphs, 14
 \mathcal{P} hypergraph property, 14
 $\mathcal{F}(\mathcal{P})$ property of \mathcal{P} -(vertex-)critical hypergraphs, 14
 $\mathbf{d}(\mathcal{P})$ minimum degree over all hypergraphs from $\mathcal{F}(\mathcal{P})$, 15
 \mathcal{O} property of edgeless hypergraphs, 14
 \mathcal{S}_k property of hypergraphs with maximum degree $\leq k$, 14
 \mathcal{D}_k property of strictly $(k + 1)$ -degenerate hypergraphs, 14
 \mathcal{H}_k specific class of hypergraphs closed under Hajós joins, 88
 \mathcal{C}_k class of $k + 1$ -critical hypergraphs \mathbf{H} with $\lambda(\mathbf{H}) \leq k$, 97
 $\mathbf{V}(\mathbf{H}, \mathcal{P}, \mathbf{L})$ set of low vertices of \mathbf{H} with respect to $(\mathcal{P}, \mathbf{L})$, 56
 $\mathbf{H}(\mathbf{V}(\mathbf{H}, \mathcal{P}, \mathbf{L}))$ low-vertex hypergraph, 56
 $\mathbf{f} : \mathbf{V}(\mathbf{H}) \rightarrow \mathbb{N}_0^p$ vector function of \mathbf{H} , 19
 \mathbf{f}_i i th coordinate of \mathbf{f} , 19
 $\mathcal{V}_p(\mathbf{H})$ set of all vector functions of \mathbf{H} , 19
 $(\mathbf{X}, \mathcal{H})$ cover of a hypergraph \mathbf{H} , 68
 $(\mathbf{H}, \mathbf{X}, \mathcal{H})$ (feasible) configuration, 71
 $(\mathbf{H}, \mathbf{f}) / (\mathbf{z}, \mathbf{j})$ reduction method for non-partitionable pairs, 26
 $(\mathbf{H}, \mathbf{X}, \mathcal{H}) / (\mathbf{v}, \mathbf{x})$ reduction method for feasible configurations, 76
 $\mathbf{H}_1 \nabla \mathbf{H}_2$ Hajós join of \mathbf{H}_1 and \mathbf{H}_2 , 88
 $\mathbf{H}_1 \boxtimes \mathbf{H}_2$ Dirac sum of \mathbf{H}_1 and \mathbf{H}_2 , 103
 $\mathbf{S}(\mathbf{H}_1, \tilde{\mathbf{e}}, \mathbf{H}_2, \tilde{\mathbf{v}}, \mathbf{s})$ splitting $\tilde{\mathbf{v}}$ into $\tilde{\mathbf{e}}$, 100

$\mathbf{S}(\mathbf{H}, \tilde{\mathbf{v}}, \tilde{\mathbf{e}}, \mathbf{s})$ splitting $\tilde{\mathbf{v}}$ into the independent set $\tilde{\mathbf{e}}$, 103

Digraph Terminology

$\mathbf{D}, \mathbf{D}', \tilde{\mathbf{D}}$ digraphs, 106

$\mathbf{V}(\mathbf{D})$ vertex set of a digraph \mathbf{D} , 106

$\mathbf{A}(\mathbf{D})$ arc set of a digraph \mathbf{D} , 106

$|\mathbf{D}|$ order of the digraph \mathbf{D} , 106

$\omega(\mathbf{D})$ clique number of \mathbf{D} , 150

$\mathbf{S}(\mathbf{D})$ symmetric part of \mathbf{D} , 150

$\mathbf{G}(\mathbf{D})$ underlying graph of \mathbf{D} , 108

$\bar{\mathbf{D}}$ complement of \mathbf{D} , 108

\mathbf{D}_C maximal subdigraph of \mathbf{D} satisfying $\mathbf{G}(\mathbf{D}_C) = \mathbf{C}$, 108

$\mathbf{D}(\mathbf{G})$ complete biorientation of \mathbf{G} , 108

$\mathbf{a} = \mathbf{uv}$ arc from \mathbf{u} to \mathbf{v} , 106

$\mathbf{N}_D^+(\mathbf{v})$ set of out-neighbors of \mathbf{v} , 107

$\mathbf{N}_D^-(\mathbf{v})$ set of in-neighbors of \mathbf{v} , 107

$\mathbf{A}_D(\mathbf{X}, \mathbf{Y})$ set of arcs with initial vertex in \mathbf{X} and terminal vertex in \mathbf{Y} , 107

$\mathbf{d}_D^+(\mathbf{v})$ out-degree of \mathbf{v} , 107

$\mathbf{d}_D^-(\mathbf{v})$ in-degree of \mathbf{v} , 107

$\mathbf{d}_D(\mathbf{v})$ total degree of \mathbf{v} , 107

$\tilde{\mathbf{d}}_D(\mathbf{v})$ geometric mean of $\mathbf{d}_D^+(\mathbf{v})$ and $\mathbf{d}_D^-(\mathbf{v})$, 166

