19th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck, T. Lahmer and F. Werner (eds.)
Weimar, Germany, 04–06 July 2012

EXTENDED ISOGEOMETRIC ANALYSIS BASED CRACK IDENTIFICATION APPLYING MULTILEVEL REGULARIZING METHODS

T. Lahmer*, **S. Sh. Ghorashi**

* *Bauhaus University Weimar*
Research Training Group „Modelquality“
Berkaer Str. 9, 99423 Weimar
E-mail: tom.lahmer@uni-weimar.de

Keywords: Crack Identification, XIGA, Inverse Problem, Regularization

Abstract. *Many structures in different engineering applications suffer from cracking. In order to make reliable prognosis about the serviceability of those structures it is of utmost importance to identify cracks as precisely as possible by non-destructive testing. A novel approach (XIGA), which combines the Isogeometric Analysis (IGA) and the Extended Finite Element Method (XFEM) is used for the forward problem, namely the analysis of a cracked material, see [1]. Applying the NURBS (Non-Uniform Rational B-Spline) based approach from IGA together with the XFEM allows to describe effectively arbitrarily shaped cracks and avoids the necessity of remeshing during the crack identification problem. We want to exploit these advantages for the inverse problem of detecting existing cracks by non-destructive testing, see e.g. [2]. The quality of the reconstructed cracks however depends on two major issues, namely the quality of the measured data (measurement error) and the discretization of the crack model. The first one will be taken into account by applying regularizing methods with a posteriori stopping criteria. The second one is critical in the sense that too few degrees of freedom, i.e. the number of control points of the NURBS, do not allow for a precise description of the crack. An increased number of control points, however, increases the number of unknowns in the inverse analysis and intensifies the ill-posedness. The trade-off between accuracy and stability is aimed to be found by applying an inverse multilevel algorithm [3, 4] where the identification is started with short knot vectors which successively will be enlarged during the identification process.*

1 INTRODUCTION

Many materials like concrete or brittle smart materials suffer from cracking caused due to intensive dynamic loadings. Cracked piezoelectric specimens have e.g. been investigated by X-Ray inspection, see Figure 1. Due to the very thin and often sharp forms of the cracks, discrete

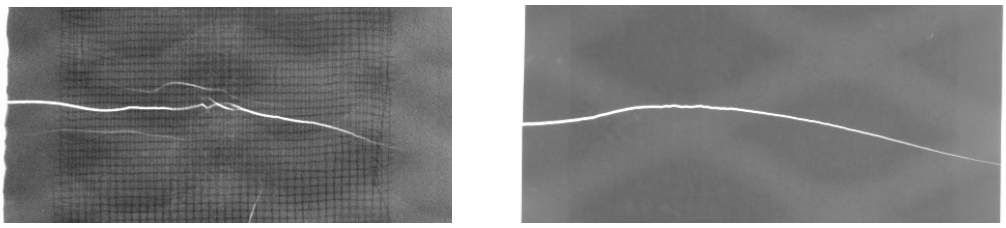

Figure 1: Cracks inside anisotropic smart materials

crack models are advantagueous to smeared crack approaches. Often, the cracks result from some extensional and rare loads, e.g. impacts due to accidents or on buildings after earthquakes while thinking of civil engineering objects. Now after a structure or specimen has been damaged, it is of high importance to know the degree of the damage (e.g. the length and shape of a crack) to derive whether the structure still has full capacity or not. By non-destructive testing, e.g. applying moderate loads, one can compare the behavior of the damaged structure to the behaviour of an undamaged one in means of displacements, accelerations, modal values and draw conclusions about the severity of the damage.

Previous works on identifying discrete cracks are e.g. reported in [2] where the authors use an XFEM approach and apply genetic algorithms to detect the cracks from some measurements on the boundaries of the specimens where the cracks are assumed to be straight lines. This may in many cases be a sufficient approximation. In some cases however, see e.g. Figure 1, the cracks exhibit some curved forms.

