Surface design based upon a combined mesh

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1. Introduction

In this talk, we consider the problem of surface design based upon a mesh that may contain triangular and quadrangular domains. Our goal is to investigate the cases when a combined mesh occurs more preferable for bivariate data interpolation than a pure triangulation.

In scattered bivariate data interpolation, one is required to design a function that fits values z_i at a given point set $P = \{(x_i, y_i), ..., (x_n, y_n)\}$ in the plane. A more general problem is to construct a surface that passes through a given point set in the space. An approach based on a piecewise model supposes two steps, namely:

1. Generate a mesh that divides the convex hull of P into nonintersecting triangular or quadrangular domains with vertices at the points of P. A mesh that contains only triangular domains is called a *triangulation*. If a mesh with exclusively quadrangular domains exists, it is called a *quadrangulation*.

2. Over every domain, compute a patch of surface that provides certain properties for the whole piecewise surface, such as continuity or smoothness.

Among all meshes that exist for a given point set, only some "well structured" ones can be used for surface design. A general heuristic criterion is that the mesh should not contain domains with very large and very small angles. There are a lot of fast algorithms for constructing a triangular mesh with good measure of quality [2]. A well-known mesh is the Delaunay triangulation that maximizes the sum of circumscribing circle areas on each triangle and some other criteria [7].

As it was shown in [1, 4], in some cases the usage of quadrangle domains is more preferable than that of triangles. This result has provoked series of investigations that have been made for the last decade, namely, constructing optimal quadrangulations. The effect of permission to add Steiner points into interior or exterior of the domains has also been studied [5, 8].

Below we present a modification of the well-known flipping algorithm that constructs a *combined* mesh that may contain triangular and quadrangular domains. Then we introduce quality measures for a single domain and compare integral interpolation errors and errors in gradients that give piecewise surface models produced by the flipping algorithm with these quality measures.

2. The flipping algorithm

Denote by t and q, respectively, a triangular and a quadrangular domain in a mesh. Let $S_1(t)$ and $S_2(q)$ be nonnegative scalar functions called the *quality measures* that satisfy the following conditions:

- $S_1(t) = S_2(q) = 0$ for degenerate domains t and q (the areas of t and q equal 0),

- $S_1(t)$, $S_2(q) \le 1$ and there exist t^* and q^* such that $S_1(t^*) = S_2(q^*) = 1$.
- $S_1(t)$ and $S_2(q)$ are scale-invariant, that is depend on the shape but not the size of the domains.

The measures with these properties are called *fair measures* [3]. The standard flipping algorithm starts with some initial triangulation with a "good" measure of quality, for example, the Delaunay triangulation. Then each edge of the mesh that is a diagonal of a convex quadrangle is inspected to satisfy the *flipping rule*: for a diagonal e_{12} and its alternative e_{34} the flipping rule holds if

$$S_1(e_{12}) < S_1(e_{34}),$$

where $S_1(e_{ij}) = \min\{S_1(t_i), S_1(t_j)\}$, and t_i and t_j are the two triangles that share the diagonal e_{ij}. If the edge e_{12} satisfies the flipping rule, it is changed for e_{34}. Such diagonal exchange is called *a flip*. The procedure is repeated for all edges until no flip yields further improvement. The meshes generated by this procedure are called *locally optimal*. It is known that the Delaunay triangulation can be obtained by the flipping algorithm with $S_1(t) = \theta_{\min}$ where $\theta_{\min}$ is the minimum angle in t.

The complexity of the flipping algorithm is $O(n^2)$ as this time may be needed to transform some source triangulation T_1 to some target triangulation T_2. But if T_1 is close to T_2 by its quality measure, the algorithm works very fast and in practice shows linear time.

To obtain a combined mesh we add another alternative to improve a convex quadrangle q: the diagonal is eliminated if the *elimination rule* holds:

$$S_2(q) > \max\{S_1(e_{12}), S_1(e_{34})\},$$

and thus a new quadrangular domain is created. Since the cycle on all edges runs only once, the effect of eliminating diagonals may be improved by sorting the edges on the value

$$S_2(q) - \max\{S_1(e_{12}), S_1(e_{34})\}.$$

A common scheme of the algorithm that constructs a combined mesh is presented below:

1. Construct the Delaunay triangulation,
2. Run the standard flipping procedure with $S_1(t)$ as the quality measure,
3. Sort the edges of the mesh that are diagonals of convex quadrangles on the value
$$S_2(q) - \max\{S_1(e_{12}), S_1(e_{34})\}$$
and put them into a queue Q,
4. Extract the first edge from Q and delete it from the mesh if it is a diagonal of a convex quadrangle and satisfies the elimination rule. Repeat this step until Q is empty.

