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RECENT DEVELOPMENT ON A THEORY OF ITERATION DYNAMICAL SYSTEMS OF DISCRETE LAPALCIANS

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Abstract. *The recent development on the mathematical theory and the computer simulations of iteration dynamical system of discrete Lapalcian on the plane lattice is reviewed and the future problem is discussed*

1. ITERATION DYNAMICAL SYSTEM OF DISCRETE LAPLACIAN

We choose the lattice L on the real plane. Each lattice point is identified with the corresponding cell $\Delta_p\,(p=(i,j))$. We consider a function f on L whose value is taken in $\{0,1\}$. The set of such functions constitute a commutative algebra F where we calculate sums and products in mod 2 calculation rule. Next we introduce neighborhoods of a lattice point $p \in L$. A set of cells which attach the referenced cell is called a neighbourhood of p which is denoted by U_p. We list several examples which will be used in this paper:

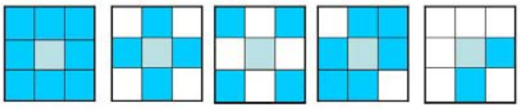
(1)Moore neighborhood : $U_M(i,j)(=\cup_{\rho,\tau=0,\pm1}\Delta_{i+\rho,j+\tau})$,

(2)von Neumann neighborhood: $U_N(i,j)(=\cup_{\rho,\tau=0,\pm1}\Delta_{i+\rho,j}\cup\Delta_{i,j+\rho})$,

(3)Diagonal Neumann neighborhood: $U_{dN}(i,j)(=\cup_{\rho,\tau=0,\pm1}\Delta_{i+\rho,j+\rho}\cup\Delta_{i+\rho,j-\rho})$

(4)Hexiagonal neighborhood: $U_H(i,j)(=U_M(i,j)-\Delta_{i-1,j+1}\cup\Delta_{i+1,j-1})$

(5)Sierpinski neighborhood: $U_S(i,j)(=\Delta_{i+1,j}\cup\Delta_{i,j-1})$



The definition is given uniquely up to orthogonal transformations.

We define the Lapacian operation for an element $f \in F$ by

$$\Delta_{U_p}f(p)=\sum_{q\in U_p}(f(q)-f(p))$$

Choosing an initial function $f_0 \in F$, we define the dynamical system defined by the iteration of the Laplacian([1],[2]):

$$\{f_n\}, f_n=\Delta_U f_{n-1}(n=1,2,...)$$

The main concern of this paper is to consider the dynamical systems by several choices of neighborhoods and initial functions. Here we give examples of computer simulations in the case of a single source:

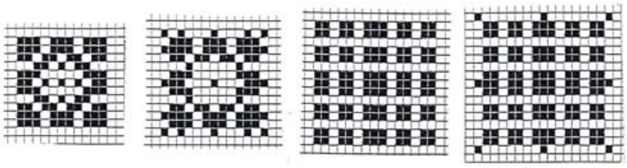


Figure 1. Moor neigh.

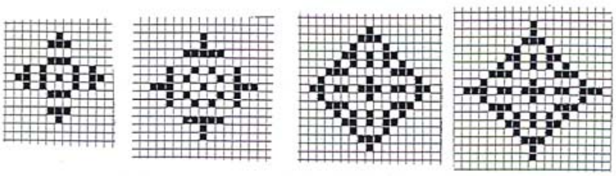


Figure 2. Neumann neigh.

