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Three Projective Problems on Finsler Surfaces

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Zusammenfassung

Zwei Finsler-Metriken auf der selben Mannigfaltigkeit heißen projektiv äquivalent, falls sie die selben unparametrisierten, orientierten Geodäten besitzen. Ein Vektorfeld auf der Mannigfaltigkeit heißt projektiv für eine Finsler-Metrik, falls sein Fluss Geodäten auf Geodäten als unparametrisierte Kurven abbildet. In dieser Dissertation werden nach einer Einführung in die allgemeine Theorie der Finsler Metriken und ihrer projektiven Aspekte, Ergebnisse zu drei projektiven Problemen für Finsler-Metriken auf Oberflächen präsentiert:

Erstens. Inspiriert durch ein von Sophus Lie gestelltes Problem wird gezeigt, dass jede Finsler-Metrik, welche drei unabhängige projektive Vektorfelder zulässt, projektiv äquivalent zu einer Randers Metrik ist. Eine explizite Liste solcher Metriken, vollständig bis auf Isometrie und projektive Äquivalenz, wird gegeben.

Zweitens. Das Problem der lokalen, Faser-globalen projektiven Metrisierung fragt, ob es zu gegebenen unparametrisierten, orientierten Kurven, eine Faser-globale Finsler Metrik gibt, deren Geodäten die vorgegebenen Kurven sind, und wenn ja, wie eindeutig diese ist. Es wird gezeigt, dass die Menge solcher Metrisierungen bis auf die triviale Freiheit in 1-zu-1 Beziehung zu Maßen mit einer Gleichgewichtseigenschaft auf dem Raum der vorgegebenen Kurven ist.

Drittens. Es wird bewiesen, dass auf einer geschlossenen Oberfläche von negativer Euler-Charakteristik zwei analytische Finsler-Metriken nur trivial projektiv äquivalent sein können: sie sind projektiv äquivalent genau dann, wenn sie sich durch Multiplikation mit einer positiven Zahl und Addition einer geschlossenen 1-Form unterscheiden.

Abstract

Two Finsler metrics on the same manifold are called projectively equivalent, if they have the same unparametrized, oriented geodesics. A vector field on the manifold is called projective for a Finsler metric, if its flow takes geodesics to geodesics as unparametrized curves. In this dissertation, after an introduction to the general theory of Finsler metrics and its projective aspects, results to three projective problems on Finsler metrics on surfaces are presented:

Firstly. Inspired by a problem posed by Sophus Lie, it is proven that every Finsler metric, admitting three independent projective vector fields, is projectively equivalent to a Randers metric. An explicit list of such metrics is given, complete up to isometry and projective equivalence.

Secondly. The problem of local, fiber-global projective metrization asks whether a given system of unparametrized, oriented curves describes the geodesics of some fiber-globally defined Finsler metric - and if yes, how unique this metric is. It is shown that the set of such metrizations is, up to the trivial freedom, in 1-to-1 correspondence with measures on the space of prescribed curves, satisfying a certain equilibrium property.

Thirdly. It is proven that on surfaces of negative Euler characteristic, two real-analytic Finsler metrics can only be trivially projectively related: they are projectively equivalent, if and only if they differ by multiplication with a positive number and addition of a closed 1-form.

Contents

Chapter 1

Introduction

1.1 Short Introduction and Results

A Finsler metric is a Lagrangian on the tangent bundle of a smooth manifold, whose unit balls in each tangent space are strictly convex bodies containing the origin. By the variation of arc-length and the Euler-Lagrange equations, every Finsler metric induces a system of curves with distinguished parametrization, that are extremal to its energy functional. The system of these curves, called geodesics, is formalized by a vector field on the tangent bundle, called a spray. The word projective in this context means to forget about the distinguished parametrization of the geodesics and to ask how much information about the Finsler metric is contained in the system of unparametrized, but oriented geodesics.

In this dissertation, results to three different projective problems in Finsler geometry on surfaces are presented:

1. The local description of Finsler metrics admitting three independent projective vector fields

2. The local, but fiber-global projective metrization problem

3. Topological obstructions to the existence of projectively equivalent Finsler metrics

Though all three problems belong truly to projective Finsler geometry, each of them is related to a different branch of differential geometry: Problem 1 is related to the classical analysis of second order ordinary differential equations, developed mainly by Sophus Lie more than one hundred years ago [44]. Problem 2 is a particular problem of the so called inverse calculus of variations, that is also studied in more general contexts [41, 42]. For Problem 3 in turn, techniques from the theory of integrable Hamiltonian systems are used.

In the choice of background material, I have tried to include enough material to make this dissertation understandable to anyone who has taken an introductory course in differential geometry.

More precisely, let M be a smooth manifold, $TM\backslash 0$ the tangent bundle with the origins removed and (x, ξ) local coordinates on TM.

Definition. *A Finsler metric is a smooth function $F : TM\backslash 0 \to \mathbb{R}_{>0}$, such that*

- *$F(x, \lambda\xi) = \lambda F(x, \xi)$ for all $\lambda > 0$.*
- *the matrix $g_{ij}|_{(x,\xi)} := \frac{1}{2}\frac{\partial^2 F^2}{\partial\xi^i\partial\xi^j}\Big|_{(x,\xi)}$ is positive definite for all $(x, \xi) \in TM\backslash 0$.*

The geodesics of F are defined as the curves $c : I \subseteq \mathbb{R} \to M$ that solve the Euler-Lagrange equation $E_i(L, c) := L_{x^i} - \frac{d}{dt}(L_{\xi^i}) = 0$ for the Lagrangian $L = \frac{1}{2}F^2$.

The two most common examples are *Riemannian* and *Randers metric*. A *Riemannian metric* is in local coordinates of the form $F(x, \xi) = \sqrt{\alpha_{ij}(x)\xi^i\xi^j}$, where $\alpha_{ij}(x)$ is a positive definite matrix varying with the point $x \in M$. A *Randers metric* is a metric that can be obtained from a Riemannian by addition of a 1-form - thus in local coordinates is of the form $F(x, \xi) = \sqrt{\alpha_{ij}(x)\xi^i\xi^j} + \beta_i(x)\xi^i$, where $\beta = \beta_i(x)\xi^i$ is a 1-form on M. To satisfy the above definition, the 1-form must be 'small' enough with respect to α in a suitable sense.

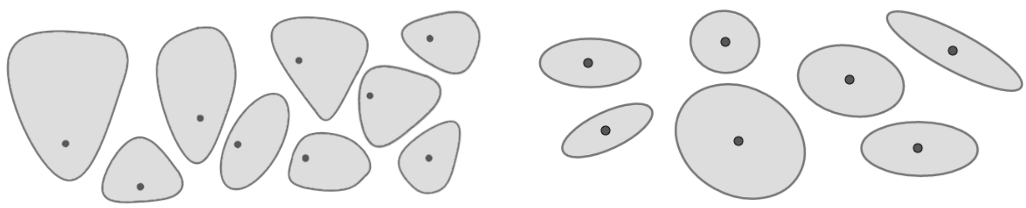


Figure 1.1: A Finsler metric is uniquely determined by the collection of its unit balls in each tangent space, each of which can be any origin enclosing, strictly convex body. For a Riemannian metric, these are ellipsoids.

Definition. *Two Finsler metrics $F, \tilde{F}$ on the same manifold are projectively equivalent, if any geodesic of F is a geodesic of $\tilde{F}$ after an orientation preserving reparametrization.*

There is always a trivial kind of projective equivalence:

Example. *Let $F, \tilde{F}$ be two Finsler metrics related by $\tilde{F} = \lambda F + \beta$, where $\lambda > 0$ and β is a closed 1-form on M. Then F and $\tilde{F}$ are projectively equivalent.*

Problem 1: Finsler metrics with three independent projective symmetries

Definition. *A vector field X on M is called projective for a Finsler metric F, if the image of each geodesic under the flow of X by a fixed time, is a geodesic after an orientation preserving reparametrization.*

It follows from the classical Lie theory of ordinary differential equations [44, 80, 81], that the set of projective vector fields $\mathfrak{p}(F)$ for a Finsler metric F forms a finite dimensional Lie algebra. If the dimension of the manifold is two, the maximal dimension of the projective algebra is eight and is obtained precisely for the metrics whose geodesics are straight lines in some local coordinates. Surprisingly, the submaximal dimension of the projective algebra that can occur is three. There, we have the following examples:

Example. *On $\mathbb{R}^2$ with coordinates (x, y), consider the Randers metric*

$$F = \sqrt{dx^2 + dy^2} + \tfrac{1}{2}(ydx - xdy).$$

Its geodesics are positively oriented circles of radius 1. Thus, any translation and rotation of $\mathbb{R}^2$ takes geodesics to geodesics up to orientation preserving reparametrization. Hence, the projective algebra of F is 3-dimensional, spanned by the three vector fields $\partial_x, \partial_y, y\partial_x - x\partial_y$ and consists out of the Killing vector fields of the Euclidean metric.

More generally, for any Riemannian metric α on a 2-dimensional manifold, let β be a 1-form, such that $d\beta$ is a multiple of the canonical volume form of α. The geodesics of the Randers metric $F := \alpha + \beta$ are curves of constant geodesic curvature with respect to α and any Killing vector field of α is projective for F. For the Riemannian metrics of constant sectional curvature ($\mathbb{R}^2$, S^2, H^2), we obtain new Finsler metrics with 3-dimensional projective algebra.

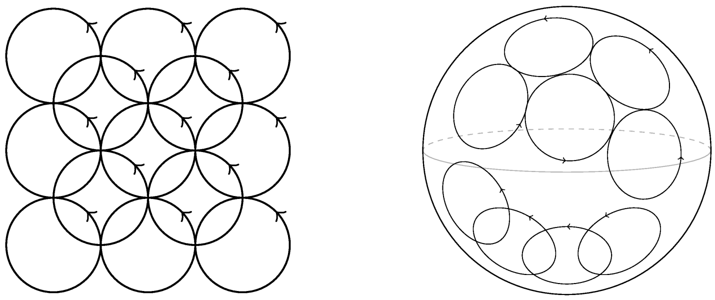


Figure 1.2: The circles of fixed radius in $\mathbb{R}^2$ or S^2 are systems of geodesics admitting three independent projective vector fields.

In 1882 Sophus Lie [43] stated the following problem:

Problem. *Describe metrics on surfaces for which the projective algebra $\mathfrak{p}$ is at least two-dimensional.*

The original problem intended for Riemannian metrics was solved in [17]. In the Finsler case, many results are known for the maximal dimensional case, that is when the geodesics are straight lines in some local coordinates. The investigation of such metrics is the fourth of the Hilbert problems stated in 1900.

The next case to study is the submaximal dimension three. In Chapter 3 we show that the above examples, complemented with the already known Riemannian metrics, exhaust all Finsler metrics with 3-dimensional projective algebra up to projective equivalence:

Theorem 3.1. *Every Finsler metric on a surface admitting at least three independent projective vector fields is projectively equivalent near any transitive point to*

- *a Randers metric $F = \alpha + \beta$, where the projective vector fields are exactly the Killing vector fields of the Riemannian metric α*
- *or to a Riemannian metric.*

In some local coordinates F is projectively equivalent to the Euclidean metric or to one of the following:

$$(a) \quad \sqrt{dx^2 + dy^2} + \tfrac{1}{2}(ydx - xdy) \qquad (b_k^{\pm}) \quad \frac{\sqrt{dx^2+dy^2} - \frac{k}{2}(ydx-xdy)}{1 \pm (x^2+y^2)}, k > 0$$

$$(c^+) \quad \sqrt{\frac{e^{3x}}{(2e^x-1)^2}dx^2 + \frac{e^x}{2e^x-1}dy^2} \qquad (c^-) \quad \sqrt{e^{3x}dx^2 + e^x dy^2}$$

Each of this metrics is strictly convex on a neighborhood of the origin and none of them is locally isometric to any Finsler metric projectively equivalent to one of the others.

Problem 2: The local, fiber-global projective metrization problem

Problem. *Given a system of unparametrized oriented curves on a manifold, does it describe the geodesics of a Finsler metric? If yes, 'how many' such metrics exist and how can we obtain them?*

This problem can be asked in different versions, depending on where one demands the metric to be defined: only *fiber-locally* (that is locally on $TM\backslash 0$), *locally* on M (that is on an open set $U \subseteq M$, but on the whole of $TU\backslash 0$) or *globally* over M.

If the dimension of M is at least three, the answer is negative: there are systems that cannot describe the geodesics of a Finsler metric even fiber-locally due to curvature obstruction (see [30] or Corollary 4.1). In dimension two, these obstruction vanish and any system describes the geodesics of some Finsler metric *fiber-locally.* The answer to the *global* version in dimension two is negative: For example, the circles of fixed geodesic curvature and orientation on the 2-sphere (Figure 1.2) cannot describe the geodesics of a globally defined Finsler metric, because by the Finslerian version of the Hopf-Rinow theorem, any two points on a closed Finsler surface can be joint by a geodesic .

The critical case is the local, but *fiber-global* one. There, the answer is positive, if the system is *reversible*, that is if every geodesic with orientation reversed is also a geodesic:

Theorem ([4, 7], Theorem 4.2)**.** *In dimension two, any reversible system of unparametrized curves describes locally, fiber-globally the geodesics of some Finsler metric.*

In fact there is a large freedom: for any positive smooth measure on the space of prescribed curves, one can produce a projective metrization. The *irreversible* case is much more troublesome and the freedom can be significantly smaller:

Example (Example)**.** *The system consisting of all positively oriented circles of radius 1 in $\mathbb{R}^2$ is irreversible (Figure 1.2). The Finsler metric (a) from Theorem 3.1 is a fiber-global projective metrization of this system. Are there any other?*

It was proven in [77] by S. Tabachnikov, that the answer is affirmative: The local, fiber-global projective metrizations are in 1-to-1 correspondence with positive functions $f : \mathbb{R}^2 \to \mathbb{R}$ on the plane, whose integral over every ball of radius one is the same constant for all balls, and the metric for such a function f is given by an integral formula.

But do such functions f exist? The obvious choice $f \equiv const \in \mathbb{R}$ corresponds to the metrization (a). Whether there are any other such functions f is a hard question and known as the Pompeiu problem. In the literature on the topic, such functions were constructed, however their construction is not particularly easy.

In Chapter 4 we review two approaches to the projective metrization problem: the first investigates the Euler-Lagrange equations to obtain a system of PDEs on the metric F that is sufficient and necessary for being a projective metrization; the second is a geometrical construction for reversible systems.

Combining the two approaches, we generalize Tabachnikov's circle result to arbitrary systems and proof that the set of local, fiber-global projective metrizations of a given system of curves is in 1-to-1 correspondence with measures on the set of unparametrized, oriented geodesics satisfying a certain equilibrium property:

Theorem 4.4 (Rough version)**.** *Given a system of unparametrized, oriented curves on $U \subseteq M$, let $\Gamma \subseteq \mathbb{R}^2$ be a parameter space for the unparametrized curves and $p : TU\backslash 0 \to \Gamma$ be a submersion, that assigns a tangent vector the unique curve tangent to it.*

Then up to the trivial freedom, the projective metrizations of the system are of the form

$$F(x, y, \phi, r) = r \int_0^{\phi} \left(f \circ p \middle| D_{T(\phi)}p \quad p_\phi \middle| \right)_{(x,y,\theta)} d\theta + a(x, y) \cos \phi + b(x, y) \sin \phi,$$

where $a_y - b_x = \left(f \circ p \middle| p_y \quad p_x \middle| \right)_{\phi=0}$ *and* $f : \Gamma \to \mathbb{R}_{>0}$ *is any smooth function, such that its integral over the connected components of* $\Gamma\backslash\{p(x, y, \theta) \mid \theta \in [0, 2\pi]\}$ *is constant under varying* (x, y).

Problem 3: Topological obstructions to the existence of projectively equivalent Finsler metrics

It is an interesting question under which conditions the projective metrization of a system of curves is rigid - that is, there is only one projective metrization up to the trivial projective equivalence. From Theorem 4.4 it is known that any Finsler metric, whose system of geodesics is reversible, admits locally a large family of non-trivially projectively equivalent metrics. Globally however, the topology of the manifold can give obstruction to the existence of pairs of projectively equivalent metrics. In Chapter 5 we proof the following:

Theorem 5.1. *Let $\mathcal{S}$ be a real-analytic surface of negative Euler characteristic with two real-analytic Finsler metrics $F, \tilde{F}$. Then the following are equivalent:*

(a) F and $\tilde{F}$ are projectively equivalent

(b) $\tilde{F} = \lambda F + \beta$ for some $\lambda > 0$ and a closed 1-form β.

This extends a result [57, Corollary 3] for Riemannian metrics (see also [58, 59, 79]), that states that, on a surface of negative Euler characteristic, two such are projectively equivalent, if and only if they differ by multiplication by a positive real number. For the Riemannian case, the assumption of real-analyticity is not necessary - for the Finslerian case however it is, as is demonstrated by Example 5.1.

Roughly speaking, Theorem 5.1 is proven by showing that the existence of two non-trivially related metrics implies integrability of their geodesic flow in the sense of integrable Hamiltonian systems. However, if the topology of the surface is complicated enough, the geodesic flow has positive entropy and cannot be integrable.

1.2 Motivation and History of (Projective) Finsler Geometry

Differential geometry has its starting point with the study of surfaces by Carl Friedrich Gauß (1777-1855). Among many other employments that he needed to finance his scientific studies, he worked as a land surveyor and had practical and theoretical interest in measuring distances on curved surfaces embedded in 3-dimensional space: on the small scale one might think of a hilly landscape (though the Kingdom of Hanover that he was to measure is rather flat), on the large scale of the surface of the entire earth. He noted that if one parametrizes a surface by two coordinates, the length of an infinitesimal displacement of the coordinates is given by the norm of an inner product - nowadays called Riemannian metric. In particular, the squared length of a vector is a quadratic polynomial in its components. Besides defining and studying curvatures of a surface, he investigated what the shortest curve (*geodesic*) between two points on a surface is - e.g. on a plane (segments of) straight lines; on a round sphere (segments of) the great circles (the intersection of the sphere with a plane that is a reflection symmetry for the sphere).

Its modern shape was given to differential geometry by Bernhard Riemann (1826-1866) in his famous Habilitationsvortrag *Über die Hypothesen, welche der Geometrie zu Grunde liegen* [69] in 1854 at the University Göttingen. He substituted the notion of an embedded surface in space by the notion of a manifold endowed with a Riemannian metric. He also mentioned the possibility of measuring length of vectors with more generalized norms that do not come from an inner product and are not quadratic in the vector components as in the Riemannian case. Unfortunately, he considered their investigation too tedious and uninteresting: '*The study of this more general class would not, it is true, require any essentially new principles, but would be time-consuming and probably throw relatively little new light on the theory of space, particularly since the results would not lend themselves to geometric form.*'.[1]

Later, in 1919, Hermann Weyl (1885-1955) published a commented version of the Habilitationsvortrag [70] and proposed to study manifolds with norms attached to each tangent space, all modelled on a fixed normed space. More precisely, all tangent spaces should all be linearly isometric to a fixed normed space - such spaces are nowadays called *monochromatic* or *generalized Berwald*, see [12] for definitions and equivalence of these notions.

At almost the same time in 1918, Paul Finsler (1894-1970), a student of Constantin Carathéodory in Göttingen, had written his doctoral dissertation [32] on manifolds where length is measured by a not necessarily quadratic norm. For a Riemannian metric, the set of vectors of length at most one is given by an solid origin-symmetric ellipsoid in each tangent space. The commonly used definition of a Finsler metric requires this set only to be a strictly convex body, containing the zero vector (see Figure 1.1). Paul Finsler developed basic notions of such metrics and discussed several differences to Riemannian geometry. Though his results are not considered to be very deep, his name remained attached to those spaces: The first publication to be found on the *MathSciNet*, in which

[1]Translation by R. Baker, C. Christenson and H. Orde in B. Riemann: *Collected papers*. German original: '*Die Untersuchung dieser allgemeinern Gattung würde zwar keine wesentlich andere Principien erfordern, aber ziemlich zeitraubend sein und verhältnissmässig auf die Lehre vom Raume wenig neues Licht werfen, zumal da sich die Resultate nicht geometrisch ausdrücken lassen;*'.

the term *Finsler space* is used, is in a paper [78] by James Henry Taylor in 1926, however more influential was the usage of this term by Élie Cartan (1869-1951) in 1933 [22].

Many modern notions of Finsler geometry were introduced and studied successfully by Ludwig Berwald (1883-1942). Among other things, he introduced a generalization of the Gaussian (sectional) curvature, nowadays known as *flag curvature*, as well as *Berwald curvature* and *projectively flat* metrics. In 1941 he was deported by the German Secret Police to the ghetto in Łódz, Poland, where he lived in inhuman conditions and died in 1942.

Since then, the interest in Finsler geometry has not declined, it rather seems to become more and more popular: in the last 15 years (2004-2018) 1399 papers with the key word '*Finsler*' have been published according to the *MathSciNet*; in the 15 years before (1989-2003) it were only 755.[2] Large Finsler geometry research groups are located especially in China, Iran, Hungary and Japan.

From a mathematical point of view, Finsler geometry is interesting as it generalizes Riemannian (and pseudo-Riemannian) geometry into a very less rigid object. Many theorems that are true in Riemannian geometry do not hold or are more complex and interesting in the Finslerian world.

But Finsler geometry also appears very naturally in real world problems. For example on an hiking trip, in that one wants to cross a mountain range to get from one valley to another. Using classical Riemannian geometry, one might find the shortest path, but it probably is not the most convenient, as it does not take the effort to climb an inclination into account. Riemannian geometry cannot model that it is easier to walk downhill than to walk uphill, because the norm of a Riemannian metric is always *symmetric*: an infinitesimal displacement in a direction is attributed the same length as the displacement in the opposite direction and the distance from A to B is the same as from B to A. Here Finsler geometry comes into play, by replacing 'length' of a vector by 'effort' that is takes to travel along it and thus using an *asymmetric* Finsler metric. Generally, Finsler geometry is an important tool in studying *asymmetric* problems - and asymmetric problems appear all over in natural science.

A very similar example is the Zermelo Navigation Problem (Problem 2.2.3): How to navigate most efficiently on a surface, where an extra force like a wind makes it easier to move in a certain direction (for example when travelling with a bike or a sail boat). It turns out that this situation can be modelled elegantly by a Finsler metric of special form, called *Randers metric*.

Besides the practical examples above, there are less obvious situations in which Finsler geometry has been applied to real world problems. In the book [5], many problems from biology and physics are tackled using Finsler geometry. Most of the biological problems are concerned with the evolution of the population in a certain biotope. There, the underlying manifold consists out of all possible configurations of how many individuals of each species exist at a moment. The Finsler metric describes how much effort/energy it costs the system to change from a certain configuration along a configuration direction. Then the real evolution is expected to follow a geodesic of the metric.

[2] This is not only due to the general growth of number of publications: the ratio of number of publications in the last 15 years, by the number for the 15 years before, is 1.85 for publications on Finsler geometry. The ratio on general publications on the *MathSciNet* for the same periods is only 1.59.

Another important, more and more popular application is in theoretical physics, where one models space(time) as a 4-dimensional manifold with a not strictly convex Finsler metric and one tries to generalize the Einstein field equations into this setting, see e.g. [8].

There are many more surprising situations in which Finsler geometry was applied to life saving real world theories: in [48] the spreading of wildfires and in [6] the evolution of seismic rays were modelled by Finsler geometry.

In all these applications, *projective* problems appear. Loosely speaking, the word 'projective' means the absence of an absolute time parameter[3], either due to the underlying theory that is used or due to the impossibility of measuring time. An example for the first is general relativity, where one postulates that no absolute time exists: Two observers travelling along different trajectories will measure time and speed of an object moving through space differently. The second case appears when one's observation is limited to trajectories of objects only, but one cannot determine a time parameter along those trajectories. Consider for example a camera pointed onto a surface on which several particles are moving to a least energy principle. The camera opens its lense for 5 seconds in which light reflected by the particles falls onto the photographic plate, and then repeats the process with a new plate. On each plate, one will be able to see the trajectories crossed by the particles in that 5 seconds, but one can not say anything about their speed. In this situation, so called projective metrization problems appear: Can one explain why the particles moved along the observed trajectories - according to which least energy principle? In other words, can one reconstruct the Finsler metric describing the energy that it takes the particles to move in a certain direction? If yes, is this metric unique? For example if all particles are moving along circles? These questions are motivational for this dissertation and certain aspects of the projective metrization problem on surfaces will be investigated in Chapters 3, 4 and 5.

[3] In more precise terms, in projective Finsler geometry one considers the geodesics of a metric without a preferred parametrization.

Chapter 2

Basic concepts and well-established theory

2.1 General Assumptions and Notation

Throughout this dissertation we work on a connected C^∞-smooth manifold M, whose dimension is usually denoted by n. After laying out the general theory, in Chapters 3, 4 and 5 we mostly restrict to the case $n = 2$, in which the manifold is denoted by $\mathcal{S}$ and is a surface. The global results from Chapter 5 are obtained for closed surfaces, that is compact and connected without boundary.

All differential objects are assumed at least C^∞-smooth, that is that their components in coordinates are infinitely many times differentiable. For a vector bundle $\pi : E \to M$ over a manifold, we denote by $E\backslash 0$ the bundle with the zero section removed. We use the word 'local' for objects defined locally on M, say on a open subset $U \subseteq M$, but fiber-globally over U, e.g. on the whole of TU for the tangent bundle - opposed to 'fiber-local', that is locally on the bundle. This distinction is crucial in Chapter 3 and 4. We write $C^\infty(M, N)$ for the space of smooth functions from M to N and $C^\infty(M)$ for $C^\infty(M, \mathbb{R})$.

For local considerations, we work on coordinate neighborhoods U identified with $\mathbb{R}^n$ with coordinates $(x^1, .., x^n)$. The induced coordinates on $TU = U \times \mathbb{R}^n$ are denoted by $(x^1, .., x^n, \xi^1, .., \xi^n)$. In the 2-dimensional case also (x, y, u, v) is used.

Einstein sum convention is used to shorten notation: whenever the same index variable appears in a term as upper and lower index, then there is a hidden summation by this index over the obvious range. For example $\beta = \beta_i dx^i$ is an abbreviation for $\beta = \sum_{i=1}^n \beta_i dx^i$. A lowered coordinate on a function denotes a partial derivative and arguments might be omitted if obvious, e.g. f_{x^i} is shorthand for $\frac{\partial f}{\partial x^i}(x)$.

2.2 Finsler metrics and its Geodesics

In this section the central objects of this dissertation are introduced, namely Finsler metrics and their geodesic spray. A Finsler metric is a positively 1-homogeneous Lagrangian on the tangent bundle of a smooth manifold, satisfying a non-degeneracy condition, and generalizes the concept of a Riemannian metric. It defines a notion of length of curves and by the variation of arc-length the locally shortest curves are given by a second order ODE, formalized by the geodesic spray. If the geodesics of two Finsler metrics coincide up to orientation preserving reparametrization, we call them projectively equivalent.

In the following we introduce the basic notions for the investigation of Finsler metrics, sprays and projective equivalence, introduce several examples and recall some standard theorems. The literature on this topic is very comprehensive and we refer to one of [10, 18, 72, 73, 74] for details and additional material.

Definition 2.1. *A Finsler metric on a smooth n-dimensional manifold M is a continuous function on the tangent bundle $F : TM \to \mathbb{R}$ with the following properties:*

(a) F is positive and smooth on $TM\backslash 0 = \bigcup_{p\in M} T_pM\backslash\{0\}$.

(b) F is positively 1-homogeneous in the fibers, that is $F(x, \lambda\xi) = \lambda F(x, \xi)$ for all $\lambda > 0$.

(c) The matrix $(g_{ij}) := (\frac{1}{2}(F^2)_{\xi^i\xi^j})$ is positive definite in all $(x, \xi) \in TM\backslash 0$ for any choice of local coordinates.

The matrix $(g_{ij}), i, j \in \{1, .., n\}$, whose entries are functions $g_{ij} : TM\backslash 0 \to \mathbb{R}$, is called fundamental tensor of the Finsler metric (with respect to the chosen coordinates).
The Finsler metric is called reversible, if F satisfies $F(x, -\xi) = F(x, \xi)$.

One cannot demand F to be smooth on the whole of TM, because then, by 1-homogeneity, F cannot be positive away from the zero vectors. Furthermore, from the 1-homogeneity it follows that F vanishes on all zero vectors, that is $F(x, 0) = 0$ for all $x \in M$.

Property (c) is sometimes called *strict convexity*, as it ensures that the unit balls in a fixed tangent space are strictly convex bodies.

Though the above definition is the most common, there are several variants present in the literature (by relaxing the assumption of positivity, smoothness, positive-definiteness, etc.), that are discussed in Section 2.2.6.

The first obvious and most familiar example of Finsler metrics are Riemannian metrics (more precisely the norm $F(x, \xi) = \sqrt{\alpha_x(\xi, \xi)}$ of a Riemannian metric α). Several additional examples and subclasses of Finsler metrics are discussed in Section 2.2.3.

2.2.1 Homogeneity, Euler's theorem and consequences

Let $k \in \mathbb{R}$. Recall that function $f : \mathbb{R}^n \to \mathbb{R}$ is called *positively k-homogeneous*, if for all $\lambda > 0$ it is $f(\lambda\xi) = \lambda^k f(\xi)$.

Theorem 2.1 (Euler's Homogeneity Theorem). *Let $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ be differentiable.*

(a) The function f is positively k-homogeneous, if and only if

$$f_{\xi^i}(\xi)\xi^i = kf(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n \backslash \{0\}.$$

(b) In this case any partial derivative $\frac{\partial f}{\partial \xi^i} : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ is positively $(k-1)$-homogeneous.

Proof. (a) For fixed $\xi \in \mathbb{R}^n \backslash \{0\}$, consider $g : (0, \infty) \to \mathbb{R}^n$ with $\lambda \mapsto f(\lambda\xi) - \lambda^k f(\xi)$. By the chain rule

$$\frac{dg}{d\lambda}(\lambda) = \frac{d}{d\lambda}\Big(f(\lambda\xi) - \lambda^k f(\xi)\Big) = f_{\xi^i}(\lambda\xi)\xi^i - k\lambda^{k-1}f(\xi).$$

If f is positively k homogeneous, g is constantly zero and putting $\lambda = 1$ in in the right side gives the claimed equality. If on the other hand the equality holds, we have

$$\frac{dg}{d\lambda}(\lambda) = \frac{k}{\lambda}f(\lambda\xi) - k\lambda^{k-1}f(\xi) = \frac{k}{\lambda}g(\lambda) \qquad \text{and} \qquad g(1) = 0.$$

This ODE is solved by the zero function with the same starting value, so that by the uniqueness theorem $g(\lambda) \equiv 0$ and f is positively k-homogeneous.

(b) By differentiating $f(\lambda\xi) = \lambda^k f(\xi)$ by ξ_i we obtain $\lambda f_{\xi^i}(\lambda\xi) = \lambda^k f_{\xi^i}(\xi)$, so if f is positively k-homogeneous, f_{ξ^i} is positively $(k-1)$-homogeneous. □

Throughout this dissertation, homogeneity is to be understood as positive homogeneity - we allow to drop the word 'positive(ly)'. A function f might also be *absolutely k-homogeneous*, that is $f(\lambda\xi) = |\lambda|^k f(\xi)$ for all $\lambda \in \mathbb{R}\backslash\{0\}$ - it is stated explicitly, when this stronger homogeneity is assumed.

The Euler theorem is used intensively in Finsler geometry as by definition the Finsler metric and hence all derived objects are homogeneous. The following are some immediate consequences for the fundamental tensor:

Corollary 2.1. *Let F be a Finsler metric on a smooth manifold M.*

(a) For $(x, \xi) \in TM\backslash 0$ the fundamental tensor $g_{ij}(x, \xi)$ defines an inner product on T_xM by

$$g_{(x,\xi)}(\nu, \eta) := g_{ij}(x, \xi)\nu^i\eta^j.$$

(b) The fundamental tensor g is 0-homogeneous, that is $g_{(x,\lambda\xi)} = g_{(x,\xi)}$.

(c) F can be recovered from g by $g_{(x,\xi)}(\xi, \xi) = g_{ij}(x, \xi)\xi^i\xi^j = F^2(x, \xi)$.

(d) The fundamental tensor might be equivalently described by

$$g_{(x,\xi)}(\nu, \eta) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0}\Big(F^2(x, \xi + t\nu + s\eta)\Big).$$

The fundamental tensor g_{ij} that plays an important role, as it appears in the geodesic equations. But also the Hessian of F itself in local coordinates, that is $h_{ij} = \frac{\partial^2 F}{\partial \xi^i \partial \xi^j}$, is important, since it appears analogously in the projective version of the geodesic equations, see Section 2.2.2 and Chapters 4 and 5.

Lemma 2.1 ([24])**.** *Let $h_{ij} = \frac{\partial^2 F}{\partial \xi^i \partial \xi^j}$ be the Hessian of a Finsler metric in local coordinates.*

(a) Each component h_{ij} is (-1)-homogeneous in the fiber coordinates and is related to the fundamental tensor by

$$g_{ij} = F h_{ij} + F_{\xi^i} F_{\xi^j}.$$

(b) The matrix (h_{ij}) is positive quasi-definite in all $(x,\xi) \in TM \backslash 0$, that is

$$h_{(x,\xi)}(\nu,\nu) = h_{ij}(x,\xi)\nu^i \nu^j \geq 0,$$

with equality if and only if $\nu = \lambda\xi$ for some $\lambda \in \mathbb{R}$.

