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INFLUENCE OF THE ABSTRACTION LEVEL IN KINEMATICAL MODELS OF FINITE ELEMENT FORMULATIONS

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Abstract. *Nonlinear analyses are characterised by approximations of the fundamental equations in different quality. Starting with a general description of nonlinear finite element formulation the fundamental equations are derived for plane truss elements. Special emphasis is placed on the determination of internal and external system energy as well as influence of different quality approaches for the displacement-strain relationship on solution quality. The different kinematics influence speed of convergence as well as exactness of solution. On a simple truss structure this influence is shown. To assess the quality of different formulations concerning the nonlinear kinematic equation three approaches are discussed. First the overall internal and external energy is compared for different kinematical models. In a second step the energy content related to single terms describing displacement-strain relationship is investigated and used for quality control following two different paths.*

1 INTRODUCTION

Geometrical nonlinear effects in structural analysis are reflected in a nonlinear displacement-strain relationship [1]. To simplify the solution procedure the nonlinear function is expanded into a Taylor series and truncated after the n-th series term. The question arises whether a criterion can be found to evaluate the solution quality related to the chosen approach without knowing the exact result. Based on this knowledge it would be beneficial to find an adaptive scheme that allows to start the calculation with a simple kinematical model and adopt the solution algorithm with increasing nonlinearity.

2 FINITE ELEMENT FORMULATION FOR NONLINEAR PROBLEMS

2.1 Newton Raphson Algorithm

The finite element procedure is derived from the weak equilibrium formulation using the principle of virtual displacement, see equation 1. Here, $\mathbf{t}_0$ denotes the boundary traction and $\delta\mathbf{r}$ represents the virtual displacements for boundary regions with applied traction.

$$-\int_V \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}dV + \int_V \delta\mathbf{v}^T\mathbf{p}dV + \int_S \delta\mathbf{r}^T\mathbf{t}_0 dS = 0 \tag{1}$$

Based on the nodal displacements $\mathbf{u}$ shape functions $\boldsymbol{\Phi}$ are used to calculate the displacements $\mathbf{v} = \boldsymbol{\Phi}\mathbf{u}$. Hence the strain is calculated using $\boldsymbol{\varepsilon} = \mathbf{D}_k\boldsymbol{\Phi}\mathbf{u} = \mathbf{B}\mathbf{u}$. $\mathbf{D}_k$ is the differential operator in the kinematic equation. For the stresses this results in $\boldsymbol{\sigma} = \mathbf{E}\mathbf{B}\mathbf{u}$. The basic equilibrium equation becomes:

$$-\mathbf{u}\underbrace{\int_V \mathbf{B}^T\mathbf{E}\mathbf{B}dV}_{\mathbf{K}} + \underbrace{\int_V \boldsymbol{\Phi}^T\mathbf{p}dV + \int_S \boldsymbol{\Phi}^T\mathbf{R}_r^T\mathbf{t}_0 dS}_{\mathbf{P}} = 0 \quad \text{using } \delta\mathbf{r} = \mathbf{R}_r\delta\mathbf{v} \tag{2}$$

The product $\mathbf{K}\mathbf{u}$ is equivalent to the internal nodal forces $\mathbf{F}$.

Taking geometric nonlinearities into account results in nonlinear relationship between external forces $\mathbf{P}$ and displacements $\mathbf{u}$. To find a solution an incremental-iterative approach is necessary. Starting with a known configuration i where the internal and external nodal forces are in equilibrium a load increment $\Delta\mathbf{P}$ is applied to the structure. Using the stiffness matrix known from the initial state the related displacement increment is calculated, see equation 3.

$$\mathbf{K}(\mathbf{u_i}) \cdot \Delta\mathbf{u} = \Delta\mathbf{P} \qquad \text{i ... No. of increments} \tag{3}$$

This procedure provides a linear approximation for the internal forces $\mathbf{F}_{i+1} = \mathbf{F}_i + \Delta\mathbf{F}$. Equilibrium with the external loads is not achieved. In an iterative procedure this difference called residual forces $\mathbf{R}$ is minimised using equation 4. Hence the unbalanced configuration $i+1$ is used as new initial configuration. Without applying a new load increments but using the residual forces the correct displacement increment is calculated within the iteration.

