18th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 07–09 July 2009

SIMULATION OF SIMPLY CROSS CORRELATED RANDOM FIELDS BY SERIES EXPANSION METHODS

M. Vořechovský[*]

[*]*Institute of Structural Mechanics, Faculty of Civil Engineering,
Brno University of Technology; Veveří 95, Brno, 602 00, Czech Republic*
E-mail: vorechovsky.m@fce.vutbr.cz

Keywords: Multivariate random field, Karhunen-Loève expansion, Orthogonal transformation of a covariance matrix, Monte Carlo simulation, SFEM

Abstract. *A practical framework for generating cross correlated fields with a specified marginal distribution function, an autocorrelation function and cross correlation coefficients is presented in the paper. The approach relies on well known series expansion methods for simulation of a Gaussian random field. The proposed method requires all cross correlated fields over the domain to share an identical autocorrelation function and the cross correlation structure between each pair of simulated fields to be simply defined by a cross correlation coefficient. Such relations result in specific properties of eigenvectors of covariance matrices of discretized field over the domain. These properties are used to decompose the eigenproblem which must normally be solved in computing the series expansion into two smaller eigenproblems. Such a decomposition represents a significant reduction of computational effort.*
Non-Gaussian components of a multivariate random field are proposed to be simulated via memoryless transformation of underlying Gaussian random fields for which the Nataf model is employed to modify the correlation structure. In this method, the autocorrelation structure of each field is fulfilled exactly while the cross correlation is only approximated. The associated errors can be computed before performing simulations and it is shown that the errors happen especially in the cross correlation between distant points and that they are negligibly small in practical situations.

1 INTRODUCTION

The random nature of many features of physical events is widely recognized both in industry and by researchers. The randomness of a gust wind, random structural features of materials, random fluctuations in temperature, humidity, and other environmental factors, all make the characterization provided by deterministic models of mechanics less satisfactory with respect to predictive capabilities. However, the entire problem of uncertainty and reliability can be addressed in a mathematically precise way and the random characteristics of nature can be addressed by computational models. For example, spatially fluctuating values of material parameters can be conveniently represented by means of random fields (e.g. strength, modulus of elasticity, fracture energy, etc). Except for the narrow class of problems that can be solved analytically, the solution to the variety of complex engineering problems involving uncertainty regarding mechanical properties and/or the excitations they are subjected to must be found by means of simulation. The only currently available universal method for accurate solution of such stochastic mechanics problems is the Monte Carlo technique.

One of the most important stages of the Monte Carlo type simulation technique as applied to systems with random properties (or systems subjected to random excitations) is the generation of sample realizations of these random properties. The generated sample functions must accurately describe the probabilistic characteristics of the corresponding stochastic processes or fields. Moreover, since the analyzed problems are usually computationally intensive (e.g. large scale nonlinear finite element computations), an analyst must select a simulation technique giving stable solutions with a small number of samples.

Simulation of non-Gaussian processes is mostly based on memoryless transforms of the standard Gaussian processes. These processes are known as translation processes [2]. The central problem is to determine the corresponding Gaussian covariance matrix (or equivalently, the Gaussian power spectral density function) that yields the target non-Gaussian covariance matrix after the memoryless transformation. Yamazaki and Shinozuka [11] proposed an iterative algorithm for generating samples of non-Gaussian random fields with prescribed spectral density and prescribed marginal distribution function based on iterative updating of the power spectral density. Recently, Sakamoto and Ghanem [8] and Puig et al. [7] utilized Hermite polynomial chaos method. In their method, the non-Gaussian processes are simulated by expanding the non-Gaussian distribution using Hermite polynomials with the standard Gaussian variable as argument. The correlation structure is decomposed according to the KLE of the underlying Gaussian process. The accuracy of this representation was studied by Field and Grigoriu [1] who pointed out some limitations of the approach. Grigoriu [3] criticize the algorithm for its computational intensity and questionable accuracy. Phoon et al. [6] have recently proposed simulation of non-Gaussian processes via Karhunen-Loéve expansion with uncorrelated non-Gaussian coefficients of zero mean and unit variance. The key feature of their technique is that the distribution of the random coefficients (random variables) is updated iteratively.

In the present paper (which is a shortened version of [9]) the well known orthogonal transformation of covariance matrix is chosen for representation of a Gaussian random field, and based on this method a simple extension to the simulation of the target type of multivariate stochastic fields is shown. After a brief review of the method in the context of univariate random fields (Sec. 2) we proceed to cross correlated Gaussian vector random fields (Sec. 3) and the proposed method. Sec. 4 shows how the presented approach can be extended for simulation of non-Gaussian vector random fields via transformations of an underlying Gaussian random field. Numerical examples and examples of applications of the method have been dropped due

to space limitations, they can be found in [9].

2   SERIES EXPANSION METHODS FOR THE SIMULATION OF A RANDOM FIELD

Suppose that the spatial variability of a random parameter is described by the Gaussian random field $H(\boldsymbol{x})$, where $\boldsymbol{x} \in \Omega$ is a continuous parameter (vector coordinate), and Ω is an open set of $\mathbb{R}^{\dim}$ describing the system geometry. The autocorrelation function $C_{HH}(\boldsymbol{x}, \boldsymbol{y})$ describes the autocorrelation structure of a random field, i.e., the the spatial variability. It is a function of some norm of two points $\boldsymbol{x}, \boldsymbol{y} \in \Omega : \|\boldsymbol{x}, \boldsymbol{y}\| = \{\|x_1, y_1\|, \ldots, \|x_{\dim}, y_{\dim}\|\}$. If the covariance function depends on distance alone, the function is said to be isotropic.

We will use the orthogonal transformation of the covariance matrix (sometimes called also the proper orthogonal transformation). The method is well-known in the simulation of univariate random fields and will provide a good basis for illustration of the proposed methodology for the simulation of multivariate random fields. The important point is that the target random functions can be suitably simulated by series expansion methods expansion using a finite number deterministic functions and random variables – coefficients. By means of these random variables $\xi_j(\theta)$, the approximated random field can be expressed as a finite summation (series expansion):

$$\mathbf{H_u}(\theta) = \sum_{j=1}^{N_{\mathrm{var}}} \xi_j(\theta) \sqrt{\lambda_j^{\mathbf{u}}} \left[\mathbf{\Phi}_j^{\mathbf{u}} \right]^T \tag{1}$$

where $\lambda_j^{\mathbf{u}}$ and $\mathbf{\Phi}_j^{\mathbf{u}}$ are the solutions of the eigenvalue problem: $\Sigma_{\mathbf{uu}} \ \mathbf{\Phi}_j^{\mathbf{u}} = \lambda_j^{\mathbf{u}} \ \mathbf{\Phi}_j^{\mathbf{u}}$, $\Sigma_{\mathbf{uu}}$ is the covariance matrix of the N (nodal) field values $\mathbf{u}$. $N_{\mathrm{var}} \leq N$ represents the truncation in the above discrete spectral representation of the field (random vector). The method is strongly related to the Karhunen-Loève expansion (KLE) method and can be extended to deliver continuous representation of a field by Kriging (the method is then known as the Expansion optimal linear estimation – EOLE).

3   CROSS CORRELATED GAUSSIAN RANDOM FIELDS

It is usual that more than one random property governs the evolution of a system. Consider for instance Young's modulus, Poisson's ratio or strength in mechanical problems, etc. In a probabilistic concept, all these quantities can be modeled by random fields.

