# On a purely real quaternionic Dirac equation, and its relations with Maxwell's equations

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## 1 Purely real quaternionic equation

From the very appearance of the Dirac equation many researchers considered that the form of writing it was unsatisfactory, because the algebra of matrices used has dimension 16, and the number of scalar equations under consideration is 4. This discrepancy explains the fact that A. Sommerfeld posed the problem of rewriting the Dirac equation in such form that the rank of the algebra of the matrices involved coincides with the number of components of the wave function, and apparently he made the first attempt to solve it (see [6]). A solution for this problem was proposed in [2] where a simple matrix transformation $A$ was obtained, which allows us to rewrite the classical Dirac operator applied to $\mathbb{C}^4$-vectors as a quaternionic operator applied to biquaternionic-valued functions.

Using the algebraic properties of $A$ and $A^{-1}$; we can rewrite the equation

$$D[\Psi] := \left( \hbar \left( \frac{\gamma_0}{c}\partial_t - \sum_{k=1}^{3} \gamma_k \partial_k \right) + im \right)[\Psi] = 0; \qquad (1)$$

in a quaternionic form

$$\left( \frac{\hbar}{c}\partial_t M^{i_1} - \hbar D + m M^{i_2} \right) F = 0; \qquad (2)$$

where $D$ is the Mosil-Theodoresco operator (see e.g. in [1]).

### 1.1 An involutive symmetry of the Dirac equation

Since the quaternionic operator of this equation is purely real, we can consider the inverse transformation of the real part of $F$ solution of (2),

$$\Psi' := A^{-1}[\mathrm{Re}[F]] = A^{-1}[\mathrm{Re}[A[\Psi]]];$$

and, by a simple calculation, we obtain from this that the function

$$\breve{\psi} := \begin{pmatrix} \psi_3^* \\ -\psi_2^* \\ -\psi_1^* \\ \psi_0^* \end{pmatrix},$$

where $\psi_k^*$ is the complex conjugation of $\psi_k$, $k = \overline{0,3}$, is a solution of (1) also.

### 1.2 Quaternionic equation of conservation of currents

Using (2) we can introduce a quaternionic equation of conservation of currents in the form

$$\frac{1}{c}\partial_t |F|^2 = -\hbar\left[(DF i_1)\overline{F} + F\left(i_1 D\overline{F}\right)\right],$$

and the equality $|F|^2 = |\psi|^2$ is valid [5].

### 1.3 Charge conjugation

Considering the problem of finding the probability density function of antiparticles (said for example the positron), let us take the quaternionic Dirac equation (obtained again by using the transformation A) for an electron moving in the electromagnetic field

$$R^{el}[F_{el}] = \left(\frac{\hbar}{c}\partial_t M^{i_1} - \hbar D + mM^{i_2} - ie\left(\hat{A} M^{i_1} + \sum_{k=1}^{3} i_k \hat{A}_k\right)\right) F_{el} = 0. \quad (3)$$

We show that the function corresponding to the positron is obtained by

$$F_{pos} = F_{el}^*.$$

where $F_{el}^*$ is simply the standard complex conjugation of $F_{el}$.

## 2 The relation between Maxwell's equations and Dirac equation

We shall show two new kinds of relations between these systems which are obtained due the reformulation (2).

### 2.1 Relation between massive Dirac equation for free particles and time-harmonic Maxwell's equations for sourceless isotropic homogeneous media

Let $F$ has the form

$$F = f e^{i\frac{E}{\hbar}t},$$

where $E$ denotes the energy of the particle. Then (2) becomes

$$\left(D + M^{\vec\alpha}\right) f = 0; \tag{4}$$

where

$$\vec\alpha := -\frac{1}{\hbar}\left(i\frac{E}{c}i_1 + m i_2\right).$$

Let us consider the Maxwell equations

$$\begin{aligned} \operatorname{rot}\vec H &= -i\omega\varepsilon\vec E; \\ \operatorname{rot}\vec E &= i\omega\mu\vec H; \\ \operatorname{div}\vec H &= 0; \\ \operatorname{div}\vec E &= 0; \end{aligned} \tag{5}$$

where $\omega$ is the frequency, $\varepsilon$ and $\mu$ are the absolute permittivity and permeability respectively. Let us denote

$$\begin{aligned} \vec\varphi &:= -i\omega\varepsilon\vec E + \kappa\vec H; \\ \vec\psi &:= i\omega\varepsilon\vec H + \kappa\vec H. \end{aligned}$$

where $\kappa := \omega\sqrt{\varepsilon\mu}$ is the wave number. One can see that $\vec\varphi$ satisfies the equation

$$(D - \kappa)\vec\varphi = 0; \tag{6}$$

and $\vec\psi$ satisfies the equation

$$(D + \kappa)\vec\psi = 0. \tag{7}$$

Then, if

$$\kappa^2 = \vec\alpha; \tag{8}$$

the following equalities hold (see [3])

$$\begin{aligned} \left(D + M^{\vec\alpha}\right) &= P^{+}(D + M^{\kappa}) + P^{-}(D - M^{\kappa}); \\ (D + k) &= P^{+}\left(D + M^{\vec\alpha}\right) + P^{-}\left(D - M^{\vec\alpha}\right); \quad (9) \\ (D - k) &= P^{-}\left(D + M^{\vec\alpha}\right) + P^{+}\left(D - M^{\vec\alpha}\right); \quad (10) \end{aligned}$$

with

$$P^{\pm} := \frac{1}{2k} M^{\kappa \pm \vec\alpha}.$$

This operator equalities give a very simple and useful relation between the Dirac bispinors and the time-harmonic electromagnetic field. Moreover, (8) agrees with the well known in quantum mechanics relation between the frequency an the impulse:

$$E^2 = p^2c^2 + m^2c^4.$$

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## References

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