Proof. (a) follows from Euler's theorem and the chain rule:

$$g_{ij} = (\tfrac{1}{2}F^2)_{\xi^i \xi^j} = (F F_{\xi^i})_{\xi^j} = F F_{\xi^i \xi^j} + F_{\xi^i} F_{\xi^j}.$$

For (b), first note that by (-1)-homogeneity it is $h_{ij}(x,\xi)\xi^i = 0$ and $h_{(x,\xi)}(\xi,\cdot) \equiv 0$. Fix a $\xi \in T_xM$ and let us show that $(h_{ij}(x,\xi))$ is positive quasi-definite. Any vector $\nu \in T_xM$ can be decomposed as $\nu = \lambda\xi + \nu^\perp$, such that ξ and $\nu^\perp$ are orthogonal with respect to the inner product $g_{(x,\xi)}$. Indeed, set $\lambda = \frac{g_{(x,\xi)}(\xi,\nu)}{F(x,\xi)^2}$ and $\nu^\perp = \nu - \lambda\xi$. Clearly,

$$\nu = \lambda\xi + \nu^\perp \qquad \text{and} \qquad g_{(x,\xi)}(\xi,\nu^\perp) = g_{(x,\xi)}(\xi,\nu) - \lambda g_{(x,\xi)}(\xi,\xi) = 0.$$

This implies $0 = g_{(x,\xi)}(\xi,\nu^\perp) = (\frac{1}{2}F^2)_{\xi^j}\nu^{\perp j} = F F_{\xi^j}\nu^{\perp j}$. Together with (a) we obtain

$$\begin{aligned} h_{(x,\xi)}(\nu,\nu) &= \lambda^2 \underbrace{h_{(x,\xi)}(\xi,\xi)}_{=0} + 2\lambda \underbrace{h_{(x,\xi)}(\xi,\nu^\perp)}_{=0} + h_{(x,\xi)}(\nu^\perp,\nu^\perp) \\ &= \frac{1}{F(x,\xi)}\Big(g(\nu^\perp,\nu^\perp) - \underbrace{(F_{\xi^i}\nu^{\perp i})^2}_{=0}\Big) \\ &= \frac{g_{(x,\xi)}(\nu^\perp,\nu^\perp)}{F(x,\xi)} \geq 0, \end{aligned}$$

and equality holds if and only if $\nu^\perp = 0$, that is, if ν is a multiple of ξ. □

Every Finsler metric restricted to a fixed tangent space satisfies two natural inequalities, in particular implying that the unit balls in the tangent spaces are strictly convex bodies.

Lemma 2.2. *(a) The closed unit balls $\overline{B_x} := \{\xi \in T_xM \mid F(x,\xi) \leq 1\} \subseteq T_xM$ are compact and strictly convex bodies containing the origin.*

(b) In each tangent space, $F(x,\cdot)$ satisfies the triangle inequality

$$F(x,\xi) + F(x,\eta) \geq F(x,\xi+\eta),$$

with equality if and only if $\eta = \lambda\xi$ or $\xi = \lambda\eta$ for some $\lambda \geq 0$.

(c) *In each tangent space, $F(x,\cdot)$ satisfies the following Cauchy-Schwarz inequality*

$$g_{(x,\xi)}(\xi,\eta) \leq F(x,\xi)F(x,\eta) \qquad \text{or equivalently} \qquad F_{\xi^i}(x,\xi)\eta^i \leq F(x,\eta)$$

with equality if and only if $\eta = \lambda\xi$ or $\xi = \lambda\eta$ for some $\lambda \geq 0$.

A complete proof can be found in [10, Section 1.2 B].
Assertion (a) follows from (b): Let $\xi, \eta \in T_xM$ with $F(x,\xi) = F(x,\eta) = 1$. Then for any convex combination of them $t\xi + (1-t)\eta$ with $t \in (0,1)$, we have

$$F(x, t\xi + (1-t)\eta) \leq tF(x,\xi) + (1-t)F(x,\eta) = 1,$$

with equality if and only ξ and η are positively proportional. The two inequalities in (c) are equivalent, because $g_{(x,\xi)}(\xi,\nu) = (\frac{1}{2}F^2)_{\xi^i}|_{(x,\xi)}\nu^i = F(x,\xi)F_{\xi^i}(x,\xi)\eta^i$ by Euler's Theorem.

It is clear that the strictly convex bodies $\overline{B_x}$ determine F. So in view of (a) a Finsler metric might equivalently be seen as a family $\overline{B_x} \subseteq T_xM$ of strictly convex bodies, each containing the origin, that vary smoothly with the base point $x \in M$.

If a Finsler metric F is reversible, that is $F(x,-\xi) = F(x,\xi)$ for all $(x,\xi) \in TM\backslash 0$, then, by the triangle inequality (b), the function $F(x,\cdot)$ is a norm on T_xM in the classical sense.

2.2.2 Geodesics and the Euler-Lagrange equations

Next, we define the length of a curve and distances on a Finsler manifold and describe 'shortest' curves by the variation of arc-length.

Definition 2.2. *Let F be a Finsler metric on a smooth manifold.*

1. *Let $c : [a,b] \to M$ be a smooth curve. We define the length $\mathcal{L}$ and energy $\mathcal{E}$ of c by*

$$\mathcal{L}(c) := \int_a^b F(\dot{c}(t))dt \qquad \text{and} \qquad \mathcal{E}(c) := \int_a^b \tfrac{1}{2}F^2(\dot{c}(t))dt.$$

2. *The induced distance function $d : M \times M \to \mathbb{R}$ is defined by*

$$d(p,q) := \inf\Big\{\mathcal{L}(c) \;\Big|\; c : [a,b] \to M \text{ smooth curve with } c(a) = p, c(b) = q\Big\}.$$

In general, the induced distance d is not symmetric: if the Finsler metric is not reversible, the length of a curve can change when its orientation is reversed and the distance from p to q might differ from the distance from q to p. However, d satisfies the definiteness property of a distance function and the triangle inequality, that is for any $p, q, r \in M$ the following hold:

$$d(p,q) \geq 0 \qquad d(p,q) = 0 \Leftrightarrow p = q \qquad d(p,r) \leq d(p,q) + d(q,r).$$

Furthermore it can be shown that the topology induced by d coincides with the topology of M and that for any $p \in M$ the map $M \to \mathbb{R}$ given by $q \mapsto d^2(p,q)$ is C^1-smooth and C^∞-smooth on $M\backslash\{p\}$. See [10, Section 6.2] for details.

We are interested in curves, which are a local minimum for the lengths functional $\mathcal{L}(c) = \int_a^b F(\dot{c}(t))dt$ and the energy functional $\mathcal{E}(c) = \int_a^b \frac{1}{2}F^2(\dot{c}(t))dt$, in the sense that no small perturbations will decrease the length or energy respectively. This is made precise in the following.

Definition 2.3. *Let M be a smooth manifold.*

(a) Let $c : [a,b] \to M$ be a curve whose trajectory is contained in a coordinate region $U \subseteq M$. A local variation of c is a map $H : [a,b] \times (-\epsilon, \epsilon) \to U$, which in the local coordinates is given by $H(t,s) = c(t) + sh(t)$, where $h : [a,b] \to \mathbb{R}^n$ is a smooth vector valued function with $h(a) = h(b) = 0$.

(b) Consider the functional $\mathcal{F} : c \mapsto \int_a^b L(\dot{c}(t))dt$, where $L : TM\backslash 0 \to \mathbb{R}$ is an smooth function. A curve $c : I \to M$ is called extremal for $\mathcal{F}$, if for every local variation H of c, we have $\frac{d}{ds}|_{s=0}\ \mathcal{F}(H(\cdot, s)) = 0$.

The extremals of a functional are given as solutions to a system of second order ODEs, the Euler-Lagrange equations:

Lemma 2.3 (Euler-Lagrange equations)**.** *A smooth curve $c : I \to M$ is extremal for $\mathcal{F} : c \mapsto \int_a^b L(\dot{c}(t))dt$, if and only if in all local coordinates it satisfies the Euler-Lagrange equations:*

$$E_i(L,c) := \frac{\partial L}{\partial x^i}|_{(c(t),\dot{c}(t))} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \xi^i}|_{(c(t),\dot{c}(t))}\Big) = 0 \qquad i = 1,..,n$$

$$\text{or in short notation:} \qquad E_i(L,c) = L_{x^i} - \frac{d}{dt}L_{\xi^i} = 0.$$

Proof. Let H be a variation of c. Then in local coordinates

$$\begin{aligned}\frac{d}{ds}|_{s=0}\mathcal{F}(H) &= \frac{d}{ds}|_{s=0}\int_a^b L\Big(c(t) + sh(t), \dot{c}(t) + s\dot{h}(t)\Big)dt \\ &= \int_a^b \frac{\partial L}{\partial x^i}|_{(c(t),\dot{c}(t))}h^i(t) + \frac{\partial L}{\partial \xi^i}|_{(c(t),\dot{c}(t))}\dot{h}^i(t)dt \\ &= \int_a^b \frac{\partial L}{\partial x^i}|_{(c(t),\dot{c}(t))}h^i(t) - \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^i}|_{(c(t),\dot{c}(t))}\right)h^i(t) + \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^i}|_{(c(t),\dot{c}(t))}h^i(t)\right)dt \\ &= \int_a^b \left(\frac{\partial L}{\partial x^i}|_{(c(t),\dot{c}(t))} - \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^i}|_{(c(t),\dot{c}(t))}\right)\right)h^i(t)dt.\end{aligned}$$

If c is a extremal for $\mathcal{F}$, this term vanishes for all possible h in all local coordinates and the bracket in the integrand must be identically 0. If on the other hand the bracket is identically 0 in all local coordinates, then c is extremal for $\mathcal{F}$, since $\frac{d}{ds}|_{s=0}L(H)$ vanishes for every variation H. □

In Finsler geometry, there are two canonical candidates for the function L: The Finsler function F itself and the energy function $E := \frac{1}{2}F^2$. If one wants to measure length, it is natural to use F. However, it turns out to be convenient to work with E instead, as its Hessian g_{ij} with respect to the fiber coordinates is by definition positive definite - the Hessian h_{ij} of F, however, is always singular, see Lemma 2.1. The relation between the Euler-Lagrange equations for F and E is explained by the next Lemma.

Lemma 2.4.

(a) Let $L : TM\backslash 0 \to \mathbb{R}$ be a 2-homogeneous smooth function and the curve c be a solution of the Euler-Lagrange equations $E_i(L, c) = 0$. Then L is constant along c, that is

$$\frac{d}{dt}\Big(L(c(t), \dot{c}(t))\Big) = 0.$$

(b) Let $L : TM\backslash 0 \to \mathbb{R}_{>0}$ be a smooth function constant along a curve $c : I \to M$. Then

$$E_i(\tfrac{1}{2}L^2, c) = LE_i(L, c).$$

(c) Let F be a Finsler metric. Then:

- *Every solution of $E_i(\frac{1}{2}F^2, c) = 0$ is a solution of $E_i(F, c) = 0$.*
- *Conversely, every solution of $E_i(F, c) = 0$, such that F is constant along c, is a solution of $E_i(\frac{1}{2}F^2, c) = 0$.*
- *If c is a solution of $E_i(F, c) = 0$, so is every orientation preserving reparametrization of c.*

(d) The Euler-Lagrange equations $E_i(L, c) = 0$ are $\mathbb{R}$-linear in L, that is

$$E_i(\lambda L + \mu\tilde{L}, c) = \lambda E_i(L, c) + \mu E_i(\tilde{L}, c) \qquad \text{for any } \lambda, \mu \in \mathbb{R}.$$

For a 1-form β on M, the Euler-Lagrange equations $E_i(\beta, c)$ vanish for all curves c, if and only if β is closed.

Proof. (a) Let c be a solution of $E_i(L, c) = 0$. Then $L_{\xi^i\xi^j}\ddot{c}^j = L_{x^i} - L_{\xi^i x^j}\dot{c}^j$ and using the Euler theorem 2.1 we obtain

$$\begin{aligned}\frac{d}{dt}(L) &= L_{x^j}\dot{c}^j + L_{\xi^j}\ddot{c}^j \\ &= L_{x^j}\dot{c}^j + L_{\xi^j\xi^k}\ddot{c}^j\dot{c}^k \\ &= L_{x^j}\dot{c}^j + L_{x^k}\dot{c}^k - L_{\xi^k x^j}\dot{c}^j\dot{c}^k = 0.\end{aligned}$$

(b) For any curve $c : I \to M$ by direct calculation using the chain rule

$$\begin{aligned}E_i(\frac{1}{2}L^2, c) &= LL_{x^i} - \frac{d}{dt}(LL_{\xi^i}) \\ &= L\Big(L_{x^i} - \frac{d}{dt}(L_{\xi^i})\Big) - \frac{dL}{dt}L_{\xi^i} \\ &= LE_i(L, c) - \frac{dL}{dt}L_{\xi^i}.\end{aligned}$$

If L is constant along c, then the last term vanishes.

(c) If c is a solution of $E_i(\frac{1}{2}F^2, c) = 0$, by (a) F is constant along c and by (b), we have

$$E_i(F, c) = \frac{1}{F}E_i(\tfrac{1}{2}F^2, c) = 0.$$

If c is a solution of $E_i(F, c) = 0$ and F constant along c, then, by (b), we have

$$E_i(\frac{1}{2}F^2) = FE_i(F) = 0.$$

If c is solution of $E_i(F,c)=0$, then for $\tilde{c}(s)=c(\varphi(s))$ with $\varphi'(s)>0$ we have by the 1-homogeneity of F and 0-homogeneity of F_{ξ^i}

$$\begin{aligned}E_i(F,\tilde{c}) &= \varphi'(s)F_{x^i}\big(c\circ\varphi(s),\dot{c}\circ\varphi(s)\big)-\frac{d}{ds}\Big(F_{\xi^i}\big(c\circ\varphi(s),\dot{c}\circ\varphi(s)\big)\Big)\\ &= \varphi'(s)\Big(F_{x^i}\big(c(t),\dot{c}(t)\big)-\frac{d}{dt}\big(F_{\xi^i}(c(t),\dot{c}(t))\big)\Big)|_{t=\varphi(s)}\\ &= \varphi'(s)E_i(F,c)|_{\varphi(s)}=0.\end{aligned}$$

(d) Linearity is obvious. Let the 1-form β be given in local coordinates by $\beta=\beta_j dx^j$. It is closed, if and only if $(\beta_j)_{x^i}-(\beta_i)_{x^j}\equiv 0$ for all $i,j\in\{1,..,n\}$. On the other hand,

$$E_i(\beta,c)=(\beta_j)_{x^i}\dot{c}^j-\frac{d}{dt}(\beta_i)=\big((\beta_j)_{x^i}-(\beta_i)_{x^j}\big)\dot{c}^j,$$

and this vanishes for all curves c, if and only if $(\beta_j)_{x^i}-(\beta_i)_{x^j}\equiv 0$ for all $i,j\in\{1,..,n\}$. □

Definition 2.4. *A geodesic of a Finsler metric F on a smooth manifold M is a curve $c:I\to M$, that is extremal for the energy functional $\mathcal{E}(c)=\int_a^b \frac{1}{2}F^2(\dot{c}(t))dt$.*

Let us write the Euler-Lagrange equations for the energy function $E=\frac{1}{2}F^2$ in local coordinates explicitly by using the chain rule, Euler's theorem for the 2-homogeneous function E and let $g^{ij}:TM\backslash 0\to\mathbb{R}$ be the entries of the inverse matrix of the fundamental tensor g_{ij}. Then

$$\begin{aligned}0 &= E_i(E,c)\\ &= E_{x^i}-E_{\xi^i x^\ell}\dot{c}^\ell-E_{\xi^i\xi^j}\ddot{c}^j\\ &= -g_{ij}\Big(\ddot{c}^j+2G^j(c(t),\dot{c}(t))\Big)\\ &\text{where } G^j := \frac{1}{2}g^{jk}\big(E_{\xi^k x^\ell}\xi^\ell-E_{x^k}\big)\\ &\qquad\quad = \frac{1}{4}g^{jk}\big(2\frac{\partial g_{kr}}{\partial x^\ell}-\frac{\partial g_{\ell r}}{\partial x^k}\big)\xi^\ell\xi^r\end{aligned}.$$

By contracting with the g^{ij}, Euler-Lagrange equations are written in normal form. Thus the geodesics of F are exactly the solutions of the ODE system

$$\ddot{c}^i(t)+2G^i(c(t),\dot{c}(t))=0.$$

Note that we cannot write the Euler-Lagrange equations for F in normal form, because its Hessian matrix h_{ij} is not invertible.

Definition 2.5. *Let M be a smooth manifold.*

(a) A spray is a smooth vector field S on $TM\backslash 0$, that in every local coordinates region $U\subseteq M$ with coordinates (x^i,ξ^i) on TU is of the form

$$S|_{(x,\xi)}=\xi^i\partial_{x^i}-2G^i(x,\xi)\partial_{\xi^i},$$

with some in ξ positively 2-homogeneous function $G^i:TM\backslash 0\to\mathbb{R}$ for $i\in\{1,..,n\}$. If the functions G^i are absolutely 2-homogeneous in ξ, the spray is called reversible.

(b) Let F be a Finsler metric on M. Its geodesic spray S_F is the spray given in local coordinates by

$$G^i(x,\xi) := \frac{1}{4}g^{ij}\Big(2\frac{\partial g_{jk}}{\partial x^\ell} - \frac{\partial g_{k\ell}}{\partial x^j}\Big)\xi^k\xi^\ell.$$

Though the geodesic spray S_F is given in terms of local coordinates, it is defined independently of the choice of coordinates, as it is the solution to a variational problem. This fact also follows from Section 2.4.2.

In the next Lemma we collect some obvious properties of sprays in general and the geodesic spray of a Finsler metric in particular.

Lemma 2.5. *Let S be a spray on a smooth manifold M.*

(a) The integral curves of a spray S and the geodesics of S, that is the curves that in local coordinates satisfy $\ddot{c} + 2G^i(c,\dot{c}) = 0$, correspond to each other under prolongation and projection:
If $\gamma : I \to TM$ is an integral curve of S, then $c = \pi \circ \gamma$ is a solution of $\ddot{c} + 2G^i(c,\dot{c}) = 0$, where $\pi : TM \to M$ is the bundle projection.
If $c : I \to M$ solves $\ddot{c} + 2G^i(c,\dot{c}) = 0$, then $\dot{c} : I \to TM$ is an integral curve of S.

(b) For every $\xi_0 \in TM$, there is a unique geodesic $c : I \to M$ with $\dot{c}(0) = \xi_0$. The unique geodesic $\tilde{c}$ with $\tilde{c}(0) = \lambda\xi_0$ is the linear reparametrization $\tilde{c}(t) = c(\lambda t)$ for all $\lambda > 0$.

(c) Any Finsler metric F is constant along its geodesic spray, that is $S_F(F) = 0$.

2.2.3 Examples and Classes of Finsler metrics

In this section, we give several examples and subclasses of Finsler metrics. These subclasses give some structure to the large variety of possible Finsler metrics on a fixed manifold, but also provide realms in which particular problems can be studied, that are too complicated for general Finsler metrics. We introduce and discuss shortly the following types of Finsler metrics:

- Riemannian metrics
- Randers metrics
- Berwald metrics
- Douglas metrics
- Minkowski metrics
- Funk and Hilbert metrics
- Projectively flat metrics (only in Section 2.2.7).

There are many more interesting types studied intensively, that will not be mentioned further on:

- Metrics of constant flag curvature
- Metrics of scalar flag curvature
- Landsberg metrics, Generalized Berwald metrics
- (Generalized) (α, β)-metrics
- Einstein metrics, Conformally flat metrics, Ricci flat metrics, and many more.

Some of the relations among those subclasses can be seen in Figure 2.1 and 2.2.

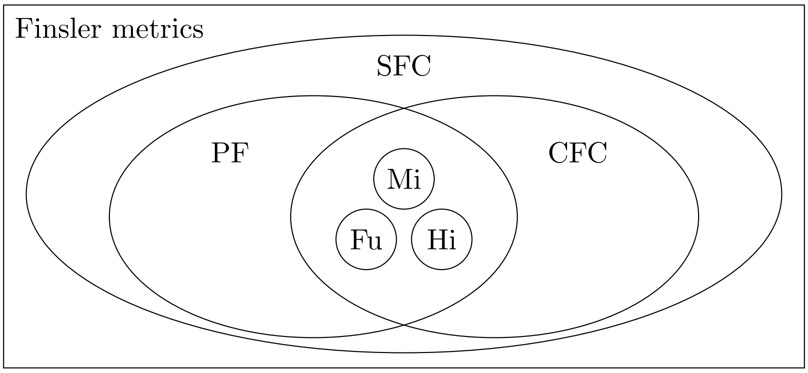


Figure 2.1: Some classes of Finsler metrics and their relations.

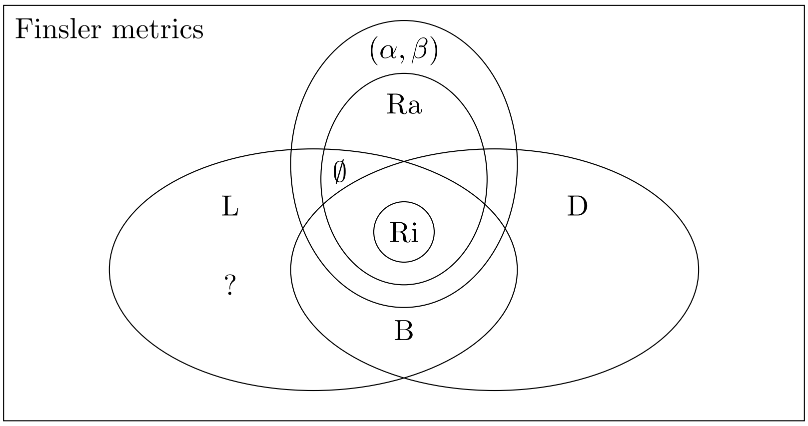


Figure 2.2: Some more classes of Finsler metrics and their relations. The '?' indicates the long-open, so called unicorn problem [9]: is there a Landberg metric that is not Berwald? A metric is Berwald, if and only if it is Landsberg and Douglas [73, Chapter 13]. The '$\emptyset$' indicates that a Randers metric is Berwald, if and only if it is Landsberg [50].

Definition 2.6. *A Riemannian metric α on a smooth manifold M is a collection of inner products $\alpha_p : T_pM \times T_pM \to M$ depending smoothly on the point $p \in M$.*

More precisely, in all local coordinates (x^i, ξ^i) it is $\alpha_x(\nu, \eta) = \alpha_{ij}(x)\nu^i\eta^j$, where the coefficients $\alpha_{ij}(x) := \alpha_x(\partial_{x^i}, \partial_{x^j})$ are the entries of the Gramian matrix of α_x and form a symmetric and positive definite matrix, with entries depending smoothly on x.

The norm induced by a Riemannian metric $F(x, \xi) = \sqrt{\alpha_x(\xi, \xi)} = \sqrt{\alpha_{ij}(x)\xi^i\xi^j}$ defines a Finsler metric. Smoothness, positivity and 1-homogeneity are obvious. The fundamental tensor g_{ij} coincides with the Gramian matrix

$$g_{ij}(x, \xi) = \tfrac{1}{2}\left(F^2\right)_{\xi^i\xi^j}(x, \xi) = \alpha_{ij}(x),$$

is independent of the fiber coordinates ξ and positive definite; thus F is indeed a Finsler metric. The closed unit balls $\overline{B_x} = \{\xi \in T_xM \mid F(\xi) \leq 1\} \subseteq T_xM$ of a Riemannian metric are origin centred ellipses for any local coordinates.

To obtain concrete examples, take any smooth submanifold M of $\mathbb{R}^N$, e.g. a sphere, and restrict the Euclidean inner product to TM (in fact, by the Nash embedding theorem every Riemannian metric arises in that way).

The geodesic spray coefficients of a Riemannian metric are given by

$$G^i = \frac{1}{4} g^{ij}\Big(2\frac{\partial g_{jk}}{\partial x^\ell} - \frac{\partial g_{k\ell}}{\partial x^j}\Big)\xi^k\xi^\ell = \frac{1}{4} g^{ij}\Big(\frac{\partial g_{jk}}{\partial x^\ell} + \frac{\partial g_{j\ell}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^j}\Big)\xi^k\xi^\ell = \tfrac{1}{2}\Gamma^i_{k\ell}\xi^k\xi^l,$$

where $\Gamma^i_{k\ell} = \Gamma^i_{k\ell}(x)$ are the Christoffel symbols of the Levi-Civita connection of α, and the geodesics of F are locally given by the equation $\ddot{c}^i + \Gamma^i_{k\ell}\dot{c}^k\dot{c}^l = 0$.

Let us also calculate the Hessian of F: it is

$$F_{\xi^i} = \Big((g_{k\ell}\xi^k\xi^\ell)^{\frac{1}{2}}\Big)_{\xi^i} = \frac{g_{i\ell}\xi^\ell}{F}$$

and thus

$$h_{ij} = F_{\xi^i\xi^j} = \tfrac{1}{F^3}\Big(g_{ij}g_{k\ell} - g_{ik}g_{j\ell}\Big)\xi^k\xi^\ell.$$

Let us also determine the missing coefficients of the Euler-Lagrange equations of F, that is $E_i(F,c) = F_{x^i} - F_{\xi^i x^j}\dot{c}^j - h_{ij}\ddot{c}^j = 0$. We have

$$F_{x^i} = \tfrac{1}{2F}(g_{rs})_{x^k}\xi^r\xi^s$$

and

$$F_{\xi^i x^k} = \tfrac{1}{F^3}\Big((g_{i\ell})_{x^k}g_{rs} - \tfrac{1}{2}g_{i\ell}(g_{rs})_{x^k}\Big)\xi^\ell\xi^r\xi^s.$$

The first non-Riemannian Finsler examples are Randers metrics:

Definition 2.7. *Let α be a Riemannian metric and β be a 1-form on a smooth manifold M. Then, if the function $F : TM \to \mathbb{R}$ defined in local coordinates by*

$$F(x,\xi) := \sqrt{\alpha_x(\xi,\xi)} + \beta_x(\xi)$$

is a Finsler metric, it is called a Randers metric.

We will use the short cut $F = \alpha + \beta$ for a Randers metric constructed by the Riemannian metric α and the 1-form β. Any function constructed in this way fulfils the smoothness and homogeneity assumptions of a Finsler metric. However, it might neither be positive, nor strictly convex, if β takes large values in comparison with α. It turns out that F is a Finsler metric, if and only if in local coordinates for all $x \in M$ the inequality

$$\alpha^{ij}\beta_i\beta_j < 1$$

holds, where $\alpha_x = \alpha_{ij}(x)dx^i dx^j$, $\beta_x = \beta_i(x)dx^i$ and (α^{ij}) is the inverse matrix of (α_{ij}) [10, Section 1.3 C].

The study of Randers metrics can be motivated by the following problem, formulated by Ernst Zermelo [87] in 1931. We follow the illustration and solution from [11]:

Problem (Zermelo Navigation Problem)**.** *Consider a smooth manifold M with a Riemannian metric α. For illustration, imagine M to be the surface of a (not necessarily round) planet covered by water. Suppose to travel on this manifold using a motorboat with constant power, so that the tangent vector of our movement curve c will be a vector of length one (with respect to the Riemannian metric α). The unit sphere S_x of the Riemannian metric in a tangent space $T_xM, x \in M$ indicates the infinitesimal displacements that we can make is an infinitesimal time unit. In order to travel from our* *position $x \in M$ to another position $y \in M$ as fast as possible, our movement curve should minimize the length functional $\mathcal{L}(c) = \int_a^b \sqrt{\alpha_{c(t)}(\dot{c}(t), \dot{c}(t))}$ - hence it should be a geodesic of the Riemannian metric α.*

Now suppose that on M there is an additional time-independent force, given by a vector field V, that shifts the set S_x of vectors reachable in an infinitesimal time unit by the vector V_x - imagine a wind blowing on the surface that drags in the direction V_x. We shall assume $\alpha_x(V_x, V_x) < 1$, to ensure that the shifted unit spheres $\tilde{S}_x = S_x + V_x$ still enclose the origins and thus movements in any direction are possible. Now under the influence of the additional force, the curve that will bring us in the least time from a point x to a point y should be a geodesic of the new Finsler metric $\tilde{F}$, whose unit spheres are given by $\tilde{S}_x = S_x + V_x$ (this defines the new metric $\tilde{F}$). Can we give an explicit formula for $\tilde{F}$, in terms of the data (α, V)?

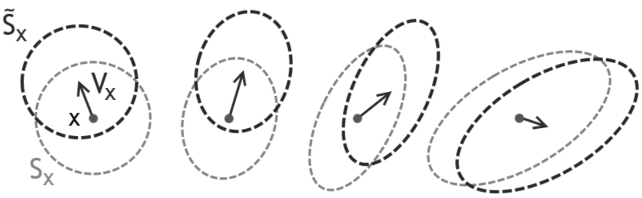


Figure 2.3: Shifting the unit spheres S_x of a Riemannian metric by a vector field V_x gives the unit balls $\tilde{S}_x$ of a Randers metric.

Theorem 2.2. *For any Riemannian metric α and a smooth vector field V satisfying $\alpha_x(V_x, V_x) < 1$ for all $x \in M$, there is a Riemannian metric $\tilde{\alpha}$ and a 1-form $\tilde{\beta}$, such that the unit spheres of the Randers metric $\tilde{F} := \tilde{\alpha} + \tilde{\beta}$ are the V-translated unit spheres S_x of α. More precisely,*

$$\Big\{\xi \in T_xM \mid \alpha_x(\xi, \xi) = 1\Big\} + V_x = \Big\{\xi \in T_xM \mid \sqrt{\tilde{\alpha}_x(\xi, \xi)} + \beta_x(\xi) = 1\Big\}.$$

Furthermore, every Randers metric arises in this way. One can give explicit formulas for $(\tilde{\alpha}, \tilde{\beta})$ in terms of (α, V) and vice versa.

By the solution of the Zermelo Navigation Problem, the unit spheres $S_x \subseteq T_xM$ of a Randers metric are shifted ellipses containing the origin. One can give a rather long explicit formula for the geodesic spray of a Randers metric $\alpha + \beta$, that we skip here.

There is an important generalization of Randers metrics, namely so called (α, β)-metrics, that are defined again using a Riemannian metric α and a 1-form β by a formula

$$F(x, \xi) = \sqrt{\alpha_x(\xi, \xi)} \cdot \phi\Big(\frac{\beta_x(\xi)}{\sqrt{\alpha_x(\xi, \xi)}}\Big),$$

where $\phi : (-s_0, s_0) \to (0, \infty)$ is a smooth function. The fact that this defines a Finsler metric can be expressed by a differential inequality on the function ϕ.

Clearly every Riemannian metric is a Randers metric, and any Randers metric is an (α, β)-metric.

Definition 2.8. *A Finsler metric F on a smooth manifold M is called Berwald metric, if its geodesic coincides with the geodesics of an affine connection, that is if in all local coordinates*

$$G^i(x,\xi) = \tfrac{1}{2}\Gamma^i_{k\ell}(x)\xi^k\xi^l,$$

for some coordinate-dependent smooth functions $\Gamma^i_{k\ell} : M \to \mathbb{R}, i,k,\ell \in \{1,..,n\}$.

Definition 2.9. *A Finsler metric F on a smooth manifold M is called Douglas metric, if there is a function $P : TM\backslash 0 \to \mathbb{R}$ such that in all local coordinates its geodesic spray coefficients are given by*

$$G^i(x,\xi) = \tfrac{1}{2}\Gamma^i_{k\ell}(x)\xi^k\xi^l + P(x,\xi)\xi^i,$$

for some coordinate-dependent smooth functions $\Gamma^i_{k\ell} : M \to \mathbb{R}, i,k,\ell \in \{1,..,n\}$.

In section 2.2.7 it will become clear that a Finsler metric is Douglas, if and only if its geodesics are up to orientation preserving reparametrization the geodesics of an affine connection.

Clearly, every Riemannian metric is Berwald, and every Berwald metric is Douglas.

Definition 2.10. *A Finsler metric F on a smooth manifold M is called Minkowski metric, if around every point there are local coordinates (x,ξ) in which $F(x,\xi) = F(\xi)$ is independent of the base coordinate x.*

Let $U \subseteq \mathbb{R}^n$ be a coordinate region with coordinates in which F is independent of x. Then the geodesic spray coefficients vanish, as $G^i(x,\xi) = \frac{1}{4}g^{ij}\Big(2\frac{\partial g_{jk}}{\partial x^\ell} - \frac{\partial g_{k\ell}}{\partial x^j}\Big)\xi^k\xi^\ell = 0$, and the geodesics are all linearly parametrized lines $c(t) = x_0 + t\xi_0$ with $p_0 \in U, \xi_0 \in \mathbb{R}^n$. A metric with such geodesics is called *projectively flat*, cf. Definition 2.16.

Furthermore, all the closed unit balls $\overline{B_x} \subseteq T_xU$ of such a metric are the same strictly convex body, seen as a subset of $T_xU = \mathbb{R}^n$ in this particular coordinates. Conversely, one can define a Minkowski metric on $\mathbb{R}^n$ by choosing any strictly convex body in $\mathbb{R}^n$ containing the origin, and impose it as the unit ball of a Finsler metric in all $T_x\mathbb{R}^n, x \in \mathbb{R}^n$.

On any strictly convex open subset of $\mathbb{R}^n$, two important Finsler metrics can be defined (see [82] for further discussion):

Definition 2.11. *Let $U \subseteq \mathbb{R}^n$ be open, not empty, strictly convex with smooth boundary.*

(a) The Funk metric[1] F_U on U is defined as the Finsler metric, whose open unit ball $B_x \subseteq T_xU = \mathbb{R}^n$ in a point $x \in U$ is the set

$$U - x = \{u - x \mid u \in U\} \subseteq T_xU.$$

(b) The Hilbert metric[2] on U is the symmetrization of the Klein metric on U, that is

$$F^H_U(x,\xi) := \tfrac{1}{2}\Big(F_U(x,\xi) + F_U(x,-\xi)\Big).$$

[1] Sometimes also called tautological Finsler metric on U.

[2] Sometimes also called Klein metric.

Both Funk metrics and Hilbert metrics are indeed Finsler metrics and the Hilbert metric is always reversible. The Funk metric can equivalently be described by

$$F_U(x,\xi) = \inf\{t > 0 \mid x + \tfrac{1}{t}\xi \in U\}.$$

Indeed, the above defined function is 1-homogeneous in ξ; and $\xi \in U - x$, if and only if $x + \xi \in U$, if and only if $\inf\{t > 0 \mid x + \frac{1}{t}\xi \in U\} < 1$.

