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AN INEQUALITY CONCERNING EDGES OF MINOR WEIGHT IN CONVEX 3-POLYTOPES

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Dedicated to Professor E. Jucovič on the occasion of his 70th birthday.

Abstract

Let e_{ij} be the number of edges in a convex 3–polytope joining the vertices of degree i with the vertices of degree j. We prove that for every convex 3-polytope there is $20e_{3,3} + 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\frac{2}{3}e_{3,7} + 5e_{3,8} + 2\frac{1}{2}e_{3,9} + 2e_{3,10} + 16\frac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\frac{2}{3}e_{4,7} + 5\frac{1}{3}e_{5,5} + 2e_{5,6} \geq 120$; moreover, each coefficient is the best possible. This result brings a final answer to the conjecture raised by B. Grünbaum in 1973.

Keywords: planar graph, convex 3-polytope, normal map.

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1. Introduction and Statement of Results

This note deals with connected planar maps. We use standard terminology and notation of graph theory, see e.g. Ore [16]. We recall, however, more specialized notions. A plane map is called *normal* if it contains neither vertices nor faces incident with less than 3 edges. Notice, however, that both loops and multiple edges can appear in a normal plane map. By the

Steinitz's theorem (see e.g. Grünbaum [11], Jucovič [13]) convex 3-polytopes are distinguished among all planar maps by the property that their graphs are 3-connected. The degree of a face ω is the number of edges incident to ω where each cut-edge is counted twice. Similarly, each loop contributes 2 to the degree of the incident vertex. Vertices and faces of degree i are called *i-vertices* and *i-faces*, respectively. Let $e_{i,j}(M) = e_{i,j}$ be the number of edges in a planar map M which join i-vertices and j-vertices. Recall that a convex 3-polytope is called simplicial if all its faces are 3-gons.

An excellent theorem of Kotzig [14] (see also [1,3,6,7,8,13,15,]) states that every convex 3-polytope contains an edge of the weight (i.e., the sum of degrees of its endvertices) at most 13; in other words $\sum_{i+j\le 13} e_{i,j} > 0$. This Kotzig's result was further developed in various directions, see e.g. Borodin [1,2,3], Grünbaum [6,7,8], Grünbaum and Shephard [9], Ivančo [10], Ivančo and Jendroľ [11], Jucovič [12,13], Zaks [17].

Grünbaum [8] has brought an idea that a relation of the type $\sum_{i+j\le 13} \alpha_{i,j} e_{i,j} \ge 1$ should hold for each convex 3-polytope ($\alpha_{i,j}$ denotes the coeficient at e_{ij}) and has conjectured that the following holds for every simplicial convex 3-polytope

$$\begin{aligned}20e_{3,3} &+ 15e_{3,4} + 12e_{3,5} + 10e_{3,6} + 6\tfrac{2}{3}e_{3,7} + 5e_{3,8} + 3\tfrac{1}{3}e_{3,9} + 2e_{3,10}\\ &+ 12e_{4,4} + 7e_{4,5} + 5e_{4,6} + 4e_{4,7} + 2\tfrac{2}{3}e_{4,8} + \tfrac{2}{3}e_{4,9}\\ &+ 4e_{5,5} + 2e_{5,6} + \tfrac{1}{3}e_{5,7}\\ &+ 12e_{6,6} \ge 120.\end{aligned}$$

Jucovič [12] proved that for each simplicial convex 3-polytope there is

$$\begin{aligned}20e_{3,3} &+ 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\tfrac{2}{3}e_{3,7} + 5e_{3,8} + 2\tfrac{1}{2}e_{3,9} + 2e_{3,10}\\ &+ 20e_{4,4} + 11e_{4,5} + 5e_{4,6} + 6e_{4,7} + 5e_{4,8} + 3e_{4,9}\\ &+ 8e_{5,5} + 2e_{5,6} + 2e_{5,7} + 2e_{5,8} \ge 120.\end{aligned}$$

Later on Jucovič, in [13], proved that this inequality holds for all convex 3-polytopes.