$$\mathbf{K}(\mathbf{u_i}) \cdot \Delta\mathbf{u} = \mathbf{P}_{i+1} - (\mathbf{F}_i + \Delta\mathbf{F}) = \mathbf{R} \tag{4}$$

The reason for geometric nonlinear behaviour is the nonlinear displacement-strain relationship $\boldsymbol{\varepsilon} = \mathbf{D}_{knl}(\mathbf{u}) \cdot \mathbf{u}$. Since for configuration $i+1$ this is not known it is approximated by a

Taylor series, see equation 5, which is than truncated after the n-th term.

$$\boldsymbol{\varepsilon}(\mathbf{u}_{i+1}) = \boldsymbol{\varepsilon}(\mathbf{u_i}) + \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{u}_{i+1}} \Bigg|_{\mathbf{u}_{i+1}=\mathbf{u}_i} \cdot \Delta\mathbf{u} + \frac{\partial^2 \boldsymbol{\varepsilon}}{\partial \mathbf{u}_{i+1}^2} \Bigg|_{\mathbf{u}_{i+1}=\mathbf{u}_i} \cdot \Delta\mathbf{u}^2 + ... \tag{5}$$

The method described above does not allow to judge on the quality of the chosen approximation for the displacement-strain relationship even if the solution converges. The solution quality can only be validated in comparison with results of more complex models. To overcome this problem the influence of different kinematic formulation is investigated. From equation 4 it becomes clear that the nonlinear kinematics influence the stiffness matrix as well as the residual forces, see chapter 4 for more details. Using better quality approximations for calculating the stiffness matrix improves the speed of convergence. Whereas more precise formulations on the right hand side of equation 4 leads to better results concerning the predicted deformation. In general for both terms equal approaches are used. In our paper we will show the described effects by applying different strain-displacement formulation for the stiffness matrix and the residual force vector.

2.2 Determination of system energy

To solve the nonlinear equation system and to determine the displacements of the structural system the modified Newton-Raphson [2] method is applied. Compared to the standard algorithm the stiffness matrix is not calculated for each iteration step but for the converged solution of a load increment. This lowers the speed of convergence because more iterations are necessary. On the other hand computing time per iteration decreases.

To determine the internal energy of geometric nonlinear structural systems it is sufficient to apply the load in one increment as long as a linear elastic material law is used. As one can see in figure 1, for linear stress-strain relationship the internal energy is independent from the number of increments. To determine the external energy correctly an incremental approach becomes necessary. Hence the increment in potential of external energy is calculated for every load step and summed up. The number of load steps is sufficient if the external energy equals the strain energy, see figure 1.

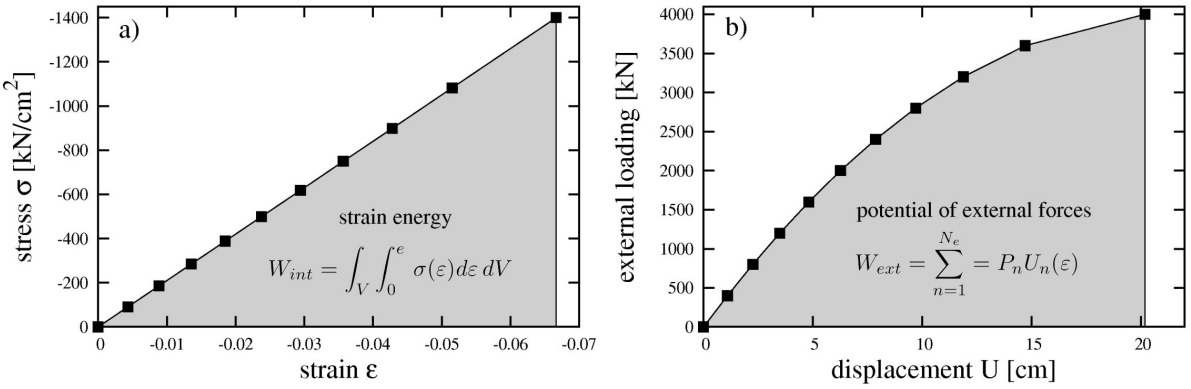
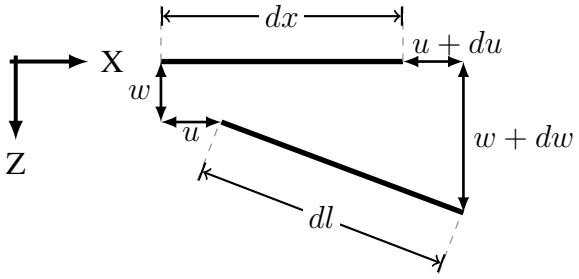


