

---

**Preprint No. M 19/09**

**Optimality conditions for set  
optimization using a directional  
derivative based on generalized  
Steiner sets**

Robert Baier, Gabriele Eichfelder, Tobias Gerlach

November 2019

**URN:** urn:nbn:de:gbv:ilm1-2019200537

---

**Impressum:**

Hrsg.: Leiter des Instituts für Mathematik  
Weimarer Straße 25  
98693 Ilmenau  
Tel.: +49 3677 69-3621  
Fax: +49 3677 69-3270  
<http://www.tu-ilmenau.de/math/>

# Optimality Conditions for Set Optimization using a Directional Derivative based on Generalized Steiner Sets

Robert Baier\*, Gabriele Eichfelder†, Tobias Gerlach‡

November 15, 2019

## Abstract

Set-optimization has attracted increasing interest in the last years, as for instance uncertain multiobjective optimization problems lead to such problems with a set-valued objective function. Thereby, from a practical point of view, most of all the so-called set approach is of interest. However, optimality conditions for these problems, for instance using directional derivatives, are still very limited. The key aspect for a useful directional derivative is the definition of a useful set difference for the evaluation of the numerator in the difference quotient.

We present here a new set difference which avoids the use of a convex hull and which applies to arbitrary convex sets, and not to strictly convex sets only. The new set difference is based on the new concept of generalized Steiner sets. We introduce the Banach space of generalized Steiner sets as well as an embedding of convex sets in this space using Steiner points. In this Banach space we can easily define a difference and a directional derivative. We use the latter for new optimality conditions for set optimization. Numerical examples illustrate the new concepts.

**Key Words:** Set optimization, Set relation, Set difference, Support function, Directional derivative, Optimality condition, Steiner point, Generalized Steiner set

**Mathematics subject classifications (MSC 2010):** 26E25, 49K99, 49J53

---

\*Department of Mathematics, Universität Bayreuth, 95440 Bayreuth, Germany, robert.baier@uni-bayreuth.de

†Institute for Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, gabriele.eichfelder@tu-ilmenau.de

‡Institute for Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, tobias.gerlach@tu-ilmenau.de

# 1 Introduction.

In set optimization one considers optimization problems with a set-valued objective function. Set optimization problems arise in many recent fields of application. Current important applications of set optimization can be found for instance in the areas of socio economics [1], welfare economics [2], and finance [3]. Such problems arise also for instance when a vector-valued objective map is considered, but (relative) errors are allowed [4], in case of robust approaches to uncertain multiobjective optimization problems [5, 6], or in bilevel optimization, if neither the optimistic nor the pessimistic approach is used [7, 8]. For a detailed introduction to set optimization see the extensive book by Khan, Tammer and Zălinescu [9].

For set optimization problems first a reasonable optimality concept has to be defined. We use in this work the so-called set approach [10, 11, 12], which has gained increasing attention in the last years as it is considered to be more realistic in many situations [13]. In the set approach, one defines binary relations in order to compare sets. There are different possibilities how to compare sets. Many such binary relations have been proposed in the literature [11]. Most prominent are the  $\ell$ -less order relation, the  $u$ -less order relation, and the set less order relation which is the combination of both.

Which of these set order relations is suitable depends on the considered application. The set approach with the  $u$ -less order relation allows for instance the treatment of decision uncertainty in multiobjective optimization [5, 6]. Moreover, the  $\ell$ -less order relation and the  $u$ -less order relation correspond to the optimistic and pessimistic approach in bilevel optimization [8], respectively. The set less order relation is an appropriate relation for many practical situations where one is interested in comparing sets by comparing the best and the worst elements of the sets at the same time.

Dealing with set optimization problems with such set relations is theoretically and numerically a challenge. Just comparing two strictly convex and compact sets is difficult, and is for instance done in [14] by solving a large number of optimization problems with a linear objective function. This is based on the idea that one can use supporting points and support functions for describing convex sets.

Recently, such supporting points have also been used by Jahn in [13] to define a set difference for strictly convex and compact sets. The advantage of this new set difference proposed in this pioneering work is that it can easily be calculated numerically as it does not contain a convex hull or similar. Moreover, it is 'sufficiently small' to allow to derive useful necessary optimality conditions: for a strictly convex set it holds that the set minus the set itself is the set containing just the zero. A property, which is for instance not satisfied by the Minkowski-difference (i.e., the set of all differences of elements of both sets), but it is needed for defining a meaningful directional derivative for set-valued maps.

This set difference by Jahn applies to strictly convex sets only. It requires that the supporting faces to the considered directions are singletons. Jahn proposes

in [13, Remark 2.1] also a generalization to arbitrary convex sets by using metric difference in case the supporting faces are not singletons, but this does not lead to satisfying results in case for instance the set-valued map is such that just the unit square is moved around in the space.

The approach by Jahn has also another drawback: it defines the set differences as sets while for numerical purposes and for defining the directional derivative, it is important to keep the direction and the associated supporting points. For these reasons we propose here a new Banach space, the Banach space of generalized Steiner sets, for which the Steiner points of supporting faces are saved with their corresponding directions. We define an embedding of sets in this space, which also allows to embed the images of set-valued maps. Based on that we can define a difference which keeps the correspondence between the directions and supporting points which would be lost if the difference would be itself a set in  $\mathbb{R}^m$ .

Thereby we use the concept of Steiner points as applied in [15, 16] for selections of set-valued maps. These allow to choose a uniquely defined point from a supporting face and this gives the possibility to extend our examinations to arbitrary convex sets which are not necessarily strictly convex. Hence, this allows for instance to work with the large class of polytopes. Another important property of the new concepts is that they allow a numerical calculation and visualization of the results. We give such numerical illustrations within this paper. We also derive strong necessary optimality conditions for set optimization problems.

The concept of the Banach space of generalized Steiner sets and the embedding of convex sets into this Banach space uses ideas presented in [17, 18, 19] for the Banach space of directed sets, which also allows an embedding of convex sets. Directed sets were already successfully applied to study derivatives of set-valued maps with convex, compact and nonempty images, see for instance [20]. However, also they have not been used for set optimization and the specific order relations so far. The concept of generalized Steiner points which we are going to use here is much simpler, from the point of view of definition but also numerical calculation. Moreover, we define embeddings of sets in such a way that they directly fit to set optimization with the set approach. Using the embedding and the difference in the new Banach space we introduce a new directional derivative.

Thus, in the next section, Section 2, we introduce the necessary concepts for set order relations as well as for characterizing convex sets by supporting faces. Then, in Section 3, we introduce the Banach space of generalized Steiner sets before we propose an embedding of convex sets in this space by using Steiner points. We give characterizations of the set order relations by using this embedding. Using so called visualizations we can associate again sets to the generalized Steiner sets. Finally, in Section 4, we use the new concepts for defining necessary optimality conditions for set optimization problems. We introduce a directional derivative by using the embedding in the Banach space. With several examples, also with numerical illustrations, we show the practical relevance of the new necessary conditions. We also relate our results to those from Jahn [13].

## 2 Notation and basic results.

At first, we list some notations and basic results used throughout the paper. For a nonempty subset  $A$  of  $\mathbb{R}^m$  we denote by  $\text{int } A$ ,  $\text{bd } A$ ,  $\text{cl } A$ ,  $\text{co } A$ , and  $\text{cone } A$  the *interior*, the *boundary*, the *closure*, the *convex hull*, and the *conical hull* of  $A$ , respectively. If nonempty sets  $A, B \subset \mathbb{R}^m$  and a scalar  $\lambda \in \mathbb{R}$  are given, then we define the *Minkowski sum* and the *Minkowski difference* of  $A$  and  $B$  by  $A + B := \{a + b \in \mathbb{R}^m \mid a \in A, b \in B\}$  and  $A - B := \{a - b \in \mathbb{R}^m \mid a \in A, b \in B\}$ . Moreover, we set  $\lambda A := \{\lambda a \in \mathbb{R}^m \mid a \in A\}$  and  $-A := \{0_{\mathbb{R}^m}\} - A = \{-a \in \mathbb{R}^m \mid a \in A\}$ , where  $0_{\mathbb{R}^m}$  is the zero element of  $\mathbb{R}^m$ . For a matrix  $M \in \mathbb{R}^{p \times m}$  we define furthermore  $MA := \{Ma \in \mathbb{R}^p \mid a \in A\}$  and we use  $I_m \in \mathbb{R}^{m \times m}$  for the *identity matrix* of dimension  $m$ .

We denote by  $\mathcal{C}(\mathbb{R}^m)$  and  $\mathcal{CS}(\mathbb{R}^m)$  the set of all nonempty, convex, and compact subsets and the set of all nonempty, strictly convex, and compact subsets of  $\mathbb{R}^m$ , respectively. Recall that a subset  $A$  of  $\mathbb{R}^m$  with  $\text{int } A \neq \emptyset$  is *strictly convex* if and only if for arbitrary  $a^1, a^2 \in A$  with  $a^1 \neq a^2$  it holds  $\lambda a^1 + (1 - \lambda)a^2 \in \text{int } A$  for all  $\lambda \in (0, 1)$ . For this definition and other equivalent characterizations we refer to [21, Definition 2.2] and Lemma 2.1. An example for a nonempty, strictly convex, and compact subset of  $\mathbb{R}^m$  is given by the *Euclidean ball* with midpoint  $c \in \mathbb{R}^m$  and radius  $r > 0$  defined by  $\mathcal{B}(c, r) := \{y \in \mathbb{R}^m \mid \|y - c\|_2 \leq r\}$ . Here,  $\|\cdot\|_2$  denotes the Euclidean norm in  $\mathbb{R}^m$ . Moreover, we define the *unit sphere* of  $\mathbb{R}^m$  by  $S_{m-1} := \{y \in \mathbb{R}^m \mid \|y\|_2 = 1\}$ .

Throughout the paper we use the following assumption:

**Assumption 1.** *The space  $\mathbb{R}^m$  is partially ordered by a closed, pointed, and convex cone  $C \neq \{0_{\mathbb{R}^m}\}$  with  $\text{int } C \neq \emptyset$ .*

Recall that a nonempty subset  $C$  of  $\mathbb{R}^m$  is a *cone* if and only if  $y \in C$  and  $\lambda \geq 0$  imply  $\lambda y \in C$ . A cone is *pointed* if and only if  $C \cap (-C) = \{0_{\mathbb{R}^m}\}$  holds. A pointed convex cone defines an *antisymmetric partial ordering*  $\leq$  on the space  $\mathbb{R}^m$  by

$$a \leq b \iff b - a \in C$$

for arbitrary elements  $a, b \in \mathbb{R}^m$ .

For a cone  $C \subset \mathbb{R}^m$  we denote by  $C^* := \{l \in \mathbb{R}^m \mid \forall c \in C : l^\top c \geq 0\}$  the *dual cone* of  $C$  and we set

$$\mathcal{T}_u := C^* \cap S_{m-1}, \quad \mathcal{T}_\ell := (-C^*) \cap S_{m-1} = -\mathcal{T}_u, \quad \text{and} \quad \mathcal{T}_s := \mathcal{T}_u \cup \mathcal{T}_\ell. \quad (1)$$

The indices  $u$ ,  $\ell$ , and  $s$  will relate to set relations used in set optimization and which we define later in this section. Moreover, we define a *sign function* for directions  $l \in \mathcal{T}_s$  by

$$\text{sign}(\cdot) : \mathcal{T}_s \rightarrow \{-1, 1\} \text{ with } l \mapsto \text{sign}(l) := \begin{cases} 1 & , \text{ if } l \in \mathcal{T}_u \\ -1 & , \text{ if } l \in \mathcal{T}_\ell \end{cases}. \quad (2)$$

Note that  $C^*$  is a closed and convex cone (see for instance [22], p. 53) and that under the Assumption 1 the dual cone  $C^*$  is pointed (see for instance [4, Lemma 1.27 (a)]) and has a nonempty interior (see again [22], p. 53). Hence it holds  $C^* \cap (-C^*) = \{0_{\mathbb{R}^m}\}$  and (2) is well defined. The defined sets  $\mathcal{T}_\diamond$  with  $\diamond \in \{\ell, u, s\}$  are compact subsets of  $\mathbb{R}^m$ .

The parametrization of sets via their support functions and supporting points is in the following of central importance. For a nonempty convex set  $A \subset \mathbb{R}^m$  the *support function* is defined by

$$\delta^*(\cdot, A): \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\} \text{ with } l \mapsto \delta^*(l, A) := \sup_{a \in A} l^\top a.$$

Based on this for a given direction  $l \in \mathbb{R}^m$  the *supporting face* of  $A$  w.r.t.  $l$  is defined by

$$Y(l, A) := \{y(l, A) \in A \mid l^\top y(l, A) = \delta^*(l, A)\}. \quad (3)$$

An element  $y(l, A)$  of the supporting face  $Y(l, A)$  is called a *supporting point*. In case  $Y(l, A) = \{y(l, A)\}$ , the singleton  $y(l, A)$  is called an *exposed point* of  $A$  w.r.t.  $l$ . Moreover, it is easy to see that  $Y(l, A) = Y(\alpha l, A)$  for all  $l \in \mathbb{R}^m$  and all  $\alpha > 0$ .

If the set  $A \subset \mathbb{R}^m$  is nonempty, closed, and convex, then it holds (see [23, Chap. V, Theorem 2.2.2, Theorem 2.2.3, Proposition 3.1.5, and Theorem 3.3.1])

$$y \in \text{int } A \Leftrightarrow l^\top y < \delta^*(l, A) \text{ for all } l \in \mathbb{R}^m \setminus \{0_{\mathbb{R}^m}\} \quad (4)$$

and

$$A = \bigcap_{l \in S_{m-1}} \{y \in \mathbb{R}^m \mid l^\top y \leq \delta^*(l, A)\}. \quad (5)$$

If  $A$  is additionally compact, then by Straszewicz's theorem (see for instance [24, Theorem 1.4.7]) it holds

$$A = \text{co} \bigcup_{l \in S_{m-1}} Y(l, A) = \text{cl co} \bigcup_{l \in S_{m-1}} \{y(l, A)\}. \quad (6)$$

Note that if for an  $l \in S_{m-1}$  the supporting face  $Y(l, A)$  is not a singleton, then the element  $y(l, A) \in Y(l, A)$  in (6) can be chosen arbitrarily. Furthermore, for  $A \in \mathcal{C}(\mathbb{R}^m)$  it follows by (4)

$$Y(l, A) \subset \text{bd } A \subset A \text{ for all } l \in \mathbb{R}^m \setminus \{0_{\mathbb{R}^m}\}. \quad (7)$$

By (6) it is also easy to see that for two nonempty, convex, and compact sets  $A, B \subset \mathbb{R}^m$  it holds

$$(A = B) \Leftrightarrow (Y(l, A) = Y(l, B) \text{ for all } l \in S_{m-1}).$$

Moreover, for directions  $l \in \mathbb{R}^m$ ,  $\eta \in \mathbb{R}^p$ , arbitrary sets  $A, B \in \mathcal{C}(\mathbb{R}^m)$ , a matrix  $M \in \mathbb{R}^{p \times m}$ , and arbitrary scalars  $\lambda, \mu \geq 0$  it holds (see for instance [23, Chap. V, Theorem 3.3.3 and Proposition 3.3.4])

$$\delta^*(l, \lambda A + \mu B) = \lambda \delta^*(l, A) + \mu \delta^*(l, B) \quad \text{and} \quad \delta^*(\eta, MA) = \delta^*(M^\top \eta, A).$$

The supporting face is by [24, Theorem 1.7.4.] a subdifferential of the support function. By applying the calculus rules for the sum and linear image of subdifferentials [23, Chap. VI, Theorems 4.1.1 and 4.2.1]) one then obtains

$$Y(l, \lambda A + \mu B) = \lambda Y(l, A) + \mu Y(l, B) \quad \text{and} \quad Y(\eta, MA) = MY(M^\top \eta, A). \quad (8)$$

The following result characterizes nonempty, strictly convex, and compact sets by the uniqueness of supporting points.

**Lemma 2.1.** [21, Corollary 3.1] *Let  $A \in \mathcal{C}(\mathbb{R}^m)$  with  $\text{int } A \neq \emptyset$ . Then  $A \in \mathcal{CS}(\mathbb{R}^m)$  if and only if the supporting face  $Y(l, A)$  is a singleton for all  $l \in \mathbb{R}^m \setminus \{0_{\mathbb{R}^m}\}$ .*

It is well known that for a given set  $A \in \mathcal{C}(\mathbb{R}^m)$  the support function is Lipschitz-continuous w.r.t. the direction (see for instance [24, Lemma 1.8.10.]) and thus by using Rademacher's theorem (see [25, Satz I], [26, Satz VII]) differentiable almost everywhere in  $\mathbb{R}^m$ . Hence, together with [24, Corollary 1.7.3] it holds for nonempty, convex, and compact sets the following result:

**Lemma 2.2.** *If  $A \in \mathcal{C}(\mathbb{R}^m)$ , then the supporting face  $Y(l, A)$  is a singleton for almost all  $l \in \mathbb{R}^m \setminus \{0_{\mathbb{R}^m}\}$ , i.e., the set of all directions  $l \in \mathbb{R}^m \setminus \{0_{\mathbb{R}^m}\}$  for which  $Y(l, A)$  is not a singleton has Lebesgue measure zero.*

Our motivation for this paper is to give optimality conditions for set optimization problems. In these optimization problems one has to work with set-valued objective functions, i.e., we have to compare sets in the image space. According to [11] practical relevant relations for comparing sets are the following three set relations [10, 12, 27] defined for arbitrary nonempty subsets  $A$  and  $B$  of  $\mathbb{R}^m$ :

- (a) the  $\ell$ -less order relation:  $A \preceq_\ell B \Leftrightarrow (\forall b \in B \exists a \in A : a \leq b) \Leftrightarrow (B \subset A + C)$ ,
- (b) the  $u$ -less order relation:  $A \preceq_u B \Leftrightarrow (\forall a \in A \exists b \in B : a \leq b) \Leftrightarrow (A \subset B - C)$ , and
- (c) the set less order relation:  $A \preceq_s B \Leftrightarrow (A \preceq_\ell B \text{ and } A \preceq_u B)$ .

