

NON-LINEAR ANALYSIS OF COMPOSITE CROSS-SECTIONS WITH PRE-DEFORMATIONS

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Abstract. *An energy method based on the LAGRANGE Principle of the minimum of total potential energy is presented to calculate the stresses and strains of composite cross-sections. The stress-strain relation of each partition of the cross-section can be an arbitrary piecewise continuous function. The strain energy is transformed into a line integral by GAUSS's integral theorem.*

The total strain of each partition of the cross-section is split into load-dependent strain and pre-strain. Pre-strains have to be taken into account when the cross-section is pre-stressed, retrofitted or influenced by shrinkage, temperature etc.

The unconstrained minimum problem can be solved for each load combination using standard software. The application of the method presented in the paper is demonstrated by means of examples.

1 INTRODUCTION

The analysis of cross-sections is usually based on equations of equilibrium of stresses, equations of compatibility of deformations and on constitutive equations relating the stresses to the strains. Alternatively, the modelling can be based on extremum principles. Fundamental aspects of this methodical conception were explained in [1, 2, 3]. The following paper shows the further development of this alternative approach insofar as it concerns pre-deformations of particular domains of the composite cross-section.

2 BASICS OF MODELLING

The stresses and strains in composite cross-sections loaded by normal force N and bending moments M_y and M_z can be calculated by the energy method based on the LAGRANGE *Principle of minimum of total potential energy*

$$\Pi = \sum_j \Pi_{i,j} - N\varepsilon_0 - M_y\kappa_z - M_z\kappa_y = \sum_j \iint_{B_j} W_j \, dydz - N\varepsilon_0 - M_y\kappa_z - M_z\kappa_y \rightarrow \text{Min} \quad (1)$$

In this equation, ε_0 is the strain in the origin of the co-ordinate system x,y . The curvatures κ_y and κ_z correspond to the slopes of the strain plane with respect to the coordinate axes y and z .

The material law can usually be defined by a non-linear stress-strain relation $\sigma = \sigma(\varepsilon)$. We assume that $\sigma(\varepsilon)$ is a piecewise continuous function. Additionally, we introduce the functions $W(\varepsilon)$, $F(\varepsilon)$ and $\Phi(\varepsilon)$, which all are partial integrals of the stress-strain relation:

$$W(\varepsilon) = \int_0^\varepsilon \sigma(\varepsilon) d\varepsilon, \quad F(\varepsilon) = \int_0^\varepsilon W(\varepsilon) d\varepsilon, \quad \Phi(\varepsilon) = \int_0^\varepsilon F(\varepsilon) d\varepsilon. \quad (2)$$

As the stress-strain relation $\sigma = \sigma(\varepsilon)$, the specific strain energy $W(\varepsilon)$ and the functions $F(\varepsilon)$ and $\Phi(\varepsilon)$ completely and equivalently describe the material behaviour.

As shown in [2], the double integral in eq. (1), representing the strain energy Π_{ij} of a partition j of the cross-section can be transformed by GAUSS's theorem into a line integral

$$\Pi_{i,j} = \iint_{B_j} W_j \, dydz = \oint_{L_j} \left(-\frac{\kappa_{zj}}{\kappa_j^2} F_j \, dy + \frac{\kappa_{yj}}{\kappa_j^2} F_j \, dz \right) = -\frac{1}{\kappa_j} \oint_{L_j} F_j \, d\eta \quad (3)$$

where

$$\kappa_j = \sqrt{\kappa_{yj}^2 + \kappa_{zj}^2}. \quad (4)$$

The strain energy of a cross-section with a piecewise linear periphery (polygonal cross-section) explicitly depends on the values σ_{ji} , W_{ji} , F_{ji} und Φ_{ji} at the corner points i of the cross-section and its partitions j .

In the case of cross-sections with *pre-deformations*, the total strains $\varepsilon_j(y,z)$ of the partitions j of the cross-section are divided into two parts: the load-dependent strain $\varepsilon(y,z)$ and the pre-strain $\varepsilon_j^{(0)}(y,z)$:

$$\varepsilon_j(y,z) = \varepsilon(y,z) + \varepsilon_j^{(0)}(y,z). \quad (5)$$

According to the BERNOULLI hypothesis, a cross-section perpendicular to the axis of the element with rigid bond between the partitions of the cross-section remains plane during the deformation process. Thus the pre-strain $\varepsilon_j^{(0)}$ of the partition B_j of the cross-section and the load-dependent strain $\varepsilon = \varepsilon(y,z)$ of the whole composite cross-section B with rigid bond between all these partitions are linear functions of the coordinates y and z

$$\varepsilon(y,z) = \varepsilon_0 + \kappa_y y + \kappa_z z \quad y, z \in B \quad (6)$$

and

$$\varepsilon_j^{(0)}(y,z) = \varepsilon_{0j}^{(0)} + \kappa_{yj}^{(0)} y + \kappa_{zj}^{(0)} z \quad y, z \in B_j \quad (7)$$

The strains ε_0 and $\varepsilon_{0j}^{(0)}$ at the origin of the coordinate system, as well as the curvatures κ_y , κ_z , $\kappa_{yj}^{(0)}$ and $\kappa_{zj}^{(0)}$, are parameters of the corresponding strain plane.