Lemma 2.6 ([64][73, Section 2.3])**.** *Any Funk metric F satisfies $F_{x^i} = FF_{\xi^i}$.*

Let us determine the geodesic spray coefficients for a Funk metric:

$$\begin{aligned} G^i &= \frac{1}{4} g^{ik}\Big((F^2)_{\xi^k x^\ell}\xi^\ell - (F^2)_{x^k}\Big) \\ &= \frac{1}{2} g^{ik}\Big((FF_{x^\ell})_{\xi^k}\xi^\ell - FF_{x^k}\Big) \\ &= \frac{1}{2} g^{ik}\Big((F^2F_{\xi^\ell})_{\xi^k}\xi^\ell - F^2F_{\xi^k}\Big) \\ &= \tfrac{1}{2} g^{ik}F^2F_{\xi^k} \\ &= \tfrac{F}{2} g^{ik}(\frac{1}{2}F^2)_{\xi^k\xi^\ell}\xi^\ell \\ &= \tfrac{F}{2}\xi^i. \end{aligned}$$

Hence the geodesics are given by the equation $\ddot{c} + F(c,\dot{c})\dot{c} = 0$ and are straight lines, though not linearly parametrized. The induced distance of the Funk metric F on U is given by

$$d(p,q) = \log\left(\frac{|a-p|}{|a-q|}\right),$$

where $a \in \partial U$ denotes the intersection of the ray $\overrightarrow{pq}$ with ∂U.

The geodesics of a Hilbert metric turn out to be straight lines as well (see Example 2.5) with spray coefficients

$$G^i(x,\xi) = \tfrac{1}{2}\Big(F(x,\xi) - F(x,-\xi)\Big)\xi^i.$$

The induced distance function is given by

$$d(p,q) = \tfrac{1}{2}\log\left(\frac{|a-p|}{|a-q|}\frac{|b-p|}{|b-q|}\right),$$

where a is the intersection of the ray $\overrightarrow{pq}$ with ∂U and b of $\overrightarrow{qp}$ with ∂U.

2.2.4 The Exponential Mapping, Hopf-Rinow and Whitehead's Theorem

In this section we state three important theorems on geodesics and the exponential mapping of sprays and Finsler metrics. Proofs can be found in [10, Chapter 6] and [72, Chapter 14].

Definition 2.12. *Let S be a spray on a smooth manifold M. We define the (forward) exponential mapping* $\exp : U \subseteq TM \to M$ *as the map that takes $\xi \in TM$ to $c_\xi(1)$, where c_ξ is the unique geodesic of S with $\dot{c}_\xi(0) = \xi$.*

The exponential mapping is not always defined on the entire TM, but in general only on an open set $U \subseteq TM$ containing the zero section, so that $\exp_p := \exp|_{T_pM}$ is defined on an open neighborhood $U_p \subseteq T_pM$ of $0 \in T_pM$ for any $p \in M$.

Geodesics of a Finsler metric are by definition the local minima of the energy functional and consequently also local minima of the length functional. The next theorem asserts that any geodesic, restricted to a small enough interval, is even an absolute minimum of the length functional.

Theorem 2.3. *Let S be a spray on a manifold M.*

(a) The exponential mapping $\exp : U \subseteq TM \to M$ *is C^∞ away from the origins and for fixed $p \in M$, the map $\exp_p : U_p \to M$ is a C^1-diffeomorphism from a neighborhood of the origin $0 \in T_pM$ onto a neighborhood of p in M.*

(b) Let F be a Finsler metric on M. For every point $p \in M$ there exists a neighborhood $U \subseteq M$, such that for every point $q \in U$ there is exactly one geodesic c_{pq} from p to q in U and every other curve from p to q is at least as long as c_{pq}.

By the next theorem, on a closed manifold shortest geodesics exist always even globally.

Theorem 2.4 (Hopf-Rinow)**.** *Let M be a closed manifold. Then for every $p \in M$, the exponential map $\exp_p : T_pM \to M$ is defined on the whole of T_pM and surjective. Furthermore, for every $p, q \in M$ there exist a geodesic from p to q which is a shortest curve from p to q.*

For a spray and two points $p, q \in M$, a geodesic from p to q is generally not unique and does not need to exist. However, any point has a small neighborhood with this properties.

Definition 2.13. *Let S be a spray on a manifold M. A set $U \subseteq M$ is called*

- *geodesically convex, if for every $p, q \in U$ there exists a geodesic of S from p to q whose trajectory lies entirely in U.*
- *geodesically simple, if for every $p, q \in U$ there exists at most one geodesic of S (up to affine reparametrization) from p to q whose trajectory lies entirely in U.*

Theorem 2.5 (Whitehead's Theorem [26, 84, 85])**.** *Let S be a spray on a manifold M. Then for every point $p \in M$ and every open neighborhood $U \subseteq M$, there is a geodesically simple and convex open neighborhood $V \subseteq U$ of p, whose boundary ∂V is a smooth submanifold diffeomorphic to S^{n-1}. Furthermore, every geodesic in V must intersect the boundary of V in exactly two distinct points.*

2.2.5 Finslerian volume forms

Let M be orientable. Recall that for a Riemannian metric g on M, there is a canonical induced volume form on M, given in local coordinates by

$$d\mu_x^{\mathrm{Riem}} := \sqrt{\det g_x}\ dx^1 \wedge .. \wedge dx^n.$$

For Finsler metrics, there is no such canonical volume form - several volume forms are used in the literature depending on what properties are needed. The most popular are the Busemann volume and the Holmes-Thompson volume - we shortly give their definitions for later use following [74].

Definition 2.14. *Let M be an orientable n-dimensional manifold and $F : TM \to \mathbb{R}$ a Finsler metric. For $x \in M$, let $B_x^F := \{\xi \in T_xM \mid F(x,\xi) < 1\} \subseteq T_xM$ be the F-unit ball,* $\mathrm{Vol}(B_x^F)$ *its Euclidean volume in the given coordinates and κ_n be the volume of a n-dimensional Euclidean unit ball.*

1. *The Busemann volume form on M is defined in local coordinates by*

$$d\mu_x^B := \frac{\kappa_n}{\mathrm{Vol}(B_x^F)}\ dx^1 \wedge .. \wedge dx^n.$$

2. *The Holmes-Thompson volume form is defined in local coordinates by*

$$d\mu_x^{HT} := \frac{\int_{B_x^F} \det\left(g_{(x,\xi)}\right)\ d\xi^1..d\xi^n}{\kappa_n}\ dx^1 \wedge .. \wedge dx^n.$$

Lemma 2.7. *The following hold:*

1. *Both volume forms are defined globally, independently of the choice of coordinates.*
2. *If the metric F is Riemannian, then both volume forms reduce to the Riemannian volume form.*

Proof. The Busemann volume form is the unique volume form for which the volumes of the unit balls $B_x^F \subseteq T_xM$ coincide with the volume of a n-dimensional Euclidean ball. Thus its defined independently of the choice of coordinates.

The well-definedness of the Holmes-Thompson volume follows from the transformation rules: if $\tilde{x}^i(x)$ are new coordinates, then $\tilde{\xi}^i(x,\xi) = \frac{\partial \tilde{x}^i}{\partial x^j}\xi^j$, $d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j}dx^j$, $d\tilde{\xi}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k}\xi^j dx^k + \frac{\partial \tilde{x}^i}{\partial x^j}d\xi^j$ and $\tilde{g}_{ij} = g_{kl}\frac{\partial \xi^k}{\partial \tilde{\xi}^i}\frac{\partial \xi^l}{\partial \tilde{\xi}^j}$, where $\frac{\partial \xi^k}{\partial \tilde{\xi}^i} = \frac{\partial x^k}{\partial \tilde{x}^i}$.

For the second assertion, if F is Riemannian, by linear algebra $\mathrm{Vol}(B_x^F) = \frac{\kappa_n}{\sqrt{\det g_x}}$, and it follows that $d\mu^{\mathrm{B}} = d\mu^{\mathrm{HT}} = d\mu^{\mathrm{R}}$. □

2.2.6 Variants of the definition of a Finsler metric

Our definition of a Finsler metric is the strictest among all definitions to be found in the literature. It might be weakened in the following ways:

- **Smoothness:** Most results and theorems in Finsler geometry demand only C^k-differentiability of the manifold and the Finsler metric for a certain k. In Chapter 5 we demand more strongly real-analyticity.

- **Dropping only positivity:** By the strict convexity property it follows by Euler's theorem that

$$\tfrac{1}{2}F^2(x,\xi) = g_{ij}\xi^i\xi^j = 0 \qquad \text{if and only if } \xi = 0.$$

  Hence a strictly convex Finsler metric can not change sign on $TM\backslash 0$. One might allow F to have have only negative values on $TM\backslash 0$, but of course this does not make any qualitative difference.

- **Replacing strict convexity by non-degeneracy of the fundamental tensor:** It is natural to weaken the assumption, that g_{ij} is a positive definite matrix, to demanding that it is non-degenerate. However, this together with positivity implies positive definiteness of g_{ij} [45], so that one has to drop positivity at the same time to really weaken the definition.

- **Dropping positivity and replacing strict convexity by non-degenerateness of the fundamental tensor:** This is probably the most common generalization of our definition: it is analogous to passing from Riemannian to pseudo-Riemannian metrics and thus is of interest to physics and general relativity (see e.g. [8]). This definition includes pseudo-Riemannian metrics. The Euler-Lagrange equations can still be written in normal form and the geodesic spray is well-defined, but much of the general theory breaks down: length and distance have strange properties in this realm and there is no easy analogue of the Hopf-Rinow theorem.

- **Replacing strict convexity by convexity of the unit balls**: Instead of demanding that the unit balls of F are strictly convex bodies (which is equivalent to g_{ij} being positive definite), one might just demand them to be convex bodies containing the origins. For example, consider $\mathbb{R}^2$, where in each tangent plane the unit ball is the unit square, corresponding to the 'maximum-norm' - possibly with smoothed vertices. Then, in a vector where the unit ball is not strictly convex, the fundamental tensor is degenerate and the Euler-Lagrange equations can not be written into normal form. As a consequence, there won't be a unique, but a large family of geodesics, tangent to that vector. In the example 'maximum-norm' example, every curve, whose tangent remains in one of the four sectors bounded by the coordinate diagonals, will be a geodesic. Such Finsler metrics are considered for example in [54].

2.2.7 Projective equivalence

In this section we introduce projective equivalence of Finsler metrics, which is crucial for this dissertation and justifies the word 'projective' in its title. Loosely speaking, two Finsler metrics are projectively equivalent, if the oriented trajectories of their geodesics coincide.

Definition 2.15. *Let M be a smooth manifold.*

(a) Two sprays $S, \tilde{S}$ on M are called projectively equivalent, if every geodesic of S can be orientation preservingly reparametrized to be a geodesic of $\tilde{S}$.

(b) Two Finsler metrics $F, \tilde{F}$ on M are called projectively equivalent, if their geodesic sprays $S_F, S_{\tilde{F}}$ are projectively equivalent.

Clearly, projective equivalence is an equivalence relation.

Lemma 2.8. *Let M be a smooth manifold and let V be the Liouville vector field on TM, given in all local coordinates by $V|_{(x,\xi)} = \xi^i \partial_{\xi^i}$. Then:*

(a) Two sprays S and $\tilde{S}$ are projectively equivalent, if and only if there is a function $f : TM\backslash 0 \to \mathbb{R}$, such that

$$\tilde{S}|_{(x,\xi)} = S|_{(x,\xi)} + f(x,\xi)V|_{(x,\xi)}.$$

(b) Two Finsler metrics F and $\tilde{F}$ are projectively equivalent, if and only if the Euler-Lagrange equations $E_i(F,c) = 0$ and $E_i(\tilde{F},c)$ admit exactly the same solutions.

Proof. (a) Suppose $S, \tilde{S}$ are projectively equivalent sprays given in local coordinates by $S = \xi^i\partial_{x^i} - 2G^i\partial_{\xi^i}$ and $\tilde{S} = \xi^i\partial_{x^i} - 2\tilde{G}^i\partial_{\xi^i}$. Let c be a geodesic of S and φ an orientation preserving reparametrization, such that $\tilde{c}(t) = c(\varphi(t))$ is a geodesic of $\tilde{S}$. Then $\dot{\tilde{c}}(t) = \dot{c}(\varphi(t))\varphi'(t)$ and

$$\begin{aligned} -2\tilde{G}^i\big(\tilde{c}(t), \dot{\tilde{c}}(t)\big) &= \ddot{\tilde{c}}^i(t) \\ &= \ddot{c}^i(\varphi(t))\varphi'(t)^2 + \dot{c}^i(\varphi(t))\varphi''(t) \\ &= -2G^i\big(\tilde{c}(t), \dot{\tilde{c}}(t)\big) + \frac{\varphi''(t)}{\varphi'(t)}\dot{\tilde{c}}^i(t). \end{aligned}$$

As for every $(x,\xi) \in TM\backslash 0$ there is a geodesic $\tilde{c}$ with $\tilde{c}(0) = x, \dot{\tilde{c}}(0) = \xi$, there is a function $f : TM\backslash 0 \to \mathbb{R}$, such that $\tilde{G}^i(x,\xi) = G^i(x,\xi) + \frac{1}{2}f(x,\xi)\xi^i$.
Conversely, suppose in local coordinates $\tilde{S} = S + f(x,\xi)\big(\xi^i\partial_{\xi^i}\big)$ for a function $f : TM\backslash 0 \to \mathbb{R}$, which then must be 1-homogeneous. Let $c(t)$ be a geodesic of S. In order for a reparametrization $\tilde{c}(t) := c(\varphi(t))$ to be a geodesic of $\tilde{S}$, we must have

$$\ddot{\tilde{c}}^i(t) = \ddot{c}^i(\varphi(t))\varphi'(t)^2 + \dot{c}^i(\varphi(t))\varphi''(t) \stackrel{!}{=} -2G^i\big(\tilde{c}(t), \dot{\tilde{c}}(t)\big) + f(\tilde{c}(t), \dot{\tilde{c}}(t))\dot{\tilde{c}}^i(t),$$

which is by $\ddot{c}^i(\varphi(t))\varphi'(t)^2 = -2G^i\big(\tilde{c}(t), \dot{\tilde{c}}(t)\big)$ equivalent to

$$\varphi''(t) = f\big(\tilde{c}(t), \dot{\tilde{c}}(t)\big)\varphi'(t),$$

which admits a solution with $\varphi' > 0$. Hence S and $\tilde{S}$ are projectively equivalent.

(b) The equivalence follows directly from Lemma 2.4 (c): The solutions of $E_i(F, c) = 0$ and $E_i(\tilde{F}, c) = 0$ are all the orientation preserving reparametrizations of geodesics of F and $\tilde{F}$ respectively. The two sets of solutions coincide, if and only if each geodesic of F can be orientation preservingly reparametrized to be a geodesic of $\tilde{F}$, that is if F and $\tilde{F}$ are projectively equivalent. $\square$

Example 2.1. *By definition, the geodesic spray coefficients of a Douglas metric are in all local coordinates of the form*

$$G^i(x, \xi) = \tfrac{1}{2}\Gamma^i_{k\ell}(x)\xi^k\xi^l + P(x, \xi)\xi^i,$$

with a globally defined function $P : TM\backslash 0 \to \mathbb{R}$*. Hence its geodesic spray is projectively equivalent to the spray given in local coordinates by*

$$\tilde{G}^i(x, \xi) = \tfrac{1}{2}\Gamma^i_{k\ell}(x)\xi^k\xi^l,$$

which is the geodesic spray of an affine connection.

Example 2.2 (Trivial projective equivalence)**.** *Let F and $\tilde{F}$ be two Finsler metrics on a smooth manifold M, such that $\tilde{F} = \lambda F + \beta$, where $\lambda > 0$ and β is a 1-form on M. Then F and $\tilde{F}$ are projectively equivalent, if and only if β is closed.*

Indeed, $E_i(\tilde{F}, c) = \lambda E_i(F, c) + E_i(\beta, c)$. If β is closed, then $E_i(\beta, c) = 0$ by Lemma 2.4 (d) and the Euler-Lagrange equations for F and $\tilde{F}$ admit exactly the same solutions. Conversely, if the metrics are projectively equivalent and $\beta = \beta_j dx^j$, for every $(x_0, \xi_0) \in TM\backslash 0$ there is a curve c_0 with $c_0(0) = x_0, \dot{c}_0(0) = \xi_0$, which solves both $E_i(F, c_0) = E_i(\tilde{F}, c_0) = 0$, so that $E_i(\beta, c_0) = 0$. Then $0 = E_i(\beta, c_0) = \big((\beta_j)_{x^i} - (\beta_i)_{x^j}\big)\dot{\xi}^j_0$ and as (x_0, ξ_0) are arbitrary, it follows $(\beta_j)_{x^i} - (\beta_i)_{x^j} \equiv 0$, that is β is closed.

In the class of essential Randers metrics (that is if $F = \alpha + \beta$ with the 1-form β not closed), the trivial projective equivalence is the only projective equivalence that can appear, as the following theorem asserts:

Theorem 2.6 ([55])**.** *Two Randers metrics $\alpha + \beta$ and $\tilde{\alpha} + \tilde{\beta}$ with $d\beta \neq 0$ are projectively equivalent, if and only if there is a $\lambda > 0$ such that $\alpha = \lambda^2\tilde{\alpha}$ and $\beta - \lambda\tilde{\beta}$ is closed.*

However, in general there is more than just trivial projective equivalence - already among Riemannian metrics, as the following classical example of Beltrami [13] shows.

Example 2.3. *Consider the standard upper half sphere S^2_+ in $\mathbb{R}^3$ with the standard Riemannian metric g under central projection coordinates φ: for $p = (p_1, p_2, p_3) \in S^2_+$, denote by $\ell(p)$ the line through p and the origin. Then the coordinate map φ takes a point $p \in S^2_+$ to the unique intersection of $\ell(p)$ with the hyperplane $\{p_3 = 1\}$.*

As the geodesics of g on the half sphere are exactly the intersections with hyperplanes through the origin, the geodesics of the push-forwarded metric $\varphi_* g$ are exactly the intersections of these hyperplanes with $\{p_3 = 1\}$ - in particular straight lines. Thus the (Riemannian) Finsler metric $\varphi_* g$ is non-trivially projectively equivalent to the Euclidean metric on $\{p_3 = 1\} = \mathbb{R}^2 \times \{1\}$.

The same construction works for higher dimension and a similar construction for the hyperbolic spaces[3], which also admit coordinates, in which the metric is projectively

[3]See [53] for a model obtained from a hyperboloid in $\mathbb{R}^n$ with 'Minkowski metric' $-dx^1 + dx^2 + .. + dx^n$. Alternatively, one can use the Beltrami-Klein model of hyperbolic space, that is the Hilbert geometry for the open unit ball in $\mathbb{R}^n$.

equivalent to the Euclidean one. Metrics with this property were studied intensively, probably because they are the subject of one of the problems posed by David Hilbert [37] in 1900 at the International Congress of Mathematics.

Problem (A Version of Hilbert's Fourth Problem). *Construct and treat systematically Finsler metrics on $\mathbb{R}^n$ for which all geodesics are straight lines.*

Example 2.4.

- *If F is a Finsler metric on $\mathbb{R}^n$ such that $F(x,\xi) = F(\xi)$ is independent of x, then the spray coefficients $G^i = \frac{1}{4}g^{ij}(2\frac{\partial g_{jk}}{\partial x^\ell} - \frac{\partial g_{k\ell}}{\partial x^j})\xi^k\xi^\ell$ vanish identically and F is projectively equivalent to the Euclidean metric (even more, their geodesic sprays coincide).*

- *Every Funk metric (and thus any Hilbert metric) is projectively equivalent to the Euclidean metric, but its geodesic spray differs from the one of the Euclidean metric (see Section 2.2.3 and Example 2.5).*

Lemma 2.8 (b) gives a characterization of metrics projectively related to the Euclidean metric:

Corollary 2.2 (Hamel's conditions [35]). *A Finsler metric F on $\mathbb{R}^n$ in projectively equivalent to the Euclidean metric, if and only if*

$$F_{\xi^i x^k}\xi^k = F_{x^i} \qquad \text{for } i = 1,..,n.$$

In this case, the following holds for $i,j \in \{1,..,n\}$:

$$F_{\xi^i\xi^j x^k}\xi^k = 0 \qquad \text{and} \qquad G^i = \frac{F_{x^k}\xi^k}{2F}\xi^i.$$

Proof. The Finsler metric F is projectively equivalent to the Euclidean metric, if and only if the equation $E_i(F,c) = 0$ holds for all linear parametrized lines $c(t) = x_0 + t\xi_0$, where $x_0, \xi_0 \in \mathbb{R}^n$. Because $\ddot{c} = 0$, the Euler-Lagrange equations for $t = 0$ are given by $F_{x^i}(x,\xi_0) - F_{\xi^i x^k}(x_0,\xi_0)\xi_0^k = 0$ and the first assertion is proven.

To see that the first equation implies the second, differentiate it by ξ^j to obtain

$$F_{\xi^i\xi^j x^k}\xi^k + F_{\xi^i x^j} = F_{x^i\xi^j}.$$

Taking the part of this equation symmetric in (i,j), that is taking (half of) the sum of this equation with the same equation with i and j interchanged, we obtain $F_{\xi^i\xi^j x^k}\xi^k = 0$.

For the geodesic spray, we have

$$\begin{aligned}
G^i &= \tfrac{1}{2}g^{i\ell}\Big((\tfrac{1}{2}F^2)_{\xi^k x^\ell}\xi^\ell - (\tfrac{1}{2}F^2)_{x^k}\Big) \\
&= \tfrac{1}{2}g^{ik}\Big(F_{\xi^k}F_{x^\ell}\xi^\ell + FF_{\xi^k x^\ell}\xi^\ell - FF_{x^k}\Big) \\
&= \tfrac{1}{2}g^{ik}F_{\xi^k}F_{x^\ell}\xi^\ell \\
&= \tfrac{1}{2}g^{ik}(\tfrac{1}{2}F^2)_{\xi^k\xi^r}\xi^r\frac{F_{x^\ell}\xi^\ell}{F} \\
&= \frac{F_{x^\ell}\xi^\ell}{2F}\xi^i.
\end{aligned}$$

□

In Chapter 4 we will obtain an integral geometric formula for Finsler metrics projectively equivalent to the Euclidean metric in dimension 2, see Example 4.2 and 4.3.

For a general Finsler metric on a smooth manifold, being projectively equivalent to the Euclidean metric is only a well defined notion if one picks a fixed coordinate chart. The coordinate independent and global version is the following:

Definition 2.16. *A Finsler metric F on a manifold M is called projectively flat, if around every point $p \in M$, there are local coordinates in which F is projectively equivalent to the Euclidean metric of the coordinates.*

Projectively flat metrics are studied intensively (see [23] for a recent, comprehensive overview).

Example 2.5.

- *Any Funk metric is projectively flat: in the usual coordinates the spray coefficients are given by $G^i = \frac{F}{2}\xi^i$, see Section 2.2.3.*
  Any Hilbert metric is projectively flat, as it satisfies the equation from Corollary 2.2:
  Recall that any Hilbert metric is of the form $\tilde{F}(x,\xi) = \frac{1}{2}\Big(F(x,\xi) + F(x,-\xi)\Big)$, where F is a Funk metric and satisfies $F_{x^k} = FF_{\xi^k}$. Hence

$$\begin{aligned}2\tilde{F}_{\xi^i x^k}(x,\xi)\xi^k &= \Big(\big(F(x,\xi)F_{\xi^k}(x,\xi)\big)_{\xi^i} + \big(F(x,-\xi)F_{\xi^k}(x,-\xi)\big)_{\xi^i}\Big)\xi^k \\ &= F_{\xi^i}(x,\xi)F(x,\xi) + F_{\xi^i}(x,-\xi)F(x,-\xi) \\ &= F_{x^i}(x,\xi) + F_{x^i}(x,-\xi) \\ &= 2\tilde{F}_{x^i}(x,\xi)\end{aligned}$$

  and the Hilbert metric $\tilde{F}$ is projectively flat with

$$\begin{aligned}\tilde{G}^i &= \frac{\tilde{F}_{x^k}\xi^k}{2\tilde{F}}\xi^i \\ &= \frac{F_{x^k}(x,\xi) + F_{x^k}(x,-\xi)}{2(F(x,\xi)+F(x,-\xi))}\xi^k\xi^i \\ &= \frac{F(x,\xi)F_{\xi^k}(x,\xi) + F(x,-\xi)F_{\xi^k}(x,-\xi)}{2(F(x,\xi)+F(x,-\xi))}\xi^k\xi^i \\ &= \frac{F(x,\xi)^2 - F(x,-\xi)^2}{2(F(x,\xi)+F(x,-\xi))}\xi^i \\ &= \tfrac{1}{2}\Big(F(x,\xi) - F(x,-\xi)\Big)\xi^i.\end{aligned}$$

- *Any Minkowski metric is projectively flat, as by definition around every point there are local coordinates, such that $F(x,\xi) = F(\xi)$ is independent of x as in Example 2.4.*

- *Beltrami's theorem [13, 53]: The Riemannian metrics of constant sectional curvature (that is the ones isometric to Euclidean $\mathbb{R}^n$, the n-sphere S^n or the hyperbolic space H^n) are all projectively flat - and for $n \geq 3$ there are no other projectively flat Riemannian metrics.*

- *By Example 2.2, a metric obtained from a projectively flat metric by scaling by some* $\lambda > 0$ *and adding a closed 1-form, is again projectively flat.*

Combining Example 2.5, Theorem 2.6 and Beltrami's theorem, we obtain that in the class of Randers metrics, the last two examples exhaust the possibilities:

Theorem 2.7. *A Randers metric* $F = \alpha + \beta$ *is projectively flat, if and only if* β *is closed and* α *is of constant sectional curvature.*

Thus Hilbert's Fourth Problem is solved in the realm of Riemannian and Randers metrics, however the problem for general Finsler metrics is more complex and only partially understood.

2.3 Lie algebras of vector fields

Before recalling the basic theory of abstact Lie algebras and Lie algebras of vector fields on a manifold, let us digress to a discussion of their importance. Differential geometric structures (e.g. Riemannian metrics, symplectic forms,...) usually come with a natural notion of symmetries (e.g. isometries, symplectic mappings). The infinitesimal version of these symmetries are vector fields, whose local flow is such a symmetry. The space of infinitesimal symmetries is usually not only a vector space, but a Lie algebra: for any two infinitesimal symmetries, their commutator vector field is also an infinitesimal symmetry. Hence the space of infinitesimal symmetries carries an additional structure, that can be used for the inverstigation of the geometrical structure in question. The instance of this for projective classes of sprays on a surface, with projective vector fields as infinitesimal symmetries, is the subject of Chapter 3. For a detailed exposition on Lie theory we refer to [65].

2.3.1 Abstract Lie algebras

Definition 2.17. *A Lie algebra* $\mathfrak{g}$ *is a real vector space together with a bilinear, antisymmetric map* $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$*, which for all* $X,Y,Z\in\mathfrak{g}$ *satisfies the Jacobi identity*

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0.$$

The vector $[X,Y]$ *is called bracket or commutator of* X *and* Y*.*

The antisymmetry of the bracket means that for any $X,Y\in\mathfrak{g}$ the relation $[X,Y]=-[Y,X]$ holds. Bilinearity means that $[\alpha X+\beta Y,Z]=\alpha[X,Z]+\beta[Y,Z]$ for any $X,Y,Z\in\mathfrak{g}$. The same relation holds in the second argument as a consequence of the antisymmetry.

Definition 2.18. *Let* $(\mathfrak{g},[\cdot,\cdot])$ *be a Lie algebra and* $\mathfrak{h}\subseteq\mathfrak{g}$ *a linear subspace. We call* $\mathfrak{h}$ *a subalgebra, if* $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$*. We call* $\mathfrak{h}$ *an ideal, if* $[\mathfrak{h},\mathfrak{g}]\subseteq\mathfrak{h}$*.*

A linear subspace $\mathfrak{h}$ is a subalgebra, if and only if $[X,Y]\in\mathfrak{h}$ for all $X,Y\in\mathfrak{h}$. It is an ideal, if and only if $[X,Y]\in\mathfrak{h}$ for all $X\in\mathfrak{h},Y\in\mathfrak{g}$. Clearly any ideal is a subalgebra.

Definition 2.19. *A map* $\varphi:(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})\to(\tilde{\mathfrak{g}},[\cdot,\cdot]_{\tilde{\mathfrak{g}}})$ *between Lie algebras is called a homomorphism, if it is linear and preserves the Lie bracket, that is* $\varphi([X,Y]_{\mathfrak{g}})=[\varphi(X),\varphi(Y)]_{\tilde{\mathfrak{g}}}$ *for all* $X,Y\in\mathfrak{g}$*. If* φ *is bijective, it is called an isomorphism.*

Let $(X_i)_{i\in I}$ be a basis of $\mathfrak{g}$. Then there are real numbers $C^{\ell}_{ij}, i,j,k\in I$, such that $[X_i,X_j]=C^{\ell}_{ij}X_\ell$. These numbers are called *structure constants* of $\mathfrak{g}$ with respect to the basis (X_i), and by bilinearity they determine the Lie bracket uniquely.

By the antisymmetry of the bracket, the structure constants satisfy $C^{\ell}_{ij}=-C^{\ell}_{ji}$ and by the Jacobi identity for X_i,X_j,X_k we have

$$\begin{aligned}0&=[X_i,C^{\ell}_{jk}X_\ell]+[X_k,C^{\ell}_{ij}X_\ell]+[X_j,C^{\ell}_{ki}X_\ell]\\&=(C^{\ell}_{jk}C^{m}_{i\ell}+C^{\ell}_{ij}C^{m}_{k\ell}+C^{\ell}_{ki}C^{m}_{j\ell})X_m,\end{aligned}$$

so that $C^{\ell}_{jk}C^{m}_{i\ell}+C^{\ell}_{ij}C^{m}_{k\ell}+C^{\ell}_{ki}C^{m}_{j\ell}=0$ holds for all $i,j,k,m\in I$. Any collection of constants with these two properties together with a choice of a basis define by bilinearity a Lie bracket on $\mathfrak{g}$.

If $\mathfrak{g}$ is a Lie algebra with a basis $(X_i)_{i\in I}$ and structure constants C_{ij}^{ℓ}, then another Lie algebra $\tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}$ if and only if it admits a basis with the same structure constants.

If $\mathfrak{g}$ is finite dimensional and $X_1, .., X_n$ is a basis. We may display the structure of the Lie algebra by its *commutator table*:

	X_1	$\dots$	X_j	$\dots$	X_n
X_1	0				
$\vdots$					
X_i			$C_{ij}^{\ell} X_\ell$		
$\vdots$					
X_n					0

.

Definition 2.20. *For $X \in \mathfrak{g}$ the map $(\operatorname{ad} X) : \mathfrak{g} \to \mathfrak{g}$ given by $Y \mapsto [X, Y]$ is called the adjoint of X.*

By bilinearity of the bracket, the adjoint $(\operatorname{ad} X)$ is a linear endomorphism of $\mathfrak{g}$. The Jacobi identity can be rewritten as $(\operatorname{ad} X)([Y, Z]) = [(\operatorname{ad} X)(Y), Z] + [Y, (\operatorname{ad} X)(Z)]$ and be interpreted as a Leibniz product rule.

2.3.2 Lie algebras of vector fields

Let M be a smooth manifold and $\mathfrak{X}(M)$ be the set of smooth vector fields on M, that is the set of smooth sections of the bundle $TM \to M$. This is a vector space with pointwise addition and scalar multiplication $(\lambda X + \mu Y)_p := \lambda X_p + \mu Y_p$ for $X, Y \in \mathfrak{X}(M)$ and $\lambda, \mu \in \mathbb{R}$.

The vector fields on M are in 1-to-1 correspondence with derivations on the space of functions on M, that is $\mathbb{R}$-linear mappings $\delta : C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz product rule $\delta(f \cdot g) = \delta(f) \cdot g + f \cdot \delta(f)$ for all $f, g \in C^\infty(M)$. For any $X \in \mathfrak{X}(M)$, the map $\delta_X : f \mapsto Xf$ is a derivation, and for any derivation δ, there is a vector field Z, such that $\delta_Z = \delta$.

Lemma 2.9.

(a) For two vector fields $X, Y \in \mathfrak{X}(M)$, there is a unique vector field $Z \in \mathfrak{X}(M)$, such that $X(Y(f)) - Y(X(f)) = Z(f)$ holds for all $f \in C^\infty(M)$. The vector field Z is called the commutator of X and Y and denoted by $[X, Y]$.

(b) In local coordinates, if $X = X^i \partial_{x^i}$ and $Y = Y^i \partial_{x^i}$, then their bracket is given by $[X, Y] = (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \partial_{x^j}$.

(c) The space of vector fields $\mathfrak{X}(M)$ with the above defined bracket forms a Lie algebra.

Proof. For (a) it is enough to check that the map $f \mapsto X(Y(f)) - Y(X(f))$ is a derivation. $\mathbb{R}$-linearity is obvious and the Leibniz rule follows from the Leibniz rule for X and Y.

For (b), one calculates $X(Yf) = X(Y^j \frac{\partial f}{\partial x^j}) = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j}$ and notices that the second term is symmetric in X and Y.

For (c), linearity and antisymmetry of the bracket are obvious and the Jacobi identity follows by direct computation. □

Definition 2.21.

- *A Lie algebra of vector fields on a smooth manifold M is a subalgebra $\mathfrak{g} \subseteq \mathfrak{X}(M)$.*
- *For a Lie algebra of vector fields $\mathfrak{g}$ and a point $p \in M$, the isotropy subalgebra at p is $\mathfrak{g}_p := \{X \in \mathfrak{g} \mid X_p = 0\}$.*

By Lemma 2.9 the isotropy subalgebra at a point p is indeed a subalgebra.

Example 2.6. *Consider the Lie algebra $\mathfrak{g}$ on $M = \mathbb{R}^2$ given as the linear span of the vectors*

$$X_0 = y\partial_x - x\partial_y \qquad X_1 = \partial_x \qquad X_2 = \partial_y.$$

This is a Lie algebra with commutator table

	X_0	X_1	X_2
X_0	0	X_2	$-X_1$
X_1	$-X_2$	0	0
X_2	X_1	0	0

.

The flows of X_1 and X_2 for fixed time are translations, the flows of X_0 are rotations around the origin. Hence the flows of $\mathfrak{g}$ are exactly the orientation preserving rigid motions of the Euclidean $\mathbb{R}^2$. The isotropy subalgebra at the origin is $\mathfrak{g}_0 = \operatorname{span}(X_0)$.

If two Lie algebras of vector fields differ by a coordinate change, then they are isomorphic as abstract Lie algebras. The converse is not true.