Figure 1: a) Internal and b) External energy of the investigated system (see figure 3), for solution the load is applied in 10 increments

3 NONLINEAR FINITE ELEMENT FORMULATIONS FOR PLANE TRUSSES

3.1 Displacement-strain relationship



$$dl = \sqrt{(dx+du)^2 + dw^2}$$
$$= \sqrt{1 + 2u' + u'^2 + w'^2}\, dx$$
$$\varepsilon = \frac{dl-dx}{dx} = \sqrt{1 + 2u' + u'^2 + w'^2} - 1$$

Figure 2: Nonlinear kinematics of a plane truss

The deformation of a plane truss element resulting from external loading is visualised in figure 2. The element strain for small strain conditions is defined as the ratio of elongation to the original length. Because this expression is to complicated for investigation of complex structures it is expanded into a Taylor series.

$$\varepsilon = \frac{1}{2}(2u' + u'^2 + w'^2) - \frac{1}{8}(2u' + u'^2 + w'^2)^2 + \frac{1}{16}(2u' + u'^2 + w'^2)^3 \ldots \tag{6}$$

In classic approaches this series expansion is truncated after the first series term. In addition the term u'^2 is neglected due to the dominant axial stiffness which normally leads to neglectable u deformations. This results in a simplified description for the strains $\varepsilon = u' + \frac{1}{2}w'^2$, with the first summand representing the linear part.

3.2 Influence of kinematical model on stiffness matrix

For a linear relationship between strains and displacements $\boldsymbol{\varepsilon} = \mathbf{u}'$ the stiffness matrix is defined with $\mathbf{K} = \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}dV$, see section 2.1. Applying the widely used simplified description for nonlinear kinematics $\varepsilon = u' + \frac{1}{2}w'^2$ results in more complicated formulations for the stiffness matrix. Rewriting the equation for the strains in matrix notation $\boldsymbol{\varepsilon} = [\mathbf{D}_k + \frac{1}{2}\mathbf{D}_{knl}(\mathbf{v})]\mathbf{v}$ and incrementing the displacements results in equation 7.

$$\begin{aligned}
\boldsymbol{\varepsilon} &= [\mathbf{D}_k + \frac{1}{2}\mathbf{D}_{knl}(\mathbf{v}_i + \Delta\mathbf{v})](\mathbf{v}_i + \Delta\mathbf{v}) \\
&= [\mathbf{D}_k + \frac{1}{2}\mathbf{D}_{knl}(\mathbf{v}_i)]\mathbf{v}_i + [\mathbf{D}_k + \mathbf{D}_{knl}(\mathbf{v}_i)]\Delta\mathbf{v} + [\frac{1}{2}\mathbf{D}_{knl}(\Delta\mathbf{v})]\Delta\mathbf{v} \\
&\text{with } \mathbf{v} = \boldsymbol{\Phi}\mathbf{u} \text{ and } \mathbf{B} = \mathbf{D}\boldsymbol{\Phi} \\
&= \mathbf{B}_L\mathbf{u_i} + \frac{1}{2}\mathbf{B}_{NL}(\mathbf{u_i})\mathbf{u_i} + \mathbf{B}_L\Delta\mathbf{u} + \mathbf{B}_{NL}(\mathbf{u_i})\Delta\mathbf{u} + \frac{1}{2}\mathbf{B}_{NL}(\Delta\mathbf{u})\Delta\mathbf{u}
\end{aligned} \tag{7}$$

