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# CAR FOLLOWING MODELS FOR PHENOMENA ON THE HIGHWAY

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**Abstract.** *Car following models are used to describe the behavior of a number of cars on the road dependent on the distance to the car in front.*

*We introduce a system of ordinary differential equations and perform a theoretical and numerical analysis in order to find solutions that reflect various traffic situations. We present three different variations of the model motivated by reality.*

## 1 INTRODUCTION

Traffic dynamics can be modeled with various mathematical techniques depending on the focus the modeler sets. For example, from a macroscopic point of view, the cars are treated as continuum with a certain density. In contrast, microscopic models describe the movement of each individuel driver. Two surveys over the literature are [6, 8]. In this paper we concentrate on a class of microscopic traffic models.

Consider a circular road of length $L$ with $N$ cars where $x_j(t)$ is the position of the $j$th driver at time $t$. The system of ordinary differential equations

$$\ddot{x}_j = \frac{1}{\tau}(V(x_{j+1} - x_j) - \dot{x}_j), \quad j = 1, \ldots, N, \tag{1}$$

describes the acceleration of each car depending on the headway $x_{j+1} - x_j$ to the car in front and its own velocity $\dot{x}_j$. With $x_{N+1} := x_1 + L$ the last car is always driving in front of the first one. Here $\tau$ models the reaction time of the drivers and $V$ is a so-called *optimal velocity* function of the headway. $V$ is always positive, monotonically increasing and satisfies $V(0) = 0$, $\lim_{x\to\infty} V(x) = V_{max}$ (Fig. 1).

System (1) is called a *car following model* and was first mentioned by Bando [2].

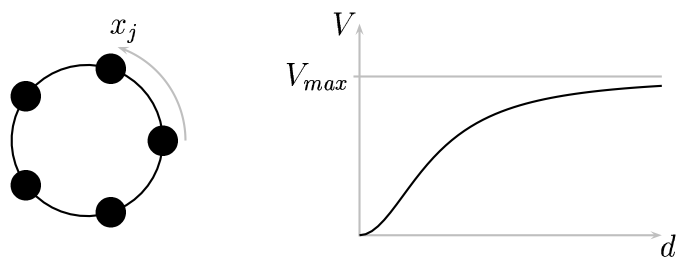


Figure 1: The general situation ($N = 5$) with an optimal velocity function $V$.

For this type of traffic model we are interested in two different classes of solutions. On the one hand the free traffic where all cars drive with the same (constant) velocity around the circle, and on the other hand the congested traffic. The second case may be observed as a kind of periodic solution where a traffic jam runs over the highway in the opposite direction to the driving cars.

How can we describe these two types of solutions mathematically?

At first we write (1) as a $2N$-dimensional first order system

$$\left\{ \begin{array}{rcl} \dot{x}_j &=& y_j \\ \dot{y}_j &=& \frac{1}{\tau}(V(x_{j+1} - x_j) - y_j) \end{array} \right\}, \quad \forall j = 1, \ldots, N. \tag{2}$$

In a car following model, where the velocity of each driver depends only on the headway to the car in front, a solution with constant velocity must also have constant headways. Therefore, we transform (2) into a system in terms of the headways:

$$\left\{ \begin{array}{rcl} \dot{\xi}_j &=& \eta_j \\ \dot{\eta}_j &=& V(\xi_{j+1}) - V(\xi_j) - \eta_j \end{array} \right\}, \quad \forall j = 1, \ldots, N, \tag{3}$$

where $\xi_j := x_{j+1} - x_j$ is the headway between drivers $j+1$ and $j$, and $\eta_j := \dot{x}_{j+1} - \dot{x}_j$ is the corresponding relative speed. The headway $\xi_{N+1}$ is equal to $\xi_1$.

Now it is easy to find solutions with constant headways and all cars driving with the same velocity. We are looking for stationary solutions of (3):

$$\left\{ \begin{array}{rcl} 0 &=& \eta_j \\ 0 &=& V(\xi_{j+1}) - V(\xi_j) - \eta_j \end{array} \right\}, \quad \forall j = 1, \ldots, N.$$

A stationary solution $(\xi^0, \eta^0)$ is given by

$$\begin{aligned} \xi_j^0 &= \frac{L}{N} \\ \eta_j^0 &= 0, \end{aligned}$$

for each $j = 1, \ldots, N$. It corresponds to a so-called *quasi-stationary* solution $x^0(t)$ in terms of system (1) with

$$x_j(t) = V\left(\frac{L}{N}\right)t + j\frac{L}{N}, \quad \forall j = 1, \ldots, N.$$

Fig. 2 shows $x_j^0(t)$ for $j = 1, \ldots, 5$, a length $L = 20$, and a reaction time $\tau = 1$.