$$\varepsilon_j(y,z) = \varepsilon(y,z) + \varepsilon_j^{(0)}(y,z) = \varepsilon_0 + \kappa_y y + \kappa_z z + \varepsilon_{0j}^{(0)} + \kappa_{yj}^{(0)} y + \kappa_{zj}^{(0)} z = \varepsilon_{0j} + \kappa_{yj} y + \kappa_{zj} z \quad (8)$$

where

$$\varepsilon_{0j} = \varepsilon_0 + \varepsilon_{0j}^{(0)}, \quad \kappa_{yj} = \kappa_y + \kappa_{yj}^{(0)}, \quad \kappa_{zj} = \kappa_z + \kappa_{zj}^{(0)} \quad (9)$$

Thus the total strain of the whole cross-section is in general a piecewise linear function.

Pre-deformations have to be taken into account when the cross-section is pre-stressed, a cross-section is retrofitted by additional partitions or when the strains are influenced by temperature, shrinkage etc.

The pre-deformations $\varepsilon_j^{(0)}$ should be related to the state $N = M_y = M_z = 0$. We assume that ILJUSHIN's criterion of simple loading is not violated, i.e. that pre-strains due to different loading can be superposed. All relevant combinations of loading should be investigated separately, taking into account the corresponding pre-deformations.

The strain energy $\Pi_{i,j}$ is determined by the total strains ε_j

$$\Pi_{i,j} = \iint_{B_j} W_j [\varepsilon_j(y,z)] dy dz \quad (10)$$

According to eq.(1), an unrestricted minimum problem can be formulated for each load combination:

$$\Pi = \sum_j \iint_{B_j} W_j [\varepsilon_j(y,z)] dy dz - N \varepsilon_0 - M_y \kappa_z - M_z \kappa_y \rightarrow \text{Min} \quad (11)$$

Variables of the problem are the parameters ε_0 , κ_y and κ_z of the corresponding strain plane. If ε_0^* , κ_y^* , κ_z^* is the solution to the problem, the function $\Pi(\varepsilon_0^*, \kappa_y^*, \kappa_z^*)$ has a minimum, and the partial derivatives of Π with respect to ε_0 , κ_y and κ_z are zero

$$\frac{\partial \Pi}{\partial \varepsilon_0} = \sum_j \frac{\partial \Pi_{i,j}}{\partial \varepsilon_0} - N = 0, \quad \frac{\partial \Pi}{\partial \kappa_y} = \sum_j \frac{\partial \Pi_{i,j}}{\partial \kappa_y} - M_z = 0, \quad \frac{\partial \Pi}{\partial \kappa_z} = \sum_j \frac{\partial \Pi_{i,j}}{\partial \kappa_z} - M_y = 0. \quad (12)$$

The double integral in eq.(12) can be transformed by the LEIBNIZ rule

$$D_t I_x = I_x D_t \quad (13)$$

where D_t is the partial derivative with respect to t and I_x is the integral operator with respect to x over a fixed domain. Thus we obtain

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial \varepsilon_0} &= \sum_j \frac{\partial}{\partial \varepsilon_0} \iint_{\mathbf{B}_j} W_j [\varepsilon_j(y, z, \varepsilon_0, \kappa_y, \kappa_z)] dydz = \sum_j \iint_{\mathbf{B}_j} \frac{\partial W_j [\varepsilon_j(y, z, \varepsilon_0, \kappa_y, \kappa_z)]}{\partial \varepsilon_0} dydz = \\
&= \sum_j \iint_{\mathbf{B}_j} \frac{dW_j(\varepsilon_j)}{d\varepsilon_j} \frac{\partial \varepsilon_j}{\partial \varepsilon_0} dydz = \sum_j \iint_{\mathbf{B}_j} \sigma_j(y, z) dydz = N
\end{aligned} \tag{14}$$

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial \kappa_y} &= \sum_j \frac{\partial}{\partial \kappa_y} \iint_{\mathbf{B}_j} W_j [\varepsilon_j(y, z, \varepsilon_0, \kappa_y, \kappa_z)] dydz = \sum_j \iint_{\mathbf{B}_j} \frac{\partial W_j [\varepsilon_j(y, z, \varepsilon_0, \kappa_y, \kappa_z)]}{\partial \kappa_y} dydz = \\
&= \sum_j \iint_{\mathbf{B}_j} \frac{dW_j(\varepsilon_j)}{d\varepsilon_j} \frac{\partial \varepsilon_j}{\partial \kappa_y} dydz = \sum_j \iint_{\mathbf{B}_j} \sigma_j(y, z) y dydz = M_z
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial \kappa_z} &= \sum_j \frac{\partial}{\partial \kappa_z} \iint_{\mathbf{B}_j} W_j [\varepsilon_j(y, z, \varepsilon_0, \kappa_y, \kappa_z)] dydz = \sum_j \iint_{\mathbf{B}_j} \frac{\partial W_j [\varepsilon_j(y, z, \varepsilon_0, \kappa_y, \kappa_z)]}{\partial \kappa_z} dydz = \\
&= \sum_j \iint_{\mathbf{B}_j} \frac{dW_j(\varepsilon_j)}{d\varepsilon_j} \frac{\partial \varepsilon_j}{\partial \kappa_z} dydz = \sum_j \iint_{\mathbf{B}_j} \sigma_j(y, z) z dydz = M_y
\end{aligned} \tag{16}$$

These three equations describe the *conditions of equilibrium*. They are used to check the results and to evaluate the numerical error of the solution.