The present paper deals with cases when all fields simulated over a region Ω share an identical autocorrelation function over Ω, and the cross correlation structure between each pair of simulated fields is simply defined by a cross correlation coefficient. Such an assumption enables one to perform the modal transformation in two "small" steps, not in one "big" step, as proposed in [12]. The advantage is a significant reduction in the dimension of the eigenvalue problem considering the fact that the modal decomposition of the given autocorrelation function (KLE) or matrix (EOLE) is done only once. An illustration of the algorithms of both methods and their comparison with a detailed description follow.

The key idea of the proposed method is that all cross correlated fields (components) are expanded using a certain of eigenfunctions/vectors, but the sets of random variables used for the expansion of each field are cross correlated. In other words, each field is expanded using a set of independent random variables, but these sets must be correlated with respect to the cross correlation matrix among all expanded random fields.

3.1 The proposed method for simulation of Gaussian cross-correlated random fields

In this section, we present some definitions needed for the problem formulation, notations and basic facts used throughout the paper. The most important properties of defined items are stated.

Cross-correlation matrix of random fields $\mathbf{C}$ is a square symmetric positive definite matrix of order N_F with elements $C^{i,j} \in (-1; 1)$ for $i \neq j$ and $C^{i,j} = 1$ for $i = j$. Matrix $\mathbf{C}$ is a cross-correlation matrix and defines the correlation structure among N_F random fields.

The cross correlation matrix $\mathbf{C}$ has N_F real, positive eigenvalues $\lambda_j^{\mathbf{C}}$, $j = 1 \ldots, N_F$ associated with N_F orthonormal eigenvectors $\mathbf{\Phi}_j^{\mathbf{C}}$, $j = 1, \ldots, N_F$. After ordering them so that $\lambda_1^{\mathbf{C}} \geq \lambda_2^{\mathbf{C}} \geq \ldots \geq \lambda_{N_F}^{\mathbf{C}}$ the eigenvector matrix reads:

$$
\begin{array}{ccccc}
\mathbf{\Phi}_1^{\mathbf{C}} & \mathbf{\Phi}_2^{\mathbf{C}} & \ldots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{C}} & \ldots & \mathbf{\Phi}_{N_F}^{\mathbf{C}}
\end{array}
$$
$$
\begin{pmatrix}
\phi_{1,1}^{\mathbf{C}} & \phi_{1,2}^{\mathbf{C}} & \ldots & \phi_{1,N_{F,r}}^{\mathbf{C}} & \ldots & \phi_{1,N_F}^{\mathbf{C}} \\
\phi_{2,1}^{\mathbf{C}} & \phi_{2,2}^{\mathbf{C}} & \ldots & \phi_{2,N_{F,r}}^{\mathbf{C}} & \ldots & \phi_{2,N_F}^{\mathbf{C}} \\
\vdots & \vdots & \ldots & \ldots & \ldots & \vdots \\
\phi_{N_F,1}^{\mathbf{C}} & \phi_{N_F,2}^{\mathbf{C}} & \ldots & \phi_{N_F,N_{F,r}}^{\mathbf{C}} & \ldots & \phi_{N_F,N_F}^{\mathbf{C}}
\end{pmatrix}
\tag{2}
$$

and the associated eigenvalues

$$
\mathbf{\Lambda}^{\mathbf{C}} = diag \begin{pmatrix} \overset{\mathbf{\Phi}_1^{\mathbf{C}}}{\lambda_1^{\mathbf{C}}} & \overset{\mathbf{\Phi}_2^{\mathbf{C}}}{\lambda_2^{\mathbf{C}}} & \ldots & \overset{\mathbf{\Phi}_{N_{F,r}}^{\mathbf{C}}}{\lambda_{N_{F,r}}^{\mathbf{C}}} & \ldots & \overset{\mathbf{\Phi}_{N_F}^{\mathbf{C}}}{\lambda_{N_F}^{\mathbf{C}}} \end{pmatrix}
$$

Each j-th eigenvector $\mathbf{\Phi}_j{}^{\mathbf{C}}$ is normalized to have an Euclidean length of 1, therefore $\left[\mathbf{\Phi}^{\mathbf{C}}\right]^T \mathbf{\Phi}^{\mathbf{C}} = \mathbf{I}$, in which $\mathbf{I}$ is an identity matrix. The spectral decomposition of correlation matrix $\mathbf{C}$ reads: $\mathbf{C}\,\mathbf{\Phi}^{\mathbf{C}} = \mathbf{\Phi}^{\mathbf{C}}\mathbf{\Lambda}^{\mathbf{C}}$. Let us denote $\mathbf{\Phi}^{\mathbf{C}} = \begin{pmatrix} \mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}} & \mathbf{\Phi}_{\mathrm{II}}^{\mathbf{C}} \end{pmatrix}$ and $\mathbf{\Lambda}^{\mathbf{C}} = \begin{pmatrix} \mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}} & \mathbf{\Lambda}_{\mathrm{II}}^{\mathbf{C}} \end{pmatrix}$, where $\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}} = \begin{pmatrix} \mathbf{\Phi}_1^{\mathbf{C}} \mathbf{\Phi}_2^{\mathbf{C}} \ldots \mathbf{\Phi}_{N_{F,r}}^{\mathbf{C}} \end{pmatrix}$ is the $(N_F \times N_{F,r})$ matrix and $\mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}} = diag\begin{pmatrix} \lambda_1^{\mathbf{C}}, \ldots, \lambda_{N_{F,r}}^{\mathbf{C}} \end{pmatrix}$ is the $(N_{F,r} \times N_{F,r})$ diagonal matrix. Partitioning of the matrices can be used later in the reduction of computational effort for the simulation of random fields. It can be shown that a large amount of computer memory can be saved at a given level of accuracy if one uses $\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}}$ instead of full $\mathbf{\Phi}^{\mathbf{C}}$ (with associated $\mathbf{\Lambda}'$s). The idea is that the largest eigenvalues and their corresponding eigenvectors dominate the foregoing transformation, so the second part of the eigenvalues/vectors can be neglected and the *approximate spectral representation of matrix* $\hat{\mathbf{C}}$ can be obtained:

$$
\hat{\mathbf{C}} = \mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}} \, \mathbf{\Lambda}_{\mathrm{I}}^{\mathbf{C}} \left[\mathbf{\Phi}_{\mathrm{I}}^{\mathbf{C}}\right]^T
\tag{3}
$$

It can be shown that for the simulation of cross correlated stochastic fields by the methods described above one needs to simulate a vector of cross correlated random variables for the expansion. These random variables have the **block cross correlation matrix $\mathbf{D}$ of random variables.** Let $\mathbf{D}$ be a squared symmetric matrix of order $(N_F\,N_{\mathrm{var}})$ assembled in this way: matrix $\mathbf{D}$ consists of $(N_F \times N_F)$ blocks (squared matrices) $C^{i,j}\,\mathbf{I}$, where $\mathbf{I}$ is the unit matrix of order N_{var}, and $C^{i,j}$ are elements of the cross-correlation matrix $\mathbf{C}$ defined previously.