The aim of this work is to develop a strategy to uniquely identify cracks of arbitrary shape. In this work we propose an efficient strategy which shall allow to identify cracks without branching. The model itself consits of an extended isogeometric analysis which allows to model moving cracks without the need of remeshing.
For the identification one has to consider the ill-posedness of the problem, which will be done by applying regularizing iterative methods in a multilevel setting.

2 XIGA ANALYSIS

Recently, two papers appeared which treat the combination of the Extended Finite Element Method with Isogeometric Analysis [1, 5]. In these works advantages of the XFEM and IGA approaches to a so called Extended Isogeometric Analysis (XIGA) are brought together. While XFEM is capable of describing discontinuities by enriching the shape functions along the crack by the Heaviside function and analytical solutions around the crack tip, the IGA allows for a very flexible geometrical description of both, the computational domain and the crack [6]. Further, the IGA has been shown to be superior to classsical FEM in terms of robustness and accuracy for may problems.

The approximated displacement field u^h in case of an XIGA is given by

$$
\begin{aligned}
u^h(\xi^1,\xi^2) &= \sum_{i=1}^{n_{en}} R_i^{p_s,q_s}(\xi^1,\xi^2)u_i + \sum_{j=1}^{n_H} R_j^{p_s,q_s}(\xi^1,\xi^2)Ha_j \\
&+ \sum_{k=1}^{n_Q} R_k^{p_s,q_s}(\xi^1,\xi^2)\sum_{\alpha=1}^{4} Q_\alpha b_k^\alpha,
\end{aligned} \tag{1}
$$

where R are the spline based ansatz functions of order p_s and q_s and (ξ^1, ξ^2) coordinates in parametric space. With $n_{en},\ n_H,\ n_Q$ the numbers of non-zero basis functions, the number of n_{en} basis functions that have a crack face (but no crack tip) and the number of n_{en} basis functions around the crack tip are denoted, respectively.
The H - Heaviside function is a common tool in XFEM analysis, taking values $+1$ if the point under consideration is above the crack and -1 in the opposite case. For the crack tip enrichment functions the following basis is appropriate

$$
Q(r,\theta) = \{Q_1,Q_2,Q_3,Q_4\} = \left\{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\theta\sin\frac{\theta}{2}, \sqrt{r}\sin\theta\cos\frac{\theta}{2}\right\},
$$

where polar coordinates are considered with $r = \sqrt{x_1^2+x_2^2}$ being the distance to the crack and $\theta = \arctan(x_2/x_1)$, see Figure 2.

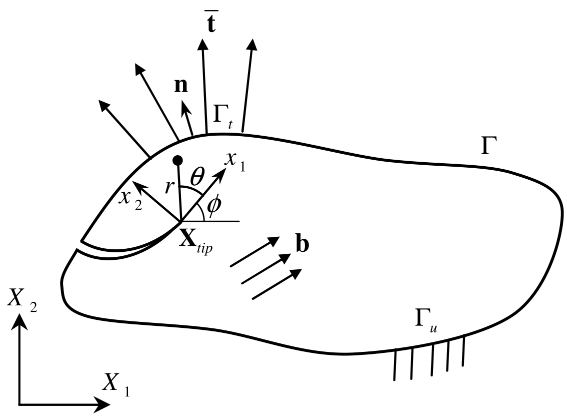

Figure 2: A two dimensional medium with a crack, previously published in [1].