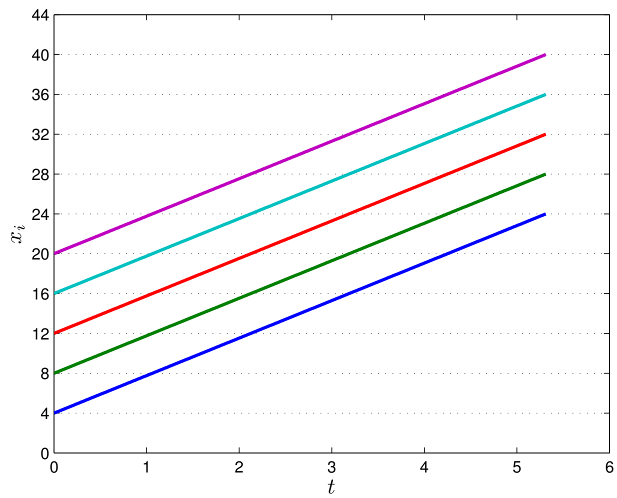


Figure 2: The quasi-stationary solution $x^0$ with $x_j^0(t) = ct + j\frac{L}{N}$ with $c = V\left(\frac{L}{N}\right)$.

Instead of simulations which are only able to find stable solutions we are going to analyse stationary solutions and their stability.

## 2 STABILITY

Let $f(\xi, \eta)$ be the right hand side of system (3). The stability of the stationary solution $(\xi^0, \eta^0)$ is connected to the eigenvalues of the matrix

$$A := Df(\xi^0, \eta^0).$$

In [4] it is shown that the spectrum of $A$ is given by the roots of the following equation:

$$\lambda^2 + \lambda + \beta(L)\left(1 - e^{ik\frac{2\pi}{N}}\right), \quad k = 1, \ldots, N, \tag{4}$$

where $\beta(L) = V\left(\frac{L}{N}\right)$. Consider that the eigenvalues as roots of Eq. (4) depend on two parameters $L$ and $N$. Varying the number of cars $N$ on the highway is a natural way, but in our case this changes the dimension of the whole system. Therefore, we change the length of the road and compute the eigenvalues of $A$ under variation of $L$.

Fig. 3 shows the eigenvalues lying in the Gaussian plane for $L = 5$ and for $L = 10$. Although there is always an eigenvalue 0 (compare (4) for $k = N$), it is shown in [4] that this eigenvalue does not influence the stability of the quasi-stationary solution.

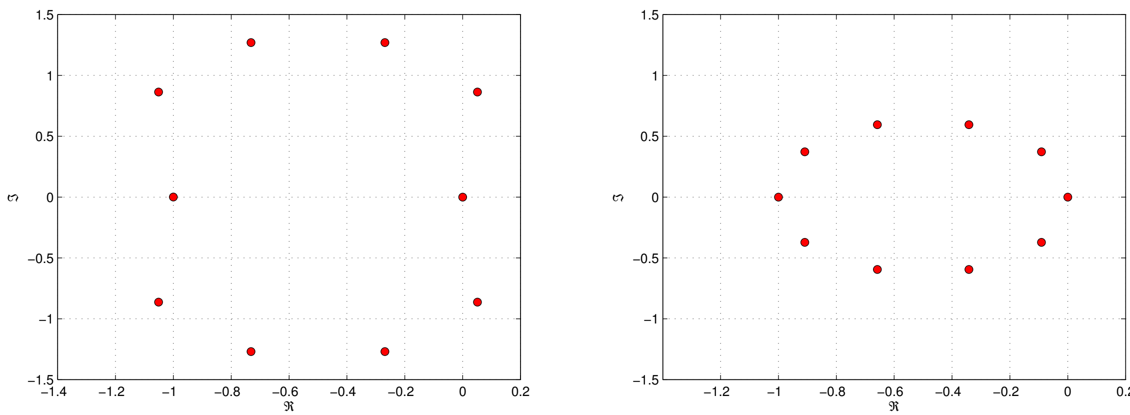


Figure 3: Eigenvalues of $Df(\xi^0, \eta^0)$ for $N = 5$, $L = 5$ (left, unstable), and $L = 10$ (right, stable).

From Fig.3 we conclude that there is a region for $L$ where at least one pair of eigenvalues has positive real parts and $(\xi^0, \eta^0)$ is unstable. Increasing $L$, these eigenvalues cross the imaginary axis stabilizing the quasi-stationary solution (Fig. 4).