Lemma 2.10. *Let $\varphi : M \to N$ be a diffeomorphism, that takes a Lie algebra of vector fields $\mathfrak{g} \subseteq \mathfrak{X}(M)$ to a Lie algebra of vector fields $\tilde{\mathfrak{g}} \subseteq \mathfrak{X}(N)$, that is that the set $\varphi(\mathfrak{g}) := \{\varphi_* X \mid X \in \mathfrak{g}\}$ coincides with $\tilde{\mathfrak{g}}$. Then*

$$\hat{\varphi} : \mathfrak{g} \to \tilde{\mathfrak{g}} \text{ defined by } X \mapsto \varphi_* X$$

is an isomorphism of abstract Lie algebras with $\hat{\varphi}(\mathfrak{g}_p) = \tilde{\mathfrak{g}}_{\varphi(p)}$ for all $p \in M$.

Example 2.7. *Consider the three Lie algebras spanned by the following vector fields:*

$$\begin{array}{lll} g^X & X_0 = x\partial_x - y\partial_y & X_1 = y\partial_x \quad X_2 = x\partial_y \\ g^Y & Y_0 = 2(\partial_x + y\partial_y) & Y_1 = \partial_y \quad Y_2 = -2y\partial_x - y^2\partial_y \\ \mathfrak{g}^Z & Z_0 = (x^2 - y^2 - 1)\partial_x + 2xy\partial_y & Z_1 = (x+1)y\partial_x + (\frac{1}{2}(-x^2 + y^2 - 1) - x)\partial_y \\ & & Z_2 = (x-1)y\partial_x + (\frac{1}{2}(-x^2 + y^2 - 1) + x)\partial_y \end{array}.$$

The flows generated by $\mathfrak{g}^X$ give the action of $SL(2)$ on the plane. The commutator tables of $\mathfrak{g}^X, \mathfrak{g}^Y, \mathfrak{g}^Z$ all coincide and are given by

	X_0	X_1	X_2
X_0	0	$-2X_1$	$2X_2$
X_1		0	$-X_0$
X_2			0

.

Thus all three are isomorphic as abstract Lie algebras. The isotropy algebras at the origin are $\mathfrak{g}_0^X = \mathfrak{g}^X$, $\mathfrak{g}_0^Y = \operatorname{span}(Y_2)$ and $\mathfrak{g}_0^Z = \operatorname{span}(Z_1 - Z_2)$. There is no diffeomorphism of $\mathbb{R}^2$ fixing the origin taking one of the Lie algebras to another. For $\mathfrak{g}^X$ this is clear, since

$\mathfrak{g}_0^X$ is 3-dimensional. For $\mathfrak{g}^Y$ and $\mathfrak{g}^Z$ consider the matrix representation of the adjoint of the isotropy vector field in the original basis:

$$\operatorname{ad} Y_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \qquad \operatorname{ad}(Z_1 - Z_2) = \begin{bmatrix} 0 & -1 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

If there was a diffeomorphism mapping $\mathfrak{g}^Y$ to $\mathfrak{g}^Z$, the matrices would be similar ($GL(n)$-conjugates) up to a scaling, which is not the case.

To measure how a tensor field changes along a vector field, we use the notion of the Lie derivative, to 'differentiate' an arbitrary tensor fields along a vector field.

Definition 2.22. *Let X be a vector field on a manifold M, Φ^t its flow and T be a tensor field. The Lie derivative of T by X is the tensor defined by $(\mathcal{L}_X T)_p := \frac{d}{dt}|_{t=0}(\Phi_t^* T)_p$.*

Lemma 2.11.

(a) If $T = f$ is a function, then $\mathcal{L}_X f = X(f)$.

(b) If $T = Y$ is a vector field, then $\mathcal{L}_X Y = [X, Y]$.

(c) Cartan's formula: If $T = \omega$ is a differential form, then $\mathcal{L}_X \omega = \iota_X(d\omega) + d(\iota_X \omega)$, where d denotes the exterior derivative and ι_X the contraction in the first argument.

2.4 Hamiltonian systems

An *Hamiltonian system* on a $2n$-dimensional manifold N is given by a symplectic form and real-valued function on the manifold, called *Hamiltonian*. This data defines a vector field, whose flow can be interpreted as the motion of particles in mechanical systems. A Hamiltonian system is *integrable*, if there exist n independent functions, which are constant along the vector field. Integrable systems are well-studied and less chaotic: for $n = 2$ under some additional assumptions, their *topological entropy* must vanish.

The geodesic flow of a Finsler metric on a manifold M is a special instance of a Hamiltonian system on the slashed tangent bundle $TM\backslash 0$. This allows us to use techniques from (integrable) Hamiltonian systems in the study of Finsler metrics in Chapter 5.

For detailed expositions on symplectic geometry we refer to [21, 33, 38] and to [28, 16, 15] for the definition and various properties of the topological entropy.

Definition 2.23. *Let N^{2n} be a $2n$-dimensional manifold.*

(a) A symplectic form ω is a smooth differential 2-form on N, that is closed and non-degenerate; that is a 2-form satisfying

$$d\omega = 0 \quad \text{and} \quad \omega_p(X, \cdot) \equiv 0 \Rightarrow X = 0 \qquad \text{for all } p, \in N, X \in T_pN.$$

(b) A Hamiltonian system is a triple (N, ω, H), where ω is a symplectic form on N and $H : N \to \mathbb{R}$ is a smooth function, called the Hamiltonian.

(c) A vector field X is called symplectic, if its flow preserves the symplectic form, that is $\mathcal{L}_X\omega = 0$.

In coordinates (x^i), the symplectic form is given pointwise by its Gramian matrix $\omega_{ij} := (\omega(\partial_{x^i}, \partial_{x^j}))_{ij}$, which is a skew-symmetric, non-degenerate $2n \times 2n$ matrix.

Example 2.8. *The cotangent bundle T^*M of any smooth manifold M carries a canonical symplectic form $\tilde{\omega}$ defined as the exterior derivative of the Poincaré 1-form $\tilde{\theta}$ on T^*M:*

In $\beta \in T^*M$ for $X \in T_\beta(T^*M)$, define the Poinaré 1-form via

$$\tilde{\theta}_\beta(X) := \beta(\pi_* X),$$

where $\pi : T^*M \to M$ is the bundle projection map. This is indeed a 1-form on T^*M and in the standard dual coordinates (x^i, p_i) on T^*M it is given by $\tilde{\theta}_{(x^i,p_i)} = p_i dx^i$.

The canonical symplectic form on T^*M is then defined as $\tilde{\omega} = d\tilde{\theta}$. This is an exact and hence closed 2-form and in coordinates given by $\tilde{\omega} = dp_i \wedge dx^i$ with Gramian matrix $\begin{bmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}$, which is not degenerate. The 2-form ω is indeed a symplectic form.

A symplectic form ω induces a bundle isomorphism $\Phi_\omega : TN \to T^*N$ by $\xi \mapsto \omega_{\pi(\xi)}(\xi, \cdot)$, that is fiber-preserving and fiber-wise a linear isomorphism $T_pN \to T_p^*N$, given in coordinates by $(x^i, \xi^i) \mapsto (x^i, \xi^i\omega_{ij})$.

Definition 2.24. *Let ω be a symplectic form on N.*

1. *The Hamiltonian vector field X_H of a smooth function $H : N \to \mathbb{R}$ is defined by $X_H|_p := -\Phi_\omega^{-1}(dH_p)$, so X_H is the unique vector field with $\omega(X_H, \cdot) = -dH(\cdot)$.*

2. *The Poisson bracket on the space of smooth functions on N is the mapping*

$$\begin{array}{rccc} \{\cdot,\cdot\}: & C^\infty(N) \times C^\infty(N) & \to & C^\infty(N) \\ & (f,g) & \mapsto & \omega(X_f, X_g) \end{array}.$$

   We say that two functions $f, g \in C^\infty(N)$ commute or are in involution, their Poisson bracket vanishes, that is $\{f, g\} = 0$.

The assignment of the Hamiltonian vector field to a function $C^\infty(N) \to \mathfrak{X}(N)$, $H \mapsto X_H$ is linear and in coordinates given by $X_H = -f_{x^i}\omega^{ij}\partial_{x^j}$, where (ω^{ij}) denotes the pointwise inverse matrix of the Gramian of ω. The function H is constant along its Hamiltonian vector field, because $X_H(H) = dH(X_H) = -\omega(X_H, X_H) = 0$.

Lemma 2.12. *Let $f, g, h \in C^\infty(N)$.*

(a) The Poisson bracket is bilinear, antisymmetric and satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Hence the Poisson bracket gives $C^\infty(N)$ a Lie algebra structure.

(b) It holds $\{f, g\} = X_f(g) = -X_g(f)$. In particular, if f and g commute, then f is constant along the Hamiltonian vector field of g and vice versa.

(c) The Poisson bracket of two functions corresponds to the Lie bracket on the level of Hamiltonian vector fields, that is $X_{\{f,g\}} = -[X_f, X_g]$.

(d) Every Hamiltonian vector field is a symplectic vector field, that is $\mathcal{L}_{X_f}\omega = 0$.

Proof. (b) By definition $\{f, g\} = \omega(X_f, X_g) = -df(X_g) = -X_g(f)$.

(d) By Cartan's formula $\mathcal{L}_{X_f}\omega = \iota_{X_f}(d\omega) + d(\iota_{X_f}\omega) = 0 + d(-df) = 0$.

(a) Bilinearity and antisymmetry are obvious. For the Jacobi identity, we might change coordinates and assume that $X_f = \partial_{x^1}$. Then by (d), the entries (ω_{ij}) do not depend on x^1. Using (b) we have

$$\begin{aligned} \{f, \{g, h\}\} &= \partial_{x^1}\left(\omega_{ij} X_g^i X_h^j)\right) \\ &= \omega_{ij}\frac{\partial X_g^i}{\partial x^1} X_h^j + \omega_{ij} X_g^i \frac{\partial X_h^j}{\partial x^1} \\ &= \{\partial_{x^1} g, h\} + \{g, \partial_{x^1} h\} \\ &= \{\{f, g\}, h\} + \{g, \{f, h\}\} \\ &= -\{h, \{f, g\}\} - \{g, \{h, f\}\}. \end{aligned}$$

(c) For any $h \in C^\infty N$, we have

$$\begin{aligned} X_{\{f,g\}}(h) &= -\{h, \{f, g\}\} \\ &= \{f, \{g, h\}\} + \{g, \{h, f\}\} \\ &= X_f(\{g, h\}) + X_g(\{h, f\}) \\ &= X_f(X_g(h)) - X_g(X_f(h)) \end{aligned},$$

hence the vector fields $X_{\{f,g\}}$ and $[X_f, X_g]$ coincide. □

Example 2.9. *On T^*M with the canonical symplectic form $\tilde{\omega}$, the Hamiltonian vector field of a function $H : T^*M \to \mathbb{R}$ is given in coordinates by $H_{p_i}\partial_{x^i} - H_{x^i}\partial_{p_i}$, since $(\tilde{\omega}^{ij}) = -(\tilde{\omega}_{ij})$.*

2.4.1 Liouville integrability and entropy

Definition 2.25. *Let (N^{2n}, ω, H) be a Hamiltonian system.*

(a) An integral is a function $f : N \to \mathbb{R}$, such that $\{H, f\} = X_H f = 0$.

(b) The system is called (Liouville) integrable, if n functions $f_1 = H, f_2, .., f_n : N \to \mathbb{R}$ exist, such that each two of them commute with respect to the Poisson bracket and their differentials $df_1|_p, .., df_n|_p$ are linearly independent for almost all $p \in N$, that is for p in an open and dense set $U \subseteq N$.

Integrability is a very strong property of a system - a general Hamiltonian system can behave chaotically; integrable systems instead behave nicely and less chaotic. One manifestation of this assertion are theorems about the topological entropy of integrable systems, which is an indicator on how chaotic a system is.

Definition 2.26. *Let (X, d) be a compact metric space with distance function $d : X \times X \to \mathbb{R}_{\geq 0}$ and let $S^t : \mathbb{R} \times X \to X$ be a flow on X.*

- *For each $t \in \mathbb{R}_{\geq 0}$ define a a new metric on X by*

$$d^t(x, y) = \max_{0 \leq \tau \leq t} d\big(S^\tau(x), S^\tau(y)\big).$$

- *Let H^t_ϵ for $\epsilon, t > 0$ be the least cardinality of ϵ-nets in the metric space (X, d^t).*
- *The topological entropy of the flow S^t is defined to be*

$$h_{\text{top}} = \lim_{\epsilon \to 0} \limsup_{t \to \infty} \tfrac{1}{t} \log(H^t_\epsilon). \tag{2.1}$$

This definition of topological entropy is due to Bowen and Dinaburg [15, 28] and needs to be commented:

- One can check that each $d^t, t \in \mathbb{R}_{\geq 0}$ is indeed a distance function and induces the same topology as d.
- The number H^t_ϵ is monotonously growing as $\epsilon \to 0$ and so is $\limsup_{t\to\infty} \frac{1}{t} \log(H^t_\epsilon)$. Thus the limit in (2.1) exists and is a number $h_{\text{top}} \in [0, \infty]$.

- A priori the topological entropy h_{top} of the flow S^t depends on the metric d; however, it can be shown that it actually depends only on the topology induced by the metric. There is another definition [1] of topological entropy that does not refer to a metric, but just the topology - in case that the topology is metrizable, both notions coincide.
- Instead of defining H^t_ϵ as the least cardinality of ϵ-nets in (X, d^t), one could use the maximal cardinality of sets in (X, d^t) that are ϵ-separated, call it $\tilde{H}^t_\epsilon$. This number will generally differ from H^t_ϵ, but the limit $\lim\limits_{\epsilon\to 0}\limsup\limits_{t\to\infty}\tilde{H}^t_\epsilon$ coincides with h_{top} defined as in (2.1).

The following theorem relates integrability of a Hamiltonian system on a 4-manifold to the topological entropy of the flow along the Hamiltonian vector field.

Theorem 2.8 ([66])**.** *Let (N^4, ω, H) be a Hamiltonian system of dimension 4, that is integrable by an integral $f : N \to \mathbb{R}$, and let $Q = \{p \in N \mid H(p) := c_0\}$ be a compact, regular level set, that is $dH|_p \neq 0$ for all $p \in Q$.*
If f is real-analytic on Q or the connected components of the set of critical points of f

$$\operatorname{Crit}(f) := \{p \in Q \mid df_p = 0\}$$

form submanifolds, then the flow of X_H has topological entropy zero.

2.4.2 The geodesic flow as a Hamiltonian system

Let F be a Finsler metric on a smooth manifold M. The flow $\Phi^t : TM\backslash 0 \to TM\backslash 0$ along the geodesic spray S_F is called the *geodesic flow* of F. In the following we define a symplectic structure ω on $TM\backslash 0$ such that the Hamiltonian vector field of the function $E := \frac{1}{2}F^2$ is the geodesic spray.

For any strictly convex smooth $f : \mathbb{R}^n \to \mathbb{R}$, that is for which

$$f(\lambda\xi + (1-\lambda)\eta) < \lambda f(\xi) + (1-\lambda)f(\eta) \qquad \text{for any } \xi, \eta \in \mathbb{R}^n, \xi \neq \eta \text{ and } t \in (0,1),$$

the mapping $\ell : \mathbb{R}^n \to (\mathbb{R}^n)^*$ defined by $\xi \mapsto df|_\xi$ is called Legendre transformation and is an injection.

Let F be a Finsler metric. The function $\frac{1}{2}F^2(x, \cdot) : T_xM \to \mathbb{R}$ is strictly convex for any $x \in M$ since $g_{ij} = (\frac{1}{2}F^2)_{\xi^i\xi^j}$ is positive definite. Therefore its (inverse) Legendre transformation

$$\ell_x : T_xM\backslash\{0\} \to T^*_xM\backslash\{0\} \qquad \xi \mapsto g_{(x,\xi)}(\xi, \cdot)$$

is a bijection, given in coordinates by $\ell_x(\xi) = g_{ij}(x,\xi)\xi^i dx^j = (\frac{1}{2}F^2)_{\xi^j}(x,\xi)dx^j$. Therefore the global Legendre transform

$$\ell : TM\backslash 0 \to T^*M\backslash 0, \qquad \ell : \xi \mapsto \pi(\xi)(\xi)$$

is a smooth bundle-isomorphism given in local coordinates by $\ell(x,\xi) = (x, g_{ij}(x,\xi)\xi^i dx^j)$.

Now pullback the Poincaré 1-form $\tilde{\theta}$ and the canonical symplectic form $\tilde{\omega}$ on $T^*M\backslash 0$ from Example 2.8 via ℓ to $TM\backslash 0$, to obtain the so called *Hilbert 1-form* $\theta = \ell^*\tilde{\theta}$ and a symplectic form $\omega := \ell^*\tilde{\omega} = d(\ell^*\tilde{\theta}) = d\theta$ on $TM\backslash 0$. It follows from the theory of

Hamiltonian systems, that the Hamiltonian vector field of $\frac{1}{2}F^2$ with respect to ω is the geodesic spray, that is $X_{\frac{1}{2}F^2} = S_F$. Let us check this in local coordinates: Firstly,

$$\begin{aligned}\theta &= \ell^*(p_j dx^j) = g_{ij}\xi^i dx^j = (\tfrac{1}{2}F^2)_{\xi^j} dx^j \\ \omega &= d(\theta) = (\tfrac{1}{2}F^2)_{x^i\xi^j} dx^i \wedge dx^j - g_{ij} dx^i \wedge d\xi^j\end{aligned}$$

and thus

$$\begin{aligned}\omega(S_F, \cdot) &= (\tfrac{1}{2}F^2)_{x^i\xi^j}\xi^i dx^j - (\tfrac{1}{2}F^2)_{x^i\xi^j}\xi^j dx^i - g_{ij}\xi^i d\xi^j - g_{ij} 2G^j dx^i \\ &= \Big(\big((g_{ik})_{x^j} - (g_{ij})_{x^k}\big)\xi^i\xi^j - \frac{2}{4}\big(2(g_{ik})_{x^j} - (g_{ij})_{x^k}\big)\xi^i\xi^j \Big) dx^k - (\tfrac{1}{2}F^2)_{\xi^k} d\xi^k \\ &= -(\tfrac{1}{2}F^2)_{x^k} dx^k - (\tfrac{1}{2}F^2)_{\xi^k} d\xi^k \\ &= -d(\tfrac{1}{2}F^2).\end{aligned}$$

So indeed, $S_F = X_{\frac{1}{2}F^2}$ with respect to ω and the geodesic spray is indeed a Hamiltonian vector field. Thus integrability of the geodesic flow of a Finsler metric is a well defined notion and it is interesting and useful to find sufficient conditions for integrability, as will be done in Chapter 5.

2.5 Topology of Closed Surfaces and Groups of Exponential Growth

Closed surfaces are completely classified from the topological point of view. The genus and the Euler characteristic are two equivalent, important topological invariants, that together with (non-)orientability distinguish the topological type of a surface. If the Euler characteristic is negative, the fundamental group is *of exponential growth.* This fact becomes important in Chapter 5 due to its connection with the entropy of the geodesic flow of a Finsler metric on such a surface. This section is an overview of definitions and properties for later use - for complete definitions and proofs, the reader is referred to [36] and [49, Chapter I-IV] for the topology of surfaces and to [34] and [61] for the definition of the growth rate of a group.

Definition 2.27. *Let $\mathcal{S}$ be a closed surface and $\mathcal{T}$ a triangulation (that is a covering of $\mathcal{S}$ by closed sets each equipped with an homeomorphism to a triangle, such that two of them intersect in an entire edge or an vertex, if they intersect at all).*
The Euler characteristic of the surface is the integer

$$\chi(\mathcal{S}) := V - E + T,$$

where V, E, T are the number of vertices/edges/triangles in $\mathcal{T}$ (where a vertex/edge that appears in several triangles is counted only once).

It is not trivial, but classically known, that every closed surface admits a triangulation and that the Euler characteristic $\chi(\mathcal{S})$ does not depend on the choice of triangulation. Furthermore, the Euler characteristic is invariant under homeomorphism and together with orientability allows to distinguish the topological type of a closed surface.

First examples of closed surfaces are the sphere S^2, the torus and the projective plane $\mathbb{R}P^2$ and more can be obtained by forming connected sums: one cuts out an open disks from each of two surfaces, takes the disjoint union of the remaining parts, and identifies the boundaries of the cut disks with each other. The classical classification theorem states that this gives all closed surfaces up to homeomorphism:

Theorem 2.9.

- *Any closed orientable surface is homeomorphic to the sphere S^2 with Euler characteristic 0 or to the connected sum of $g \geq 1$ tori with Euler characteristic $2 - 2g$.*
- *Any closed non-orientable surface is homeomorphic to the connected sum of $g \geq 1$ projective planes with Euler characteristic $2 - g$.*

No two of these surfaces are homeomorphic to each other, as they are distinguished by their Euler characteristic or equivalently, by the number g, which is called *genus* of the surface (the genus of the sphere is set to be 0). Hence the genus and the Euler characteristic of a closed surface $\mathcal{S}$ are related by $\begin{cases} \chi(\mathcal{S}) = 2 - 2g & \text{if } \mathcal{S} \text{ is orientable} \\ \chi(\mathcal{S}) = 2 - g & \text{if } \mathcal{S} \text{ is non-orientable} \end{cases}.$

Definition 2.28. *The fundamental group $\pi(X, x_0)$ of a topological space X at a point $x_0 \in X$ is defined as the set*

$$\{\gamma : [0,1] \to X \text{ continuous with } \gamma(0) = \gamma(1) = x_0\}$$

of loops at x_0, modulo the homotopy equivalence relation, that is $\gamma_0 \sim \gamma_1$, if there is a continuous map $H : [0,1]^2 \to X$ satisfying

$$H(0,t) = \gamma_0(t), \quad H(1,t) = \gamma_1(t) \quad \text{and} \quad H(s,0) = H(s,1) = x_0.$$

The concatenation of (representative) paths defines a multiplication on $\pi(X, x_0)$ and gives it the structure of a group.

If the space X is pathwise connected, the isomorphism class of the fundamental group does not depend on the choice of the basepoint x_0 and is denoted by $\pi(X)$.

Definition 2.29. *Let S be a set.*

(a) The free group over S, denoted by $\langle S \rangle$, is the set of finite words in the alphabet S, that is formal products $g_1^{p_1} \cdot \ldots \cdot g_n^{p_n}$ with $g_1, .., g_n \in S$, $g_i \neq g_{i+1}$ and $p_1, .., p_n \in \mathbb{Z} \backslash \{0\}$, endowed with the concatenation (followed by an obvious reduction to ensure the above conditions) as group multiplication.

(b) Let $R \subseteq \langle S \rangle$. The group generated by S with relations R, denoted by $\langle S \mid R \rangle$, is the quotient of $\langle S \rangle$ by the smallest normal subgroup containing R, that is, quotient by the subgroup $\{grg^{-1} \mid g \in \langle S \rangle, r \in R\}$.

Via the Seifert-Van Kampen theorem one can calculate the fundamental group for all closed surfaces:

Lemma 2.13.

- *The fundamental group of the sphere is isomorphic to the trivial group $\{1\}$.*
- *The fundamental group of an orientable closed surface of genus g is isomorphic to the group $\langle a_1, b_1, .., a_g, b_g \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \rangle$.*
- *The fundamental group of a non-orientable closed surface of genus g is isomorphic to the group $\langle a_1, .., a_g \mid \prod_{i=1}^{g} a_i^2 \rangle$.*

Definition 2.30. *Let G be a group finitely generated by $S \subseteq G$.*

(a) The S-word norm of an element $g \in G$ is the length of the shortest word representing g, that is

$$|g| := \min\{\sum_{i=1}^{\ell} |p_i| \mid \ell \in \mathbb{N}, s_i \in S, p_i \in \mathbb{Z}, g = s_{i_1}^{p_1} s_{i_2}^{p_2} ... s_{i_\ell}^{p_\ell}, \}.$$

(b) *The growth rate of the group G is the asymptotic behaviour of $\#B_n$ for $n \to \infty$, where $B_n := \{g \in G \mid |g| \leq n\}$ are the closed balls in the S-word norm.*
The group is said to be of

(i) *exponential growth, if for some $C > 1$ it holds $\#B_n \geq C^n$ for all $n \in \mathbb{N}$.*

(ii) *polynomial growth $d \geq 0$, if for some $C > 0$ it holds $\#B_n \leq Cn^d$ for all $n \in \mathbb{N}$.*

The number $\#B_n$ counts how many elements of G can be written as a word with at most n letters in the alphabet S. The growth rate does not depend on the choice of S: If S' is another finite generator, there is a number k, such that every element of S can be written as a S'-word of length at most k and vice versa. Hence the two norms satisfy $\frac{1}{k}|g|_S \leq |g|_{S'} \leq k|g|_S$ and it is $B^{S'}_{\frac{n}{k}} \subseteq B^S_n \subseteq B^{S'}_{kn}$. It follows that the growth rate of a group is defined independently of the choice of generator S.

Being of exponential growth and being of polynomial growth for some d exclude each other. However there are groups which do not fall in one of those categories.

Example 2.10.

1. *A finite group is of constant growth, as $\#B_n = |G|$ for n large enough.*

2. *The group $(\mathbb{Z}, +)$ is isomorphic to the free group on one element and is of linear growth, as $\#B_n = 2n + 1$ for $S = \{1\}$. Any Abelian group generated by d elements $s_1, .., s_d$, e.g. $(\mathbb{Z}^d, +)$, is of polynomial growth of order d, because all elements are of the form $s_1^{p_1} \cdot .. \cdot s_d^{p_d}$ with $p_1, .., p_d \in \mathbb{Z}$, so that $\#B_n \leq (2n + 1)^d$.*

3. *A free group on $k \geq 2$ elements is of exponential growth: for $n \geq 1$ there are $2k(2k - 1)^{n-1}$ elements of norm n and this number is exponential in n.*

Corollary 2.3. *The fundamental group of a closed surface is of exponential growth if and only if it is of negative Euler characteristic $\chi < 0$, and of polynomial growth otherwise.*

Up to homeomorphism, the only closed surfaces with non-negative Euler characteristic are the sphere, the torus, the projective plane and the Klein bottle.

Proof. If the genus is zero or one, the fundamental group is Abelian and of polynomial growth. If the surface is homeomorphic to the connected sum of at least two tori or at least three projective planes, the fundamental group contains a free subgroup over two elements and is of exponential growth. The remaining tricky case is the connected sum of two projective planes, namely the Klein bottle, for which by a theorem of Milnor [61] the fundamental group is of polynomial growth 2. We give an elementary computational proof, by establishing rules for reduced forms of words and then counting, how many of this reduced words of length n can appear at most.

Let the fundamental group be generated by $S = \{a, b\}$ with the only relation $a^2b^2 = 1$. We might assume that in a reduced word no generator is next to its inverse and that none of the following ten expressions appears:

$$b^2,\ b^{-2},\ a^{-1}b^{-1},\ b^{-1}a^{-1},\ ba^2,\ a^2b,\ a^{-1}ba,\ b^{-1}ab,\ ba^{-2},\ b^{-1}a^2$$

There are only eight words of length two in reduced form:

$$aa,\ ab,\ ab^{-1},\ ba,\ ba^{-1},\ a^{-1}a^{-1},\ a^{-1}b,\ b^{-1}a.$$

Every reduced word of length $n+1$ is a reduced word W of length n with a letter attached at the end, and the two last letters of W restrict the possibilities by the above rules. Let $\lambda^n = (\lambda_i^n)_{i=1..8}$ be a column vector whose entries are the numbers of reduces words of length n ending with the i-th reduced word of length two from above, in particular $\lambda^2 = (1, .., 1)^T$. Then the rules for reduced words imply $\lambda^{n+1} \leq A\lambda^n$, where A is the following matrix with Jordan form J.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The number of reduced words of length $n \geq 2$ is not greater than $(1, .., 1) \cdot A^{n-2} \cdot (1, .., 1)^T$, which is a linear expression in n, and we conclude that $\#B_n$ is bounded by a polynomial in n of degree 2. $\square$

Chapter 3

Finsler metrics with 3-dimensional projective algebra in dimension 2

A vector field is called *projective* for a Finsler metric on a smooth manifold, if its flow preserves the projective class of the metric. The projective vector fields of a fixed Finsler metric form a finite dimensional Lie algebra $\mathfrak{p}$. On a surface, the maximal dimension of the *projective algebra* is 8 and is obtained exactly by the projectively flat metrics. Surprisingly, the submaximal dimension is 3. In this chapter, we explain the above assertions, recall a problem of Sophus Lie about describing metrics admitting projective vector fields and give a partial answer in the submaximal case: the following theorem gives an explicit list of all Finsler metrics with 3-dimensional projective algebra up to projective equivalence and isometry near a transitive point, that is in which the projective vector fields span the whole tangent plane.

Theorem 3.1. *Every Finsler metric on a surface admitting at least three independent projective vector fields is projectively equivalent near any transitive point to*

- *a Randers metric $F = \alpha + \beta$, where the projective vector fields are exactly the Killing vector fields of the Riemannian metric α*
- *or to a Riemannian metric.*

In some local coordinates F is projectively equivalent to the Euclidean metric or to one of the following:

$$\begin{array}{llll} (a) & \sqrt{dx^2+dy^2}+\frac{1}{2}(ydx-xdy) & (b_k^{\pm}) & \frac{\sqrt{dx^2+dy^2}-\frac{k}{2}(ydx-xdy)}{1\pm(x^2+y^2)},\, k>0 \\ (c^+) & \sqrt{\frac{e^{3x}}{(2e^x-1)^2}dx^2+\frac{e^x}{2e^x-1}dy^2} & (c^-) & \sqrt{e^{3x}dx^2+e^x dy^2} \end{array}$$

Each of this metrics is strictly convex on a neighborhood of the origin and none of them is locally isometric to any Finsler metric projectively equivalent to one of the others.

3.1 A problem by Sophus Lie

3.1.1 Projective, affine and Killing symmetries of Finsler metrics

Let F be a Finsler metric on a smooth manifold M and $\Phi : M \to M$ a smooth map. There are three natural notions when to call Φ a symmetry of F, according to how much information about F is preserved by Φ - this could be the metric itself, the geodesic spray or only the projective class of the geodesic spray.

Definition 3.1. *Let F be a Finsler metric on M with geodesic spray S_F. A smooth bijection $\Phi : M \to M$ is called*

- *an isometry of F, if $\Phi^*F = F$.*
- *an affine symmetry of F, if $\Phi^*(S_F) = S_F$.*
- *a projective symmetry of F, if $\Phi^*(S_F)$ is projectively equivalent to S_F.*

The infinitesimal version of such symmetries are vector fields, whose local flow for every fixed time is a symmetry in the above sense.

Definition 3.2. *Let F be a Finsler metric on M with geodesic spray S_F. A smooth vector field $X \in \mathfrak{X}(M)$ with local flow $\Phi_X^t : M \to M$ is called*

- *a Killing vector field of F, if $(\Phi_X^t)^*F = F$ for all t.*
- *an affine vector field of F, if $(\Phi_X^t)^*(S_F) = S_F$ for all t.*
- *a projective vector field of F, if $(\Phi_X^t)^*(S_F)$ is projectively equivalent to S_F for all t.*

We denote by $\mathfrak{iso}(F)$, $\mathfrak{a}(F)$ and $\mathfrak{p}(F)$ the set of Killing, affine and projective vector fields of a fixed Finsler metric F.

In other words, a vector field is Killing, if its flow preserves the metric; it is affine, if its flow takes each parametrized geodesic to a parametrized geodesic; it is projective, if its flow takes each geodesic to a geodesic with a possibly different parametrization. Clearly, every Killing vector field is affine, and every affine vector field is projective, thus

$$\mathfrak{iso}(F) \subseteq \mathfrak{a}(F) \subseteq \mathfrak{p}(F).$$

Lemma 3.1. *Let F be a Finsler metric on a smooth manifold M.*

1. *Let X be a smooth vector field on M with prolongation $\hat{X}$ to TM. Then:*
   (a) *X is Killing, if and only if $\mathcal{L}_{\hat{X}}F = 0$.*
   (b) *X is affine, if and only if $\mathcal{L}_{\hat{X}}S_F = 0$.*
   (c) *X is projective, if and only if $\mathcal{L}_{\hat{X}}S_F = \lambda \cdot V$ for some function $\lambda : TM\backslash 0 \to \mathbb{R}$, where V is the Liouville vector field, given in all local coordinates by $V = \xi^i\partial_{\xi^i}$.*
2. *The set of Killing, affine and projective vector fields $\mathfrak{iso}(F)$, $\mathfrak{a}(F)$ and $\mathfrak{p}(F)$ form finite dimensional Lie algebras.*

Here, the prolongation of a vector field X on M is the vector field $\hat{X}$ on TM, whose flow is the differential mapping of the flow of X. More precisely, if $\Phi_X^t : M \to M$ denotes the flow of X, we have the mappings $d\Phi_X^t : TM \to TM$ and the prolongation vector field $\hat{X}$ is defined by $\hat{X}|_\xi = \frac{d}{dt}|_{t=0}\big(d\Phi_X^t(\xi)\big)$. If X is given by $X^i\partial_{x^i}$ in local coordinates, then $\hat{X}$ is given in a vector $(x, \xi) \in TM$ by $X^i\partial_{x^i} + (X^i)_{x^j}\xi^j\partial_{\xi^i}$.

That the three sets form Lie algebras follows from (1.). Firstly, the Lie derivative and the prolongation of vector fields are linear operations, so if X, Y are Killing/affine/projective, so is any linear combination of them. That the bracket of two Killing/affine fields is Killing/affine is due to the identity $\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$ and because prolongation and taking the bracket commutes. For the same fact on projective vector fields, one additionally needs that $[\hat{X}, V] = 0$ for any prolongation $\hat{X}$, where V is again the Liouville vector field.

The finite dimensionality of $\mathfrak{p}$ and hence of the other two Lie algebras was shown in the two-dimensional case by Sophus Lie in [44, Chapter 17 §3] by a geometric argument. A proof for the general case can be found in [65, Theorem 6.38].