$$
\mathbf{D} = \begin{array}{c} \\ H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_{N_F} \end{array}
\begin{array}{c}
\begin{array}{ccccc} H_1 & H_2 & H_3 & \ldots & H_{N_F} \end{array} \\
\begin{pmatrix}
\mathbf{I} & C^{1,2}\,\mathbf{I} & C^{1,3}\,\mathbf{I} & \ldots & C^{1,N_F}\,\mathbf{I} \\
\vdots & \mathbf{I} & C^{2,3}\,\mathbf{I} & \ldots & C^{2,N_F}\,\mathbf{I} \\
\vdots & \vdots & \mathbf{I} & \ldots & C^{3,N_F}\,\mathbf{I} \\
\vdots & sym. & \vdots & \ddots & \vdots \\
\ldots & \ldots & \ldots & \ldots & \mathbf{I}
\end{pmatrix}
\end{array}
$$

D is a correlation matrix having nonzero elements on sub-diagonals of partial square blocks. The fact that each square block matrix on the diagonal of **D** is the $(N_{\text{var}} \times N_{\text{var}})$ unit matrix can be simply interpreted: random variables needed for the expansion of one random field H_i, $i = 1 \ldots N_F$ are uncorrelated (and also independent since we will work with Gaussian random variables). The off-diagonal square blocks (diagonal matrices) represent cross correlation between each two sets of random variables used for expansion of the fields H_i and H_j, $i \neq j$; $i,j = 1, \ldots, N_F$. The key property for the proposed method is the spectral property of the correlation matrix **D**. Cross correlation matrix **D** has $N_F \, N_{\text{var}}$ real, positive eigenvalues $\lambda_j^{\mathbf{D}}$, $j = 1, \ldots, (N_F \, N_{\text{var}})$ associated with orthogonal eigenvectors. Obviously matrix **D** has the same eigenvalues as matrix **C**, but these are N_{var}-multiple. Similarly the eigenvectors of **D** are associated with the eigenvectors of **C**. The space described by $\mathbf{\Phi}^{\mathbf{C}}$ is enriched so that the dimension is N_{var}-times higher, but the components of the orthogonal eigenvectors $\mathbf{\Phi}^{\mathbf{C}}$ remain.

After ordering the eigenvalues so that $\lambda_1^{\mathbf{D}} \geq \lambda_2^{\mathbf{D}} \geq \cdots \geq \lambda_{N_F \times N_{\text{var}}}^{\mathbf{D}}$, one can assemble the eigenvectors/eigenvalue matrices using a block-matrix with squared block submatrices: $\mathbf{\Phi}^{\mathbf{D}} =$

$$
\begin{array}{ccccc}
\mathbf{\Phi}_1^{\mathbf{D}} & \ldots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{D}} & \ldots & \mathbf{\Phi}_{N_F}^{\mathbf{D}}
\end{array}
$$
$$
\begin{pmatrix}
\phi_{1,1}^{\mathbf{C}}\mathbf{I} & \ldots & \phi_{1,N_{F,r}}^{\mathbf{C}}\mathbf{I} & \ldots & \phi_{1,N_F}^{\mathbf{C}}\mathbf{I} \\
\phi_{2,1}^{\mathbf{C}}\mathbf{I} & \ldots & \phi_{2,N_{F,r}}^{\mathbf{C}}\mathbf{I} & \ldots & \phi_{2,N_F}^{\mathbf{C}}\mathbf{I} \\
\vdots & \ldots & \vdots & \ldots & \vdots \\
\phi_{N_F,1}^{\mathbf{C}}\mathbf{I} & \ldots & \phi_{N_F,NF,r}^{\mathbf{C}}\mathbf{I} & \ldots & \phi_{N_F,N_F}^{\mathbf{C}}\mathbf{I}
\end{pmatrix}
\tag{4}
$$

and the eigenvalue matrices corresponding to vector blocks $\left(\mathbf{\Phi}_1^{\mathbf{D}}, \ldots, \mathbf{\Phi}_{N_F}^{\mathbf{D}}\right)$: $\mathbf{\Lambda}^{\mathbf{D}} =$

$$
\begin{array}{ccccc}
\mathbf{\Phi}_1^{\mathbf{D}} & \ldots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{D}} & \ldots & \mathbf{\Phi}_{N_F}^{\mathbf{D}}
\end{array}
$$
$$
= diag \begin{pmatrix} \lambda_1^{\mathbf{C}}\mathbf{I} & \ldots & \lambda_{N_{F,r}}^{\mathbf{C}}\mathbf{I} & \ldots & \lambda_{N_F}^{\mathbf{C}}\mathbf{I} \end{pmatrix}
\tag{5}
$$

where $\mathbf{I}$ is the unit matrix of order N_{var}. Matrices **C** and **D** are positive definite. Similarly to Eq. (3) the second part of the eigenvalues/vectors can be neglected and the *approximate spectral representation of (cross) correlation matrix* $\hat{\mathbf{D}}$ can be obtained as:

$$
\hat{\mathbf{D}} = \mathbf{\Phi}_{\mathbf{I}}^{\mathbf{D}} \, \mathbf{\Lambda}_{\mathbf{I}}^{\mathbf{D}} \, \left[\mathbf{\Phi}_{\mathbf{I}}^{\mathbf{D}}\right]^T
\tag{6}
$$

where the matrix $\mathbf{\Phi}_{\mathbf{I}}^{\mathbf{D}}$, again, contains only the respective eigenvectors to the $N_{F,r}$ eigenvalues.

It might be important to know how the correlation matrix of all N_F fields, each discretized into the same set of N points $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$, looks like. We call it **full-block correlation matrix F**. Let **F** be a squared symmetric matrix of order $N_F \, N$ assembled as follows. Matrix **F** consists of $N_F \times N_F$ blocks (squared matrices) $\mathbf{F}_{\mathbf{uu}}^{i,j}$ which are correlation matrices of order N. Each submatrix $\mathbf{F}_{\mathbf{uu}}^{i,j} = \mathbf{F}_{\mathbf{uu}}^{j,i} =$

$$
\begin{array}{cccccc}
& \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_l & \ldots & \boldsymbol{x}_N
\end{array}
$$
$$
\begin{array}{c}
\boldsymbol{x}_1 \\
\vdots \\
\boldsymbol{x}_k \\
\vdots \\
\boldsymbol{x}_N
\end{array}
\begin{pmatrix}
F_{1,1}^{i,j} & \ldots & F_{1,l}^{i,j} & \ldots & F_{1,N}^{i,j} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
F_{k,1}^{i,j} & \ldots & F_{k,l}^{i,j} & \ldots & F_{k,N}^{i,j} \\
\vdots & \ldots & \vdots & \ddots & \vdots \\
F_{N,1}^{i,j} & \ldots & F_{N,l}^{i,j} & \ldots & F_{N,N}^{i,j}
\end{pmatrix}
\tag{7}
$$

is symmetric and the general entry $F_{k,l}^{i,j} = F_{l,k}^{i,j} = \text{Corr}\left[H_i\left(x_k\right), H_l\left(x_l\right)\right]$ has the meaning of correlation between two field's (i,j) nodal values at points x_k, x_l $(k,l = 1,\ldots,N)$. Matrix $\mathbf{F}$ can be obtained using the autocorrelation matrix $\mathbf{F_{uu}} = \mathbf{F}_{uu}^{i,i}$ and using the cross-correlation matrix $\mathbf{C}$ among random fields (vectors) $\mathbf{H}_1,\ldots,\mathbf{H}_{N_F}$ simply by multiplying the autocorrelation by the cross correlation: $F_{k,l}^{i,j} = C^{i,j}F_{k,l}$. Matrix $\mathbf{F}$ can be written using the squared $(N \times N)$ blocks $\mathbf{F}_{uu}^{i,j} = C^{i,j}\mathbf{F_{uu}}$ as: $\mathbf{F} =$

