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ADAPTIVE EIGENVALUE COMPUTATION FOR ELLIPTIC OPERATORS

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Abstract. *This article is concerned with recent developments of adaptive wavelet solvers for elliptic eigenvalue problems. We describe the underlying abstract iteration scheme of the preconditioned perturbed iteration. We apply the iteration to a simple model problem in order to identify the main ideas which a numerical realization of the abstract scheme is based upon. This indicates how these concepts carry over to wavelet discretizations. Finally we present numerical results for the Poisson eigenvalue problem on an L-shaped domain.*

1 INTRODUCTION

The Poisson eigenvalue problem with homogeneous Dirichlet boundary conditions on the L-shaped domain $\Omega \subset \mathbb{R}^2$

$$-\Delta u = \lambda u \quad \text{on } \Omega \tag{1a}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{1b}$$

is the prototype of the problems considered in this article. More precisely we want to determine a few eigenvalues λ in the lower part of the spectrum along with corresponding eigenfunctions.

A standard approach is to discretize first the infinite dimensional problem, for instance, by means of Finite Element Methods (FEM). The resulting finite dimensional algebraic eigenvalue problem can then be treated by suitable iterative methods that take advantage of the sparsity of the discretization. However, due to the reentrant corner of the L-shaped domain and resulting possible singularities of the eigensolutions a priori chosen grids may either lead to large discretization errors and hence to inaccurate eigenvalue approximations or to an unnecessarily large problem size rendering the overall computation very inefficient. Instead one might try to combine the iterative eigensolver with suitably adapted discretizations in order to achieve a desired accuracy of eigenvectors and eigenvalues while keeping the size of the discrete problems as small as possible. One expects that such adapted meshes exhibit higher refinement levels near the reentrant corner in order to resolve the singularity. A possible strategy is to use adaptive FE methods which generate a sequence of grids based on local error estimators, see for example [1]. There an iterative scheme is described which can be proved to converge, however, complexity estimates are not available up to now [9] so that the actual advantage of such methods over much simpler uniform discretizations is not clear. A main problem is the tolerance up to which the intermediate problems have to be solved.

In the present paper we tackle the problem from a different perspective inspired by the work of [2]. Instead of solving a sequence of finite dimensional eigenvalue problems we stick to the infinite dimensional formulation. On this level we can construct first an (ideal) iterative algorithm which can be shown to converge to the lowest eigenvalues and corresponding eigenvectors by generalizing the preconditioned inverse iteration [7] to the operator case. Moreover, we show that the iteration is robust under perturbations as long as their magnitude is kept at most proportional to the current error.

When carrying out the iteration numerically one ultimately has to discretize the problem in some way. In order to facilitate adaptation one can use suitable stable bases for the (infinite dimensional) energy space in order to transform first the eigenvalue problem into an equivalent one on the infinite dimensional sequence space ℓ_2. For a wide class of eigenvalue problems this can be done by means of wavelet bases. The unknown eigenvector being now represented by an element of ℓ_2, the idea is now to confine actual numerical computations to possibly good *finitely supported* approximations. On account of the robustness of the ideal iteration under perturbations one only has to make sure that the finitely supported approximations stay sufficiently close to the exact solutions. Of course, this requires identifying among other things suitable tolerances that are dynamically updated in the course of the iteration. Another central issue is to apply the generally almost fully populated matrix representations of the involved operators within the desired tolerances in an efficient way. It turns out that for a wide range of operators wavelet representations are *quasi-sparse*, i.e. they can be approximated up to any tolerance by a

matrix with only finitely many entries in each row and column. These features, namely adaptive accuracy controlled finitely supported outputs from approximate operator applications give rise to a computable algorithm.

Finally, the main point is to show that the iteration can be carried out in *optimal linear complexity* by which we mean the following. For any target accuracy the number of operations stays proportional to the degrees of freedom which are needed at least to represent an approximation to the solution within that same accuracy tolerance. This latter benchmark brings in the theory of best N-term approximation [6]. In order to keep approximations always close to best N-term approximations suitable coarsening steps are employed from time to time in the iteration.

This article summarizes the results obtained by previous work in [4] and [10]. It highlights the motivation and the key issues of the proposed iteration and presents a numerical example. In order to bring out the main ideas without going into too many technicalities we first perform the iteration for toy eigenvalue problem using the Fourier basis instead of wavelets.

The article is organized as follow. First, in Section 2 we formulate the eigenvalue problem in a more abstract way in terms of a Gelfand triple and operators. The perturbed preconditioned iteration is then analyzed in Section 3. In Section 4 a toy realization of the abstract iteration scheme is presented which clarifies the properties needed for the application of wavelet bases in Section 5. Finally we present a numerical example for the wavelet context.