3.1.2 The dimension of the projective algebra on surfaces and the Lie problem

In 1882 Sophus Lie [43] stated the following problem[1]:

Problem. *Describe metrics on surfaces for which the projective algebra $\mathfrak{p}$ is at least two dimensional.*

The original problem was probably intended for pseudo-Riemannian metrics and was solved in [17], where local normal forms for metrics with $\dim \mathfrak{p} \geq 2$ were given. We recall their results shortly in Section 3.1.3. It is interesting to study the generalization of this problem for Finsler metrics, where many additional examples appear.

By Lemma 3.1, the dimension of the projective algebra $\mathfrak{p}$ of a fixed Finsler metric is finite, but in dimension two one can say even more.

Theorem 3.2. *If the dimension of the projective algebra of a Finsler metric on a surface is larger than three, then it is eight and the Finsler metric is projectively flat.*

In other words, the maximal dimension of the projective algebra is eight, and is attained exactly by the projectively flat metrics. The submaximal dimension is three - the metrics from Theorem 3.1 are examples. The above theorem follows directly from a corresponding and rather classical result on second order ODEs, namely Lemma 3.2 as explained below.

Thus the above problem asks to study Finsler metrics on surfaces for which the projective algebra is two-, three- or eight-dimensional. The latter case is the one of projectively flat metrics, which was studied intensively and a systematic investigation of such metrics is carried out in the literature, e.g. [2, 3, 62, 75, 76]. We have reviewed several properties in Section 2.2.7 and refer to [23] or [72, Chapter 6.2] for comprehensive overviews on the topic. In the following we investigate the *next easiest* case, namely when the dimension of the projective algebra is three (see also Example 4.1 for $\dim \mathfrak{p} = 2$).

Consider a Finsler metric F with geodesic spray S_F in some local coordinates (x, y) on a surface. The portion of curves of S_F for which $\dot{x} > 0$ and the portion with $\dot{x} < 0$ may be each described by a second order ODE

$$\begin{aligned} \ddot{y}(x) &= f_+\big(x, y(x), \dot{y}(x)\big) \quad \dot{x} > 0 \\ \ddot{y}(x) &= f_-\big(x, y(x), \dot{y}(x)\big) \quad \dot{x} > 0\text{'} \end{aligned} \tag{3.1}$$

[1] The original formulation in German is the following: '*Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Curven mehrere infinitesimale Transformationen gestatten.*'

and these ODEs coincide for projectively equivalent Finsler metrics (for formulas, see (3.5) and Lemma 3.7). We call them *induced ODEs*. They have been studied at least since the time of Beltrami in the case that the geodesics come from an affine connection and are sometimes referred to as *projective connection*.

The induced ODEs are the main tool for our classification of Finsler metrics (and sprays) with projective symmetries, due to the following fact: if X is a projective vector field of F, then its local flow preserves the induced ODEs. Such vector fields are classically known as infinitesimal point symmetries of an ODE. It is classical that the infinitesimal point symmetries of an ODE form a Lie algebra and ODEs admitting point symmetries were studied intensively since the time of Lie. For our particular interest, we have the following result (see [44, 80, 81] for classical references and [39] for a modern review):

Lemma 3.2 ([71, Theorem 2, Proposition 1])**.** *If the algebra of infinitesimal point symmetries of an ODE $\ddot{y} = f(x, y, \dot{y})$ is more than 3-dimensional, then it is 8-dimensional and in some local coordinates the ODE is given as $\ddot{y} = 0$. This is the case if and only if it is of the form*

$$\ddot{y} = A + B\ \dot{y} + C\ \dot{y}^2 + D\ \dot{y}^3$$

with functions A, B, C, D depending just on x, y satisfying

$$\begin{aligned} -A_{yy} + \tfrac{2}{3}B_{xy} - \tfrac{1}{3}C_{xx} - DA_x - 2AD_x + CA_y + AC_y + \tfrac{1}{3}BC_x - \tfrac{2}{3}BB_y &= 0 \\ \tfrac{2}{3}C_{xy} - \tfrac{1}{3}B_{yy} - D_{xx} + AD_y + 2DA_y - DB_x - BD_x - \tfrac{1}{3}CB_y + \tfrac{2}{3}CC_x &= 0 \end{aligned}.$$

If a Finsler metric admits a more than three-dimensional projective algebra, then both induced equations $\ddot{y} = f_\pm(x, y, \dot{y})$ admit them as point symmetries, and there are local coordinates in which $f_+ \equiv 0$. In that case, the projective algebra consists of the generators of line preserving local transformations of the plane. It is not hard to see (cf. Lemma 3.6) that the only 2nd order ODE invariant under those is $\ddot{y} = 0$, so that in this case also $f_- \equiv 0$ and the metric is projectively flat. This proves Theorem 3.2.

3.1.3 The pseudo-Riemannian problem as a starting point

For a pseudo-Riemannian metric or more generally an affine connection, the two induced ODEs (3.1) coincide, as every orientation-reversed geodesic is again a geodesic. Moreover, they are a polynomial of degree 3 in $\dot{y}$, explicitly

$$\ddot{y} = K^0(x, y) + K^1(x, y)\dot{y} + K^2(x, y)\dot{y}^2 + K^3(x, y)\dot{y}^3, \tag{3.2}$$

where the coefficients are given in terms of the Christoffel symbols of the (Levi-Civita) connection by

$$K^0 = -\Gamma^2_{11} \qquad K^1 = \Gamma^1_{11} - 2\Gamma^2_{12} \qquad K^2 = -(\Gamma^2_{22} - 2\Gamma^1_{12}) \qquad K^3 = \Gamma^1_{22}.$$

The general strategy in [17] and for our Finslerian version, is to first find all projective classes of sprays admitting two or three independent projective vector fields - and afterwards find all pseudo-Riemannian or Finsler metrics, with the corresponding geodesics.

Regarding the first step, a peculiarity to the pseudo-Riemannian case is, that the system of geodesics is described by a single 2nd order ODE $\ddot{y} = f_+(x, y, \dot{y})$, which is polynomial in $\dot{y}$, and all information on the projective class of the metric is contained in the function f_+. Because of possible irreversibility of the geodesic sprays, in the Finslerian case we need the two induced ODEs $\ddot{y} = f_\pm(x, y, \dot{y})$ to collect all information about the projective class. Furthermore, because the metric is not a quadratic form, the functions $f_\pm$ are generally not polynomial in the fiber coordinates. Consequently, in the Finsler case the variety of projective classes of sprays to consider is much larger.

The second step of finding a metric for a given projective class of sprays is called *projective metrization* and is in its general form subject to Chapter 4. In the pseudo-Riemannian case, this problem is linearised by the Theorem below, so that the projective metrizations of a fixed projective class of sprays form a finite dimensional vector space - and one has an effective tool to find *all* projective metrizations. A similar results for the Finslerian setting is Theorem 4.1 to be explained later: it gives a linear PDE system on the Hessian of F that is necessary and sufficient to be a projective metrization of a fixed spray. However, this PDE lives on the tangent bundle TM and not only on M as in the pseudo-Riemannian case. This makes it difficult to find *all* Finsler metrics for a given system of geodesics - in this chapter we are satisfied to find one projective metrization for every spray with 3-dimensional projective algebra.

Theorem 3.3 ([17, Lemma 5]). *The induced ODEs (3.5) of a pseudo-Riemannian metric $g = (g_{ij})$ are given by (3.2), if and only if the coefficients of the matrix $a = (\det g)^{-2/3} g$ satisfy the linear PDE system*

$$\left.\begin{array}{rcl} \partial_x a_{11} - \frac{2}{3}K^1 a_{11} + 2K^0 a_{12} & = & 0 \\ \partial_y a_{11} + 2\partial_x a_{12} - \frac{4}{3}K^2 a_{11} + \frac{2}{3}K^1 a_{12} + 2K^0 a_{22} & = & 0 \\ 2\partial_y a_{12} + \partial_x a_{22} - 2K^3 a_{11} - \frac{2}{3}K^2 a_{12} + \frac{4}{3}K^1 a_{22} & = & 0 \\ \partial_y a_{22} - 2K^3 a_{12} + \frac{2}{3}K^2 a_{22} & = & 0 \end{array}\right\}. \tag{3.3}$$

This approach allowed to give all pseudo-Riemannian metrics admitting a 2- or 3-dimensional projective algebra locally up to isometry only, which is the main result of the paper [17]:

Theorem 3.4. *[17, Theorem 1] Let g be a Riemannian metric admitting at least two independent projective vector fields. If $\dim \mathfrak{p} > 3$, then the metric is projectively flat and by the Beltrami theorem of constant sectional curvature. Otherwise, around any point where the projective algebra is transitive, there are local coordinates where g is given as follows:*

- *If* $\dim \mathfrak{p} = 3$*:*
  - *(a)* $\epsilon_1 e^{3x} dx^2 + \epsilon_2 e^x dy^2$, *where* $\epsilon_i \in \{\pm 1\}$
  - *(b)* $a\left(\frac{e^{3x}}{(e^x+\epsilon_2)^2} dx^2 + \epsilon_1 \frac{e^x}{e^x+\epsilon_2} dy^2\right)$, *where* $a \in \mathbb{R}\backslash\{0\}$ *and* $\epsilon_i \in \{\pm 1\}$
  - *(c)* $a\left(\frac{1}{(cx+2x^2+\epsilon_2)^2 x} dx^2 + \epsilon_1 \frac{x}{cx+2x^2+\epsilon_2} dy^2\right)$, *where* $a > 0$, $\epsilon \in \{\pm 1\}$ *and* $c \in \mathbb{R}$
- *If* $\dim \mathfrak{p} = 2$*:*
  - *(a)* $\epsilon_1 e^{(b+2)x} dx^2 + \epsilon_2 e^{bx} dy^2$, *where* $b \in \mathbb{R}\backslash\{-2, 0, 1\}$ *and* $\epsilon_i \in \{\pm 1\}$
  - *(b)* $a\left(\frac{e^{(b+2)x}}{(e^{bx}+\epsilon_2)^2} dx^2 + \epsilon_1 \frac{e^{bx}}{e^{bx}+\epsilon_2} dy^2\right)$, *where* $a \in \mathbb{R}\backslash\{0\}$, $b \in \mathbb{R}\backslash\{-2, 0, 1\}$, $\epsilon_i \in \{\pm 1\}$
  - *(c)* $a\left(\frac{e^{2x}}{x^2} dx^2 + \frac{\epsilon}{x} dy^2\right)$, *where* $a \in \mathbb{R}\backslash\{0\}$ *and* $\epsilon \in \{\pm 1\}$

No two distinct metrics from this list are isometric.

3.2 Finsler metrics with 3-dimensional projective algebra

In this section we prove Theorem 3.1 and thus describe all Finsler metrics with 3-dimensional projective algebra on surfaces around a transitive point up to projective equivalence and isometry.

Structure of the proof. As the projective algebra of a metric is defined in terms of its geodesic spray, the projective algebra is defined for any spray and coincides for projectively equivalent sprays. Our first goal is to find all sprays up to projective equivalence and coordinate change, that admit a 3-dimensional projective algebra. To that end, in Section 3.2.1 we give local normal forms for general 3-dimensional Lie algebras of vector fields around a transitive point. They allow us to deduce the list below of normal forms of 2nd order ODEs, that admit three independent infinitesimal point symmetries in Section 3.2.2. The techniques to obtain this list were developed by Sophus Lie more than 100 years ago [44].

Lemma 3.3. *Let $\ddot{y} = f\big(x, y(x), \dot{y}(x)\big)$ be a second order ODE and X_1, X_2, X_3 three linearly independent vector fields on the plane, transitive at the origin, whose flow preserves the ODE. Then in some local coordinates around the origin, the equation takes the form $\ddot{y} = 0$ or one of the following:*

$$\begin{array}{ll} D1 & \ddot{y} = C(y^2 - 2\dot{y})^{3/2} - y^3 + 3y\dot{y} \\ D2 & \ddot{y} = C\dot{y}^{\frac{\lambda-2}{\lambda-1}} \\ J1 & \ddot{y} = C\dot{y}^3 e^{-1/\dot{y}} \\ J2 & \ddot{y} = \frac{1}{2}\dot{y} + Ce^{-2x}\dot{y}^3 \\ C1 & \ddot{y} = C(\dot{y}^2+1)^{3/2} e^{-\lambda \arctan(\dot{y})} \\ C2 & \ddot{y} = \frac{C(\dot{y}^2+1)^{3/2} \pm 2(x\dot{y}-y)(\dot{y}^2+1)}{1 \pm (x^2+y^2)} \end{array} , \qquad C \neq 0, \lambda \geq 0.$$

Not all of these ODEs can describe the portion of curves with $\dot{x} > 0$ of a *fiber-globally* defined spray. By sorting out those, we obtain in Section 3.2.3 normal forms for sprays with three independent projective vector fields up to projective equivalence.

Lemma 3.4. *Let S be a spray on the plane with* $\dim \mathfrak{p} \geq 3$. *Then there are local coordinates (x, y) with induced fiber coordinates (u, v), in which the geodesics of S are straight lines, or S is projectively equivalent to one of:*

$$\begin{array}{ll} (a) & u\partial_x + v\partial_y - \sqrt{u^2+v^2}(v\partial_u - u\partial_v) \\ (b_k^{\pm}) & u\partial_x + v\partial_y - \frac{k\sqrt{u^2+v^2} \pm 2(yu-xv)}{1 \pm (x^2+y^2)}(v\partial_u - u\partial_v) \quad k > 0 \\ (c^{\pm}) & u\partial_x + v\partial_y - \frac{1}{2}(3u^2 \pm e^{-2x}v^2)\partial_u - uv\partial_v \end{array}$$

None of these sprays can be transformed into one projectively equivalent to one of the others by a local coordinate change.

The sprays (a) and $(b_k^{\pm})$ are very geometric: The curves of the spray (a) are positively oriented circles of radius 1 in the Euclidean plane. Similarly, the curves of (b_k^{+}) and (b_k^{-}) are the positively oriented curves of constant geodesic curvature k on the two-sphere S^2 in stereographic coordinates and in the Poincaré disk model of the hyperbolic plane.

We remark that the sprays $(c^{\pm})$ are *geodesically reversible*, meaning that the unique geodesics tangent to the vectors v and $-v$ have the same trajectories on M. The sprays (a) and $(b_k^{\pm})$ are *geodesically irreversible.*

In section 3.2.4 we explain how one can calculate the induced ODEs of a Finsler metric directly. This allows to check quickly, that the geodesic sprays of the metrics from Theorem 3.1 are projectively equivalent to the sprays from Lemma 3.4. This finishes the proof of Theorem 3.1.

Additionally we explain how we found the metrics from Theorem 3.1. For the sprays $(c^{\pm})$ we apply the method from [17] to find Riemannian metrics; for the sprays (a) and $(b_k^{\pm})$ we construct *Randers metrics* by adding an appropriate 1-form to the Riemannian metrics of constant sectional curvature.

3.2.1 3-dimensional Lie algebras of vector fields in the plane

Let $\mathfrak{g}, \tilde{\mathfrak{g}}$ be 3-dimensional Lie algebras of vector fields on the plane and $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid X|_0 = 0\}$ be the *isotropy subalgebra* at the origin. We assume that both $\mathfrak{g}, \tilde{\mathfrak{g}}$ are *transitive* at the origin, i.e. $\dim \mathfrak{g}_0 = \dim \tilde{\mathfrak{g}}_0 = 1$ and that no $X \neq 0$ vanishes on an entire open neighborhood of the origin.

Lemma 3.5.

1. *There is a local coordinate transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ taking each vector field from $\mathfrak{g}$ to one from $\tilde{\mathfrak{g}}$ and fixing the origin, if and only if there is an isomorphism of Lie algebras $\psi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ which takes $\mathfrak{g}_0$ to $\tilde{\mathfrak{g}}_0$.*

2. *$\mathfrak{g}_0$ is not an ideal, i.e. there is a vector field $X \in \mathfrak{g}$ with $[X, \mathfrak{g}_0] \not\subseteq \mathfrak{g}_0$.*

Proof. If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism with $\varphi(0) = 0$ that takes the vector fields from $\mathfrak{g}$ to vector fields from $\tilde{\mathfrak{g}}$, then $\psi : \mathfrak{g} \to \tilde{\mathfrak{g}}, X \mapsto \varphi_* X$ is an isomorphism of Lie algebras, and $(\varphi_* X)|_0 = 0$ if and only if $X|_0 = 0$. Hence $\psi(\mathfrak{g}_0) = \tilde{\mathfrak{g}}_0$.

For the other direction, assume $\psi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is an isomorphism of Lie algebras with $\psi(\mathfrak{g}_0) = \tilde{\mathfrak{g}}_0$. Let $X_0 \in \mathfrak{g}_0$ and $X_1, X_2 \in \mathfrak{g}$ form a basis of $\mathfrak{g}$ and set $\tilde{X}_i = \psi(X_i)$ for $i = 1, 2, 3$. Let $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ be the local diffeomorphism defined by $(s, t) \mapsto \Phi^s_{X_1} \circ \Phi^t_{X_2}(0)$ and $\tilde{\rho}$ accordingly, where Φ_X denotes the flow along X. We show that for $i = 1, 2, 3$ the vector fields $Y_i := \rho^*(X_i)$ and $\tilde{Y}_i := \tilde{\rho}^*(\tilde{X}_i)$ on $\mathbb{R}^2$ coincide, so that $\phi := \tilde{\rho} \circ \rho^{-1}$ is the sought coordinate transformation.

It is obvious that $Y_1 = \partial_s$, $Y_2|_{(0,t)} = \partial_t$ and $Y_0|_{(0,0)} = 0$. Let C^k_{ij} be the structure constants of the basis X_0, X_1, X_2, so that $[Y_i, Y_j] = C^k_{ij} Y_k$. For points $(0, t)$ consider the equations $[Y_0, Y_2] = C^k_{02} Y_k$. The only unknowns in the right side are the components of Y_0. In the left side, we can replace all derivatives by s in terms of components of the Y_i's using the commutation relations with $Y_1 = \partial_s$. Since $Y_2|_{(0,t)} = \partial_t$, in each equation exactly one derivative by t survives with coefficient 1. Hence we have a system of two ODEs in normal form on the components of Y_0 in points $(0, t)$ with starting value $Y_0|_{(0,0)} = 0$, so that the vectors $Y_0|_{(0,t)}$ are uniquely defined by the structure constants. Now for fixed t_0 the four equations $[Y_0, \partial_s] = C^k_{01} Y_k$ and $[\partial_s, Y_2] = C^k_{12} Y_k$ in points (s, t_0) again form a system of ODEs with already determined starting values, so that Y_0, Y_1, Y_2 are determined on a neighborhood of the origin only by the structure constants. But the same holds for $\tilde{Y}_0, \tilde{Y}_1, \tilde{Y}_2$, since they have the same structure constants, so we have $\tilde{Y}_i = Y_i$.

For the second statement, suppose that the commutator of X_0 with X_1, X_2 is a multiple of X_0. Then the first system of ODEs $[Y_0, Y_2] = C^0_{02} X_0$ in points $(0, t)$ has the obvious solution $Y_0|_{(0,t)} \equiv 0$ and by $[Y_0, Y_1] = C^0_{01} X_0$ we would have $Y_0 \equiv 0$ on a neighborhood of the origin. $\square$

Lemma 3.5 explains how we can obtain a complete list of 3-dimensional Lie algebras of vector fields around a transitive point up to coordinate transformation: For every pair $(\mathfrak{g}, \mathfrak{h})$ of an abstract 3-dimensional Lie algebra and 1-dimensional subalgebra, which is not an ideal, we should find one representative Lie algebra of vector fields isomorphic to $\mathfrak{g}$ such that the isotropy subalgebra corresponds to $\mathfrak{h}$.

Let us describe all such abstract pairs $(\mathfrak{g}, \mathfrak{h})$ up to isomorphisms preserving the subalgebra. Let $(\mathfrak{g}, \mathfrak{h})$ be fixed and $X_0 \in \mathfrak{h}\backslash\{0\}$. We might choose X_1, X_2 such that the map $\operatorname{ad} X_0 : \mathfrak{g} \to \mathfrak{g}, X \mapsto [X_0, X]$ is given in the basis X_0, X_1, X_2 by a matrix

$$\operatorname{ad} X_0 = \begin{bmatrix} 0 & * \\ 0 & A \end{bmatrix},$$

where $A \neq 0$ is a (2×2) Jordan block, and by scaling X_0 we might scale this matrix by any nonzero constant.

For the diagonal case $A = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$ we can restrict to $|\lambda| \geq 1$. For the Jordan case $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ we only need to consider $\lambda = 0$ and $\lambda = 1$. For the complex case $A = \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix}$ we can restrict to $\lambda \geq 0$.

Let $[X_0, X_1] = \sum_{i=0}^2 \alpha_i X_i$,$[X_0, X_2] = \sum_{i=0}^2 \beta_i X_i$ and $[X_1, X_2] = \sum_{i=0}^2 \gamma_i X_i$. The Jacobi identity $[X_0, [X_1, X_2]] + [X_1, [X_2, X_0]] + [X_2, [X_0, X_1]] = 0$ is given by

$$\begin{aligned} \alpha_0\gamma_1 + \beta_0\gamma_2 - \beta_2\gamma_0 - \alpha_1\gamma_0 &= 0 \\ \beta_1\gamma_2 + \alpha_1\beta_0 - \beta_2\gamma_1 - \alpha_0\beta_1 &= 0. \\ \alpha_2\gamma_1 + \alpha_2\beta_0 - \alpha_0\beta_2 - \alpha_1\gamma_2 &= 0 \end{aligned}$$

In the diagonal case, the Lie bracket table is of the form

$$\begin{bmatrix} \alpha_0 X_0 + X_1 & \beta_0 X_0 + \lambda X_2 \\ & \gamma_0 X_0 + \gamma_1 X_1 + \gamma_2 X_2 \end{bmatrix}$$

Let us manipulate the basis without changing the span of X_0, such that the Lie bracket table has simple form. By adding a multiple of X_0 to X_1, we can assume $\alpha_0 = 0$. By the Jacobi identity we get $(1 + \lambda)\gamma_0 = 0$, $\beta_0 = \lambda\gamma_1$ and $\gamma_2 = 0$. If $\lambda \neq 0$, we might add a multiple of X_0 to X_2 to assume $\beta_0 = 0$, so that we can assume this in any case. If $\gamma_0 \neq 0$, we have $\lambda = -1$ and $\gamma_1 = 0$ and by scaling X_1 we can obtain $\gamma_0 = 1$. If $\gamma_0 = 0$ either $\gamma_1 = 0$ holds already (if $\lambda \neq 0$), or replacing X_2 by $X_2 + \gamma_1 X_0$ achieves this without changing the other relations (if $\lambda = 0$). Hence, up to isomorphism of pairs, we only need to consider two tables:

$$\begin{bmatrix} X_1 & -X_2 \\ & X_0 \end{bmatrix} \begin{bmatrix} X_1 & \lambda X_2 \\ & 0 \end{bmatrix}$$

Each describes an equivalence class of a pair $(\mathfrak{g}, \mathfrak{h})$. To find corresponding Lie algebras of vector fields one could choose X_1 arbitrary (e.g. $X_1 = \partial_x$) and determine X_0 and X_2 by solving ODEs, following the proof of Lemma 3.5. However we just present a solution for every pair having the correct Lie bracket table.

$D1$	$-x\partial_x + y\partial_y$	∂_x	$-\frac{1}{2}x^2\partial_x + (xy + 1)\partial_y$
$D2$	$-x\partial_x - \lambda y\partial_y$	∂_x	∂_y

In the Jordan case, the Lie bracket table is of the form

$$\begin{bmatrix} \alpha_0 X_0 + \lambda X_1 & \beta_0 X_0 + X_1 + \lambda X_2 \\ & \gamma_0 X_0 + \gamma_1 X_1 + \gamma_2 X_2 \end{bmatrix}$$

If $\lambda = 1$, again by adding multiples of X_0 to X_1 and X_2, we can assume $\alpha_0 = \beta_0 = 0$. From the Jacobi identity we get $\gamma_0 = \gamma_1 = \gamma_2 = 0$. If $\lambda = 0$, we can assume $\beta_0 = 0$ and get from the Jacobi identity $\alpha_0 = \gamma_2$ and $\gamma_1\gamma_2 = 0$. In the case that $\gamma_2 \neq 0$, we have $\gamma_1 = 0$ and can assume $\gamma_2 = 1$ by scaling X_1 and X_2 by $\frac{1}{\gamma_2}$. Then by replacing X_2 by $X_2 + \frac{\gamma_0}{2} X_0$, we can also assume $\gamma_0 = 0$. The case that $\gamma_2 = 0$, also $\alpha_0 = 0$ and we could assume $\gamma_0, \gamma_1 \in \{0, 1\}$, but we can give a Lie algebra of vector fields with parameters that covers all cases. Hence, up to isomorphism of pairs, we have three cases:

$$\begin{bmatrix} X_1 & X_1 + X_2 \\ & 0 \end{bmatrix} \begin{bmatrix} X_0 & X_1 \\ & X_2 \end{bmatrix} \begin{bmatrix} 0 & X_1 \\ & \gamma_0 X_0 + \gamma_1 X_1 \end{bmatrix}$$

$J1$	$-(x+y)\partial_x - y\partial_y$	∂_x	∂_y	
$J2$	$-y\partial_x - \frac{1}{2}y^2\partial_y$	$-\partial_x - y\partial_y$	$-\partial_y$	$(c^{\pm})$
$J3$	$y\partial_x$	∂_x	$(\gamma_0 y + \gamma_1)x\partial_x + (\gamma_0 y^2 + \gamma_1 y - 1)\partial_y$	

In the complex case the Lie bracket table is of the form

$$\begin{bmatrix} \alpha_0 X_0 + \lambda X_1 - X_2 & \beta_0 X_0 + X_1 + \lambda X_2 \\ & \gamma_0 X_0 + \gamma_1 X_1 + \gamma_2 X_2 \end{bmatrix}$$

By replacing X_1 by $X_1 + rX_0$ and X_2 by $X_2 + sX_0$, the conditions that the new coefficients of X_0 in the first row become vanish the linear equations $\lambda r + s = \alpha_0$ and $r + \lambda s$, which admit a solution. Hence we can assume $\alpha_0 = \beta_0 = 0$. The Jacobi identity gives $\lambda\gamma_0 = 0$ and $\gamma_1 = \gamma_2 = 0$. If $\gamma_0 = 0$, we are left with $\lambda \geq 0$ as a parameter. Otherwise $\lambda = 0$ and by scaling X_1, X_2 by $\frac{1}{\sqrt{\gamma_0}}$ we can achieve $\gamma_0 = \pm 1$. Hence we have two cases:

$$\begin{bmatrix} \lambda X_1 - X_2 & X_1 + \lambda X_2 \\ & 0 \end{bmatrix} \begin{bmatrix} -X_2 & X_1 \\ & \pm X_0 \end{bmatrix}$$

$C1$	$-(\lambda x - y)\partial_x - (x + \lambda y)\partial_y$	∂_x	$-\partial_y$	(a)
$C2$	$y\partial_x - x\partial_y$	$\frac{1}{2}(x^2 - y^2 \pm 1)\partial_x + xy\partial_y$	$xy\partial_x + \frac{1}{2}(-x^2 + y^2 \pm 1)$	$(b_k^{\pm})$

Summing up, we have shown that for every 3-dimensional Lie algebra of vector fields in the plane around a transitive point, there are coordinates in which it is given by one of the seven types $D1$, $D2$, $J1$, $J2$, $J3$, $C1$ and $C2$.

3.2.2 Second order ODEs with three independent infinitesimal point symmetries and proof of Lemma 3.3

Recall that a vector field X on $\mathbb{R}^2$ is called *infinitesimal point symmetry* of an ODE $\ddot{y}(x) = f(x, y(x), \dot{y}(x))$, if its local flow preserves the equation, or equivalently, it takes each trajectory of a solution $(x, y(x))$ to some trajectory of a solution. Sophus Lie proved that the infinitesimal point symmetries form a Lie algebra of dimension at most eight [44, Chapter 17 §2,3].

With a second order ODE one has associated a 1-dimensional distribution $\langle D\rangle$ on the space $J\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}$ of tangent directions not parallel to ∂_y, induced by the vector field $D|_{(x,y,z)} = \partial_x + z\partial_y + f(x, y, z)\partial_z$, where (x, y, z) are the coordinates for the direction $\mathrm{span}(\partial_x + z\partial_y)$ in the point (x, y).

A curve $c : I \to \mathbb{R}^2$ is prolonged naturally to $J\mathbb{R}^2$ by $\hat{c}(t) := (c^1(t), c^2(t), \frac{\dot{c}^2(t)}{\dot{c}^1(t)})$. If the prolongation of a curve is tangent to $\langle D\rangle$, then the same is true for any reparametrization. Moreover the lift of a curve $c(x) = (x, y(x))$ is tangent to $\langle D\rangle$, if and only if $y(x)$ is a solution to $\ddot{y} = f(x, y(x), \dot{y}(x))$.

If $X = a(x, y)\partial_x + b(x, y)\partial_y$ is a vector field on the plane, its induced flow on $J\mathbb{R}^2$ is generated by the vector field $\hat{X}|_{(x,y,z)} = a(x, y)\partial_x + b(x, y)\partial_y + c(x, y, z)\partial_z$, where $c := b_x + zb_y - z(a_x + za_y)$. Here the subscripts denote partial derivatives and arguments are suppressed.

Lemma 3.6.

1. *A vector field $X = a(x, y)\partial_x + b(x, y)\partial_y$ is an infinitesimal point symmetry of $\ddot{y} = f(x, y, \dot{y})$, if and only if $[\hat{X}, D]$ is proportional to D in every point (x, y, z). This condition is given by the equation*

$$af_x + bf_y + cf_z = (c_z - a_x - za_y)f + c_x + zc_y. \tag{3.4}$$

2. *A prescribed 3-dimensional algebra of infinitesimal point symmetries transitive at the origin for an ODE $\ddot{y} = f(x, y, \dot{y})$ together with an initial value $f(0, 0, z_0)$ for some $(0, 0, z_0) \in J\mathbb{R}^2$ determine the function f uniquely.*

Proof. (1.) The vector field X is an infinitesimal symmetry, if and only if its flow on $J\mathbb{R}^2$ preserves the distribution $\langle D\rangle$, i.e. if $\mathcal{L}_{\hat{X}}D = [\hat{X}, D]$ is a multiple of D. By direct calculation

$$[\hat{X}, D] = -(a_x + za_y)\partial_x - z(a_x + za_y)\partial_y + (af_x + bf_y + cf_z - c_x - zc_y - fc_z)\partial_z$$

can only be a multiple of D with factor $-(a_x + za_y)$, which is the case if and only if

$$af_x + bf_y + cf_z - c_x - zc_y - fc_z = -(a_x + za_y)f.$$

(2.) Let $X_i = a^i\partial_x + b^i\partial_y$ for $i = 0, 1, 2$ be a basis of the Lie algebra $\mathfrak{g}$, such that $X_0 \in \mathfrak{g}_0$. If the ODE $\ddot{y} = f(x, y, \dot{y})$ admits $\mathfrak{g}$ as infinitesimal point symmetries, it must solve the corresponding three equations (3.4). If the matrix $\begin{pmatrix} a^0 & b^0 & c^0 \\ a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \end{pmatrix}$ is regular in some point $(0, 0, z_0)$, this system might be written in normal form in a neighborhood and has a unique solution. Otherwise the matrix is singular in all points $(0, 0, z)$, and we must have $c^0(0,0,z) \equiv 0$, which gives $b^0_x(0,0) = a^0_y(0,0) = 0$ and $a^0_x(0,0) = b^0_y(0,0)$. By direct calculation it follows that $[X_0, X] = -a^0_x(0,0)X$ for all $X \in \mathfrak{g}$, that is $\mathfrak{g}_0 \subseteq \mathfrak{g}$ is an ideal, contradicting Lemma 3.5. □

We are now ready to proof Lemma 3.3: For each of the seven 3-dimensional algebras of vector fields from the last section, we find the 1-parameter family of ODEs admitting it as its infinitesimal point symmetry algebra. In each case we have to solve the three equations (3.4) on f, that by Lemma 3.6 determine f up to a parameter. In each case, the system can be solved by elementary methods. The solutions are given below, where $C \neq 0$ is a constant (for $C = 0$, all ODEs fulfil the assumptions of Lemma 3.2 and can be turned into the equation $\ddot{y} = 0$). Thus, any ODE admitting a 3-dimensional algebra of infinitesimal point symmetries can be transformed by a coordinate change into one of:

	$f(x,y,z)$
$D1$	$C(y^2-2z)^{3/2} - y^3 + 3yz$
$D2$	$Cz^{\frac{\lambda-2}{\lambda-1}}$
$J1$	$Cz^3e^{-1/z}$
$J2$	$\frac{1}{2}z + Ce^{-2x}z^3$
$C1$	$C(z^2+1)^{3/2}e^{-\lambda \arctan(z)}$
$C2$	$\frac{C(z^2+1)^{3/2} \pm 2(xz-y)(z^2+1)}{1 \pm (x^2+y^2)}$

The case $J3$ is not listed, because the equations for X_0 and X_1 force the equation to be of the form $h(y)z^3$ for some function h and by Lemma 3.2 its algebra of infinitesimal point symmetries is 8-dimensional. All the others do not fulfil the assumptions from Lemma 3.2 and hence their algebra of infinitesimal point symmetries is exactly 3-dimensional.

3.2.3 Sprays with 3-dim. projective algebras and proof of Lemma 3.4

Recall that a *spray* S on a smooth manifold M is a vector field $S \in \mathfrak{X}(TM\backslash 0)$, where $TM\backslash 0$ is the tangent bundle with the origins removed, which in all local coordinates (x^i, ξ^i) on TM is given as $S = \xi^i\partial_{x^i} - 2G^i(x^i,\xi^i)\partial_{\xi^i}$, where the G^i are smooth and positively 2-homogeneous in ξ. Two sprays on M are *projectively equivalent* if the projections to M of their integral curves, called *geodesics*, coincide as oriented point sets. This is the case if and only if $S - \tilde{S} = \lambda(x,\xi)V$ for some function $\lambda : TM\backslash 0 \to \mathbb{R}$, where V is the Liouville vector field.

A vector field X on M is called *projective* for a spray S if its flow Φ^X_t maps geodesics to geodesics as point sets, that is if S and $(\Phi^{\hat{X}}_t)_*S$ are projectively equivalent for all t, where $\hat{X}$ is the prolongation of X to TM. This is the case if and only if $\mathcal{L}_{\hat{X}}S = \lambda(x,\xi)V$ for some function $\lambda : TM\backslash 0 \to \mathbb{R}$. By the Jacobi identity the projective vector fields form a Lie algebra $\mathfrak{p}(S)$.

