COMPARISON OF SOME VARIANTS OF THE FINITE STRIP METHOD FOR ANALYSIS OF COMPLEX SHELL STRUCTURES

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1. INTRODUCTION

The finite strip method (FSM) is a powerful and effective contemporary technique mainly for general linear elastic static, dynamic and stability analyses of structures with a constant geometry and stiffnesses in one direction (usually in length) and simple (one-type) boundary conditions at the cross ends. First of all the long single-span or multiple-spans (continuous) thin-walled prismatic or curve structures are appropriate for this analyses. Many of thin-walled bridge structures, some roof and floor structures, some reservoirs, channels, tunnels, subways, layered plates, shear-walls, shells and others can be studied by the FSM. This method is especially effective if two opposite ends of the structure are simply supported, for example by the ideal rigid diaphragms in its plane and infinitely flexible out of it.

Three basic variants of the FSM are known: semi-analytical [1], [2], analytical [3]-[6] and numerical [7]. Review of these versions for analysis of thin-walled structures is given in [8]. The semi-analytical FSM and the finite element method (FEM) in displacements are compared in [1]. The semi-analytical FSM, the numerical (spline) FSM and the conventional FEM are compared in quality and in quantity in [7]. Some mainly quantitative comparisons of the analytical FSM and the FEM, as well of the semi-analytical and analytical FSM in displacements are presented in [4] and [5] in analysis of simply supported prismatic shell structures. The main purpose of this paper is to explore, compare in quality and quantity and evaluate the three basic versions of the FSM for linear elastic analysis of complex thin-walled structures.

2. BASIC APPROACHES AND VARIANTS OF THE FINITE STRIP METHOD

The FSM combines the idea of the analytical Kantorovich-Vlassov's method and the FEM technique. A discretized-continual model is used. The thin-walled structure is divided by longitudinal sections (called linear nodes or nodal lines) only in one (transverse) direction into finite number right or curved finite strips (Fig. 1). The separate finite strip is shown in Fig. 2. In the general case the strips receive both membrane and bending internal forces.

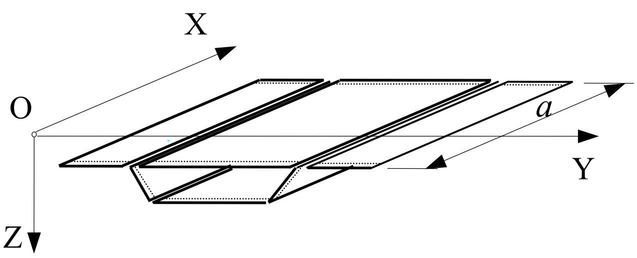

Fig. 1. A model of a Sample Structure

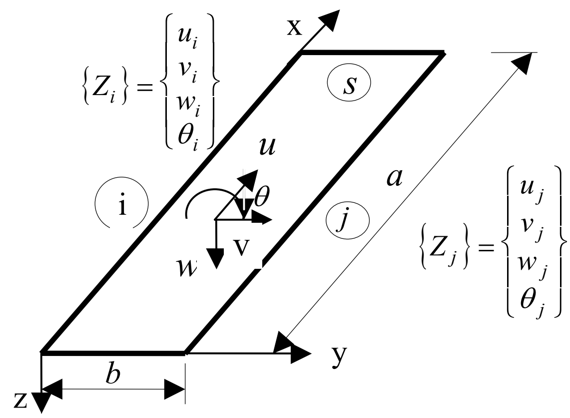

Fig. 2. A Finite Strip of the Prismatic Shell

The basic unknown quantities in the FSM are involved at the nodal lines. Depending on these quantities several approaches of the method can be distinguished. The FSM by directly determination of the displacements is a well developed stiffness approach. It is effective and widely applied in practice. Most of all in this concept the three displacements *u*, *v* and *w* at the points on the nodal lines and the rotation $\theta = \partial w / \partial y$ about these lines are accepted as basic unknowns (Figs. 1, 2). In some cases the unknown nodal parameters can include also strains, curvatures etc. This approach in complex joints leads to the smallest and constant number of the nodal DOF. The flexibility, mixed and hybrid approaches of the FSM are more slightly developed and rarely used. Therefore only the FSM in displacements will be discussed further.