Applying a variation on the displacement parameters and inserting the term for $\boldsymbol{\varepsilon}$ in the

principle of virtual displacement, equation 1, results in:

$$\begin{aligned}\delta\Delta\mathbf{u}^T\Bigg\{ & \int_V \mathbf{N}^T\mathbf{p}_i dV + \int_V \mathbf{N}^T\Delta\mathbf{p}dV \\ & - \int_V [\mathbf{B}_L + \mathbf{B}_{NL}(\mathbf{u}_i)]^T \cdot \mathbf{E} \cdot [\mathbf{B}_L + \frac{1}{2}\mathbf{B}_{NL}(\mathbf{u}_i)] \cdot \mathbf{u}_i dV \\ & - \int_V \mathbf{B}_{NL}^T(\Delta\mathbf{u}) \cdot \mathbf{E} \cdot [\mathbf{B}_L + \frac{1}{2}\mathbf{B}_{NL}(\mathbf{u}_i)] \cdot \mathbf{u}_i) dV \\ & - \int_V [\mathbf{B}_L + \mathbf{B}_{NL}(\mathbf{u}_i)]^T \cdot \mathbf{E} \cdot [\mathbf{B}_L + \mathbf{B}_{NL}(\mathbf{u}_i)] \cdot \Delta\mathbf{u}) dV \Bigg\} = 0\end{aligned} \tag{8}$$

The first and second integral represent the external forces. The third term embodies the internal forces that are already in equilibrium for the initial state. From the last two integrals the different parts of the stiffness matrix can be derived. For plane trusses with linear shape functions the stiffness matrix becomes:

$$\mathbf{K} = \underbrace{\frac{EA}{l}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{K}_E} + \underbrace{\frac{EAw_i'}{l}\begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{K}_{uL}} + \underbrace{\frac{EAw_i'^2}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\mathbf{K}_{uNL}} + \underbrace{\frac{N_i}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\mathbf{K}_{\sigma L}} \tag{9}$$

Taking more series terms for the description of ε into account results in even more complex formulations for the stiffness matrix. Hence more detailed formulation of $\mathbf{K}$ only effect the speed of convergence but not the solution quality, see table 2, additional effort is not reasonable.

3.3 Influence of kinematical model on internal forces

For trusses the member forces depend directly on the chosen ε-formulation. After calculating the additional displacement due to the applied load increment, using equation 4, the overall displacement $\mathbf{u}_{i+1}$ is transferred from the global to the local coordinate system. Afterwards the displacement derivate u' and w' are determined. Depending on the degree of nonlinearity ε is calculated with required accuracy. The axial force for one element results then from $N = EA\varepsilon(u)$. This force needs to be transferred backwards to global coordinate system, see figure 3. Also here the different kinematics needs to be considered, see table 1.

Table 1: Transformation of local internal forces into global forces

ε	Determination of internal nodal forces
u'	The member force N is transformed in internal nodal forces $\mathbf{F}$ using the initial geometry $\mathbf{F} = \mathbf{T}^{-1}N$ with $\mathbf{T} = \begin{bmatrix} cos\alpha_0 & sin\alpha_0 \end{bmatrix}$
$u' + \frac{1}{2}w'^2$	From the third integral in equation 8 the description for the internal nodal forces is derived. Applying the transformation condition results in: $\mathbf{F} = N \begin{bmatrix} -cos\alpha_0 \\ -sin\alpha_0 \\ cos\alpha_0 \\ sin\alpha_0 \end{bmatrix} + Nw' \begin{bmatrix} -sin\alpha_0 \\ cos\alpha_0 \\ sin\alpha_0 \\ -cos\alpha_0 \end{bmatrix}$
$\sqrt{1 + 2u' + u'^2 + w'^2} - 1$	The transformation is achieved using the same equation like for $\varepsilon = u'$. The difference is in the transformation matrix. A geometry update is made for every iteration and the updated angle α_{upd} is used for calculation of internal nodal forces.

