Hyperbolic Laplace Operator and the Weinstein Equation in $\mathbb{R}^3$

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Abstract

We study the Weinstein equation

$$\Delta u - \frac{k}{x_2}\frac{\partial u}{\partial x_n} + \frac{l}{x_2^2}u = 0,$$

on the upper half space $\mathbb{R}^3_+ = \left\{(x_0, x_1, x_2) \in \mathbb{R}^3\right\}$ for $4l \leq (k+1)^2$. If $l = 0$, the operator $x_2^{2k}\left(\Delta u - \frac{k}{x_2}\frac{\partial u}{\partial x_2}\right)$ is the Laplace-Beltrami operator with respect to the Riemannian metric $ds^2 = x_2^{-2k}\left(\sum_{i=0}^{2} dx_i^2\right)$. In case $k = 1$ the Riemannian metric is the hyperbolic distance of Poincaré upper half space. The Weinstein equation is connected to the axially symmetric potentials. We compute solutions of the Weinstein equation depending on the hyperbolic distance and x_2. These results imply the explicit mean value properties. We also compute the fundamental solution. The main tools are the hyperbolic metric and its invariance properties.

1 Introduction

Weinstein introduced axially symmetric potential theory in [12]. The idea was to consider the following simple elliptic differential equation with variable coefficients in the neighborhood of the singular plane $x_n = 0$

$$x_n \triangle h + p\frac{\partial h}{\partial x_n} = 0,$$

where as usual

$$\triangle h = \frac{\partial^2 h}{\partial x_0^2} + ... + \frac{\partial^2 h}{\partial x_n^2}.$$

Note that if p is an integer then a axially symmetric harmonic function in $p+2$-dimensional space satisfies the preceding equation in the meridian plane (see for example [9])

We consider the solutions of the generalized Weinstein equation

$$x_2^2 \triangle h - kx_2\frac{\partial h}{\partial x_2} + lh = 0 \tag{1}$$

in an open domain whose closer is contained in the upper half space

$$\mathbb{R}^3_+ = \{(x_0, x_1, x_2) \mid x_0, x_1, x_2 \in \mathbb{R},\ x_2 > 0\}.$$

Our general technical assumption is that the constants $l, k \in \mathbb{R}$ satisfy $4l \leq (k+1)^2$. This equation has been researched for example by Leutwiler and Akin in [10] and in [1]. We transfer solutions of this equation to solutions of Laplace-Beltrami equation of the hyperbolic metric in the Poincaré upper half space. In the main result, we present the fundamental solution of the equation (1) in terms of the hyperbolic distance function.

We recall that the operator

$$\Delta_h f = x_2^2 \Delta f - x_2\frac{\partial f}{\partial x_2}$$

is the hyperbolic Laplace-Beltrami operator with respect to the hyperbolic Riemannian metric

$$ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2}{x_2^2}$$

in the Poincaré upper half space model.

The hyperbolic distance may be computed as follows (see the proof for example in [11]).

Lemma 1 *The hyperbolic distance $d_h(x,a)$ between the points $x = (x_0, x_1, x_2)$ and $a = (a_0, a_1, a_2)$ in $\mathbb{R}^3_+$ is*

$$d_h(x,a) = \operatorname{arcosh} \lambda(x,a),$$

where

$$\lambda(x,a) = \frac{(x_0-a_0)^2 + (x_1-a_1)^2 + x_2^2 + a_2^2}{2x_2a_2} = \frac{|x-a|^2}{2x_2a_2} + 1$$

and $|x-a|$ is the usual Euclidean distance between the points a and x.

We also apply the simple calculation rules of the hyperbolic distance stated next.

Lemma 2 *If $x = (x_0, x_1, x_2)$ and $a = (a_0, a_1, a_2)$ are points in $\mathbb{R}^3_+$ then*

$$|x - a|^2 = 2x_2 a_2 \left(\lambda(x, a) - 1\right), \tag{2}$$

$$|x - \hat{a}|^2 = 2x_2 a_2 \left(\lambda(x, a) + 1\right), \tag{3}$$

$$\frac{|x - a|^2}{|x - \hat{a}|^2} = \frac{\lambda(x, a) - 1}{\lambda(x, a) + 1} = \tanh^2\left(\frac{d_h(x, a)}{2}\right), \tag{4}$$

where $\widehat{a} = (a_0, a_1, -a_2)$.

We also note the relation between the Euclidean and hyperbolic balls.

Proposition 3 *The hyperbolic ball $B_h(a, r_h)$ with the center $a = (a_0, a_1, a_2)$ and the radius r_h is the same as the Euclidean ball with the Euclidean center $(a_0, a_1, a_2 \cosh r_h)$ and the Euclidean radius $r_e = a_2 \sinh r_h$.*

2 The hyperbolic Laplace operator depending on the hyperbolic distance in $\mathbb{R}^3_+$

We need the computations of the hyperbolic Laplace operator of functions depending on λ, computed in [2].

Lemma 4 *If f is twice continuously differentiable depending only on $\lambda = \lambda(x, e_n)$ then*

$$\triangle_h f(x) = \left(\lambda^2 - 1\right) \frac{\partial^2 f}{\partial \lambda^2} + 3\lambda \frac{\partial f}{\partial \lambda}.$$

Using this it is relatively easy to compute the result.