2. DISCRETE LAPLACIAN AND USUAL LAPLACIAN

Here we discuss the relationship between discrete Lapalcian and the usual (continuous) Lapalcian. We recall the usual continuous Laplacian on the plane:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

We notice that the discertization of the Lpalcian is nothing but the Laplacian for Neumann neighborhood. This can be seen by putting

$$\frac{\partial f}{\partial x} \Rightarrow \frac{f(x+h,y)-f(x,y)}{h}, \ \frac{\partial f}{\partial y} \Rightarrow \frac{f(x,y+h)-f(x,y)}{h},$$

Hence putting

$$\frac{\partial^2 f}{\partial x^2} \Rightarrow \frac{\{f(x+h,y)-f(x,y)\}-\{f(x,y)-f(x-h,y)\}}{h^2},$$
$$\frac{\partial^2 f}{\partial y^2} \Rightarrow \frac{\{f(x,y+h)-f(x,y)\}-\{f(x,y)-f(x,y-h)\}}{h^2},$$

and $h=1$, we have the discrete Lapalcian of the Neumann neighborhood. Hence we may understand our discrete dynamical system is the discrete version of the diffusion equation:

$$\frac{\partial f}{\partial t} = \Delta f \ \Rightarrow \{\Delta^n f : n = 1,2,...\}$$

In fact, we can simulate ecosystems and dynamics of human sociological behaviors following the analogous results obtained by the diffusion equations. Here we want to notice that our discrete Laplacian takes the value {0, 1} and the behaviors of the dynamical systems have rather different characters from those of the diffusion equation. We can find new proper phenomena for this Laplacian. In fact, we may expect to simulate the evolution of the universe by use of the Laplacians which are defined by odd neighborhoods(see 5). The Laplacian of the odd neighborhoods can not find its continuous counterpart.

3. MATHMATICAL STRUCTURE OF DISCRTE LAPLACIAN

Here we recall some basic notations on the dynamical systems and state assertions on mathematical structures ([1], [4]). At first we notice that we consider dynamical systems under the periodic condition. Namely, choosing an integer M, which is called the size, we consider the following periodic functions:

$$F(M) = \{f \in F \mid f(x+mM, y+nM) = f(x,y)(n.m \in Z)\}$$

Choosing a neighborhood under the periodic condition, we can define the discrete Laplacian and we can consider the iteration dynamical system. We prepare several basic notations:

(1)A dynamical system is called stable if $^{\exists}k \in N\ s.t.\ f_n = f_k(^{\forall}n \geq k)$

(2)A dynamical system is called periodic, if $^{\exists}n,^{\exists}l \in N\ s.t.\ f_n = f_{n+kl}(^{\forall}k \in N)$ f_n is called recurrence state and n is called recurrence step.

(3)A point $p \in L$ is called a source of a dynamical system, if $f_n(p) = 1$ for $^{\forall}n \in N$.

Conjecture ([1], [2])

By use of computer simulations, we may have the following conjectures:

(1) In the case $M = 2^p$ and a single source, we have the following results:

(α) If the neighborhood is even, we see that the dynamical system is stable and its stability speed is 2^{p-1} for Moor, hexagonal, and Neumann neighborhoods.

(β) If the neighborhood is odd, we see that the dynamical system is periodic, period is different depending on neighborhoods.

(2) In the case where M is odd, we see that the dynamical system is periodic in the case of a single source. We give the table of periods for smaller M (see Table 1).

M	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
Period	1	5	6	13	30	62	29	30	511	126	2046	2045	1021	16384	61
Recurrence point	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 1 Periods for smaller odd sizes

THEOREM (STABILITY THEOREM FOR EVEN NEIGHBORHOODS)

In the case where $M = 2^p$, neighborhood is Sierpinski type (resp. Neumann type), and it has one point source, the dynamical system is stable with the stability speed 2^p (resp. 2^{p-1}).

Proof of the assertion for Sierpinski neighborhood

We give an idea of the proof of Proposition in the case p=2. Making an observation only in this simple case, we may understand that our assertion holds (see Figure 3).