The basic unknown nodal quantities R(x) as well the external loads are presented by the following generalized series (1a), but the displacements (internal forces) S(x, y) at the internal points of the strip as well the external plane loads and actions are developed into the generalized series (1b) [1]-[7]:

$$R(x) \approx \sum_{k=1}^{n} R_k X_k(x)\,, \quad \text{(1a)} \qquad S(x,y) \approx \sum_{k=1}^{n} Y_k(y) X_k(x)\,, \quad \text{(1b)} \qquad (k = 1,2,\ldots,n)\,. \quad (1)$$

Here $X_k(x)$ are longitudinal coordinate functions, $Y_k(y)$ are transverse functions (generalized coordinates) of the strip quantities, R_k are constant coefficients of the nodal quantities (Figs. 1, 2). The function series $X_k(x)$ are adequately continuously differentiable or discontinuous and linearly independent functions of the longitudinal coordinate x, which produce a complete set; they are chosen to satisfy the boundary conditions at the cross ends of the structure a priori. Depending on the choice of the longitudinal and transverse functions of the displacements three basic stiffness versions of the FSM have been developed: semi-analytical, analytical and numerical. The kind of the displacement functions in the longitudinal and transverse directions of the strip appears the basic difference between them.

3. COMPARISONS OF THE SEMI-ANALYTICAL, ANALYTICAL AND NUMERICAL FSM

Comparisons of the three basic versions of the finite strip method are presented in Table 1.

Table 1. Comparisons in quality and in quantity of the semi-analytical, analytical and numerical FSM

Semi-analytical FSM (SFSM)	Analytical FSM (AFSM)	Numerical FSM (NFSM)
Trigonometric series (for simple supporting) [1], beam eigen functions of vibration or stability [1], products of polynomials and trigonometric functions or eigen functions of vibration [2], orthogonal polynomial series and other analytical functions are used as longitudinal functions of the displacements. These function series are continuously differentiable. But for some bending problems involving abruptly changing parameters (properties, loads, supports) the second or the third derivatives of such series should be discontinuous.	So far this version is elaborated only in the case of simple supporting and then trigonometric series are applied [3]-[6].	B_3-spine expressions (a piecewise cubic polynomial interpolation of equal section length) are used as longitudinal functions of the displacements [7]. A suitable spline function with the required continuity or discontinuity conditions can always be found [7]. A variety of spline functions is available. The B_3-splines as displacement functions ensure continuity up to the second order (C_2-continuity) for both in-plane and out-of-plane states achieved by only four DOF per node [7]. To achieve the same continuity conditions for the conventional FEM, it is necessary to have three times as many nodal unknowns [7].
Simple approximating polynomials or directly interpolating polynomials (conventional transverse shape functions) of different kind (power, lagrangian, hermitian etc.) and power (1-3 for *u* and *v*, 3-5 for *w* etc.) [1] are used as transverse functions of the displacements. They satisfy the	The precise solutions of the corresponding differential equations of the plane state and of the bending plate are applied as two accurate transverse functions of the plane stresses and of the strip normal displacement [3]-[6]. They include hyperbolic functions and satisfy the boundary conditions at the	The conventional transverse shape functions of the displacements are the same as in the semi-analytical FSM [7].