The NURBS - functions $\{R_i^p\}$ are defined as

$$
R_i^p(\xi) = \frac{N_{i,p}(\xi)w_i}{\sum_{ii=1}^{n} N_{ii,p}(\xi)w_{ii}} \tag{2}
$$

with the recursive definition of the B-spline basis functions

$$
N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi \le \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{3}
$$

and

$$
N_{i,p}(\xi) = \frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi) + \frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi), \quad \text{for } p = 1,2,3,.... \tag{4}
$$

For more details we refer to [1]. The physical model is a two-dimensional elastostatic problem derived from the following equilibrium equations

$$\mathcal{B}^T\sigma + b = 0 \quad \text{in } \Omega,$$

together with natural boundary conditions

$$\sigma n = \overline{T}, \quad \text{on } \Gamma_t$$

and essential boundary conditions

$$u = \overline{u}, \quad \text{on } \Gamma_u$$

compare with Fig. 2
A point on a crack is modelled with the help of NURBS curves C. Those are defined by a set of N control points $\pi_i := (X_{i,1}, X_{i,2})$, i.e.

$$C(\xi) = \sum_{i=1}^{N} R_i(\xi)(\pi_i)$$

The vector of control points handling the geometrical description of the crack is denoted by $p_N := (\pi_1, ..., \pi_i, ...\pi_N)$. This vector will the sought-for quantity in the inverse crack analysis.

3 CRACK IDENTIFICATION

In the sequel it will be assumed that some structure suffered from cracking due to an extreme load. The task is now, to identify the position and form of the crack, i.e. the exact determination of the entries of the set of control points p_N of the crack by measurements on the boundary of the structure which is assumed to be stressed by moderate loads which do not cause the crack further to grow. We define the following forward operator which maps the control points to displacements on the boundary of the structure

$$F : X \rightarrow Y \tag{5}$$

$$p_N \mapsto (u_1, u_2)_{|\Gamma}. \tag{6}$$

Assuming that y^δ contains measured data with data error level δ, the inverse problems corresponds to solving for p_N in

$$F(p_N) = y^\delta. \tag{7}$$

In (5) X denotes the parameter space, i.e. the space of all possible crack forms and Y the space of measurements (here displacements) which are assumed to in $L^2(\Omega)$. In the sequel, the forward operator or parameter-to-solution map F is assumed to be continuous, differentiable and its Frechét derivative F' to be Lipschitz continuous and normalized such that $||F'(p)|| \leq 1, \ \forall\, p \in D(F)$. For the measured data we further assume, that the following condition holds

$$||y - y^\delta|| \leq \delta,$$

where y are exact data and δ is an upper bound of the data error, assumed to be estimable.
Solving equations of type (7) is not only an inverse but also an ill-posed problem, as existence and uniqueness of the solution are not guaranteed. Additionally, high instability may be introduced due to errors in the data, i.e. small perturbations in the data lead to large perturbations and oscillations in the solution.

This requires the three strategies:

- As existence is not guaranteed, a solutions in the sense of „least-squares" is sought-for,
- As uniqueness is not guaranteed, a close initial guess of the solution needs to be provided,
- As stability is not guaranteed, regularizing methods have to be applied.

3.1 Identification procedure and regularization

In order to solve (7) iterative regularizing methods will be applied. The regularization follows two strategies

1. Regularization by discretization, see e.g. [7, 8]
2. Regularization by early stopping (a posteriori discrepancy principles), see e.g. [8, 9].

Now, for the solution of (7) modified Landweber Methods are run on finite dimensional subspaces X_N of X, i.e. $X_N := Proj_N X$ of X, where $Proj_N$ denotes an orthogonal projection, i.e.

$$X_N \subseteq X, \quad \text{with} \quad X_0 \subseteq X_1 \subseteq ... \subseteq X_N \quad \text{and} \quad \overline{\bigcup_{N \in \mathbb{N}} X_N} = X. \tag{8}$$

Let the union $\cup_{N\in\mathbb{N}} X_N$ be dense in X. The initial guess is denoted by p_0^0 and the first iterate at level N by $p_N^{0,\delta}$, $p^\dagger$ denotes the exact solution. The following local condition is assumed to hold on each discrete subspace X_N

$$||F(p) - F(Proj_N p^\dagger) - F'(p)(p - Proj_N p^\dagger)|| \leq \eta_N ||F(p) - F(Proj_N p^\dagger)|| \tag{9}$$
$$\text{for all } p \in X_N \cap \mathcal{B}_{\rho/2}(Proj_N p^\dagger) \subseteq D(F) \text{ with } \eta_N \leq \frac{1}{4}.$$

