The theory of random matrices, or random matrix theory, RMT in what follows, has been developed at the beginning of the fifties to describe the statistical properties of energy levels of complex quantum systems, [1], [2], [3]. In the early eighties it has enjoyed renewed interest since it has been recognized as a very useful tool in the study of numerous physical systems. Specifically, it is very useful in the analysis of chaotic quantum systems. In fact, in the last years many papers appeared about the problem of quantum chaos which implies the quantization of systems whose underlying classical dynamics is irregular (i.e. chaotic). The simplest models considered in this field are billiards of various shapes. From the the classical point of view, a point particle in a 2-dimensional billiard displays regular or irregular motion depending on the shape of the billiard; for instance motion in a rectangular or circular billiard is regular thanks to the symmetries of the boundary. On the other hand, billiards of arbitrary shapes imply chaotic motion, i.e. exponential divergence of initially nearby trajectories. In order to study quantum billiards we have to consider the Schroedinger equation in various 2-dimensional domains. The eigenvalues of the Schroedinger equation represent the allowed energy levels of our quantum particle in the billiard under consideration, while the eigenfunction norms represent the probability density of finding the particle in a certain position. The question of quantum chaos is whether the character of the classical motion (regular or chaotic) can influence some properties

of the corresponding quantum system. The connection with random matrix theory came in 1984, in the wake of many numerical experiments suggesting that the eigenvalues of strongly chaotic systems possess the same statistical properties of the eigenvalues of a certain matrix ensemble (see below). Thus, the random matrix theory applies not only in the study of complex quantum systems but also in investigating simple quantum systems whose underlying classical dynamics is chaotic. Actually, the latter are the most common in nature since integrability, as opposite to chaotic behavior, represents the exception rather than the rule. This is the reason for which random matrices are now considered to be a universal tool in quantum mechanics. The fundamental assumption lies in the fact that physical systems exhibiting the same properties in terms of symmetries, although differing in the details and in global properties, do possess the same statistical local properties.

Technically speaking, we consider an $N \times N$ matrix H together with a probability distribution function (pdf) assigned on its elements. If we assume that H is invariant with respect to a certain symmetry operation O, then $H' = OHO^{-1}$. Now the pdf $p(H)$ of the entries of H is determined from the condition $p(H) = p(H')$, together with the requirement that the probabilities of the entries be independent. This statistical independence choice leads to the so called Gaussian systems, respectively called *GOE*, for $\beta = 1$, *GUE*, for $\beta = 2$, *GSE*, for $\beta = 4$, see (1) below. The latter are characterized by the following eigenvalue joint pdf

$$P_{N,\beta} = C_{N,\beta} \exp \left(-\frac{\beta}{2} \sum_{i=1}^{\infty} x_i^2 \right) \prod_{j>i} |x_i - x_j|^{\beta} . \tag{1}$$

Should the simplifying assumption of statistical independence fail, the joint pdf is given by assigning the function $V(x)$ in the following expression

$$P(x_1, ..., x_N) = C_N \exp \left(- \sum V(x_i) \right) \prod_{j>i} |x_i - x_j|^{\beta} .$$

In the development of the theory an extremely important quantity is the correlation function of n points,

$$R_n(x_1, ..., x_n) = \int P_{N,\beta}(x_1, ..., x_N) \, dx_{n+1} ... dx_N . \tag{2}$$

The simplest case is given by $\beta = 2$. In such case, all the correlation functions can be written in terms of a single function

$$R_n(x_1, ..., x_n) = \det\left[K_N(x_i, x_j)\right]_{i,j=1,...,n}$$

with

$$K_N(x_i, x_j) = \exp\ -\frac{1}{2}\left(V(x) + V(y)\right) \sum_{n=0}^{N-1} P_n(x) P_n(y),$$

where $P_n(x)$ represents the orthogonal polynomial of degree n with respect to the weight $\exp(-V(x))$.

The universality assumption of RMT says that in the limit $N \longrightarrow \infty$ the correlation functions and their related statistical quantities are independent of $V(x)$ and tend to "universal" functions. This statement holds provided that by a change of variable the average density $\rho_V(x) \equiv K_N(x, x)$ is normalized to 1. A simple example of such a universal function is given by the *sinc* function

$$K(x, y) = \frac{\sin\left[\pi(x - y)\right]}{\pi(x - y)} = sinc(x),$$

for every $V(x)$. This result is analytically proven for certain potential classes, by means of the theory of orthogonal polynomials, and has been numerically verified in numerous other instances.