Lemma 5 *If f is twice continuously differentiable depending only on $r_h = d_h(x, e_n)$ then the hyperbolic Laplace in $\mathbb{R}^3_+$ is given by*

$$\triangle_h f(r_h) = \frac{\partial^2 f}{\partial r_h^2} + 2 \coth r_h \frac{\partial f}{\partial r_h}.$$

Proof. Using $r_h = \operatorname{arcosh} \lambda(x, e_n)$, we compute

$$\frac{\partial r_h}{\partial \lambda} = \frac{1}{\sinh r_h}$$

and

$$\frac{\partial^2 r_h}{\partial \lambda^2} = -\frac{\cosh r_h}{\sinh^3 r_h}.$$

Hence applying the chain rule we obtain

$$\begin{aligned}
\frac{\partial f(r_h)}{\partial \lambda} &= \frac{\partial f}{\partial r_h}\frac{\partial r_h}{\partial \lambda} = \frac{\partial f}{\partial r_h}\frac{1}{\sinh r_h}, \\
\frac{\partial^2 f(\lambda)}{\partial \lambda^2} &= \frac{\partial^2 f}{\partial r_h^2}\left(\frac{\partial r_h}{\partial \lambda}\right)^2 + \frac{\partial f}{\partial r_h}\frac{\partial^2 r_h}{\partial \lambda^2} \\
&= \frac{\partial^2 f}{\partial r_h^2}\frac{1}{\sinh^2 r_h} - \frac{\partial f}{\partial r_h}\frac{\cosh r_h}{\sinh^3 r_h},
\end{aligned}$$

completing the proof by the preceding lemma. ■

If we know one strictly positive solution depending on r_h, we may compute all the solutions depending on r_h.

Theorem 6 *If μ is a strictly positive solution of the equation*

$$\triangle_h f + \gamma f = \frac{\partial^2 f}{\partial r_h^2} + 2\frac{\partial f}{\partial r_h}\frac{\cosh r_h}{\sinh r_h} + \gamma f = 0 \tag{5}$$

depending on $r_h = d_h(x, e_n)$ then the general solution of this equation is

$$f(r_h) = \left(C\int_{r_0}^{r_h} \sinh^{-2} u \mu^{-2}(u)\, du + C_0\right)\mu(r_h)$$

for some real constants C and C_0.

Proof. Assume that $\mu(r_h)$ is a particular positive solution of (5). Setting $f(r_h) = g(r_h)\mu(r_h)$ we obtain

$$\begin{aligned}
0 &= \mu\frac{d^2 g}{dr_h^2} + 2\frac{d\mu}{dr_h}\frac{dg}{dr_h} + g\frac{d^2\mu}{dr_h^2} \\
&\quad + 2\frac{\cosh r_h}{\sinh r_h} g\frac{d\mu}{dr_h} + 2\frac{\cosh r_h}{\sinh r_h}\mu\frac{dg}{dr_h} + \gamma\mu g \\
&= \mu\frac{d^2 g}{dr_h^2} + 2\frac{d\mu}{dr_h}\frac{dg}{dr_h} + 2\frac{\cosh r_h}{\sinh r_h}\mu\frac{dg}{dr_h}.
\end{aligned}$$

Denoting $\frac{dg}{dr_h} = h$, we deduce

$$\mu\frac{dh}{dr_h} + \left(2\frac{d\mu}{dr_h} + 2\frac{\cosh r_h}{\sinh r_h}\mu\right)h = 0.$$

Hence we solve

$$\frac{d}{dr_h}\left(\log h + 2\log\mu + 2\log(\sinh r_h)\right) = 0$$

and therefore

$$\frac{\partial g}{\partial r_h} = h = C\sinh^{-2} r_h \mu^{-2}(r_h).$$

Consequently, the general solution is$\frac{n^2-(k+1)^2}{4}$

$$f\left(r_h\right)=\left(C\int_{r_0}^{r_h}\sinh^{-2}u\mu^{-2}\left(u\right)du+C_0\right)\mu\left(r_h\right).$$

■

We recall the relation between solutions of (1) and eigenfunctions of the hyperbolic Laplace-Beltrami operator.

Proposition 7 ([10]) *Let $\Omega\subset\mathbb{R}^3_+$ be an open subset Ω of $\mathbb{R}^3_+$. If u is a solution of (1) in Ω, then $f(x)=x_2^{\frac{1-k}{2}}u(x)$ is an eigenfunction of the hyperbolic Laplace operator corresponding to the eigenvalue $\frac{1}{4}\left((k+1)^2-4l-4\right)$. Conversely, if f is the an eigenfunction of the hyperbolic Laplace operator corresponding to the eigenvalue γ in Ω then $u(x)=x_2^{\frac{k-1}{2}}f(x)$ is the solution of the equation (1) in Ω with $l=\frac{1}{4}\left((k+1)^2-4\gamma-4\right)$.*

The mean value property for the solutions of (1) can be stated in terms of the hypergeometric functions. We recall their definition:

$${}_2F_1\left(a,b;c;x\right)=\sum_{m=0}^{\infty}\frac{\left(a\right)_m\left(b\right)_m}{\left(c\right)_m}\frac{x^m}{m!},$$

where $\left(a\right)_m=a\left(a+1\right)...\left(a+m-1\right)$ and $\left(a\right)_0=1$. This series converges for x satisfying $|x|<1$. We recall also an important Euler's integral formula valid for $a,b,c\in\mathbb{C}$ satisfying $0<\operatorname{Re}b<\operatorname{Re}c$

$${}_2F_1(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt.$$

where the Beta funtion has the representation

$$B\left(x,y\right)=\frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x+y\right)}.$$

The mean value property for solutions of (1) with respect to the hyperbolic surface measure was proved in [8].