For an illustration of the defined set relations see Figure 1. Note that by definition it holds

$$A \preceq_\ell B \Leftrightarrow -B \preceq_u -A, \quad A \preceq_u B \Leftrightarrow -B \preceq_\ell -A, \quad \text{and} \quad A \preceq_s B \Leftrightarrow -B \preceq_s -A.$$

These set order relations are in general not antisymmetric. Instead, it holds (cf. [11, Proposition 3.1])

$$\begin{aligned} (A \preceq_\ell B \text{ and } B \preceq_\ell A) &\Leftrightarrow (A + C = B + C), \\ (A \preceq_u B \text{ and } B \preceq_u A) &\Leftrightarrow (A - C = B - C), \text{ and} \\ (A \preceq_s B \text{ and } B \preceq_s A) &\Leftrightarrow (A + C = B + C \text{ and } A - C = B - C). \end{aligned}$$

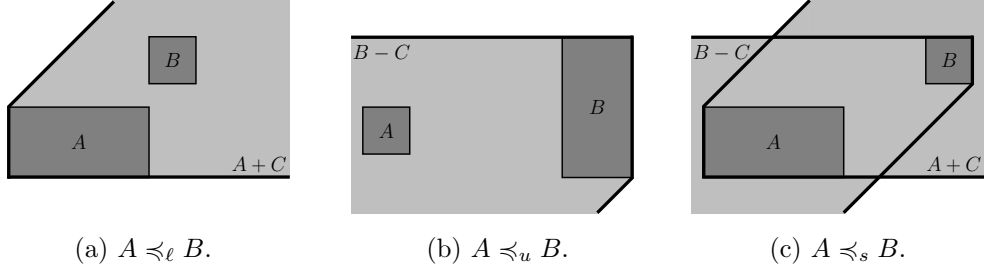


Figure 1: Illustration of the set relations  $\prec_\ell$ ,  $\prec_u$ , and  $\prec_s$  for  $A, B \subset \mathbb{R}^2$  and ordering cone  $C = \{x \in \mathbb{R}^2 \mid x_1 \geq x_2 \geq 0\}$ .

As before let Assumption 1 be satisfied, i.e.,  $C$  is a closed cone. If  $A$  and  $B$  are nonempty, convex, and compact subsets, then for instance by [4, Remark 1.5 and Lemma 1.34] the sets  $A + C$  and  $B - C$  are nonempty, convex, and closed. Thus the sets  $A \pm C$  and  $B \pm C$ , which appear in the above definition of the three set relations, can also be described by the intersection of half-spaces and using support functions (cf. (5)). We obtain for all  $l \in S_{m-1}$  by [23, Sec. V, Corollary 3.1.2 and Theorem 3.3.3]

$$\delta^*(l, A + C) = \delta^*(l, A) + \delta^*(l, C) \text{ and } \delta^*(l, B - C) = \delta^*(l, B) + \delta^*(l, -C).$$

By the definition of the dual cone and as  $0_{\mathbb{R}^m} \in C \cap (-C)$  it holds

$$\delta^*(l, C) = \sup_{c \in C} l^\top c = \begin{cases} 0 & , \text{ if } l \in \mathcal{T}_\ell \subset -C^* \\ \infty & , \text{ if } l \in S_{m-1} \setminus \mathcal{T}_\ell \end{cases}$$

and

$$\delta^*(l, -C) = \sup_{c \in -C} l^\top c = \begin{cases} 0 & , \text{ if } l \in \mathcal{T}_u \subset C^* \\ \infty & , \text{ if } l \in S_{m-1} \setminus \mathcal{T}_u \end{cases}.$$

Thus it holds for  $A + C$ , which appears in the  $\ell$ -less order relation,

$$\delta^*(l, A + C) = \delta^*(l, A) \text{ for all } l \in \mathcal{T}_\ell$$

and for  $B - C$ , which appears in the  $u$ -less order relation,

$$\delta^*(l, B - C) = \delta^*(l, B) \text{ for all } l \in \mathcal{T}_u.$$

This also explains our choice of indices when defining the sets  $\mathcal{T}_\ell$  and  $\mathcal{T}_u$  (cf. (1)). We obtain by using (5)

$$A + C = \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \leq \delta^*(l, A)\} \tag{9}$$



and

$$B - C = \bigcap_{l \in \mathcal{T}_u} \{y \in \mathbb{R}^m \mid l^\top y \leq \delta^*(l, B)\}. \quad (10)$$

We give the definition of the set optimization problems and the corresponding solution concepts using the set approach at the beginning of the forthcoming Section 4.

### 3 Generalized Steiner sets as a tool in set optimization

In this section we introduce the new concept of generalized Steiner sets. These generalized Steiner sets form a Banach space in which we will embed arbitrary convex and compact sets by using the known concept of Steiner points. This embedding allows us to use the arithmetic of the Banach space to define concepts like a directional derivative for set-valued maps (see the forthcoming subsection 4.2).

From the basic structure we follow the guideline of directed sets in [18, 19]. That is due to the fact that generalized Steiner sets and directed sets are related. However, directed sets are much more complicated to define and to calculate. They are based on the definition of directed intervals. The definition for subsets of  $\mathbb{R}^m$  with  $m \geq 2$  is then done recursively by a definition for  $\mathbb{R}^{m-1}$ . In contrary to directed sets, generalized Steiner sets are not recursively defined. They are simple families of vectors parameterized by directions  $l$  of the unit sphere. In analogy to directed sets the arithmetics for two generalized Steiner sets are done only for elements with a common unit direction and the norm of a generalized Steiner set is just the supremum norm of its elements. Following [19] we define for visualization purposes the boundary part of a generalized Steiner set and the positive, the negative, and the mixed-type part. As the embedding of a convex and compact set  $A$  into the Banach space of all generalized Steiner sets, the vectors parameterized by the directions  $l$  will be defined in direction of unique support by the exposed point  $y(l, A)$  and in directions of non-unique support by the Steiner point of the supporting face  $Y(l, A)$ .

#### 3.1 The Banach spaces of generalized Steiner sets.

All our further considerations are related to the concept of generalized Steiner sets, which we introduce in the following:

**Definition 3.1.** Let  $\mathcal{T}$  be a nonempty subset of the unit sphere  $S_{m-1}$ . A *generalized Steiner set* w.r.t.  $\mathcal{T}$  is defined by

$$\mathcal{F}_{GS, \mathcal{T}} := (\mathcal{F}(l))_{l \in \mathcal{T}},$$

where  $\mathcal{F}: \mathcal{T} \rightarrow \mathbb{R}^m$  is a function such that the set  $\mathcal{F}(\mathcal{T}) := \bigcup_{l \in \mathcal{T}} \{\mathcal{F}(l)\} \subset \mathbb{R}^m$  is bounded. The set of all generalized Steiner sets w.r.t.  $\mathcal{T}$  is denoted by  $\mathcal{GS}(\mathbb{R}^m, \mathcal{T})$ .

We will use this definition in most cases with a special setting: We choose  $\mathcal{T} := \mathcal{T}_\diamond$  with  $\diamond \in \{\ell, u, s\}$  (cf. (1)), fix a convex and compact set  $A$ , and define the function  $\mathcal{F}: \mathcal{T} \rightarrow \mathbb{R}^m$  using the concept of Steiner points by  $\mathcal{F}(l) := \text{St}(Y(l, A))$  for all  $l \in \mathcal{T}$  (cf. the forthcoming Definition 3.9). This will imply in the case of unique support for an  $l \in \mathcal{T}$  that  $\mathcal{F}(l) = y(l, A)$ , i.e., we associate to  $l$  the exposed point of  $A$  w.r.t.  $l$ . Thus to each convex and compact set  $A$  we define a generalized Steiner set.

Nevertheless we first analyze the structure of the set of all generalized Steiner sets w.r.t. any given nonempty subset  $\mathcal{T}$  of  $S_{m-1}$ . We are motivated to do so, since we believe that generalized Steiner sets w.r.t. more general sets  $\mathcal{T}$  are a versatile tool also in other areas of set-valued analysis as for instance for nonconvex subdifferentials.

We start by introducing an addition  $\oplus_{\mathcal{T}}$  and a scalar multiplication  $\odot_{\mathcal{T}}$  on  $\mathcal{GS}(\mathbb{R}^m, \mathcal{T})$ .

**Definition 3.2.** Let two generalized Steiner sets  $\mathcal{F}_{\mathcal{GS}, \mathcal{T}} = (\mathcal{F}(l))_{l \in \mathcal{T}}$  and  $\mathcal{G}_{\mathcal{GS}, \mathcal{T}} = (\mathcal{G}(l))_{l \in \mathcal{T}}$  w.r.t.  $\mathcal{T}$  and a scalar  $\alpha \in \mathbb{R}$  be given. Then we define

$$\mathcal{F}_{\mathcal{GS}, \mathcal{T}} \oplus_{\mathcal{T}} \mathcal{G}_{\mathcal{GS}, \mathcal{T}} := (\mathcal{F}(l) + \mathcal{G}(l))_{l \in \mathcal{T}} \quad \text{and} \quad \alpha \odot_{\mathcal{T}} \mathcal{F}_{\mathcal{GS}, \mathcal{T}} := (\alpha \mathcal{F}(l))_{l \in \mathcal{T}}. \quad (11)$$

It is easy to verify that the following holds:

**Proposition 3.3.** Let  $\mathcal{T}$  be a nonempty subset of  $S_{m-1}$  and let the arithmetic operations  $\oplus_{\mathcal{T}}$  and  $\odot_{\mathcal{T}}$  be defined as in (11). Then  $(\mathcal{GS}(\mathbb{R}^m, \mathcal{T}), \oplus_{\mathcal{T}}, \odot_{\mathcal{T}})$  is a real linear space with zero element

$$0_{\mathcal{GS}, \mathcal{T}} := (0_{\mathbb{R}^m})_{l \in \mathcal{T}}$$

and additive inverse

$$\odot_{\mathcal{T}} \mathcal{F}_{\mathcal{GS}, \mathcal{T}} := (-\mathcal{F}(l))_{l \in \mathcal{T}} \quad \text{for all } \mathcal{F}_{\mathcal{GS}, \mathcal{T}} \in \mathcal{GS}(\mathbb{R}^m, \mathcal{T}).$$

Furthermore,

$$\|\mathcal{F}_{\mathcal{GS}, \mathcal{T}}\| := \sup_{l \in \mathcal{T}} \|\mathcal{F}(l)\|_2 \quad (12)$$

defines a norm on  $(\mathcal{GS}(\mathbb{R}^m, \mathcal{T}), \oplus_{\mathcal{T}}, \odot_{\mathcal{T}})$  based on the Euclidean norm of the elements  $\mathcal{F}(l)$  of the corresponding generalized Steiner set w.r.t.  $\mathcal{T}$ .

Note that for convergence studies in the following the definition of the norm in (12) with the supremum over all  $l \in \mathcal{T}$  will guarantee a uniform convergence w.r.t. the directions  $l$  (see for instance the proof of the following theorem). The set of all generalized Steiner sets forms also a complete linear space.

**Theorem 3.4.** Let  $\mathcal{T}$  be a nonempty subset of  $S_{m-1}$  and let the arithmetic operations  $\oplus_{\mathcal{T}}$  and  $\odot_{\mathcal{T}}$  be defined as in (11). Then  $(\mathcal{GS}(\mathbb{R}^m, \mathcal{T}), \oplus_{\mathcal{T}}, \odot_{\mathcal{T}})$  equipped with the norm defined as in (12) is a Banach space.

*Proof.* Let  $(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}^{(k)})_{k \in \mathbb{N}} = ((\mathcal{F}^{(k)}(l))_{l \in \mathcal{T}})_{k \in \mathbb{N}} \subset \mathcal{G}\mathcal{S}(\mathbb{R}^m, \mathcal{T})$  be a Cauchy sequence of generalized Steiner sets w.r.t.  $\mathcal{T}$ , i.e., for all  $\varepsilon > 0$  there exists a constant  $k_0 = k_0(\varepsilon) \in \mathbb{N}$  such that

$$\|\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}^{(i)} \ominus_{\mathcal{T}} \mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}^{(j)}\| = \sup_{l \in \mathcal{T}} \|\mathcal{F}^{(i)}(l) - \mathcal{F}^{(j)}(l)\|_2 \leq \varepsilon \quad \text{for all } i, j \geq k_0 = k_0(\varepsilon),$$

and thus

$$\|\mathcal{F}^{(i)}(l) - \mathcal{F}^{(j)}(l)\|_2 \leq \varepsilon \quad \text{for all } l \in \mathcal{T} \text{ and all } i, j \geq k_0 = k_0(\varepsilon). \quad (13)$$

Hence, for all  $l \in \mathcal{T}$  the sequence  $(\mathcal{F}^{(k)}(l))_{k \in \mathbb{N}} \subset \mathbb{R}^m$  is a Cauchy sequence in  $\mathbb{R}^m$  and therefore convergent. We define  $\mathcal{F}(l) := \lim_{k \rightarrow \infty} \mathcal{F}^{(k)}(l)$  for all  $l \in \mathcal{T}$  and based on this  $\mathcal{F}_{\mathcal{T}} := (\mathcal{F}(l))_{l \in \mathcal{T}}$ .

First we show by using the triangle inequality that  $\mathcal{F}(\mathcal{T}) = \bigcup_{l \in \mathcal{T}} \{\mathcal{F}(l)\}$  is bounded and thus  $\mathcal{F}_{\mathcal{T}}$  is a generalized Steiner set. If we choose  $\varepsilon := 1$  in (13), then we obtain for the corresponding  $k_0 = k_0(1)$

$$\|\mathcal{F}^{(k_0+j)}(l)\|_2 \leq \|\mathcal{F}^{(k_0+j)}(l) - \mathcal{F}^{(k_0)}(l)\|_2 + \|\mathcal{F}^{(k_0)}(l)\|_2 \leq 1 + \|\mathcal{F}^{(k_0)}(l)\|_2$$

for all  $l \in \mathcal{T}$  and all  $j \in \mathbb{N}$  and thus by the continuity of the Euclidean norm it holds

$$\|\mathcal{F}(l)\|_2 = \lim_{j \rightarrow \infty} \|\mathcal{F}^{(k_0+j)}(l)\|_2 \leq 1 + \|\mathcal{F}^{(k_0)}(l)\|_2 \leq 1 + \sup_{l \in \mathcal{T}} \|\mathcal{F}^{(k_0)}(l)\|_2 = 1 + \|\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}^{(k_0)}\|$$

for all  $l \in \mathcal{T}$ . Consequently, the set  $\mathcal{F}(\mathcal{T})$  is bounded and it holds  $\mathcal{F}_{\mathcal{T}} \in \mathcal{G}\mathcal{S}(\mathbb{R}^m, \mathcal{T})$ . Finally, we show that  $(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}^{(k)})_{k \in \mathbb{N}}$  converges to  $\mathcal{F}_{\mathcal{T}}$  w.r.t. the norm defined in (12). By (13) it holds for arbitrary  $\varepsilon > 0$  and  $i \geq k_0 = k_0(\varepsilon)$

$$\|\mathcal{F}^{(i)}(l) - \mathcal{F}^{(i+j)}(l)\|_2 \leq \varepsilon \quad \text{for all } l \in \mathcal{T} \text{ and all } j \in \mathbb{N}$$

with  $\lim_{j \rightarrow \infty} \mathcal{F}^{(i+j)}(l) = \mathcal{F}(l)$  for all  $l \in \mathcal{T}$ . Moreover, again by the continuity of the norm we obtain

$$\|\mathcal{F}^{(i)}(l) - \mathcal{F}(l)\|_2 = \lim_{j \rightarrow \infty} \|\mathcal{F}^{(i)}(l) - \mathcal{F}^{(i+j)}(l)\|_2 \leq \varepsilon \quad \text{for all } l \in \mathcal{T},$$

i.e., the convergence of the sequences  $(\mathcal{F}^{(k)}(l))_{k \in \mathbb{N}} \subset \mathbb{R}^m$  is uniform w.r.t.  $l \in \mathcal{T}$ . Hence, it follows

$$\|\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}^{(i)} \ominus_{\mathcal{T}} \mathcal{F}_{\mathcal{T}}\| = \sup_{l \in \mathcal{T}} \|\mathcal{F}^{(i)}(l) - \mathcal{F}(l)\|_2 \leq \varepsilon$$

for all  $\varepsilon > 0$  and all  $i \geq k_0 = k_0(\varepsilon)$ . Thus we obtain  $\lim_{k \rightarrow \infty} \mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}^{(k)} = \mathcal{F}_{\mathcal{T}} \in \mathcal{G}\mathcal{S}(\mathbb{R}^m, \mathcal{T})$  and we are done.  $\square$

Note that  $\ominus_{\mathcal{T}}$  is defined in such a way that for any generalized Steiner set  $\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}} = (\mathcal{F}(l))_{l \in \mathcal{T}}$  it holds  $\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}} \ominus_{\mathcal{T}} \mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}} = 0_{\mathcal{G}\mathcal{S},\mathcal{T}} = (0_{\mathbb{R}^m})_{l \in \mathcal{T}}$ .

Generalized Steiner sets are related to bounded subsets of  $\mathbb{R}^m$ , what can be seen by the assumed boundedness of  $\mathcal{F}(\mathcal{T}) = \bigcup_{l \in \mathcal{T}} \{\mathcal{F}(l)\} \subset \mathbb{R}^m$ . But they contain more information: Generalized Steiner sets save an association of each  $l \in \mathcal{T}$  to the point  $\mathcal{F}(l) \in \mathcal{F}(\mathcal{T})$ . Nevertheless, sometimes one is only interested in the set  $\mathcal{F}(\mathcal{T})$ , what we will call the boundary part of the generalized Steiner sets. Moreover, the points  $\mathcal{F}(l)$  will later be defined using the supporting faces  $Y(l, A)$  for convex and compact subsets  $A$  of  $\mathbb{R}^m$ . By (5) such subsets  $A$  can be characterized by the intersection of the sets  $\{y \in \mathbb{R}^m \mid l^\top y \leq \delta^*(l, A)\}$  for all  $l \in S_{m-1}$ , where  $\delta^*(l, A) = l^\top y(l, A)$  for any  $y(l, A) \in Y(l, A)$ . For the generalized Steiner sets we only use a subset  $\mathcal{T}$  of the unit sphere  $S_{m-1}$ . Hence, we obtain by the intersection of the sets  $\{y \in \mathbb{R}^m \mid l^\top y \leq \delta^*(l, A)\}$  for all  $l \in \mathcal{T}$  in general only a superset of  $A$ . We will call this superset the positive part of the visualization for a generalized Steiner set. Note that the visualization defined in the following consists of two convex (positive and negative part) and an in general nonconvex set (mixed-type part). We will use the positive and the negative part in the formulation of our optimality conditions in Section 4.

**Definition 3.5.** Let  $\mathcal{T}$  be a nonempty subset of  $S_{m-1}$  and let a generalized Steiner set  $\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}} = (\mathcal{F}(l))_{l \in \mathcal{T}} \in \mathcal{G}\mathcal{S}(\mathbb{R}^m, \mathcal{T})$  be given. Then the *boundary part* of  $\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}$  is defined by

$$B_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) := \mathcal{F}(\mathcal{T}) = \bigcup_{l \in \mathcal{T}} \{\mathcal{F}(l)\},$$

and based on this the *visualization* of  $\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}$  is defined by

$$V_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) := P_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) \cup N_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) \cup M_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}})$$

where

$$\begin{aligned} P_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) &:= \bigcap_{l \in \mathcal{T}} \{y \in \mathbb{R}^m \mid l^\top y \leq l^\top \mathcal{F}(l)\}, \\ N_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) &:= \bigcap_{l \in \mathcal{T}} \{y \in \mathbb{R}^m \mid l^\top y \geq l^\top \mathcal{F}(l)\}, \text{ and} \\ M_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) &:= B_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) \setminus (P_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}}) \cup N_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}})) \end{aligned}$$

are denoted as the *positive part*, the *negative part*, and the *mixed-type part* of  $V_{\mathcal{T}}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}})$ , respectively.