Iterations on every level are defined as follows

$$p_N^{k+1,\delta} = p_N^{k,\delta} + \omega_N^{k,\delta} s_N^{k,\delta}, \qquad s_N^{k,\delta} := Proj_N F'(p_N^{k,\delta})^*(y^\delta - F(p_N^k)) \tag{10}$$

where the step-size $\omega_N^{k,\delta}$ is chosen as

$$\omega_N^{k,\delta} := \frac{||y^\delta - F(p_N^{k,\delta})||^2}{||s_N^{k,\delta}||^2}, \tag{11}$$

which renders the discrete classical Landweber iteration a discrete version of the minimal error method, see also [10, 11, 12]. The method can be regarded as a gradient-type iteration with optimal step-length choice. Convergence results and regularizing properties are worked out in [4, 13] under the given conditions, where an application of the same strategy to recover material nonlinearities approximated by cubic splines is reported. Instead of the minimal error update $s_N^{k,\delta}$ an update according to quasi Newton - Methods (Broyden, BFGS) is easily implemented (as we work in finite dimensional spaces). A convergence theory for Broyden's method for ill-posed problems is given in [8], for the BFGS method corresponding results are not known to the author.

The multilevel algorithm is now implemented as follows

1. Start the iterations (10) with an initial guess on the coarsest level (e.g. 2 control points for a straight line approximation of the crack).

2. Run the iterations until an inner discrepancy principle of the type

$$||F(p_N^{k+1,\delta}) - y^\delta|| \leq \tilde{C}_1(\delta + ||(Proj_N - I)p^\dagger||) \tag{12}$$

   becomes active or until a maximal number of iterations is reached.

3. Refine, i.e. include additional control points for the crack description and switch to the next finer level.

4. Use the last iterate from the preceding level as initial value, i.e. $p_{N+1}^{0,\delta} := p_N^{k_*(N,\delta),\delta}$.

5. Continue the iterations (10) on the new level.

6. Stop, if on level N the outer discrepancy principle

$$||y^\delta - F(p_N^{k_*(N,\delta),\delta})|| \leq C\delta \leq ||y^\delta - F(p_N^{k,\delta})||, \quad C > 2\frac{1+\eta_N}{1-2\eta_N} > 2.$$

   tells to stop or when a maximal number of iterations is reached.

The inner discrepancy principle (12) contains both, an estimate on the data error δ and an estimate of the approximation error as we work on finite dimensional subspace of the true crack forms. For details, see [10, 13].
Quantities like the number of control points, the degree of the splines and measures on the curvature of $p^\dagger$ allow to estimate the term $||(Proj_N - I)p^\dagger||$.
Additional refinements are possible, e.g. the refinement of the number of control points used for the XIGA analysis and increase in the spline orders p_s, q_s.
Remark: The assumed error in the data are genereally not only due to measurement errors but also due to shortcomings of the model. However, in the language of inverse problem theory both measurement and model error are simultaneously estimated by the data error δ.

4 NUMERICAL RESULTS

First numerical results are presented in the following. The shape and loading of the double cantilever beam is according to Fig. 3, where a plane stress state is considered with elasticity modulus $E = 3.0 \times 10^7 \mathrm{N/mm}^2$ and Possion's ratio $\nu = 0.3$. Vertical tensile loads are applied for $X = 0$, $X = \frac{1}{4}L$ and $X = \frac{1}{2}L$. Derivatives of the forward operator with respect to changes in the input parameters (the coordinates of the control points) are computed numerically. A derivation of the linearization and the adjoint operator of it, is work in progress, with which the parameter-updates s_N^k can be implemented more efficiently. Nevertheless, the numerical approach already helps to get an understanding of model-based reconstruction of crack having arbirtrary shapes. For the inner iterations a maximal number of 10 is prescribed. After four refinements, the algorithm stops in case of no data noise, else according to the discrepancy principle. The initially assumed crack, the exact crack and its reconstruction are shown in Figure 4. Figure 5 replots Figure 4 under different scaling together with intermediate solutions at the switch of a coearser to the next finer level.
The following table summarizes error norms ($l2$) for the reconstructed crack and the corresponding residuals for different data noise levels δ.