Theorem 8 *Let l and k be real numbers satisfying $4l\leq\left(k+1\right)^2$ and $U\subset\mathbb{R}^3_+$ be open. If*

$$\psi_{2,k,l}(r_h)=e^{-\frac{1+\sqrt{(k+1)^2-4l}}{2}r_h}{}_2F_1(1+\frac{\sqrt{(k+1)^2-4l}}{2},1;2;1-e^{-2r_h})$$

then $\psi_{2,k,l}(r_h)$ is an eigenfunction of the hyperbolic Laplace operator corresponding to the eigenvalue $\frac{1}{4}\left((k+1)^2-4l-4\right)$. Moreover, if $u:U\to\mathbb{R}$ is a solution of the Weinstein equation

$$\Delta u-\frac{k}{x_2}\frac{\partial u}{\partial x_2}+\frac{l}{x_2^2}u=0$$

in U *then*

$$u(a)=\frac{a_2^{\frac{k-1}{2}}}{4\pi\sinh^2(r_h)\psi_{2,k,l}(r_h)}\int_{\partial B_h(a,r_h)}u(x)\frac{d\sigma}{x_2^{\frac{3+k}{2}}}$$

for all hyperbolic balls satisfying $\overline{B_h(a,r_h)}\subset U$.

In our special case $\mathbb{R}^3_+$, we can give a simple formula for the function $\psi_{2,k,l}(r_h)$ as follows.

Theorem 9 *Let* l *and* k *be real numbers satisfying* $4l\leq(k+1)^2$ *and* $U\subset\mathbb{R}^3_+$ *be open. Denote*

$$a=1+\frac{\sqrt{(k+1)^2-4l}}{2}$$

and $r_h=d_h(x,e_n)$. *Then*

$$\begin{aligned}\psi_{2,k,l}(r_h)&=e^{-ar_h}{}_2F_1(a,1;2;1-e^{-2r_h})\\&=\begin{cases}\frac{\sinh(r_h(a-1))}{(a-1)\sinh r_h}, & \text{if } 4l\neq(k+1)^2,\\ \frac{r_h}{\sinh r_h}, & \text{if } 4l=(k+1)^2,\end{cases}\end{aligned}$$

is the eigenfunction of the hyperbolic Laplace operator corresponding to the eigenvalue $\frac{1}{4}\left((k+1)^2-4l-4\right)$.

For the sake of completeness, we first prove the lemma.

Lemma 10 *If* $|x|<1$ *then*

$${}_2F_1(a,1;2;x)=\begin{cases}\frac{1-(1-x)^{-a+1}}{x(-a+1)} & \text{if } a\neq1,\\ -\frac{\log(1-x)}{x} & \text{if } a=1.\end{cases}$$

Proof. If we replace t with $1-s$ in Euler's integral we obtain

$$\begin{aligned}{}_2F_1(a,b;c;x)&=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1(1-x+xs)^{-a}(1-s)^{b-1}s^{c-b-1}ds\\&=\frac{(1-x)^{-a}\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\left(1+\frac{xs}{1-x}\right)^{-a}(1-s)^{b-1}s^{c-b-1}ds.\end{aligned}$$

In case $a\neq1$ we infer

$$\begin{aligned}{}_2F_1(a,1;2;x)&=(1-x)^{-a}\int_0^1\left(1+\frac{xs}{1-x}\right)^{-a}dt\\&=(1-x)^{-a}\frac{1-x}{x(-a+1)}\left(\left(1+\frac{x}{1-x}\right)^{-a+1}-1\right)\\&=(1-x)^{-a}\frac{x-1}{x(-a+1)}\left(\frac{1}{(1-x)^{-a+1}}-1\right)\\&=\frac{1-(1-x)^{-a+1}}{x(-a+1)}.\end{aligned}$$

If $a = 1$ we compute

$$
\begin{aligned}
{}_2F_1(a,1;2;x) &= (1-x)^{-a}\int_0^1 \left(1+\frac{xs}{1-x}\right)^{-1} dt \\
&= (1-x)^{-1}\frac{1-x}{x}\log\left(1+\frac{x}{1-x}\right) \\
&= -\frac{\log(1-x)}{x}.
\end{aligned}
$$

■

We are ready to verify the preceding theorem.