Note that the first edge in Q may occur a border edge between a triangle and a newly created quadrangle and thus cannot be deleted. The above remarks concerning complexity of the standard flipping algorithm can be applied for the modified version as well, that is $O(n^2)$ for the worst-case performance and $O(n \log n)$ in average.

3. Quality measure functions

Let now define quality measure functions S_1 and S_2 that can be applied to the both triangular and quadrangular domains. Since the structure of the mesh completely describes the shape of the piecewise linear surface, the quality of the mesh can be associated with the error of piecewise linear interpolation. Let f be a continuous scalar function and g be its linear approximation, both defined over a single domain t. Consider the Hessian matrix

$$H(p)=\begin{pmatrix} \frac{\partial^2}{\partial x^2}f(p) & \frac{\partial^2}{\partial x\partial y}f(p) \\ \frac{\partial^2}{\partial x\partial y}f(p) & \frac{\partial^2}{\partial y^2}f(p) \end{pmatrix},$$

and define the norms $E(t)=\| f-g\|_\infty$ and $E_g(t)=\|\nabla f-\nabla g\|_\infty$. Then the following upper bounds hold [6]:

$$E(t)\le c_t\frac{l_{\max}^2}{6},\quad E_g(t)\le c_t R, \tag{1}$$

where $l_{\max}$ is the length of the longest side of t, R is the radius of its circumscribing circle and c_t is the upper bound for the directional curvature of f:

$$\| \mathrm{d}^T H(p)\mathrm{d}\| \le c_t$$

for any $p\in t$ and any unit direction vector d. As it follows from (1), the interpolation error correlates with the size of the domain, whereas the error of the gradient is defined by its shape and approaches infinity when the largest angle of t approaches 180°. Similar upper bounds hold for a single quadrangle domain q and bilinear function g [1].

Consider two pairs of quality measure functions [$S_1(t)$, $S_2(q)$] and [$S_3(t)$, $S_4(q)$] that can be applied to domains of the both types*. (A denotes the area of the domain).

$$S_1(t)=\frac{2}{\sqrt{3}}\min\begin{cases}\sin\theta, & 0\le\theta\le 90^\circ\\ 1+\cos\theta, & 90<\theta<180^\circ\end{cases},\quad S_2(q)=\min\begin{cases}\sin\theta, & 0\le\theta\le 90^\circ\\ 1+\cos\theta, & 90<\theta<180^\circ\end{cases}, \tag{2}$$

$$S_3(t)=\left(\frac{4}{3\sqrt{3}}\frac{A}{R^2}\right)^{\frac{1}{3}},\quad S_4(q)=\left(\frac{A}{2R_{\max}^2}\right)^{\frac{1}{3}}. \tag{3}$$

Here $R_{\max}$ denotes the radius of the maximum circumscribing circle on all four triples of vertices of q. The measure $S_1(t)$ in (2) is similar to $S_1(t)=\theta_{\min}$ that produces the Delaunay triangulation. We use this form here to take more care about excluding large angles in triangular and quadrangular domains. The measure $S_3(t)$ in (3) is mentioned in [6] in the equivalent form

$$S_3(t)=\frac{4}{\sqrt{3}}\frac{A}{(l_1l_2l_3)^{2/3}}$$

as the one of the most efficient measures that correlate with the errors of the gradients of an interpolated piecewise linear function. A similar function $S_4(q)$ is used for a quadrangle domain where $R_{\max}^2$ in the denominator controls the value of large angles.

Contour plots on Figure 2 illustrate dependence of the measure value on the domain shape. For a triangle, one side (displayed by the bold line) is fixed whereas the third vertex varies along with the