4 NUMERICAL EXAMPLE

4.1 System and Solution

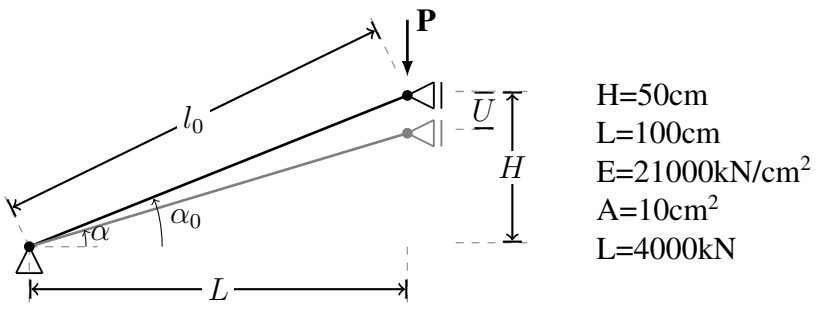


Figure 3: Investigated plane truss system

In figure 3 the investigated plane truss system is shown. Table 2 shows the calculated deflections for different strain approximations. As expected with refined kinematical model the solution quality improves. The same information results from investigating the different energy terms, see figure 4. Additionally the effect described in chapter 2.2 can be seen. To determine the external energy the loading needs to be applied in multiple intervals. The question arises how the solution quality can be assessed by quality measures without necessity to compare different kinematical models.

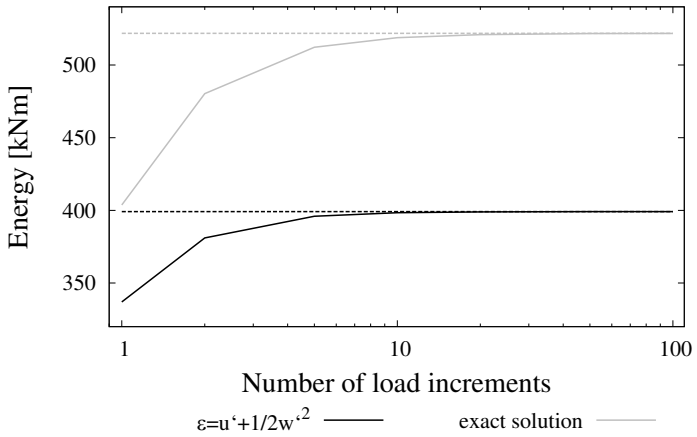


Figure 4: Comparison of strain energy (dashed line) and potential of external forces (continuous line) for different formulations of ε, Energy is summed up over the number of load increments

Table 2: Global deformation of the truss structure using different approaches for ε and number of iteration subject to the displacement-strain formulation used to calculate **K**; No. of load increments: 10

ε	U [cm]	No. of iterations over all load increments for **K** based on	
		$\varepsilon = u'$	$\varepsilon = u' + \frac{1}{2}w'^2$
u'	10,6	-	-
$u' + \frac{1}{2}w'^2$	16,8	179	81
$\sqrt{1 + 2u' + u'^2 + w'^2} - 1$	20,2	317	137

4.2 Criterion 1 - overall system energy

The first approach is to change the solution algorithm with respect to the transformation of internal forces from local to global coordinate system. Different approaches for ε effect the calculation of the residual forces twice. First when the member forces N are calculated and second when they are transferred from local into global coordinates. Using always the exact expression $F = sin(\alpha_{updated}) \cdot N$, this transformation leads to results for the converged solution of external energy as shown in table 3. For the different ε-approximations a difference between the internal and external energy terms can be detected, even for converged solution. This difference is becoming smaller if the kinematic equation is improved. Therewith this energy difference might be taken as a first error measure for the quality of the kinematic formulation.