Proof. Setting $a = 1 + \frac{\sqrt{(k+1)^2-4\ell}}{2} \neq 1$ we obtain

$$
\begin{aligned}
\psi_{2,k,l}(r_h) &= e^{-ar_h}{}_2F_1(1+\frac{\sqrt{(k+1)^2-4\ell}}{2},1;2;1-e^{-2r_h}) \\
&= e^{-ar_h}\left(\frac{e^{2r_h(a-1)}-1}{(1-e^{-2r_h})(a-1)}\right) \\
&= e^{-ar_h}e^{r_h(a-1)}e^{r_h}\frac{\frac{e^{r_h(a-1)}-e^{-r_h(a-1)}}{2}}{\frac{(e^{r_h}-e^{-r_h})}{2}(a-1)} \\
&= \frac{\sinh(r_h(a-1))}{(a-1)\sinh r_h}.
\end{aligned}
$$

If $a = 1$ then

$$
\psi_{2,k,l}(r_h) = -e^{-r_h}\frac{\log e^{-2r_h}}{(1-e^{-2r_h})} = \frac{2r_h}{(e^{r_h}-e^{-r_h})} = \frac{r_h}{\sinh r_h}.
$$

Note also that

$$
\lim_{r_h\to 0}\psi_{2,k,l}(r_h) = 1
$$

and with this extension $\psi_{2,k,l}$ is a continuously differential function. ■

Substituting the values of $\psi_{2,k,l}$ for the mean value theorem we immediately obtain the result.

Theorem 11 *Let k be a real number and $U \subset \mathbb{R}^3_+$ be open. If $u : U \to \mathbb{R}$ is a solution of the Weinstein equation*

$$
x_2^2\Delta u - kx_2\frac{\partial u}{\partial x_n} = 0
$$

in U then

$$
u(a) = \frac{a_n^{\frac{k-1}{2}}|k+1|}{8\pi\sinh(r_h)\sinh\left(\frac{r_h|k+1|}{2}\right)}\int_{\partial B_h(a,r_h)} x_2^{-\frac{k+3}{2}}u(x)d\sigma
$$

in case $k \neq -1$and in case $k = -1$

$$
u(a) = \frac{1}{4\pi a_2 r_h\sinh(r_h)}\int_{\partial B_h(a,r_h)} x_n^{-1}u(x)d\sigma.
$$

Similarly, the general solution of the equation (5) has the representation.

Theorem 12 *If* $\gamma = \frac{1}{4}(4 - (k+1)^2)$ *and* $k \neq -1$ *the general solution of the equation (5) is*

$$f(r_h) = C_1 \frac{\cosh\left(\frac{|k+1|r_h}{2}\right)}{\sinh r_h} + C_0 \frac{\sinh\left(\frac{r_h|k+1|}{2}\right)}{\sinh r_h}$$

for some real constants C_1 *and* C_0*. If* $k = -1$ *the general solution for* $\gamma = 1$ *is*

$$f(r_h) = C_1 \frac{1}{\sinh r_h} + C_0 \frac{r_h}{\sinh r_h}.$$

Proof. Assuming $k \neq -1$ and substituting

$$\mu(r_h) = \frac{\sinh\left(\frac{r_h|k+1|}{2}\right)}{\sinh r_h}$$

in

$$f(r_h) = \left(C \int_{r_0}^{r_h} \sinh^{-2} u \mu^{-2}(u)\, du + C_0\right) \mu(r_h),$$

we obtain

$$\begin{aligned} f(r_h) &= \left(C \int_{r_0}^{r_h} \sinh^{-2} u \frac{(k+1)^2 \sinh^{-2}\left(\frac{u|k+1|}{2}\right)}{4\sinh^{-2} u} du + C_0\right) \frac{2\sinh\left(\frac{r_h|k+1|}{2}\right)}{|k+1| \sinh r_h} \\ &= \left(C \int_{\frac{|k+1|r_0}{2}}^{\frac{|k+1|r_h}{2}} \sinh^{-2}(s)\, ds + C_0\right) \frac{\sinh\left(\frac{r_h|k+1|}{2}\right)}{\sinh r_h} \\ &= \left(C \coth\left(\frac{|k+1|\, r_0}{2}\right) - C \coth\left(\frac{|k+1|\, r_h}{2}\right) + C_0\right) \frac{\sinh\left(\frac{r_h|k+1|}{2}\right)}{\sinh r_h}. \end{aligned}$$

completing the proof, if we choose the constants properly. The case $k = -1$ is proved similarly. ∎

Corollary 13 *The particular solution of (5) with* $\gamma = \frac{1}{4}(4 - (k+1)^2)$ *outside the point* e_2 *is*

$$F(x) = \frac{\cosh\left(\frac{|k+1|d_h(x,e_2)}{2}\right)}{\sinh d_h(x, e_2)} = \frac{\cosh\left(\frac{|k+1|d_h(x,e_2)}{2}\right)}{|x - \cosh d_h(x, e_2)\, e_2|}$$

and $x_2^{\frac{k-1}{2}} F(r_h)$ *is* k*-hyperbolic harmonic.*

Denote

$$F(x, a) = \frac{\cosh\left(\frac{|k+1|d_h(x,a)}{2}\right)}{\sinh d_h(x, a)}.$$

We obtain this function by transforming the preceding function with the transformation $\tau(x) = a_2 x + Pa$.

Corollary 14 *The function $F_h(x,a)$ satisfies the equation*

$$\frac{\partial^2 f}{\partial r_h^2} + 2\frac{\partial f}{\partial r_h}\frac{\cosh r_h}{\sinh r_h} + \gamma f = 0$$

with $\gamma = \frac{1}{4}(4-(k+1)^2)$ outside $x = a$ and $x_2^{\frac{k-1}{2}} F_h(x,a)$ is k-hyperbolic harmonic outside $x = a$.