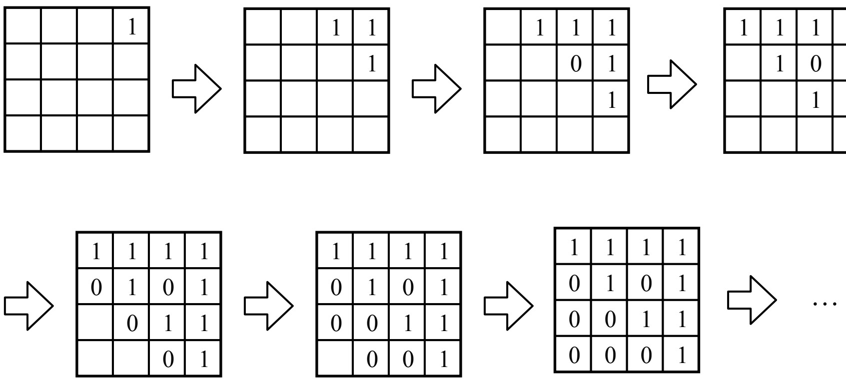


Figure 3 Proof for Sierpinski'case (p=2)

We introduce a coordinate of cell $\{(i, j)\}$ so that the origin is $(0,0)$ at the corner in the upper and right side of the rectangle and choose the coordinate low and left directions from right to left and from upper to lower. We denote the support (or locus) of n th generation by N_n : $N_n = \{(i, j) : i + j = n\}$. Also we put $M_n = \cup_{k=0}^{n} N_k$. We can prove the following proposition which prove the assertion in the case of general . 2^p

PROPOSITION

For the dynamical system $\{f_n\}$ with the source at the origin, we see that

(1) $f_n(i, j) = f_n(j,i)$ on N_n,

(2) $f_n(n,0) = f_n(0,n) = 1(0 \le n \le M-1)$

(3) $f_n(i, j) = f_n(i-1, j) + f_n(i, j-1)$ on $N_n(\mathrm{mod}\,2)$

(4) The Laplacian preserves the invariance on $M_n : f_{n+1|_{M_n}} = f_n$

By this proposition we see the following facts: (i) We see that the Pascal triangle mod 2 appears in the upper triangle part. (ii) At the 2^p step, every element in the diagonal is 1. (iii) Then the lower triangle is filled by 0 (see Figure 3).

Proof of the assertion for von Neumann neighborhood

We give a proof of Proposition in the case p=2. Making an observation only in this simple case, we may understand that our assertion holds (see Figure 4).

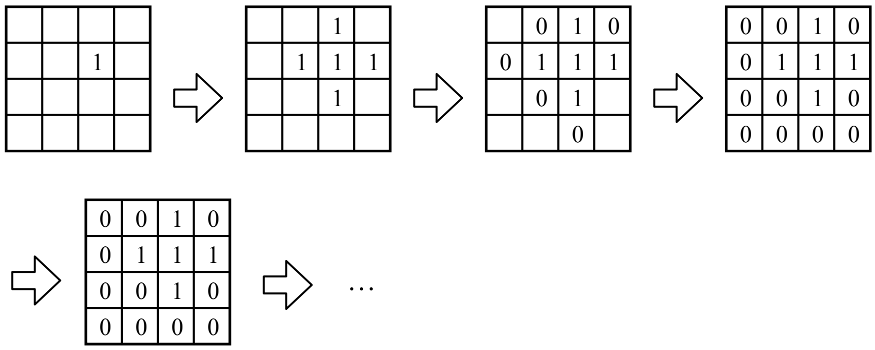


Figure 4 Proof for von Neumann neighborhood (p=4)

We introduce a coordinate of cells $\{(i,j)\}$ so that the origin is (0,0) at the center one cell right and one cell up and choose the coordinate as in the usual manner. We denote the n-th generation by $N_n : N_n = \{(i,j) : i + j = \pm n\}$ and we put $M_n = \cup_{k=0}^{n} N_k$. Then we can prove the assertion by the following proposition:

PROPOSITION

Let $\{f_n\}$ be a dynamical system with a source at the origin. Then we can prove the following for an integer n of a form $n = 2^q (0 \le q \le p-1)$:

(1) The Laplacian Δ brings the support of M_n to that of M_{n+1},

(2) The Laplacian preserves the function f_n on M_n, i.e., $f_{n+1}|_{M_n} = f_n$,

(3) $f_n(i,j) = 1 : (i,j) \in N_n$, (4) $f_n(i,j) = 0$ the outside of M_n,

(4) $f_{n+1}(n+1, 0) = 1$, $f_{n+1}(0, n+1) = 1$ on N_n.