$$
\begin{array}{c}
\begin{array}{cccc}
\mathbf{H}_1 & \mathbf{H}_2 & \ldots & \mathbf{H}_{N_F}
\end{array}\\
\begin{array}{c}
\mathbf{H}_1\\
\mathbf{H}_2\\
\vdots\\
\mathbf{H}_{N_F}
\end{array}
\left(
\begin{array}{cccc}
\mathbf{F_{uu}} & C^{1,2}\,\mathbf{F_{uu}} & \ldots & C^{1,N_F}\,\mathbf{F_{uu}}\\
\vdots & \mathbf{F_{uu}} & \ldots & C^{2,N_F}\,\mathbf{F_{uu}}\\
\vdots & sym. & \ddots & \vdots\\
\ldots & \ldots & \ldots & \mathbf{F_{uu}}
\end{array}
\right)
\end{array}
\tag{8}
$$

This illustrates the simple cross correlation relationships between the vector fields $\mathbf{H}_i, \mathbf{H}_j$ (single correlation coefficients $C^{i,j}$). Matrix $\mathbf{F}$ is the target cross-correlation matrix of discretized random fields (random vectors) $\mathbf{H}_1,\ldots,\mathbf{H}_{N_F}$, each discretized into the same set of points $x_i, \; (i = 1,\ldots,N)$.

It is not difficult to show that if the correlation matrix $\mathbf{F}$ consists of blocks (autocorrelation matrices $\mathbf{F_{uu}}$, each multiplied by a cross correlation coefficient $C^{i,j}$), the eigenvector matrix denoted $\mathbf{\Phi^u}$ can be assembled as a block-matrix with block submatrices $\mathbf{\Phi}_1^{\mathbf{F}},\ldots,\mathbf{\Phi}_{N_F}^{\mathbf{F}}$: $\mathbf{\Phi^F} =$

$$
\begin{array}{c}
\begin{array}{ccccc}
\mathbf{\Phi}_1^{\mathbf{F}} & \ldots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{F}} & \ldots & \mathbf{\Phi}_{N_F}^{\mathbf{F}}
\end{array}\\
\left(
\begin{array}{ccccc}
\phi_{1,1}^{\mathbf{C}}\mathbf{\Phi^u} & \ldots & \phi_{1,N_{F,r}}^{\mathbf{C}}\mathbf{\Phi^u} & \ldots & \phi_{1,N_F}^{\mathbf{C}}\mathbf{\Phi^u}\\
\phi_{2,1}^{\mathbf{C}}\mathbf{\Phi^u} & \ldots & \phi_{2,N_{F,r}}^{\mathbf{C}}\mathbf{\Phi^u} & \ldots & \phi_{2,N_F}^{\mathbf{C}}\mathbf{\Phi^u}\\
\vdots & \ldots & \vdots & \ldots & \vdots\\
\phi_{N_F,1}^{\mathbf{C}}\mathbf{\Phi^u} & \ldots & \phi_{N_F,NF,r}^{\mathbf{C}}\mathbf{\Phi^u} & \ldots & \phi_{N_F,N_F}^{\mathbf{C}}\mathbf{\Phi^u}
\end{array}
\right)
\end{array}
\tag{9}
$$

and $\mathbf{\Lambda^F} =$

$$
\begin{array}{c}
\begin{array}{ccccc}
\mathbf{\Phi}_1^{\mathbf{F}} & \ldots & \mathbf{\Phi}_{N_{F,r}}^{\mathbf{F}} & \ldots & \mathbf{\Phi}_{N_F}^{\mathbf{F}}
\end{array}\\
diag\left(
\begin{array}{ccccc}
\lambda_1^{\mathbf{C}}\mathbf{\Lambda^u} & \ldots & \lambda_{N_{F,r}}^{\mathbf{C}}\mathbf{\Lambda^u} & \ldots & \lambda_{N_F}^{\mathbf{C}}\mathbf{\Lambda^u}
\end{array}
\right)
\end{array}
\tag{10}
$$

where $\mathbf{\Lambda^u}$ is the (reduced) eigenvalue matrix of $\mathbf{F}$ of order N_{var} and $\lambda_i^{\mathbf{C}}$ $(i = 1,\ldots,N_F)$ are the eigenvalues of cross correlation matrix $\mathbf{C}$. Note that the eigenvalues $\lambda^{\mathbf{F}}$ are not sorted automatically even if the eigenvalues of both $\mathbf{\Lambda^u}$ and $\mathbf{\Lambda^C}$ are sorted. The partitioning of $\mathbf{\Phi^F}$ and $\mathbf{\Lambda^F}$ in the case that only the reduced number of eigenmodes $N_{F,r}$ of matrix $\mathbf{C}$ are available is obvious.

Block sample matrix $\mathbf{\chi^D}$. Let $\mathbf{\chi^D}$ be a $(N_{\text{var}}\,N_F)$-dimensional jointly normally distributed random vector with correlation matrix $\mathbf{D}$. The vector consists of N_F blocks. Each block (subvector) $\mathbf{\chi}_j^{\mathbf{D}}$, $j = 1,\ldots,N_F$ represents a Gaussian random vector with N_{var} standard Gaussian independent (and therefore also non-correlated) random variables (marginals) while the vectors $\mathbf{\chi}_i^{\mathbf{D}}$, $\mathbf{\chi}_j^{\mathbf{D}}$ are cross correlated.

For a given number of realizations N_{sim} the vector $\mathbf{\chi^D}$ is represented by an $(N_{\text{var}}\,N_F) \times N_{\text{sim}}$ random matrix. Each of the N_{sim} columns is one realization of a Gaussian random vector. The random vector $\mathbf{\chi^D}$ is partitioned into N_F vectors each with the dimension N_{var}:

$$
\mathbf{\chi^D} = \left[\; \left[\mathbf{\chi}_1^{\mathbf{D}}\right]^T \quad \left[\mathbf{\chi}_2^{\mathbf{D}}\right]^T \quad \left[\mathbf{\chi}_3^{\mathbf{D}}\right]^T \; \cdots \; \left[\mathbf{\chi}_{N_F}^{\mathbf{D}}\right]^T \; \right]^T
$$

Simulation of the matrix $\boldsymbol{\chi}^{\mathbf{D}}$ is the most important step in the method. The matrix $\mathbf{D}$ is targeted in simulation of $\boldsymbol{\chi}^{\mathbf{D}}$ as the correlation matrix. The key idea of the method is the utilization of spectral decomposition of correlation matrix $\mathbf{D}$ as this decomposition is very easy to perform (Eq. 4). Therefore, the orthogonal transformation of correlation matrix will be used. The utilization of the equivalence with prescribed correlation matrix $\mathbf{C}$ among fields has a significant computational impact: instead of solving the $N_{\mathrm{var}} \times N_F$ eigenvalue problem of $\mathbf{D}$, one needs to solve the N_F eigenvalue problem of prescribed correlation matrix $\mathbf{C}$. In cases when the number of random variables utilized in the expansion of one random field is large (thousands), the reduction is significant. By partitioning the matrix $\boldsymbol{\chi}^{\mathbf{D}}$ into N_{var}-dimensional blocks, one obtains an independent standard Gaussian random vector for the simulation of each of the N_F random fields.

