Pseudorigidity method (PRM) for Solving the Problem of Limit Equilibrium of Rigid-plastic Constructions

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Limit equilibrium calculations are a broad class of problems arising during investigation of strength of constructions. When solving such problems, a maximal level of loading is instituted on the given structure, which it can resist without its destruction. Today, most universal procedures used for solving such problems are methods of linear and non-linear programming [1] and [2].

They allow to examine all types of models both for linearized and non-linearized yield criteria. However, reduction of the limit equilibrium problem to standard ratios of linear and non-linear programming is very complex and requires special skills from the investigator.

The Pseudorigidity Method (PRM) described hereinafter is a universal tool. At the same time, it can be easily automated with existing software for calculation of elastic constructions. The designer can naturally and simply use such software.

1. TASK DEFINITION

If the construction is in deep plastic deformation conditions, then its elastic deformation can be discarded. In this case, the model of the construction is a rigid-plastic body. Such a body is described by equilibrium equations and Drucker's postulate [3].

In the space of macro factors (generalized stress R_i and generalized transposition U_i) yield surface equation and Drucker's postulate are written as follows [2]

$$f(R_i) = 1. \tag{1a}$$

$$\dot{U}_i = \begin{cases} \lambda \dfrac{\partial f}{\partial R_i}, \lambda \geq 0, если f = 1, \\ 0, если f < 1. \end{cases} \tag{1b}$$

So, for example, if $f(R_i)$ is a homogeneous quadratic form, then

$$\{\dot{U}\} = \lambda[\Pi]\{R\}. \tag{2}$$

In (2) $\{\dot{U}\}, \{R\}$ are matrixes-columns, the elements of which are $\dot{U}_i$ and R_i, $[\Pi]$ is a square matrix, the elements of which are defined according to (1b) if substituting $\{\dot{U}\}$ with $\{U\}$.

Relations (2) are similar to dependencies between deformation and force factors for elastic task definition. The subsequently described PRM is based on this similarity.

Let us specify the mentioned similarity for rod models. We consider a rod in flat bend and torsion conditions. For this case of loading in [4], on the basis of Huber-Mizes criterion, condition (1b) is obtained in the following form

$$\sum_{i=1}^{2} \alpha_i(S) M_i^2(S) = 1 \tag{3}$$

In (3) $M_i(S)$ are the bending momentum (i=1) and torque (i=2) in cross sections of the rods; S is the coordinate of the cross section. Coefficients $\alpha_r(S) = \frac{1}{M_{iT}^2(S)}$, where M_{iT} is the limit momentum of cross section at the "i" type of deformation. From (1b) and (3) we get that in (2)

$$\{\dot{U}\} = \{\dot{\chi}_i(S)\}; \{R\} = \{M_i(S)\}; [\Pi] = \begin{bmatrix} 2\alpha_1(S) & 0 \\ 0 & 2\alpha_2(S) \end{bmatrix} \tag{4}$$

The elastic analog (2) in this case is the relation

$$\{\chi(S)\} = [D]^{-1}\{M(S)\} \tag{5}$$

where [D] $= \begin{bmatrix} E_1 I_1 & 0 \\ o & E_2 I_2 \end{bmatrix}$ is the rigidity matrix, I_1 and I_2 are axial and polar momentum of inertia, $E_1 = E$ is Young's modulus; $E_2 = G$ is the shift modulus.

Using this analogy we can build an elastic scheme, the internal stresses in which under limit loading will be the same, as in the discussed rigid-plastic system. Let us demonstrate this.

Let us assume that the rigid-plastic solution for the considered problem is known. We retain the geometrical dimensions, boundary conditions and system loading, and assume that its material has not rigid-plastic, but elastic properties. Let us assume the following distribution of rigidities of sections:

- on sections of the rigid-plastic model, where $\sum_{i=1}^{2} \alpha_i M_i^2 < 1$

$$D_i = E_i I_i = \infty; \tag{6a}$$

- in each deforming part, where $\sum \alpha_i M_i^2 = 1$

$$D_i = E_i I_i = \frac{1}{2t\lambda(S)\alpha_i(S)} = K(S)\frac{1}{\alpha_i(S)}. \tag{6b}$$

In (6b) t is the coefficient with a time dimension, $K(S)$ is the unknown function, which is alike for all types of deformation (bending, torsion).