In this section we give a complete list of projective classes of sprays on $\mathbb{R}^2$ with $\dim \mathfrak{p} = 3$ up to local coordinate change and proof Lemma 3.4. Let (x,y) be coordinates on $\mathbb{R}^2$ and (x,y,u,v) the induced coordinates on $T\mathbb{R}^2$. To each projective class of sprays, we associate the two *induced second order ODEs*, whose solutions $y(x)$ are the reparametrizations by the parameter x of the geodesics with $\dot{x} > 0$ and $\dot{x} < 0$ respectively. This is independent of the choice of a representative S for the projective class.

Let us determine the induced ODEs in terms of the spray coefficients G^i. Let $(x, y(x))$ be a curve, such that $(\varphi(t), y(\varphi(t)))$ is a geodesic for S for a certain reparametrization $\varphi : \mathbb{R} \to \mathbb{R}$. Then

$$\begin{array}{rcl} & \varphi''(t) & = -2G^1(\varphi(t), y(\varphi(t)), \varphi'(t), \dot{y}(\varphi(t))\varphi'(t)) \\ \ddot{y}(\varphi(t))\varphi'(t)^2 + \dot{y}(\varphi(t))\varphi''(t) & = \frac{d^2(y(\varphi(t)))}{dt^2} & = -2G^2(\varphi(t), y(\varphi(t)), \varphi'(t), \dot{y}(\varphi(t))\varphi'(t)) \end{array}.$$

By 2-homogeneity of G^i we see that the x-reparametrizations of geodesics with $\dot{x} \neq 0$ are given as the solutions to the 2nd order ODEs

$$\begin{array}{lll} (X_+) & \ddot{y} = 2G^1(x, y, +1, +\dot{y})\dot{y} - 2G^2(x, y, +1, +\dot{y}) & (\dot{x} > 0) \\ (X_-) & \ddot{y} = 2G^1(x, y, -1, -\dot{y})\dot{y} - 2G^2(x, y, -1, -\dot{y}) & (\dot{x} < 0) \end{array}. \tag{3.5}$$

Note that the *induced ODEs* (3.5) determine the spray S up to projective equivalence. Furthermore, the flow of every projective vector field of S preserves this equations and is an infinitesimal point symmetry. If a spray admits a 3-dimensional algebra of point symmetries, then the induced ODE must be of the form (up to coordinate change) as in Lemma 3.3.

Let S be a spray with $\dim \mathfrak{p} > 3$. Then there are local coordinates where both induced ODEs (3.5) must be of the form $\ddot{y} = 0$ and hence S is projectively equivalent to the flat spray $u\partial_x + v\partial_y$.

Let S be a spray with $\dim \mathfrak{p} = 3$. Then we might assume that after a coordinate change $\mathfrak{p}$ is one of the in Section 3.2.1 obtained Lie algebras of vector fields and the two induced ODEs (3.5)

$$\begin{array}{ll} (X_+) & \ddot{y} = f_+(x, y, \dot{y}) \\ (X_-) & \ddot{y} = f_-(x, y, \dot{y}) \end{array}. \tag{3.6}$$

have the corresponding form from Lemma 3.3 with possibly different constants C_+, λ_+ and C_-, λ_-.

To understand whether a system (3.6) is induced by a spray, we associate to a spray two more ODEs describing its geodesics reparametrized by the parameter y:

$$\begin{array}{ll} (Y_+) & \ddot{x} = g_+(x, y, \dot{x}) \\ (Y_-) & \ddot{x} = g_-(x, y, \dot{x}) \end{array}.$$

By a similar calculation as for (3.5), one finds that they are given by

$$\begin{array}{ll} \ddot{x} = 2G^2(x, y, +\dot{x}, +1)\dot{x} - 2G^1(x, y, +\dot{x}, +1) & (\dot{y} > 0) \\ \ddot{x} = 2G^2(x, y, -\dot{x}, -1)\dot{x} - 2G^1(x, y, -\dot{x}, -1) & (\dot{y} < 0) \end{array}.$$

and hence

$$g_\pm(x, y, z) = \begin{cases} -z^3 f_\pm(x, y, \frac{1}{z}) & \text{if } z \geq 0 \\ -z^3 f_\mp(x, y, \frac{1}{z}) & \text{if } z \leq 0 \end{cases}.$$

The functions $f_\pm, g_\pm$ must be defined and smooth at least on $U \times \mathbb{R}$ for some open subset $U \subseteq \mathbb{R}^2$ containing the origin. This excludes several possible ODEs:

For $D1$, already $f_\pm(x, y, z)$ are not defined for all z.

For $D2$ we have $f_+(x, y, z) = Cz^k$. If $k \in \{0, 1, 2, 3\}$ again the assumptions of Lemma 3.2 are fulfilled and the equations can be transformed to $\ddot{y} = 0$. If $k \neq 0, 1, 2, 3$, f_+ or its z-derivatives have a singularity at $z = 0$ unless k is a natural number. But then $g_+(x, y, z) = -Cz^{-k+3}$ for $z \geq 0$ has a necessary singularity at $z = 0$ and the same holds for $J1$.

For $J2$ we have $g_+(x,y,z) = -\frac{1}{2}z^2 - C_+e^{-2x}$ whenever $z \geq 0$ and $g_+(x,y,z) = -\frac{1}{2}z^2 - C_-e^{-2x}$ for $z \leq 0$, so that $C_+ = C_-$. Furthermore by the coordinate change $(x,y) \mapsto (x, \sqrt{2|C|}y)$ we can assume $C = \pm\frac{1}{2}$, and this ODEs are the induced ODEs of the sprays $(c^\pm)$ respectively.

For $C1$ by evaluating g_+ and g_- in $z = 0$ one finds $C_- = -C_+e^{\pi\lambda}$ and $C_+ = -C_-e^{\pi\lambda}$, which is only possible if $\lambda = 0$. By the coordinate change $(x,y) \mapsto (Cx, Cy)$ we can then assume $C_+ = -C_- = -1$ and the ODEs are exactly the induced ODEs of the spray (a).

For $C2$ similarly we find $C_- = -C_+$ and by $(x,y) \mapsto (x,-y)$ we can assume $C > 0$. The ODEs are exactly the ones induced by the spray $(b_k^\pm)$.

To end the proof of Lemma 3.4, two projective classes of sprays from the Lemma can not be transformed into each other by a coordinate transformation: This is obvious when the structure of the projective algebra $\mathfrak{p}$ with isotropy subalgebra $\mathfrak{p}_0$ is not isomorphic. We only need to distinguish (c^+) from (c^-) and (b_k^+) (and b_k^- respectively) for different $k > 0$. One can either do this by direct calculations or using the invariants for the induced ODEs from [29].

3.2.4 Construction of the metrics and end of the proof of Theorem 3.1

In this subsection we finish the proof of Theorem 3.1 by showing that the geodesic sprays of the Finsler metrics $(a, b_k^\pm, c^\pm)$ are projectively equivalent to the sprays from Lemma 3.4. That each metric is not isometric to any projectively equivalent to one of the others follows from the additional statement of Lemma 3.4.

Lemma 3.7. *The induced ODEs (3.5) of the geodesic spray of a Finsler metric F are given by*

$$\begin{array}{ll} \ddot{y} = \left.\dfrac{\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial v} - v\frac{\partial^2 F}{\partial y \partial v}}{\frac{\partial^2 F}{\partial v \partial v}}\right|_{(x,y,1,\dot{y})} & \dot{x} > 0 \\ \ddot{y} = \left.\dfrac{\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial v} - v\frac{\partial^2 F}{\partial y \partial v}}{\frac{\partial^2 F}{\partial v \partial v}}\right|_{(x,y,-1,-\dot{y})} & \dot{x} < 0 \end{array}.$$

Proof. If a curve $c : I \to \mathbb{R}^2$ is a geodesic of F, then by Lemma 2.4 (c) it is extremal for the functional $c \mapsto \int_a^b \frac{1}{2}F^2(c,\dot{c})dt$ and a solution to the Euler-Lagrange equations for the energy $E_i(\frac{1}{2}F^2, c) = 0$ and any orientation preserving reparametrization is a solution of the Euler-Lagrange equations for the metric itself, namely $E_i(F,c) = 0$.

Let $\tilde{c}(t) = (t, \tilde{y}(t))$ or $\tilde{c}(t) = (-t, \tilde{y}(-t))$ be an orientation preserving reparametrization of c. Then the Euler-Lagrange equation $E_2(F,\tilde{c}) = 0$ is satisfied, that is $\frac{\partial F}{\partial y} - \frac{\partial F}{\partial v \partial x}\dot{x} - \frac{\partial F}{\partial v \partial y}\dot{y} - \frac{\partial F}{\partial v \partial u}\ddot{x} - \frac{\partial F}{\partial v \partial v}\ddot{y} = 0$. Substituting the curve $\tilde{c}$ gives the equations. □

We now can calculate easily the induced ODEs of the Finsler metrics from Theorem 3.1 and see that they coincide with the ones induced by the sprays from 3.4 - thus the metrics are projective metrization of the corresponding sprays. Theorem 3.1 is proven. In the following we explain how these Finsler metrics were constructed.

Riemannian metrics

Using Lemma 3.7, one can easily calculate that the induced ODEs of a Riemannian metric are a polynomial of degree 3 in $\dot{y}$ with coefficients expressed by the Christoffel symbols of the Levi-Civita connection as in formula (3.2). Thus only the two sprays $(c^{\pm})$ can be projectively equivalent to the geodesic spray of a Riemannian metric.

In order to find such metrics g, we may use Lemma 3.3, which for a fixed induced ODE

$$\ddot{y} = K^0(x,y) + K^1(x,y)\dot{y} + K^2(x,y)\dot{y}^2 + K^3(x,y)\dot{y}^3$$

gives linear PDEs on the coefficients of the matrix $a = (\det g)^{-2/3} g$, that are necessary and sufficient for g to have the system of geodesics.

Note that one can reconstruct g from a by $g = \frac{1}{(\det a)^2} a$. Using the above Lemma, one can describe explicitly the 4-dimensional space of pseudo-Riemannian metrics whose geodesic spray is projectively related to the spray $(c^{\pm})$, in particular one finds the two locally Riemannian metrics

$$(c^+)\ \sqrt{\frac{e^{3x}}{(2e^x-1)^2}dx^2 + \frac{e^x}{2e^x-1}dy^2} \qquad (c^-)\ \sqrt{e^{3x}dx^2 + e^x dy^2},$$

whose geodesic spray is projectively equivalent to the sprays (c^+) and (c^-) respectively.

Randers metrics

Recall that a *Randers metric* $F = \alpha + \beta$ is given as the sum of a Riemannian norm and a 1-form, i.e. $F(x,\xi) := \sqrt{\alpha_x(\xi,\xi)} + \beta_x(\xi)$, where α is a Riemannian metric and β a 1-form.

We now explain how to construct Finsler metrics whose geodesic spray is projectively equivalent to the remaining (a) and $(b_k^{\pm})$. Starting with a Riemannian metric α of constant sectional curvature and hence with 3-dimensional Killing algebra $\mathfrak{iso}(\alpha)$, we construct a Randers metric F whose geodesics are curves of constant geodesic curvature k with respect to α. Their geodesic spray will be projectively equivalent to (a) (for the Euclidean metric and $k = 1$), to (b_k^+) for the standard metric on the two-sphere (in stereographic coordinates) and to (b_k^-) for the metric of the hyperbolic plane (in the Poincare disk model).

Let $\alpha = \alpha_{ij} dx^i dx^j$ be a Riemannian metric on the plane and choose a multiple of the Riemannian volume form $\Omega_x = -k\sqrt{\det \alpha_x}\, dx^1 \wedge dx^2$ with $k > 0$. Since we may work on a simply connected neighborhood, any 2-form is exact and we can choose a 1-form $\beta = \beta_j dx^j$, whose exterior derivative is Ω.

Lemma 3.8. *The projective algebra of the Randers metric* $F(x,\xi) := \sqrt{\alpha_x(\xi,\xi)} + \beta_x(\xi)$ *contains the Killing algebra of* α*. Its geodesics are exactly the positively oriented curves of constant geodesic curvature* k *with respect to* α*.*

Proof. Both α and Ω induce a natural bundle isomorphism $\phi_\alpha, \phi_\Omega : T\mathbb{R}^2 \to T^*\mathbb{R}^2$, which in coordinates are given by $(x,\xi) \mapsto (x, \alpha\xi)$ and $(x,\xi) \mapsto (x, \Omega\xi)$, where $\alpha = (\alpha_{ij})$ and $\Omega = (\Omega_{ij})$ are the Gramian matrices wrt. the fixed coordinates. The map $J := \phi_\alpha^{-1} \circ \phi_\Omega$ is given by $\xi \mapsto \alpha^{-1}\Omega\xi$ and is a bundle automorphism $T\mathbb{R}^2 \to T\mathbb{R}^2$ with $J^2 = -k^2 \mathrm{Id}$. Indeed, it is $\Omega\alpha^{-1}\Omega = -k^2 \det(\alpha)\det(\alpha^{-1})(\alpha^{-1})^{-1} = -k^2\alpha$.

Consider the Euler-Lagrange equations for $L(x,\xi) = \frac{1}{2}\alpha_x(\xi,\xi) + \beta_x(\xi)$. We first calculate $E_i(\beta,c) = \frac{\partial\beta_j}{\partial x^i}\dot{c}^j - \frac{\partial\beta_i}{\partial x^j}\dot{c}^j = (\Omega\dot{c})_i$. Contracting the equations $E_j(L,c)=0$ with α^{ij} and using the Levi-Civita connection ∇ of α, gives

$$\alpha^{ij}E_j(L) = \alpha^{ij}E_i(\tfrac{1}{2}\alpha^2, c) + \alpha^{ij}E_i(\beta,c) = -(\nabla_{\dot{c}}\dot{c})^i + \alpha^{ij}\Omega_{jk}\dot{c}^k,$$

so that the system $E_i(L,c)=0$ is equivalent to

$$\nabla_{\dot{c}}\dot{c} = \alpha^{-1}\Omega\dot{c} = J\dot{c}. \tag{3.7}$$

Note that the solutions c of equation (3.7) have constant α-velocity, since $\frac{1}{2}\frac{d}{dt}\alpha(\dot{c},\dot{c}) = \alpha(\dot{c},\nabla_{\dot{c}}\dot{c}) = \alpha(\dot{c},J\dot{c}) = \Omega(\dot{c},\dot{c}) = 0$. Furthermore the equation is preserved under α-isometries and the flow of any Killing vector field takes each α-unit speed solution to a α-unit speed solution.

Now consider the Euler-Lagrange equation of the Randers metric

$$F(x,\xi) = \sqrt{\alpha_x(\xi,\xi)} + \beta_x(\xi).$$

We claim that every α-unit speed solutions of equation (3.7) if up to orientation preserving reparametrization a geodesic of F. Indeed, if c is such a curve, then as in Lemma 2.4

$$E_i(\tfrac{1}{2}\alpha^2, c) = \alpha\cdot(\partial_{x^i}\alpha) - \frac{d}{dt}\Big(\alpha\cdot(\partial_{\xi^i}\alpha)\Big) = E_i(\alpha,c)$$

and hence

$$E_i(F,c) = E_i(\alpha,c) + E_i(\beta,c) = E_i(\frac{1}{2}\alpha^2,c) + E_i(\beta,c) = E_i(L,c) = 0.$$

Since the family of solutions to $E_i(F,c)=0$ are exactly all orientation preserving reparametrizations of the solutions of $E_i(\frac{1}{2}F^2,c)=0$, we see that the geodesics of F are exactly the α-unit speed solutions to equation (3.7) reparametrized to F-arc length. In particular every isometry of α preserves the geodesics of F as oriented point sets and we have $\mathfrak{iso}(\alpha)\subseteq\mathfrak{p}(F)$.

The geodesics have constant geodesic curvature $\kappa_\alpha = k$, since for their α-unit speed parametrization c we have

$$\kappa_\alpha(c)^2 = \alpha(\nabla_{\dot{c}}\dot{c},\nabla_{\dot{c}}\dot{c}) = \alpha(J\dot{c},J\dot{c}) = -\dot{c}^t\Omega\alpha^{-1}\Omega\dot{c} = k^2\cdot\alpha(\dot{c},\dot{c}) = k^2.$$

□

To produce Randers metrics with three dimensional projective algebra, we may choose α as a metric of constant curvature, since each of them admits a 3-dimensional Killing algebra. For the Euclidean metric dx^2+dy^2 and volume form $-dx\wedge dy$, we might choose $\beta = \frac{1}{2}(ydx - xdy)$. For the spherical and hyperbolic metric $\frac{dx^2+dy^2}{(1\pm(x^2+y^2))^2}$ with volume form $\frac{-k}{(1\pm(x^2+y^2))^2}dx\wedge dy$, we might choose $\beta = \frac{k}{2}\frac{ydx-xdy}{1\pm(x^2+y^2)}$.

The result are the Randers metrics (a) and $(b_k^\pm)$ from Theorem 3.1. By Lemma 3.8, we have $\mathfrak{iso}(\alpha)\subseteq\mathfrak{p}(F)$. In fact we have $\mathfrak{iso}(\alpha)=\mathfrak{p}(F)$: otherwise $\dim\mathfrak{p}(F)>3$ and F would be projectively equivalent to the Euclidean metric by Lemma 3.4 and in particular geodesically reversible, which obviously is not the case.

3.2.5 Discussion of rigidity

Theorem 3.1 rises the question how rigid the found Finsler metrics are - in other words, are there other Finsler metrics whose geodesic spray is projectively equivalent to the ones from Lemma 3.4? This problem of *projective metrization* will be discussed at length in Chapter 4.

Recall that there is always a trivial freedom of scaling and adding a closed 1-form: Suppose two Finsler metrics $F, \tilde{F}$ are related by $\tilde{F} = \lambda F + \beta$ for some $\lambda > 0$ and a 1-form β with $d\beta = 0$, then F and $\tilde{F}$ are projectively equivalent (see Example 2.2).

We will see that for the *geodesically reversible* Finsler metrics $(c^{\pm})$ there are many non-trivially projectively equivalent Finsler metrics: at least for every function on the space of unoriented geodesics one can construct a non-trivial Finsler metric projectively equivalent to the original one, cf. Theorem 4.4.

For Finsler metrics with irreversible geodesics the situation is quite different: already for the Finsler metric (a) it is not easy to find a non-trivially projectively equivalent Finsler metric. In [55] (see also Section 2.2.7) it was proven that two proper Randers metrics are projectively equivalent if and only if they are trivially related. However it was noted in [77] that every smooth density function on the plane (seen as the space of geodesics oriented geodesics of the spray (a)) such that every ball of radius 1 has measure 1, gives rise to a non-trivial Finsler metric projectively equivalent to the metric (a). This fact is recalled in Section 4.3.1 and generalized for arbitrary sprays by Theorem 4.4.

The question whether non-constant measures on the plane exist, whose integral over every ball of radius 1 is 1, is the so called Pompeiu problem and has a positive answer, hence giving rise to new Finsler metrics projectively equivalent to the metric (a). However, the solutions are not particularly easy to construct.

Chapter 4

The problem of local projective metrization in dimension two

Given a spray and a point on a manifold, is there a Finsler metric defined *fiber-globally* over a neighborhood of the point whose geodesic spray is projectively equivalent to the prescribed spray? If yes, how unique is this metric? This is the problem of local projective Finsler metrization. In this chapter we consider this problem locally and in dimension two.

We recall two approaches present in the literature: the first is a rather analytical investigation of the Euler-Lagrange equations for a Finsler metric and reformulates the condition of being a projective metrization in terms of linear PDEs on the metric, which can be reduced to *first order linear* PDEs on the Hessian of the metric; the second is a geometric construction specific to the 2-dimensional case, which implicitly produces a family of projective metrizations for any *reversible* spray.

Combining them allows us to generalize the geometric approach to the irreversible situation: we show that the local projective metrizations of a fixed spray are in 1-to-1 correspondence to measures on the space of oriented geodesics (Theorem 4.4) satisfying an equilibrium condition. It follows that every reversible spray is projectively metrizable and there is a large freedom in the choice of the metric. In the irreversible case, the situation is more complex and it remains unknown, whether any spray is locally projectively metrizable.

4.1 The PDE for projective metrization

4.1.1 The general case

Let S be a spray on a n-dimensional manifold M given by $S = \xi^i \partial_{x^i} - 2G^i \partial_{\xi^i}$ in local coordinates and denote for a Finsler metric $F : TM\backslash 0 \to \mathbb{R}$ its Hessian with respect to the local coordinates by

$$h_{ij} : TM\backslash 0 \to \mathbb{R}, \qquad h_{ij} := F_{\xi^i \xi^j} \qquad i, j \in \{1, .., n\}.$$

The following important second order linear PDEs on F for the projective metrization problem were obtained by A. Rapcsák [68]:

Lemma 4.1 (Projective Metrization PDEs)**.** *A Finsler metric F is a projective metrization of the spray S, if and only if for all $i \in \{1, .., n\}$ it holds*

$$F_{x^i} - F_{\xi^i x^\ell}\xi^\ell + 2G^\ell F_{\xi^i \xi^\ell} = 0. \tag{4.1}$$

This is the case, if and only if for all $i, j \in \{1, ..n\}$ it holds

$$F_{\xi^i x^j} - G^\ell_{\xi^j} h_{i\ell} = F_{\xi^j x^i} - G^\ell_{\xi^i} h_{j\ell}. \tag{4.2}$$

Proof. By Lemma 2.4, a Finsler metric F is a projective metrization of a spray S, if and only if every geodesic c of S, that is which satisfies in local coordinates

$$\ddot{c}^i + 2G^i(c, \dot{c}) = 0,$$

is a solution to the Euler-Lagrange equations $E_i(F, c) = 0$. More explicitly, this equation is given by

$$E_i(F, c) = F_{x^i} - F_{\xi^i x^\ell}\dot{c}^\ell - F_{\xi^i \xi^\ell}\ddot{c}^\ell = 0.$$

Suppose F is a projective metrization of S. As every $(x, \xi) \in TM\backslash 0$ is attained by some geodesic c of S, we obtain equation (4.1). On the other hand, if this equation is fulfilled, then every geodesic of S solves the Euler-Lagrange equations $E_i(F, c) = 0$. Thus F is a projective metrization of S, if and only if it fulfils (4.1).

To see that (4.1) implies (4.2), differentiate the first by ξ^j and change sign to obtain

$$\begin{aligned} 0 &= -\Big(F_{x^i\xi^j} - F_{\xi^i\xi^j x^\ell}\xi^\ell - F_{\xi^i x^j} + 2G^\ell (h_{i\ell})_{\xi^j} + 2G^\ell_{\xi^j} h_{i\ell}\Big) \\ &= (F_{\xi^i x^j} - F_{x^i \xi^j}) + (h_{ij})_{x^\ell}\xi^\ell - 2G^\ell (h_{ij})_{\xi^\ell} - 2G^\ell_{\xi^j} h_{i\ell}. \end{aligned} \tag{4.3}$$

The part antisymmetric in (i, j) gives equation (4.2). On the other hand, if F satisfies (4.2), contracting with ξ^j gives (4.1), as by homogeneity we have the relations $G^\ell_{\xi^j}\xi^j = 2G^\ell$, $F_{\xi^j x^i}\xi^j = F_{x^i}$ and $h_{jl}\xi^j = 0$. □

In [26], from these two equations a system of algebraic conditions and first order linear PDEs on the Hessian h_{ij} were obtained, that are necessary, and sufficient for F being a projective metrization of S up to addition of a 1-form. The theorem below is their result in a modified form. Denote by $G^i_j := (G^i)_{\xi^j}$ and $G^i_{jk} := (G^i)_{\xi^j \xi^k}$ the derivatives of the spray coefficients in local coordinates with respect to the fiber coordinates and define the quantities

$$R^\ell_{jk} = (G^\ell_k)_{x^j} - (G^\ell_j)_{x^k} - G^r_j G^\ell_{kr} + G^r_k G^\ell_{jr}.$$

These quantities can also be calculated from the components of the Riemann curvature tensor R^i_j of the spray via $R^i_{jk} := \frac{1}{3}\big((R^i_j)_{\xi^k} - (R^i_k)_{\xi^j}\big)$ and are antisymmetric in j and k.

Theorem 4.1. *Let F be a Finsler metric and S be a spray. There exists a 1-form β such that $F + \beta$ is a local projective metrization of the spray S, if and only if in all local coordinates the Hessian h_{ij} of F satisfies for all $i, j, k \in \{1, ..n\}$ the equations*

$$S(h_{ij}) - G^\ell_i h_{\ell j} - G^\ell_j h_{\ell i} = 0 \tag{4.4}$$

$$h_{i\ell}R^\ell_{jk} + h_{j\ell}R^\ell_{ki} + h_{k\ell}R^\ell_{ij} = 0. \tag{4.5}$$

Here $S(h_{ij})$ denotes the derivative of the function h_{ij} in the direction of S, so that equation (4.4) reads more explicitly $\xi^\ell (h_{ij})_{x^\ell} - 2G^\ell (h_{ij})_{\xi^\ell} - G^k_i h_{kj} - G^k_j h_{ki} = 0$. In [26] it is shown that equation (4.5) is equivalent to the equation $h_{i\ell} W^\ell_j = h_{j\ell} W^\ell_i$, where W^i_k is the Weyl curvature of the spray S - an important projective invariant of sprays.

Proof of Theorem 4.1. In the proof we will use the so called *horizontal vector fields* $H_k := \partial_{x^k} - G^r_k \partial_{\xi^r}$, which together with the vector fields ∂_{ξ^k} form a basis of $T_{(x,\xi)}(TM\backslash 0)$ at each $(x, \xi) \in TM\backslash 0$. First note that the Lie bracket of the horizontal fields is given by the quantities R^ℓ_{jk} via

$$\begin{aligned} [H_j, H_k] &= -\Big((G^\ell_k)_{x^j} - G^r_j G^\ell_{kr} - (G^\ell_j)_{x^k} + G^r_k G^\ell_{jr}\Big) \\ &= -R^i_{jk} \partial_{\xi^i}. \end{aligned} \tag{4.6}$$

Suppose that $\tilde{F} = F + \beta$ is a projective metrization of S. The Hessians of F and $\tilde{F}$ coincide; denote them by h_{ij}. Then $\tilde{F}$ satisfies (4.1) and the derivatives of this equation by the fiber coordinates ξ^j are given by (4.3). The part symmetric in (i, j) gives (4.4):

$$\begin{aligned} 0 &= (h_{ij})_{x^\ell} \xi^\ell - 2G^\ell (h_{ij})_{\xi^\ell} - G^\ell_i h_{j\ell} - G^\ell_j h_{i\ell} \\ &= S(h_{ij}) - G^\ell_i h_{j\ell} - G^\ell_j h_{i\ell}. \end{aligned}$$

The metric $\tilde{F}$ also satisfies (4.2) which using the horizontal vector fields becomes

$$H_i(\tilde{F}_{\xi^j}) - H_j(\tilde{F}_{\xi^i}) = 0.$$

Apply H_k to this equation and take the cyclic sum over (i, j, k) to obtain

$$[H_k, H_i](\tilde{F}_{\xi^j}) + [H_j, H_k](\tilde{F}_{\xi^i}) + [H_i, H_j](\tilde{F}_{\xi^k}) = 0$$

and by (4.6) we obtain the desired equation (4.5).

Suppose on the other hand that the Hessian of a Finsler metric F satisfies (4.4) and (4.5). Then the projective metrization PDE (4.1) is satisfied up to a term independent of the fiber coordinates ξ^i by trivial index symmetry observations: Indeed, if we denote by A_i the left hand side of (4.1), then by (4.4) we have $(A_i)_{\xi^j} + (A_j)_{\xi^i} = 0$. Equation (4.2) is equivalent to $(A_i)_{\xi^j} - (A_j)_{\xi^i} = 0$ and $\big((A_i)_{\xi^j} - (A_j)_{\xi^i}\big)_{\xi^k} = -(A_k)_{\xi^i \xi^j} + (A_k)_{\xi^i \xi^j} = 0$.

Thus we have for some functions $c_{ij} : M \to \mathbb{R}$

$$F_{\xi^i x^j} - F_{\xi^j x^i} - G^\ell_j h_{i\ell} + G^\ell_i h_{j\ell} = c_{ij}(x).$$

Adding a 1-form $\beta = \beta_\ell(x) dx^\ell$ to F will add $(\beta_i)_{x^j} - (\beta_j)_{x^i}$ to the left hand side. Thus we can find locally on M a 1-form, such that $\tilde{F} = F + \beta$ satisfies the projective metrization equation (4.2), if and only if the 2-form $\gamma = c_{ij}(x) dx^i \wedge dx^j$ is closed. This is the case since $c_{ij} = (A_i)_{\xi^j} - (A_j)_{\xi^i} = H_i(F_{\xi^j}) - H_j(F_{\xi^i})$ and the components of $d\gamma$ are given by

$$\begin{aligned} &\partial_{x^k} c_{ij} + \partial_{x^j} c_{ki} + \partial_{x^i} c_{jk} \\ &= H_k\Big(H_i(F_{\xi^j}) - H_j(F_{\xi^i})\Big) + H_j\Big(H_k(F_{\xi^i}) - H_i(F_{\xi^k})\Big) + H_i\Big(H_j(F_{\xi^k}) - H_k(F_{\xi^j})\Big) \\ &= [H_k, H_i](F_{\xi^j}) + [H_j, H_k](F_{\xi^i}) + [H_i, H_j](F_{\xi^k}), \end{aligned}$$

which vanishes by the relation of H_i and R^i_{jk} and (4.5) as before.

Thus, $\tilde{F}$ satisfies the PDEs for being a projective metrization and it remains to ensure positivity and strict convexity of $\tilde{F}$ around a fixed point $x_0 \in M$. To that end, we show that we can add a closed 1-form to $\tilde{F}$, to make the sum positive on $T_{x_0}M\backslash 0$ and thus on TU for a neighborhood U of x_0. This operation does not change the Euler-Lagrange equations $E_i(F, c) = 0$ by Example 2.2 and thus does not effect equation (4.5).

By the inequality (c) from Lemma 2.2 we have

$$F_{\xi^i}(x_0, \eta)\xi^i \leq F(x_0, \xi)$$

for all $\xi, \eta \in T_{x_0}M\backslash 0$ with equality if and only if ξ is a non-negative multiple of η and the same holds for $\tilde{F}$. Thus for fixed $\eta_0 \in T_{x_0}M\backslash 0$ the function $\xi \mapsto \tilde{F}(x_0, \xi) - \tilde{F}_{\xi^i}(x_0, \eta_0)\xi^i$ restricted to $T_{x_0}M$ is positive for all ξ that are not non-negative multiples of η_0 and zero for those. Now it is easy to see that there are constants $b_i \in \mathbb{R}$, such that

$$\hat{F}(x, \xi) := \tilde{F}(x, \xi) - \tilde{F}_{\xi^i}(x_0, \eta_0)\xi^i + b_i\xi^i$$

is positive on $T_{x_0}M\backslash 0$ and thus on a neighborhood U of x_0.

The new $\hat{F}$ differs from $\tilde{F}$ by a closed 1-form and $\hat{F}$ still solves the projective metrization PDEs. By Lemma 2.1 (b), the Hessian of F and thus the Hessian of $\hat{F}$ is positive quasi-definite and by Lemma 2.1 (a) it is $\hat{g}_{ij} = \hat{F}_{\xi^i}\hat{F}_{\xi^j} + \hat{F}h_{ij}$, which together with positivity implies that the fundamental tensor $\hat{g}$ of $\hat{F}$ is positive definite on $TU\backslash 0$. $\square$

We call a collection of functions (h_{ij}) *admissible* for a spray S, if they satisfy (4.4) and (4.5) together with the obvious properties of a Hessian of a Finsler metric, namely

$$\begin{gathered} h_{ij} = h_{ji} \qquad (h_{ij})_{\xi^k} = (h_{ik})_{\xi^j} \qquad h_{ij}\xi^j = 0 \\ h_{ij}(x, \xi)\nu^i\nu^j \geq 0 \text{ with equality only if } \eta \text{ is a multiple of } \xi. \end{gathered} \tag{4.7}$$

In dimension $n \geq 3$, existence of an admissible collection (h_{ij}) is not only necessary for local projective metrizability, but also sufficient. Indeed, as a fixed tangent space $T_xM\backslash 0$ is simply connected, by the first two additional properties there exists a function $F : TM \to \mathbb{R}$, whose Hessian is given by (h_{ij}). By the third condition, it can be chosen to be 1-homogeneous, and by Theorem 4.1, there exists a 1-form whose addition will turn it into a solution of the projective metrization PDEs. By the fourth condition and an argument similar to the one in the proof, one can ensure local positivity by addition of a closed 1-form, which together with quasi-definiteness of h implies local strict convexity.

The same is not true in dimension two, because each $T_xM\backslash 0$ is homotopy equivalent to S^1 and thus not simply connected. An additional integral condition discussed in the next section must be imposed to ensure that (h_{ij}) is fiber-globally the Hessian of some Finsler metric.

Corollary 4.1. *In dimension $n \geq 3$, there exist sprays that are even fiber-locally not projectively metrizable.*

Proof. The equations (4.4) are a system of ODEs along each integral curve and define h completely given a starting value in a point (x, ξ). Equation (4.5) does give for $n \geq 3$ at least one algebraic relation among the components of h that is generically not compatible with the system.

For a concrete example, consider the following spray (borrowed from [31]) over $\mathbb{R}^n$:

$$S = \xi^i\partial_{x^i} + \Big((x^2)^2 + (x^3)^2\Big)(\xi^1)^2\partial_{\xi^2}.$$

Then by (4.4) we have $S(h_{22}) = S(h_{23}) = 0$. By (4.5) we obtain $h_{2k} \equiv 0$ for $k \geq 4$ and $x_2 h_{23} = x_3 h_{22}$. Applying S to both sides, we obtain $\xi^2 h_{23} = \xi^3 h_{22}$ and it follows that $x_2 \xi^3 h_{22} = x_2 \xi^2 h_{23} = x_3 \xi^2 h_{22}$, so that $h_{22} \equiv h_{23} \equiv 0$, and thus $h_{12} \equiv 0$ by homogeneity. Thus h cannot satisfy (4.7) and S is not projectively metrizable on any open, non-empty subset of $T\mathbb{R}^n$. □

4.1.2 The 2-dimensional case

Let us now specify to dimension two. In this case, condition (4.5) is always satisfied. Indeed, two of the three indices must coincide and by the antisymmetry of R^i_{jk} the equation vanishes identically.