2 OPERATOR FORMULATION

In this section we introduce the notation, state the basic assumptions and pose the problem in terms of operators. This is based on the abstract setting of a Gelfand triple which will simplify the later analysis.

The Poisson eigenvalue problem, see equation (1) can be generalized to an abstract eigenvalue problem in the weak formulation

$$a(u, v) = \lambda(u, v) \quad \text{for all } v \in V, \tag{2}$$

where a is a bilinear form on some Hilbert space V and $(\cdot, \cdot)$ is the inner product on some Hilbert space H which contains V as a densely embedded subspace. We denote the norm on H by $|\cdot|$ and the norm on V by $\|\cdot\|$, respectively. Furthermore we assume that a is symmetric, strongly elliptic and bounded. In the model example of the Poisson eigenvalue problem, see equation (1), we have

$$a(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x, \quad (u, v) = \int_\Omega u(x)v(x)\, \mathrm{d}x,$$

$V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

However iteration schemes can be stated more intuitively in terms of operators. To this end, the eigenvalue problem of (2) can be written as

$$Au = \lambda Eu \tag{3}$$

where A maps V into its dual V^* while E is a mass matrix in nature corresponding to an inner product on H. In fact, in our model example A is the Laplacian seen as a linear operator

mapping $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$ while E stems from the inner product on the space $L^2(\Omega)$. Denoting by $\langle \cdot, \cdot \rangle$ the dual pairing on V^* and V the operators A and E are defined (through the Riesz representation Theorem) by $\langle Au, v\rangle = a(u, v)$ and $\langle Eu, v\rangle = (u, v)$ for all $u, v \in V$, respectively.

Now $\lambda \in \mathbb{R}$ is an *eigenvalue* if there exists a $v \in V \setminus \{0\}$, such that $Av = \lambda Ev$ and v is called an *eigenvector*. The *Rayleigh quotient* is given by

$$\mu(v) = \frac{\langle Av, v\rangle}{\langle Ev, v\rangle} = \frac{\langle Av, v\rangle}{(v, v)}, \quad v \in V. \tag{4}$$

For simplicity we shall assume in the present paper from now on that the lowest eigenvector is single and isolated from the rest of the spectrum although one may also deal with degenerated or multiple eigenvalues .

In summary, the problem we will treat in the sequel can be formulated as follows: Find the smallest eigenvalue $\lambda_1 \in \mathbb{R}$ and a corresponding eigenvector $u_1 \in V \setminus \{0\}$ such that

$$Au_1 = \lambda_1 Eu_1.$$

3 PERTURBED PRECONDITIONED ITERATION

The task of the current section is to develop an iteration scheme which solves the abstract eigenvalue problem (3) directly in the (possibly infinite dimensional) space V. One possibility would be to apply the inverse iteration, which under the above assumptions can be shown to converge also in the infinite dimensional setting. For the Poisson eigenvalue problem of equation (1) the successor v' of the current iterate v is then given by

$$v' = -\Delta^{-1} v,$$

that is one has to solve the Poisson boundary value problem in each iteration step.

In order to avoid this costly task we propose a different algorithm. It is based on the steepest descent of the Rayleigh quotient

$$\mu(v) = \frac{(-\Delta v, v)}{(v, v)}.$$

since all eigenvectors to the lowest eigenvalue minimize μ. Furthermore, as in the case of conjugate gradient iteration, the convergence speeds up if we use an appropriately scaled preconditioner P^{-1}. Then the iteration reads

$$v' = v - P^{-1}(-\Delta v - \mu(v)v).$$

Note that in the case of a perfect preconditioner $P^{-1} = -\Delta^{-1}$ the iteration coincides with the inverse iteration. The finite dimensional version of the iteration is well known in linear algebra, see [7] and references therein.

Regarding a numerically realization it is clear that such an iteration can not be carried out exactly. For that reason we define the iteration scheme for abstract eigenvalue problems (3) allowing a perturbation in each step.

Definition 1. Let the starting vector $v^0 \in V$, $v^0 \neq 0$, be given and denote its associated Rayleigh quotient by $\mu^0 = \mu(v^0)$. A *perturbed preconditioned inverse iteration (PPINVIT)* is a sequence of vectors $(v^n)_{n\geq 0}$ and associated Rayleigh quotients $(\mu^n)_{n\geq 0}$ generated by

$$\begin{aligned} \tilde{v}^{n+1} &= v^n - P^{-1}(Av^n - \mu(v^n)Ev^n) + \xi^n, \\ v^{n+1} &= |\tilde{v}^{n+1}|^{-1}\tilde{v}^{n+1}, \\ \mu^{n+1} &= \mu(v^{n+1}), \end{aligned}$$

where $(\xi^n)_{n\geq 0} \in V$ are perturbations.