Note that the boundary part is already used in the definition of a generalized Steiner set and there its boundedness was required.

**Example 3.6.** Let  $A := \mathcal{B}(0_{\mathbb{R}^2}, 1)$  be the unit ball in  $\mathbb{R}^2$ ,  $C = C^* = \mathbb{R}_+^2$ ,  $\mathcal{T} := \mathcal{T}_u$ , and  $\mathcal{F}: \mathcal{T}_u \rightarrow \mathbb{R}^2$  with

$$\mathcal{F}(l) := y(l, A) = \operatorname{argmax}\{l^\top a \mid a \in A\} \text{ for all } l \in \mathcal{T}_u.$$

Then we obtain for the generalized Steiner set  $\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}_u} = (\mathcal{F}(l))_{l \in \mathcal{T}_u}$  the following visualization parts (cf. the forthcoming Proposition 3.15):

$$\begin{aligned} B_{\mathcal{T}_u}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}_u}) &= \{y \in \mathbb{R}^2 \mid \|y\|_2 = 1, y_1 \geq 0, y_2 \geq 0\}, \\ P_{\mathcal{T}_u}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}_u}) &= A - \mathbb{R}_+^2, \\ N_{\mathcal{T}_u}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}_u}) &= \{(1, 1)^\top\} + \mathbb{R}_+^2, \text{ and} \\ M_{\mathcal{T}_u}(\mathcal{F}_{\mathcal{G}\mathcal{S},\mathcal{T}_u}) &= \emptyset. \end{aligned}$$

### 3.2 Embeddings of convex sets via Steiner points.

So far, we have introduced the concept of generalized Steiner sets in a very general way. In this subsection we suggest a definition for a generalized Steiner set for a given convex and compact set, i.e., we answer the question of how to embed such a set in the Banach space of all generalized Steiner sets. Thereby and as already stated above we will restrict ourselves in the following to the cases  $\mathcal{T} := \mathcal{T}_\diamond$  with  $\diamond \in \{\ell, u, s\}$ , as only these cases are of interest for our set optimization problems. Then the suggested embeddings save for given directions  $l \in \mathcal{T}_\diamond$  the exposed points in the case of unique support. For directions  $l$  with non-unique support we use the concept of Steiner points:

**Definition 3.7.** [28] The *Steiner point* for a set  $A \in \mathcal{C}(\mathbb{R}^m)$  is defined as

$$\text{St}(A) := \frac{1}{\mathcal{V}_m} \int_{\mathcal{B}(0_{\mathbb{R}^m}, 1)} m(Y(l, A)) dl,$$

where  $dl$  denotes the Lebesgue measure,  $\mathcal{V}_m$  is the Lebesgue measure of the unit ball  $\mathcal{B}(0_{\mathbb{R}^m}, 1) \subset \mathbb{R}^m$  and  $m(Y(l, A))$  denotes the (unique) norm-minimal point of the set  $Y(l, A)$ .

Note that in view of Lemma 2.2, the norm-minimal point  $m(Y(l, A))$  in Definition 3.7 coincides almost everywhere with the unique exposed point  $y(l, A)$  so that the choice for the non-unique case does not matter for the Lebesgue integral (and the Steiner point). Moreover, for sets  $A, B \in \mathcal{C}(\mathbb{R}^m)$ , scalars  $\alpha, \beta \in \mathbb{R}$ , and an orthogonal matrix  $R \in \mathbb{R}^{m \times m}$  it holds (see for instance [24, Sec. 1.7 and 5.4])

$$\text{St}(A) \in A, \quad \text{St}(\alpha A + \beta B) = \alpha \text{St}(A) + \beta \text{St}(B), \quad \text{and} \quad \text{St}(RA) = R \text{St}(A). \quad (14)$$

By using (14) we obtain for a singleton  $x \in \mathbb{R}^m$  immediately  $\text{St}(\{x\}) = x$  and  $\text{St}(\{-x\}) = -\text{St}(\{x\})$ . Further examples for which the Steiner point can easily be calculated either directly from the definition or from basic properties are listed in the following example.

**Example 3.8.** (i) For a line segment  $\text{co}\{z^1, z^2\}$  between two points  $z^1, z^2 \in \mathbb{R}^m$  the Steiner point is given by its midpoint, i.e.,  $\text{St}(\text{co}\{z^1, z^2\}) = \frac{1}{2}(z^1 + z^2)$ .

- (ii) For an Euclidean ball  $\mathcal{B}(c, r)$  with  $c \in \mathbb{R}^m$  and  $r > 0$  it holds  $\text{St}(\mathcal{B}(c, r)) = c$ .  
(iii) For an  $m$ -dimensional Box  $[r^1, r^2]_m := [r_1^1, r_1^2] \times [r_2^1, r_2^2] \times \dots \times [r_m^1, r_m^2] \subset \mathbb{R}^m$  with  $r^1, r^2 \in \mathbb{R}^m$  and  $r_i^1 \leq r_i^2$  for all  $i \in \{1, \dots, m\}$  it holds  $\text{St}([r^1, r^2]_m) = \frac{1}{2}(r^1 + r^2)$ .

Note that for a general polytope in  $\mathbb{R}^2$ , we can calculate the Steiner point with external angles as a convex combination of vertices, see [29] for more details. Moreover, for  $A \in \mathcal{C}(\mathbb{R}^m)$  it holds by using (3), (7), and (14)

$$\text{St}(Y(l, A)) \in Y(l, A) \subset \text{bd } A \subset A \text{ for all } l \in \mathbb{R}^m \setminus \{0_{\mathbb{R}^m}\}$$

and

$$l^\top \text{St}(Y(l, A)) = \delta^*(l, A) \text{ for all } l \in \mathbb{R}^m. \quad (15)$$

Steiner points are uniquely defined. This allows us to define the embeddings of nonempty, convex, and compact sets into the Banach space of generalized Steiner sets as follows:

**Definition 3.9.** Let Assumption 1 be fulfilled, let  $\diamond \in \{\ell, u, s\}$ , and let  $A \in \mathcal{C}(\mathbb{R}^m)$ . Then the *embedding* of the set  $A$  into the Banach space  $(\mathcal{GS}(\mathbb{R}^m, \mathcal{T}_\diamond), \oplus_{\mathcal{T}_\diamond}, \odot_{\mathcal{T}_\diamond})$  equipped with the norm of (12) is defined by

$$J_\diamond(A) := (\text{St}(Y(l, A)))_{l \in \mathcal{T}_\diamond}.$$

These embeddings are positive homogeneous of degree 1 and additive:

**Lemma 3.10.** *Let Assumption 1 be fulfilled and let  $\diamond \in \{\ell, u, s\}$ . Then it holds*

$$J_\diamond(\alpha A + \beta B) = (\alpha \odot_{\mathcal{T}_\diamond} J_\diamond(A)) \oplus_{\mathcal{T}_\diamond} (\beta \odot_{\mathcal{T}_\diamond} J_\diamond(B))$$

for  $A, B \in \mathcal{C}(\mathbb{R}^m)$  and  $\alpha, \beta \geq 0$ .

*Proof.* By definition and using (8) and (14) it holds

$$\begin{aligned} J_\diamond(\alpha A + \beta B) &= (\text{St}(Y(l, \alpha A + \beta B)))_{l \in \mathcal{T}_\diamond} \\ &= (\text{St}(\alpha Y(l, A) + \beta Y(l, B)))_{l \in \mathcal{T}_\diamond} \\ &= (\alpha \text{St}(Y(l, A)) + \beta \text{St}(Y(l, B)))_{l \in \mathcal{T}_\diamond} \\ &= (\alpha \odot_{\mathcal{T}_\diamond} J_\diamond(A)) \oplus_{\mathcal{T}_\diamond} (\beta \odot_{\mathcal{T}_\diamond} J_\diamond(B)). \end{aligned}$$

□

The embeddings defined in Definition 3.9 are in general not injective. This fact is not surprising, since the embeddings take into account only directions from  $\mathcal{T}_\diamond$  with  $\diamond \in \{\ell, u, s\}$ . On the other hand, if two sets  $A, B \in \mathcal{C}(\mathbb{R}^m)$  have an identical embedding in the space  $(\mathcal{GS}(\mathbb{R}^m, \mathcal{T}_\diamond), \oplus_{\mathcal{T}_\diamond}, \odot_{\mathcal{T}_\diamond})$ , then a necessary condition must be fulfilled which uses the set order relation  $\preceq_\diamond$  (cf. Lemma 3.12). For the proof of this lemma we need the following result:

**Lemma 3.11.** [14, Lemma 2.1, Remark 2.1, Remark 2.2, and Theorem 2.1] Let Assumption 1 be fulfilled, let  $A, B \in \mathcal{C}(\mathbb{R}^m)$ , and let the sign function be defined as in (2). Then for  $\diamond \in \{\ell, u, s\}$  it holds

$$(A \preceq_{\diamond} B) \Leftrightarrow (\forall l \in \mathcal{T}_{\diamond}: \text{sign}(l)\delta^*(l, A) \leq \text{sign}(l)\delta^*(l, B))$$

and

$$(A \preceq_{\diamond} B \text{ and } B \preceq_{\diamond} A) \Leftrightarrow (\forall l \in \mathcal{T}_{\diamond}: \delta^*(l, A) = \delta^*(l, B)).$$

The necessary condition for two nonempty, convex, and compact subsets with an identical embedding can now be stated and proved :

**Lemma 3.12.** Let Assumption 1 be fulfilled and let  $A, B \in \mathcal{C}(\mathbb{R}^m)$ . Then for  $\diamond \in \{\ell, u, s\}$  it holds

$$(J_{\diamond}(A) = J_{\diamond}(B)) \Rightarrow (A \preceq_{\diamond} B \text{ and } B \preceq_{\diamond} A). \quad (16)$$

*Proof.* By definition, using (15) and Lemma 3.11 it holds

$$\begin{aligned} J_{\diamond}(A) = J_{\diamond}(B) &\Leftrightarrow \forall l \in \mathcal{T}_{\diamond}: \text{St}(Y(l, A)) = \text{St}(Y(l, B)) \\ &\Rightarrow \forall l \in \mathcal{T}_{\diamond}: l^{\top} \text{St}(Y(l, A)) = l^{\top} \text{St}(Y(l, B)) \\ &\Leftrightarrow \forall l \in \mathcal{T}_{\diamond}: \delta^*(l, A) = \delta^*(l, B) \\ &\Leftrightarrow A \preceq_{\diamond} B \text{ and } B \preceq_{\diamond} A. \end{aligned}$$

□

Note that in general  $A \preceq_{\diamond} B$  and  $B \preceq_{\diamond} A$  does not imply  $J_{\diamond}(A) = J_{\diamond}(B)$  with  $\diamond \in \{\ell, u, s\}$  as the following example shows.

**Example 3.13.** Let  $C = C^* = \mathbb{R}_+^2$  and  $\diamond \in \{\ell, u, s\}$ . For the sets  $A, B \in \mathcal{C}(\mathbb{R}^2)$  with

$$A := [-1, 1] \times [-1, 1] \text{ and } B := A \cap \{y \in \mathbb{R}^2 \mid y_1 - 1 \leq y_2 \leq y_1 + 1\},$$

it holds  $A \preceq_{\diamond} B$  and  $B \preceq_{\diamond} A$  for all  $\diamond \in \{\ell, u, s\}$ ,

$$y(l, A) = y(l, B) = \begin{cases} (-1, -1)^{\top} & \text{if } l \in \mathcal{T}_{\ell} \setminus \{(-1, 0)^{\top}, (0, -1)^{\top}\} \\ (1, 1)^{\top} & \text{if } l \in \mathcal{T}_u \setminus \{(1, 0)^{\top}, (0, 1)^{\top}\} \end{cases},$$

$$\text{St}(Y(l, A)) = l \text{ if } l \in \{(-1, 0)^{\top}, (0, -1)^{\top}, (1, 0)^{\top}, (0, 1)^{\top}\},$$

$$\text{St}(Y(l, B)) = \begin{cases} (-1, -\frac{1}{2})^{\top} & \text{if } l = (-1, 0)^{\top} \\ (-\frac{1}{2}, -1)^{\top} & \text{if } l = (0, -1)^{\top} \\ (1, \frac{1}{2})^{\top} & \text{if } l = (1, 0)^{\top} \\ (\frac{1}{2}, 1)^{\top} & \text{if } l = (0, 1)^{\top} \end{cases},$$

and thus  $J_{\diamond}(A) \neq J_{\diamond}(B)$  for all  $\diamond \in \{\ell, u, s\}$ .

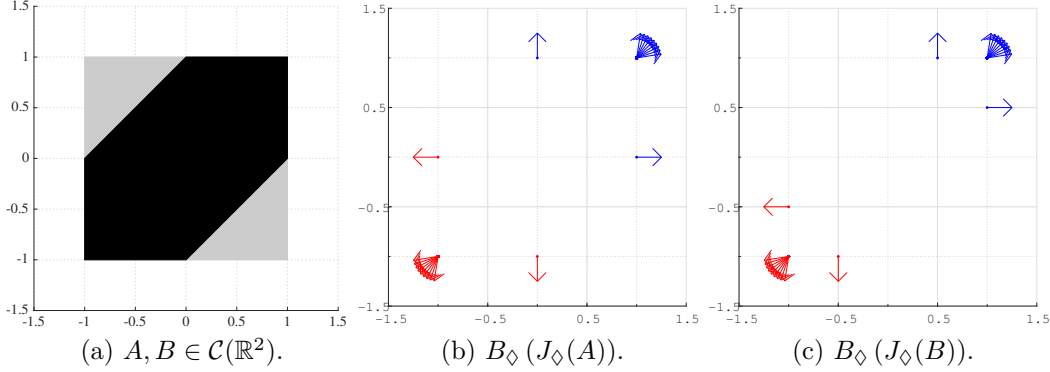


Figure 2: The sets  $A, B \in \mathcal{C}(\mathbb{R}^2)$  and the boundary parts of their embeddings for  $\diamond \in \{\ell, u, s\}$  for Example 3.13.

For an illustration see Figure 2. In subfigure 2a the sets  $A$  (the gray filled square) and  $B$  (overlapping in black) are shown. Moreover, the boundary parts  $B_\diamond(J_\diamond(A))$  and  $B_\diamond(J_\diamond(B))$  are depicted in subfigure 2b and 2c. Note that in both subfigures the blue points represent the points  $\text{St}(Y(l, A))$  and  $\text{St}(Y(l, B))$  for  $l \in \mathcal{T}_u$  while the red points represent the points  $\text{St}(Y(l, A))$  and  $\text{St}(Y(l, B))$  for  $l \in \mathcal{T}_\ell$ . Additionally, small arrows attached to these points illustrate some of the corresponding (scaled) directions  $l \in \mathcal{T}_s = \mathcal{T}_\ell \cup \mathcal{T}_u$ .

Hence, the reverse implication in (16) can only hold under additional assumptions. One needs the uniqueness of the supporting points, which is fulfilled for instance if  $A$  and  $B$  are strictly convex (cf. Lemma 2.1).

**Lemma 3.14.** *Let Assumption 1 be fulfilled, let  $A, B \in \mathcal{C}(\mathbb{R}^m)$ , and let  $\diamond \in \{\ell, u, s\}$ . If additionally for all  $l \in \mathcal{T}_\diamond$  the supporting faces  $Y(l, A)$  and  $Y(l, B)$  are singletons, i.e.,  $Y(l, A) = \{y(l, A)\}$  and  $Y(l, B) = \{y(l, B)\}$  for all  $l \in \mathcal{T}_\diamond$ , then it holds*

$$(J_\diamond(A) = J_\diamond(B)) \Leftrightarrow (A \preceq_\diamond B \text{ and } B \preceq_\diamond A). \quad (17)$$

*Proof.* We restrict ourselves to the case  $\diamond = \ell$  and use ideas of the proof of [30, Corollary 2.1]. Let  $A, B \in \mathcal{C}(\mathbb{R}^m)$  with  $A \preceq_\ell B$  and  $B \preceq_\ell A$ . Then by Lemma 3.11 it holds  $\delta^*(l, A) = l^\top y(l, A) = l^\top y(l, B) = \delta^*(l, B)$  for all  $l \in \mathcal{T}_\ell$ . Assume, there exists  $\hat{l} \in \mathcal{T}_\ell$  such that  $y(\hat{l}, A) \neq y(\hat{l}, B)$ . By the definition of  $A \preceq_\ell B$  it follows for  $y(\hat{l}, B)$  the existence of  $\hat{a} \in A$  and  $\hat{k} \in C$  such that  $y(\hat{l}, B) = \hat{a} + \hat{k}$  and hence

$$\hat{l}^\top y(\hat{l}, A) = \hat{l}^\top y(\hat{l}, B) = \hat{l}^\top (\hat{a} + \hat{k}) = \hat{l}^\top \hat{a} + \hat{l}^\top \hat{k}.$$

Using  $\hat{l} \in \mathcal{T}_\ell \subset (-C^*)$  it holds  $\hat{l}^\top \hat{k} \leq 0$  and by  $\hat{l}^\top \hat{a} \leq \hat{l}^\top y(\hat{l}, A)$  we obtain  $\hat{l}^\top \hat{k} = 0$  and  $\hat{l}^\top y(\hat{l}, A) = \hat{l}^\top \hat{a}$ . Thus using the uniqueness of the supporting points for all  $l \in \mathcal{T}_\diamond$  it holds  $y(\hat{l}, A) = \hat{a}$  and consequently  $y(\hat{l}, B) = y(\hat{l}, A) + \hat{k}$ . Analogously, using  $B \preceq_\ell A$  it can be shown that there exist  $\bar{k} \in C$  with  $y(\hat{l}, A) = y(\hat{l}, B) + \bar{k}$ . Due to  $y(\hat{l}, A) \neq y(\hat{l}, B)$  it holds  $\hat{k} \neq 0_{\mathbb{R}^m}$  and  $\bar{k} \neq 0_{\mathbb{R}^m}$  and we obtain  $-\hat{k} = y(\hat{l}, A) - y(\hat{l}, B) = \bar{k} -$  contradicting the pointedness of  $C$ . Finally, by using Lemma 3.12 we are done.  $\square$



It is easy to see that for all  $A, B \in \mathcal{C}(\mathbb{R}^m)$  and  $\diamond \in \{\ell, u, s\}$  it holds

$$(B_{\mathcal{T}_\diamond}(J_\diamond(A) \ominus_{\mathcal{T}_\diamond} J_\diamond(B)) = \{0_{\mathbb{R}^m}\}) \Leftrightarrow (J_\diamond(A) = J_\diamond(B)), \quad (18)$$

and thus under the additional assumption of Lemma 3.14 we can extend (17) and (18) to

$$\begin{aligned} (B_{\mathcal{T}_\diamond}(J_\diamond(A) \ominus_{\mathcal{T}_\diamond} J_\diamond(B)) = \{0_{\mathbb{R}^m}\}) &\Leftrightarrow (J_\diamond(A) = J_\diamond(B)) \\ &\Leftrightarrow (A \preceq_\diamond B \text{ and } B \preceq_\diamond A). \end{aligned}$$

To conclude this section, we study the visualization introduced in Definition 3.5 in more detail for the generalized Steiner sets, which we obtain by the embeddings of nonempty, convex, and compact sets given by Definition 3.9. Note that for instance the mixed-type part is empty in this case. These results will be used in Section 4 for the formulation of optimality conditions for set optimization problems.