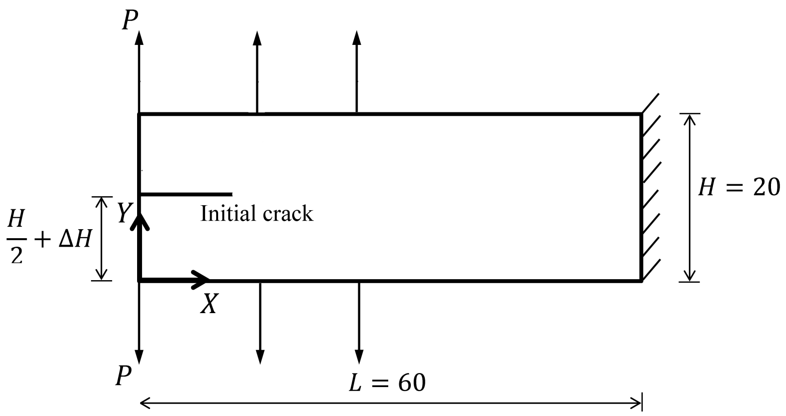

Figure 3: Geometry and loading of the double cantilever beam, previously published in [1].

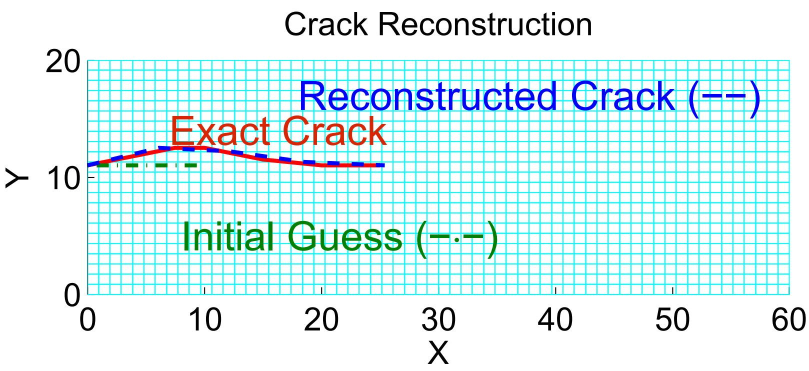

Figure 4: Initial, Exact and Reconstructed Crack

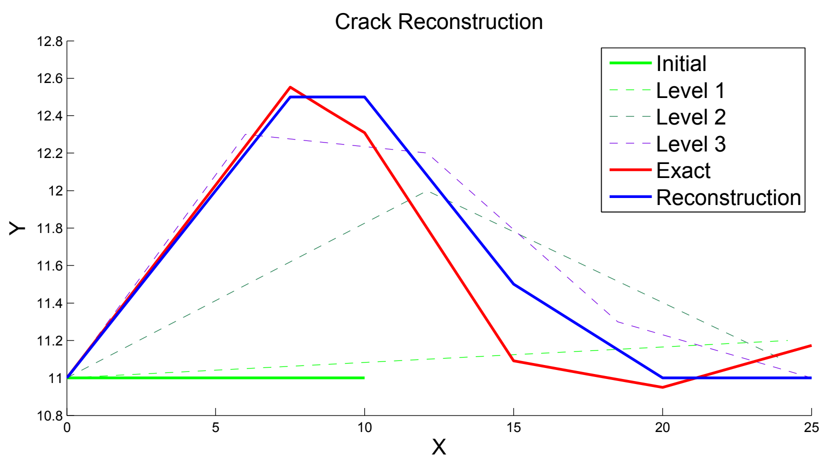

Figure 5: Same as Figure 4 with reconstructed crack a level 4 with intermediate solutions at the end of the iterations on levels 1, 2, 3.