If the partition j of the cross-section is pre-loaded or pre-stressed, the corresponding parameters of deformation $\varepsilon_{0j}^{(p)}$, $\kappa_{yj}^{(p)}$ and $\kappa_{zj}^{(p)}$ represent the pre-deformations $\varepsilon_{0j}^{(0)}$, $\kappa_{yj}^{(0)}$ and $\kappa_{zj}^{(0)}$ with respect to the following load cases

$$\varepsilon_{0j}^{(0)} = \varepsilon_{0j}^{(p)} \quad \kappa_{yj}^{(0)} = \kappa_{yj}^{(p)} \quad \kappa_{zj}^{(0)} = \kappa_{zj}^{(p)}. \tag{17}$$

If the partition j is subjected to temperature and the deformations $\varepsilon_{0j}^{(T)}$, $\kappa_{yj}^{(T)}$ and $\kappa_{zj}^{(T)}$ are prohibited by the bond between the partitions, the pre-deformations are defined by

$$\varepsilon_{0j}^{(0)} = -\varepsilon_{0j}^{(T)} \quad \kappa_{yj}^{(0)} = -\kappa_{yj}^{(T)} \quad \kappa_{zj}^{(0)} = -\kappa_{zj}^{(T)}. \tag{18}$$

The deformations due to shrinkage $\varepsilon_{0j}^{(sh)}$, $\kappa_{yj}^{(sh)}$ and $\kappa_{zj}^{(sh)}$ are taken into account by the pre-deformations

$$\varepsilon_{0j}^{(0)} = -\varepsilon_{0j}^{(sh)} \quad \kappa_{yj}^{(0)} = -\kappa_{yj}^{(sh)} \quad \kappa_{zj}^{(0)} = -\kappa_{zj}^{(sh)}. \tag{19}$$

Similarly, deformations induced by creep can be incorporated into the model for approximate calculations. For each separate time interval, the deformations due to creep are calculated by iteration.