In [4], the concept of the linearization of the discrete Laplacian and the iteration dynamical system is considered. The comparison theorem on the stability and periodicity between these operators are given and the interesting similarities are observed.

5. COMPUTER SIMULATIONS (I) (EVOLUTION, SELF ORGANIZATION

In this section we show that our dynamical system is an important candidate of

cellular automata which supply computer simulations of self organization and evolution in a unified manner:

At first we notice that it is identical with the #90 of Wolfram's automata on the real line ([11]). We know that it supplies one of four chaotic dynamical systems in his classification. Hence we may say that our cellular automaton is a plane lattice version of the #90 automaton of Wolfram. We may expect to develop a generalization of Wolfram's considerations to the plane lattice and we may expect our cellular automata can describe chaotic dynamical systems on the plane lattice.

At first we notice that our cellular automaton is defined as a discretization of the usual Laplace operator in the case of von Neumann neighborhood. We know that the Laplacian is very important and we can not describe physics without the operator. Hence we may expect that our cellular automata can describe the discrete version of the analysis connected to the Laplace operator. In fact, our cellular automata can describe the discrete version of the diffusion equation including the growth of crystals of water and others (Figure 5).

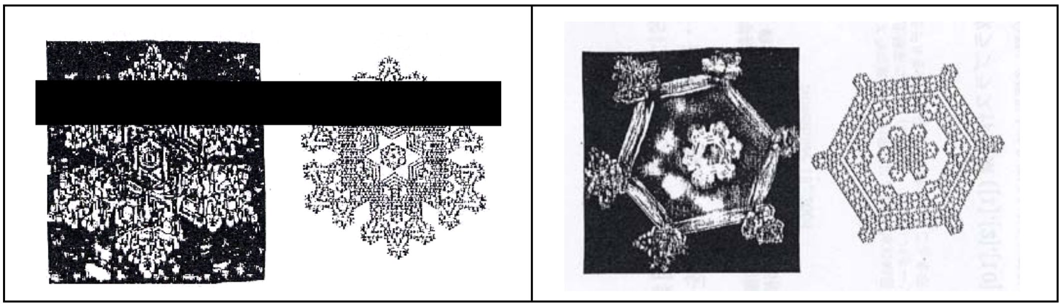


Figure 5, Simulations of crystals of water

Next we shall show that the time change of the numbers of families of extinct animals can be simulated quite well and we can discuss the mutation in terms of the changes of neighborhoods and sources. Here we give an example . As for systematic treatment see([7]) .

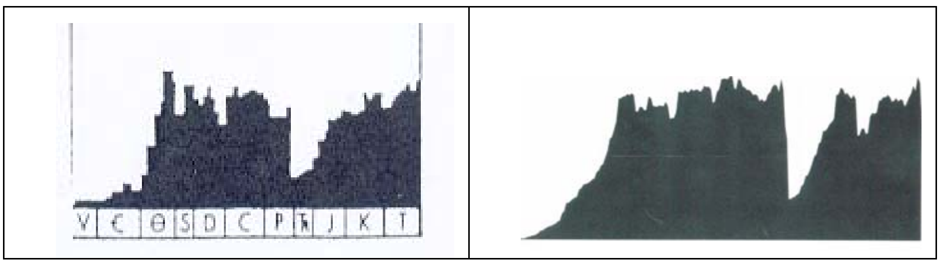


Figure 6, Evolution of an extinct animal

We give a simulation of distribution of galaxies in a fixed direction([3]). Although the results are partial ones, they may temptate us to the research in the future(Figure 7).