Theorem 4.1 then implies that there is always a fiber-local Finsler function satisfying the projective metrization equation (4.1), as (4.4) is just an ODE along the integral curves of S. However, we are interested in local, but fiber-global projective metrizations, and it is unknown whether in dimension two every spray admits a local, fiber-global projective metrization.

Note that the equations (4.1) are always dependent by homogeneity and thus in dimension two, there is only one independent equation. Indeed, contracting (4.1) with ξ^i gives

$$\Big(F_{x^i} - F_{\xi^i x^\ell}\xi^\ell + 2G^\ell F_{\xi^i \xi^\ell}\Big)\xi^i = F_{x^i}\xi^i - F_{x^\ell}\xi^\ell = 0.$$

Let (x, y, u, v) be local coordinates on TM and introduce polar coordinates $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$ for the fibers of $TM\backslash 0$. We have the standard relations

$$\partial_u = -\tfrac{1}{r}\sin\phi\,\partial_\phi + \cos\phi\,\partial_r \qquad \text{and} \qquad \partial_v = \tfrac{1}{r}\cos\phi\,\partial_\phi + \sin\phi\,\partial_r.$$

Then, the Liouville vector field and the spray S are expressed in the polar coordinates by

$$\begin{aligned} V|_{(x,y,\phi,r)} &= r\partial_r \\ S|_{(x,y,\phi,r)} &= r\cos\phi\,\partial_x + r\sin\phi\,\partial_y + \frac{2G^1\sin\phi - 2G^2\cos\phi}{r}\partial_\phi - (2G^1\cos\phi + 2G^2\sin\phi)\partial_r. \end{aligned}$$

Define $G : TM\backslash 0 \to \mathbb{R}$ as the 0-homogeneous function $G := \frac{2G^1\sin\phi - 2G^2\cos\phi}{r^2}$. As the addition of a multiple of the Liouville vector field does not change the projective class of S, we might assume that the spray that we want to metrize projectively is given by

$$S = r(\cos\phi\,\partial_x + \sin\phi\,\partial_y + G\partial_\phi).$$

Let us write down the PDE for projective metrization. The first equation of (4.1) is

$$\begin{aligned} 0 &= F_x - F_{ux}u - F_{uy}v + 2G^1 F_{uu} + 2G^2 F_{uv} \\ &= (F_{vx} - F_{uy})v + 2G^1 F_{uu} + 2G^2 F_{uv}. \end{aligned}$$

Using $\partial_r F = \frac{1}{r}F$, this is rewritten in polar coordinates as

$$\sin\phi\Big(\sin\phi F_x - \cos\phi F_y + \cos\phi F_{x\phi} + \sin\phi F_{y\phi} + \big(\frac{2G^1}{r^2}\sin\phi - \frac{2G^2}{r^2}\cos\phi\big)(F + F_{\phi\phi})\Big) = 0.$$

Introducing the vector fields

$$T(\phi) := \cos\phi\partial_x + \sin\phi\partial_y \qquad \text{and} \qquad N(\phi) := T(\phi + \tfrac{\pi}{2}) = -\sin\phi\partial_x + \cos\phi\partial_y$$

and denoting by $D_X f := X(f)$ the derivative of the function F in the direction of a vector field X, we obtain a single projective metrization PDE in polar coordinates:

Lemma 4.2 (Projective Metrization PDE in dimension 2)**.** *A Finsler metric $F : T\mathcal{S} \to \mathbb{R}$ on a surface is a projective metrization of a spray S, if and only if in all local polar coordinates the following equation holds in all points $(x, y, \phi, r) \in T\mathcal{S}\backslash 0$:*

$$-D_{N(\phi)}F + D_{T(\phi)}F_\phi + (F + F_{\phi\phi})G = 0. \tag{4.8}$$

Differentiating the PDE (4.8) on F by ϕ similarly to the proof of Theorem 4.1, we obtain the following PDE on the term $F+F_{\phi\phi}$, which is necessary for F to be a projective metrization of S:

$$D_{T(\phi)}(F + F_{\phi\phi}) + G(F + F_{\phi\phi})_\phi + (F + F_{\phi\phi})G_\phi = 0,$$

or in similarity with (4.4) using the above obtained expression for S in polar coordinates

$$S(F + F_{\phi\phi}) + (F + F_{\phi\phi})G_\phi = 0. \tag{4.9}$$

The significance of the term $F + F_{\phi\phi}$ is that it determines the Hessian. Calculating $h_{ij} = F_{\xi^i\xi^j}$ in polar coordinates gives

$$(h_{ij})|_{(x,y,r,\phi)} = \frac{F + F_{\phi\phi}}{r^2}\begin{pmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{pmatrix} \tag{4.10}$$

and we note that $\mathrm{tr}(h) = h_{11} + h_{22} = \frac{F+F_{\phi\phi}}{r^2}$.

Also strict convexity of the metric can be expressed in terms of $F + F_{\phi\phi}$:

Lemma 4.3. *Let $\mathcal{S}$ be a surface and $F : T\mathcal{S}\backslash 0 \to \mathbb{R}$ be a smooth, positively 1-homogeneous function. Then F is a Finsler metric, if and only if $F > 0$ and $F + F_{\phi\phi} > 0$.*

Proof. By Lemma 2.1 (a), we have $g_{ij} = F_{\xi^i}F_{\xi^j} + Fh_{ij}$ and as a consequence the determinant of the fundamental tensor is $\det(g_{ij}) = \frac{1}{r^4}F^3(F + F_{\phi\phi})$. Indeed, we have

$$\begin{aligned}\det(g_{ij}) = \det(F_{\xi^i}F_{\xi^j} + Fh_{ij}) &= F(F_{\xi^1}^2 h_{22} - 2F_{\xi^1}F_{\xi^2}h_{12} + F_{\xi^2}^2 h_{11}) + F^2\det(h_{ij}) \\ &= \tfrac{1}{r^4}F^3(F + F_{\phi\phi}).\end{aligned}$$

The matrix g_{ij} is positive definite, if and only if $\det(g_{ij})$ and g_{11} are positive. Thus assuming F is positive and using (4.10), F is strictly convex if and only if $F + F_{\phi\phi}$ is positive. ☐

To produce a projective metrization of a spray S, we could first try to solve the linear, first order equation

$$S(\tau) + \tau G_\phi = 0. \tag{4.11}$$

for a positive, 1-homogeneous solution $\tau : T\mathcal{S}\backslash 0 \to \mathbb{R}$ and then find the function $F : T\mathcal{S}\backslash 0 \to \mathbb{R}$, such that $F + F_{\phi\phi} = \tau$, which then by (4.9) is a projective metrization of S.

However, having a solution τ to equation (4.11) does not ensure local, fiber-global projective metrizability - to ensure existence of a function $F : T\mathcal{S}\backslash 0 \to \mathbb{R}$, such that $F + F_{\phi\phi} = \tau$, it is necessary and sufficient that

$$\int_0^{2\pi} \sin(\phi - \theta)\tau(x, y, \theta, r)d\theta = 0 \qquad \text{for all } (x, y, \phi, r) \in T\mathcal{S}\backslash 0.$$

Indeed, if $\tau = F + F_{\phi\phi}$, by

$$\Big(F(\theta)\cos(\phi - \theta) + F_\phi(\theta)\sin(\phi - \theta)\Big)_\theta = \sin(\phi - \theta)\big(F(\theta) + F_{\phi\phi}(\theta)\big)$$

the integral vanishes. If on the other hand the integral condition is satisfied, then the function $F(x, y, \phi, r) := \int_0^\phi \sin(\phi-\theta)\tau(x, y, \theta, r)d\theta$ is well-defined, has the correct Hessian and can be turned into a projective metrization by the addition of a 1-form:

The left side of the projective metrizability equation (4.8) for F is a term $r \cdot k(x, y)$ independent of ϕ, which can be taken care of by addition of a closed 1-form similarly as in Theorem 4.1: As the equation is linear, the addition of a 1-form $\beta = b(x, y) \cdot r \cos\phi$ to F adds the term

$$r\Big(\sin\phi b_x \cos\phi - \cos\phi b_y \cos\phi + \cos\phi b_x(-\sin\phi) + \sin\phi b_y(-\sin\phi)\Big) = -rb_y$$

to the left side, so that for choosing $b := \int k(x, y)dy$, the function $\tilde{F} := F + \beta$ satisfies the projective metrizability equation (4.8). If $\tilde{F}$ is positive, then F is a Finsler metric by Lemma 4.3. Otherwise, a closed 1-form can be added (similarly to the last part of the proof of Theorem 4.1) to ensure positivity locally, without affecting the projective metrization PDE (4.8).

Anyway, in general it is difficult to find a fiber-global solution τ to (4.11) satisfying the integral condition to produce a fiber-global projective metrization - and it is not clear whether this is possible locally for any spray.

Example 4.1 (Projective metrization of sprays with 2-dimensional projective algebra)**.** *Let $S = u\partial_x + v\partial_y - 2G^1\partial_u - 2G^2\partial_v$ be a spray on the plane that admits exactly two independent, transitive projective vector fields (see Section 3.1.1). As there are only two abstract 2-dimensional Lie algebras, by Lemma 3.5 we might choose coordinates in which*

$$\mathfrak{p}(S) = \langle\partial_x, \partial_y\rangle \qquad \text{or} \qquad \mathfrak{p}(S) = \langle\partial_x, x\partial_x + y\partial_y\rangle.$$

By Lemma 3.1, $\mathcal{L}_{\hat{X}}S$ is collinear to the Liouville vector field $u\partial_u + v\partial_v$ in each point $(x, \xi) \in T\mathbb{R}^2\backslash 0$. In the Abelian case, this implies that for some function $\lambda : T\mathbb{R}^2\backslash 0 \to \mathbb{R}$

$$\begin{pmatrix} G^1(x, y, u, v) \\ G^2(x, y, u, v) \end{pmatrix} = \lambda(x, y, u, v) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \tilde{G}^1(u, v) \\ \tilde{G}^2(u, v) \end{pmatrix},$$

and thus, S is projectively equivalent to a spray whose spray coefficients are independent of (x, y) - assume this is already the case for S as we are interested in projective metrizations.

Similarly for the non-Abelian case, we might assume that the coefficients are independent of x and by the condition for the second vector field, we get that $\begin{pmatrix} G^1_y y + G^1 \\ G^2_y y + G^2 \end{pmatrix}$

must be proportional to the Liouville vector field. By addition of a multiple of the Liouville vector field to S, we might assume that the above vector vanishes, that is that

$$\begin{pmatrix} G^1(x,y,u,v) \\ G^2(x,y,u,v) \end{pmatrix} = \frac{1}{y} \begin{pmatrix} H^1(u,v) \\ H^2(u,v) \end{pmatrix}.$$

Let us now try to find some easy projective metrizations. For the Abelian case, we might try to find a metrization, whose Hessian described by $\tau = F + F_{\phi\phi}$ is independent of (x,y). In this case, equation (4.11) is simply

$$(G\tau)_\phi = 0.$$

We have a fiber-global solution, if and only if G does not vanish (or vanishes identically) and the solution is given by $\tau = \frac{r}{G}$ and can be made positive by changing sign if necessary. The condition that G does not vanish means that the spray does not contain a line as a geodesic in the chosen coordinates.

Thus the spray is projectively metrized by a Minkowski metric plus a 1-form, if and only if it does not contain a line in the chosen coordinates and $\int_0^{2\pi} \frac{\sin(\phi-\theta)}{G(\theta)} d\theta = 0$ for all (x,y,ϕ). One such metric is given by

$$F := r\Big(\int_0^\phi \frac{\sin(\phi-\theta)}{G(\theta)} d\theta + y \cdot \cos\phi \Big)$$

and any other has the same Hessian, thus differs by a 1-form, which by Example 2.2 must be closed.

Similarly, in the non-Abelian case we have $\tau = \frac{r}{y \cdot G}$ as a solution to equation (4.11), which is fiber-globally defined, as long as the spray does not contain a straight line as a geodesic in the chosen coordinates, implying the following projective metrization $F = r\Big(\int_0^\phi \frac{\sin(\phi-\theta)}{yG(y,\theta)} d\theta + \ln(y)\cos\phi \Big)$, as long as this function is 2π-periodic in ϕ. This F solves (4.8) and is again a Minkowski metric plus a 1-form.

However, we have not projectively metrized all sprays admitting a two-dimensional projective algebra - it is still possible that the sprays, for which the construction fails, can be metrized by a more complicated metric.

4.2 A geometric method for reversible sprays

The method of projective metrization to be presented in this section is specific to dimension two and was originally used by Herbert Busemann [20] to produce solutions to Hilberts Fourth problem of finding metrics whose geodesics are straight lines (see Section 2.2.7). As an application to the projective metrization problem for arbitrary *geodesically reversible* sprays, it was used in [4, 7]. There, the space of *unoriented* geodesics was used and thus *reversible* Finsler metrics were produced. We are going to present their approach in a slightly *generalized* form by working with *oriented* geodesics to produce also *irreversible* projective metrization.

Recall that we call a spray S *geodesically reversible*, if any orientation-reversed geodesic is - up to orientation preserving reparametrization - again a geodesic. Equivalently, S is geodesically reversible, if it is projectively equivalent to a reversible spray.

Let S be a geodesically reversible spray on $U \subseteq \mathbb{R}^2$ and assume that

- U is geodesically simple and convex, that is for every pair $x, y \in U$ there is exactly one geodesic γ_{xy} from x to y up to affine reparametrization,
- the set of unparametrized oriented geodesics Γ is a smooth 2-dimensional manifold,
- μ is a smooth positive measure on Γ with density function $f : \Gamma \to \mathbb{R}_{>0}$
- and $p : TU\backslash 0 \to \Gamma$ a map that assigns to a vector $\xi \in TU\backslash 0$ the unique unparametrized oriented geodesic γ that is tangent to ξ. We assume p to be a smooth submersion, such that $dp|_\xi$ has full rank at every $\xi \in TU\backslash 0$.

Locally on any surface $\mathcal{S}$, such quadruples (U, Γ, μ, p) exist for any spray: By Whitehead's Theorem 2.5, around every point we can find an open neighborhood U,

- that is geodesically simple and convex,
- whose boundary ∂U is a smooth submanifold diffeomorphic to S^1,
- and such that every geodesic in U intersects ∂U in exactly two points.

Thus, we can identify an unoriented geodesic in U by its first and second intersection with the boundary ∂U, and for every pair $(a, b) \in \Gamma := \{(c, d) \in \partial U \times \partial U \mid c \neq d\}$, there is a unique geodesic from a to b. This turns the space of oriented geodesics Γ into a 2-dimensional smooth manifold diffeomorphic to the cylinder $(S^1 \times S^1)\backslash\Delta$, where Δ is the diagonal $\Delta := \{(c, d) \in S^1 \times S^1 \mid c = d\}$. We can assume Γ to be embedded in $\mathbb{R}^2$.

Theorem 4.2. *For $x, y \in U$ let $[x, y] \subseteq \Gamma$ be the open set of geodesics $\gamma \in \Gamma$ that intersect the segment on the unique geodesic $\gamma_{x,y}$ from x to y between the two points with positive orientation. Then:*

1. *The function $d : U \times U \to \mathbb{R}$ defined by $d(x, y) := \mu([x, y])$ is a (not necessarily symmetric) distance function, for which the curves $\gamma \in \Gamma$ are exactly the shortest paths.*
2. *If d is smooth in all $(x, y) \in U \times U$ with $x \neq y$ and $F(x, \xi) := \frac{d}{dt}|_{t=0}\Big(d(x, x + t\xi)\Big)$ is a Finsler metric, then F is a projective metrization of the initial spray S.*

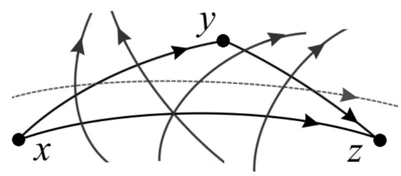


Figure 4.1: The (strict) triangle inequality for the distance function d.

Proof. Clearly, $d(x, y) \geq 0$ as μ is positive, and $d(x, y) = 0$ if and only if $[x, y]$ is a set of measure zero. But if $x \neq y$, the set $[x, y]$ is open and not empty. Thus $d(x, y) = 0$ if and only if $x = y$.

For the triangle inequality, let $x, y, z \in U$ be three points. We next show that if $\gamma \in [x, z]$, then $\gamma \in [x, y]$ or $\gamma \in [y, z]$ and thus

$$d(x, z) = \mu([x, z]) \leq \mu([x, y]) + \mu([y, z]) = d(x, y) + d(y, z),$$

with equality if and only if y lies on the segment of $\gamma_{x,z}$ between the two points. This will show that d is a distance function for which the geodesics of S are the shortest paths.

Indeed, as U is geodesically simple and convex and the spray is reversible, for two fixed points (p, q) the only geodesics γ that passing through both are $\gamma_{p,q}$ and $\gamma_{q,p}$, which have the same trajectory. As a consequence, two geodesics with different trajectories intersect in at most one point, and every $\gamma \in [x, z]$ must for obvious topological reasons intersect either the segment of $\gamma_{x,y}$ or $\gamma_{y,z}$ with positive orientation (see Figure 4.1).

Furthermore, if y lies on $\gamma_{x,z}$ on the segment between x and z, then we have $d(x, z) = d(x, y) + d(y, z)$, again because two geodesics with different trajectories intersect in at most one point. If y doesn't lie on the segment of $\gamma_{x,z}$ between the two points, by geodesical convexity the open and non-empty set $[y, x] \cap [y, z]$ does not intersect $[x, z]$ for obvious topological reasons and is contained in $[y, z]$. It follows that $d(x, z) < d(x, y) + d(y, z)$. □

Theorem 4.2 does not only imply that every *reversible* spray on a surface is locally, fiber-globally Finsler metrizable, but also that there is a large freedom in finding projective metrizations - namely choosing a positive smooth function f on the space of unoriented geodesics Γ.

Formula. Let us obtain a more concrete formula for the Finsler metric constructed in Theorem 4.2 (2). Consider $TU\backslash 0$ with polar coordinate for the fibers (x, y, ϕ, r) and assume $\Gamma \subseteq \mathbb{R}^2$ with coordinates (v, w). The submersion $p : TU\backslash 0 \to \Gamma$ satisfies $p(x, y, \phi, \lambda r) = p(x, y, \phi, r)$ for all $\lambda > 0$, so that we might suppress the last argument and see p as defined on the unit tangent bundle $SU := (TU\backslash 0)/\mathbb{R}_{>0}$.

Let $c : I \to U$ be a geodesic of the spray S, whose tangent $\dot{c}(t)$ is given by $\begin{pmatrix}\cos(\phi_c(t))\\ \sin(\phi_c(t))\end{pmatrix}$ and set $A_t := \Big\{\big(c(s),\theta\big) \mid s \in (0,t), \theta \in (\phi_c(s), \phi_c(s)+\pi)\Big\} \subseteq SU$. Then

$$\begin{aligned} d\Big(c(0), c(t)\Big) &= \mu\Big([c(0), c(t)]\Big) = \mu\Big(p(A_t)\Big) \\ &= \int_{p(A_t)} f(v,w)\; dvdw \\ &= \int_0^t \int_{\phi_c(s)}^{\phi_c(s)+\pi} f\Big(p\big(c(s),\theta\big)\Big) \Big| D_{\dot{c}(s)}p \;\; p_\phi \Big|_{(c(s),\theta)} d\theta ds, \end{aligned}$$

where $\Big|X \;\; Y\Big|$ denotes the determinant of the 2×2 matrix whose columns are X and Y. By differentiating we obtain the following formula for the constructed Finsler metric, where we use as before $T(\phi) = \cos\phi\partial_x + \sin\phi\partial_y$:

$$F(x,y,\phi,r) = r\int_\phi^{\phi+\pi} f\Big(p(x,y,\theta)\Big)\Big| D_{T(\phi)}p \;\; p_\phi\Big|_{(x,y,\theta)} d\theta. \tag{4.12}$$

Strict convexity. How are the spray S and the submersion p related? Recall from Section 4.1.2 that we can assume the spray S to be given in the polar coordinates by $S|_{(x,y,\phi,r)} = r\big(T(\phi) + G|_{(x,y,\phi)}\partial_\phi\big)$. Because $S(p) = 0$, the kernel of the differential $dp_{(x,y,\phi,r)}$ is spanned by $S|_{(x,y,\phi,r)}$ and in all (x,y,ϕ) it holds

$$D_{T(\phi)}p = -Gp_\phi. \tag{4.13}$$

As a consequence, the term $\delta(x,y,\phi,\theta) := \Big|D_{T(\phi)}p \;\; p_\phi\Big|_{(x,y,\theta)}$ vanishes, if and only if $\theta - \phi$ is an integer multiple of π, and in particular has constant sign for all $x, y \in U$ and $\theta \in (\phi, \phi+\pi)$ and opposite sign for $\theta \in (\phi+\pi, \phi+2\pi)$. By possibly changing the orientation of Γ, we might assume it to be positive for $\theta \in (\phi, \phi+\pi)$. Thus the metrics (4.12) are positive, and for strict convexity it is necessary and sufficient, that $F + F_{\phi\phi}$ is positive by Lemma 4.3. Using $\delta(x,y,\phi,\phi+\pi) = \delta(x,y,\phi,\phi) = 0$ we have

$$F_\phi = r\int_\phi^{\phi+\pi} f\Big(p(x,y,\theta)\Big)\Big|D_{N(\phi)}p \;\; p_\phi\Big|_{(x,y,\theta)} d\theta$$

and

$$F + F_{\phi\phi} = r\Big[f\Big(p(x,y,\theta)\Big)\Big|D_{N(\phi)}p \;\; p_\phi\Big|_{(x,y,\theta)}\Big]_{\theta=\phi}^{\theta=\phi+\pi}. \tag{4.14}$$

Because of

$$\Big|D_{N(\phi)}p \;\; p_\phi\Big|_{(x,y,\theta)} = \delta(x,y,\phi+\tfrac{\pi}{2},\theta), \tag{4.15}$$

it follows that $F + F_{\phi\phi} > 0$ and F is indeed a Finsler metric.

The Projective Metrization PDE (4.8). Theorem 4.2 claims that the constructed Finsler metrics (4.12) are projective metrizations of S, that is they solve the projective metrization PDE (4.8)

$$-D_{N(\phi)}F + D_{T(\phi)}F_\phi + (F + F_{\phi\phi})G = 0.$$

Let us see, why this is the case. We will use that in (x, y, θ) we have

$$\begin{aligned}
&D_{T(\phi)}(f \circ p)|D_{N(\phi)}p \ \ p_\phi| - D_{N(\phi)}(f \circ p)|D_{T(\phi)}p \ \ p_\phi| \\
&= df \circ dp\Big(|D_{N(\phi)}p \ \ p_\phi|T(\phi) - |D_{T(\phi)}p \ \ p_\phi|N(\phi)\Big) \\
&= df \circ dp\Big(|D_{N(\theta)}p \ \ p_\phi|T(\theta) - |D_{T(\theta)}p \ \ p_\phi|N(\theta)\Big) \\
&= df \circ dp\Big(|D_{N(\theta)}p \ \ p_\phi|(-G)\partial_\theta\Big) \qquad (4.16)\\
&= df \circ dp\Big(|D_{N(\theta)}p \ \ D_{T(\theta)}p|\partial_\theta\Big) \\
&= df \circ dp\Big(|D_{N(\phi)}p \ \ D_{T(\phi)}p|\partial_\theta\Big) \\
&= \partial_\theta(f \circ p)|D_{N(\phi)}p \ \ D_{T(\phi)}p|.
\end{aligned}$$

Using this, we can check that in $(x, y, \phi, r) \in TU\backslash 0$, we have

$$\begin{aligned}
-D_{N(\phi)}F + D_{T(\phi)}F_\phi &= r\int_\phi^{\phi+\pi} -D_{N(\phi)}(f \circ p)\left|D_{T(\phi)}p \quad p_\phi\right| - (f \circ p)\left|D_{T(\phi)}p \quad D_{N(\phi)}p_\phi\right| \\
&\quad + D_{T(\phi)}(f \circ p)\left|D_{N(\phi)}p \quad p_\phi\right| + (f \circ p)\left|D_{N(\phi)}p \quad D_{T(\phi)}p_\phi\right| d\theta \\
&\overset{(4.16)}{=} \int_\phi^{\phi+\pi} \partial_\theta\Big(f \circ p\big|D_{N(\phi)}p \quad D_{T(\phi)}p\big|\Big) d\theta \\
&= r\Big[f \circ p \cdot \big|D_{N(\phi)}p \quad D_{T(\phi)}p\big|\Big]_{\theta=\phi}^{\theta=\phi+\pi} \\
&= r\Big(G \cdot f \circ p \cdot \Big|D_{N(\phi)}p \ \ p_\phi\Big|\Big)_{\theta=\phi+\pi} + r\Big(G \cdot f \circ p \cdot \Big|D_{N(\phi)}p \ \ p_\phi\Big|\Big)_{\theta=\phi} \\
&= -rG|_{\theta=\phi}\Big[f \circ p\big|D_{N(\phi)}p \ \ p_\phi\big|\Big]_{\theta=\phi}^{\theta=\phi+\pi} \\
&= -(F + F_{\phi\phi})G.
\end{aligned}$$

Thus the PDE for projective metrization is indeed fulfilled. In the penultimate step we have crucially used that the spray S is reversible and thus $G|_{\phi+\pi} = -G|_\phi$. When the spray is irreversible, the terms corresponding to $\phi + \pi$ do not match - we will adapt the construction to deal with this in Section 4.3.2.

Generality of the metrizations. Note that by (4.14) the Hessian of the constructed is always reversible, in the sense that $(F + F_{\phi\phi})|_{(x,y,\phi+\pi,r)} = -\big(F + F_{\phi\phi}\big)|_{(x,y,\phi,r)}$, which for no reason is a necessary condition for projective metrizations of a reversible spray. Hence the metrics constructed do not exhaust the whole possibility of projective metrizations. For example, every Minkowski metric is a projective metrization of the flat spray, but its Hessian is generally not reversible in the above sense.

Example 4.2 (Metrics whose geodesics are straight lines I)**.** *For the flat spray $S = u\partial_x + v\partial_y$ on $U = \mathbb{R}^2$, whose geodesics are straight lines, a quadruple $(\mathbb{R}^2, \Gamma, \mu, p)$ satisfying the assumptions of Theorem 4.2 is as follows:*

- $\Gamma = S^1 \times \mathbb{R}$, *where the first component θ stands for the angle of a line with the x-axis and the second component s for the distance to the origin,*
- $p(x, y, \theta) = \Big(\theta, \Big\langle \begin{pmatrix} x \\ y \end{pmatrix}, N(\theta)\Big\rangle\Big)$ *using polar-coordinates (x, y, ϕ) for $S\mathbb{R}^2$,*
- μ *any positive, smooth measure given by a density function $f : S^1 \times \mathbb{R} \to \mathbb{R}_{>0}$ with respect to the volume form $d\theta \wedge ds$.*

We have

$$D_{T(\phi)}p|_{(x,y,\theta)} = \begin{pmatrix} 0 \\ \langle T(\phi), N(\theta)\rangle \end{pmatrix} \qquad \textit{and} \qquad p_\phi|_{(x,y,\theta)} = \begin{pmatrix} 1 \\ * \end{pmatrix},$$

so that we obtain from formula (4.12) the large family of Finsler metrics

$$F(x,y,\phi,r) = r\int_{\phi}^{\phi+\pi} f\Big(\theta, -x\sin\theta + y\cos\theta\Big)\sin(\theta-\phi)d\theta,$$

whose geodesics are straight lines.

For another application, see Example 5.1 from Chapter 5.

4.3 Local projective metrization of irreversible sprays

In Section 4.1, we have seen that in dimension two, every spray is *fiber-locally* projectively metrizable. For the *global* version of this question, a spray on S^2 whose geodesics are positively oriented circles of fixed non-zero geodesic curvature (that is spray (b_k^+) from Lemma 3.4) provides a counter-example (see also [73, Example 14.1.1]): Suppose there is a global projective metrization defined on the whole TS^2. By the Hopf-Rinow Theorem 2.4, the exponential mapping is surjective at each $p \in S^2$. But as the geodesics are 'small' circles, this is not the case. Thus, there cannot be a globally defined projective metrization.

The *local, but fiber-global* situation is between the fiber-local and global one: By Section 4.2, for any *geodesically reversible* spray, fiber-global projective metrizations exist. In this section we consider the much more complicated case of *geodesically irreversible* sprays.

4.3.1 The circle example and the results by Tabachnikov

Among other things, the paper [77] investigates Finsler metrics defined over the plane, whose geodesics are circles of radius 1 (the number 1 is of course not important). This is a special instance of the projective metrization problem, namely to describe projective metrizations of the spray (a) from Lemma 3.4

$$S|_{(x,y,u,v)} = u\partial_x + v\partial_y - \sqrt{u^2+v^2}(v\partial_u - u\partial_v).$$

From Chapter 3 we know that the Randers metric (a) from Theorem 3.1

$$F(x,y,u,v) = \sqrt{dx^2+dy^2} + \tfrac{1}{2}(ydx - xdy)$$

and all trivially projectively equivalent metrics (see Example 2.2) are projective metrizations of this spray. Note that this spray is not geodesically reversible, so that the method from Section 4.2 does not apply directly. Nonetheless, the following characterization of projective metrizations was obtained, using polar coordinates $u = r\cos\phi$ and $v = r\sin\phi$ for the fibers of $T\mathbb{R}^2$:

Theorem 4.3 ([77, Theorem 6]). *Every Finsler metric on the plane, whose geodesics are positively oriented circles of radius one, is of the form*

$$F(x,y,r,\phi) = r\left(\int_0^{\phi+\pi/2} \cos(\phi-\theta)g(x+\cos\theta, y+\sin\theta)d\theta + a(x,y)\cos\phi + b(x,y)\sin\phi\right),$$

where $g : \mathbb{R}^2 \to \mathbb{R}_{>0}$ is a positive density function in the plane, such that the integral over each ball of radius one is constant and $a_y - b_x = g(x+1,y)$.

The integral condition on g is only to ensure that the defined metric F is 2π-periodic with respect to ϕ. Theorem 4.3 is a special case of our Theorem 4.4 from the next section, see Example 4.4. There, the geometric role of the balls of radius 1 as a subset of the space of oriented, unparametrized geodesics will become clear: in fact their boundary describes the set of curves passing through a point $(x,y) \in \mathbb{R}^2$, if one identifies a circle by its midpoint.

Plugging $g \equiv const > 0$ into the formula of Theorem 4.3 gives all metrics that are trivially projectively equivalent to the metric (a) from Theorem 3.1. A priori it is not clear, whether the theorem produces any other projective metrizations, i.e. whether there exist non-constant functions g satisfying the assumptions of the theorem.

Problem (Pompeiu problem)**.** *Is there any non-constant, smooth function on the plane, such that the integral over every ball of radius 1 is the same for all balls?*

This named problem has a long and curious history and many results were obtained, see [86] for a recent overview. The answer is yes: such functions do exist, however the solutions are not particularly easy to construct.

4.3.2 Reformulation in terms of a measure on the space of geodesics

Let us now adapt the geometric method from Section 4.2 to the geodesically irreversible case. Though the geometric argument breaks down in this situation, the PDE point of view is not too bad: the problems with the irreversible case when checking the projective metrization PDE (4.8) in the last calculation of Section 4.2 came from the term corresponding to the integration boundary $\phi + \pi$ - we thus shall try to change the integration boundaries. In fact, the main theorem of this chapter below says that by this we obtain exactly all the solutions to the projective metrization PDE - even if the spay is geodesically irreversible:

Theorem 4.4. *Let S be a spray over $U \subseteq \mathcal{S}$, $\Gamma \subseteq \mathbb{R}^2$ be open and $p : SU \to \Gamma$ be a submersion, such that $\left|p_y \;\; p_\phi\right|_{\phi=0}$ is positive for some (x, y) and $p(\xi) = p(\eta)$ if and only if ξ and η are tangent to the same oriented curve of S. Then the solutions of the projective metrization PDE (4.8) of S over U are exactly of the form*

$$F(x, y, \phi, r) = r \int_0^\phi \left(f \circ p \middle| D_{T(\phi)}p \quad p_\phi \middle| \right)_{(x,y,\theta)} d\theta + a(x, y) \cos\phi + b(x, y) \sin\phi, \quad (4.17)$$

where $f : \Gamma \to \mathbb{R}$ is any density function and $a_y - b_x = \left(f \circ p \middle| p_y \quad p_x \middle| \right)_{\phi=0}$.
The such defined function F is

- *a Finsler metric, if and only if f and F are positive,*
- *2π-periodic, if and only if the measure of the connected components of $\Gamma \backslash \{p(x, y, \theta) \mid \theta \in [0, 2\pi]\}$ is constant under varying (x, y).*

Proof. As the formula is similar to the geometric formula from Section 4.2, the calculations are almost identical. Firstly for F defined by (4.17) we have

$$F_\phi = r \int_0^\phi f \circ p \left| D_{N(\phi)}p \quad p_\phi \right|_{(x,y,\theta)} d\theta - a(x, y) \sin\phi + b(x, y) \cos\phi$$

and

$$F + F_{\phi\phi} = r \Big[f \circ p \Big| D_{N(\phi)}p \quad p_\phi \Big| \Big]_{\theta=\phi}.$$

Recall that as before (see Equation (4.15)), the term $|D_{N(\phi)}p \;\; p_\phi|_{\theta=\phi}$ has constant sign for all (x, y, ϕ) and is thus positive, because of the assumption that $\left|p_y \; p_\phi\right|_{\phi=0}$ is positive for some (x, y). Hence by Lemma 4.3, the function is a Finsler metric, if and only if f is positive and the function F itself is positive (which locally can be achieved by addition of a closed 1-form as in the proof of Theorem 4.1).