In the wavelet realization [4] and also in the realization of section 4 of the given algorithm the perturbations $\xi^{(n)}$ account for the fact that the operators P^{-1}, A, E are not applied exactly. That means we have

$$r^n = P^{-1}(Av^n - \mu(v^n)Ev^n) + \xi^n,$$

is the actually computed finitely supported approximation to the exact preconditioned residual $P^{-1}(Av^n - \mu(v^n)Ev^n)$.

The iteration resembles the preconditioned Richardson iteration. Here the residual for an iterate v is given by $Av - \mu(v)Ev$. As in the case of linear equations the magnitude of the residual can be used to estimate the error. For that purpose we define

$$\rho(v) = \|A - \lambda(v)Ev\|_{A^{-1}}/\|v\|_A,$$

where $\|\cdot\|_A$ denotes the norm defined by $\langle A\cdot,\cdot\rangle^{1/2}$. $\|\cdot\|_{A^{-1}}$ is defined analogously..

Along the lines of the proof in the algebraic case [7] combined with a perturbation argument convergence can be shown in the case where the perturbations are sufficiently small, see [10].

Theorem 2. *Let $v \in V$, $v \neq 0$, such that the associated Rayleigh quotient $\mu = \mu(v)$ fulfills $\mu < \lambda_2$, where λ_2 is the second lowest eigenvalue. Assume that the preconditioner P and perturbation ξ fulfill*

$$\|\mathrm{Id} - P^{-1}A\|_A \leq \gamma_p, \quad \|\xi\|_A/\|v\|_A \leq \gamma_\xi\, \rho(v) \tag{5}$$

where $\gamma = \gamma_P + \gamma_\xi < 1$. Then the result v' of the next step of PPINVIT (cf. Definition 1) with starting vector v and the associated Rayleigh quotient $\mu' = \mu(v')$, satisfy

$$\frac{\mu' - \lambda_1}{\lambda_2 - \mu'} \leq q^2(\gamma, \lambda_1, \lambda_2)\frac{\mu - \lambda_1}{\lambda_2 - \mu}.$$

Here q is given by

$$q(\gamma, \lambda_1, \lambda_2) = 1 - (1 - \gamma)(1 - \lambda_1/\lambda_2).$$

Therefore, the rate of decay is only governed by the eigenvalue gap and the quality of the preconditioner. Note that the presence of a perturbation has the same effect as applying a preconditioner with a constant γ instead of γ_P.

Aside from the Rayleigh quotient one is also interested in convergence to the eigenspace, which is best described by the convergence of the angle between the iterand and the eigenspace.

Theorem 3. *Let $v \in V$, $v \neq 0$, such that the associated Rayleigh quotient fulfills $\mu(v) < \lambda_2$. Denote the angle with respect to the scalar product $\langle A\cdot, \cdot\rangle$ between v and the eigenspace spanned u_1 by ϕ. Then*

$$\sin\phi \leq \sqrt{\frac{\lambda_2}{\lambda_1} \cdot \frac{\mu(x) - \lambda_1}{\lambda_2 - \mu(x)}}.$$

Moreover the eigenvector residual controls the angle, i.e.

$$\frac{\lambda_1}{3\mu(v)}\rho(v) \leq \sin\phi \leq \frac{\lambda_2}{\lambda_2 - \mu(v)}\rho(v).$$

A proof of this theorem can be found in [10]. Convergence of the Rayleigh quotient towards λ_1 assures convergence of the angle between the iterands and the corresponding eigenvector u_1. Obviously the size of the perturbation need to be kept proportional to the current error in the subspaces which in practice has to be ensured by sufficiently accurate operator applications.

4 MODEL REALIZATION

In the previous section an iteration scheme was presented that operates in the abstract spaces introduced for the eigenvalue problem (3). However, it is still not obvious how such a scheme can be turned into a numerically realizable algorithm. The purpose of this section is to explain prerequisites which lead to a numerical algorithm. In order not to obscure the main ideas by the somewhat higher level of technicality encountered with the wavelet context, we shall discuss in the following setting first a toy problem discretized by Fourier series. However, already this simple example comprises all the ingredients which are later necessary for applying the algorithm in a more realistic wavelet setting.