**Proposition 3.15.** *Let Assumption 1 be fulfilled, let  $\diamond \in \{\ell, u, s\}$ , and let  $A \in \mathcal{C}(\mathbb{R}^m)$ . Then it holds*

$$P_{\mathcal{T}_\ell}(J_\ell(A)) = A + C, \quad (19)$$

$$P_{\mathcal{T}_u}(J_u(A)) = A - C, \quad (20)$$

$$P_{\mathcal{T}_s}(J_s(A)) = (A + C) \cap (A - C), \text{ and} \quad (21)$$

$$M_{\mathcal{T}_\diamond}(J_\diamond(A)) = \emptyset. \quad (22)$$

If additionally  $C = C^* = \mathbb{R}_+^m$  and the points  $\underline{a}, \bar{a} \in \mathbb{R}^m$  are defined by

$$\underline{a}_i := \min_{a \in A} a_i \text{ and } \bar{a}_i := \max_{a \in A} a_i \text{ for all } i \in \{1, \dots, m\},$$

then it holds

$$N_{\mathcal{T}_\ell}(J_\ell(A)) = \{\underline{a}\} - \mathbb{R}_+^m \text{ and } N_{\mathcal{T}_u}(J_u(A)) = \{\bar{a}\} + \mathbb{R}_+^m. \quad (23)$$

*Proof.* By definition and (9) we have

$$\begin{aligned} P_{\mathcal{T}_\ell}(J_\ell(A)) &= \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \leq l^\top \text{St}(Y(l, A))\} \\ &= \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \leq \delta^*(l, A)\} \\ &= A + C \end{aligned}$$

and (20) can be proved in an analogous way. Thus (21) follows by

$$P_{\mathcal{T}_s}(J_s(A)) = P_{\mathcal{T}_\ell}(J_s(A)) \cap P_{\mathcal{T}_u}(J_s(A)).$$

By using the previously proven results and

$$B_{\mathcal{T}_\diamond}(J_\diamond(A)) \subset A \subset (A + C) \cap (A - C) \subset (A \pm C)$$

we obtain (22) by definition.

Let now  $C := \mathbb{R}_+^m$  and thus  $C = C^* = \mathbb{R}_+^m = \text{cone}(\text{co}(\{e^i \mid i \in \{1, \dots, m\}\}))$  where  $e^i$ ,  $i \in \{1, \dots, m\}$  denotes the  $i$ -th unit vector of  $\mathbb{R}^m$ . For the defined points  $\underline{a}, \bar{a} \in \mathbb{R}^m$  we obtain by the definition of the support function

$$\underline{a}_i = \min_{a \in A} a_i = -\max_{a \in A} (-e^i)^\top a = -\delta^*(-e^i, A) = (e^i)^\top \text{St}(Y(-e^i, A)) \quad (24)$$

and

$$\bar{a}_i = \max_{a \in A} a_i = \max_{a \in A} (e^i)^\top a = \delta^*(e^i, A) = (e^i)^\top \text{St}(Y(e^i, A))$$

for all  $i \in \{1, \dots, m\}$ . Moreover, it follows

$$l^\top a = \sum_{i=1}^m l_i a_i \leq \sum_{i=1}^m l_i \underline{a}_i = l^\top \underline{a} \quad \text{for all } l \in \mathcal{T}_\ell \subset (-\mathbb{R}_+^m) \text{ and all } a \in A$$

and

$$l^\top a = \sum_{i=1}^m l_i a_i \leq \sum_{i=1}^m l_i \bar{a}_i = l^\top \bar{a} \quad \text{for all } l \in \mathcal{T}_u \subset \mathbb{R}_+^m \text{ and all } a \in A.$$

Thus we obtain

$$l^\top \text{St}(Y(l, A)) \leq l^\top \underline{a} \quad \text{for all } l \in \mathcal{T}_\ell \subset (-\mathbb{R}_+^m) \quad (25)$$

and

$$l^\top \text{St}(Y(l, A)) \leq l^\top \bar{a} \quad \text{for all } l \in \mathcal{T}_u \subset \mathbb{R}_+^m.$$

In the following we restrict ourselves to the proof of the left equation in (23). It follows by definition and (24)

$$\begin{aligned} N_{\mathcal{T}_\ell}(J_\ell(A)) &= \bigcap_{\ell \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \geq l^\top \text{St}(Y(l, A))\} \\ &\subset \bigcap_{i \in \{1, \dots, m\}} \{y \in \mathbb{R}^m \mid (-e^i)^\top y \geq (-e^i)^\top \text{St}(Y(-e^i, A))\} \\ &= \bigcap_{i \in \{1, \dots, m\}} \{y \in \mathbb{R}^m \mid (e^i)^\top y \leq (e^i)^\top \text{St}(Y(-e^i, A))\} \\ &= \bigcap_{i \in \{1, \dots, m\}} \{y \in \mathbb{R}^m \mid y_i \leq \underline{a}_i\} \\ &= \{\underline{a}\} - \mathbb{R}_+^m. \end{aligned}$$

Finally, we obtain by (10), as  $\mathcal{T}_u = -\mathcal{T}_\ell$ , and by (25)

$$\begin{aligned} \{\underline{a}\} - \mathbb{R}_+^m &= \bigcap_{\ell \in \mathcal{T}_u} \{y \in \mathbb{R}^m \mid l^\top y \leq l^\top \underline{a}\} \\ &= \bigcap_{\ell \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid -l^\top y \leq -l^\top \underline{a}\} \\ &= \bigcap_{\ell \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \geq l^\top \underline{a}\} \\ &\subset \bigcap_{\ell \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \geq l^\top \text{St}(Y(l, A))\} \\ &= N_{\mathcal{T}_\ell}(J_\ell(A)) \end{aligned}$$

and we are done.  $\square$

Note that the defined points  $\underline{a}, \bar{a} \in \mathbb{R}^m$  in Proposition 3.15 are often denoted as the *ideal point* and as the *antiideal point* of the set  $A \in \mathcal{C}(\mathbb{R}^m)$ , respectively. By using  $0_{\mathbb{R}^m} \in C$  and (19), (20), and (21) it follows

$$A \subset P_{\diamond}(J_{\diamond}(A)) \text{ for } A \in \mathcal{C}(\mathbb{R}^m) \text{ and } \diamond \in \{\ell, u, s\}. \quad (26)$$

## 4 Optimality conditions for set optimization

The aim of this paper are optimality conditions for set optimization problems. In view of this we need the following assumption:

**Assumption 2.** *Let Assumption 1 be fulfilled. Moreover, let  $S \subset \mathbb{R}^n$  with  $S \neq \emptyset$ ,  $\hat{S}$  be an open superset of  $S$ , and let  $F: \hat{S} \rightrightarrows \mathbb{R}^m$  be a set-valued map with  $F(x) \in \mathcal{C}(\mathbb{R}^m)$  for all  $x \in \hat{S}$ .*

Under this assumption we study the set optimization problem

$$\min_{x \in S} F(x). \quad (\text{SOP}_{F,S})$$

A point  $\bar{x} \in S$  is called a *minimal solution* of  $(\text{SOP}_{F,S})$  w.r.t. the order relation  $\preceq_{\diamond}$  and  $\diamond \in \{\ell, u, s\}$  if

$$F(x) \preceq_{\diamond} F(\bar{x}), x \in S \implies F(\bar{x}) \preceq_{\diamond} F(x) \quad (27)$$

holds. Obviously, this is equivalent to the fact that there exists no  $x \in S$  such that

$$F(x) \preceq_{\diamond} F(\bar{x}) \text{ and } F(\bar{x}) \not\preceq_{\diamond} F(x).$$

Moreover, by using Lemma 3.11 one obtains for the set optimization problem  $(\text{SOP}_{F,S})$  the following characterization of minimal solutions (cf. [13, Theorem 5.3]):

**Lemma 4.1.** *Let Assumption 2 be fulfilled and let  $\diamond \in \{\ell, u, s\}$ . Then  $\bar{x} \in S$  is a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_{\diamond}$  if and only if there is no  $x \in S$  such that*

$$\begin{aligned} \forall l \in \mathcal{T}_{\diamond}: \quad & \text{sign}(l)\delta^*(l, F(x)) \leq \text{sign}(l)\delta^*(l, F(\bar{x})) \quad \text{and} \\ \exists \hat{l} \in \mathcal{T}_{\diamond}: \quad & \text{sign}(\hat{l})\delta^*(\hat{l}, F(x)) < \text{sign}(\hat{l})\delta^*(\hat{l}, F(\bar{x})). \end{aligned}$$

We illustrate the definition of a minimal solution of a set optimization problem with the following example, which we will also use in the following to illustrate our new optimality conditions.

**Example 4.2.** Let  $\hat{S} = \mathbb{R}$ ,  $S = [0, 1]$ ,  $C = C^* = \mathbb{R}_+^2$ , and  $A := [0, 1] \times [0, 1] \subset \mathbb{R}^2$ . Moreover, we define the set-valued map  $F: \hat{S} \rightrightarrows \mathbb{R}^2$  by

$$F(x) := A \cap \{y \in \mathbb{R}^2 \mid y_1 + y_2 \geq x\}.$$

It is obvious from the geometrical construction of the sets  $F(x)$  or by Lemma 4.1 that the unique minimal solution of  $(\text{SOP}_{F,S})$  w.r.t. the order relations  $\preceq_{\ell}$  and  $\preceq_s$  is given by  $\bar{x} = 0$ , and all feasible points are minimal solutions of  $(\text{SOP}_{F,S})$  w.r.t. the order relation  $\preceq_u$ .

## 4.1 Optimality conditions for set optimization problems based on visualization results

In this subsection we formulate optimality conditions for set optimization problems based on visualization results for generalized Steiner sets. One possible approach is the application of Proposition 3.15 to reformulate the definitions of the set order relations. Thereby we obtain for  $A, B \in \mathcal{C}(\mathbb{R}^m)$  for instance the following equivalences:

**Lemma 4.3.** *Let Assumption 1 be fulfilled and let  $A, B \in \mathcal{C}(\mathbb{R}^m)$ . Then it holds*

$$\begin{aligned} (A \preceq_\ell B) &\Leftrightarrow (P_{\mathcal{T}_\ell}(J_\ell(B)) \subset P_{\mathcal{T}_\ell}(J_\ell(A))) \text{ and} \\ (A \preceq_u B) &\Leftrightarrow (P_{\mathcal{T}_u}(J_u(A)) \subset P_{\mathcal{T}_u}(J_u(B))). \end{aligned}$$

*Proof.* We restrict ourselves to the proof of the first equivalence. Thus, let  $A \preceq_\ell B$ . Then by definition and Proposition 3.15 we obtain

$$\begin{aligned} A \preceq_\ell B &\Leftrightarrow B \subset A + C \\ &\Rightarrow B + C \subset A + (C + C) = A + C \\ &\Leftrightarrow P_{\mathcal{T}_\ell}(J_\ell(B)) \subset P_{\mathcal{T}_\ell}(J_\ell(A)). \end{aligned}$$

If  $P_{\mathcal{T}_\ell}(J_\ell(B)) \subset P_{\mathcal{T}_\ell}(J_\ell(A))$ , then  $A \preceq_\ell B$  follows immediately by Proposition 3.15 and (26):

$$B \subset P_{\mathcal{T}_\ell}(J_\ell(B)) \subset P_{\mathcal{T}_\ell}(J_\ell(A)) = A + C.$$

□

As a consequence of Lemma 4.3 we obtain

$$\begin{aligned} (A \preceq_\ell B \text{ and } B \preceq_\ell A) &\Leftrightarrow (P_{\mathcal{T}_\ell}(J_\ell(A)) = P_{\mathcal{T}_\ell}(J_\ell(B))) \text{ and} \\ (A \preceq_u B \text{ and } B \preceq_u A) &\Leftrightarrow (P_{\mathcal{T}_u}(J_u(A)) = P_{\mathcal{T}_u}(J_u(B))), \end{aligned}$$

and thus by definition also

$$\begin{aligned} (A \preceq_s B \text{ and } B \preceq_s A) &\Leftrightarrow ((P_{\mathcal{T}_\diamond}(J_\diamond(A)) = P_{\mathcal{T}_\diamond}(J_\diamond(B))) \text{ for } \diamond \in \{\ell, u\}) \\ &\Rightarrow (P_{\mathcal{T}_s}(J_s(A)) = P_{\mathcal{T}_s}(J_s(B))). \end{aligned}$$

The following lemma gives additional reformulations of the set relations:

**Lemma 4.4.** *Let Assumption 1 be fulfilled and let  $A, B \in \mathcal{C}(\mathbb{R}^m)$ . Then it holds*

$$(A \preceq_\ell B) \Leftrightarrow (0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(A)) \ominus_{\mathcal{T}_\ell} J_\ell(B))$$

and

$$(A \preceq_u B) \Leftrightarrow (0_{\mathbb{R}^m} \in P_{\mathcal{T}_u}(J_u(B)) \ominus_{\mathcal{T}_u} J_u(A))$$

*Proof.* Again we restrict ourselves to the proof of the first equivalence. By Lemma 3.11 and the definition of the positive part we obtain

$$\begin{aligned}
A \preceq_\ell B &\Leftrightarrow \forall l \in \mathcal{T}_\ell: -\delta^*(l, A) \leq -\delta^*(l, B) \\
&\Leftrightarrow \forall l \in \mathcal{T}_\ell: 0 \leq \delta^*(l, A) - \delta^*(l, B) \\
&\Leftrightarrow \forall l \in \mathcal{T}_\ell: l^\top 0_{\mathbb{R}^m} \leq l^\top \text{St}(Y(l, A)) - l^\top \text{St}(Y(l, B)) \\
&\Leftrightarrow \forall l \in \mathcal{T}_\ell: l^\top 0_{\mathbb{R}^m} \leq l^\top (\text{St}(Y(l, A)) - \text{St}(Y(l, B))) \\
&\Leftrightarrow 0_{\mathbb{R}^m} \in \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \leq l^\top (\text{St}(Y(l, A)) - \text{St}(Y(l, B)))\} \\
&\Leftrightarrow 0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}((\text{St}(Y(l, A)) - \text{St}(Y(l, B)))_{l \in \mathcal{T}_\ell}) \\
&\Leftrightarrow 0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}((\text{St}(Y(l, A)))_{l \in \mathcal{T}_\ell} \ominus_{\mathcal{T}_\ell} (\text{St}(Y(l, B)))_{l \in \mathcal{T}_\ell}) \\
&\Leftrightarrow 0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B)).
\end{aligned}$$

□

Using these reformulations the following optimality conditions for set optimization problems follow immediately:

**Corollary 4.5.** *Let Assumption 2 be fulfilled.*

(a) *Then  $\bar{x} \in S$  is a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_\ell$  if and only if for all  $x \in S$  the following implication holds*

$$(0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(F(x)) \ominus_{\mathcal{T}_\ell} J_\ell(F(\bar{x})))) \implies (0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(F(\bar{x})) \ominus_{\mathcal{T}_\ell} J_\ell(F(x)))).$$

(b) *Then  $\bar{x} \in S$  is a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_u$  if and only if for all  $x \in S$  the following implication holds*

$$(0_{\mathbb{R}^m} \in P_{\mathcal{T}_u}(J_u(F(\bar{x})) \ominus_{\mathcal{T}_u} J_u(F(x)))) \implies (0_{\mathbb{R}^m} \in P_{\mathcal{T}_u}(J_u(F(x)) \ominus_{\mathcal{T}_u} J_u(F(\bar{x}))).$$

Finally, using the statements of Lemma 4.4 further characterizations of the set relations can be given:

**Lemma 4.6.** *Let Assumption 1 be fulfilled and let  $A, B \in \mathcal{C}(\mathbb{R}^m)$ . Then it holds*

$$\begin{aligned}
&(A \preceq_\ell B \text{ and } B \preceq_\ell A) \\
&\Leftrightarrow (P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B)) = P_{\mathcal{T}_\ell}(J_\ell(B) \ominus_{\mathcal{T}_\ell} J_\ell(A)) = C), \\
&(A \preceq_u B \text{ and } B \preceq_u A) \\
&\Leftrightarrow (P_{\mathcal{T}_u}(J_u(A) \ominus_{\mathcal{T}_u} J_u(B)) = P_{\mathcal{T}_u}(J_u(B) \ominus_{\mathcal{T}_u} J_u(A)) = -C), \tag{28}
\end{aligned}$$

and

$$\begin{aligned}
&(A \preceq_s B \text{ and } B \preceq_s A) \\
&\Leftrightarrow (P_{\mathcal{T}_s}(J_s(A) \ominus_{\mathcal{T}_s} J_s(B)) = P_{\mathcal{T}_s}(J_s(B) \ominus_{\mathcal{T}_s} J_s(A)) = \{0_{\mathbb{R}^m}\}).
\end{aligned}$$

*Proof.* If  $A \preceq_\ell B$  and  $B \preceq_\ell A$ , then by definition of the positive part, Lemma 3.11, and (9) it holds

$$\begin{aligned}
P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B)) &= P_{\mathcal{T}_\ell}((\text{St}(Y(l, A)))_{l \in \mathcal{T}_\ell} \ominus_{\mathcal{T}_\ell} (\text{St}(Y(l, B)))_{l \in \mathcal{T}_\ell}) \\
&= P_{\mathcal{T}_\ell}((\text{St}(Y(l, A)) - \text{St}(Y(l, B)))_{l \in \mathcal{T}_\ell}) \\
&= \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \leq l^\top (\text{St}(Y(l, A)) - \text{St}(Y(l, B)))\} \\
&= \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \leq l^\top \text{St}(Y(l, A)) - l^\top \text{St}(Y(l, B))\} \\
&= \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \leq \delta^*(l, A) - \delta^*(l, B)\} \\
&= \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^m \mid l^\top y \leq 0 = \delta^*(l, \{0_{\mathbb{R}^m}\})\} \\
&= \{0_{\mathbb{R}^m}\} + C \\
&= C
\end{aligned}$$

and by analogous conclusions  $P_{\mathcal{T}_\ell}(J_\ell(B) \ominus_{\mathcal{T}_\ell} J_\ell(A)) = C$ .

Otherwise, if  $P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B)) = P_{\mathcal{T}_\ell}(J_\ell(B) \ominus_{\mathcal{T}_\ell} J_\ell(A)) = C$ , then  $0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B))$  and  $0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(B) \ominus_{\mathcal{T}_\ell} J_\ell(A))$  are true and by Lemma 4.4 it follows  $A \preceq_\ell B$  and  $B \preceq_\ell A$ .