Table 3: Difference between internal and external energy for different approaches for ε using the exact formulation for coordinate system transformation of internal forces

ε	Internal energy [kNm]	Converged solution for external energy [kNm]	ΔEnergy [%]
u'	373	313	16,1
$u' + \frac{1}{2}w'^2$	482	472	2,1
$\frac{1}{2}(2u' + u'^2 + w'^2)$	532	534	0,4
$-\frac{1}{8}(2u' + u'^2 + w'^2)^2$			
$\sqrt{1 + 2u' + u'^2 + w'^2} - 1$	522	522	0,0

This strategy has several disadvantages. First of all the principle of energy equilibrium, whereon the whole solution scheme is based, is violated. Additionally for simple kinematical models that according to table 2 do not provide a satisfying solution the difference of the two energies is comparably low. The user would expect a much lower difference in the displacement as it really is. At least for this simple example this approach has to be judged critically.

4.3 Criterion 2 - energy related to single epsilon terms

Another option is the evaluation of energy related to single terms of epsilon, see figure 5. For a calculation with exact kinematical model the additional energy per load increment is plotted in percent related to the exact value. It becomes clear that with increasing load the structure behaves more nonlinear. The portion of higher order ε-terms to total energy increases. For the first load increments using only $\varepsilon = \frac{1}{2}(2u' + u'^2 + w'^2)$ would be a good approximation. But starting from the 4th/5th load step the influence of the second series term can no longer be neglected.

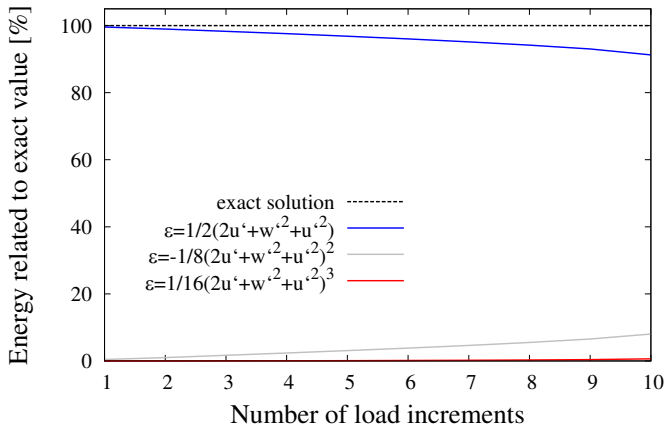


Figure 5: Internal energy related to single terms of epsilon for each load increment compared with exact energy

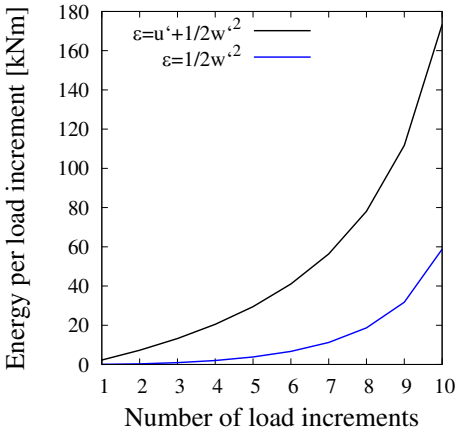


Figure 6: For $\varepsilon = u' + \frac{1}{2}w'^2$ comparison of internal energy content related to nonlinear part to total energy for each load increment

Figure 6 shows the same approach for a simpler approximation of ε. For the first two load steps a linear approximation of the strain-displacement relationship seems to be sufficient. By increasing the load the nonlinear part grows rapidly. The high value of energy related to the term $\varepsilon = \frac{1}{2}w'^2$ for higher load steps might be used as an indicator that even more exact formulations for describing ε needs to be considered.

For this approach it is always necessary to include one simple nonlinear term in addition to the linear ε-formulation to have the opportunity of comparison. Furthermore the question arises which amount of deviation between the energy terms indicates that more series terms needs to be taken into account.