In the so formed elastic system we create deformation corresponding to the form of destruction of the elastoplastic system: $\chi = t\dot{\chi}_i$.

Then $M_i(S) = D_i \chi_i = \frac{\dot{\chi}_i(S)}{2\lambda(S)\alpha_i(S)}$ (7)

If we compare (7) with (2) and (4), we can see that values $M_i(S)$ in deformed parts coincide in elastic and rigid-plastic systems. The values of internal momentum in places of formation of plastic joints completely define the limit loading and the values of internal momentum on all sections of the system with nonzero velocity [2]. This is why formed elastic and initial rigid-plastic systems have a comparable loading and distribution of internal stresses. Thus, the considered rigid-plastic problem is reduced to searching for distribution of rigidities D_i in an "equivalent" rigid system. Such values as D_i shall be referred to as pseudorigidities.

2. ITERATION ALGORITHM OF THE PSEUDORIGIDITY METHOD

For finding *K(S)* we can propose an iteration algorithm. The result of (6b), (7) and (3) is:

$$K(S) = \frac{1}{\sqrt{\dfrac{\chi_1^2}{\alpha_1(S)} + \dfrac{\chi_2^2}{\alpha_2(S)}}} \tag{8}$$

Relation (8) is true for all sections of the rod system, including non-deformed sections (in this case $\chi_i = 0$ and $K(S)=\infty$).

Specifying in zero approximation $K^{(0)}(S) \neq \infty$ we can acquire from the solution of the elastic problem $\chi_i^{(0)}(S)$, then using (8) $K^{(1)}(S)$, etc.

So, the scheme of the iteration process is as follows (*n* is the number of iteration):

$$K^{(n)}(S) = \frac{1}{\sqrt{\sum_{i=1}^{2} \dfrac{1}{\alpha_i(S)} \left[\chi_i^{(n-1)}(S)\right]^2}} = \frac{K^{(n-1)}(S)}{K^{(n-1)}(S)\sqrt{\sum_{i=1}^{2} \dfrac{1}{\alpha_i(S)} \left[\chi_i^{(n-1)}(S)\right]^2}} =$$

$$= \frac{K^{(n-1)}(S)}{\sqrt{\sum_{i=1}^{2} \alpha_i(S)\left[D_i^{(n-1)}(S)\chi_i^{(n-1)}(S)\right]^2}} = \frac{K^{(n-1)}(S)}{\sqrt{\sum_{i=1}^{2} \alpha_i(S)\left[M_i^{(n-1)}(S)\right]^2}} \tag{9(a)}$$

In other words, $K^{(n)}(S) = \dfrac{K^{(n-1)}(S)}{\tilde{f}(R_i^{(n-1)}(S))}$, (9b)

where $\tilde{f}(R_i)$ is the yield criterion reduced to a linear homogeneous function.

At every iteration step the loading applied to the system changes according to the formula

$$\vec{F}^{(n)}(S) = \beta^{(n)} \vec{F}^{(n-1)}(S), \tag{10a}$$

where $\beta^{(n)} = \min\limits_{s} \dfrac{1}{\tilde{f}(R_i^{(n-1)}(S))}$. (10b)

The system destruction scheme is defined by values of $K_{\lim}(S)$: in plastic joints $K_{\lim}(S)$ are finite; in rigid zones $K_{\lim}(S) = \infty$. Convergence of iterations (9) and (10) to a precise solution of the rigid-plastic problem is proved in [5].

3. PSEUDOELASTIC DEPENDENCIES FOR DIFFERENT MODELS

It has been shown before, that with a quadratic yield criteria for realization of PRM (finding the distribution of rigidities in an "equivalent" elastic system) it is necessary instead of (2) to form a pseudoelastic model:

$$\{R\} = K[\Pi]^{-1}\{U\} \tag{11}$$

Next, the scalar multiplier K is determined using an iteration process similar to (9). The iteration procedure can be performed on the basis of existing software tools, if a certain standard elastic model with a rigidity matrix $[\Pi]^{-1}$ will correspond to the matrix $[D]=K[\Pi]^{-1}$. For the major problems of limited equilibrium such a correspondence exists.