The equivalence in (28) can be proved by using similar arguments.

Let now  $A \preceq_s B$  and  $B \preceq_s A$ , then it follows by the pointedness of  $C$

$$\begin{aligned}
&A \preceq_s B \text{ and } B \preceq_s A \\
\Leftrightarrow &A \preceq_\ell B \text{ and } B \preceq_\ell A \text{ and } A \preceq_u B \text{ and } B \preceq_u A \\
\Leftrightarrow &P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B)) = P_{\mathcal{T}_\ell}(J_\ell(B) \ominus_{\mathcal{T}_\ell} J_\ell(A)) = C \text{ and} \\
&P_{\mathcal{T}_u}(J_u(A) \ominus_{\mathcal{T}_u} J_u(B)) = P_{\mathcal{T}_u}(J_u(B) \ominus_{\mathcal{T}_u} J_u(A)) = -C \\
\Rightarrow &P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B)) \cap P_{\mathcal{T}_u}(J_u(A) \ominus_{\mathcal{T}_u} J_u(B)) = \{0_{\mathbb{R}^m}\} \text{ and} \\
&P_{\mathcal{T}_\ell}(J_\ell(B) \ominus_{\mathcal{T}_\ell} J_\ell(A)) \cap P_{\mathcal{T}_u}(J_u(B) \ominus_{\mathcal{T}_u} J_u(A)) = \{0_{\mathbb{R}^m}\} \\
\Leftrightarrow &P_{\mathcal{T}_s}(J_s(A) \ominus_{\mathcal{T}_s} J_s(B)) = P_{\mathcal{T}_s}(J_s(B) \ominus_{\mathcal{T}_s} J_s(A)) = \{0_{\mathbb{R}^m}\}.
\end{aligned}$$

Finally, if  $P_{\mathcal{T}_s}(J_s(A) \ominus_{\mathcal{T}_s} J_s(B)) = P_{\mathcal{T}_s}(J_s(B) \ominus_{\mathcal{T}_s} J_s(A)) = \{0_{\mathbb{R}^m}\}$ , then it holds by Lemma 4.4

$$\begin{aligned}
&P_{\mathcal{T}_s}(J_s(A) \ominus_{\mathcal{T}_s} J_s(B)) = P_{\mathcal{T}_s}(J_s(B) \ominus_{\mathcal{T}_s} J_s(A)) = \{0_{\mathbb{R}^m}\} \\
\Leftrightarrow &P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B)) \cap P_{\mathcal{T}_u}(J_u(A) \ominus_{\mathcal{T}_u} J_u(B)) = \{0_{\mathbb{R}^m}\} \text{ and} \\
&P_{\mathcal{T}_\ell}(J_\ell(B) \ominus_{\mathcal{T}_\ell} J_\ell(A)) \cap P_{\mathcal{T}_u}(J_u(B) \ominus_{\mathcal{T}_u} J_u(A)) = \{0_{\mathbb{R}^m}\} \\
\Rightarrow &0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(A) \ominus_{\mathcal{T}_\ell} J_\ell(B)) \text{ and } 0_{\mathbb{R}^m} \in P_{\mathcal{T}_u}(J_u(A) \ominus_{\mathcal{T}_u} J_u(B)) \text{ and} \\
&0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(B) \ominus_{\mathcal{T}_\ell} J_\ell(A)) \text{ and } 0_{\mathbb{R}^m} \in P_{\mathcal{T}_u}(J_u(B) \ominus_{\mathcal{T}_u} J_u(A)) \\
\Leftrightarrow &A \preceq_\ell B \text{ and } B \preceq_u A \text{ and } B \preceq_\ell A \text{ and } A \preceq_u B \\
\Leftrightarrow &A \preceq_s B \text{ and } B \preceq_s A.
\end{aligned}$$

□

For deriving now optimality conditions for  $(\text{SOP}_{F,S})$  note that the implication (27) can also be reformulated as

$$F(x) \preceq_\diamond F(\bar{x}), x \in S \implies (F(x) \preceq_\diamond F(\bar{x}) \text{ and } F(\bar{x}) \preceq_\diamond F(x)).$$

Hence, we obtain by Lemma 4.4 and Lemma 4.6 the following optimality condition:

**Corollary 4.7.** *Let Assumption 2 be fulfilled.*

(a) *Then  $\bar{x} \in S$  is a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_\ell$  if and only if for all  $x \in S$  the following implication holds*

$$\begin{aligned} & (0_{\mathbb{R}^m} \in P_{\mathcal{T}_\ell}(J_\ell(F(x)) \ominus_{\mathcal{T}_\ell} J_\ell(F(\bar{x})))) \\ \implies & (P_{\mathcal{T}_\ell}(J_\ell(F(x)) \ominus_{\mathcal{T}_\ell} J_\ell(F(\bar{x}))) = P_{\mathcal{T}_\ell}(J_\ell(F(\bar{x})) \ominus_{\mathcal{T}_\ell} J_\ell(F(x))) = C). \end{aligned}$$

(b) *Then  $\bar{x} \in S$  is a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_u$  if and only if for all  $x \in S$  the following implication holds*

$$\begin{aligned} & (0_{\mathbb{R}^m} \in P_{\mathcal{T}_u}(J_u(F(\bar{x})) \ominus_{\mathcal{T}_u} J_u(F(x)))) \\ \implies & (P_{\mathcal{T}_u}(J_u(F(\bar{x})) \ominus_{\mathcal{T}_u} J_u(F(x))) = P_{\mathcal{T}_u}(J_u(F(x)) \ominus_{\mathcal{T}_u} J_u(F(\bar{x}))) = -C). \end{aligned}$$

In conclusion of this subsection we apply our formulated optimality conditions on the following example:

**Example 4.8.** We consider again the set optimization problem  $(\text{SOP}_{F,S})$  defined in Example 4.2. We verify the statements regarding the minimal solution made there by our optimality conditions for  $\diamond \in \{\ell, u\}$ .

Thereby it holds

$$\text{St}(Y(l, F(0))) = \begin{cases} (1, \frac{1}{2})^\top & , \text{ if } l = (1, 0)^\top \\ (1, 1)^\top & , \text{ if } l \in \mathcal{T}_u \setminus \{(1, 0)^\top, (0, 1)^\top\} \\ (\frac{1}{2}, 1)^\top & , \text{ if } l = (0, 1)^\top \\ (0, \frac{1}{2})^\top & , \text{ if } l = (-1, 0)^\top \\ (0, 0)^\top & , \text{ if } l \in \mathcal{T}_\ell \setminus \{(-1, 0)^\top, (0, -1)^\top\} \\ (\frac{1}{2}, 0)^\top & , \text{ if } l = (0, -1)^\top \end{cases}$$

and

$$\text{St}(Y(l, F(x))) = \begin{cases} (1, \frac{1}{2})^\top & , \text{ if } l = (1, 0)^\top \\ (1, 1)^\top & , \text{ if } l \in \mathcal{T}_u \setminus \{(1, 0)^\top, (0, 1)^\top\} \\ (\frac{1}{2}, 1)^\top & , \text{ if } l = (0, 1)^\top \\ (0, \frac{1+x}{2})^\top & , \text{ if } l \in \mathcal{T}_\ell \text{ and } l_1 < l_2 \\ (\frac{x}{2}, \frac{x}{2})^\top & , \text{ if } l = -\frac{1}{\sqrt{2}}(1, 1)^\top \\ (\frac{1+x}{2}, 0)^\top & , \text{ if } l \in \mathcal{T}_\ell \text{ and } l_1 > l_2 \end{cases}$$

for all  $x \in (0, 1]$ .

First, we consider the minimal solutions of  $(\text{SOP}_{F,S})$  w.r.t. the  $\ell$ -less order relation  $\preceq_\ell$ . In this case, for all  $x \in (0, 1]$  we obtain

$$P_{\mathcal{T}_\ell}(J_\ell(F(x)) \ominus_{\mathcal{T}_\ell} J_\ell(F(0))) = \bigcap_{l \in \mathcal{T}_\ell} \{y \in \mathbb{R}^2 \mid l^\top y \leq l^\top (\text{St}(Y(l, F(x))) - \text{St}(Y(l, F(0))))\}$$

with

$$\text{St}(Y(l, F(x))) - \text{St}(Y(l, F(0))) = \begin{cases} (0, \frac{x}{2})^\top & , \text{ if } l = (-1, 0)^\top \\ (0, \frac{1+x}{2})^\top & , \text{ if } l \in \mathcal{T}_\ell \text{ and } -1 < l_1 < l_2 \\ (\frac{x}{2}, \frac{x}{2})^\top & , \text{ if } l = -\frac{1}{\sqrt{2}}(1, 1)^\top \\ (\frac{1+x}{2}, 0)^\top & , \text{ if } l \in \mathcal{T}_\ell \text{ and } l_1 > l_2 > -1 \\ (\frac{x}{2}, 0)^\top & , \text{ if } l = (0, -1)^\top \end{cases}.$$

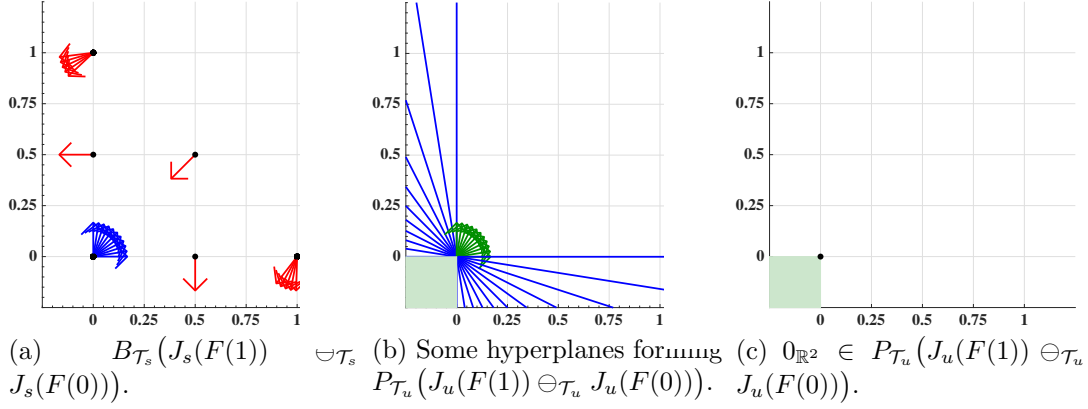


Figure 3: Visualization results in Example 4.8.

For  $x = 1$ , the differences  $\text{St}(Y(l, F(1))) - \text{St}(Y(l, F(0))) \in \mathbb{R}^2$  are illustrated in subfigure 3a with black dots, where the directions  $l \in \mathcal{T}_\ell$  are shown scaled as red arrows. The union of these differences is then the boundary part  $B_{\mathcal{T}_\ell}(J_\ell(F(1)) \ominus_{\mathcal{T}_s} J_\ell(F(0)))$ .

Moreover, for  $l = -\frac{1}{\sqrt{2}}(1, 1)^\top \in \mathcal{T}_\ell$  it holds

$$\{y \in \mathbb{R}^2 \mid l^\top y \leq l^\top (\text{St}(Y(l, F(x))) - \text{St}(Y(l, F(0))))\} = \{y \in \mathbb{R}^2 \mid y_1 + y_2 \geq x\}$$

and thus

$$0_{\mathbb{R}^2} \notin P_{\mathcal{T}_\ell}(J_\ell(F(x)) \ominus_{\mathcal{T}_\ell} J_\ell(F(0))) \quad (29)$$

for all  $x \in (0, 1]$ . Hence, by using Corollary 4.5 (a) or Corollary 4.7 (a) we obtain that  $\bar{x} = 0 \in S$  is a minimal solution of the set optimization problem  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_\ell$ .

Note that by (29) it also follows

$$P_{\mathcal{T}_\ell}(J_\ell(F(x)) \ominus_{\mathcal{T}_\ell} J_\ell(F(0))) \neq C = \mathbb{R}_+^2$$

for all  $x \in (0, 1]$ . Moreover, it is easy to see that  $\text{St}(Y(l, F(0))) - \text{St}(Y(l, F(x))) \in -\mathbb{R}_+^2$  and thus  $0 \leq l^\top (\text{St}(Y(l, F(0))) - \text{St}(Y(l, F(x))))$  hold for all  $x \in (0, 1]$  and all  $l \in \mathcal{T}_\ell \subset -\mathbb{R}_+^2$ .

If we now consider the minimal solutions of  $(\text{SOP}_{F,S})$  w.r.t. the  $u$ -less order relation  $\preceq_u$ , then for all  $x^1, x^2 \in [0, 1]$  and  $l \in \mathcal{T}_u$  it holds  $\text{St}(Y(l, F(x^1))) = \text{St}(Y(l, F(x^2)))$  and we get

$$J_u(F(x^1)) \ominus_{\mathcal{T}_u} J_u(F(x^2)) = (\text{St}(Y(l, F(x^1))) - \text{St}(Y(l, F(x^2))))_{l \in \mathcal{T}_u} = 0_{\mathcal{G}S, \mathcal{T}_u}$$

(cf. subfigure 3a for the case  $x^1 = 1$  and  $x^2 = 0$ , the directions  $l \in \mathcal{T}_u$  are shown scaled as blue arrows) and thus

$$0_{\mathbb{R}^2} \in P_{\mathcal{T}_u}(J_u(F(x^1)) \ominus_{\mathcal{T}_u} J_u(F(x^2))) = \bigcap_{l \in \mathcal{T}_u} \{y \in \mathbb{R}^2 \mid l^\top y \leq 0\} = -\mathbb{R}_+^2 = -C.$$



By Corollary 4.5 (b) or Corollary 4.7 (b) we obtain that all  $x \in [0, 1]$  are minimal solutions of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_u$ .

In subfigure 3b, the positive part  $P_{\mathcal{T}_u}(J_u(F(1)) \ominus_{\mathcal{T}_u} J_u(F(0)))$  is given in light green. The blue lines illustrate some of the hyperplanes

$$\{y \in \mathbb{R}^2 \mid l^\top y = l^\top (\text{St}(Y(l, F(1))) - \text{St}(Y(l, F(0)))) = 0\}$$

for  $l \in \mathcal{T}_u$  which define the half-spaces forming the positive part. The green arrows give the (scaled) directions  $l$  which are normal vectors to these hyperplanes. Finally, subfigure 3c shows again the positive part  $P_{\mathcal{T}_u}(J_u(F(1)) \ominus_{\mathcal{T}_u} J_u(F(0)))$ , which contains here the zero element  $0_{\mathbb{R}^2}$ .

## 4.2 A new directional derivative for set optimization

To formulate in the following a necessary condition for a minimal solution of the set optimization problem  $(\text{SOP}_{F,S})$  we use a new directional derivative for set-valued maps embedded in the spaces of generalized Steiner sets. Before we do this, we first recall the concept of a directional derivative for a vector-valued map. Let  $\hat{S}$  be an open and nonempty subset of  $\mathbb{R}^n$ , let  $f: \hat{S} \rightarrow \mathbb{R}^m$  be a vector-valued map, let  $x \in \hat{S}$ , and let  $d \in \mathbb{R}^n$ . Then the map  $f$  is called *directionally differentiable at  $x$  in direction  $d$*  if the limit

$$D[f(\cdot)](x; d) := \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (f(x + \lambda d) - f(x))$$

exists. Note that we will use in the following this concept for the special setting  $f(\cdot) := \text{St}(Y(l, F(\cdot)))$  with  $l \in \mathcal{T}_\diamond$  and  $\diamond \in \{\ell, u, s\}$ .

We define the proposed new directional derivative for set-valued maps in the Banach spaces of generalized Steiner sets as follows:

**Definition 4.9.** Let Assumption 2 be fulfilled, let  $x \in \hat{S}$ , let  $d \in \mathbb{R}^n$ , and let  $\diamond \in \{\ell, u, s\}$ . Then the set-valued map  $F$  is called  *$\mathcal{GS}_\diamond$ -directionally differentiable at  $x$  in direction  $d$*  if in the Banach space  $(\mathcal{GS}(\mathbb{R}^m, \mathcal{T}_\diamond), \oplus_{\mathcal{T}_\diamond}, \odot_{\mathcal{T}_\diamond})$  equipped with the norm of (12) the limit

$$DF_\diamond^{\mathcal{GS}}(x; d) := \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(F(x + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(F(x)))$$

exists.

A map  $F$  is by definition  $\mathcal{GS}_s$ -directionally differentiable at  $x$  in direction  $d$  if and only if  $F$  is  $\mathcal{GS}_\ell$ -directionally differentiable and  $\mathcal{GS}_u$ -directionally differentiable at  $x$  in direction  $d$ . The advantage of this directional derivative is that it can be calculated numerically in many cases and that it is also applicable to set-valued maps with images which are not strictly convex. What is more, we can give optimality conditions based on it. Nevertheless, the assumption of  $\mathcal{GS}_\diamond$ -directional differentiability for a set-valued map is already a strong assumption, as the following example illustrates. Later, we also give examples for set-valued maps where such a derivative exists.

*Remark 1.* In case a set-valued map is  $\mathcal{GS}_\diamond$ -directionally differentiable at some  $x$  in direction  $d$ , then this implies that the maps  $f_l: \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $f_l(x) := \text{St}(Y(l, F(x)))$  are directional differentiable at  $x$  in direction  $d$  for all  $l \in \mathcal{T}_\diamond$ , i.e., the limits

$$\lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x))))$$

exist for all  $l \in \mathcal{T}_\diamond$ .

Remark 1 gives only a necessary condition. For a full characterization we refer to Theorem 4.11.

**Example 4.10.** We consider again the set optimization problem  $(\text{SOP}_{F,S})$  defined in Example 4.2 with the unique minimal solution w.r.t. the order relations  $\preceq_\ell$  and  $\preceq_s$  given by  $\bar{x} = 0$ . Let  $x \in (0, 1]$  be arbitrarily chosen and set  $d := x - \bar{x} = x$ . Based on the statements in Example 4.8 it holds

$$\frac{1}{\lambda} (\text{St}(Y(l, F(\lambda x))) - \text{St}(Y(l, F(0)))) = \begin{cases} 0_{\mathbb{R}^2} & , \text{ if } l \in \mathcal{T}_u \\ (0, \frac{x}{2})^\top & , \text{ if } l = (-1, 0)^\top \in \mathcal{T}_\ell \\ (0, \frac{1+\lambda x}{2\lambda})^\top & , \text{ if } l \in \mathcal{T}_\ell \text{ and } -1 < l_1 < l_2 \\ (\frac{x}{2}, \frac{x}{2})^\top & , \text{ if } l = -\frac{1}{\sqrt{2}}(1, 1)^\top \in \mathcal{T}_\ell \\ (\frac{1+\lambda x}{2\lambda}, 0)^\top & , \text{ if } l \in \mathcal{T}_\ell \text{ and } l_1 > l_2 > -1 \\ (\frac{x}{2}, 0)^\top & , \text{ if } l = (0, -1)^\top \in \mathcal{T}_\ell \end{cases}$$

for all  $\lambda \in (0, \frac{1}{x}]$ . Thus the set-valued map  $F$  is  $\mathcal{GS}_u$ -directionally differentiable at  $\bar{x} = 0$  in direction  $d = x$  for all  $x \in (0, 1]$ .