Proof. Since the hyperbolic distance is invariant under Möbius transformation mapping the upper half space onto itself, applying $\tau(x) = a_2 x + Pa$ we infer

$$d_h(\tau(x), a) = d_h(x, e_2)$$

and

$$F(x) = \frac{\cosh\left(\frac{|k+1| d_h(\tau(x),a)}{2}\right)}{\sinh d_h(\tau(x), a)}.$$

Since the hyperbolic Laplace operator is invariant under Möbius transformation mapping the upper half space onto itself the function

$$F\left(\tau^{-1}(x)\right) = \frac{\cosh\left(\frac{|k+1| d_h(x,a)}{2}\right)}{\sinh d_h(x,a)}$$

is the eigenfunction of the hyperbolic Laplace operator with the eigenvalue with $\gamma = \frac{1}{4}(4-(k+1)^2)$, completing the proof. ■

Lemma 15 *The function $F_h(x,a)$ is Lebesgue integrable in the hyperbolic ball $B_h(a, r_h)$ and*

$$\int_{B_h(a,r_h)} x_2^{-\frac{5k+1}{2}} F_h(x,a)\, dx \le M(a, r_h)\left(\frac{(\cosh r_h - 1)^2}{6} + \frac{\sinh^2 r_h}{2}\right),$$

for some function $M(a, r_h) > 0$ with a bounded limit when $r_h \to 0$.

Proof. It is enough to prove the statement for $a = e_2$. Note that

$$\frac{|x|^2 + 1}{2x_2} = \lambda(x, e_2) = \cosh d_h(x, e_2).$$

Since $e^{-r_h} < x_2 < e^{r_h}$ in $B_h(e, r_h) = B(\cosh r_h e_2, \sinh r_h)$ we obtain

$$\begin{aligned}
\frac{x_2^{-\frac{1+5k}{2}} \cosh\left(\frac{|k+1| d_h(x,a)}{2}\right)}{\sinh d_h(x, e_2)} &= \frac{x_2^{-\frac{1+5k}{2}} \cosh\left(\frac{|k+1| d_h(x,a)}{2}\right)}{\sqrt{\lambda - 1}\sqrt{\lambda + 1}} \\
&\le \frac{x_2^{-\frac{1+5k}{2}} \cosh\left(\frac{|k+1| r_h}{2}\right)}{\sqrt{\lambda - 1}} \\
&\le \frac{\sqrt{2} e^{\frac{5k}{2} r_h} \cosh\left(\frac{|k+1| r_h}{2}\right)}{\sqrt{|x|^2 + 1 - 2x_2}}
\end{aligned}$$

in $B_h(a, r_h)$ it is enough to consider the integral

$$\int_{B_h(e_r, r_h)} \frac{dx}{\sqrt{|x|^2 + 1 - 2x_2}}$$

$$\int_{B(\cosh r_h e_2, \sinh r_h)} \frac{dx}{\sqrt{|x|^2 + 1 - 2x_2}}.$$

Denote $c = \cosh r_h e_2$. Changing the variables

$$\begin{aligned} x_0 &= r \sin\theta \cos\phi, \\ x_1 &= r \sin\theta \sin\phi, \\ x_2 &= r \cos\theta + c, \end{aligned}$$

we obtain

$$\begin{aligned} &\int_{B(\cosh r_h e_2, \sinh r_h)} \frac{dx}{\sqrt{|x|^2 + 1 - 2x_2}} \\ &= \int_0^{\sinh r_h} \int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin\theta d\theta d\phi dr}{\sqrt{r^2 + 2r(c-1)\cos\theta + (c-1)^2}} \\ &= 2\pi \int_0^{\sinh r_h} -\frac{1}{c-1} \left(r\left(|r - c + 1|\right) - r(r + c - 1) \right) dr \\ &= 2\pi \int_0^{c-1} -\frac{1}{c-1} \left(r(c - 1 - r) - r(r + c - 1) \right) dr \\ &+ 2\pi \int_{c-1}^{\sinh r_h} -\frac{1}{c-1} \left(r(r - c + 1) - r(r + c - 1) \right) dr \\ &= 2\pi \int_0^{c-1} \frac{2r^2}{c-1} dr + 2\pi \int_{c-1}^{\sinh r_h} r dr \\ &= 2\pi \left(\frac{2}{3}(c-1)^2 + \frac{\sinh^2 r_h}{2} - \frac{1}{2}(c-1)^2 \right) \\ &= 2\pi \left(\frac{1}{6}(c-1)^2 + \frac{\sinh^2 r_h}{2} \right), \end{aligned}$$

completing the proof. ■

We recall the Green formula in some Riemannian manifolds.

Proposition 16 ([1]) *Let $R \subset \mathbb{R}^3_+$ be a bounded open set with the smooth boundary contained $\mathbb{R}^3_+$ and denote the volume element corresponding to the Riemannian metric*

$$ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2}{x_2^{2k}}$$

by $dm_{(k)} = x_2^{-3k} dm$, the surface elements by $d\sigma_{(k)} = x_2^{-2k} d\sigma$ and the outer normal $\frac{\partial u}{\partial n_{(k)}} = x_2^k \frac{\partial u}{\partial n}$. where n is the outer normal to the the surface ∂R. Then the Laplace-Beltrami operator is

$$\triangle_k = x_2^{2k} \left(\triangle - \frac{k}{x_2} \frac{\partial}{\partial x_2} \right)$$

and
$$\int_R \left(u\triangle_k v dm_{(k)} - v\triangle_k u dm_{(k)}\right) = \int_{\partial R}\left(u\frac{\partial v}{\partial n_k} - v\frac{\partial u}{\partial n_k}\right) d\sigma_{(k)}$$
for any functions u and v that are twice continuously differentiable functions in an open set containing the closure $\overline{\Omega}$ of Ω.