Table 1: Strain energy and internal forces

Cross-section Type A	Cross-section Type B
$\Pi_{i,j,i}^Q = \begin{cases} -\frac{1}{\kappa_j} \Delta\eta_i \frac{\Delta\Phi_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0, \Delta\varepsilon_i \neq 0) \\ -\frac{1}{\kappa_j} \Delta(\eta F)_i & (\kappa_j \neq 0, \Delta\varepsilon_i = 0) \\ \frac{W_{im}}{2} (y_i z_{i+1} - y_{i+1} z_i) & (\kappa_j = 0). \end{cases}$	$\Pi_{ii}^Q = \begin{cases} \frac{\Delta F_i}{\Delta\varepsilon_i} A_i & (\Delta\varepsilon_i \neq 0) \\ W_{im} A_i & (\Delta\varepsilon_i = 0). \end{cases}$
$N_{j,i} = \begin{cases} -\frac{1}{\kappa_j} \Delta\eta_i \frac{\Delta F_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0, \Delta\varepsilon_i \neq 0) \\ -\frac{1}{\kappa_j} \Delta\eta_i W_{im} & (\kappa_j \neq 0, \Delta\varepsilon_i = 0) \\ \frac{\sigma_{im}}{2} (y_i z_{i+1} - y_{i+1} z_i) & (\kappa_j = 0) \end{cases}$	$N_i = \begin{cases} \frac{\Delta W_i}{\Delta\varepsilon_i} A_i & (\Delta\varepsilon_i \neq 0) \\ \sigma_{im} A_i & (\Delta\varepsilon_i = 0) \end{cases}$
$M_{y,i} = \begin{cases} -\frac{1}{\kappa_j} \Delta\eta_i \frac{\Delta(Fz)_i}{\Delta\varepsilon_i} + \frac{1}{\kappa_j^2} C_i \frac{\Delta\Phi_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0, \Delta\varepsilon_i \neq 0) \\ -\frac{1}{\kappa_j} \Delta\eta_i (Wz)_{im} + \frac{\kappa_{zj}}{\kappa_j^3} \Delta\eta_i F_{im} & (\kappa_j \neq 0, \Delta\varepsilon_i = 0) \\ \frac{\sigma_{im}}{6} (y_i z_{i+1} - y_{i+1} z_i) (z_{i+1} + z_i) & (\kappa_j = 0) \end{cases}$	$M_{y_i} = \begin{cases} \left(\frac{\Delta(Wz)_i}{\Delta\varepsilon_i} - \frac{\Delta F_i}{\Delta\varepsilon_i} \cdot \frac{\Delta z_i}{\Delta\varepsilon_i} \right) A_i & (\Delta\varepsilon_i \neq 0) \\ \sigma_{im} z_{im} A_i & (\Delta\varepsilon_i = 0) \end{cases}$
$M_{z,i} = \begin{cases} -\frac{1}{\kappa_j} \Delta\eta_i \frac{\Delta(Fy)_i}{\Delta\varepsilon_i} + \frac{1}{\kappa_j^2} D_i \frac{\Delta\Phi_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0, \Delta\varepsilon_i \neq 0) \\ -\frac{1}{\kappa_j} \Delta\eta_i (Wy)_{im} + \frac{\kappa_{yj}}{\kappa_j^3} \Delta\eta_i F_{im} & (\kappa_j \neq 0, \Delta\varepsilon_i = 0) \\ \frac{\sigma_{im}}{6} (y_i z_{i+1} - y_{i+1} z_i) (y_{i+1} + y_i) & (\kappa_j = 0) \end{cases}$	$M_{z_i} = \begin{cases} \left(\frac{\Delta(Wy)_i}{\Delta\varepsilon_i} - \frac{\Delta F_i}{\Delta\varepsilon_i} \cdot \frac{\Delta y_i}{\Delta\varepsilon_i} \right) A_i & (\Delta\varepsilon_i \neq 0) \\ \sigma_{im} y_{im} A_i & (\Delta\varepsilon_i = 0) \end{cases}$
Cross-section Type C	
$M_{z,i} = \begin{cases} -\frac{1}{\kappa_j} \Delta\eta_i \frac{\Delta(Fy)_i}{\Delta\varepsilon_i} + \frac{1}{\kappa_j^2} D_i \frac{\Delta\Phi_i}{\Delta\varepsilon_i} & (\kappa_j \neq 0, \Delta\varepsilon_i \neq 0) \\ -\frac{1}{\kappa_j} \Delta\eta_i (Wy)_{im} + \frac{\kappa_{yj}}{\kappa_j^3} \Delta\eta_i F_{im} & (\kappa_j \neq 0, \Delta\varepsilon_i = 0) \\ \frac{\sigma_{im}}{6} (y_i z_{i+1} - y_{i+1} z_i) (y_{i+1} + y_i) & (\kappa_j = 0) \end{cases}$	$\Pi_{i,i}^Q = \iint_B W(y,z) dydz = A_i W_i$ $N_i = \sigma_i A_i$ $M_{y_i} = \sigma_i z_i A_i$ $M_{z_i} = \sigma_i y_i A_i$
$C_i = \frac{\kappa_{zj}}{\kappa_j} \Delta\eta_i - \frac{\kappa_{yj}}{\kappa_j} \Delta\zeta_i + \frac{\kappa_j \Delta\eta_i}{\Delta\varepsilon_i} \Delta z_i$	$\eta_i = y_i \cos \varphi + z_i \sin \varphi = \frac{\kappa_{zj}}{\kappa_j} y_i - \frac{\kappa_{yj}}{\kappa_j} z_i$
$D_i = \frac{\kappa_{yj}}{\kappa_j} \Delta\eta_i + \frac{\kappa_{zj}}{\kappa_j} \Delta\zeta_i + \frac{\kappa_j \Delta\eta_i}{\Delta\varepsilon_i} \Delta y_i.$	$\zeta_i = -y_i \sin \varphi + z_i \cos \varphi = \frac{\kappa_{yj}}{\kappa_j} y_i + \frac{\kappa_{zj}}{\kappa_j} z_i$
$(\kappa_j \neq 0)$	

3 APPLICATION

The application of the energy method proposed in Chapter 2 will now be demonstrated by means of two examples. We assume that the geometry of the composite cross-section and its partitions correspond to the basic types shown in Fig. 1.

The strain energy Π_i and the internal forces N , M_y and M_z can be calculated by the formulae compiled in Tab. 1.

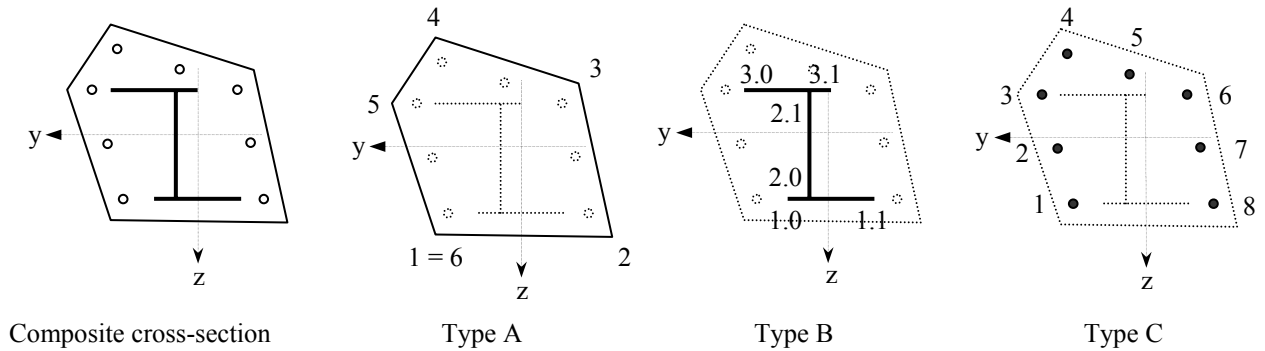


Fig. 1: Composite cross-section and basic types of cross-sections

The numerical solution to the nonlinear optimisation problem can be realised by a programme system for spread-sheet analysis.

3.1 Example 1

The composite cross-section shown in Tab.2 is produced in two stages. One part of the cross-section consists of concrete and is reinforced by a steel element and four bars in the corners of the cross-section. This part is pre-loaded by the bending moment $M_y = 22,000$ kNcm.