The cases $\beta = 1$ and $\beta = 4$ are the most complicated ones. One of the techniques used for their study is based on the use of quaternionic self-dual matrices.

Let us consider the case of the Gaussian unitary ensemble, GUE, for which $V(x) = -\frac{x^2}{2}$, $\beta = 1$. A method has been developed [1] to write the joint pdf as

$$P_{N_1}(x_1, ..., x_N) = \frac{1}{N!} \det\left[\sigma_{N_1}(x_j, x_k)\right]_{j,k=1,...,N}$$

and the $n-$point correlation functions as

$$R_n(x_1, ..., x_N) = \det\left[\sigma_{N_1}(x_j, x_k)\right]_{j,k=1,...,N}.$$

Here the matrix $Q_{N_1} = \left[\sigma_{N_1}(x_j, x_k)\right]_{j,k=1,...,N}$ is a self-dual quaternionic matrix defined by

$$\sigma_{N_1}(x, y) = \begin{matrix} S_N + \alpha & DS_N \\ JS_N & S_N^T + \alpha^T \end{matrix} \equiv \begin{matrix} S_N(x, y) + \alpha(x) & DS_N(x, y) \\ JS_N(x, y) & S_N(y, x) + \alpha(y) \end{matrix}.$$

In the above expressions,

$$S_N(x,y) = \sum_{j=0}^{N-1} \phi_j(x)\,\phi_j(y) + \frac{\overline{N}}{2}\phi_{N-1}(x) \int_{-\infty}^{\infty} \epsilon(y-t)\,\phi_N(t)\,dt,$$

$$\epsilon(x) = \begin{array}{cc} -\frac{1}{2} & x<0 \\ 0 & x=0 \\ \frac{1}{2} & x>0 \end{array}$$

Moreover the operators appearing in the other entries are

$$DS_N(x,y) = -\frac{d}{dy}S_N(x,y),\ IS_N(x,y) = \int_{-\infty}^{\infty} \epsilon(x-t)\,S_N(t,y)\,dt,$$

$$\alpha(x) = \begin{array}{cc} \hat{\phi}_{2m}(x) & N=2m+1 \\ 0 & N=2m \end{array},\ \hat{\phi}_{2m}(x) = \frac{\phi_{2m}(x)}{\int_{-\infty}^{\infty}\phi_{2m}(t)\,dt},$$

$$u(x) = \int_0^x \alpha(t)\,dt,\ JS_N(x,y) = IS_N(x,y) - \epsilon(x-y) + u(x) - u(y).$$

Another interesting statistical quantity is the probability that n points fall in a given randomly chosen interval of length s. The latter is connected to the probability that no eigenvalue lies in the interval of length s, and this probability is given by

$$E_1(0,s) = \prod_{i=0}^{\infty}(1-\lambda_{2i}),\quad \lambda_j = \frac{s}{4}|\mu_j|^2.$$

Here, μ_j and $f_j(x)$ represent the eigenpairs of the following integral equation

$$\mu f(x) = \int_{-1}^{1} \exp(i\pi xys/2)\,f(y)\,dy,$$

or, taking real parts, μ_{2j} and f_{2j} represent the eigenvalues and eigenfunctions of

$$\mu f(x) = 2\int_0^1 \cos\left(\frac{\pi xys}{2}\right) f(y)\,dy. \tag{3}$$

The eigenvalue problem for integral equations has been recently considered in some cases, see e.g. [4], [5], [6], [7].

The asymptotic properties of the integral equation (3) are known, but these depend on the particular situation discussed above. For more general situations, corresponding to arbitrary potentials $V(x)$, there are two possibilities of proceeding. One can attempt either to find analytically the asymptotics corresponding of the choice for $V(x)$; alternatively, if a numerical method is available, the latter can be used to study many physical situations, for many forms of $V(x)$. The former direction has the problem of being dependent on the particular choice for $V(x)$. The second direction instead is a more versatile tool for the study of various situations. We would like to pursue the latter, although at the present stage the construction of a general purpose routine might be a difficult task to achieve. Some preliminary results do however show that it is a feasible path of research.

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