Having Eq. (4) for the correlation matrix $\mathbf{D}$ at hand the simulation of the block sample matrix $\boldsymbol{\chi}^{\mathbf{D}}$ is straightforward (orthogonal transformation of the correlation matrix):

$$\boldsymbol{\chi}^{\mathbf{D}} = \boldsymbol{\Phi}^{\mathbf{D}} \left(\boldsymbol{\Lambda}^{\mathbf{D}}\right)^{1/2} \boldsymbol{\xi} \tag{11}$$

where $\boldsymbol{\xi} = \{\xi_i, \ i = 1, \ldots, N_F \times N_{\mathrm{var}}\}$ forms a vector of independent standard Gaussian random variables. Of course, the (sparse) matrices $\boldsymbol{\Phi}^{\mathbf{D}}$ and $\boldsymbol{\Lambda}^{\mathbf{D}}$ do not need to be assembled and stored in computer memory. They can be used in the form of an algorithm, and only the eigen-matrices $\boldsymbol{\Phi}^{\mathbf{C}}$ and $\boldsymbol{\Lambda}^{\mathbf{C}}$ must be solved (or at least their dominating parts $\boldsymbol{\Phi}_{\mathrm{I}}^{\mathbf{C}}$ and $\boldsymbol{\Lambda}_{\mathrm{I}}^{\mathbf{C}}$).

Yamazaki nad Shinozuka [12] proposed the universal simulation of discretized multivariate stochastic fields by one orthogonal transformation of (block) covariance matrix $\mathbf{F}$. The modal matrix of matrix $\mathbf{F}$ is then used for the transformation of random vector $\boldsymbol{\xi}$ composed of $N \times N_F$ independent Gaussian random variables. The main difference from the method proposed here is that they need to solve an eigenvalue problem of matrix $\mathbf{F}$ that has a large order $(N \times N_F)$ while in this paper the problem is decomposed into two separate modal solutions, namely (i) the autocovariance structure (order N in EOLE; a reduced number of N_{var} eigenmodes must be solved) and (ii) the cross-correlation matrix of order N_F ($N_{F,r}$ modes). A simple illustration with a comparison of the approaches is given in Fig. 1. The figure illustrates a) the expansion of a univariate random field using the random vector $\boldsymbol{\xi}$ and the eigenvalue matrix $\boldsymbol{\Lambda}$ with associated eigenfunctions [eigenvectors] in KLE [EOLE], b) the simulation procedure employing one "huge" orthogonal transformation of the correlation matrix $\mathbf{F}$ [12]:

$$\boldsymbol{H} = \boldsymbol{\Phi}^{\mathbf{F}} \left(\boldsymbol{\Lambda}^{\mathbf{F}}\right)^{1/2} \boldsymbol{\xi} \tag{12}$$

This procedure is general. In our case the correlation matrix $\mathbf{F}$ can be assembled using the products of the cross-correlation matrix $\mathbf{C}$ and autocorrelation matrix $\mathbf{F_{uu}}$. We have shown that the eigenvector and eigenvalue matrices of $\mathbf{C}$ and $\mathbf{F_{uu}}$ solved separately can be used to compute the required matrices $\boldsymbol{\Phi}^{\mathbf{F}}$ and $\boldsymbol{\Lambda}^{\mathbf{F}}$ (see Eqs. 9,10) and therefore computational effort can be saved. It will be shown later that such a technique yields identically good results as the proposed scheme depicted in the third part c) of the figure: decomposition into (i) the preparation of a vector of cross correlated random variables $\boldsymbol{\chi}^{\mathbf{D}}$ and (ii) the expansion of each random field H_i using a subset $\boldsymbol{\chi}_i^{\mathbf{D}}$ and always the same orthogonal base as in a). The advantage of the proposed procedure c) is that the simulation of each random field can be done separately using either a KLE or EOLE base while the cross correlated random variables $\boldsymbol{\chi}^{\mathbf{D}}$ are prepared in advance. Incorporation into an existing algorithm for simulation of univariate fields is therefore simple and transparent.
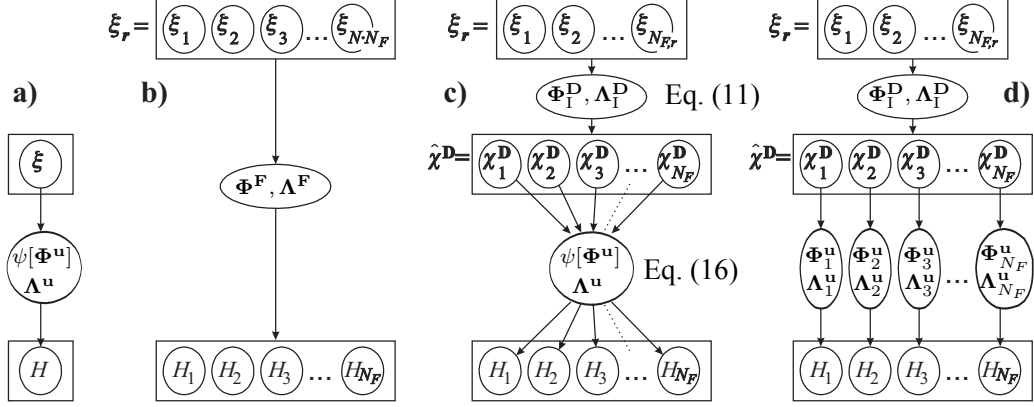
Figure 1: a) Simulation of a univariate random field using N_{var} eigenmodes; b) illustration of the method due to Yamazaki and Shinozuka (1990); c) proposed method for simulation of cross correlated fields in two steps when components share the same distribution; d) proposed method for components with different distributions, where eigenanalysis of each field is performed separately.

4 TRANSFORMATION TO NON-GAUSSIAN RANDOM FIELDS

In most applications, the Gaussian random field $\mathbf{H}$ is used to model uncertainties with spatial variability because of convenience and a lack of alternative models. However, the Gaussian model is not applicable in many situations. For example, it cannot be used to model Young's modulus or the strength of a material, which is always positive.

Let us denote the marginal cumulative [probability] distribution function (cdf) of each component $\widetilde{H}_i$ of the non-Gaussian vector random field $\widetilde{\mathbf{H}}$ by $G_i[g_i]$. In the discretized version, one can assemble the target correlation matrix $\widetilde{\mathbf{F}}$ of all random fields by computing the entries $\widetilde{F}_{k,l}^{i,j}$ as a product of the autocorrelation coefficient $\widetilde{F}_{\mathbf{uu}}^{i,j}$ (depending only on the positions of each pair of points) and the target cross correlation $\widetilde{C}^{i,j}$. It would be convenient to find an underlying Gaussian random field $\mathbf{H}$ (with some cross correlation matrix $\mathbf{C}$ studied earlier) that can be easily transformed into the target field $\widetilde{\mathbf{H}}$ while keeping the target cross correlation matrix between these components denoted by $\widetilde{\mathbf{C}}$. The univariate nonlinear transformation of the Gaussian variables, called the translation process by Grigoriu [2] is the mapping $h_i(\cdot)$:

$$\widetilde{H}_i(x_k) = h_i[H_i(x_k)] = G_i^{-1}\{\Phi[H_i(x_k)]\} \tag{13}$$

where $\Phi(\cdot)$ is the standard cumulative Gaussian probability function and $i = 1, \ldots, N_F$; $k = 1, \ldots, N$.