Table 2: Parameters of geometry and material (Example 1)

Cross-Section (CS)		Geometry:					Material	Pre-deformation	Effective Cross-section				
j	i	b_{ji} [cm]	h_{ji} [cm]	y'_{jis} [cm]	z'_{jis} [cm]	A_{ji} [cm ²]	E_j [kN/cm ²]	$\varepsilon_{j10}^{(0)}$	LC1	LC2	LC3	LC4	LC5
1	Concrete 1	40,0	48,0	0,00	0,00	1.920,0	2.880	0,00	1	1	1	1	1
2	Concrete 2	40,0	10,0	0,00	-29,00	400,0	2.880	0,00	0	1	1	1	1
3	Steel	1	31,0	3,9		120,9	20.000	0,00	1	1	1	1	1
		2	2,1	26,2	0,00	0,00	55,0						
		3	31,0	3,9		120,9							
		n_{ji}	D_{ji} [cm]	d_{1y} [cm]	d_{1z} [cm]	A_{ji} [cm ²]	$\varepsilon_{ji}^{(0)}$						
4	Reinforcement 1	4	2,0	4,00	4,00	12,6	20.000	0,00	1	1	1	1	1
5	Reinforcement 2	4	1,2	4,00	3,00	4,5	20.000	0,00	0	1	1	1	1

Load Case	N [kN]	M_y [kNcm]	M_z [kNcm]
LC1	0	22.000	0
LC2	0	50.000	20.000
...			

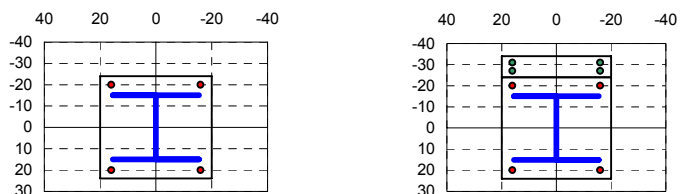


Table 3: Calculation scheme

Load case: 1										Potentials				
Parameters of deformation														
X1: ϵ_1	-3,92E-04	ϵ_0	4,82E-05	κ [cm ⁻¹]	1,29E-05					Π_0 [kN]	-2,85E-01			
X2: ϵ_2	-3,92E-04	κ_y [cm ⁻¹]	0,00E+00					Π_1 [kN]	1,42E-01					
X3: ϵ_3	3,59E-04	κ_z [cm ⁻¹]	1,29E-05					Π [kN]	-1,42E-01					
CS	$\epsilon_{0j}^{(0)}$	ϵ_{0j}	$\kappa_{yj}^{(0)}$	κ_{yj}	$\kappa_{zj}^{(0)}$	κ_{zj}	κ_j	Optimisation			N	M_y	M_z	
j											[kN]	[kNm]	[kNm]	
1	1	0,00E+00	4,82E-05	0,00E+00	0,00E+00	1,29E-05	1,29E-05	Objective function:	$OF = \Pi :$	-1,42E-01	prov	0	22.000	0
2	0	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	Variables:	$X_1 = \epsilon_1$	-3,92E-04	cal	0	22.000	0
3	1	0,00E+00	4,82E-05	0,00E+00	0,00E+00	1,29E-05	1,29E-05		$X_2 = \epsilon_2$	-3,92E-04	Difference:	9,04E-04	9,10E-03	0,00E+00
4	1	0,00E+00	4,82E-05	0,00E+00	0,00E+00	1,29E-05	1,29E-05		$X_3 = \epsilon_3$	3,59E-04				
5	0	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	Restrictions:	no					