Let us check that the constructed F satisfies the PDE for projective metrization (4.8):

$$\begin{aligned}
\tfrac{1}{r}(-D_{N(\phi)}F + D_{T(\phi)}F_\phi) &= \int_0^\phi -D_{N(\phi)}(f\circ p)\left|D_{T(\phi)}p \quad p_\phi\right| - (f\circ p)\left|D_{T(\phi)}p \quad D_{N(\phi)}p_\phi\right| \\
&\quad + D_{T(\phi)}(f\circ p)\left|D_{N(\phi)}p \quad p_\phi\right| + (f\circ p)\left|D_{N(\phi)}p \quad D_{T(\phi)}p_\phi\right| d\theta \\
&\quad + b_x - a_y \\
&\overset{(4.16)}{=} \int_0^\phi \partial_\theta\Big(f\circ p\big|D_{N(\phi)}p \quad D_{T(\phi)}p\big|\Big)d\theta + b_x - a_y \\
&= \Big[f\circ p\big|D_{N(\phi)}p \quad D_{T(\phi)}p\big|\Big]_0^\phi + b_x - a_y \\
&= G|_\phi\Big(f\circ p\Big|D_{N(\phi)}p \quad p_\phi\Big|\Big)_\phi - \Big(f\circ p\Big|p_y \quad p_x\Big|\Big)_{\theta=0} + b_x - a_y \\
&= -\tfrac{1}{r}(F + F_{\phi\phi})G.
\end{aligned}$$

Let us show, that the formula gives the most general solution to the PDE: Let F_1 be the constructed function with $f \equiv 1$ and h_1 its fiber-Hessian and F_2 be any other projective metrization with fiber-Hessian h_2. By Equation (4.9), it is $h_2 = gh_1$ for some 0-homogeneous function $g : TM\backslash 0 \to \mathbb{R}$ constant along the integral curves of the spray. Thus, h_2 is the Hessian of the function F_3 produced by formula (4.17) with $f = g$. Then F_2 and F_3 differ by a 1-form, and as they are projectively equivalent, the form is closed (see Example 2.2) and also F_2 is of the form (4.17).

The function F from (4.17) is 2π-periodic, if for all (x, y, ϕ) the following holds

$$\int_0^{2\pi}\Big(f\circ p\Big|D_{T(\phi)}p \quad p_\phi\Big|\Big)_{(x,y,\theta)}d\theta = 0.$$

Let $A_{(x,y)}$ be one of the two components of $\Gamma\backslash\{p(x,y,\theta) \mid \theta\in[0,2\pi]\}$. Then the rate of change of the integral over $A_{(x,y)}$ in the direction $T(\phi)$ is given by

$$D_{T(\phi)}\Big(\int_{A_{(x,y)}} f(v,w)dvdw\Big) = \pm\int_0^{2\pi} f(p(x,y,\theta))|D_{T(\phi)}p \quad \partial_\theta p|_{(x,y,\theta)}d\theta.$$

Hence F is 2π-periodic, if and only if the integral of f over all $A_{(x,y)}$ is the same. □

Corollary 4.2. *A spray is locally, fiber-globally projectively metrizable, if and only if there exists a smooth, positive measure on* Γ*, such that the curves* $\theta \mapsto p(x,y,\theta)$ *separate* Γ *into components of constant measure under varying* (x,y)*.*

Note that the freedom of choosing a density function f is the same as the freedom of choosing a different parametrization $\tilde p$ of the curves. Indeed, if $f \equiv 1$ and $\tilde p = \varphi\circ p$, then

$$\tilde F = \int_0^\phi |D_{T(\phi)}\tilde p \quad \tilde p_\phi|d\theta = \int_0^\phi |d\varphi_p||D_{T(\phi)}p \quad p_\phi|d\theta.$$

Let us apply the construction to Hilbert's Fourth problem:

Example 4.3 (Metrics whose geodesics are straight lines II)**.** *For the flat spray* $S = u\partial_x + v\partial_y$ *on* $U = \mathbb{R}^2$*, whose geodesics are straight lines, as in Example 4.2 let*

- $\Gamma = S^1 \times \mathbb{R}$ *with coordinates* (θ, s)*,*
- $p : S\mathbb{R}^2 \to \Gamma$ *with* $p(x, y, \phi) = (\theta, -x \sin\theta + y \cos\theta)$*.*

They satisfy the assumptions form Theorem 4.2. As $|p_y \quad p_x| = 0$*, up to addition of a closed 1-form, the Finsler metric on* $\mathbb{R}^2$*, whose geodesics are straight lines, are exactly of the form*

$$F(x, y, \phi, r) = r \int_0^\phi f\Big(\theta, -x \sin\theta + y \cos\theta\Big) \sin(\theta - \phi) d\theta,$$

where $f : \Gamma \to \mathbb{R}_{>0}$ *is any function whose integral over the connected components of* $\Gamma \backslash \{(\theta, -x \sin\theta + y \cos\theta) \mid \theta \in [0, 2\pi]\}$ *with respect to the volume form* $f(\theta, s) d\theta \wedge ds$ *is constant under varying* (x, y)*.*

Example 4.4 (Reobtaining Tabachnikovs circle formula)**.** *Consider once more the spray (a) from Lemma 3.4, given as*

$$S|_{(x,y,u,v)} = u\partial_x + v\partial_y - \sqrt{u^2 + v^2}(v\partial_u - u\partial_v)$$

on $\mathbb{R}^2$*, all of whose geodesics are positively oriented circles of radius 1.*

As we can identify a circle by its midpoint, we choose U *to be the ball* $B_1(0)$ *of radius 1 around the origin,* $\Gamma = B_2(0)\backslash 0$ *and* $p(x, y, \theta) = (x - \sin\theta, y + \cos\theta)$*. We obtain*

$$F(x, y, \phi, r) = r \int_0^\phi f(x - \sin\theta, y + \cos\theta) \sin(\phi - \theta) d\theta + a(x, y) \cos\phi + b(x, y) \sin\phi,$$

where $f : B_2(0)\backslash 0 \to \mathbb{R}_{>0}$ *and* $b_x - a_y = -f(x, y+1)$ *as the most general solution to the projective metrization PDE.*

The constructed function is 2π*-periodic, if and only if the integral of* f *over balls of radius* 1 *in* $B_0(2)\backslash\{0\}$ *is constant. Thus, every solution of the Pompeiu problem gives a non-trivial projective metrization of the spray* S*.*

Note however, that the function f *must only be defined on* $\Gamma = B_2(0)\backslash 0$ *and is allowed to have a singularity at the origin. Thus, possibly additional solutions (that are not solutions to the Pompeiu problem) exist, giving Finsler metrics defined locally over* $B_1(0)$*.*

Chapter 5

Topological obstructions to projective equivalence

The existence of two non-trivially projectively equivalent, real-analytic metrics on a surface implies that their geodesic flow is Liouville integrable and has topological entropy zero. In contrast, if the topology of the surface is complicated enough, i.e. if it is of negative Euler characteristic, any geodesic flow must have positive topological entropy - hence there is a topological obstruction to the existence of non-trivially projectively equivalent metrics. This chapter is devoted proving the following Theorem.

Theorem 5.1. *Let $\mathcal{S}$ be a real-analytic surface of negative Euler characteristic with two real-analytic Finsler metrics $F, \tilde{F}$. Then the following are equivalent:*

(a) F and $\tilde{F}$ are projectively equivalent

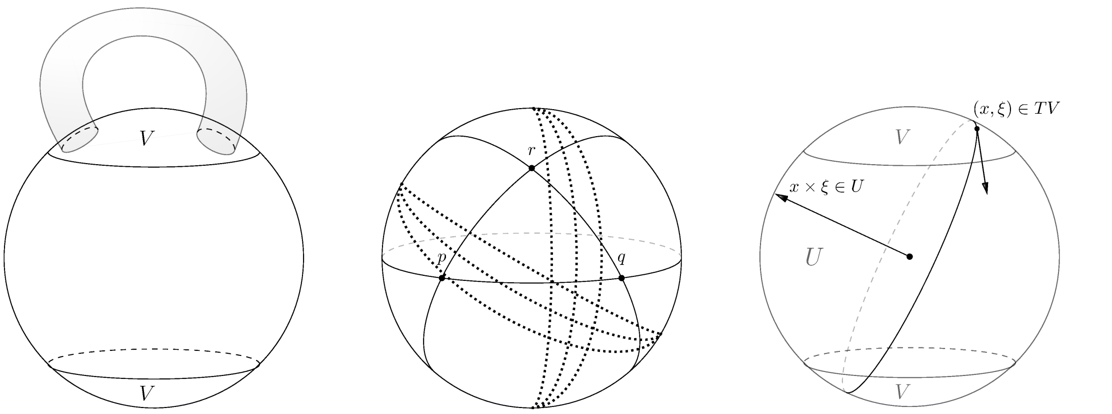
(b) $\tilde{F} = \lambda F + \beta$ for some $\lambda > 0$ and a closed 1-form β.

The proof is based on slightly generalized classical theorems from integrable system and the observation that for two projectively equivalent metrics $F, \tilde{F}$, the ratio of the traces of their fiber-Hessians is a globally defined integral for the geodesic flow of both metrics. We show that this integral must be constant, if the Euler characteristic of the surface is negative, which implies that the two metrics are trivially related. A similar argument to show that an integral must be constant was used in [67]. Alternatively, one can obtain the integral by a general construction for *trajectory equivalent* Hamiltonian systems, similarly to [56]. The corresponding result for Riemannian metrics has been obtained in [57, Corollary 3] (see also [58, 59, 79]), where the assumption of analyticity is not necessary.

The assumption of real-analyticity is necessary in the Finsler case: On any closed surface there are (non real-analytic) projectively equivalent metrics that are not related by scaling and addition of a closed 1-form:

Example 5.1. *Let $\hat{F}$ be the round metric on S^2. We claim that there is a smooth metric $\check{F}$ on S^2 projectively equivalent to $\hat{F}$, such that*

- *$\hat{F}$ and $\check{F}$ coincide over an open, non-empty set $V \subseteq S^2$,*
- *but are not related by $\hat{F} = \lambda F + \beta$ for any $\lambda > 0$ and any 1-form β.*



(a) Attaching handles in the region over which the metrics coincide.

(b) The distance $d(p,q)$ is the measure of all curves (dotted) intersecting the great circle segment.

(c) The metrics $\check{F}$ and $\hat{F}$ coincide on TV, because $\check{\lambda}|_U = \hat{\lambda}|_U \equiv 1$.

Figure 5.1: Construction of non-trivially projectively equivalent metrics on any closed surface.

Then by attaching orientable or non-orientable handles to S^2 (Figure 5.1a) in the set V, we obtain projectively equivalent metrics on any closed surface, that are not related by scaling and addition of a closed 1-form.

The metric $\check{F}$ can be constructed by the method from Section 4.2: Suppose the space of unparametrized geodesics of a reversible metric on a surface forms a smooth manifold endowed with a positive measure. Define the distance $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ of two points on the surface as the measure of curves intersecting the unique shortest geodesic segment connecting the points (Figure 5.1b). Then the unparametrized geodesics of the original metric are shortest for the constructed distance function. Then the function $F(x,\xi) := \frac{d}{dt}|_{t=0} d(x, c(t))$, where c is any curve such that $c(0) = x$ and $\dot{c}(0) = \xi$, is a Finsler metric projectively equivalent to the original metric (see Theorem 4.2).

Applying this procedure to the round sphere with a density function $\lambda : S^2 \to \mathbb{R}_{>0}$ satisfying $\lambda(-x) = \lambda(x)$ and identifying an oriented great circle by its normal (Figure 5.1c), one obtains the following family of Finsler metrics

$$F(x,\xi) := \frac{1}{4} \int_{\substack{\eta \in T_x S^2 \\ |\eta|=1}} \lambda\big(x \times \eta\big)|\nu_\xi| \qquad \text{with } \nu_\xi(\cdot) = \langle \xi, \cdot \rangle, \tag{5.1}$$

all projectively equivalent to the round metric, where $x \times \eta$ denotes the cross-product in $\mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ the Euclidean inner product; see (4.12).

For $\hat{\lambda} \equiv 1$, we obtain the round metric $\hat{F}$. Let $U = \{(x_1, x_2, x_3) \in S^2 \mid |x_3| \leq \frac{1}{\sqrt{2}}\}$, denote by V its complement and choose any $\check{\lambda} : S^2 \to \mathbb{R}_{>0}$ such that $\check{\lambda}|_U \equiv 1$, but

$\check{\lambda}|_V > 1$. *The obtained metric* $\check{F}$ *coincides with* $\hat{F}$ *over* V, *because for* $(x, \xi) \in TV$ *with* $|\xi| = 1$, *the cross product* $x \times \xi$ *is in* U *(Figure 5.1c), but is not related to it by scaling and addition of a closed 1-form.*

5.1 Entropy of the geodesic flow on surfaces of negative Euler characteristic

Lemma 5.1. *The geodesic flow of any Finsler metric on a closed manifold* M, *whose fundamental group* $\pi(M)$ *is of exponential growth, has positive topological entropy.*

This result was proven by Dinaburg in [28, Section 4] and [47] for the geodesic flow of Riemannian metrics. We give a straight-forward generalization for the geodesic flow of a (not necessarily reversible) Finsler metric. Instead of using the canonical Riemannian volume form on M, we use any volume form invariant under isometries (e.g. the Holmes-Thompson or Busemann volume, cf. Section 2.2.5). To deal with the irreversibility of the metrics, we shall use the reversibility number $\lambda_F := \sup_{\xi \in TM} \frac{F(-\xi)}{F(\xi)} \geq 1$, which is finite, if M is compact, as its unit sphere bundle is compact. Furthermore, the reversibility number of the induced distance function d_F (see Definition 2.2) defined by $\lambda_{d_F} := \sup_{x,y \in M} \frac{d(y,x)}{d(x,y)}$ is at most λ_F.

Proof. Let $(\widetilde{M}, p : \tilde{M} \to M)$ be the universal cover of M. Let $d\mu$ be a volume form on M invariant under F-isometries and $d : M \times M \to \mathbb{R}$ the distance function induced by F. Let $\tilde{F}(\tilde{\xi}) := F(d\pi(\tilde{\xi}))$ be the lift of the Finsler metric to the universal cover $\widetilde{M}$, so that p is a local isometry. Denote by $d\tilde{\mu}$ the corresponding volume form of $\tilde{F}$ and by $\tilde{d}$ the induced distance function on $\widetilde{M}$.

Firstly, we show that the volume of closed forward balls $B_r(\tilde{x}) = \{\tilde{y} \in \widetilde{M} \mid \tilde{d}(\tilde{x}, \tilde{y}) \leq r\}$ in $\widetilde{M}$ grows exponentially with their radius as a consequence of the exponential growth of the fundamental group, that is for any $\tilde{x} \in \widetilde{M}$ we have

$$\exists s_0, d_0, \mu_0, k > 0 \;\; \forall n \in \mathbb{N} : \quad \tilde{\mu}\big(B_{s_0 n + d_0}(\tilde{x})\big) \geq \mu_0 e^{kn}. \tag{5.2}$$

By compactness and definition of the universal covering, there is a finite family of open, connected, simply connected subsets $U_i \subseteq M$, that covers M and such that $p^{-1}(U_i)$ is the union of open, disjoint subsets $V_{ij} \subseteq \widetilde{M}$, such that $p : (V_{ij}, \tilde{F}) \to (U_i, F)$ is an isometry.

Fix $\tilde{x} \in \widetilde{M}$ and set $x := p(\tilde{x})$. Let $S = \{a_1, .., a_\ell\}$ be a set of closed, smooth curves through x generating the fundamental group $\pi(M, x)$ and assume that S is closed under inversion. Let $\#B_n$ the number of elements that can we written as a word of length at most n on S. By assumption, there is $k > 0$, such that $\#B_n \geq e^{kn}$ for all $n \in \mathbb{N}$. Set

$$s_0 := \max_i \Big(\operatorname{length}_F(a_i)\Big) \qquad d_0 := \max_i \Big(\operatorname{diam}_d(U_i)\Big) \qquad \mu_0 := \min_i \Big(\mu(U_i)\Big).$$

All three are positive. Let $\tilde{x} \in V_{ij}$ and note that $\tilde{\mu}(V_{ij}) = \mu(U_i) \geq \mu_0$. For $a \in \pi(M, x)$, let $|a|$ be the smallest number of elements from S whose product gives a. Consider the covering transformation $\Gamma(a) : V_{ij} \to \widetilde{M}$, that maps a $\tilde{y} \in V_{ij}$ to the endpoint of the unique lift of the curve cac^{-1} starting at $\tilde{y}$, where c is any curve from $y := p(\tilde{y})$ to x inside U_i. Then

$$\tilde{d}(\tilde{x}, \Gamma(a)\tilde{y}) \leq \inf_{\substack{c \text{ curve in } U_i \\ \text{from } y \text{ to } x}} \Big(\text{length}(\tilde{a}) + \text{length}(\tilde{c}^{-1})\Big) \leq s_0|a| + d_0,$$

where $\tilde{a}$ and $\tilde{c}^{-1}$ are the unique lifts of a and c^{-1} to $\widetilde{M}$ starting at $\tilde{x}$ and $\Gamma(a)\tilde{x}$ respectively, and have the same length as a and c^{-1}, because p is a local isometry. Furthermore, for different a, the sets $\Gamma(a)V_{ij}$ are disjoint and we have $p\big(\Gamma(a)V_{ij}\big) = U_i$, hence $\tilde{\mu}(\Gamma(a)V_{ij}) = \mu(U_i) \geq \mu_0$. It follows that

$$\tilde{\mu}(B_{s_0 n + d_0}) \geq \tilde{\mu}\Big(\bigcup_{|a| \leq n} \Gamma(a)V_{ij}\Big) \geq \mu_0 \cdot \#B_n \geq \mu_0 e^{kn}.$$

Secondly, let ρ and $\tilde{\rho}$ be the symmetrizations of d and $\tilde{d}$ on M and $\widetilde{M}$ respectively, that is $\rho(y_1, y_2) := \frac{d(y_1,y_2)+d(y_2,y_1)}{2}$. and $\tilde{\rho}(\tilde{y}_1, \tilde{y}_2) := \frac{\tilde{d}(\tilde{y}_1,\tilde{y}_2)+\tilde{d}(\tilde{y}_2,\tilde{y}_1)}{2}$. Choose $\epsilon > 0$ such that any ρ-ball of radius 2ϵ in M is contained in a set U_i, for example quarter of the Lebesgue number of the covering U_i for the distance ρ. Then any $\tilde{\rho}$-ball of radius 2ϵ in $\widetilde{M}$ is contained in one of the sets V_{ij}. In particular, the $\tilde{\mu}$-measure of $\tilde{\rho}$-balls of radius 2ϵ is bounded from above by a finite number $c_0 > 0$.

For fixed $\tilde{x} \in \widetilde{M}$, we show existence of a sequence $r_i \to \infty$, such that for each r_i there are at least $\frac{1}{c_0} e^{\frac{k}{2} r_i}$ unit speed geodesics $\tilde{\gamma}_j^{r_i}$ of length r_i starting from $\tilde{x}$, whose endpoints are ϵ-separated for the symmetrized distance $\tilde{\rho}$, where $k > 0$ is as in the first part of the proof.

Let $\delta > 0$ and consider the $\tilde{d}$-annuli $U_r := B_{r+\delta}(\tilde{x}) \backslash B_r(\tilde{x})$. There is a sequence $r_i \to \infty$ such that $\tilde{\mu}(U_{r_i}) \geq e^{\frac{k}{2} r_i}$. Indeed, suppose the inequality is violated for all but finitely many members of the sequence $r_i = i\delta$. Then $\tilde{\mu}(B_{n\delta}(\tilde{x})) = \sum_{i=0}^{n-1} \tilde{\mu}(U_{i\delta}) \leq \frac{2}{k\delta} e^{\frac{k}{2}\delta n} + C$, where C is a constant independent of n. This contradicts (5.2).

Let Q_{r_i} be a maximal 2ϵ-separated set for $\tilde{\rho}$ in U_{r_i}. Then the $\tilde{\rho}$-balls of radius 2ϵ with center $\tilde{q} \in Q_{r_i}$ must cover U_{r_i} and hence

$$c_0 \cdot \#Q_{r_i} \geq \tilde{\mu}(U_{r_i}) \geq e^{\frac{k}{2} r_i}.$$

As (M, F) is forward complete, so is $(\widetilde{M}, \tilde{F})$ and for each $\tilde{q} \in Q_{r_i}$ we may choose a unit speed geodesic from $\tilde{x}$ to $\tilde{q}$ of length between r_i and $r_i + \delta$. Let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be two such geodesics ending at $\tilde{q}_1, \tilde{q}_2$. Then using $\tilde{\rho}(\tilde{y}_1, \tilde{y}_2) \leq \frac{1+\lambda}{2} \tilde{d}(\tilde{y}_1, \tilde{y}_2)$, where λ is the reversibility number of F, we have

$$\begin{aligned} \tilde{\rho}(\gamma_1(r), \gamma_2(r)) &\geq \tilde{\rho}(\tilde{q}_1, \tilde{q}_2) - \tilde{\rho}(\tilde{\gamma}_1(r), q_1) - \tilde{\rho}(\tilde{\gamma}_1(r), q_2) \\ &\geq 2\epsilon - \frac{1+\lambda}{2}\big(\tilde{d}(\gamma_1(r), q_1) + \tilde{d}(\gamma_2(r), q_1)\big) \\ &\geq 2\epsilon - \frac{1+\lambda}{2} \cdot 2\delta \end{aligned}$$

Thus, choosing $\delta = \frac{\epsilon}{1+\lambda}$ gives the desired sequence r_i and geodesics $\gamma_j^{r_i}$.

Finally, let $\hat{\rho}$ be any symmetric distance on TM, such that $\hat{\rho}\big(\xi, \nu\big) \geq \rho\big(\pi(\xi), \pi(\nu)\big)$, where $\pi : TM \to M$ is the bundle projection. Recall that the topological entropy of the geodesic flow can be defined by $h_{\text{top}} = \lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(H^t_\epsilon)$, where H^t_ϵ is the maximal cardinality of an ϵ-separated set with respect to the distance $\hat\rho^t$ on TM defined by $\hat\rho^t(\xi,\nu) = \max_{0\le\tau\le t}\hat\rho\big(S^\tau\xi, S^\tau\nu\big)$, where S^τ is the geodesic flow of F.

Let r_i and $\gamma_j^{r_i}$ as before. Then the starting vectors of the projected geodesics $\gamma_j^{r_i} := p(\tilde\gamma_j^{r_i})$ are ϵ-separated with respect to $\hat\rho^{r_i}$. Indeed, let γ_1, γ_2 be two such geodesics and $t \in (0, r_i]$ the smallest value, such that $\tilde\rho\big(\tilde\gamma_1(t), \tilde\gamma_2(t)\big) = \epsilon$. Then $\tilde\gamma_1(t)$ and $\tilde\gamma_2(t)$ lie in the same V_{ij} and as p is a local isometry also for the symmetrized distances, we have $\rho\big(\gamma_1(t), \gamma_2(t)\big) = \epsilon$. Because H^t_ϵ is monotonously increasing as $\epsilon \to 0$, it follows that

$$h_{\text{top}} \ge \limsup_{r_i\to\infty}\frac{1}{r_i}\log\big(\tfrac{1}{c_0}e^{\frac{k}{2}r_i}\big) \ge \tfrac{k}{2} > 0.$$

□

Recall that the fundamental group of a closed surface is of exponential growth, if and only if it has negative Euler characteristic, see Corollary 2.3. Thus we have the following:

Corollary 5.1. *The geodesic flow of any Finsler metric on a closed surface $\mathcal{S}$ of negative Euler characteristic has positive topological entropy.*

5.2 Integrability of projective equivalent metrics and proof of Theorem 5.1

Lemma 5.2. *If $\hat F$ and $\check F$ are two projectively equivalent Finsler metrics on a surfaces $\mathcal{S}$, then the function $I : T\mathcal{S}\backslash 0 \to \mathbb{R}$ defined in local coordinates by*

$$I(x,\xi) := \frac{\operatorname{tr}\hat h}{\operatorname{tr}\check h}\Big|_{(x,\xi)} = \frac{\hat F_{\xi^1\xi^1} + \hat F_{\xi^2\xi^2}}{\check F_{\xi^1\xi^1} + \check F_{\xi^2\xi^2}}\Big|_{(x,\xi)}$$

is a globally defined integral for the geodesic flow of both $\hat F$ and $\check F$.

This fact is a consequence of Theorem 4.1. We give the proof independently for completeness. Note that $I(x,\xi)$ is the factor of proportionality of the fiber-Hessians, that is $\hat h = I\check h$.

Proof. For a Finsler metric F, denote as before by $h_{ij} := F_{\xi^i\xi^j}$ the Hessian of F. By the 1-homogeneity of F we have $h_{ij}\xi^j = 0$ and it follows that the Hessian has only one independent component and

$$h|_{(x,\xi)} = \frac{\operatorname{tr} h|_{(x,\xi)}}{(\xi^1)^2 + (\xi^2)^2}\begin{pmatrix} (\xi^2)^2 & -\xi^1\xi^2 \\ -\xi^1\xi^2 & (\xi^1)^2 \end{pmatrix}.$$

Firstly, recall that $g_{ij} = Fh_{ij} + F_{\xi^i}F_{\xi^j}$ and that $h_{ij}|_{(x,\xi)}\nu^i\nu^j = 0$ if and only if ν is a multiple of ξ (see Lemma 2.1). Thus $\det h = 0$ and $\operatorname{tr} h \neq 0$, as otherwise h would vanish.

Let $S = \xi^i\partial_{x^i} - 2G^i\partial_{\xi^i}$ be the geodesic spray of $\hat F$. By Lemma 4.1 both $\hat F$ and $\check F$ satisfy

$$F_{x^i} - F_{\xi^i x^\ell}\xi^\ell + 2G^\ell h_{i\ell} = 0,$$

and thus by differentiating by ξ^i and changing sign

$$S(h_{ii}) - 2G_i^\ell h_{i\ell} = 0.$$

Adding the two equations gives

$$S(\operatorname{tr} h) = 2G_1^\ell h_{1\ell} + 2G_2^\ell h_{2\ell} = \underbrace{2\frac{G_1^1(\xi^2)^2 - (G_2^1 + G_1^2)\xi^1\xi^2 + G_2^2(\xi^1)^2}{(\xi^1)^2 + (\xi^2)^2}}_{c(x,\xi):=} \operatorname{tr} h.$$

As this is a linear ODE along the integral curves of S, any two solutions must be a constant multiple of each other along the integral curves. Let $\operatorname{tr}\hat{h} = I(x,\xi)\operatorname{tr}\check{h}$. Then

$$c\operatorname{tr}\hat{h} = S(\operatorname{tr}\hat{h}) = S(I\operatorname{tr}\check{h}) = S(I)\operatorname{tr}\check{h} + cI\operatorname{tr}\check{h} = S(I)\operatorname{tr}\check{h} + c\operatorname{tr}\hat{h},$$

and thus $S(I) = 0$ as claimed.

Now let us show that the function I is well-defined. The value $I(x,\xi)$ is defined such that $\hat{F}_{\xi^i\xi^j}|_{(x,\xi)} = I(x,\xi)\check{F}_{\xi^i\xi^j}|_{(x,\xi)}$. Let $\overline{x}^i(x)$ be a change of coordinates. Then $\overline{\xi}^i(x,\xi) = \frac{\partial\overline{x}^i}{\partial x^j}\xi^j$ and $F_{\overline{\xi}^i\overline{\xi}^j} = F_{\xi^k\xi^\ell}\frac{\partial\xi^k}{\partial\overline{\xi}^i}\frac{\partial\xi^\ell}{\partial\overline{\xi}^j} = F_{\xi^k\xi^\ell}\frac{\partial x^k}{\partial\overline{x}^i}\frac{\partial x^\ell}{\partial\overline{x}^j}$. As a consequence, we have $\hat{F}_{\overline{\xi}^i\overline{\xi}^j}|_{(\overline{x},\overline{\xi})} = I(\overline{x},\overline{\xi})\check{F}_{\overline{\xi}^i\overline{\xi}^j}|_{(\overline{x},\overline{\xi})}$. Thus $I : T\mathcal{S}\backslash 0 \to \mathbb{R}$ is defined independently of the choice of coordinates. □

Now we have all the ingredients to proof the main result of this Section:

Proof of Theorem 5.1. Recall that by Theorem 2.8, the geodesic flow of a Hamiltonian system on a 4-dimensional manifold, integrable by a real-analytic integral, has zero entropy. It follows in combination with Corollary 5.1, that on a closed surface of negative Euler characteristic, the geodesic flow cannot be integrable by a real-analytic integral.

However, if F and $\tilde{F}$ are projectively equivalent real-analytic metrics, the function I from Lemma 5.2 is a real-analytic integral for the geodesic flow of F. Thus the set of points $(x,\xi) \in T\mathcal{S}\backslash 0$, where the differentials of F and I are linearly dependent admits an accumulation point. But in all those points, the differentials of F and I must be proportional and as $V(F)|_{(x,\xi)} = F(x,\xi) \neq 0$ and $V(I)|_{(x,\xi)} = 0$, where $V = \xi^i\partial_{\xi^i}$ is the Liouville vector field, the differential of I must vanish in all these points and thus I must be a constant λ on $T\mathcal{S}\backslash 0$.

This implies that $\operatorname{tr}(\tilde{h}) = \lambda\operatorname{tr}(h)$. As h has only one independent component (see Equation 4.10), it follows that $\tilde{h}_{ij} = \lambda h_{ij}$ and thus $\tilde{F} = \lambda F + \beta$ for some 1-form β on $\mathcal{S}$. But as $F, \tilde{F}$ are projectively equivalent, so are $\lambda F, \tilde{F}$, which differ by the 1-form β, which thus must be closed (by Example 2.2). □

Chapter 6

Outlook

In this closing chapter we collect several open problems and perspectives for further research on the three problems considered in this dissertation.

Finsler metrics admitting many projective vector fields

In Chapter 3 we have found locally all Finsler metrics on 2-dimensional manifolds admitting a projective algebra of the submaximal dimension 3 up to isometry and projective equivalence.

i) What can be said about the case, when the projective algebra is 2-dimensional? We have seen in Example 4.1 two large families of such metrics. However, they do not exhaust all projective classes of sprays with 2-dimensional projective algebra - they are only the ones not containing a straight line as a geodesic in the coordinates from Example 4.1, and satisfying an integral condition. What can be said about the remaining projective classes of sprays?

ii) Consider the same problem on a n-dimensional manifold with $n \geq 3$. Again, the maximal dimension of the projective algebra is obtained by the projectively flat metrics and is $(n+1)^2 - 1 = (n+2)n$. What is the submaximal dimension? What can be said about metrics with submaximal dimensional projective algebra?

Local projective Finsler metrizability

In Chapter 4 we have reformulated the local, fiber-global projective metrization problem on surfaces in terms of the existence of a measure on the space of geodesics having an equilibrium property. However it is still not clear whether every geodesically irreversible spray is locally projectively metrizable. We have seen that this is the case for geodesically reversible sprays and that there is always a large family of projective metrizations. On the other hand, in the irreversible circle example from Section 4.3.1, it was much harder to find projective metrizations, though several instances can be found.

iii) Prove that in dimension two, every spray is locally, fiber-globally projectively metrizable or find a spray, that is not.

iv) Investigate the rigidity of projective metrizations for irreversible sprays. Are there sprays that admit, up to trivial freedom, only one projective metrization even locally?

v) Assume existence of projective vector fields. Does it imply that the family of curves is well behaved and a measure with the equilibrium property can be found?

Integrability of the geodesic flow and topological obstructions to existence of non-trivially projectively equivalent metrics

In Chapter 5 we have proven that, if on a surface there are two non-trivially projectively related real-analytic Finsler metrics, then the geodesic flow of both of them is Liouville integrable. As a consequence, we have seen that on a surface of negative Euler characteristic, any two projectively equivalent, real-analytic metrics must be trivially related. By an example we have demonstrated, that the assumption of real-analyticity is necessary.

vi) Prove that existence of two real-analytical, non-trivially projectively equivalent Finsler metrics implies existence of several independent integrals of the geodesic flow also on manifolds of dimension three and higher.

vii) We conjecture that in higher dimension, if a real-analytic Finsler metric has an ergodic geodesic flow, then it does not admit any non-trivially projectively equivalent metrics.

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Symbols and Abbreviations

Symbol	Explanation	Homogeneity
F	Finsler metric	1
g_{ij}	Fundamental tensor $\frac{1}{2}(F^2)_{\xi^i\xi^j}$	0
g^{ij}	Inverse of (g_{ij})	0
h_{ij}	Hessian of F: $F_{\xi^i\xi^j}$	-1
S	Spray, $S = \xi^i\partial_{x^i} - 2G^i\partial_{\xi^i}$	
G^i	Spray coefficients	2
G^i_j	$(G^i)_{\xi^j}$	1
G^i_{jk}	$(G^i)_{\xi^j\xi^k}$	0
R^ℓ_{jk}	$(G^\ell_k)_{x^j} - (G^\ell_j)_{x^k} - G^r_jG^\ell_{kr} + G^r_kG^\ell_{jr}$	1
H_k	Horizontal vector fields $H_k = \partial_{x^k} - G^r_k\partial_{\xi^r}$	
V	Liouville vector field $V = \xi^i\partial_{\xi^i}$	
S_F	Geodesic spray of F with $G^i = \frac{1}{4}g^{ik}(2\frac{\partial g_{kr}}{\partial x^\ell} - \frac{\partial g_{\ell r}}{\partial x^k})\xi^\ell\xi^r$	
$E_i(L,c)$	Euler-Lagrange equations with Lagrangian L	
$\tilde{\theta}$	Poincaré 1-form: canonical 1-form on T^*M	
θ	Hilbert 1-form on TM	
$\mathcal{S}$	A Surface	
$\pi(\mathcal{S})$	Fundamental group of $\mathcal{S}$	
$\langle S\rangle$	Free group over a set S	
$\#B_n$	Number of elements in a group that can be written as a word of length at most n on some finite generator	
I	Integral $I = \frac{\operatorname{tr} h}{\operatorname{tr}\tilde{h}}$ for the geodesic flow, when $F, \tilde{F}$ projectively equivalent	0

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Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts wurde ich von Prof. Dr. Vladimir Matveev unterstützt.

Julius Lang, Jena, am 10. Februar 2020