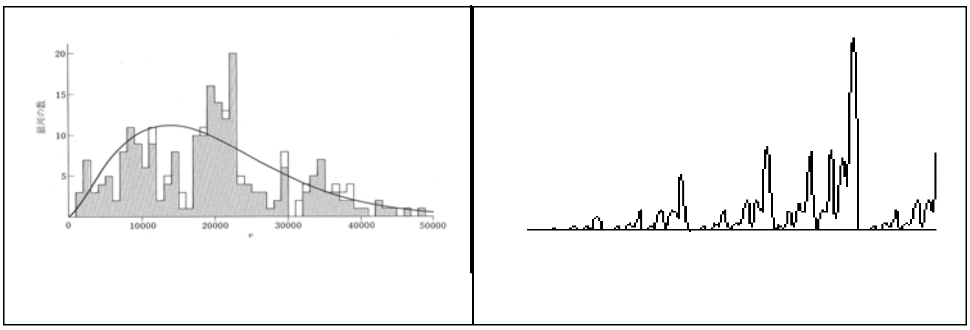


Figure 7, Evolution of the universe.

From these facts we see that our simulation has not only mathematical interest but also has possibility of self-organization and evolutions in a unified manner. Finally we want to make a stress the following fact: Although we have many kinds of cellular automata in [5], we can not find automata defined by our discrete Laplacian in its reference. Moreover only one kind of simulations can describe wide classes of evolutions and organizations in a unified manner. At present we must say that simulations are main and we have a not so rich mathematical theory till now .

6. COMPUTER SIMULATIONS(II)(DESIGNS)

We can produce a lot of design-patterns choosing neighborhoods, sources, steps. We notice that their characters depend on oddness and evenness of neighborhoods strongly.:

We give several examples. As for details see [8],[9].

Design-pattern I - even neighborhood with a single source

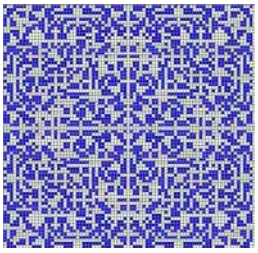
We can observe the following kinds of designs:(1) Floral pattern,(2)Diamond pattern,(3)Mosaic pattern,(4)Bordered pattern (5)Crystal pattern, (6)Braid pattern and (7)Gasket pattern:

Design-pattern II - odd neighborhood with plural sources -

We can obtain other kinds of design patterns with different taste :(1)stripe pattern, (8)mosaic pattern, (9)check pattern,(10)undulate pattern and(11)line pattern:

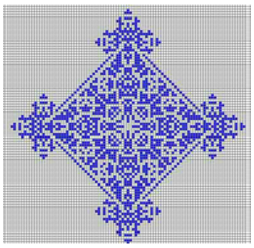
We can make chaotic patterns and designs of periodic characters choosing plural sources. When we put the sources without symmetries, we may obtain chaotic patterns easily.