For bending plates and beams-walls $K[\Pi]^{-1}=[D]_{\nu=0,5}$, where $[D]_{\nu=0,5}$ is a rigidity matrix for the elastic model with Poisson's ratio $\nu = 0,5$ and a variable modulus of elasticity E(x,y).

For shells $K[\Pi]^{-1} = [H]$, where $[H]$ is a rigidity matrix with variable E for an elastic multilayer shell at $\nu = 0,5$.

For a 3D problem at $\nu = 0,5$ there is no matrix $[D]$. In this case for forming pseudoelastic dependencies we should use models describing non-compressible materials. However, the PRM provides a high accuracy if we accept $K[\Pi]^{-1} = [D]_{\nu=0,49}$, where $[D]$ is the rigidity matrix for a 3D stressed state.

The previously described relations are based on the Huber-Mizes yield criterion. In [4] it is shown that the PRM can be applied also for linearized yield criteria.

Let us call the program performing elastic calculation with the rigidity matrix $D=K[\Pi]^{-1}$ - the basic program (BP). The iteration process is described by relations (9b) and (10).

Interaction between the iteration procedure and the BP is realized as follows:

- a strength model of the examined system is formed within the BP;
- the elastic problem is solved for this model, whereas the level of the given type of loading can be selected randomly;
- internal stresses of the model, determined by elastic calculation, enter the program realizing the PRM as an output file of the BP. PRM processes this file and changes the rigidity parameters of the examined model. After that the new rigidity parameters enter the BP as a new file of initial data for the problem.

4. TEST EXAMPLES

The author has calculated a number of test models: beams, frames, plates, shells, 3D bodies. Results produced with the PRM differ from known solutions not more than by 2% [4]. For example, arc calculation results (Hodgee's problem) are shown in Table 1 (where $k = \frac{M_T}{2RN_T} = 0.01$, $p = \frac{PR}{M_T}$, R is the arc radius, 2φ is the aperture angle, 2P is the force in the center of the arc.)

Table 1
Limit loading of the arc considering the longitudinal force

Limit loading of the arc	φ=10°	φ=20°	φ=30°	φ=40°	φ=50°	φ=60°
P (according to [2])	5.6	6.6	7.0	6.6	5.9	5.2
p (PRM)	5.75	6.6	7.0	6.55	5.9	5.2

Comparison of MPR with results given in [2] for a spherical segment is shown in Fig.1.

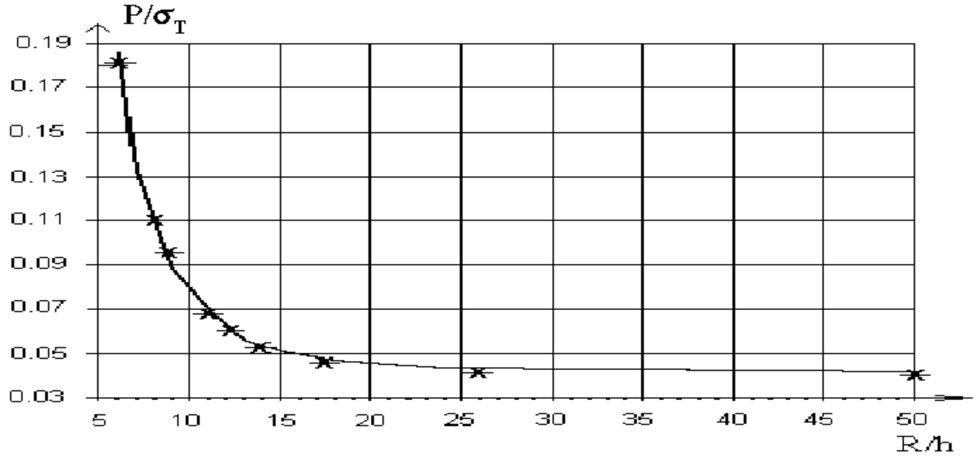


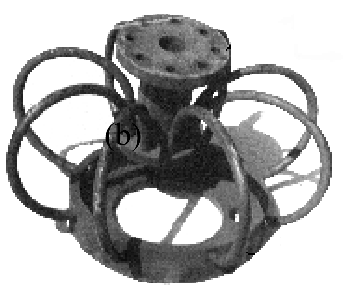
Fig.1. Comparison of calculations using MPR with the results of [2].
* - MPR, —— - results of [2].