The Nataf [5] model has been proposed in [4] for transforming non-Gaussian multivariate distribution into standardized Gaussian distribution. We will show how the Nataf model can be used within the presented framework for effective simulation of cross correlated Gaussian random fields in order to model non-Gaussian fields with prescribed marginal distributions G_i, the autocorrelation function and cross correlated via $\mathbf{C}$. For application of the Nataf model, the correlation coefficient $\widetilde{\rho}_{i,j}$ of each pair (i,j) of non-Gaussian variables must be adjusted to form the correlation coefficient $\rho_{i,j}$ of a pair of Gaussian variables. The adjustment has been shown in [4] to be a unique solution of a certain two-fold integral (Eq. 12 in [4]) :

$$\tilde{\rho}_{i,j} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\widetilde{H}_i - \mu_i}{\sigma_i}\frac{\widetilde{H}_j - \mu_j}{\sigma_j}\varphi(H_i, H_j, \rho_{i,j})\ \mathrm{d}H_i\ \mathrm{d}H_j \tag{14}$$

8

where the values of the original variables $\widetilde{H}_i, \widetilde{H}_j$ (with means μ_i, μ_j and standard deviations σ_i, σ_j) are expressed in terms of the the standard Gaussian variables in the spirit of Eq. (13) via $\widetilde{H}_i = G_i^{-1} [\Phi(H_i)]$. Function $\varphi(H_i, H_j, \rho_{i,j})$ is the standard bivariate Gaussian density. Due to the uniqueness of the solution, the relationship in Eq. (14) can be expressed as a correction to the original correlation [4]:

$$\rho_{i,j} = \kappa \, \widetilde{\rho}_{i,j} \tag{15}$$

In general, the correction factor κ (satisfying $\kappa \geq 1$) is a function of both marginal distributions and their correlation: $\kappa = \kappa(G_i, G_j, \widetilde{\rho}_{i,j})$. For some pairs of distributions κ becomes just a constant or a function of only some of the three types of information. Other important properties are that (i) $\rho_{i,j} = 0$ for $\widetilde{\rho}_{i,j}$, (ii) $|\widetilde{\rho}_{i,j}| \leq |\rho_{i,j}|$ and that (iii) $\widetilde{\rho}_{i,j}$ is a strictly increasing function of $\rho_{i,j}$.

In application of the Nataf model, we seek the corresponding correlation matrix $\mathbf{F}$ of a Gaussian random vector field $\mathbf{H}$. The correct method is to solve the correction factor for each entry $F_{k,l}^{i,j} = \kappa_{kl}^{i,j} \widetilde{F}_{k,l}^{i,j}$ depending on G_i, G_j and $\widetilde{F}_{k,l}^{i,j}$. Unfortunately, the full block correlation matrix $\mathbf{F}$ does not follow the simple pattern from Eq. (8) any more. In particular, the diagonal blocks are not identical anymore, because the distributions G_i may differ in general, and the off-diagonal blocks are not a simple multiple of the diagonal block (for the same reasons). Even if the distributions G_i were identical, the corrections κ would prevent each off-diagonal block from being a simple multiple of the diagonal block, because in general $C^{i,j}$s are not zeros (and also generally are not all the same).

We remark also that not every combination of the autocorrelation structure with the non-Gaussian marginal distribution can be admissible for the mapping via underlying Gaussian random field. There are two possible incompatibilities. The first one arises when the autocorrelation functions of the non-Gaussian fields do not have a corresponding admissible correlations in the Gaussian space (this happens often in cases of high negative correlations combined with strongly non-Gaussian marginals). The second incompatibility arises when the auto-correlation function (or matrix) in the Gaussian space becomes non-positive definite and, therefore, not admissible. The second problem can sometimes be remedied by ignoring negative eigenvalues and corresponding eigenmodes.

From the preceding paragraphs, it becomes clear that the presented approach for simulation of Gaussian vector random fields can not generally be employed for simulation of vector random fields with arbitrary marginals. However, it is known that for the majority of commonly used continuous distributions the correction factors κ are only slightly greater than one [4]. Therefore, the difference between correlation matrices $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ is usually very small. The difference actually depends on the "non-Gaussianity" of the distributions G_i. The closer the component distributions G_i are to the elliptical distributions (Gaussian inclusive), the closer these two matrices are. In the paragraphs after the following summary of the method, we will try to find an approximation $\mathbf{F}'$ of the correct Nataf full correlation matrix $\mathbf{F}$ in order to be able to profit from the presented framework for Gaussian fields.

The proposed procedure for the simulation of random fields can be itemized as follows (see Fig. 1c,d):

1. Given the common autocorrelation function in original (non-Gaussian) space, a Nataf correction function $\kappa_i(\widetilde{\rho}) = \kappa(G_i, G_i, \widetilde{\rho})$ must be found for each field $i = 1, \ldots, N_F$ over the whole range of autocorrelation coefficients $\widetilde{\rho}$. The set of functions κ_i transform the original correlations into Gaussian space. Then, the spectral analysis of the auto-correlation structure for each underlying Gaussian field and the choice of the common

number of eigenmodes N_{var} is made, i.e., Based on the discretization (grid of points) the autocorrelation matrices $\mathbf{F}^i_{\mathbf{uu}}$ are assembled and the corresponding sets of N_{var} largest eigenvalues and the corresponding eigenvectors ($\mathbf{\Phi}^{\mathbf{u}}_i$ and $\mathbf{\Lambda}^{\mathbf{u}}_i$ matrices) are determined.

In most cases, the Nataf's correction will be only slightly greater than one over the whole range of possible autocorrelations and thus the eigenvalues and eigenfunctions [vectors] will be very similar for each Gaussian field. Therefore, one can solve one field only and use iterations to refine the eigenmodes for the other fields.

2. Find a corrected cross correlation matrix $\mathbf{C}$ given the target matrix $\widetilde{\mathbf{C}}$ and marginals $G_1, \ldots, G_{N_F}$ using the Nataf model. Each off-diagonal entry is obtained as $C^{i,j} = \kappa^{i,j} \ \widetilde{C}_{i,j}$, $(i, j = 1, \ldots, N_F)$. Then, eigenvalues must be computed with corresponding orthogonal eigenvectors $\left(\mathbf{\Phi}^{\mathbf{C}}_{\mathbf{I}} \ \mathbf{\Lambda}^{\mathbf{C}}_{\mathbf{I}}\right)$ of the target cross correlation matrix $\mathbf{C}$ among random fields. The choice of number of eigen-modes $N_{F,r} \leq N_F$ is made.

3. Simulation of Gaussian random vector $\boldsymbol{\xi}$ of $N_r = N_{F,r} \cdot N_{\text{var}}$ independent standard Gaussian variables ξ_j. For a given number of simulations N_{sim} a random vector becomes an $N_r \cdot N_{\text{sim}}$ random matrix, where N_{sim} is a sample size for each random variable. The LHS technique is recommended for the simulation of the random vector [10].

4. The simulation of cross correlated random vector $\boldsymbol{\chi}^{\mathbf{D}}$ by matrix multiplication (Eq. 11). Matrices from Eqs. (4 and 5) of the matrix $\mathbf{D}$ (an enlarged matrix from step 2) and a random matrix from step (3) are utilized.