For a wider class of planar maps which also includes convex 3-polytopes Borodin [3] has obtained.

Theorem 1. *For each normal planar map there holds*

$$\begin{aligned}40e_{3,3} &+ 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\tfrac{2}{3}e_{3,7} + 5e_{3,8} + 2\tfrac{1}{2}e_{3,9} + 2e_{3,10}\\ &+ 16\tfrac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\tfrac{2}{3}e_{4,7}\\ &+ 5\tfrac{1}{3}e_{5,5} + 2e_{5,6} \ge 120;\end{aligned}\tag{1}$$

moreover, each coefficient of this inequality is the best possible.

In the same paper Borodin [3] proves that for simplicial convex 3-polytopes (1) is the best possible if we put $\alpha_{3,3} = 20$ instead of $\alpha_{3,3} = 40$. For other results of this type see Borodin [1,2,3], Borodin and Sanders [5], Jucovič [13].

The main purpose of the present note is to give a final answer to the above mentioned conjecture by Grünbaum [8]. We prove the following

Theorem 2. *For each convex* 3*-polytopes there holds*

$$\begin{aligned} 20e_{3,3} &+ 25e_{3,4} + 16e_{3,5} + 10e_{3,6} + 6\tfrac{2}{3}e_{3,7} + 5e_{3,8} + 2\tfrac{1}{2}e_{3,9} + 2e_{3,10} \\ &+ 16\tfrac{2}{3}e_{4,4} + 11e_{4,5} + 5e_{4,6} + 1\tfrac{2}{3}e_{4,7} \\ &+ 5\tfrac{1}{3}e_{5,5} + 2e_{5,6} \geq 120; \end{aligned} \tag{2}$$

moreover, each coefficient of this inequality is the best possible.

2. Proof of Theorem 2

We prove our Theorem 2 in a dual form. It is well known that the dual of a 3-connected planar map is also 3-connected, see e.g. Ore [16, Chapter 3] and, due to Steinitz's theorem, it is also true for convex 3-polytopes. It is easy to check that the dual of a normal map is again normal.

For the purposes of this proof an edge h is called an (i,j)-edge when it is incident with an i-gon and a j-gon. Let $g_{i,j}(M) = g_{ij}$ denote the number of (i,j)-edges in a map M. If M^d is the dual to a normal map M, then clearly $e_{ij}(M^d) = g_{ij}(M)$. Let $V(M), E(M)$ and $F(M)$ denote the set of vertices, edges and faces of the map M, respectively.

The proof is by contradiction. Replace e_{ij} with g_{ij} in the left part of (2) and denote it by $\sum$. We want to prove that for every 3-connected planar M there is $\sum(M) \geq 120$. Suppose M be a counterexample having a minimum number of faces.

To obtain a contradiction we are going to look for a suitable configuration in M which will be changed locally to obtain a new 3-connected plane map M^* with $\sum(M^*) \leq \sum(M) < 120$ and with a fewer number of faces than in M. During this transformation of the map M into M^* some edges and vertices of M are deleted, some edges change their types (an edge is of the type (i,j) if it is an (i,j)-edge) and some new edges and vertices can appear in M^*.

Asociate with an (i,j)-edge h of the map M the charge $\alpha(h,M) = \alpha_{ij}$, where α_{ij} is as in (2) or $\alpha_{i,j} = 0$ for $i = 3, j \geq 11$ or $i = 4, j \geq 8$ or

$i = 5, j \geq 7$ or $i \geq 6, j \geq 6$. Hence $\sum(M) = \sum_{h \in E(M)} \alpha(h, M)$. Let $\Delta(h) = \alpha(h, M) - \alpha(h, M^*)$.
Since every 3-connected plane map is also normal Theorem 1 yields $g_{3,3}(M) \neq 0$, i.e., M contains a $(3,3)$-edge $h_0 = uv$. Denote by s and t the vertices incident to triangles incident with h_0 and different from u and v, see Figure 1. Let $h_1 = us, h_2 = sv, h_3 = vt$ and $h_4 = ut$ be edges of M.