4.1 Model problem

Consider the eigenvalue problem

$$-u'' + gu = \lambda u \quad \text{on } (0, 2\pi) \tag{6a}$$

$$u(0) = u(2\pi) = 0 \tag{6b}$$

for a n-times continuous differentiable function $g \in C^n(0, 2\pi)$ with $g \geq 0$. This eigenvalue problem satisfies the assumptions of the general case: By setting $H = (L^2(0, 2\pi), \|\cdot\|_0)$ and $V = (H_0^1(0, 2\pi), |\cdot|_1)$ the differential operator given by

$$Au = -u'' + gu$$

with homogeneous Dirichlet boundary conditions is a symmetric mapping between V and its dual. By Friedrich's inequality A is strongly positive and the fact that g is also bounded from above shows the boundedness of a. Moreover the spectrum of A is purely discrete since $(0, 2\pi)$ is a bounded domain.

Now the objective is to apply the perturbed preconditioned inverse iteration of Definition 1 numerically. For that purpose we to chose a representation of functions in V in terms of the sine

functions

$$\psi_k(x) = \pi^{-1/2} \frac{\sin(kx)}{k} \quad k = 1, 2, \ldots.$$

which form an orthonormal basis for V. Therefore we have the following *norm equivalence* which is the first key ingredient:

$$|u|_1^2 = \sum_{k=1}^{\infty} |x_k|^2 \quad u = \sum_{k=1}^{\infty} x_k \psi_k \tag{7}$$

establish a norm isomorphism between $H_0^1(0, 2\pi)$ and ℓ_2. Using this representation, the eigenvalue problem (6) then reads as

$$Sx = \lambda Mx,$$

where the infinite matrix representations are given by

$$S = \left[\langle A\psi_k, \psi_{k'}\rangle\right]_{k,k'=1,2,\ldots} \quad M = \left[\langle \psi_k, \psi_{k'}\rangle\right]_{k,k'=1,2,\ldots}$$

Note that now we have transformed the original eigenvalue problem (3) to an equivalent one in the sequence space ℓ_2.

For the construction of a preconditioner we use the special structure of the discretization matrix S. In particular since ψ_k are orthonormal with respect to the Laplacian it follows that $S = \mathrm{Id} + G$ where

$$G = [G_{k,k'}]_{k,k'=1,2,\ldots} \quad G_{k,k'} = \frac{1}{kk'\pi} \int_0^{2\pi} \sin(kx) g(x) \sin(k'x)\, \mathrm{d}x.$$

We choose the preconditioner to be a scaled version of the inverse of the Laplacian, which in the above representation is just the identity. Then

$$\langle x, x\rangle \leq \langle Ax, x\rangle \leq (1 + \|g\|_\infty)\langle x, x\rangle.$$

for all $x \in \ell_2$. It can be shown that $P^{-1} = \alpha \cdot \mathrm{Id}$ with

$$\alpha = \frac{2}{1 + \|g\|_\infty}$$

satisfies equation (5) with $\gamma_P = \|g\|_\infty/(2 + \|g\|_\infty)$.

4.2 Approximate iteration

In the previous part of the section we assembled all ingredients for applying the preconditioned inverse iteration of Definition 1 on the sequence space ℓ_2:

$$x' = x - \alpha(Sx - \mu(x)Mx), \quad \mu(x) = \langle Sx, x\rangle/\langle Mx, x\rangle. \tag{8}$$

In order to compute the iteration numerically all $x^{(n)}$ created by this iteration have to have only a finite number of nonvanishing entries. Let us fix the following notation.

Definition 4. Let $x = (x_k)_{k=1,2,\dots}$ be a sequence in ℓ_2. The *support* of x is given by $\operatorname{supp}(x) = \{i \mid x_i \neq 0\}$. The sequence x is said to be *finitely supported* if its support is finite. Furthermore let $\sharp x = \sharp \operatorname{supp}(x)$, which is the number of non-zero entries of the sequence x.

Clearly one has to make sure that all operations in (8) preserve the the finiteness of support. The mass matrix M is just a diagonal matrix since the functions $\{\psi_k \mid k = 1, 2, \dots\}$ are also orthogonal with respect to the L^2 inner product . As a consequence Mx has the same number of entries as x. However, the stiffness matrix S is in general fully populated. Therefore Sx has in general an infinite number of entries and the iteration can not be carried out exactly.

However, it has been pointed out in Section 3 that the iteration need to be carried out only up to a perturbation ξ. Therefore the application Sx has to be carried out only approximately. One possibility is to find an operators S_j with only finite many entries in each row which approximate S to any given tolerance as j increases – the number of non-zero entries in $S_j x$ then stays finite if x has only finite support.