δ in %	$\|\|p_N^k - Proj_N p^\dagger\|\|$	$\|\|F(p_N^k) - y^\delta\|\|$
0	0.49	0.00075
0.5	0.51	0.00114
1	0.53	0.0015
2	0.62	0.0018
5	1.1	0.0024

Table 1: Errors in parameter and image space between reconstructed and exact solution for different noise level terms

5 SUMMARY, CONCLUSIONS AND OUTLOOK

The combination of the Isogeometric Analysis with the Extended Finite Element Method provides a tool which is extremely useful for inverse crack analysis. The flexibility in geometric descriptions of crack and computational domain and the missing need of remeshing make it a very efficient tool. From the numerical examples, instability during the reconstruction of crack is a critical issue. More pronounced seems however the question of unique identifiability. Analytic answers to this are of utmost importance, in particular for designing the experiment properly. Application of BFGS Newton's method to ill-posed problems needs to be analysed.

6 ACKNOWLEDGMENT

This research is supported by the German Research Institute (DFG) via Research Training Group "Evaluation of Coupled Numerical Partial Models in Structural Engineering (GRK 1462)", which is gratefully acknowledged by the authors.

REFERENCES

[1] S. Sh. Ghorashi, N. Valizadeh, and S. Mohammadi. Extended isogeometric analysis for simulation of stationary and propagating cracks. *Int. J. Numer. Meth. Engng*, 89:1069–1101, 2012.

[2] D. Rabinovich, D. Givoli, and S. Vigdergauz. XFEM-based crack detection scheme using a genetic algorithm. *International Journal for Numerical Methods In Engineering*, (71):1051–1080, 2007.

[3] O. Scherzer. An iterative multi level algorithm for solving nonlinear illposed problems. *Numerische Mathematik*, 80:579–600, 1998. 10.1007/s002110050379.

[4] T. Lahmer. Modified landweber iterations in a multilevel algorithm applied to inverse problems in piezoelectricity. *Journal of Inverse and Ill-Posed Problems*, 17(1), 2009.

[5] E. De Luycker, D. J. Benson, T. Belytschko, Y. Bazilevs, and M. C. Hsu. X-fem in isogeometric analysis for linear fracture mechanics. *International Journal for Numerical Methods in Engineering*, 87(6):541–565, 2011.

[6] S. Sh. Ghorashi, N. Valizadeh, S. Mohammadi, and T. Rabczuk. Extended isogeometric analyis of plates with curved cracks. In *The Eighth International Conference on Engineering Computational Technology, ECT2012, Dubrovnik, Croatioa, 2012*.

[7] B. Kaltenbacher. Regularization by projection with a posteriori discretization level choice for linear and nonlinear ill-posed problems. *Inverse Problems*, 16:1523–1539, 2000. DOI:10.1088/0266-5611/16/5/322.

[8] B. Kaltenbacher, A. Neubauer, and O. Scherzer. *Iterative Regularization Methods for Nonlinear Ill-Posed Problems*. Radon Series on Computational and Applied Mathematics 6. de Gruyter, Berlin - New York, May 2008.

[9] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*. Kluwer, Dordrecht, 1996.

[10] O. Scherzer. An iterative multi level algorithm for solving nonlinear ill-posed problems. *Numer. Math.*, pages :579–600, 1998.

[11] A. Neubauer and O. Scherzer. A convergence rate result for a steepest descent method and a minimal error method for the solution of nonlinear ill-posed problems. *Z. Anal. Anwend.*, 14:369–377, 1995.

[12] O. Scherzer. A convergence analysis of a method of steepest descent and a two-step algorithm for nonlinear ill-posed problems. *Numer. Funct. Anal. and Optimiz.*, pages 197–214, 1996.

[13] T. Lahmer. *Forward and Inverse Problems in Piezoelectricity*. PhD thesis, Friedrich-Alexander University Erlangen-Nuremberg, Germany, 2008.