4.4 Criterion 3 - prediction of energy related to single epsilon terms

To overcome the problems of the previous described criterion the calculation is started based on the linear term. After the iteration for one load step is completed the percentage increase of energy related to the next higher series term is calculated. If the simple formulation of ε is sufficient the value should not exceed a predefined limit, for example 1% of the total system

energy. Otherwise for the next load increment a more precise formulation for the displacement-strain relationship is used. This procedure is repeated after each load step. For the simple truss structure using 10 load increments and a limit of 1% results in an adaptive process as shown in figure 7.

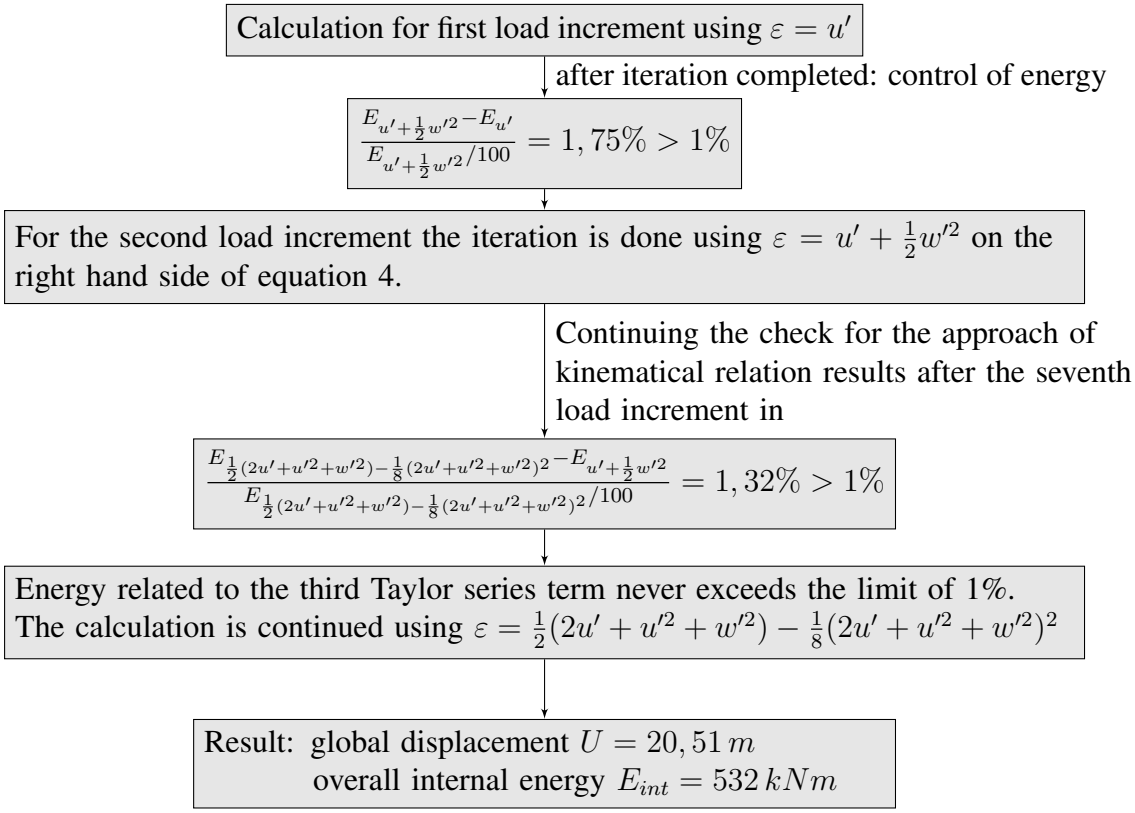


Figure 7: Adaption of kinematical formulation subjected to chancing degree of nonlinearity of the structure

5 PERSPECTIVE

Based on a simple example of a truss structure the idea of model quality assessment using energy terms is presented. Calculation of the overall energy and relations of different energy terms can be used as an indicator for the solution quality. Based on this knowledge an adaptive scheme was used to change the kinematical model depending on increasing nonlinearity of the structure. The solution quality has turned out satisfactory compared to the exact result. More detailed investigations are necessary to find criteria for the threshold values for the iterative process as well as for decision on number and stepsize of incremental load steps.

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