ANALYSIS AND DETERMINATION OF STRENGTH IN PLASTIC STAGE OF FREE FORM STEEL SHAPES

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Introduction

The steel structure design codes [1,2,3] require to check up the member strength when evaluating plastic deformations. The model of perfectly plastic material is accepted. The strength criteria for simple cross-sections (I section, etc.) of steel members are given in design codes. However, evaluating complicated cross sections (open thin walled, etc.) that are used in contemporary steel constructions, it is necessary to extend the given strength criterion range in design codes or to use modern methods of checking carrying capacity of the cross-sections. One of the ways for checking the carrying capacity of cross-sections is the use of methods that are applied for defining strain-deformed state of elastic perfectly plastic systems.

In this article, two formulations for checking cross-section carrying capacity are presented. First, the carrying capacity of cross-section is verified according to extremum principle of plastic fail under monotonically loading [4]. Second, the strain-deformed state of cross-section is defined and in the same time carrying capacity of cross-section is checked according to extremum energy principals of elastic potential of residual stresses and complementary work of residual displacements [4]. Principal differences between the two approaches is that according to the first one, there can be obtained parameter of carrying capacity for given distribution of internal forces and the actual distribution of stresses in limit state of cross-section. According to the second approach, there can be verified the carrying capacity of cross-section for given internal forces as well as stress and strain distribution in cross-section. It is very important to define the deformed state, because in some design codes, for example [2], the strength criteria are obtained when limiting the residual deformations in cross-section boundary fiber.

The methods offered here could be used not only for cross-sections of steel members, but also for the analysis of cross-sections combined from different materials.

Formulating the problem

The carrying capacity of cross-section is obtained according to the principle of simple plastic fail and the mathematical expression of this principal for discrete system, using linear yield conditions, as follows:

Static formulation	Kinematics formulation
$\nu \to \min,$	$\boldsymbol{\lambda}^T \boldsymbol{S_0} \to \max,$
$[\boldsymbol{\Phi}]\boldsymbol{S} \le \boldsymbol{S_0},$	$[A]^T \dot{\boldsymbol{u}} - [\boldsymbol{\Phi}]^T \boldsymbol{\lambda} = \boldsymbol{0},$
$[A]\boldsymbol{S} - \nu \boldsymbol{F} = \boldsymbol{0}.$	$\boldsymbol{\lambda} \ge \boldsymbol{0}, \qquad \dot{\boldsymbol{u}}^T \boldsymbol{F} = 1.$

(1)

The actual strain-deformed state is defined according to extremum energy principals of elastic potential of residual stresses and complementary work of residual displacements. The mathematical expression of these principals for discrete system, using linear yield conditions, is as follows:

Static formulation | Kinematics formulation

$$\frac{1}{2}\boldsymbol{S}_r[\boldsymbol{D}]\boldsymbol{S}_r \rightarrow \min, \qquad \left\{-\frac{1}{2}\boldsymbol{S}_r^T[\boldsymbol{D}]\boldsymbol{S}_r^T + \boldsymbol{\lambda}^T[\boldsymbol{\Phi}]\boldsymbol{S}_e - \boldsymbol{\lambda}^T\boldsymbol{S_0}\right\} \rightarrow \max,$$

$$[\boldsymbol{\Phi}]\boldsymbol{S}_r \leq \boldsymbol{S_0} - [\boldsymbol{\Phi}]\boldsymbol{S}_e, \qquad [\boldsymbol{D}]\boldsymbol{S}_r + [\boldsymbol{\Phi}]^T\boldsymbol{\lambda} - [\boldsymbol{A}]^T\boldsymbol{u}_r = \boldsymbol{0}, \qquad (2)$$

$$[\boldsymbol{A}]\boldsymbol{S}_r = \boldsymbol{0}. \qquad \boldsymbol{\lambda} \geq \boldsymbol{0}.$$

In mathematical models (1) and (2) $\boldsymbol{S}, \boldsymbol{S}_r, \boldsymbol{S}_e, \boldsymbol{S_0}$ - vectors respectively of actual, residual, elastic and limit generalized stresses, $[\boldsymbol{A}]$- coefficient matrix of equilibrium equations, $[\boldsymbol{\Phi}]$- coefficient matrix of linear yield conditions, $[\boldsymbol{D}]$- flexibility matrix of discrete system, $\boldsymbol{\lambda}$ - the vector of plastic multiplying, $\boldsymbol{u}_r$ - vector of residual displacements, $\dot{\boldsymbol{u}}$ - vector of displacement velocities, ν - parameter of limit load and $\boldsymbol{F}$ - given vector of cross-section forces.