j	i	y_i	z_i	A_{ji}	η_i	ζ_i	ϵ_i	$\epsilon_{ip}^{(0)}$	ϵ_{ip}	σ_{ip}	W_{ip}	F_{ip}	Φ_{ip}	$\Pi_{i,j}$	N_{ip}	M_{yp}	M_{zp}
		[cm]	[cm]	[cm ²]	[cm]	[cm]	[-]	[-]	[-]	[kN/cm ²]	[kN/cm ²]	[kN/cm ²]	[kN/cm ²]	[kN/cm ²]	[kN]	[kNm]	[kNm]
1	Concrete 1	1	20,0	24,0	20,00	24,00	3,59E-04	0,00E+00	3,59E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		2	-20,0	24,0	-20,00	24,00	3,59E-04	0,00E+00	3,59E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	262
		3	-20,0	-24,0	-20,00	-24,00	-2,62E-04	0,00E+00	-2,62E-04	-7,56E-01	9,92E-05	-8,68E-09	5,69E-13	2,68E-02	-306	5,284	0
		4	20,0	-24,0	20,00	-24,00	-2,62E-04	0,00E+00	-2,62E-04	-7,56E-01	9,92E-05	-8,68E-09	5,69E-13	0,00E+00	0	0	-262
		5=1	20,0	24,0	20,00	24,00	3,59E-04	0,00E+00	3,59E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
2	Concrete 2 nicht vorh	1	20,0	-24,0	20,00	-24,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		2	-20,0	-24,0	-20,00	-24,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		3	-20,0	-34,0	-20,00	-34,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		4	20,0	-34,0	20,00	-34,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		5=1	20,0	-24,0	20,00	-24,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
3	Steel	1.0	15,5	15,1	120,90	15,50	15,05	2,43E-04	0,00E+00	2,43E-04	4,86E+00	5,91E-04	4,79E-08	7,14E-02	588	8,845	0
		1.1	-15,5	15,1	63,21	-15,50	15,05	2,43E-04	0,00E+00	2,43E-04	4,86E+00	5,91E-04	4,79E-08	9,47E-03	61	1,236	0
		2.0	0,0	15,1	63,21	0,00	15,05	2,43E-04	0,00E+00	2,43E-04	4,86E+00	5,91E-04	4,79E-08	9,47E-03	61	1,236	0
		2.1	0,0	-15,1	63,21	0,00	-15,05	-1,47E-04	0,00E+00	-1,47E-04	-2,93E+00	2,15E-04	-1,05E-08	2,60E-02	-354	5,334	0
		3.0	15,5	-15,1	120,90	15,50	-15,05	-1,47E-04	0,00E+00	-1,47E-04	-2,93E+00	2,15E-04	-1,05E-08	2,60E-02	-354	5,334	0
		3.1	-15,5	-15,1	120,90	-15,50	-15,05	-1,47E-04	0,00E+00	-1,47E-04	-2,93E+00	2,15E-04	-1,05E-08	2,60E-02	-354	5,334	0
4	Reinforcement 1	1	16,0	20,0	3,14	16,00	20,00	3,07E-04	0,00E+00	3,07E-04	6,14E+00	9,43E-04	2,96E-03	19	386	309	
		2	-16,0	20,0	3,14	-16,00	20,00	3,07E-04	0,00E+00	3,07E-04	6,14E+00	9,43E-04	2,96E-03	19	386	-309	
		3	-16,0	-20,0	3,14	-16,00	-20,00	-2,11E-04	0,00E+00	-2,11E-04	-4,21E+00	4,44E-04	1,39E-03	-13	265	212	
		4	16,0	-20,0	3,14	16,00	-20,00	-2,11E-04	0,00E+00	-2,11E-04	-4,21E+00	4,44E-04	1,39E-03	-13	265	-212	
5	Reinforcement 2 nicht vorh	1	16,0	-27,0	1,13	16,00	-27,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0	
		2	-16,0	-27,0	1,13	-16,00	-27,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0	
		3	-16,0	-31,0	1,13	-16,00	-31,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0	
		4	16,0	-31,0	1,13	16,00	-31,00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0	

Load case: 2										Potentials					
Parameters of deformation															
X1: ϵ_1	-7,44E-04	ϵ_0	9,91E-05	κ [cm ⁻¹]	2,54E-05					Π_0 [kN]	#####				
X2: ϵ_2	1,64E-04	κ_y [cm ⁻¹]	2,27E-05					Π_1 [kN]	7,83E-01						
X3: ϵ_3	-8,07E-05	κ_z [cm ⁻¹]	1,14E-05					Π [kN]	-2,42E-01						
CS	$\epsilon_{0j}^{(0)}$	ϵ_{0j}	$\kappa_{yj}^{(0)}$	κ_{yj}	$\kappa_{zj}^{(0)}$	κ_{zj}	κ_j	Optimisation			N	M_y	M_z		
j											[kN]	[kNm]	[kNm]		
1	1	4,82E-05	1,47E-04	0,00E+00	2,27E-05	1,29E-05	2,44E-05	3,33E-05	Objective function:	$OF = \Pi :$	-2,42E-01	prov	0	50.000	20.000
2	1	0,00E+00	9,91E-05	0,00E+00	2,27E-05	0,00E+00	1,14E-05	2,54E-05	Variables:	$X_1 = \epsilon_1$	-7,44E-04	cal	0	50.000	20.000
3	1	4,82E-05	1,47E-04	0,00E+00	2,27E-05	1,29E-05	2,44E-05	3,33E-05		$X_2 = \epsilon_2$	1,64E-04	Difference:	8,22E-04	1,03E-02	9,47E-03
4	1	4,82E-05	1,47E-04	0,00E+00	2,27E-05	1,29E-05	2,44E-05	3,33E-05		$X_3 = \epsilon_3$	-8,07E-05				
5	1	0,00E+00	9,91E-05	0,00E+00	2,27E-05	0,00E+00	1,14E-05	2,54E-05	Restrictions:	no					