The following definition states the second key ingredient for the numerical realization of the PPINVIT, namely *compressibility*.

Definition 5. Let $s^* > 0$. A matrix A is *s^*-compressible* if for every $j \in \mathbb{N}$ there exists a matrix A_j with at most 2^j entries in each row an column such that $\|A - A_j\| \leq \alpha_j 2^{-s^* j}$ where the sequence $(a_j)_{j\in\mathbb{N}}$ is in ℓ_1.

In order to construct such an approximation S_j for S we shall exploit the structure of the matrix S. In particular, the contribution of the Laplacian is already in diagonal form whereas the multiplication with g fills up the entries of S. This matrix G can be simplified further using

$$g_k = \frac{1}{2\pi} \int_0^{2\pi} \cos(kx) g(x) \, \mathrm{d}x$$

and the trigonometric identity $\sin(x)\sin(y) = 1/2 \ \cos(x-y) - 1/2 \cos(x+y)$ which leads to

$$G_{k,k'} = \frac{1}{kk'}(g_{|k-k'|} - g_{k+k'}).$$

From Fourier series theory it follows that the more regular g the faster the coefficients g_k decay as $k \to \infty$. Therefore the leading term in matrix entry $G_{k,k'}$ stems from the term $g_{|k-k'|}$, which decays away from the diagonal. Therefore it is reasonable to approximate S by a banded matrix.

Theorem 6. *Let $g \in C^n$ with $n \in \mathbb{N}$. Then the discretization matrix D is s^*-compressible for all $s^* < n$. In particular the approximation is given by*

$$(G_j)_{k,k'} = \begin{cases} G_{k,k'} & |k-k'| \leq 2^j \\ 0 & \textit{otherwise}, \end{cases}$$

i.e. the original matrix G keeping only the banded diagonal part with width 2^j.

Proof. For Sobolev spaces $H_0^n(0, 2\pi)$ we have the norm equivalence between the weighted Fourier coefficients of some function u and the corresponding norm, i.e.

$$|u|_{H^n}^2 = \sum_{k=1}^{\infty} k^{2n} |u_k|^2, \quad u_k = (2\pi)^{-1/2} \int_0^{2\pi} u(x) e^{ikx} \, \mathrm{d}x.$$

In particular, every coefficient has to satisfy $|u_k| \leq k^{-n}|u|_{H^n}$ and hence also $|g_k| \lesssim k^{-n}$ by definition of g_k. The symbol $\lesssim$ means less than or equal up to a fixed constant factor. As a consequence one also has

$$|G_{k,k'}| \lesssim \frac{1}{k\ k'}|k-k'|^{-n}, \quad k-k' \neq 0.$$

Now the norm of $G - G_j$ can be estimated by

$$\|G - G_j\| \leq \sup_{k=1,2,\dots} \sum_{k'=1}^{\infty} \left|(G-G_j)_{k,k'}\right|$$

which follows from the Schur Lemma and the symmetry of G. Inserting the estimate of $|G_{k,k'}|$ gives

$$\sup_{k=1,2,\dots} \sum_{k':|k-k'|>2^j} \frac{1}{kk'}|k-k'|^{-n} \lesssim 2^{-jn}.$$

Here we have also used the fact that we only have to sum up contributions which are more than j positions away from the diagonal. This proves the assertion. □

The previous result lays the basis for performing the perturbed preconditioned inverse iteration. Instead of calculating the exact residual $r = Sx - \mu(x)Mx$ we calculate

$$r_j = S_j x - \mu_j(x)Mx, \quad \mu_j(x) = \langle S_j x, x\rangle / \langle Mx, x\rangle$$

which has only finite number of non-zero entries. The error can be estimated by

$$\|r - r_j\| \leq \|S - S_j\|\|x\| + |\mu(x) - \mu_j(x)|\|x\| \lesssim 2^{-jn}\|x\|$$

where Theorem 6 was used. Finally we choose j sufficiently large such that the error bound in Theorem 2 for the perturbation is satisfied.

4.3 Complexity

Theorem 2 ensures that the error between the iterand $x^{(i)}$ and the solution decreases in each step of PPINVIT. However, in every iteration step the support of $x^{(n)}$ grows thereby increasing the active degrees of freedom. Now the question arises whether these degrees of freedom are really necessary to achieve the current error. In other words one should compare the number of entries of the current iterate with the sparsest possible approximation realizing the same error.