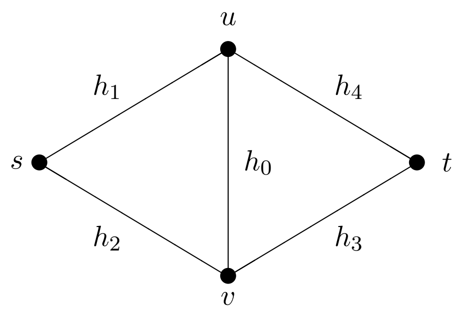

Figure 1

To finish our proof several cases have to be considered

Case 1. deg $u \geq 4$ and deg $v \geq 4$.

1.1. Let deg $s = 3$ or deg $t = 3$. The required map M^* is obtained by deleting the edge h_0 from M, i.e., $M^* = M - h_0$. Because M is 3-connected and at least one of the vertices s and t is a 3-vertex also M^* is 3-connected. We can easily see that $|F(M^*)| = |F(M)| - 1$ and $\Delta(\sum) = \sum(M) - \sum(M^*) = \alpha(h_0, M) + \sum_{i=1}^{4}(\alpha(h_i, M) - \alpha(h_i, M^*)) = \alpha_{3,3} + \sum_{i=1}^{4} \Delta(h_i) \geq 20 + 4 \cdot (-5) = 0$. The last inequality is due to the fact that if a $(3,k)$-edge h is transformed into a $(4,k)$-edge, its charge always decreases or is the same except of the case $k = 3$. We also refer to the fact that $\Delta(h_i) \geq -5$ for any edge $h_i \in E(M)$.

1.2. deg $s \geq 4$ and deg $t \geq 4$. In this case we transform M into M^* as shown in Figure 2. We delete the edge h_0 from M and split the vertex t of M into two new vertices t_1 and t_2 such that we obtain, in M^*, deg $t_1 = 3$ and deg $t_2 =$ deg $t - 1$. (The reason for this transformation of M into M^* is to preserve 3-connectivity also in M^*.) Let h', h_1, h_2, h_3 and h_4 be edges and ω_1 and ω_2 be faces of M^* as depicted in Figure 2. Without loss of generality we can assume that $4 \leq$ deg $\omega_1 \leq$ deg ω_2. Put $\Delta^* = \sum_{x \in E(M) - \{h_0, h_3, h_4\}} \Delta(x)$. Then we have $|F(M^*)| = |F(M)| - 1$ and $\Delta(\sum) = \alpha(h_0, M) + \Delta(h_3) +$

$\Delta(h_4) + \Delta^* - \alpha(h', M^*) \geq 0$. To check it use $\alpha(h_0, M) = \alpha_{3,3} = 20$ and for the values $\Delta(h_3), \Delta(h_4), \alpha(h', M^*)$ and Δ^* see Table 1 below. To count Δ^* we also refer to the fact that $g_{3,3}(M) \leq 5$ (because M is a counterexample) and consider the "worst" case.

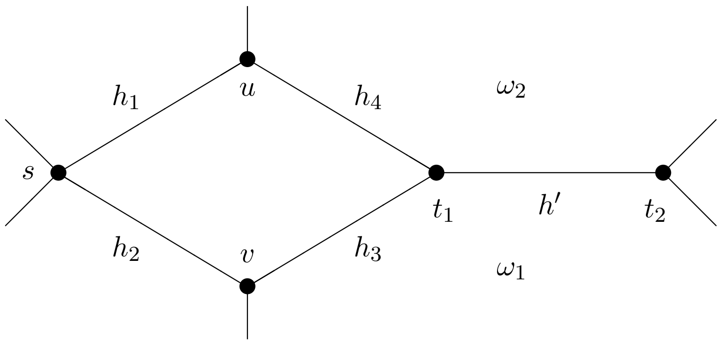

Figure 2

Case 2. deg $u = 3$ and deg $v \geq 4$.