The mathematical model (1) is linear convex mathematical programming problem and the solution of this problem is the parameter of limit load ν and the vector of actual stresses $\boldsymbol{S}$. The strength of cross-section is satisfied, if $\nu \geq 1$. The cross-section limit load vector is $\nu\boldsymbol{F}$.

Solving the quadratic programming problem (2), vectors $\boldsymbol{S}_r, \boldsymbol{\lambda}, \boldsymbol{u}_r$ are obtained. The strength of cross-section is sufficient if the mathematical programming problem (2) has the solution. The actual strain-deformed state of cross-section is obtained in the following way:

$$\boldsymbol{S} = \boldsymbol{S}_e + \boldsymbol{S}_r, \quad \boldsymbol{u} = \boldsymbol{u}_e + \boldsymbol{u}_r, \quad \boldsymbol{q} = \boldsymbol{q}_e + \boldsymbol{q}_r, \qquad (3)$$

here $\boldsymbol{u}, \boldsymbol{u}_e, \boldsymbol{u}_r, \boldsymbol{q}, \boldsymbol{q}_e, \boldsymbol{q}_r$ are the vectors respectively of actual, elastic and residual generalized stresses, displacements and deformations.

The following residual deformations are obtained:

$$\boldsymbol{q}_r = [\boldsymbol{D}]\boldsymbol{S}_r + [\boldsymbol{\Phi}]\boldsymbol{\lambda}. \qquad (4)$$

The elastic solution is calculated according to well-known formulas:

$$\boldsymbol{u}_e = \left([\boldsymbol{A}][\boldsymbol{D}]^{-1}[\boldsymbol{A}]^T\right)^{-1}\boldsymbol{F}, \quad \boldsymbol{S}_e = [\boldsymbol{D}]^{-1}[\boldsymbol{A}]^T\boldsymbol{u}_e, \quad \boldsymbol{q}_e = [\boldsymbol{D}]\boldsymbol{S}_e. \qquad (5)$$

Discrete relationships

In this article the relationships of finite elements for static formulation of the problem (1,2) are formed so, that kinematics formulation relationships could be obtained in a formal way using the theory of duality [5].

The cross-section of steel member using finite element method is divided into free form plane elements. The constant distribution of stresses along the finite element is accepted. The constant distribution of stresses does not require the continuity between the elements and enables to evaluate possible breaks of stresses, which can occur using the model of elastic perfectly plastic material.

The cross-section is analysed in *x,y,z* coordinate system. All relationships for finite elements given below are for the actual stress-strained state. These relationships have the same form for elastic and residual stress-strained states.

The approximating stress matrix for element *k*, when distribution of stresses along the element is constant, has the following expression:

$$[\boldsymbol{N}_k(y,z)] = [\boldsymbol{E}], \tag{6}$$

here $[\boldsymbol{E}]$ - unit matrix.

The forces of element is defined by a vector:

$$\boldsymbol{S}_k = \int_{A_k} [\boldsymbol{N}_k(y,z)]\boldsymbol{\sigma} da = A_k \boldsymbol{\sigma}\,, \tag{7}$$

here $\boldsymbol{S}_k = |N_{kx}, Q_{ky}, Q_{kz}|^T$ -vector of finite element forces, N_{kx} and Q_{ky}, Q_{kz} -the axial and shear forces added at the gravity center of element, $\boldsymbol{\sigma} = |\sigma_{xx}, \tau_{xy}, \tau_{xz}|^T$ - vector of stresses, A_k - area of finite element.

The flexibility matrix of the element, evaluating transition from stresses to forces of element is obtained in the following way:

$$[\boldsymbol{D}_k] = \int_{A_k} \frac{1}{A_k}[\boldsymbol{D}]\frac{1}{A_k} da\,, \tag{8}$$

here the flexibility matrix has this expression:

$$[\boldsymbol{D}] = \begin{bmatrix} \frac{1}{E} & & \\ & \frac{1}{\psi G} & \\ & & \frac{1}{\psi G} \end{bmatrix}, \tag{9}$$

where E - module of elasticity, G - module of shear, ψ - coefficient of cross section form.