compatibility conditions of the displacements at the nodal lines between the adjacent strips and make possible the constant strains of the strip, similarly to the FEM [1].	longitudinal fixed ends of the strip. They ensure also a complete (differential) compatibility of the displacements and of the internal forces (stresses) at the boundary between the adjoining strips.	
The finite strips of lower order LO2 [1] lead to satisfactory results for displacements but don't ensure a good compatibility of the stresses (internal forces) at the boundary between the adjoining strips. In most cases this method produce a more wavy solution than the true one because of the excitation of the higher terms in the function series.	The exact transverse functions of the displacements keep a complete compatibility of the displacements and of the stresses (internal forces) at the boundary between the adjoining strips.	The B-spline expression yields a smoother curve than the actual on it tries to model [7].
To increase the accuracy and the convergence of the results in the longitudinal direction, the term number can be increase but in the transverse direction it is needed to thicken the mesh of the finite strips or to use the strips of higher order (HO2, HO3 [1]). They have got a larger number DOF and enable a good compatibility of the stresses (internal forces) at the nodal lines. The strip HO3 has got an additional internal nodal line (two-times subdivision) and it is more adapted than the HO2 strip for structures with a step-like cross variation of the stiffnesses. The unknown displacements at this line usually are excluded by a static condensation [1]	The precision and the convergence of the results in the longitudinal direction can be enhanced increasing the number of the kept terms. There is no need to get measures for increase of the accuracy and the convergence of the results in the transverse direction. The mesh density (the strip width) does not effect on the precision and the convergence of the results. One-step discretization is adequate in the transverse direction. The number of the DOF at the nodal lines is constant and it is corresponding to the LO2 strip.	The exactness and the convergence of the results in the longitudinal direction can be improved to a certain extent increasing the number of subdividing sections (the section knots) in length into the B_3-splines. The results indicates that in the most cases strips with B_3-spline representations for the displacement functions by subdivisions from 4- to 7-sections provide admissible results [7]. In the transverse direction it is needed to refine the mesh of the finite strips or to use the strips of higher order (HO2, HO3 [1]).
The calculation of the particular finite strip is carried out entirely in displacements (*u*, *v* and *w*) [1].	The solution of the separate strip is realized in a mixed form so that the cross displacement *w* and the stress function are unknowns. Till now the solution of the strip entirely in displacements is not well developed.	The computation of the separate finite strip is held entirely in displacements (*u*, *v* and *w*) [7].
This method leads to results of a lower accuracy in the same discretization and using the same longitudinal functions.	This method yields refined results in the same discretization and using the same longitudinal functions.	This method provides results of high accuracy.
This method is closer to the FEM.	This method is closer to the analytical methods and first of all to the trigonometric series method for analysis of the 2D structures.	This method is closest to the FEM.
For the present versions of this method in internal forces as well mixed and hybrid versions are not known.	This method is occasionally in displacements and semi-hybrid too. Stiffness, mixed and hybrid versions are not developed. A semi-hybrid version partially in internal forces is	Up to now versions of this method in internal forces as well mixed and hybrid versions are not known.

	slightly developed by B. Ulitzki (1962) but a version partially in displacements is slightly developed by A. Aleksandrov (1963).	
The stiffness, mass and load matrices of the strip are obtained by the principle of the minimum total potential strain energy or by the principle of the virtual displacements [1]. These matrices are derived in closed analytical forms for a right isotropic and for orthotropic strip [1].	The stiffness matrix of the strip is derived directly by the consecutively specification of the unity displacements at the nodal lines. The plane distributed loads should satisfy the Dirichlet's conditions. The nodal distributed loads should de self-equilizing at the separate nodal line [4]. For an isotropic strips the stiffness and load matrices are derived in a closed form [3]-[5] but for an orthotropic strip they are obtained in an implicit form [6].	The stiffness, mass and load matrices of the strip are obtained by the principle of the minimum total potential strain energy or by the principle of the virtual displacements, similarly to the FEM.
This method is developed for prismatic, curved, skew, layered thin-walled structures from isotropic and from orthotropic material. Intermediate supports (columns, frames, shear-walls), flexible end supports (frames, beams), different ways of supporting at the cross and longitudinal ends, longitudinal beams, local loads, interior cross diaphragms and beams, strips on an elastic foundation can be considered. The method is generalized for dynamic and stability analyses [2].	Till now this method is elaborated only for static analysis of prismatic thin-walled structures. The internal column supports, longitudinal beams, local loads, temperature actions can be taken into account. The method is expanded for analysis of thin-walled structures on a Winkler's elastic foundation, for analysis of a stress concentration in the prestressed structures (in combination with the FEM).	So far this method is worked up only for static analysis of right thin-walled structures. The various types of end and interior supports (columns), any prescribed external and internal boundary conditions, local (point and patch) loads, multiple spans can be considered [7]. This method overcomes the difficulties experienced in the two rest versions. It is more flexible in the boundary conditions treatment.
This method is well developed for analysis of wider class of thin-walled structures and it is widely applied. It is more general, versatile and powerful.	So far this method is elaborated for analysis of a smaller class thin-walled structures and it is used more rarely. The method is more accurate and can be used for a verification.	Till now this method is elaborated for structural analysis of right box girder bridges and it is used more rarely [3].