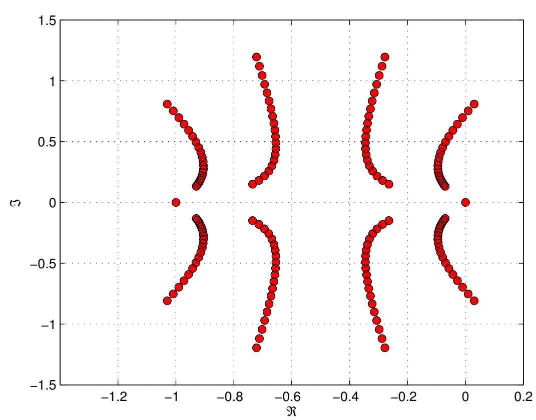


Figure 4: Two eigenvalues crossing the imaginary axis under variation of $L = 5 \ldots 10$.

In general an eigenvalue crossing the imaginary axis changes the stability of a stationary solution. If there is pair of complex conjugated eigenvalues with zero real parts this leads to Hopf bifurcations [7]. Under appropriate conditions, the theorem of Hopf guarantees the existence of *periodic solutions* when stability changes take place under variation of a parameter like $L$. As an example, see Fig. 5 (left) which shows a periodic orbit in phase space together with the unstable stationary point $(\xi^0, \eta^0)$.

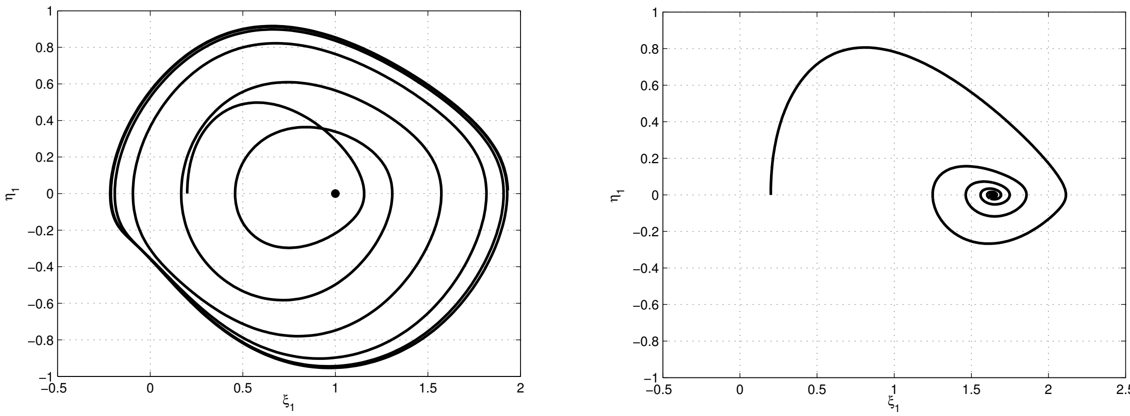

Figure 5: Two Phaseportraits $\eta_1$ over $\xi_1$: a stable stationary solution for $L = 10$ (right) and a stable periodical solution for $L = 5$ (left).

These solutions being periodic in the headways of the cars and in the relative speeds, describe a kind of congested traffic as in a stop-and-go situation. In fact, for traffic dynamics it is very interesting to show the existence of periodic solutions analytically. For an analysis of their stability see [4].

## 3 VARIATIONS OF THE MODEL

There are many possibilities to vary system (1) to cope with different phenomena arising on the highway. It is our goal to keep the model as simple as possible in order to obtain not only numerical but also theoretical results.

### 3.1 Aggressive Driving Behavior

The drivers in (1) change their velocity very 'moderately' dependent on an optimal velocity at a certain distance to the car in front. In reality everybody knows the situation driving on the highway with an aggressive driver behind, blinking and changing his velocity immediately with oneself's velocity.

This driving behavior can be modeled with an extra term in the model equation:

$$\ddot{x}_j = \frac{1}{\tau}\Big(V\big(x_{j+1} - x_j\big) - \dot{x}_j + \alpha \cdot \big(\dot{x}_{j+1} - \dot{x}_j\big)\Big), \quad j = 1, \ldots, N. \tag{5}$$

Here, $\alpha > 0$ is an *aggressive driving parameter*. The more aggressive the driver the bigger is $\alpha$.