j	i	y_i	z_i	A_{ji}	η_i	ζ_i	ϵ_i	$\epsilon_{ip}^{(0)}$	ϵ_{ip}	σ_{ip}	W_{ip}	F_{ip}	Φ_{ip}	$\Pi_{i,j}$	N_{ip}	M_{yp}	M_{zp}
		[cm]	[cm]	[cm ²]	[cm]	[cm]	[-]	[-]	[-]	[kN/cm ²]	[kN/cm ²]	[kN/cm ²]	[kN/cm ²]	[kN/cm ²]	[kN]	[kNm]	[kNm]
1	Concrete 1	1	20,0	24,0	-1,72	31,19	8,28E-04	3,59E-04	1,19E-03	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		2	-20,0	24,0	-30,99	3,93	-8,07E-05	3,59E-04	2,78E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	6,37E-02	-286	1,446	5,914
		3	-20,0	-24,0	1,72	-31,19	-6,29E-04	-2,62E-04	-8,92E-04	-2,57E+00	1,15E-03	-3,40E-07	7,59E-11	7,34E-02	-329	7,691	348
		4	20,0	-24,0	30,99	-3,93	2,79E-04	-2,62E-04	1,64E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		5	20,0	24,0	-1,72	31,19	8,28E-04	3,59E-04	1,19E-03	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
2	Concrete 2	1	20,0	-24,0	30,43	7,07	2,79E-04	0,00E+00	2,79E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	-1,47E-02	93	-3,002	-188
		2	-20,0	-24,0	12,44	-28,66	-6,29E-04	0,00E+00	-6,29E-04	-1,81E+00	5,70E-04	-1,20E-07	1,88E-11	5,49E-02	-239	5,053	3,339
		3	-20,0	-34,0	21,37	-33,15	-7,44E-04	0,00E+00	-7,44E-04	-2,14E+00	7,96E-04	-1,97E-07	3,67E-11	2,86E-02	-154	6,720	-193
		4	20,0	-34,0	39,36	2,57	1,64E-04	0,00E+00	1,64E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
		5	20,0	-24,0	30,43	7,07	2,79E-04	0,00E+00	2,79E-04	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0	0	0
3	Steel	1.0	15,5	15,1	120,90	1,08	21,58	6,23E-04	2,43E-04	8,66E-04	1,73E+01	7,50E-03	2,17E-06	3,70E-01	1,243	18,712	4,396
		1.1	-15,5	15,1	63,21	-21,60	0,45	-8,08E-05	2,43E-04	1,62E-04	3,25E+00	2,63E-04	1,42E-08	4,21E-02	186	2,327	0
		2.0	0,0	15,1	63,21	-10,26	11,01	2,71E-04	2,43E-04	5,14E-04	1,03E+01	2,64E-03	4,53E-07	4,21E-02	186	2,327	0
		2.1	0,0	-15,1	63,21	10,26	-11,01	-7,30E-05	-1,47E-04	-2,20E-04	-4,39E+00	4,82E-04	-3,53E-08	4,21E-02	186	2,327	0
		3.0	15,5	-15,1	120,90	21,60	-0,45	2,79E-04	-1,47E-04	1,32E-04	2,65E+00	1,75E-04	7,73E-09	1,08E-01	-531	7,989	4,396
		3.1	-15,5	-15,1	120,90	-21,60	0,45	-8,08E-05	-1,47E-04	-5,71E-04	-1,14E+01	3,27E-03	-6,22E-07	1,08E-01	-531	7,989	4,396
4	Reinforcement 1	1	16,0	20,0	3,14	-1,92	25,54	6,91E-04	3,07E-04	9,98E-04	2,00E+01	9,96E-03	3,13E-02	63	1,254	1,003	
		2	-16,0	20,0	3,												

The second part of the composite cross-section consists of concrete and four reinforcement bars and is added later. The whole cross-section is loaded by biaxial bending ($M_y = 50,000\text{kNcm}$ and $M_z = 20,000\text{kNcm}$). The concrete and the reinforcement behave in a linear elastic manner, whereas the concrete of the tension zone is cracked and cannot transmit tensile stresses. Parameters of geometry and material are given in Tab.2

The calculation scheme is shown in Tab.3. The variables of the unconstrained optimisation problem are the strains ε_1 , ε_2 and ε_3 , which in turn define the parameters ε_0 , κ_y und κ_z . The geometry of all partitions j of the cross-section is described by the co-ordinates y_i , z_i and the corresponding areas A_{ji} . The co-ordinates η_i , ζ_i are calculated by the formulae of Tab.1. The strains ε_i and $\varepsilon_{ji}^{(0)}$ are obtained from eq.(6) and (7), and the strains ε_{ji} from eq.(5). The stresses σ_{ji} , the values W_{ji} , F_{ji} , Φ_{ji} and the strain energy $\Pi_{i,ji}$ are determined by Tab.1. By minimising the total potential energy Π , we find the solution to the problem. The differences $N_{\text{prov}} - N_{\text{calc}}$ etc. indicate the numerical error in satisfying the equations of equilibrium.

3.2 Example 2

The cross-section of a statically-determined reinforced concrete beam shown in Fig.2 is subjected to temperature. The distribution of the temperature across the height of the cross-section is described by a piecewise linear function. Both concrete and reinforcement behaves in a linear-elastic manner.