However, the limits

$$\lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\text{St}(Y(l, F(\lambda x))) - \text{St}(Y(l, F(0))))$$

do not exist for  $l \in \mathcal{T}_\ell \setminus \{(-1, 0)^\top, -\frac{1}{\sqrt{2}}(1, 1)^\top, (0, -1)^\top\}$  and thus (cf. Remark 1)  $F$  is not  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\bar{x} = 0$  in direction  $d = x$  for all  $x \in (0, 1]$  with  $\diamond \in \{\ell, s\}$ .

The following theorem provides an equivalent characterization of  $\mathcal{GS}_\diamond$ -directional differentiability:

**Theorem 4.11.** *Let Assumption 2 be fulfilled, let  $x \in \hat{S}$ , let  $d \in \mathbb{R}^n$ , and let  $\diamond \in \{\ell, u, s\}$ . Then the set-valued map  $F$  is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $x$  in direction  $d$ , i.e., the generalized Steiner set  $DF_\diamond^{\mathcal{GS}}(x; d) := (DF_\diamond^{\mathcal{GS}}(x; d)(l))_{l \in \mathcal{T}_\diamond}$  w.r.t.  $\mathcal{T}_\diamond$  exists, if and only if the following two conditions (a) and (b) hold:*

- (a) *The vector-valued maps  $\text{St}(Y(l, F(\cdot))): \mathbb{R}^n \rightarrow \mathbb{R}^m$  are directionally differentiable at  $x$  in direction  $d$  for all  $l \in \mathcal{T}_\diamond$  and the set  $\bigcup_{l \in \mathcal{T}_\diamond} \{D[\text{St}(Y(l, F(\cdot)))](x; d)\}$  is bounded.*

(b) For all  $\varepsilon > 0$  there exists  $\bar{\lambda} = \bar{\lambda}(\varepsilon) > 0$  such that for all  $\lambda \in (0, \bar{\lambda})$  it holds

$$\left\| D[\text{St}(Y(l, F(\cdot)))](x; d) - \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x)))) \right\|_2 \leq \varepsilon \quad (30)$$

for all  $l \in \mathcal{T}_\diamond$ .

Moreover, if  $F$  is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $x$  in direction  $d$ , then for all  $l \in \mathcal{T}_\diamond$  it holds

$$DF_\diamond^{\mathcal{GS}}(x; d)(l) = D[\text{St}(Y(l, F(\cdot)))](x; d). \quad (31)$$

*Proof.* Let the set-valued map  $F$  be  $\mathcal{GS}_\diamond$ -directionally differentiable at  $x$  in direction  $d$ . This is by definition and by the definition of the norm equivalent to the fact that for all  $\varepsilon > 0$  there exists  $\bar{\lambda} = \bar{\lambda}(\varepsilon) > 0$  such that for all  $\lambda \in (0, \bar{\lambda})$  it holds

$$\begin{aligned} & \left\| DF_\diamond^{\mathcal{GS}}(x; d) \ominus_{\mathcal{T}_\diamond} \left( \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(F(x + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(F(x))) \right) \right\| \\ &= \left\| (DF_\diamond^{\mathcal{GS}}(x; d)(l) - \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x)))) \right)_{l \in \mathcal{T}_\diamond} \right\| \\ &= \sup_{l \in \mathcal{T}_\diamond} \left\| DF_\diamond^{\mathcal{GS}}(x; d)(l) - \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x)))) \right\|_2 \\ &\leq \varepsilon. \end{aligned}$$

This in turn is equivalent to the fact that for all  $\varepsilon > 0$  there exists  $\bar{\lambda} = \bar{\lambda}(\varepsilon) > 0$  such that for all  $\lambda \in (0, \bar{\lambda})$  it holds

$$\left\| DF_\diamond^{\mathcal{GS}}(x; d)(l) - \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x)))) \right\|_2 \leq \varepsilon$$

for all  $l \in \mathcal{T}_\diamond$ . This implies

$$DF_\diamond^{\mathcal{GS}}(x; d)(l) = \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x)))) = D[\text{St}(Y(l, F(\cdot)))](x; d)$$

for all  $l \in \mathcal{T}_\diamond$ . Thus (a), (b), and (31) hold.

Let now on the other hand (a) and (b) be fulfilled and define the generalized Steiner set  $\mathcal{F}_{\mathcal{GS}, \mathcal{T}_\diamond} = (\mathcal{F}(l))_{l \in \mathcal{T}_\diamond}$  by  $\mathcal{F}(l) := D[\text{St}(Y(l, F(\cdot)))](x; d)$  for all  $l \in \mathcal{T}_\diamond$ . Then we obtain by (30) that for all  $\varepsilon > 0$  there exists  $\bar{\lambda} = \bar{\lambda}(\varepsilon) > 0$  such that for all  $\lambda \in (0, \bar{\lambda})$  it holds

$$\begin{aligned} \varepsilon &\geq \sup_{l \in \mathcal{T}_\diamond} \left\| D[\text{St}(Y(l, F(\cdot)))](x; d) - \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x)))) \right\|_2 \\ &= \left\| (\mathcal{F}(l) - \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x)))) \right)_{l \in \mathcal{T}_\diamond} \right\| \\ &= \left\| \mathcal{F}_{\mathcal{GS}, \mathcal{T}_\diamond} \ominus_{\mathcal{T}_\diamond} \left( \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(F(x + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(F(x))) \right) \right\| \end{aligned}$$

and thus  $\lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(F(x + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(F(x))) = \mathcal{F}_{\mathcal{GS}, \mathcal{T}_\diamond}$ . Hence, by definition the set-valued map  $F$  is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $x$  in direction  $d$  and we are done.  $\square$

Note that if a set-valued map  $F$  is  $\mathcal{GS}_\diamond$ -directionally differentiable at a point  $x$  in direction  $d$ , then it holds by (31)

$$DF_\diamond^{\mathcal{GS}}(x; d) = (D[\text{St}(Y(l, F(\cdot)))](x; d))_{l \in \mathcal{T}_\diamond},$$

and we obtain for all  $l \in \mathcal{T}_\diamond$

$$\begin{aligned} l^\top DF_\diamond^{\mathcal{GS}}(x; d)(l) &= l^\top D[\text{St}(Y(l, F(\cdot)))](x; d) \\ &= l^\top \left( \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x)))) \right) \\ &= \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (l^\top \text{St}(Y(l, F(x + \lambda d))) - l^\top \text{St}(Y(l, F(x)))) \quad (32) \\ &= \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\delta^*(l, F(x + \lambda d)) - \delta^*(l, F(x))) \\ &= D[\delta^*(l, F(\cdot))](x; d), \end{aligned}$$

where  $D[\delta^*(l, F(\cdot))](x; d) := \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\delta^*(l, F(x + \lambda d)) - \delta^*(l, F(x)))$  is the directional derivative of the scalar valued function  $\delta^*(l, F(\cdot))$  at  $x$  in the direction  $d$ .

Next, we give some calculations rules for the new directional derivative. The following proposition provides a sum rule and a product rule for the directional derivative.

**Proposition 4.12.** *Let Assumption 1 be fulfilled, let  $\hat{S}$  be an open and nonempty subset of  $\mathbb{R}^n$ , let  $g: \hat{S} \rightarrow \mathbb{R}$  be a scalar-valued function, let  $G, H: \hat{S} \rightrightarrows \mathbb{R}^m$  be two set-valued maps with  $G(x) \in \mathcal{C}(\mathbb{R}^m)$  and  $H(x) \in \mathcal{C}(\mathbb{R}^m)$  for all  $x \in \hat{S}$ , let  $\hat{x} \in \hat{S}$  and  $d \in \mathbb{R}^n$ , and let  $\diamond \in \{\ell, u, s\}$ . Then it holds:*

- (a) *If  $G$  and  $H$  are  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d$ , then for all  $\alpha, \beta \geq 0$  the set-valued map  $F: \hat{S} \rightrightarrows \mathbb{R}^m$  with*

$$F(x) := \alpha G(x) + \beta H(x) \text{ for all } x \in \hat{S}$$

*is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d$  with*

$$DF_\diamond^{\mathcal{GS}}(\hat{x}; d) = (\alpha \odot_{\mathcal{T}_\diamond} DG_\diamond^{\mathcal{GS}}(\hat{x}; d)) \oplus_{\mathcal{T}_\diamond} (\beta \odot_{\mathcal{T}_\diamond} DH_\diamond^{\mathcal{GS}}(\hat{x}; d)).$$

- (b) *If  $g$  is directionally differentiable at  $\hat{x}$  in direction  $d$ ,  $G$  is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d$ , and if there exists  $\bar{\lambda} > 0$  such that  $g(\hat{x} + \lambda d) \geq 0$  for all  $\lambda \in [0, \bar{\lambda}]$ , then the set-valued map  $F: \hat{S} \rightrightarrows \mathbb{R}^m$  with*

$$F(x) := g(x)G(x) \text{ for all } x \in \hat{S}$$

*is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d$  with*

$$DF_\diamond^{\mathcal{GS}}(\hat{x}; d) = (D[g(\cdot)](\hat{x}; d) \odot_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))) \oplus_{\mathcal{T}_\diamond} (g(\hat{x}) \odot_{\mathcal{T}_\diamond} DG_\diamond^{\mathcal{GS}}(\hat{x}; d)).$$

*Proof.* For the proof of statement (a) it holds by the calculation rules for the supporting faces and the Steiner points for sufficiently small  $\lambda > 0$

$$\begin{aligned}
& J_\diamond(F(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(F(\hat{x})) \\
&= (\text{St}(Y(l, \alpha G(\hat{x} + \lambda d) + \beta H(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond} \ominus_{\mathcal{T}_\diamond} (\text{St}(Y(l, \alpha G(\hat{x}) + \beta H(\hat{x}))))_{l \in \mathcal{T}_\diamond} \\
&= (\text{St}(\alpha Y(l, G(\hat{x} + \lambda d)) + \beta Y(l, H(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond} \\
&\quad \ominus_{\mathcal{T}_\diamond} (\text{St}(\alpha Y(l, G(\hat{x})) + \beta Y(l, H(\hat{x}))))_{l \in \mathcal{T}_\diamond} \\
&= (\alpha \odot_{\mathcal{T}_\diamond} (\text{St}(Y(l, G(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond}) \oplus_{\mathcal{T}_\diamond} (\beta \odot_{\mathcal{T}_\diamond} (\text{St}(Y(l, H(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond}) \\
&\quad \ominus_{\mathcal{T}_\diamond} (\alpha \odot_{\mathcal{T}_\diamond} (\text{St}(Y(l, G(\hat{x}))))_{l \in \mathcal{T}_\diamond}) \ominus_{\mathcal{T}_\diamond} (\beta \odot_{\mathcal{T}_\diamond} (\text{St}(Y(l, H(\hat{x}))))_{l \in \mathcal{T}_\diamond}) \\
&= \alpha \odot_{\mathcal{T}_\diamond} ((\text{St}(Y(l, G(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond} \ominus_{\mathcal{T}_\diamond} (\text{St}(Y(l, G(\hat{x}))))_{l \in \mathcal{T}_\diamond}) \\
&\quad \oplus_{\mathcal{T}_\diamond} \beta \odot_{\mathcal{T}_\diamond} ((\text{St}(Y(l, H(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond} \ominus_{\mathcal{T}_\diamond} (\text{St}(Y(l, H(\hat{x}))))_{l \in \mathcal{T}_\diamond}) \\
&= \alpha \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))) \oplus_{\mathcal{T}_\diamond} \beta \odot_{\mathcal{T}_\diamond} (J_\diamond(H(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(H(\hat{x})))
\end{aligned}$$

and we obtain

$$\frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(F(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(F(\hat{x}))) = \alpha \odot_{\mathcal{T}_\diamond} \Gamma(\lambda) \oplus_{\mathcal{T}_\diamond} \beta \odot_{\mathcal{T}_\diamond} \Lambda(\lambda)$$

with

$$\Gamma(\lambda) := \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x})))$$

and

$$\Lambda(\lambda) := \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(H(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(H(\hat{x}))).$$

Then it holds

$$\lim_{\lambda \rightarrow 0^+} \Gamma(\lambda) = DG_\diamond^{\mathcal{GS}}(\hat{x}; d) \quad \text{and} \quad \lim_{\lambda \rightarrow 0^+} \Lambda(\lambda) = DH_\diamond^{\mathcal{GS}}(\hat{x}; d)$$

and thus (a) is proven.

For the proof of (b) it holds by similar arguments for sufficiently small  $\lambda > 0$

$$\begin{aligned}
& J_\diamond(F(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(F(\hat{x})) \\
&= (\text{St}(Y(l, g(\hat{x} + \lambda d)G(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond} \ominus_{\mathcal{T}_\diamond} (\text{St}(Y(l, g(\hat{x})G(\hat{x}))))_{l \in \mathcal{T}_\diamond} \\
&= (g(\hat{x} + \lambda d) \text{St}(Y(l, G(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond} \ominus_{\mathcal{T}_\diamond} (g(\hat{x}) \text{St}(Y(l, G(\hat{x}))))_{l \in \mathcal{T}_\diamond} \\
&= ((g(\hat{x} + \lambda d) - g(\hat{x})) \text{St}(Y(l, G(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond} \\
&\quad \oplus_{\mathcal{T}_\diamond} (g(\hat{x}) (\text{St}(Y(l, G(\hat{x} + \lambda d))))_{l \in \mathcal{T}_\diamond} \ominus_{\mathcal{T}_\diamond} (\text{St}(Y(l, G(\hat{x}))))_{l \in \mathcal{T}_\diamond}) \\
&= ((g(\hat{x} + \lambda d) - g(\hat{x})) \odot_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x} + \lambda d))) \\
&\quad \oplus_{\mathcal{T}_\diamond} (g(\hat{x}) \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x})))) \\
&= ((g(\hat{x} + \lambda d) - g(\hat{x})) \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))) \oplus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))) \\
&\quad \oplus_{\mathcal{T}_\diamond} (g(\hat{x}) \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x})))) \\
&= ((g(\hat{x} + \lambda d) - g(\hat{x})) \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x})))) \\
&\quad \oplus_{\mathcal{T}_\diamond} ((g(\hat{x} + \lambda d) - g(\hat{x})) \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x})))) \\
&\quad \oplus_{\mathcal{T}_\diamond} (g(\hat{x}) \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))))
\end{aligned}$$

and we obtain

$$\frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(F(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(F(\hat{x}))) = \Theta(\lambda) \oplus_{\mathcal{T}_\diamond} \Phi(\lambda) \oplus_{\mathcal{T}_\diamond} \Psi(\lambda)$$

with

$$\begin{aligned}\Theta(\lambda) &:= \lambda \left( \frac{1}{\lambda} (g(\hat{x} + \lambda d) - g(\hat{x})) \right) \odot_{\mathcal{T}_\diamond} \left( \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))) \right), \\ \Phi(\lambda) &:= \frac{1}{\lambda} (g(\hat{x} + \lambda d) - g(\hat{x})) \odot_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x})), \text{ and} \\ \Psi(\lambda) &:= g(\hat{x}) \odot_{\mathcal{T}_\diamond} \left( \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))) \right).\end{aligned}$$

Then it holds

$$\begin{aligned}\lim_{\lambda \rightarrow 0^+} \Theta(\lambda) &= \left( \lim_{\lambda \rightarrow 0^+} \lambda \right) D[g(\cdot)](\hat{x}; d) \odot_{\mathcal{T}_\diamond} DG_\diamond^{\mathcal{GS}}(\hat{x}; d) = 0_{\mathcal{GS}, \mathcal{T}_\diamond}, \\ \lim_{\lambda \rightarrow 0^+} \Phi(\lambda) &= D[g(\cdot)](\hat{x}; d) \odot_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x})), \text{ and} \\ \lim_{\lambda \rightarrow 0^+} \Psi(\lambda) &= g(\hat{x}) \odot_{\mathcal{T}_\diamond} DG_\diamond^{\mathcal{GS}}(\hat{x}; d),\end{aligned}$$

and we are done.  $\square$

In the following we apply Proposition 4.12 (a) to the special case of a set-valued map defined by  $\{f(x)\} + A$  with a convex and compact set  $A$ , and we obtain that the directional derivative depends on the directional derivative of  $f$  only, as one would expect. We will shortly come back to this aspect in subsection 4.3 again.

**Lemma 4.13.** *Let Assumption 1 be fulfilled, let  $\hat{S}$  be an open and nonempty subset of  $\mathbb{R}^n$ , let  $\hat{x} \in \hat{S}$ , let  $d \in \mathbb{R}^n$ , let  $f: \hat{S} \rightarrow \mathbb{R}^m$  be a vector-valued function which is directionally differentiable at  $\hat{x}$  in direction  $d$ , and let  $\diamond \in \{\ell, u, s\}$ . Then for all  $A \in \mathcal{C}(\mathbb{R}^m)$  the set-valued map  $F: \hat{S} \rightrightarrows \mathbb{R}^m$  defined by*

$$F(x) := \{f(x)\} + A \text{ for all } x \in \hat{S}$$

is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d$  with

$$DF_\diamond^{\mathcal{GS}}(\hat{x}; d)(l) = D[f(\cdot)](\hat{x}; d) \text{ for all } l \in \mathcal{T}_\diamond.$$

*Proof.* To apply Proposition 4.12 (a) we set  $\alpha = \beta = 1$ ,  $G(x) := \{f(x)\}$  for all  $x \in \hat{S}$ , and  $H(x) := A$  for all  $x \in \hat{S}$ . Then it is sufficient to show that  $G$  and  $H$  are both  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d$  with  $DG_\diamond^{\mathcal{GS}}(\hat{x}; d)(l) = D[f(\cdot)](\hat{x}; d)$  for all  $l \in \mathcal{T}_\diamond$  and  $DH_\diamond^{\mathcal{GS}}(\hat{x}; d) = 0_{\mathcal{GS}, \mathcal{T}_\diamond}$ .