(1) Floral pattern



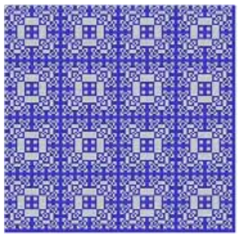
moor〔4〕step=63

(2)Diamond pattern



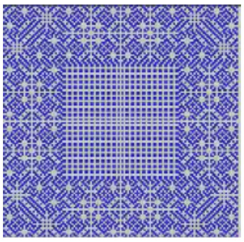
neuman 〔4〕 step=42

(3)Mosaic pattern



diag neuman 〔4〕 step=114

(4)Bordered pattern

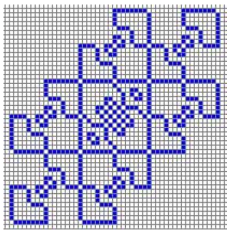


moor 〔4〕 step=94

(5)Crystal pattern

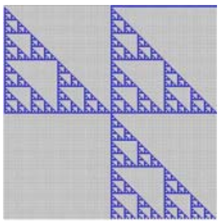
moor 〔4〕 step=221

(6)Braid pattern



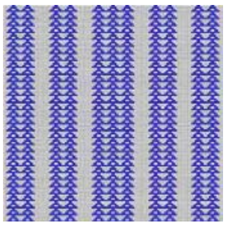
hexagonal 〔4〕 step=24

(7) Gasket pattern



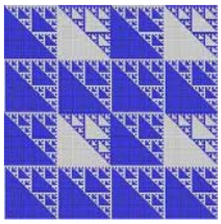
sierpinski 〔1〕 step=128

(8)Stripe pattern



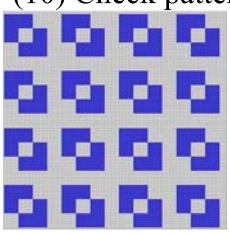
N E SE SW W〔4〕step=124

(9)Mosaic pattern



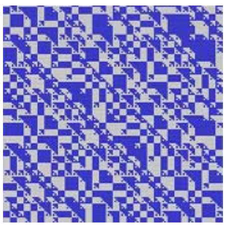
W NW NE SE S 〔1〕step=96

(10) Check pattern



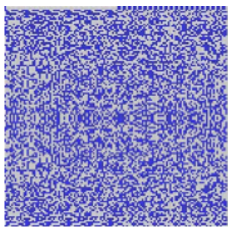
N NE E〔2〕 step=112

(11) Undulate pattern



NW N E SE SW〔5〕step=120

(12) Chaotic pattern



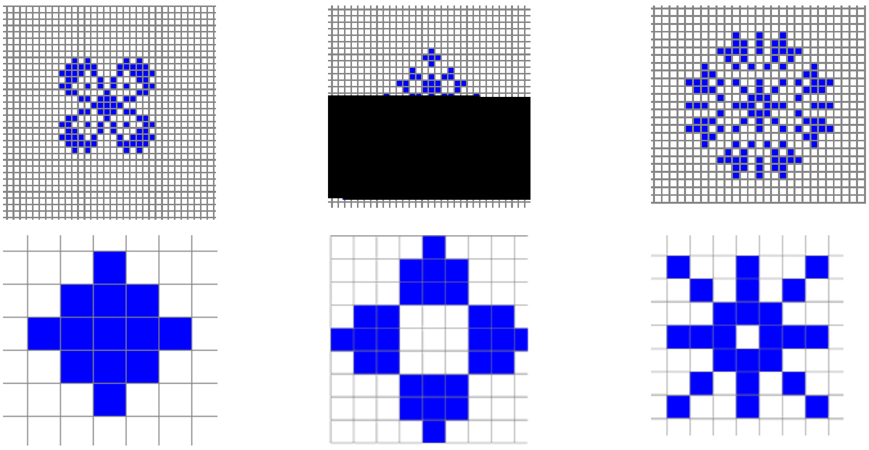
7. THE FUTURE PROBLEM

In this section we will find a possibility of producing the patterns of wings of butterfly by use of patterns of flower type.

Several type of patterns of flower type

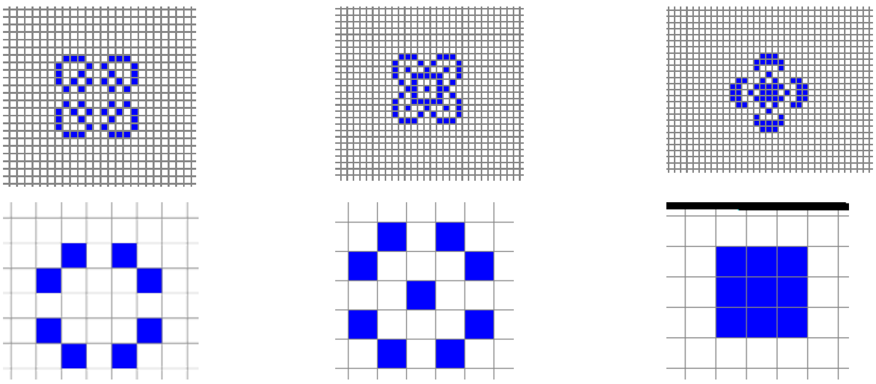
At first we give several simulations which may describe how flowers can be produced in nature. The upper figures are simulations and the lower figures are sources of the simulations:

(i)Radiation type (ii)Uniform-radiation type



Diagonal Neuman step 5 Neuman step=5 Neuman Step 11

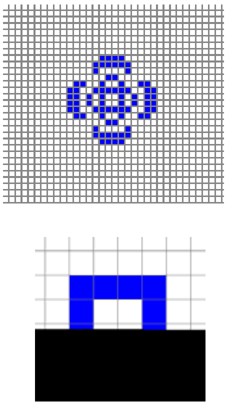
(iii)Corner-type (iv)Border type



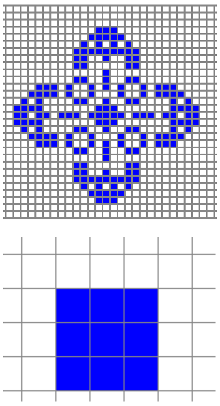
Moor step= 3 Moor step=3 Moor Step=5

(iv)Border type

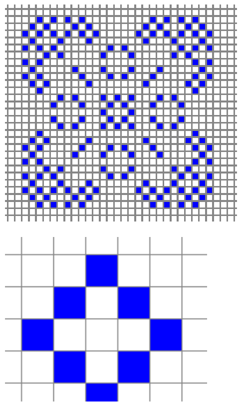
(v)Circle pattern



Neuman step=5

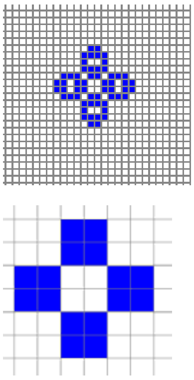


Neuman step=11

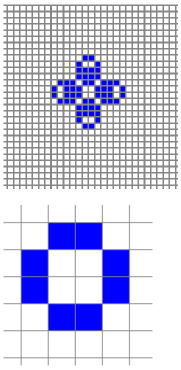


Neuman step 11

(v)Spot pattern



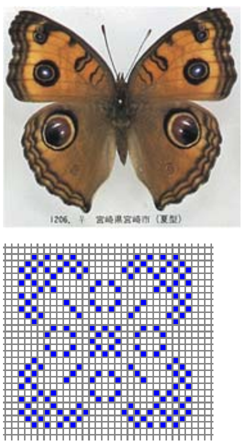
Neuman Step=27

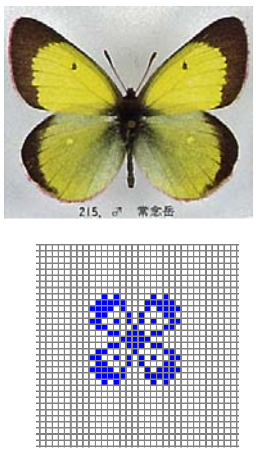


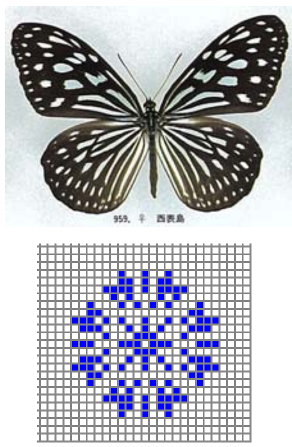
Neuman Step=4

(2)Butterfly pattern

Next we proceed to the realization of designs of butterfly wings. Here we suggest that our dynamical systems have a chance of describing the various patterns of butterfly patterns, especially the circle pattern which is called “snake eye pattern” in Japanese.

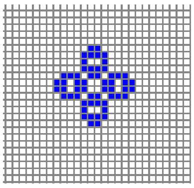
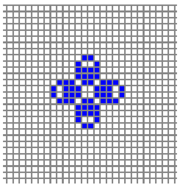
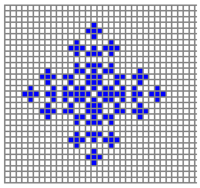






Snake eye pattern | Bordered pattern | Uniform radiation pattern



Snake eye pattern | Bordered pattern | Uniform radiation pattern

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