A function $f : \Omega \to \mathbb{R}$ is called *k-hyperbolic harmonic* if

$$\triangle_k f = 0$$

in Ω. The theory of *k-hyperbolic harmonic functions* was developed in [3]. Denote

$$H(x,y) = y_2^{\frac{k-1}{2}} x_2^{\frac{k-1}{2}} \frac{\cosh\left(\frac{|k+1| d_h(x,y)}{2}\right)}{\sinh d_h(x,y)}.$$

We will show that $H(x,y)$ is the fundamental k-hyperbolic harmonic functions with a pole in x. We need following lemma.

Lemma 17 *Let $\Omega \subset \mathbb{R}^3_+$ be open and x a point with $\overline{B_h(x,r_h)} \subset \Omega$. Then*

$$\lim_{r_h\to 0} \frac{\int_{\partial B_h(x,r_h)} u \frac{\partial H(x,y)}{\partial n_k} d\sigma_{(k)}(y)}{4\pi} = -u(x)$$

for any hyperbolic balls $B_h(x,r_h)$ satisfying $\overline{B_h(x,r_h)} \subset \Omega$.

Proof. Using Proposition 3 we infer that in $\partial B_h(x,r_h)$ the outer normal at y is

$$n = (n_0,n_1,n_2) = \frac{(y_0 - x_0, y_1 - x_1, y_2 - x_2\cosh r_h)}{x_2 \sinh r_h}.$$

Denote $r_h = d(x,y)$. We first compute

$$\begin{aligned}
\frac{\partial H(x,y)}{\partial n_k} &= y_2^k \frac{\partial H(x,y)}{\partial n} = y_2^k (n, \operatorname{grad} v). \\
&= y_2^{\frac{3k-1}{2}} x_2^{\frac{k-1}{2}} \frac{\partial}{\partial r_h} \frac{\cosh\left(\frac{|k+1| r_h}{2}\right)}{\sinh r_h} \sum_{i=1}^{2} n_i \frac{\partial r_h}{\partial y_i} + \frac{k-1}{2} y_2^{k-1} n_2 H(x,y) \\
&= y_2^k H(x,y)\left(\frac{|k+1|}{2}\tanh\left(\frac{|k+1| r_h}{2}\right) - \coth r_h\right)\sum_{i=1}^{2} n_i \frac{\partial r_h}{\partial y_i} \\
&+ \frac{k-1}{2} y_2^{k-1} n_2 H(x,y).
\end{aligned}$$

Applying Lemma 1 we infer

$$\frac{\partial r_h}{\partial y_i} = \frac{\partial \operatorname{arcosh} \lambda(x,y)}{\partial y_i} = \frac{y_i - x_i - x_2(\cosh r_h - 1)\delta_{in}}{y_2 x_2 \sinh r_h},$$

and therefore we conclude

$$\sum_{i=1}^{2} n_i \frac{\partial r_h}{\partial y_i} = \frac{1}{y_2}.$$

Hence we have

$$\frac{\partial H}{\partial n_k}(x,y) = y_2^{k-1} H(x,y)\left(\frac{|k+1|}{2}\tanh\left(\frac{|k+1|\, r_h}{2}\right) - \coth r_h + \frac{k-1}{2} n_2\right).$$

Since $B_h(x,r_h) = B(x_e, x_2 \sinh r_h)$ for $x_e = (x_0, x_1, x_2 \cosh r_h)$ and $u y_2^{\frac{3k-3}{2}}$ is continuous we obtain

$$\lim_{r_h\to 0} \frac{|k+1|\, x_2^{\frac{k-1}{2}}}{8\pi} \frac{\sinh\left(\frac{|k+1| r_h}{2}\right)}{\sinh r_h} \int_{\partial B_h(x,r_h)} u(y)\, y_2^{\frac{3k-3}{2}} d\sigma_{(k)}(y) = 0.$$

Similarly we deduce that

$$\begin{aligned} &\lim_{r_h\to 0} \frac{k-1}{8\pi} \int_{\partial B_h(x,r_h)} \frac{y_2^{k-1} u(y)\, n_2 H(x,y)}{x_2 \sinh r_h} d\sigma_{(k)}(y) \\ &= \lim_{r_h\to 0} \frac{(k-1)\, x_2^{\frac{k+1}{2}}}{8\pi x_2^2 \sinh^2 r_h} \int_{\partial B_h(x,r_h)} u(y)\, y_2^{\frac{3k-1}{2}} (y_2 - x_2 \cosh r_h) \cosh \frac{|k+1|\, r_h}{2} d\sigma_{(k)} = 0. \end{aligned}$$