This ties into the concept of best N-term approximation, see [6] and references therein. Let us denote

$$\Sigma_N = \{x \in \ell_2 \mid \sharp x \leq N\},$$

consisting of all sequences with at most N non-zero entries. For an element $x \in \ell_2$ we define the error of the best approximation in Σ_N as

$$\sigma_N(x) = \inf_{y \in \Sigma_N} \|x - y\|.$$

The decay of $\sigma_N(x)$ as $N \to \infty$ now characterizes the approximability of the vector x. Therefore we introduce approximation spaces

$$\mathcal{A}^s = \big\{x \in \ell_2 \mid \|x\|_{\mathcal{A}^s} := \|x\| + |x|_{\mathcal{A}^s} < \infty\big\}, \quad |x|_{\mathcal{A}^s} = \sup_{N\in\mathbb{N}} N^s \sigma_N(x),$$

i.e. the set of all sequences such that the error of the best approximation with at most N non-zero coefficients decays at least as N^{-s}. Another way to view this is the following: it takes at most $N_\varepsilon = \varepsilon^{-1/s}|x|_{\mathcal{A}^s}$ number of entries in order to approximate $x \in \mathcal{A}^s$ with accuracy ε. Thus, in what follows optimal complexity can be expressed by the relation

$$\text{accuracy } \varepsilon \quad \longleftrightarrow \quad \text{degrees of freedom } \varepsilon^{-1/s}|\cdot|_{\mathcal{A}^s}.$$

Now for any approximation x to $u \in \mathcal{A}^s$ with error ε we can compare the number of non-zero entries of x with the best possible bound $\varepsilon^{-1/s}|u|_{\mathcal{A}^s}$ (over the whole class). If the support of x is to large we can calculate a more efficient approximation at the cost of a slightly bigger error by judiceously coarsening the current approximation.

Theorem 7. *Let $x \in \mathcal{A}^s$ for some $s > 0$ and $y \in \ell_2$ such that $\|x - y\| \le \varepsilon$. For a fixed $b > 0$ define*

$$w = \operatorname{argmin}_{\|y-z\|\le(1+b)\varepsilon} \sharp z$$

Then one has

$$\|w - x\| \le (2+b)\varepsilon, \quad \sharp w \lesssim \varepsilon^{-1/s}\|x\|_{\mathcal{A}^s}, \quad \|w\|_{\mathcal{A}^s} \lesssim \|x\|_{\mathcal{A}^s}$$

where the involved constants are independent of ε. That is w is a quasi-optimal approximation of x with tolerance $(2+b)\varepsilon$.

The presented procedure is called coarsening and a proof can be found in [2]. For any finitely supported y the calculation of w can be done easily: first sort all entries of y according to their magnitude and then take the largest ones (in absolute value) until the desired accuracy is reached. Since this procedure involves sorting of the coefficients it takes $\log(\sharp y)\sharp y$ operations. However, at the expense of a slightly worse target accuracy one can get away with quasi-sorting while removing the $\log$-term. This coarsening procedure therefore provides a means to create a more efficient approximation from a given one.

The aim for an iterative scheme can now be stated as follows: for any given target accuracy the iteration calculates a corresponding approximation such that the number of required operations stays proportional to the minimum number of degrees of freedom needed to represent the solution with the same accuracy.

To achieve this linear complexity, however, one has to make sure that the support of all intermediate vectors stays proportional to the best possible one. This boils down to the following assertion: let $x \in \mathcal{A}^s$ for some $s > 0$ and a tolerance ε be given. One can calculate an approximation w_ε of Sx such that

$$\|Sx - w_\varepsilon\| \le \varepsilon \quad \sharp w_\varepsilon \le \varepsilon^{-1/s}|x|_{\mathcal{A}^s}.$$

A direct approach would be the following: First x can be coarsened up to some tolerance $c\varepsilon$ with $c < 1$ to achieve x_ε with $\sharp x_\varepsilon \lesssim \varepsilon^{-1/s}|x|_{\mathcal{A}^s}$ number of non-zero entries. Then apply S_j with j small enough such that $2^{-j(n+2)} \leq c\varepsilon$. Then

$$\|Sx - S_j x_\varepsilon\| \leq \|S\| \, \|x - x_\varepsilon\| + \|S - S_j\| \, \|x_\varepsilon\| \lesssim \varepsilon \|x\|.$$

However the number of entries in $y_\varepsilon = S_j x_\varepsilon$ can be as high as

$$\sharp(S_j x_\varepsilon) \leq 2^j \cdot \sharp x_\varepsilon \lesssim \varepsilon^{-1/n} \cdot \varepsilon^{-1/s}|x|_{\mathcal{A}^s},$$

as the number of non-zero entries in each row is at most 2^j. Therefore the support of the result $S_j x_\varepsilon$ is too large and hence also the number of operations scale more than linearly.