We start with the map  $H$ . By definition it holds for sufficiently small  $\lambda > 0$

$$J_\diamond(H(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(H(\hat{x})) = J_\diamond(A) \ominus_{\mathcal{T}_\diamond} J_\diamond(A) = 0_{\mathcal{GS}, \mathcal{T}_\diamond}.$$

Hence, the set-valued map  $H$  is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d$  and it holds

$$DH_\diamond^{\mathcal{GS}}(\hat{x}; d) = \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(H(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(H(\hat{x}))) = 0_{\mathcal{GS}, \mathcal{T}_\diamond}.$$

Finally, for the set-valued map  $G$  it holds for sufficiently small  $\lambda > 0$

$$(J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))) (l) = f(\hat{x} + \lambda d) - f(\hat{x}) \text{ for all } l \in \mathcal{T}_\diamond.$$

By using the directional differentiability of  $f$  we can define a generalized Steiner set  $\mathcal{F}_{\mathcal{GS}, \mathcal{T}_\diamond} = (\mathcal{F}(l))_{l \in \mathcal{T}_\diamond}$  by  $\mathcal{F}(l) := D[f(\cdot)](\hat{x}; d)$  for all  $l \in \mathcal{T}_\diamond$ . Now it holds

$$\begin{aligned} & \lim_{\lambda \rightarrow 0^+} \|\mathcal{F}_{\mathcal{GS}, \mathcal{T}_\diamond} \ominus_{\mathcal{T}_\diamond} (\frac{1}{\lambda} \odot_{\mathcal{T}_\diamond} (J_\diamond(G(\hat{x} + \lambda d)) \ominus_{\mathcal{T}_\diamond} J_\diamond(G(\hat{x}))))\| \\ &= \lim_{\lambda \rightarrow 0^+} \sup_{l \in \mathcal{T}_\diamond} \|D[f(\cdot)](\hat{x}; d) - \frac{1}{\lambda} (f(\hat{x} + \lambda d) - f(\hat{x}))\|_2 \\ &= \lim_{\lambda \rightarrow 0^+} \|D[f(\cdot)](\hat{x}; d) - \frac{1}{\lambda} (f(\hat{x} + \lambda d) - f(\hat{x}))\|_2 \\ &= 0. \end{aligned}$$

Hence, the set-valued map  $G$  is also  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d$  with

$$DG_\diamond^{\mathcal{GS}}(\hat{x}; d) = \mathcal{F}_{\mathcal{GS}, \mathcal{T}_\diamond} = (D[f(\cdot)](\hat{x}; d))_{l \in \mathcal{T}_\diamond}$$

and we are done.  $\square$

We now formulate a necessary optimality condition for set optimization problems as the main result of this subsection:

**Theorem 4.14.** *Let Assumption 2 be fulfilled and let  $\diamond \in \{\ell, u, s\}$ . Moreover, let  $\bar{x} \in S$  such that there exists  $\bar{\lambda} > 0$  such that for all  $x \in S \setminus \{\bar{x}\}$  and all  $\lambda \in [0, \bar{\lambda}]$  it holds  $\bar{x} + \bar{\lambda}(x - \bar{x}) \in S$  and such that for all  $x \in S$  the map  $F$  is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\bar{x}$  in direction  $d := x - \bar{x}$ , i.e., the generalized Steiner sets  $DF_\diamond^{\mathcal{GS}}(\bar{x}; x - \bar{x}) := (DF_\diamond^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l))_{l \in \mathcal{T}_\diamond}$  w.r.t.  $\mathcal{T}_\diamond$  exist for all  $x \in S$ . Then the following holds:*

*If  $\bar{x} \in S$  is a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t. the order relation  $\preceq_\diamond$ , then for all  $x \in S$  there exists  $l \in \mathcal{T}_\diamond$  such that*

$$\text{sign}(l)l^\top (DF_\diamond^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l)) \geq 0. \quad (33)$$

*Proof.* Let  $\bar{x} \in S$  be a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_\diamond$  and let  $x \in S$ . Since the map  $F$  is  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\bar{x}$  in direction  $d = x - \bar{x}$  we obtain by Theorem 4.11 that for all  $\varepsilon > 0$  there exists  $\tilde{\lambda} \in (0, \bar{\lambda}]$  such that for all  $\lambda \in (0, \tilde{\lambda}]$  it holds

$$\left\| \frac{1}{\lambda} (\text{St}(Y(l, F(\bar{x} + \lambda(x - \bar{x})))) - \text{St}(Y(l, F(\bar{x})))) - DF_\diamond^{\mathcal{GS}}(\bar{x}, x - \bar{x})(l) \right\|_2 \leq \varepsilon$$

for all  $l \in \mathcal{T}_\diamond$  and thus by the Cauchy-Schwarz inequality it follows

$$\begin{aligned} & \text{sign}(l) \frac{1}{\lambda} (\delta^*(l, F(\bar{x} + \lambda(x - \bar{x}))) - \delta^*(l, F(\bar{x}))) - \text{sign}(l)l^\top DF_\diamond^{\mathcal{GS}}(\bar{x}, x - \bar{x})(l) \\ &= \text{sign}(l)l^\top \left( \frac{1}{\lambda} (\text{St}(Y(l, F(\bar{x} + \lambda(x - \bar{x})))) - \text{St}(Y(l, F(\bar{x})))) - DF_\diamond^{\mathcal{GS}}(\bar{x}, x - \bar{x})(l) \right) \\ &\leq \left\| \text{sign}(l)l \right\|_2 \left\| \frac{1}{\lambda} (\text{St}(Y(l, F(\bar{x} + \lambda(x - \bar{x})))) - \text{St}(Y(l, F(\bar{x})))) - DF_\diamond^{\mathcal{GS}}(\bar{x}, x - \bar{x})(l) \right\|_2 \\ &= \left\| \frac{1}{\lambda} (\text{St}(Y(l, F(\bar{x} + \lambda(x - \bar{x})))) - \text{St}(Y(l, F(\bar{x})))) - DF_\diamond^{\mathcal{GS}}(\bar{x}, x - \bar{x})(l) \right\|_2 \\ &\leq \varepsilon \end{aligned} \quad (34)$$

for all  $l \in \mathcal{T}_\diamond$ . Note that the definition of the norm in (12) by using the supremum over all  $l \in \mathcal{T}$  does again guarantee the uniform convergence w.r.t. the directions  $l$ . Assume now that for all  $l \in \mathcal{T}_\diamond$  it holds  $\text{sign}(l)l^\top (DF_\diamond^{\mathcal{GS}}(\bar{x}, x - \bar{x})(l)) < 0$ . Then we obtain by definition and (32) that for all  $l \in \mathcal{T}_\diamond$  it holds

$$0 > \text{sign}(l)l^\top DF_\diamond^{\mathcal{GS}}(\bar{x}, x - \bar{x})(l) = \text{sign}(l) D[\delta^*(l, F(\cdot))](\bar{x}; x - \bar{x}), \quad (35)$$

where  $D[\delta^*(l, F(\cdot))](\bar{x}; x - \bar{x}) := \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\delta^*(l, F(\bar{x} + \lambda(x - \bar{x}))) - \delta^*(l, F(\bar{x})))$  is the directional derivative of  $\delta^*(l, F(\cdot))$  at  $\bar{x}$  in the direction  $d = x - \bar{x}$ .

Using Assumption 2 the support function  $\delta^*(\cdot, F(\bar{x}))$  is Lipschitz-continuous (see for instance [24, Lemma 1.8.10.]) and by [23, Chap. VI, Subsec. 1.1, Remark 1.1.3.] it follows that the directional derivative  $D[\delta^*(l, F(\cdot))](\bar{x}; x - \bar{x})$  is also Lipschitz-continuous w.r.t. the direction  $l$ . Since  $\mathcal{T}_\diamond$  is a compact subset of  $\mathbb{R}^m$  we obtain by the Weierstraß Theorem that

$$\kappa := \max_{l \in \mathcal{T}_\diamond} \text{sign}(l) D[\delta^*(l, F(\cdot))](\bar{x}; x - \bar{x})$$

exists and by (35) it holds

$$\kappa = \max_{l \in \mathcal{T}_\diamond} \text{sign}(l) D[\delta^*(l, F(\cdot))](\bar{x}; x - \bar{x}) = \max_{l \in \mathcal{T}_\diamond} \text{sign}(l)l^\top (DF_\diamond^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l)) < 0.$$

If we now choose  $\varepsilon := -\frac{\kappa}{2} > 0$  in (34), then there exists  $\tilde{\lambda} \in (0, \bar{\lambda}]$  such that for all  $\lambda \in (0, \tilde{\lambda}]$  it holds

$$\begin{aligned} & \text{sign}(l) \frac{1}{\lambda} (\delta^*(l, F(\bar{x} + \lambda(x - \bar{x}))) - \delta^*(l, F(\bar{x}))) - \kappa \\ & \leq \text{sign}(l) \frac{1}{\lambda} (\delta^*(l, F(\bar{x} + \lambda(x - \bar{x}))) - \delta^*(l, F(\bar{x}))) - \text{sign}(l)l^\top DF_\diamond^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l) \\ & \leq -\frac{\kappa}{2} \end{aligned}$$

which leads to

$$\text{sign}(l)\delta^*(l, F(\bar{x} + \lambda(x - \bar{x}))) - \text{sign}(l)\delta^*(l, F(\bar{x})) \leq \lambda \frac{\kappa}{2} < 0$$

for all  $l \in \mathcal{T}_\diamond$ . Thus we obtain for instance by setting  $\lambda := \tilde{\lambda}$

$$\text{sign}(l)\delta^*(l, F(\bar{x} + \tilde{\lambda}(x - \bar{x}))) < \text{sign}(l)\delta^*(l, F(\bar{x}))$$

for all  $l \in \mathcal{T}_\diamond$  – contradicting by Lemma 4.1 that  $\bar{x} \in S$  is a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t. the order relation  $\preceq_\diamond$ .  $\square$

Note that we can simplify (33) in the case of  $l \in \mathcal{T}_\ell$  to  $l^\top (DF_\diamond^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l)) \leq 0$  and in the case of  $l \in \mathcal{T}_u$  to  $l^\top (DF_\diamond^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l)) \geq 0$ . This also leads to the following necessary optimality conditions using the positive and the negative part of the visualization of  $DF_\ell^{\mathcal{GS}}(\bar{x}; x - \bar{x})$  and  $DF_u^{\mathcal{GS}}(\bar{x}; x - \bar{x})$ , respectively:



**Lemma 4.15.** *Let all assumptions of Theorem 4.14 be fulfilled and let  $\bar{x} \in S$  be a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t. the order relation  $\preceq_\diamond$ . Then for all  $x \in S$  it holds:*

- (a) *If  $\diamond = \ell$ , then  $0_{\mathbb{R}^m} \notin \text{int } P_{\mathcal{T}_\ell}(DF_\ell^{\mathcal{G}S}(\bar{x}; x - \bar{x}))$ .*
- (b) *If  $\diamond = u$ , then  $0_{\mathbb{R}^m} \notin \text{int } N_{\mathcal{T}_u}(DF_u^{\mathcal{G}S}(\bar{x}; x - \bar{x}))$ .*
- (c) *If  $\diamond = s$ , then  $0_{\mathbb{R}^m} \notin (\text{int } P_{\mathcal{T}_\ell}(DF_\ell^{\mathcal{G}S}(\bar{x}; x - \bar{x}))) \cap (\text{int } N_{\mathcal{T}_u}(DF_u^{\mathcal{G}S}(\bar{x}; x - \bar{x})))$ .*

*Proof.* Let  $\diamond = \ell$  and let  $\bar{x} \in S$  be a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_\ell$ . Then we obtain by Theorem 4.14 that for all  $x \in S$  there exists  $l \in \mathcal{T}_\ell$  such that

$$l^\top (DF_\ell^{\mathcal{G}S}(\bar{x}; x - \bar{x})(l)) \leq 0. \quad (36)$$

Assume now that there exists  $\tilde{x} \in S$  such that  $0_{\mathbb{R}^m} \in \text{int } P_{\mathcal{T}_\ell}(DF_\ell^{\mathcal{G}S}(\bar{x}; \tilde{x} - \bar{x}))$ . Hence, there exist  $\tilde{\varepsilon} > 0$  such that

$$\mathcal{B}(0_{\mathbb{R}^m}, \tilde{\varepsilon}) \subset P_{\mathcal{T}_\ell}(DF_\ell^{\mathcal{G}S}(\bar{x}; \tilde{x} - \bar{x}))$$

and thus

$$\tilde{\varepsilon}\eta \in P_{\mathcal{T}_\ell}(DF_\ell^{\mathcal{G}S}(\bar{x}; \tilde{x} - \bar{x})) \text{ for all } \eta \in S_{m-1}.$$

By the definition of the positive part it follows for all  $l \in \mathcal{T}_\ell$  that

$$\tilde{\varepsilon}l^\top \eta = l^\top (\tilde{\varepsilon}\eta) \leq l^\top DF_\ell^{\mathcal{G}S}(\bar{x}; \tilde{x} - \bar{x})(l) \text{ for all } \eta \in S_{m-1}.$$

Hence, we obtain for all  $l \in \mathcal{T}_\ell$  by setting  $\eta := l$

$$0 < \tilde{\varepsilon} = \tilde{\varepsilon}l^\top l \leq l^\top DF_\ell^{\mathcal{G}S}(\bar{x}; \tilde{x} - \bar{x})(l)$$

– contradicting (36).

The corresponding statements for  $\diamond \in \{u, \ell\}$  can be proved by similar arguments.  $\square$

The usefulness of the optimality conditions formulated in Theorem 4.14 and Lemma 4.15 will be illustrated with the following example. Thereby, this necessary condition is applied to all feasible points and it succeeds to exclude all points which are not a minimal solution of the set optimization problems.

**Example 4.16.** Let  $\hat{S} = \mathbb{R}$ ,  $S = [0, 1]$ ,  $C = C^* = \mathbb{R}_+^2$ ,  $g, h: \hat{S} \rightarrow \mathbb{R}^m$  with  $g(x) := x$  and  $h(x) := 1 - x$  for all  $x \in \hat{S}$ ,  $A := [0, 1] \times [0, 1] \subset \mathbb{R}^2$ ,  $r > 0$ ,  $B := \mathcal{B}(0_{\mathbb{R}^2}, r)$ ,  $G, H: \hat{S} \rightrightarrows \mathbb{R}^m$  with  $G(x) := g(x)A \in \mathcal{C}(\mathbb{R}^m)$  and  $H(x) := h(x)B \in \mathcal{C}(\mathbb{R}^m)$  for all  $x \in \hat{S}$ , and  $\hat{x} \in S$ . Moreover, we define the set-valued map  $F: \hat{S} \rightrightarrows \mathbb{R}^2$  with  $F(x) := G(x) + H(x) = g(x)A + h(x)B$  for all  $x \in \hat{S}$ . Thus we have

$$F(x) = [0, x] \times [0, x] + \mathcal{B}(0_{\mathbb{R}^2}, (1 - x)r) \text{ for all } x \in \hat{S}.$$

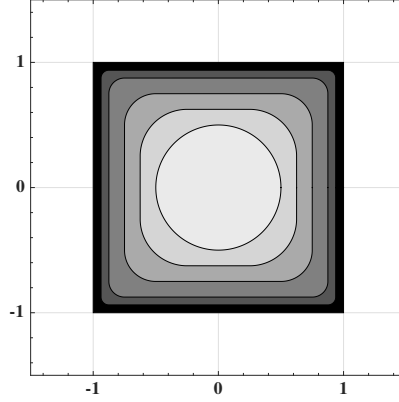


Figure 4: The images of  $F(x)$  for  $r = \frac{1}{2}$  and  $x \in \{\frac{i}{8} \mid i = 0, 2, 4, 6, 7, 8\}$  in Example 4.16.

For a visualization of the map  $F$  for  $r = \frac{1}{2}$  see Figure 4. By Proposition 4.12 (b) the set-valued maps  $G$  and  $H$  are  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d := x - \hat{x}$  for all  $x \in S \setminus \{\hat{x}\}$  with

$$DG_\diamond^{\mathcal{GS}}(\hat{x}; x - \hat{x}) = D[g(\cdot)](\hat{x}; x - \hat{x}) \odot_{\mathcal{T}_\diamond} J_\diamond(A) = (x - \hat{x}) \odot_{\mathcal{T}_\diamond} (\text{St}(Y(l, A)))_{l \in \mathcal{T}_\diamond}$$

where

$$\text{St}(Y(l, A)) = \begin{cases} (1, 0)^\top & , \text{ if } l = (1, 0)^\top \\ (1, 1)^\top & , \text{ if } l \in \mathcal{T}_u \setminus \{(1, 0)^\top, (0, 1)^\top\} \\ (0, 1)^\top & , \text{ if } l = (0, 1)^\top \\ (-1, 0)^\top & , \text{ if } l = (-1, 0)^\top \\ (-1, -1)^\top & , \text{ if } l \in \mathcal{T}_\ell \setminus \{(-1, 0)^\top, (0, -1)^\top\} \\ (0, -1)^\top & , \text{ if } l = (0, -1)^\top \end{cases} ,$$

and

$$DH_\diamond^{\mathcal{GS}}(\hat{x}; x - \hat{x}) = D[h(\cdot)](\hat{x}; x - \hat{x}) \odot_{\mathcal{T}_\diamond} J_\diamond(B) = -(x - \hat{x}) \odot_{\mathcal{T}_\diamond} (rl)_{l \in \mathcal{T}_\diamond} .$$

Thus by Proposition 4.12 (a) the set-valued map  $F$  is also  $\mathcal{GS}_\diamond$ -directionally differentiable at  $\hat{x}$  in direction  $d := x - \hat{x}$  for all  $x \in S \setminus \{\hat{x}\}$  with

$$DF_\diamond^{\mathcal{GS}}(\hat{x}; x - \hat{x}) = (DF_\diamond^{\mathcal{GS}}(\hat{x}; x - \hat{x})(l))_{l \in \mathcal{T}_\diamond} ,$$

$$DF_\diamond^{\mathcal{GS}}(\hat{x}; x - \hat{x})(l) = \begin{cases} (x - \hat{x})(1 - r)(1, 0)^\top & , \text{ if } l = (1, 0)^\top \\ (x - \hat{x})((1, 1)^\top - rl) & , \text{ if } l \in \mathcal{T}_u \setminus \{(1, 0)^\top, (0, 1)^\top\} \\ (x - \hat{x})(1 - r)(0, 1)^\top & , \text{ if } l = (0, 1)^\top \\ (x - \hat{x})(1 - r)(-1, 0)^\top & , \text{ if } l = (-1, 0)^\top \\ (x - \hat{x})((-1, -1)^\top - rl) & , \text{ if } l \in \mathcal{T}_\ell \setminus \{(-1, 0)^\top, (0, -1)^\top\} \\ (x - \hat{x})(1 - r)(0, -1)^\top & , \text{ if } l = (0, -1)^\top \end{cases} ,$$

and  $\diamond \in \{\ell, u, s\}$ . We restrict ourselves in the following to the case  $0 < r < 1$ . In this case it is easy to see that

- $\bar{x} = 1$  is the unique minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_\ell$ ,
- $\bar{x} = 0$  is the unique minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_u$ , and
- all  $x \in S = [0, 1]$  are minimal solutions of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_s$ .