Let w be a face incident to the edges h_1 and h_4, see Figure 1.

2.1. If deg $w = 3$ then M^* is obtained by removing the vertex u from M, i.e. $M^* = M - \{u\}$. We have $\Delta(\sum) = \alpha(h_0, M) + \alpha(h_1, M) + \alpha(h_4, M) = 60 > 0$ and $|F(M^*)| = |F(M)| - 2$.

2.2. Let deg $w = k \geq 4$. If we delete the vertex u from M and then insert a new edge $h^* = st$ we obtain a required map $M^*, M^* = M - \{u\} + \{h^*\}$. In this case $|F(M^*)| = |F(M)| - 1$ and we can check that $\Delta(\sum) = \alpha(h_0, M) + \alpha(h_1, M) + \alpha(h_4, M) - \alpha(h^*, M^*) + \tilde{\Delta} \geq 0$. To see it, take $\alpha(h_0, M) = \alpha_{3,3} = 20$ and the values $\alpha(h_1, M), \alpha(h_4, M), \alpha(h^*, M^*)$ and $\tilde{\Delta}$ from the Table 2 below; here $\tilde{\Delta} = \sum \Delta(g)$, where the sum is taken over all edges g incident to the face $\omega, g \neq h_1, h_4$. Note that during this transformation the edge g changes its type (n, k) into the type $(n, k-1)$ and in the counting we consider the worst case, that is $\tilde{\Delta} \geq (k-2)(\alpha_{3,k} - \alpha_{3,k-1})$.

Case 3. deg $u =$ deg $v = 3$.

This assumption leads immediately to the graph of the tetrahedron or to a 2-connected planar map. In both cases we get a contradiction.

The proof that a 3-connectivity of M implies a 3-connectivity of M^* is easy and is left to the reader.

The coefficient $\alpha_{3,3} = 20$ cannot be improved as we can see from the tetrahedron. The above mentioned examples by Borodin [3] also show the impossibility to improve the other coefficient $\alpha_{i,j}$ in Theorem 2.

Table 1

deg ω_1	deg ω_2	$\Delta(h_3)$	$\Delta(h_4)$	$\alpha(h', M^*)$	$\Delta^* \geq$
4	4	$3\frac{1}{3}$	$3\frac{1}{3}$	$16\frac{2}{3}$	-10
4	5	$3\frac{1}{3}$	14	11	-20
4	6	$3\frac{1}{3}$	11	5	-20
4	7	$3\frac{1}{3}$	$8\frac{1}{3}$	$1\frac{2}{3}$	-20
4	≥ 8	$3\frac{1}{3}$	≥ 0	0	-20
5	5	14	14	$5\frac{1}{3}$	-10
5	6	14	11	2	-10
5	≥ 7	14	≥ 0	0	-10
≥ 6	≥ 6	≥ 0	≥ 0	0	-10

Table 2

deg ω	$d(h_1, M)$	$d(h_4, M)$	$d(h^*, M^*)$	$\tilde{\Delta} \geq$
4	25	25	20	$2 \cdot (-8\frac{1}{3})$
5	16	16	25	$3 \cdot (-9)$
6	10	10	16	$4 \cdot (-6)$
7	$6\frac{2}{3}$	$6\frac{2}{3}$	10	$5 \cdot (-3\frac{1}{3})$
8	5	5	$6\frac{2}{3}$	$6 \cdot (-1\frac{2}{3})$
9	$2\frac{1}{2}$	$2\frac{1}{2}$	5	$7 \cdot (-2\frac{1}{2})$
10	2	2	$2\frac{1}{2}$	$8 \cdot (-\frac{1}{2})$
11	0	0	2	$9 \cdot (-2)$
≥ 12	0	0	0	0

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