A more elaborate version however achieves linear complexity.

Theorem 8. *Let A be a s^* compressible matrix and $x \in \mathcal{A}^s$ with $s < s^*$. Then for any tolerance $\varepsilon > 0$ there exits an algorithm which computes w_ε such that*

$$\|Ax - w_\varepsilon\| \leq \varepsilon, \quad \sharp w_\varepsilon \lesssim \varepsilon^{-1/s}\|x\|_{\mathcal{A}^s}$$

where the number of operations stays bounded by a multiple of $\varepsilon^{-1/s}\|x\|_{\mathcal{A}^s}$.

The algorithm given in [2] uses the following strategy. The matrix vector multiplication Ax is done by multiplying each component of x with the corresponding column and then summing up. Now depending on the magnitude of a component x_i the corresponding column of an approximation A_k with $k = k(x_i)$ is chosen. The bigger $|x_i|$ is the better the approximation has to be. In this way one can constructively prove the preceding theorem.

5 WAVELET REALIZATION

In the last section we presented a toy problem which was discretized using the Fourier basis. However it is clear that the scope of this method is limited. In contrast wavelet discretization allow one to treat a variety of different problems. For an introduction to such concepts see for example [3].

In the first part of this section we state the key assumptions necessary for using PPINVIT in the case of a wavelet discretization. In the second part we present numerical results for the Poisson eigenvalue problem (1) on a L-shaped domain.

5.1 Prerequisites

Let the set $\Psi = \{\psi_\lambda \mid \lambda \in \nabla\}$ of wavelets ψ_λ for some index set ∇ be a Riesz-basis for V. What we mean by Riesz-basis will be explained in a moment below. In the case of $V = H^s(\Omega)$ for some domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary concrete realizations of such wavelet bases are known.

The first key ingredient for the realization of PPINVIT in the last section was the norm equivalence (7) which was actually an equality. Here we require that the collection Ψ is a

Riesz-basis, i.e. the mapping $\{v_\lambda\}_{\lambda\in\nabla} \to \sum_{\lambda\in\nabla} v_\lambda\psi_\lambda$ is injective and there exit constants c, C such that

$$c|u|^2 \leq \sum_{\lambda\in\nabla} |u_\lambda|^2 \leq C|u|^2, \quad u = \sum_{\lambda\in\nabla} u_\lambda\psi_\lambda.$$

Note that $|\cdot|$ denotes the norm on V. This means that the importance of a contribution $u_\lambda\psi_\lambda$ of some basis function ψ_λ in the whole expansion is reflected by the magnitude of the coefficient u_λ up to a constant C/c, the condition number of the basis. In the case of Sobolev spaces $V = H^s(\Omega)$ for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ such a construction is always possible.

Now an equivalent formulation for the eigenvalue problem (3) is given by

$$Sx = \lambda Mx$$

in $\ell_2(\nabla)$. Here the infinite dimensional matrix representations of the operators A and E are given by

$$S = [\langle A\psi_\lambda, \psi_{\lambda'}\rangle]_{\lambda,\lambda'\in\nabla}, \quad M = [\langle E\psi_\lambda, \psi_{\lambda'}\rangle]_{\lambda,\lambda'\in\nabla},$$

respectively. These matrix representations are usually not sparse in the strict sense. For a given row there exists in general an infinite number of entries. However, as in the case of Section 4 for a wide variety of operators these matrices are quasi-sparse in the sense of Definition 5. This is due to the fact that the wavelets have vanishing moments up to some appropriate order m, which means that for any polynomial p of order at most m one has

$$\int_\Omega \psi_\lambda(x)\, p(x)\, \mathrm{d}x = 0.$$

Therefore for a wide range of eigenvalue problems wavelet provide a framework for adaptive solvers that are able to compute an approximate solution up to any tolerance in (asymptotically) optimal complexity.

5.2 Numerical example

Finally we wish to present some numerical experiments for the Poisson eigenvalue problem (1) an the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1]^2$. From classical regularity theory [8], the Sobolev regularity is restricted by the biggest interior angle of a polygonal domain. In particular, for the L-shaped domain the lowest eigenfunction can only be shown to be in H^s for $s < 5/3$. This means that even for piecewise linear hat functions the convergence rate of for a uniform refinement will be at best $N^{-1/3}$ in the H^1 norm, where N is the number of degrees of freedom.