First, we check the necessary condition of Theorem 4.14 for the case  $\diamond = \ell$ . Let therefore  $\hat{x} \in [0, 1)$  and  $x = 1$ . Then it holds

$$DF_\ell^{\mathcal{GS}}(\hat{x}; x - \hat{x})(l) = \begin{cases} (1 - \hat{x})(1 - r)(-1, 0)^\top & , \text{ if } l = (-1, 0)^\top \\ (1 - \hat{x})((-1, -1)^\top - rl) & , \text{ if } l \in \mathcal{T}_\ell \setminus \{(-1, 0)^\top, (0, -1)^\top\} \\ (1 - \hat{x})(1 - r)(0, -1)^\top & , \text{ if } l = (0, -1)^\top \end{cases}$$

and thus  $l^\top (DF_\ell^{\mathcal{GS}}(\hat{x}; x - \hat{x})(l)) > 0$  for all  $l \in \mathcal{T}_\ell$ . Hence, by Theorem 4.14 all  $\hat{x} \in [0, 1)$  are not a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_\ell$ .

Now, let  $\bar{x} = 1$  and  $x \in S$  be arbitrarily chosen. Then it holds

$$DF_\ell^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l) = \begin{cases} (x - 1)(1 - r)(-1, 0)^\top & , \text{ if } l = (-1, 0)^\top \\ (x - 1)((-1, -1)^\top - rl) & , \text{ if } l \in \mathcal{T}_\ell \setminus \{(-1, 0)^\top, (0, -1)^\top\} \\ (x - 1)(1 - r)(0, -1)^\top & , \text{ if } l = (0, -1)^\top \end{cases}$$

(cf. subfigure 5a in the case  $r = \frac{1}{2}$ ,  $\bar{x} = 1$ , and  $x = 0$ ) and thus  $l^\top (DF_\ell^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l)) \leq 0$  for all  $l \in \mathcal{T}_\ell$ . Hence,  $\bar{x} = 1$  fulfills the necessary condition (33) of Theorem 4.14 for a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_\ell$  for all  $l \in \mathcal{T}_\ell$ . Moreover, the necessary condition of Lemma 4.15 (a) is also fulfilled (cf. subfigure 5b and subfigure 5c).

If  $\diamond = u$ ,  $\hat{x} \in (0, 1]$ , and  $x = 0$ , then it follows

$$DF_u^{\mathcal{GS}}(\hat{x}; x - \hat{x})(l) = \begin{cases} -\hat{x}(1 - r)(1, 0)^\top & , \text{ if } l = (1, 0)^\top \\ -\hat{x}((1, 1)^\top - rl) & , \text{ if } l \in \mathcal{T}_u \setminus \{(1, 0)^\top, (0, 1)^\top\} \\ -\hat{x}(1 - r)(0, 1)^\top & , \text{ if } l = (0, 1)^\top \end{cases}$$

and thus  $l^\top (DF_u^{\mathcal{GS}}(\hat{x}; x - \hat{x})(l)) < 0$  for all  $l \in \mathcal{T}_u$ . Hence, by Theorem 4.14 all  $\hat{x} \in (0, 1]$  are not a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_u$ .

Finally, let  $\bar{x} = 0$  and  $x \in S$ . Then we obtain

$$DF_u^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l) = \begin{cases} x(1 - r)(1, 0)^\top & , \text{ if } l = (1, 0)^\top \\ x((1, 1)^\top - rl) & , \text{ if } l \in \mathcal{T}_u \setminus \{(1, 0)^\top, (0, 1)^\top\} \\ x(1 - r)(0, 1)^\top & , \text{ if } l = (0, 1)^\top \end{cases}$$

and thus  $l^\top (DF_u^{\mathcal{GS}}(\bar{x}; x - \bar{x})(l)) \geq 0$  for all  $l \in \mathcal{T}_u$ . Hence,  $\bar{x} = 0$  fulfills the necessary condition (33) for a minimal solution of  $(\text{SOP}_{F,S})$  w.r.t.  $\preceq_u$  for all  $l \in \mathcal{T}_u$ .

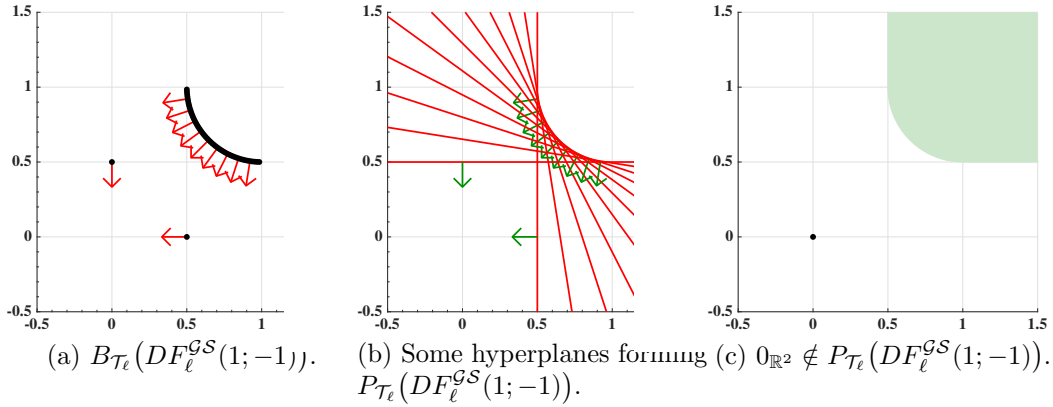


Figure 5: Visualization results for  $r = \frac{1}{2}$ ,  $\bar{x} = 1$ , and  $x = 0$  in Example 4.16.

### 4.3 Relation to Jahn's set difference and directional derivative

The new set difference and directional derivative using the embeddings of nonempty, convex, and compact subsets of  $\mathbb{R}^m$  into the Banach space of generalized Steiner sets are related to the concepts introduced by Jahn in [14, 13]. We discuss the relations (and differences) in more detail in the following.

Jahn also bases his results on Lemma 4.1 (which he states originally under weaker assumptions). He only makes use of the set  $\mathcal{T}_u$ , also for the  $\ell$ -less order relation, and not of the set  $\mathcal{T}_\ell$ . For that reason he needs to solve maximization as well as minimization problems, while we only have the maximization problems from the definition of the support functions. This is only a matter of notation but leads to a notation using vectors in  $\mathbb{R}^2$  in some of the results by Jahn:

$$\begin{pmatrix} \inf_{y \in F(x)} l^\top y \\ \sup_{y \in F(x)} l^\top y \end{pmatrix} = \begin{pmatrix} -\delta^*(-l, F(x)) \\ \delta^*(l, F(x)) \end{pmatrix}.$$

We propose a unified notation by using the sign function and the sets  $\mathcal{T}_u$ ,  $\mathcal{T}_\ell$ , and  $\mathcal{T}_s$ .

A more significant difference between our approach and the approach of Jahn is the fact that he considers only the values of the support functions and we consider (and save) the values and an element of the supporting face at the same time by the embedding. This is for instance important for defining a meaningful directional derivative. In [13, Definition 2.2] Jahn defines a set difference for two suitable nonempty subsets  $A$  and  $B$  of  $\mathbb{R}^m$  by

$$A -_J B := \bigcup_{l \in \mathcal{T}_s} \{y(l, A) - y(l, B)\}. \quad (37)$$

For being suitable, he requires that the supporting faces of the sets are singletons for

the functionals  $l \in \mathcal{T}_s$ . For instance strictly convex sets have such unique supporting points. Then, the elements  $y(l, A)$  and  $y(l, B)$  in (37) are uniquely defined.

Obviously, the set difference is a set again, and the set  $A -_J B$  does not reveal which elements have been obtained by which functionals  $l \in \mathcal{T}_s$ . For overcoming this issue, we define the set difference in the Banach space of generalized Steiner sets after an embedding of the sets. As we use Steiner points in this context we can also handle sets with non-unique supporting points.

Using the concept of Steiner points one could also easily extend the set difference of Jahn and one can omit the uniqueness requirements of the supporting points:

$$A -_{st} B := \bigcup_{l \in \mathcal{T}_s} \{\text{St}(Y(l, A)) - \text{St}(Y(l, B))\}. \quad (38)$$

This new set difference (without an embedding) is directly related to our difference which uses the embedding for  $A$  and  $B$ . Using the visualization and the boundary part, we obtain

$$A -_{st} B = B_{\mathcal{T}_s}(J_s(A) \ominus_{\mathcal{T}_s} J_s(B)).$$

The set difference in (38) is well defined for arbitrary sets  $A, B \in \mathcal{C}(\mathbb{R}^m)$ . For the new difference  $-_{st}$  it still holds  $A -_{st} A = \{0_{\mathbb{R}^m}\}$ , which is an important property for further results in set optimization.

If, additionally, the supporting faces  $Y(l, A)$  and  $Y(l, B)$  are singletons for all  $l \in \mathcal{T}_s$ , then it holds

$$A -_J B = A -_{st} B = B_{\mathcal{T}_s}(J_s(A) \ominus_{\mathcal{T}_s} J_s(B)).$$

Hence, in the special case of strictly convex sets, the set difference by Jahn equals the boundary part of our new set difference in the Banach space of generalized Steiner sets.

Also Jahn has proposed in [13, Remark 2.1] a generalization to arbitrary convex sets by using metric difference in case the supporting faces are not singletons. However, this does not lead to satisfying results. To see this, one might have a look on set-valued maps defined by  $F(x) = \{f(x)\} + A$  where  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a vector-valued map and  $A$  is some polyhedral set, for instance the unit cube. Then the difference between two sets  $F(x^1)$  and  $F(x^2)$  should obviously depend on the difference  $f(x^1) - f(x^2)$  only. This is true for  $A -_{st} B$  but not for the modification proposed by Jahn using metric differences. For the study of such specific set-valued maps we also refer to [31].

Moreover, Jahn introduced in [13], again under the additional assumption of unique supporting points, a directional derivative for set-valued maps: if the supporting faces  $Y(l, F(x))$  are singletons for all  $x \in \hat{S}$  and all  $l \in \mathcal{T}_s$  and if the maps  $y(l, F(\cdot))$  are directionally differentiable at  $x$  in direction  $d$  for all  $l \in \mathcal{T}_s$ , i.e., the limits

$$D[y(l, F(\cdot))](x; d) := \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (y(l, F(x + \lambda d)) - y(l, F(x))) \quad (39)$$

exist for all  $l \in \mathcal{T}_s$ , then Jahn's directional derivative of  $F$  at  $x$  in direction  $d$  is defined by

$$DF_J(x; d) := \bigcup_{l \in \mathcal{T}_s} \{D[y(l, F(\cdot))](x; d)\}.$$

One can again observe that the limit is calculated for each direction  $l$  individually. Thus, one cannot just define the directional derivative as one is used to by using a difference quotient and by replacing the difference of the sets  $F(x + \lambda d)$  and  $F(x)$  by Jahn's difference. Instead, one defines a directional derivative for each  $l$  individually and combines those by taking the union. The concept of generalized Steiner sets allows us to define a directional derivative in a classical way by using a difference quotient and a suitable difference in the denominator. However, we can only do this in the space of generalized Steiner sets.

Again, by using the concept of Steiner points, we can also generalize this concept of Jahn directly. This allows to omit the uniqueness requirements:

$$DF_{St}(x; d) := \bigcup_{l \in \mathcal{T}_s} \{D[\text{St}(Y(l, F(\cdot)))](x; d)\}$$

where the limits

$$D[\text{St}(Y(l, F(\cdot)))](x; d) = \lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} (\text{St}(Y(l, F(x + \lambda d))) - \text{St}(Y(l, F(x))))$$

have to exist for all  $l \in \mathcal{T}_s$ . Thus, we obtain, if the set-valued map  $F$  is  $\mathcal{GS}_s$ -directionally differentiable at  $x$  in direction  $d$ , by Theorem 4.11 that

$$DF_{St}(x; d) = B_{\mathcal{T}_s}(DF_s^{\mathcal{GS}}(x; d)).$$

Finally, in [13, Theorem 5.2] Jahn formulates in analogy to Theorem 4.14 a necessary optimality conditions for set optimization problems using his original concept of the directional derivative. Please note that in the original proof the additional assumption that the convergence of the limits in (39) is uniform w.r.t.  $l$  is required. Thus, the assumptions imposed by Jahn for his necessary optimality conditions and the assumptions needed for our results are similar strong in case one has already assumed that the supporting faces are singletons. He needs to assume the uniform convergence additionally, while it is a consequence from our definition of the directional derivative, i.e., of  $\mathcal{GS}_s$ -directional differentiability, see Theorem 4.11.

So to sum up, we have been able, using the key ideas of Jahn, to extend his set difference and his directional derivative to the case where the supporting points are not unique. Moreover, we have found a way to keep track of the associated functionals  $l$  to the supporting points in a mathematical rigorous manner.

## Acknowledgements

The scientific exchange on which this paper is based was in parts funded by the German Research Foundation (DFG) under Project ID 392195690.

## References

- [1] Norman Neukel. Order relations of sets and its application in socio-economics. *Appl. Math. Sci.*, 7:5711–5739, 2013.
- [2] Truong Q. Bao and Boris S. Mordukhovich. Set-valued optimization in welfare economics. In Shigeo Kusuoka and Toru Maruyama, editors, *Advances in Mathematical Economics*, volume 13 of *Adv. Math. Econ.*, pages 113–153. Springer Japan, Tokyo, 2010.
- [3] Andreas H. Hamel, Frank Heyde, Andreas Löhne, Birgit Rudloff, and Carola Schrage. Set Optimization – A Rather Short Introduction. In Andreas H. Hamel, Frank Heyde, Andreas Löhne, Birgit Rudloff, and Carola Schrage, editors, *Set Optimization and Applications – The State of the Art*, volume 151 of *Springer Proc. Math. Stat.*, pages 65–141. Springer-Verlag, Berlin, Heidelberg, 2015.
- [4] Johannes Jahn. *Vector Optimization – Theory, Applications, and Extensions*. Springer-Verlag, Berlin, Heidelberg, 2004.
- [5] Gabriele Eichfelder, Corinna Krüger, and Anita Schöbel. Decision uncertainty in multiobjective optimization. *J. Global Optim.*, 69(2):485–510, 2017.
- [6] Gabriele Eichfelder, Julia Niebling, and Stefan Rocktäschel. An algorithmic approach to multiobjective optimization with decision uncertainty. *J. Global Optim.*, 2019.
- [7] Stephan Dempe. *Foundations of Bilevel Programming*. Kluwer Academic Publishers, Dordrecht-Boston-London, 2002.
- [8] Maria Pilecka. *Set-valued optimization and its application to bilevel optimization*. PhD thesis, Technische Universität Bergakademie Freiberg, February 2016.
- [9] Akhtar A. Khan, Christiane Tammer, and Constantin Zălinescu. *Set-valued Optimization – An Introduction with Applications*. Springer-Verlag, Berlin, Heidelberg, 2015.
- [10] Gabriele Eichfelder and Johannes Jahn. Vector Optimization Problems and Their Solution Concepts. In Qamrul Hasan Ansari and Jen-Chih Yao, editors, *Recent Developments in Vector Optimization*, pages 1–27. Springer-Verlag, Berlin, Heidelberg, 2012.
- [11] Johannes Jahn and Truong Xuan Duc Ha. New Order Relations in Set Optimization. *J. Optim. Theory Appl.*, 148(2):209–236, 2011.

- [12] Daishi Kuroiwa. The Natural Criteria in Set-Valued Optimization. *RIMS Kokyuroku*, 1031:85–90, 1998.
- [13] Johannes Jahn. Directional Derivatives in Set Optimization with the Set Less Order Relation. *Taiwanese J. Math.*, 19(3):737–757, 2015.
- [14] Johannes Jahn. Vectorization in Set Optimization. *J. Optim. Theory Appl.*, 167(3):783–795, 2015.
- [15] Darinka Dentcheva. Differentiable Selections and Castaing Representations of Multifunctions. *J. Math. Anal. Appl.*, 223(2):371–396, 1998.
- [16] Robert Baier and Elza M. Farkhi. Regularity and Integration of Set-Valued Maps Represented by Generalized Steiner Points. *Set-Valued Anal.*, 15(2):185–207, 2007.
- [17] Robert Baier and Elza M. Farkhi. Directed Sets and Differences of Convex Compact Sets. In M. P. Polis, A. L. Dontchev, P. Kall, I. Lasiecka, and A. W. Olbrot, editors, *Systems modelling and optimization, Proceedings of the 18<sup>th</sup> IFIP TC7 Conference held in Detroit, Michigan, July 22–25, 1997*, volume 396 of *Research Notes in Mathematics*, pages 135–143, Boca Raton–London–New York–Washington, D.C., 1999. Chapman and Hall/CRC.
- [18] Robert Baier and Elza M. Farkhi. Differences of Convex Compact Sets in the Space of Directed sets, Part I: The Space of Directed Sets. *Set-Valued Anal.*, 9(3):217–245, 2001.
- [19] Robert Baier and Elza M. Farkhi. Differences of Convex Compact Sets in the Space of Directed sets, Part II: Visualization of Directed Sets. *Set-Valued Anal.*, 9(3):247–272, 2001.
- [20] Robert Baier and Elza M. Farkhi. Directed Derivatives of Convex Compact-Valued Mappings. In N. Hadjisavvas and P. M. Pardalos, editors, *Proceedings of the Conference "Advances in Convex Analysis and Global Optimization: Honoring the Memory of C. Caratheodory (1873-1950)" held at Pythagorion, Samos, Greece, June 5–9, 2000*, volume 54 of *Nonconvex Optimization and its Applications*, pages 501–514, Dordrecht-Boston-London, 2001. Kluwer Academic Publishers.
- [21] Truong Xuan Duc Ha and Johannes Jahn. Characterizations of strictly convex sets by the uniqueness of support points. *Optimization*, 68(7):1321–1335, 2019.
- [22] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, New York (NY), 2004.
- [23] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Convex Analysis and Minimization Algorithms I*. Springer-Verlag, Berlin, Heidelberg, 1993.



- [24] Rolf Schneider. *Convex Bodies: The Brunn-Minkowski Theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.
- [25] Hans Rademacher. Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale. *Math. Ann.*, 79(4):340–359, 1919.
- [26] Hans Rademacher. Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln. II. *Math. Ann.*, 81(1):52–63, 1920.
- [27] Rosalind Cecily Young. The algebra of many-valued quantities. *Math. Ann.*, 104(1):260–290, 1931.
- [28] Jakob Steiner. Von dem Krümmungs-Schwerpunkte ebener Curven. *J. Reine Angew. Math.*, 21:33–63, 1840.
- [29] Geoffrey Colin Shephard. The Steiner Point of a Convex Polytope. *Canad. J. Math.*, 18:1294–1300, 1966.
- [30] Stephan Dempe and Maria Pilecka. Optimality Conditions for Set-Valued Optimization Problems Using a Modified Demyanov Difference. *J. Optim. Theory Appl.*, 171(2):402–421, 2016.
- [31] Gabriele Eichfelder and Tobias Gerlach. On classes of set optimization problems which are reducible to vector optimization problems and its impact on numerical test instances. In Akhtar A. Khan, Elisabeth Köbis, and Christiane Tammer, editors, *Variational Analysis and Set Optimization*, pages 265–290. Taylor & Francis, Boca Raton, 2019.