5. APPLICATION OF MPR FOR SOLVING PRACTICAL PROBLEMS

The author has used MPR for resolving a whole number of practical problems. Below you can see the results of calculating a plastic damper and protective ring frame support of a metal-concrete container. Plastic dampers are used for protecting different technical objects from shock loading [6].

The damper is designed as a connection of curvilinear metallic rods with a round section. The axis of each of the rods is a semicircle. The calculation scheme of such a damper with applied to it loading is shown on Fig.1. The results of calculation of the yield surface of the plastic damper are shown in Table 2.

(a) (b)



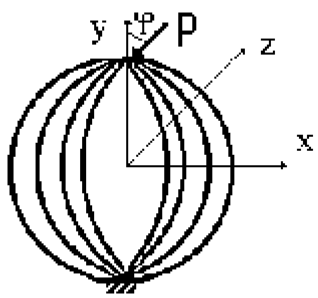


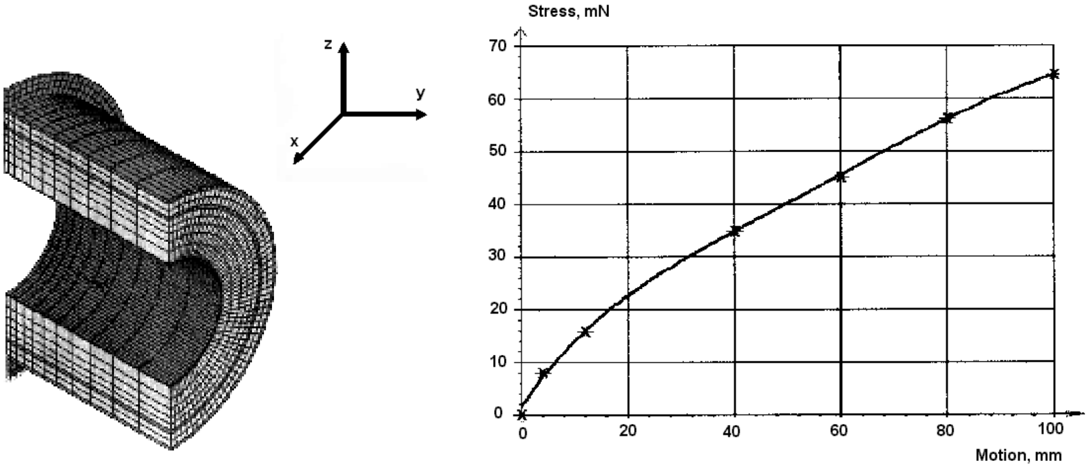
Fig.1. Plastic Damper (a) and its calculation scheme (b).

Table 2
Yield surface of a three-dimensional plastic damper

φ(°)	0	10	20	30	40	50	60	70	80	90

Py/Pyo	1	0.881	0.731	0.587	0.462	0.344	0.250	0.156	0.081	0
Px/Pyo	0	0.156	0.262	0.337	0.387	0.412	0.431	0.437	0.444	0.45

In Table 2 Py, Px are components of limit loading; φ is the angle between axis Y and loading applied to the damper laying in the XY plane; Pyo is the value of limit loading at φ = 0; P_z=0.



(a) (b)

Fig.2. FE model of MCC (a) and calculation of stress in ring support (b).

A metal-concrete container (MCC) has been developed for storing processed nuclear fuel. According to existing standards, MCC must preserve its durability after falling on a rigid basement from a height H=9 meters. For protecting MCC from shock loading plastic deforming ring frame supports are used. A finite element model of MCC and ring frame support is shown on Fig. 2.a. Results of calculating the stress of interaction between MCC and the foundation using MPR are shown on Fig. 2.b.

6. CONCLUSION

The pseudorigidity method is a new technology for solving limit equilibrium problems. The technology is efficient (insignificant time losses for computer processing and problem preparation) and compact (requires little computer memory). Its important advantage is the ability to function on the basis of an elastic calculation program. Design and research companies usually always possess such software. At the same time the pseudorigidity method is highly universal.

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