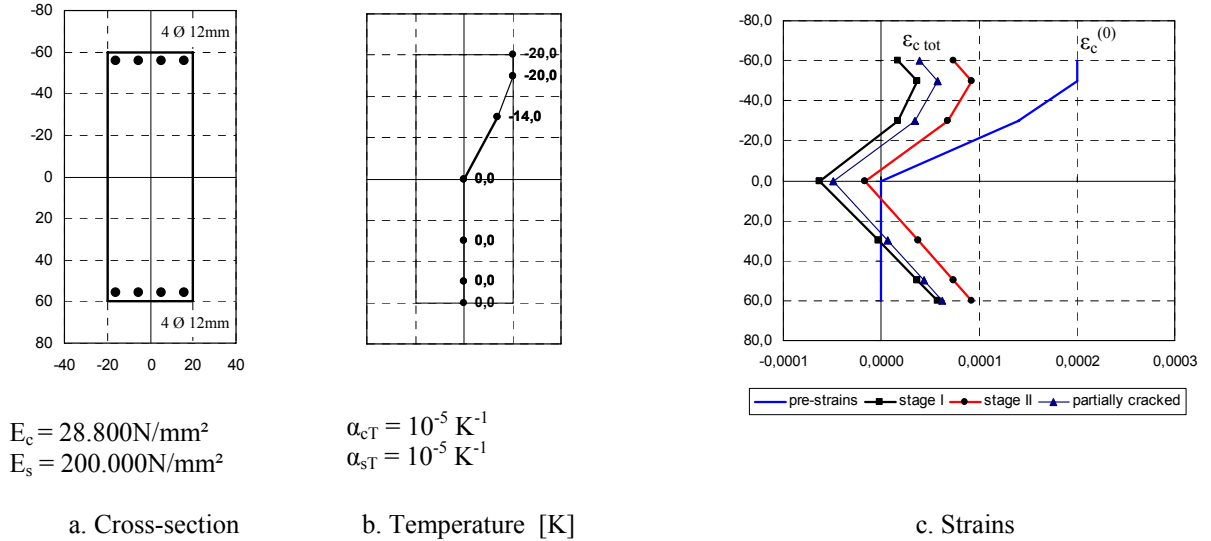


Fig. 2: Example 2

Three cases are investigated with respect to the tensile zone:

- where the concrete is uncracked
- where the concrete transmits no tensile stresses (see Tab.5)
- where the stresses in the effective zone around the reinforcement bars are limited to $\beta_{ct}f_{ct}$ ($\beta_{ct} = 0.4$, $f_{ct} = 2.8\text{N/mm}^2$).

Table 5: Calculation schema (Example 2)

Load case:		T		Stage II		Potentials:		Internal forces					
Parameters of Deformation:						II_{ij} [kN]		0					
X1: ε_1	-1,26E-04	ε_0	-1,70E-05			II [kN]		7,09E-04					
X2: ε_3	9,19E-05	$\kappa=\kappa_z$ [cm ⁻¹]	1,82E-06			II [kN]		7,09E-04					
				s_{-1}	1,00E-08			prov					
								cal					
								Difference:					
								3,08E-05 5,46E-04					
j	i	z_i [cm]	A_i [cm ²]	ε_i [-]	$\varepsilon_{ji}^{(0)}$ [-]	ε_{ji} [-]	σ_{ji} [kN/cm ²]	W_{ji} [kN/cm ²]	F_{ji} [kN/cm ²]	II_{ij} [kN/cm ²]	N_{ji} [kN]	M_{yji} [kNcm]	
1	Concrete	1.0 1)	-60,0	400,0	-1,26E-04	2,00E-04	7,41E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	
		1.11)	-50,0		-1,08E-04	2,00E-04	9,22E-05	0,00E+00	0,00E+00	0,00E+00			
		2.0	-50,0	800,0	-1,08E-04	2,00E-04	9,22E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	
		2.1	-30,0		-7,15E-05	1,40E-04	6,85E-05	0,00E+00	0,00E+00	0,00E+00			
		3.0	-30,0	1200,0	-7,15E-05	1,40E-04	6,85E-05	0,00E+00	0,00E+00	0,00E+00	3,31E-05	-5,84E+00	1,16E+01
		3.1	0,0		-1,70E-05	0,00E+00	-1,70E-05	-4,90E-02	4,16E-07	-2,36E-12			
		4.0	0,0	1200,0	-1,70E-05	0,00E+00	-1,70E-05	-4,90E-02	4,16E-07	-2,36E-12	5,20E-05	-9,17E+00	-2,86E+01
		4.1	30,0		3,75E-05	0,00E+00	3,75E-05	0,00E+00	0,00E+00	0,00E+00			
		5.0	30,0	800,0	3,75E-05	0,00E+00	3,75E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00
		5.1	50,0		7,38E-05	0,00E+00	7,38E-05	0,00E+00	0,00E+00	0,00E+00			
		6.0 2)	50,0	400,0	7,38E-05	0,00E+00	7,38E-05	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00	0,00E+00
		6.1 2)	60,0		9,19E-05	0,00E+00	9,19E-05	0,00E+00	0,00E+00	0,00E+00			
2	Reinforcement 1	-56,0	4,5	-1,19E-04	2,00E-04	8,13E-05	1,63E+00	6,61E-05		2,99E-04	7,36E+00	-4,12E+02	
3	Reinforcement 2	56,0	4,5	8,47E-05	0,00E+00	8,47E-05	1,69E+00	7,17E-05		3,24E-04	7,66E+00	4,29E+02	

The corresponding strains ε_c are shown in Fig. 2c. The behaviour of concrete has a significant influence on the distribution of strains and stresses.

4 CONCLUSIONS

The analysis of strains and stresses can be realised very effectively by the energy method, using an integral formulation of material law. This method is applicable without principal modification for cross-sections with pre-deformations. Strains due to pre-loading, temperature shrinkage etc. are integrated into the model by the decomposition of the total strains of each partition of the composite cross-section.

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