5. Simulation of all underlying Gaussian fields $i = 1, \ldots, N_F$ one-by-one using the correct portion of $\boldsymbol{\chi}^{\mathbf{D}}$ and eigenmodes from step 1 (see Fig. 1d for illustration). Each random field i is expanded using the i-th block $\boldsymbol{\chi}^{\mathbf{D}}_i$ of random vector $\boldsymbol{\chi}^{\mathbf{D}}$ $\left(\hat{\boldsymbol{\chi}}^{\mathbf{D}}\right)$ and the N_{var} eigenmodes from step (1):

$$\boldsymbol{H}_i = \mathbf{\Phi}^{\mathbf{u}}_i \ (\mathbf{\Lambda}^{\mathbf{u}}_i)^{1/2} \ \chi^{\mathbf{D}}_i \tag{16}$$

6. The last step is the transformation of standardized underlying Gaussian random fields $i = 1, 2, \ldots, N_F$ into non-Gaussian via Eq. (13).

One randomly chosen realization of the three fields is plotted in Fig. 2a,b and c. In the same figure it can be seen how the cross correlation of fields influences the shape similarity of corresponding realizations. Fig. 2d illustrates the typical plot of the mean and variance profiles of the field over the target domain Ω. Such a plot serves as visual check for the accuracy of simulations of field.

The plot also illustrates that the multivariate 2D random field over a plate can be viewed as a 3D random fields discretized using three two-dimensional plates. These three plates are parallel and therefore the autocorrelation of a pair of two points k, l from two different plates i and j is, in fact, cross correlation $F^{k,l}_{k,l}$. This gives a clear hint on how simply cross correlated vector random fields could be treated as cuts through a single random field with a higher dimension.

In the procedure, we have made a certain simplification of the consistent approach described above, so it is important to assess the error of the approximation. Assume that the distribution of the underlying Gaussian random field is simulated correctly. Then the non-Gaussian field obtained by the memoryless transformation has no error in the marginal distributions. The only error can arise is in the correlation structure of the fields. Obviously, every field alone has a correct autocorrelation structure, because it is expanded using independent Gaussian variables via orthogonal transformation of correct correlation matrices. Let us now take a look at the cross-correlations obtained with the suggested approach.
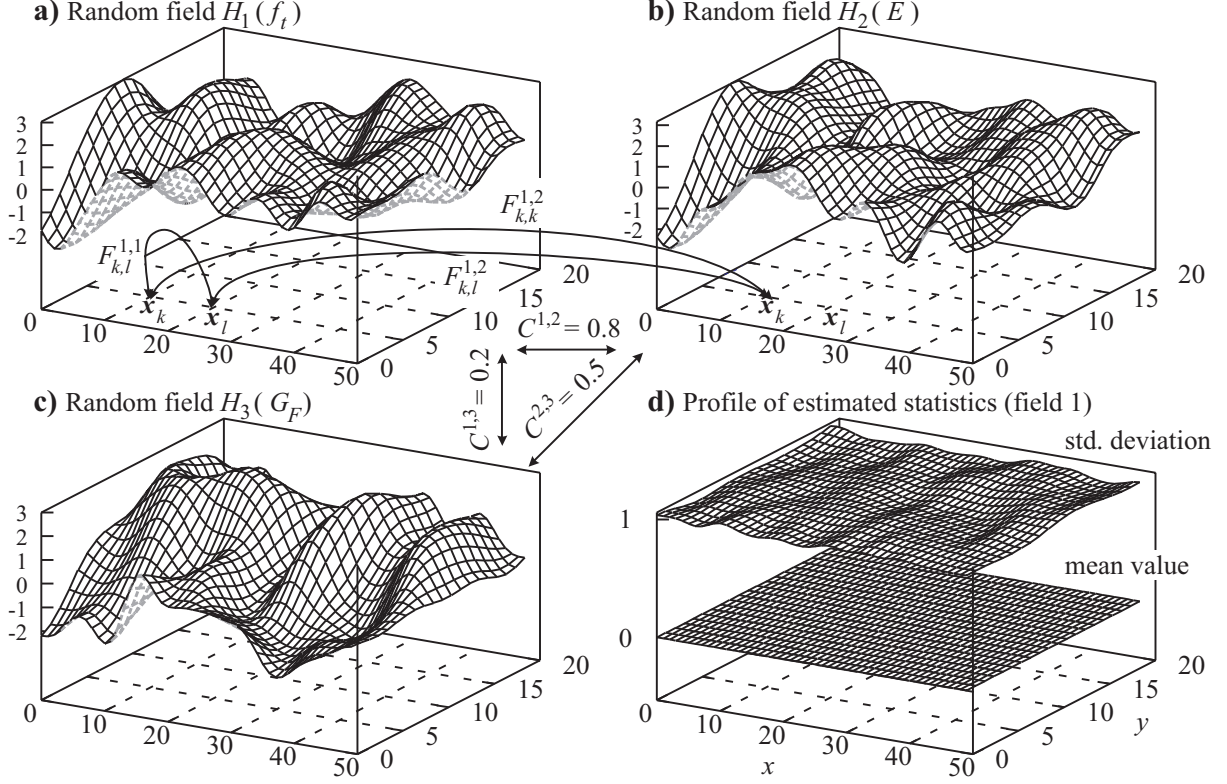
Figure 2: a), b) and c) Random realization of simulated three-variate Gaussian random field, illustration of the meaning of correlation coefficients $F_{k,l}^{i,j}$. d) Profile of the estimated mean value and standard deviation of f_t-field ($N_{\text{sim}} = 1000$).

The simulation of the all nodal point values of all fields (step 5) can be written simply as:

$$\boldsymbol{H} = \boldsymbol{\Phi^E} \left(\boldsymbol{\Lambda^E}\right)^{1/2} \boldsymbol{\chi^D} \tag{17}$$

where $\boldsymbol{\Phi^E}$ and $\boldsymbol{\Lambda^E}$ are the eigenvector and eigenvalue matrices of a (block-diagonal) correlation matrix $\mathbf{E}$ that is constructed as follows. Matrix $\mathbf{E}$ consists of diagonal blocks $\mathbf{F_{uu}^{i,i}}$; all off-diagonal blocks are zero matrices. Therefore, the eigenvalue [eigenvector] $\boldsymbol{\Lambda^E}$ [$\boldsymbol{\Phi^E}$] matrices have the matrices $\boldsymbol{\Lambda_i^u}$ [$\boldsymbol{\Phi_i^u}$] on the diagonal blocks and zeros elsewhere. By substituting Eq. (11) into Eq. (17) we obtain the fields in terms of transformation of independent variables $\boldsymbol{\xi}$:

$$\boldsymbol{H} = \underbrace{\boldsymbol{\Phi^E} \left(\boldsymbol{\Lambda^E}\right)^{1/2} \boldsymbol{\Phi^D} \left(\boldsymbol{\Lambda^D}\right)^{1/2}}_{\boldsymbol{\Psi^F}} \boldsymbol{\xi} = \boldsymbol{\Psi^F} \boldsymbol{\xi} \tag{18}$$