In contrast, employing piecewise linear wavelet bases with two vanishing moments in the adaptive algorithm PPINVIT will lead to an optimal convergence rate $N^{-1/2}$ for the eigenfunction in the H^1-norm [10, 4]. Therefore, adaptive solution concepts for the Poisson eigenvalue problem will outperform uniform grid refinement.

The implementation of our eigenvalue solver is based on the adaptive wavelet code described in [11]. The wavelet basis is constructed along the lines of [5]. In Figure 1 the error in the Rayleigh quotient for the smallest eigenvalue is shown. As a reference value we used $\lambda_1 =$

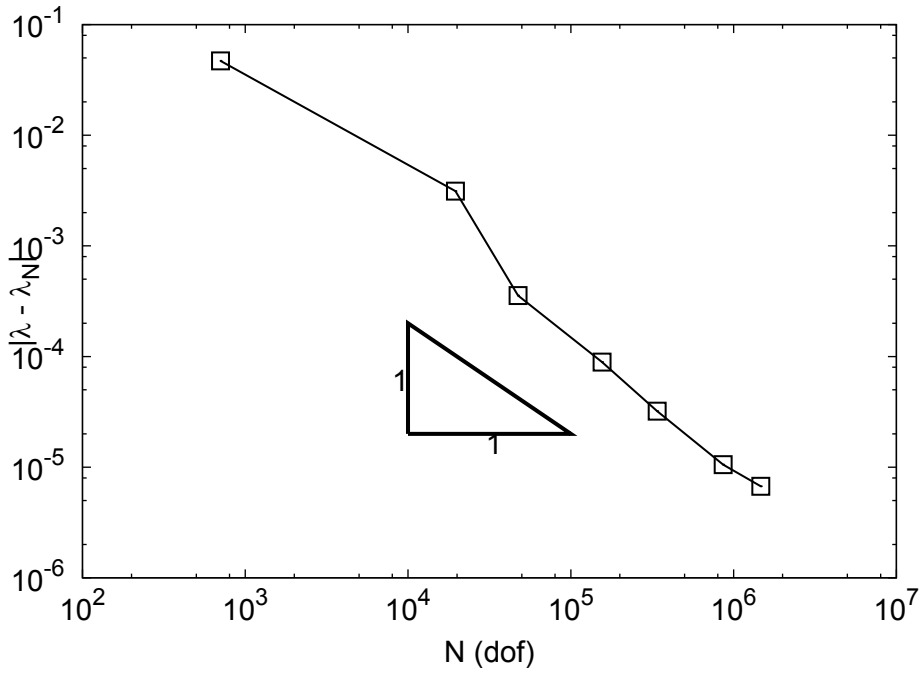

Figure 1: Convergence of the Rayleigh quotient for the adaptive algorithm with respect to the degrees of freedom.

9.639723844, see [1]. It can be seen that the error decreases like N^{-1}, which is as expected twice as high as the rate for the corresponding eigenfunction.

Also of interest is the structure of the chosen wavelets which is shown in Figure 2. The plot shows the center of the active ansatz functions during the sixth step for two different zoom levels. There one can see the self similarity in the two scales.

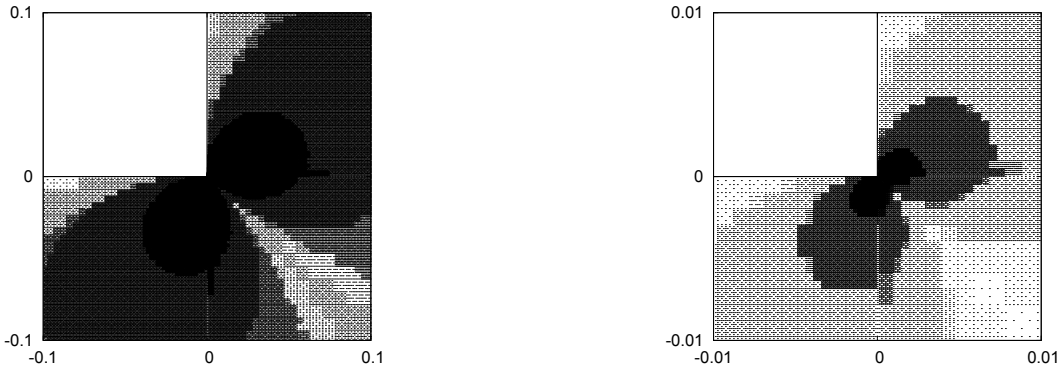

Figure 2: Plot of the active wavelets in the sixth step for two different zoom levels. Each dot corresponds to a center of support of an active wavelet.

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