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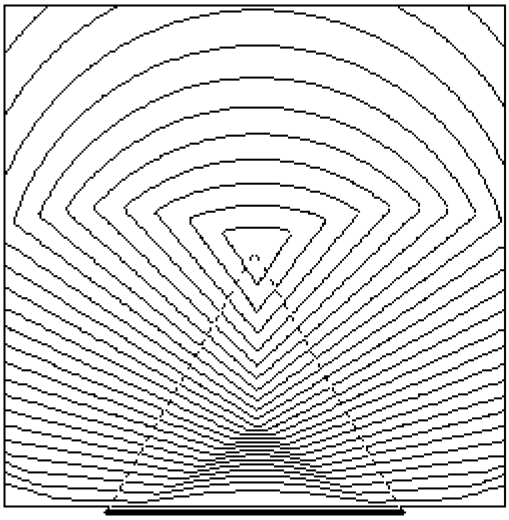
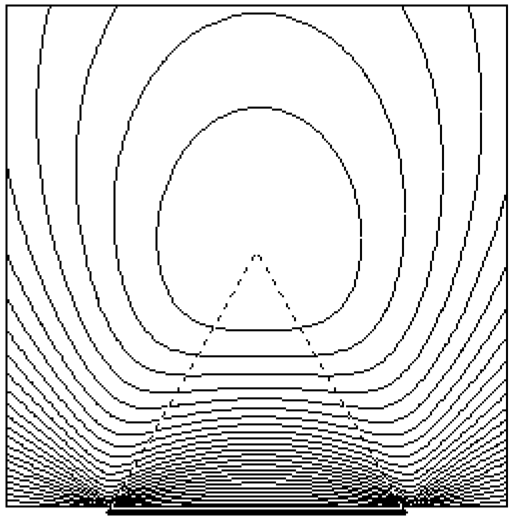
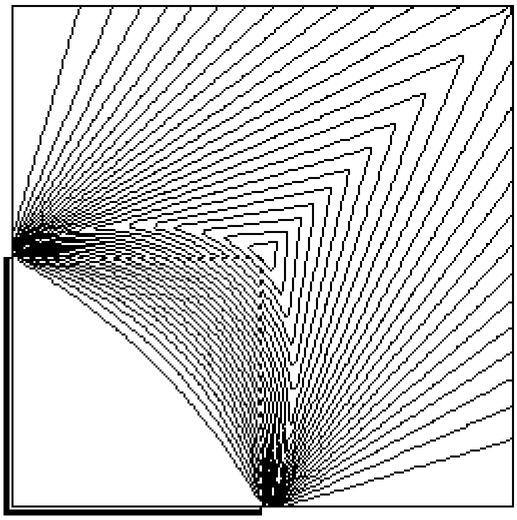
* See [6] for a detailed review of quality measures for triangular domains.



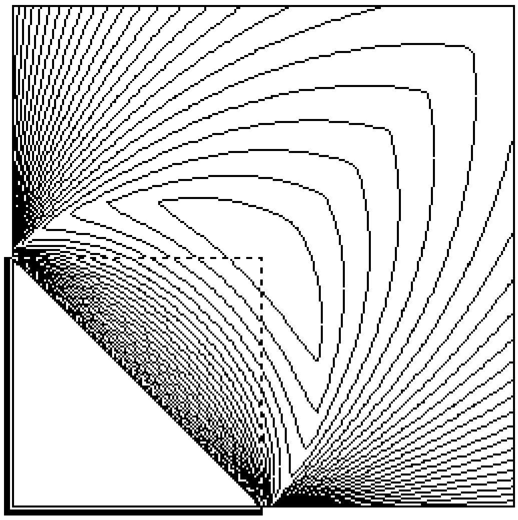
$S_1(t)$



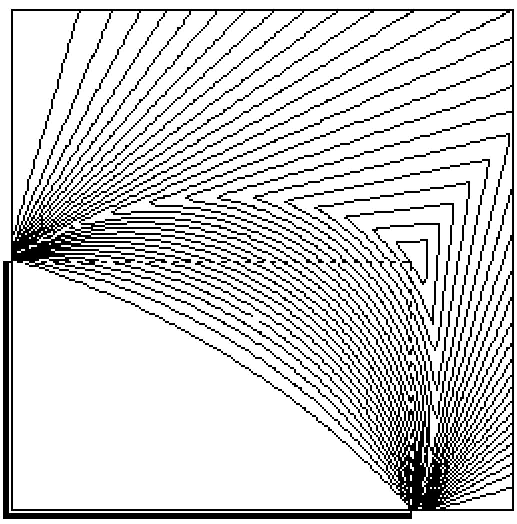
$S_3(t)$



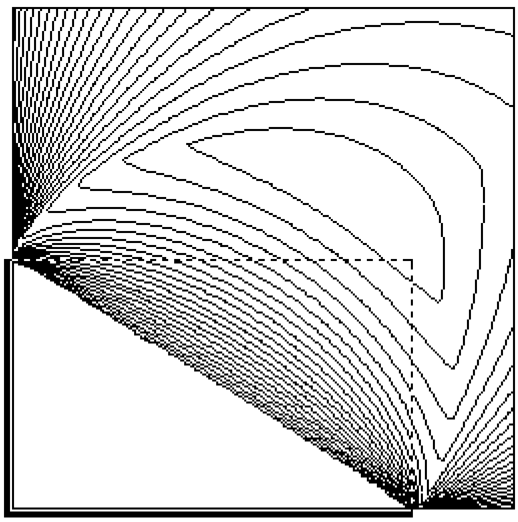
$S_2(q)$, square



$S_4(q)$, square



$S_2(q)$, rectangle



$S_4(q)$, rectangle

Figure 2: Quality measures for triangular and quadrangular domains

shape of the triangle. The dotted line shows the optimal shape. The measure of a quadrangle is presented for a square and a rectangle with two sides fixed.