Therefore, the resulting full (block) correlation matrix can be computed as $\mathbf{F'} = \boldsymbol{\Psi^F} \left[\boldsymbol{\Psi^F}\right]^T$:

$$
\begin{aligned}
\mathbf{F'} &= \boldsymbol{\Phi^E} \cdot \left(\boldsymbol{\Lambda^E}\right)^{1/2} \boldsymbol{\Phi^D} \cdot \left(\boldsymbol{\Lambda^D}\right)^{1/2} \\
&\quad \left[\boldsymbol{\Phi^E} \cdot \left(\boldsymbol{\Lambda^E}\right)^{1/2} \boldsymbol{\Phi^D} \cdot \left(\boldsymbol{\Lambda^D}\right)^{1/2}\right]^T \\
&= \boldsymbol{\Phi^E} \left(\boldsymbol{\Lambda^E}\right)^{1/2} \cdot \mathbf{D} \cdot \left(\boldsymbol{\Lambda^E}\right)^{1/2} \left[\boldsymbol{\Phi^E}\right]^T
\end{aligned} \tag{19}
$$

where $\mathbf{D} = \boldsymbol{\Phi^D} \cdot \left(\boldsymbol{\Lambda^D}\right)^{1/2} \cdot \left(\boldsymbol{\Lambda^D}\right)^{1/2} \left[\boldsymbol{\Phi^D}\right]^T$. By this construction, the $\mathbf{F'}$ matrix can be written

11

in square blocks (each of order N): $\mathbf{F}' =$

$$
\begin{array}{cccccc}
 & \mathbf{H}_1 & \ldots & \mathbf{H}_j & \ldots & \mathbf{H}_{N_F} \\
\mathbf{H}_1 & \begin{pmatrix} \mathbf{F}'_{1,1} & \ldots & \mathbf{F}'_{1,j} & \ldots & \mathbf{F}'_{1,N_F} \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ \mathbf{F}'_{i,1} & \ldots & \mathbf{F}'_{i,j} & \vdots & \mathbf{F}'_{i,N_F} \\ \vdots & \ldots & \vdots & \ddots & \vdots \\ \mathbf{F}'_{N_F,1} & \ldots & \mathbf{F}'_{N_F,j} & \ldots & \mathbf{F}'_{1,N_F} \end{pmatrix}
\end{array} \tag{20}
$$

Of course, the matrix is symmetric as a whole, but the blocks are not symmetric in general. Using Eq. (19), each block i, j can be written as

$$
\mathbf{F}'_{i,j} = C^{i,j} \underbrace{\mathbf{\Phi}^{\mathbf{u}}_i \left(\mathbf{\Lambda}^{\mathbf{u}}_i \right)^{1/2}}_{\mathbf{\Psi}^{\mathbf{u}}_i} \underbrace{\left(\mathbf{\Lambda}^{\mathbf{u}}_j \right)^{1/2} \left[\mathbf{\Phi}^{\mathbf{u}}_j \right]^T}_{\left[\mathbf{\Psi}^{\mathbf{u}}_j \right]^T} \tag{21}
$$

The $\mathbf{F}'$ matrix (consisting of blocks $\mathbf{F}'_{i,j} = C^{i,j} \mathbf{\Psi}^{\mathbf{u}}_i [\mathbf{\Psi}^{\mathbf{u}}_j]^T$) represents a good approximation of $\mathbf{F}$ in most cases (see [9] for a numerical example with an estimation of error). The diagonal blocks are equal to the autocorrelation of each field $\mathbf{F}'_{i,i} = \mathbf{F}^i_{\mathbf{uu}}$. The off-diagonal blocks $\mathbf{F}'_{i,j}$, in a certain sense, inherit a combination of the autocorrelations of the i-th and j-th random field (a product of the eigenmodes of both). Note that if a pair of fields i, j follow an identical autocorrelation structure, the corresponding cross-correlation block is just a $C^{i,j}$ multiple of it (compare to Eq. 8).

The $\mathbf{F}'$ can be computed and compared to $\mathbf{F}$ before performing any simulations. If the difference (cross correlation errors) is not acceptable for the analyst and he wants to return to the consistent procedure employing the correct Nataf transformation for $\mathbf{F}$ in the orthogonal transformation via Eq. (12), we recommend to use Eq. (18) to find a very good approximation of the eigenmodes of $\mathbf{F}$ needed in Eq. (12). The eigenmodes can be refined iteratively.

5 CONCLUSIONS

The main result of this paper is the utilization of the spectral properties (eigen-properties) of defined block correlation matrices. These can be advantageously utilized for the simulation of multivariate stochastic fields with a simple cross correlation structure and a common distribution of components. If all fields share the same distribution shape, the decomposition of the autocovariance structure is done only once for all univariate fields. For Gaussian vector random fields, the resulting distribution and correlation properties are correct. For non-Gaussian fields the autocorrelation structure is correct for all fields, but taking full advantage of the computational simplification brings about small errors in cross correlations. These errors can be predicted without any simulations. The reduction of computational effort is often significant.

6 ACKNOWLEDGMENT

REFERENCES

[1] R.V. Field and M. Grigoriu. On the accuracy of the polynomial chaos approximation. *Probab. Eng. Mech.*, **19**(1-2):65–80, 2004.

[2] Mircea Grigoriu. *Applied non–Gaussian processes: Examples, theory, simulation, linear random vibration and MATLAB solutions*. Prentice-Hall, Englewoods Cliffs, N.J., 1995.

[3] Mircea Grigoriu. Evaluation of Karhunen-Loève, spectral, and sampling representations for stochastic processes. *Journal of Engineering Mechanics*, **132**(2):179–189, 2006.

[4] P. Liu and A. Der Kiureghian. Multivariate distribution models with prescribed marginals and covariances. *Probabilistic Engineering Mechanics*, **1**(2):105–111, 1986.

[5] A. Nataf. Détermination des distributions de probabilités dont les marges sont donnés. *Comptes Rendus de L'Académie des Sciences*, **225**:42–43, 1962.

[6] K.K. Phoon, H.W. Huang, and S.T. Quek. Simulation of strongly non–Gaussian processes using Karhunen-Loève expansion. *Probabilistic Engineering Mechanics*, **20**(2):188–198, 2005.

[7] B. Puig, F. Poirion, and Christian Soize. Non–Gaussian simulation using Hermite polynomial expansion: convergences and algorithms. *Probabilistic Engineering Mechanics*, **17**(3):253–264, 2002.

[8] S. Sakamoto and Roger Ghanem. Polynomial chaos decomposition for the simulation of non–Gaussian nonstationary stochastic processes. *Journal of Engineering Mechanics*, **128**(2):190–201, 2002.

[9] M. Vořechovský. Simulation of simply cross correlated random fields by series expansion methods. *Structural safety (Elsevier)*, **30**(4):337–363, 2008.

[10] M. Vořechovský and D. Novák. Simulation of random fields for stochastic finite element analyses. In G. Augusti, G. I. Schuëller, and M. Ciampoli, editors, *ICoSSaR '05 the 9th International Conference on Structural Safety and Reliability, held in Rome, Italy*, pages 2545–2552, Rotterdam, Netherlands, 2005. Millpress.

[11] F. Yamazaki and M. Shinozuka. Digital generation of non–Gaussian stochastic fields. *Journal of Engineering Mechanics*, **114**(7):1183–1197, 1988.

[12] F. Yamazaki and M. Shinozuka. Simulation of stochastic fields by statistical preconditioning. *Journal of Engineering Mechanics*, **116**(2):268–287, 1990.