Lastly we infer

$$\lim_{r_h\to 0} -\frac{x_2^{\frac{k+3}{2}} \cosh r_h \cosh\left(\frac{|k+1| r_h}{2}\right)}{4\pi x_2^2 \sinh^2 r_h} \int_{\partial B_h(x,r_h)} \frac{u(y)}{y_2^{\frac{k+3}{2}}} d\sigma(y) = -u(x),$$

completing the proof. ■

Theorem 18 *Let $\Omega \subset \mathbb{R}^3_+$ be open and R a bounded open set with a smooth boundary satisfying $\overline{R} \subset \Omega$. If u is twice continuously differentiable functions in Ω and $x \in R$ then*

$$u(x) = \frac{1}{4\pi} \int_{\partial R} \left(u \frac{\partial H}{\partial n_k} - H \frac{\partial u}{\partial n_k}\right) d\sigma_k(y) - \frac{1}{4\pi} \int_R H \triangle_k u\, dm_{(k)}$$

where $d\sigma_k$, $dm_{(k)}$ and $\frac{\partial}{\partial n_k}$ are the same as in Lemma 16. Moreover, if $u \in \mathcal{C}_0^2(R)$ then

$$u(x) = -\frac{1}{4\pi} \int_R H \triangle_k u\, dm_{(k)}.$$

Proof. Applying Green formula in the set $R \backslash B_h(x,r_h)$ we obtain

$$\begin{aligned} \int_{R\backslash B_h(x,r_h)} (H \triangle_k u - u \triangle_k H)\, dm_{(k)} &= \int_{\partial R\backslash B_h(x,r_h)} \left(H \frac{\partial u}{\partial n_k} - u \frac{\partial H}{\partial n_k}\right) d\sigma_k \\ &= \int_{\partial R} \left(H \frac{\partial u}{\partial n_k} - u \frac{\partial H}{\partial n_k}\right) d\sigma_k \\ &- \int_{\partial B_h(x,r_h)} \left(H \frac{\partial u}{\partial n_k} - u(y) \frac{\partial H}{\partial n_k}\right) d\sigma_k. \end{aligned}$$

Since H is k-hyperbolic harmonic in $R\backslash B_h(x, r_h)$ we obtain

$$\begin{aligned}\int_{R\backslash B_h(x,r_h)} H(x,y) \triangle_k u(y)\, dm_{(k)}(y) &= \int_{\partial R} \left(H\frac{\partial u}{\partial n_k} - u\frac{\partial H}{\partial n_k}\right) d\sigma_k(y) \\ &\quad - \int_{\partial B_h(x,r_h)} \left(H\frac{\partial u}{\partial n_k} - u\frac{\partial H}{\partial n_k}\right) d\sigma_k.\end{aligned}$$

Since

$$\begin{aligned}&\int_{\partial B_h(x,r_h)} H\frac{\partial u}{\partial n_k} d\sigma_k \\ &= x_2^{\frac{k-1}{2}} \frac{\cosh\left(\frac{|k+1| r_h}{2}\right)}{\sinh r_h} \int_{\partial B_h(x,r_h)} \frac{\partial u}{\partial n_k} y_2^{-\frac{3k+1}{2}} d\sigma\end{aligned}$$

and $\frac{\partial u}{\partial n_k} y_2^{-\frac{3k+1}{2}}$ is bounded in $\partial B_h(x, r_h)$ we obtain

$$\int_{\partial B_h(x,r_h)} \left|H\frac{\partial u}{\partial n_k}\right| d\sigma_k \leq m(x, r_h)\, x_2^{\frac{k-1}{2}}\, 4\cosh\left(\frac{|k+1|\, r_h}{2}\right) \pi \sinh r_h$$

for some founction $m > 0$ with bounded limit when $r_h \to 0$ and therefore

$$\lim_{r_h \to 0} \int_{\partial B_h(x,r_h)} H\frac{\partial u}{\partial n_k} d\sigma_k = 0.$$

Since the function $\triangle_k u(y)$ is a continuous function and by Lemma 15 $H(x,y)$ is integrable in a bounded set R we obtain

$$\lim_{r_h \to 0} \int_{R\backslash B_h(x,r_h)} H \triangle_k u y dm_{(k)} = \int_R H \triangle_k u dm_{(k)}.$$

Combining all the preceding steps and applying Lemma 17 we conclude the result.
■

Corollary 19 *Let $\Omega \subset \mathbb{R}^3_+$ be open and R a bounded open set with a smooth boundary satisfying $\overline{R} \subset \Omega$. If u is k-hyperbolic harmonic in Ω and $x \in R$ then*

$$u(x) = \frac{1}{4\pi} \int_{\partial R} \left(u\frac{\partial H}{\partial n_k} - H\frac{\partial u}{\partial n_k}\right) d\sigma_k(y)$$

where $d\sigma_k$, $dm_{(k)}$ and $\frac{\partial}{\partial n_k}$ are the same as in Lemma 16.

Note that if $k = 1$, then

$$H(x,y) = \coth(d_h(x,y)) = \int_{d_h(x,y)}^{\infty} -\frac{du}{\sinh^2 u}$$

and if $k = -1$, then

$$H(x,y) = \frac{1}{x_2 y_2 \sinh d_h(x,y)} = \frac{1}{x_2 y_2 \sqrt{\lambda^2 - 1}} = \frac{2}{|x-y|\,|x-\widehat{y}|}.$$

These kernels were already used in integral formulas for hypermonogenic functions, see for example in [2] and [6]. Mean value properties for hyperbolic harmonic functions were verified in [7] .