4. NUMERICAL EXAMPLE, COMPARISON IN QUANTITY AND DISCUSSION OF RESULTS

For some quantitative comparisons a simply supported isotropic square plate in side *a* and with a cylindrical stiffness *D* subjected to uniform vertical load of an intensity *q* is calculated. The Poisson's ratio is as ν=0,3.

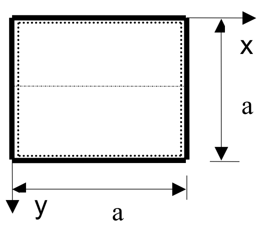

Fig.1.Simply supported square plate

This example is solved by the semi-analytical FSM applying the LO2 strip and by the FEM in [1] using different meshes and different number of the series terms. In view of the double symmetry one half of the plate is computed by the semi-analytical FSM an a quarter plate is calculated by the FEM. The whole plate is solved also as one strip using the same number of the series terms by the Navier's method as well by the M. Levy's method which practically coincides with the analytical FSM for a separate strip. The maximal absolute values of the deflections w_{max} and bending moments $M_{x,max}$ and $M_{y,max}$ in the plate center are given in Table 2.

Table 2. Comparison of some results for a square plate obtained by the semi-analytical and analytical FSM

Number of terms	Semi-analytical FSM			M. Levy's method (Analytical FSM)			Navier's method		
	w_{max} $.\frac{qa^4}{D}.10^{-2}$	$M_{x\max}$ $.qa^2.10^{-2}$	$M_{y\max}$ $.qa^2.10^2$	w_{max} $.\frac{qa^4}{D}.10^{-2}$	$M_{x\max}$ $.qa^2.10^{-2}$	$M_{y\max}$ $.qa^2.10^2$	w_{max} $.\frac{qa^4}{D}.10^{-2}$	$M_{x\max}$ $.qa^2.10^{-2}$	$M_{y\max}$ $.qa^2.10^2$
1	0.414	5.40	5.61	0.411	5.16	4.92	0.416	5.34	5.34
2	0.409	4.94	5.46	0.406	4.72	4.78	0.405	4.69	4.69
3	0.409	5.04	5.49	0.406	4.82	4.81	0.406	4.86	4.94
4	0.409	5.04	5.49	0.406	4.78	4.80	0.407	4.81	4.90
Precisely	0.406	4.79	4.79	0.406	4.79	4.79	0.406	4.79	4.79

The agreement in the displacements by the three methods is observed very good and the convergence is fast. Even only the first term of the series is sufficient. This term is sufficient also for the bending moments by the analytical methods. The results by the M. Levy's method are more accurate and more quickly converging than by the Navier's method. However in the semi-analytical FSM a larger number of the series terms and also of the finite strips are necessary (these results are not included in Table 2) to reach an satisfactory accuracy in the moments since the first term does not ensure this. In each of the three methods the values of the two bending moments differ each other. For the Navier's method these distinctions are smallest but for the other two methods they are more significant and can be explained by the different approaches in the two directions of the plate (different displacement functions, different degree in satisfying of the boundary conditions etc.). It can be assumed that in the presence of the point and patch loads significantly larger number of the series terms will be need to achieve an acceptable precision and convergence in the results. By the semi-analytical FSM the values of the $M_{y,max}$ are more inexact than for the $M_{x,mzx}$ and it is probably due to the more inaccurate approximation in cross direction of the strip. The values of the $M_{x,mzx}$ by the M. Levy's method are more inexact due to the more inaccurate expressions of the displacements in longitudinal direction.