An equilateral triangle is the optimal shape for $S_1(t)$ and $S_3(t)$ as well as a square is optimal for $S_2(q)$ and $S_4(q)$. It seems a surprising fact that a rectangle is not the optimal shape for $S_4(q)$ if two perpendicular sides are fixed. Indeed, the area of the upper right triangle of q with a vertex on the fixed circumscribing circle takes its maximum when its upper and right sides have the same length.

Now let us try to determine which of the two pairs, $[S_1(t), S_2(q)]$ or $[S_3(t), S_4(q)]$, is more preferable for the piecewise combined linear and bilinear interpolation. Tables 1-3 present the results of the computational experiment that was carried out to compare interpolation errors and errors in gradients for the above four measures and three types of test functions, namely, strictly convex, saddle-shaped and randomly shaped. The latter function was generated so that the number of local extremes did not exceed the number of the vertex in the mesh. The total interpolation error and the total error in gradients were computed as an approximated value of the L_2-norm over all triangular and quadrangular domains in the mesh. The tests were made for 500 points distributed uniformly in a circle. For each test function there were generated 20 point distributions, the average error values were taken as the result. The values of interpolation errors and errors in gradients for a convex test function are presented in Table 1. Similar results for a saddle-shaped and random test functions are presented in Tables 2 and 3. The minimum and the maximum error values in a column are denoted respectively by '+' and '–'.

Table 1: Summary error values for convex test function

Measure	$\| f - g \|_2$	$\| \nabla f - \nabla g \|_2$
$S_1(t)$	$1.47488^{(+)}$	$1.36158^{(+)}$
$S_2(q)$	1.48296	1.36387
$S_3(t)$	$1.49133^{(-)}$	$1.36252^{(-)}$
$S_4(q)$	1.47829	1.36528

Table 2: Summary error values for saddle-shaped test function

Measure	$\| f - g \|_2$	$\| \nabla f - \nabla g \|_2$
$S_1(t)$	0.43240	0.39430
$S_2(q)$	0.41975	0.38893
$S_3(t)$	$0.44917^{(-)}$	$0.39478^{(-)}$
$S_4(q)$	$0.41348^{(+)}$	$0.38847^{(+)}$

Table 3: Summary error values for randomly shaped test function

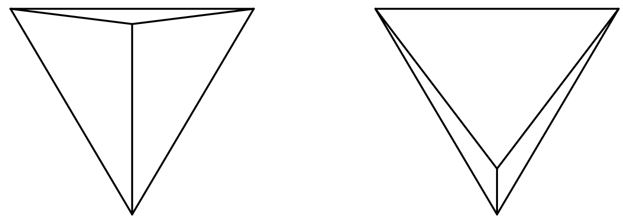
Measure	$\| f - g \|_2$	$\| \nabla f - \nabla g \|_2$
$S_1(t)$	19.53332	$7.16701^{(-)}$
$S_2(q)$	$19.21260^{(+)}$	$7.00942^{(+)}$
$S_3(t)$	$19.62211^{(-)}$	7.15842
$S_4(q)$	19.21331	7.02106

The experiment shows that a triangular mesh with the shape quality $S_1(t)$ provides the best quality of interpolation only if the interpolated function is strictly convex, as well as a saddle-shaped function is well interpolated by bilinear patches within a combined mesh. The measure $S_4(q)$ works a little bit better than $S_2(q)$, but they both give an essential gain to a combined mesh in compare with a pure triangulation. These two experimental results correlate with the theoretical ones [1].

The results in Table 3 demonstrate smaller error values provided by combined meshes and, what is more interesting, better stability for randomly shaped test functions that are good models of real terrain surfaces.

4. Some other resources for mesh improvement

The elimination rule allows decreasing the number of domains with large angles in a combined mesh. Nevertheless, any well-shaped triangulation or combined mesh may occur impossible for point configurations such as presented below:



One remedy from this disease is adding into the mesh interior or exterior Steiner points [2]. However, in some applications, no acceptable procedure can automatically evaluate or approximate the interpolated function at the additional points. Thus, another aspect of the mesh generation problem arises: is it possible to improve a locally optimal mesh without using Steiner points? The method that we propose acts in the opposite direction: we continue rarefying the mesh and exclude some "bad" input points from the mesh generating procedure. Because the function values at these points should not be lost, the surface must now contain nonlinear patches such that the resulting surface passes through the excluded points. The goal for further investigations can be finding a compromise between simplicity of the patch model and requested accuracy.

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