$\Delta^+(\mathbf{D})$ maximum out-degree of \mathbf{D} , 107

$\Delta^-(\mathbf{D})$ maximum in-degree of \mathbf{D} , 107

$\Delta(\mathbf{D})$ maximum total degree of \mathbf{D} , 107

$\tilde{\Delta}(\mathbf{D})$ maximum geometric mean over all vertices of \mathbf{D} , 167

$\delta^+(\mathbf{D})$ minimum out-degree of \mathbf{D} , 107

$\delta^-(\mathbf{D})$ minimum in-degree of \mathbf{D} , 107

$\delta(\mathbf{D})$ minimum total degree of \mathbf{D} , 107

$\mathbf{D}' \subseteq \mathbf{D}$ \mathbf{D}' subdigraph of \mathbf{D} , 107

$\mathbf{D}' \subset \mathbf{D}$ \mathbf{D}' proper subdigraph of \mathbf{D} , 107

$\mathbf{D}[\mathbf{X}]$ subdigraph induced by \mathbf{X} , 107

$\mathbf{D}-\mathbf{X}$ subdigraph of \mathbf{D} induced by $\mathbf{V}(\mathbf{D}) \setminus \mathbf{X}$, 107

\mathbf{P} (directed) path, 107

\mathbf{C} (directed) cycle, 107

\mathbf{M} matching, 114

\mathbf{B} Block of \mathbf{D} , 108

$\mathcal{B}(\mathbf{D})$ set of all blocks of \mathbf{D} , 108

$\mathcal{B}_v(\mathbf{D})$ set of blocks of \mathbf{D} containing \mathbf{v} , 108

$\vec{\chi}(\mathbf{D})$ dichromatic number of \mathbf{D} , 109

$\vec{\chi}_\ell(\mathbf{D})$ list-dichromatic number of \mathbf{D} , 116

$\vec{\chi}_{DP}(\mathbf{D})$ DP-dichromatic number of \mathbf{D} , 115

$\vec{\chi}_m(\mathbf{D})$ m -degenerate dichromatic number of \mathbf{D} , 167

$(\mathbf{X}, \mathcal{D})$ cover of \mathbf{D} , 114

$(\mathbf{D}, \mathbf{X}, \mathcal{D})$ feasible configuration, 117

$(\mathbf{X}, \mathcal{D})/\mathbf{v}$ reduction method for feasible configurations, 121

$\mathbf{u} \rightarrow \mathbf{v}$ shifting the color from $x_u \in X_u$ to $x_v \in X_v$ (DP-case), 125

$\mathbf{u} \rightarrow \mathbf{v}$ shifting the color from \mathbf{u} to \mathbf{v} , 143

$\vec{\mathbf{CRI}}(\mathbf{k})$ class of k -critical digraphs, 140

$\vec{\mathbf{ext}}(\mathbf{k}, \mathbf{n})$ minimum number of arcs in a k -critical digraph of order \mathbf{n} , 169

$\vec{\mathbf{Ext}}(\mathbf{k}, \mathbf{n})$ class of k -critical digraphs of order \mathbf{n} with minimum number of arcs, 169

\mathbf{D}_L low vertex subdigraph of \mathbf{D} , 140

$\mathbf{D}_1 \boxtimes \mathbf{D}_2$ Dirac sum of \mathbf{D}_1 and \mathbf{D}_2 , 147

$\mathbf{D}_1 \nabla \mathbf{D}_2$ Hajós join of \mathbf{D}_1 and \mathbf{D}_2 , 147

$\mathbf{D}_1 \overset{\leftrightarrow}{\nabla} \mathbf{D}_2$	bidirected Hajós join of \mathbf{D}_1 and \mathbf{D}_2 , 153	\mathcal{O}_k^*	class of Ore- k -constructible digraphs containing $\mathbf{D}(\mathbf{K}_k)$, 155
$(\mathbf{D}_1, \mathbf{v}_1, \mathbf{u}_1) \nabla_i^o (\mathbf{D}_2, \mathbf{v}_2, \mathbf{u}_2)$	directed Ore join of \mathbf{D}_1 and \mathbf{D}_2 , 154	$\mathbf{D}(\mathbf{K}_k) \overset{\rightarrow}{+} \mathbf{v}$	results from $\mathbf{D}(\mathbf{K}_k)$ by adding \mathbf{v} plus one arc \mathbf{uv} , 155
$(\mathbf{D}_1, \mathbf{v}_1, \mathbf{u}_1) \overset{\leftrightarrow}{\nabla}_i^o (\mathbf{D}_2, \mathbf{v}_2, \mathbf{u}_2)$	bidirected Ore join of \mathbf{D}_1 and \mathbf{D}_2 , 154	$\mathbf{D}(\mathbf{K}_k) \overset{\leftarrow}{+} \mathbf{v}$	results from $\mathbf{D}(\mathbf{K}_k)$ by adding \mathbf{v} plus one arc \mathbf{vu} , 155
\mathcal{O}_k	class of Ore- k -constructible digraphs, 155	$\mathbf{D}(\mathbf{K}_k) + \mathbf{a}$	results from $\mathbf{D}(\mathbf{K}_k)$ by adding new vertices \mathbf{u}, \mathbf{v} and the arc \mathbf{uv} , 155