5. CONCLUSIONS, RECOMMENDATIONS AND PROPOSALS

The FSM is especially effective for analysis of the structures with a constant geometry and stiffnesses in longitudinal direction and simply supported at their cross ends by ideal diaphragms, since the decomposition of the series terms including longitudinal functions and a separate solution for each term become possible. This is an important and a wide-spread case in practice, for example for bridge superstructures supported at its end by ideal diaphragms. This reduces the problem dimension by one and yields real simplification of the algorithm and in the computer program, decreases the number of the unknown quantities by one order, diminishes the size and the width of the matrix band of the structure basic set of linear algebraic equations, leads to small amount of the input data, the results and the necessary computer resources. As a final result all these strongly eases the solution by the FSM. If the structure is not simply supported, the series terms are connected and the FSM effectiveness decreases.

The FSM deals hard with the concentrated loads but they are usual for some thin-walled structures. This method is not effective also for analysis of nonlinear problems. The accuracy and the convergence of the solution by the FSM depend on the kind of the longitudinal and transverse functions of the displacements as well on the number of the kept terms in the series of the longitudinal functions. In the semi-analytical and numerical FSM they depend also on the number of the strips, especially in the zones where a rapid change in the stresses and strains is expected. For point and patch loads it is necessary to combine the FSM with the FEM in these zones or to apply methods for accelerate the convergence of the solution since it is very slow in these places and it is slightly effected by an increase of the number of the detained series terms [4]. The convergence of the displacements is relatively fast but for the internal forces it is slower and a larger number of the series terms is need to reach the desirable exactness and convergence of the results. The slowest convergence is available for the shear forces caused by concentrated loads and their Fourier's coefficients have got the smallest possible order of n^{-1}. Although the functions of the moments are continuous and their Fourier's coefficients have got an order of n^{-2} but these functions contain singular points at which the convergence is very slow. For the most-spread distributed loads 1-5 non-zero terms are usually necessary by the semi-analytical and analytical FSM but for point and patch loads 15-50 terms are normally used. Practically the vertical loads are most important and usual for this class structures. These loads on bridge superstructures are of clearly local type since they are caused by the wheels of the vehicles. The series convergence can be improved as the infinite sums are represented in an closed form [4]. The eigenvalue dynamic and stability problems of the structure can be solved by the usual methods. The approximating series in longitudinal direction and the satisfying of the boundary conditions at the cross ends of the strip in the semi-analytical and analytical FSM are identical. The number of the input data and of the results, as well the kind and size of the matrix are closed.

6. FINAL REMARKS

The semi-analytical, analytical and numerical variants of the FSM are compared in quality and in quantity. The FSM combines till a great extent the accuracy of the analytical methods (first of all of the variational Kantorovich-Vlassov's method) and the generality and the technique of the numerical methods, especially of the FEM. The FSM is not so universal, powerful and versatile than the FEM. But for linear elastic analysis of long thin-walled structures the FSM is more effective, simple and economical than the traditional FEM since the 2D problem is reduced to 1D problem. The three basic versions of the FSM are rational, reliable and they are widely used in practice and research. The analytical FSM provides practically precise solution in comparison with the semi-analytical and the numerical FSM, especially for the wider strips since there is no approximation in transverse direction of the strip. Simultaneously the generalization of the analytical FSM is difficult and even practically impossible for more complex cases: curved, skew, layered and others structures, supporting of different type at one end, nonlinear problems etc. The accuracy and the convergence of the solution as well the possibility for satisfaction of different boundary conditions in a great extent depend on the displacement functions and first of all on the longitudinal functions of the displacement. The precision depends on the number of the retained terms of the series too. In the three versions of the FSM carrying out standard folded-plate analysis, the plane stress and bending strips are formulated individually and later combined to form a shell strip. A narrow band matrix of a small to moderate size, little computational efforts and a small amount of input data are available in all cases. All these methods don't deal acceptable with the abrupt transverse changes in the stiffnesses.

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