Having integrated expression (8), we obtain the flexibility matrix of element:

$$[\boldsymbol{D}_k] = [\boldsymbol{D}]/A_k\,. \tag{10}$$

The equilibrium equations express equilibrium between the stresses of elements and forces of cross-section. During the stresses these equations are written the following way:

$$\int_A [\boldsymbol{A}(x,y,w)]\boldsymbol{\sigma} da = \boldsymbol{F}\,, \tag{11}$$

here $\boldsymbol{F} = |N_x, M_y, M_z, M_x, Q_y, Q_z|^T$ - vector of cross-section forces, the elements of this vector are axial force, bending moments, moment of rotation and shear forces. Matrix

$$
\left[\boldsymbol{A}(x,y,z)\right]=\begin{bmatrix}1 & z & y & & \\ & & & -1 & 1 \\ & & & 1 & 1\end{bmatrix}^{T}. \tag{12}
$$

Having used the expression (11), the equilibrium equations for discrete cross-section are described as follows:

$$
\sum_{k}[\boldsymbol{A_k}]\boldsymbol{S_k}=\boldsymbol{F}, \tag{13}
$$

here matrix $\left[\boldsymbol{A}_k\right]$ is obtained from matrix $\left[\boldsymbol{A}(x,y,w)\right]$, using the coordinates of element gravity center and warping area center of element *k*. It is necessary to note that the relationship between shear stresses and shear forces of cross-section depending on member type could be different.

The yield conditions should be formed for every node surrounding of finite element using stress approximating functions. The type of yield conditions, when the stress distribution is accepted as constant, is written:

$$
\int_{A_k}\left([\boldsymbol{\Phi}][\boldsymbol{N}(y,z)]\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}'\right)-\boldsymbol{\sigma}_0\right)da\leq\boldsymbol{0}, \tag{14}
$$

here $\boldsymbol{\sigma}'$ - residual stresses, arising in the member section during manufactory process. Performing analysis according to steel structures design codes, yield strain must be changed to yield strength, for example $\sigma_0=f_y/\gamma_M$ [1], $\sigma_0=R_y$ [2].

Having integrated the expression (14), the yield conditions for element k are described as follows:

$$
[\boldsymbol{\Phi_k}]\boldsymbol{S_k}\leq\boldsymbol{S}_{0k}, \tag{15}
$$

here $[\boldsymbol{\Phi_k}]=[\boldsymbol{\Phi}]$, $\boldsymbol{S}_{0k}=A_k(\boldsymbol{\sigma_0}-[\boldsymbol{\Phi}]\boldsymbol{\sigma}')$.

The formulation of finite elements with equal stress distribution described here allows us to get the kinematics formulation of the problem using theory of duality. Then the elements of deformation vector are axial and shear deformations in the gravity center of element and the elements of displacement vector are the axial deformation, curvatures and shear deformations in the center of cross-section.

Numerical example

The I cross-section with extended upper flange (Fig. 1) is taken to demonstrate the given method. The cross-section forces vector is $\boldsymbol{F}=\left|N_x,M_y,M_z\right|^T$. Yield strength for bottom flange is $\sigma_0=250\,MPa$ and for other elements of cross-section it is $\sigma_0=210\,MPa$. Module of elasticity is E = 206000 MPa.

The cross-section is covered with rectangular shape element mesh: upper flange – 4x1, 2x12, 4x1; web – 16x2; bottom flange – 2x8. The convergence of solution depending on density of element mesh is not presented in this paper.

Determination of cross-section strength. The cross-section force vector magnitudes are $\boldsymbol{F} = \left|N_x, M_y, M_z\right|^T =$ $= \left|1070.0(kN), 70.1(kNm), 407.0(kNm)\right|^T$. While solving the problem of checking the strength of cross-section it is necessary to apply the static formulation of mathematical models (1).

The solution of linear programming problem is limit load parameter $\nu = 1.052$ and forces in finite elements for limit state of cross-section. It should be noted that for given force vector element forces is $\boldsymbol{S}_k \equiv N_{kx}$. The value of limit load parameter indicates that the strength of cross-section is sufficient for given magnitudes of force vector. The limit magnitudes of cross-section force vector can be obtained multiplying it by parameter ν.