In [5] we found out that the new $\alpha$-term does not change the (stability) analysis of the model qualitatively — especially there are still quasi-stationary solutions $(\xi^0, \eta^0)$ and Hopf bifurcation points for a certain parameter constellation. Further more we showed, that for fixed $L$ an increasing $\alpha$ can stabilize $(\xi^0, \eta^0)$ (see also [9]).

### 3.2 Variable Reaction Times

One major problem of (microscopic) car following models is the possibility of negative headways which can be interpreted as a consequence of collisions in our model. In [5] we tried to reduce this risk by introducing a new reaction time $\tau = \tau(x_{j+1} - x_j)$, depending on the headways as shown in Fig. 6:

$$\ddot{x}_j = \frac{1}{\tau\big(x_{j+1} - x_j\big)} \Big(V\big(x_{j+1} - x_j\big) - \dot{x}_j\Big), \quad j = 1, \ldots, N. \tag{6}$$

For the drivers on a highway the justification for this behavior is a more realistic modeling of the reaction for short distances — the closer the car in front, the better the concentration.

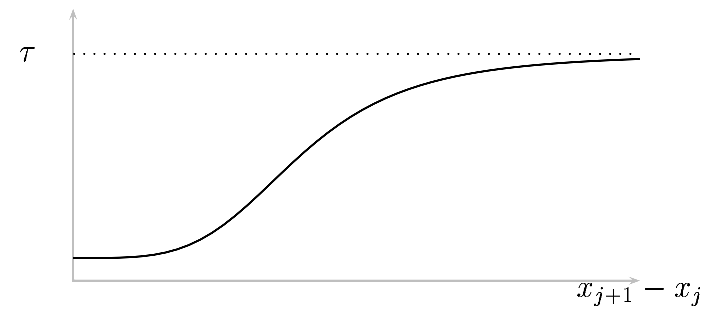


Figure 6: The variable reaction time $\tau(x_{j+1} - x_j)$.

### 3.3 Road Works

Most of the car following models in literature have in common a model equation depending on the headway independent of the certain *position* $x_j$ of the $j$th driver. Our current interest focusses on *road works* on the highway.

Road works mathematically can be modeled by a maximum (optimal) velocity $V_{max}$ that now depends on the position: there is a region on the circle, where the cars have to reduce their speed (Fig. 7).

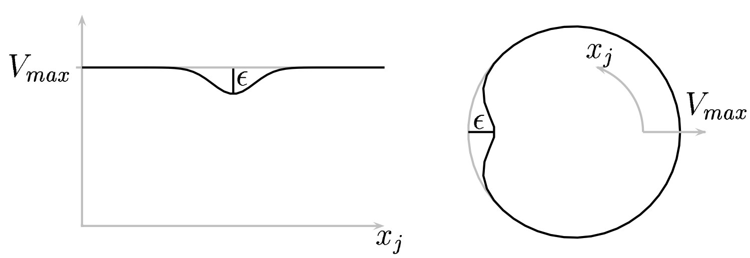


Figure 7: A region of reduced maximum velocity.

The parameter $\epsilon$ describes the 'size' of the road works which changes model (1) in the following way:

$$\ddot{x}_j = \frac{1}{\tau} \left(V_{max}(x_j, \epsilon) \cdot V(x_{j+1} - x_j) - \dot{x}_j\right). \tag{7}$$

For $\epsilon = 0$ the maximum optimal velocity $V_{max}$ is constant as in (1), so (7) is a perturbation of the model without road works.

Anyway, this new model is qualitatively different from the models in sections 3.1 and 3.2 as the symmetry of the circle is broken. We can no longer hope to find stationary or quasi-stationary solutions as each car changes its velocity and headway to the next car whenever it passes the road works region — even for small $\epsilon$.

A way out of this technical problem is to formulate a new (more general) solution type that describes the movement of the cars on a circular highway with road works properly. In [1] there is suggested a type of solutions called *ponies on a merry-go-round* (POM) that we adapted to our model.

A POM solution is a solution that fullfills the following two conditions for a period $T > 0$.

1. $x_i(t+T) = x_i(t) + L \quad (i = 1, \ldots, N)$
2. $x_i(t) = x_{i-1}\left(t + \frac{T}{N}\right) \quad (i = 1, \ldots, N)$.

For this solution type it is possible (but mathematically more complex) to perform an analysis analog to section 2.

## 4 CONCLUSIONS

The car following models can help to understand the behavior of a finite number of cars on the road from a microscopic point of view. The more drivers are involved into the system the more equations we need to describe their dynamics, increasing the complexity of numerical simulations.

The main mathematical tool is the analysis of stability and the use of bifurcation theory. The models can be understood easily in terms of one's own experiences on the highway.

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