We may prove also similar results for eigenfunctions of the hyperbolic Laplace operator.

Theorem 20 *Let $\Omega \subset \mathbb{R}^3_+$ be open and R a bounded domain with a smooth boundary satisfying $\overline{R} \subset \Omega$. Denote $\gamma = \frac{1}{4}(4-(k+1)^2)$. If u is twice continuously differentiable functions in Ω and $x \in R$ then*

$$u(x) = \frac{1}{4\pi}\int_{\partial R}\left(u\frac{\partial F}{\partial n_h} - F\frac{\partial u}{\partial n_h}\right) d\sigma_h(y) - \frac{1}{4\pi}\int_R F(\triangle_h u - \gamma u)\, dm_h,$$

where $d\sigma_h = \frac{d\sigma}{y_2^2}$, $dm_h = \frac{dm}{y_2^3}$ and $\frac{\partial}{\partial n_h} = y_2\frac{\partial}{\partial n}$. Moreover. if $u \in \mathcal{C}^2_0(R)$ then

$$u(x) = -\frac{1}{4\pi}\int_R F(\triangle_h u - \gamma u)\, dm_h.$$

Proof. Using Green formula in the set $R\backslash B_h(x,r_h)$ we obtain

$$\begin{aligned}\int_{R\backslash B_h(x,r_h)} (F(\triangle_h u - \gamma u) - u(\triangle_h F - \gamma F))\, dm_h &= \int_{\partial R\backslash B_h(x,r_h)}\left(F\frac{\partial u}{\partial n_h} - u\frac{\partial F}{\partial n_h}\right) d\sigma_h \\ &= \int_{\partial R}\left(F\frac{\partial u}{\partial n_h} - u\frac{\partial F}{\partial n_h}\right) d\sigma_h \\ &\quad - \int_{\partial B_h(x,r_h)}\left(F\frac{\partial u}{\partial n_h} - u\frac{\partial F}{\partial n_h}\right) d\sigma_h.\end{aligned}$$

From $\triangle_h F(x,y) - \gamma F(x,y) = 0$ in $R\backslash B_h(x,r_h)$, it follows that

$$\begin{aligned}\int_{R\backslash B_h(x,r_h)} F(\triangle_h u - \gamma u)\, dm_h &= \int_{\partial R}\left(F\frac{\partial u}{\partial n_h} - u\frac{\partial F}{\partial n_h}\right) d\sigma_h \\ &\quad - \int_{\partial B_h(x,r_h)}\left(F\frac{\partial u}{\partial n_h} - u\frac{\partial F}{\partial n_h}\right) d\sigma_h.\end{aligned}$$

Since $\frac{\partial u}{\partial n_k}$ is bounded in $\partial B_h(x,r_h)$ we obtain

$$\int_{\partial B_h(x,r_h)}\left|F(x,y)\frac{\partial u(y)}{\partial n_h}\right| d\sigma_h(y) \le 4m(x,r_h)\cosh\frac{|k+1|\,r_h}{2}\pi x_2 \sinh r_h$$

for some function $m > 0$ with a bounded limit when $r_h \to 0$ and therefore

$$\lim_{r_h\to 0}\int_{\partial B_h(x,r_h)} F(x,y)\frac{\partial u(y)}{\partial n_h} d\sigma_h(y) = 0.$$

Since the function $\triangle_h u - \gamma u$ is a continuous function and by Lemma 15 $F(x,y)$ is integrable in a bounded set R we obtain

$$\lim_{r_h\to 0}\int_{R\backslash B_h(x,r_h)} F(x,y)(\triangle_h u - \gamma u)\, dm_h(y) = \int_R F(x,y)(\triangle_h u - \gamma u)\, dm_h(y).$$

The proof is completed when we verify that

$$\lim_{r_h\to 0}\frac{\int_{\partial B_h(x,R_h)}u\frac{\partial F(x,y)}{\partial n_h}d\sigma_h\left(y\right)}{4\pi}=-u\left(x\right).$$

This follows from the preceding calculations similarly as earlier proof, since

$$\frac{\partial F}{\partial n_h}=F\left(x,y\right)\left(\frac{|k+1|}{2}\tanh\left(\frac{|k+1|\,r_h}{2}\right)-\coth r_h\right)$$

and

$$\begin{aligned}\lim_{r_h\to 0}\int_{\partial B_h(x,r_h)}u\frac{\partial F\left(x,y\right)}{\partial n_h}d\sigma_h&=-\lim_{r_h\to 0}\frac{\cosh\left(\frac{|k+1|r_h}{2}\right)\cosh r_h}{\sinh^2 r_h}\int_{\partial B_h(x,r_h)}\frac{ud\sigma}{y_2^2}\\&=-4\pi u\left(x\right).\end{aligned}$$

■

Corollary 21 *Let $\Omega\subset\mathbb{R}^3_+$ be open and R a bounded open set with a smooth boundary satisfying $\overline{R}\subset\Omega$. If u is an eigenfunction corresponding to the eigenvalue $\gamma=\frac{1}{4}(4-(k+1)^2$ in Ω and $x\in R$ then*

$$u\left(x\right)=\frac{1}{4\pi}\int_{\partial R}\left(u\frac{\partial F\left(x,y\right)}{\partial n_h}-F\left(x,y\right)\frac{\partial u}{\partial n_h}\right)d\sigma_h.$$

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