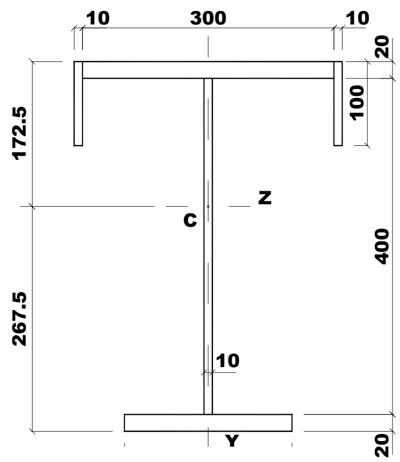

Fig. 1. Cross-section

Composition of interaction curves. The static formulation of mathematical models (1) can be used to obtain the interaction curves according to standard curve fitting procedure. The points of the interaction curves are obtained changing the ratio between the forces of cross-section and solving problem (1). The interaction curves for axial force and bending about strong and weak axis are shown in Fig. 2.

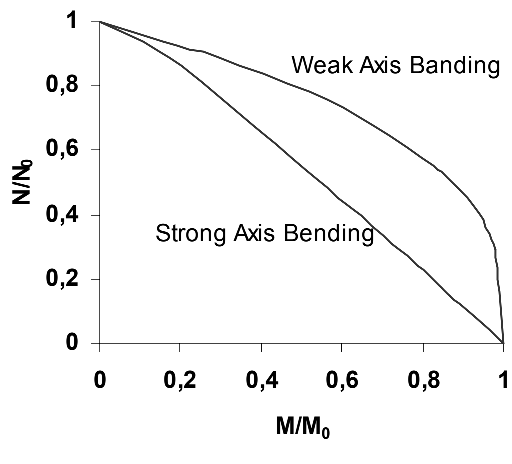

Fig. 2. Interaction curves for Cross-section, Fig.1.

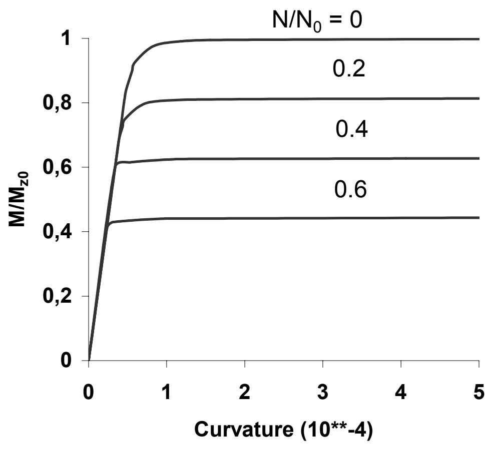

Fig. 3. Moment-Curvature Relationships for Cross-section, Fig.1

Analysis of cross-section strength. The actual stresses, deformations and displacements are obtained by solving dual pair of quadratic programming problems (2) and expression (3). Fig.3 shows the relationships between the moment about strong axis and curvature for different level of axial force. Depending on magnitudes of bending moment and axial force the cross-section remains purely elastic (vector of residual forces $\boldsymbol{S}_r = \boldsymbol{0}$, vector of residual deformations $\boldsymbol{q}_r = \boldsymbol{0}$ and residual curvature $\phi_r = 0$) or become elastic-plastic.

Here, it should be noticed that numerical values of deformations are important to know, because analytical strength criteria in some codes are obtained by limiting residual deformations in boundary fiber of cross-section. For example in code [3], strength criteria are obtained when limit value of residual deformation is $\bar{\varepsilon} = \varepsilon(E / R_y) = 3$.

Conclusions

The proposed methods allow using numerical procedures to check the strength and to analyses strain-deformed state of cross-sections when the behaviour of material is perfectly elastoplastic. This approach is highly efficient for complicated form of cross-sections and could be successfully used CAD systems. One of attractiveness of proposed approach that no iteration
s are needed to compare with stepwise methods.

The element with constant distribution of strains allows achieving sufficient numerical results for practicable design needs.

References

1. Eurocode No 3. Design of Steel Structures. Part 1. General Rules and Rules of Buildings. Vol. 1. Brussels, 1993.
2. DC II-23-81. Steel Structures. Moscow: CITP Gosstroja USSR, 1990. 96 p. (in Russian)
3. DIN 18800. Part 1. Steel Structures Analysis and Design. Berlin: Beuth Verlag GmbH. 1990. 49 p. (in German)
4. A.Čyras. Analysis and optimization of Elastoplastic Systems. N.Y.:John Willy & Sons. 1983. 121 p.
5. A.Daniunas. Analysis of Complicated Form Sections of Steel Members //Civil Engineering, 1997, No. 1(9), Vilnius, 34-38 p.