On the factorization of the Schrödinger operator and its applications for studying some first order systems of mathematical physics

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Let D denote the well known Moisil-Theodoresco operator acting on biquaternion valued functions f according to the rule $Df = \sum_{k=1}^{3} e_k \partial_k f$, where $\partial_k = \frac{\partial}{\partial x_k}$, e_k are standard quaternionic imaginary units (see, e.g., [3], [6]) and the function f of real variables x_1, x_2, x_3 has the form $f = \sum_{k=0}^{3} f_k e_k$, where $f_k \in \mathbb{C}$, $k = \overline{0;3}$ are continuously differentiable functions.

Consider the Schrödinger operator $-\Delta + v$ applied to a scalar function φ. Let $\vec{\alpha}$ be a purely vectorial biquaternion valued function such that $D\vec{\alpha} + (\vec{\alpha})^2 = -v$. Then, as was shown in [1], [2], the following equality is valid

$$(-\Delta + v)\varphi = (D + M^{\vec{\alpha}})(D - M^{\vec{\alpha}})\varphi; \qquad (1)$$

where $M^{\vec\alpha}$ stands for the operator of multiplication by $\vec\alpha$ from the right-hand side: $M^{\vec\alpha} f = f \cdot \vec\alpha$.

The operator $D + M^{\vec\alpha}$ is closely related to the static Maxwell system, to the classical Dirac operator as well as to the so called Beltrami or force-free fields (see [4]). In [5] the factorization (1) was used in order to obtain integral representations for solutions of the equations $(D + M^{\vec\alpha})f = 0$, $(D + \gamma)f = 0$ and $\mathrm{rot}\, \vec f + \gamma \vec f = 0$, where $\vec\alpha = \alpha_1(x_1)e_1$ and $\gamma = \gamma(x_1)$ is a scalar function. These three equations were reduced to a set of Schrödinger equations.

In the present work we study the case when $\vec\alpha$ has the form

$$\vec\alpha = \alpha_1(x_1)e_1 + \alpha_2(x_2)e_2 + \alpha_3(x_3)e_3. \tag{2}$$

For example, in a particular case when $\vec\alpha = -((i\omega + \psi(x_1))e_1 + me_2)$ with ω and m being constants, the operator $D + M^{\vec\alpha}$ represents the Dirac operator for a particle of mass m, frequency ω moving in an electric field with the potential ψ [4].

Moreover, let the permittivity ε of a medium be of the form $\varepsilon(x) = \varepsilon_1(x_1) \cdot \varepsilon_2(x_2) \cdot \varepsilon_3(x_3)$. Then the static Maxwell system

$$\mathrm{div}(\varepsilon(x)\vec E(x)) = 0 \quad \text{and} \quad \mathrm{rot}\, \vec E(x) = 0$$

is equivalent to the equation

$$(D + M^{\vec\alpha(x)})E(x) = 0,$$

where $E = \sqrt{\varepsilon}\vec E$ and $\vec\alpha$ has the form (2) with $\alpha_k = \partial_k \varepsilon_k/(2\varepsilon_k)$, $k = 1, 2, 3$.

Denote by $\vec\alpha^{(k)}$ the result of the following involution:

$$\vec\alpha^{(k)} = e_k \vec\alpha\, \overline{e_k}, \quad k = 0, 1, 2, 3,$$

where the bar stands for the quaternionic conjugation.

The following proposition is valid.

Proposition 1 *Let f be a solution of the equation*

$$(D + M^{\vec\alpha})f = 0. \tag{3}$$

Then the components f_k are solutions of the Schrödinger equations $(-\Delta + w_k)f_k = 0$, $k = 0, 1, 2, 3$, where $w_k = D\vec\alpha^{(k)} - (\vec\alpha^{(k)})^2$.

The following fact gives us a method for constructing exact solutions of (3) having obtained solutions of the corresponding Schrödinger equations.

Proposition 2 Let four scalar functions g_k, $k = 0, 1, 2, 3$ satisfy the following equations $(-\Delta + v_k)g_k = 0$, where $v_k = -D\vec{\alpha}^{(k)} - (\vec{\alpha}^{(k)})^2$. Then the function

$$f = (D - M^{\vec{\alpha}}) \sum_{k=0}^{3} g_k e_k \tag{4}$$

is a solution of (3).

Moreover, we prove the following theorem which guarantees that under certain conditions any solution of (3) has the form (4).

Theorem 3 Let Ω be some domain in R^3 which can coincide with the whole space, $F(\Omega)$ and $G(\Omega)$ some functional spaces such that the equation

$$(-\Delta + w_k(x))u(x) = \mu(x); \quad x \in \Omega; \quad k = 0, 1, 2, 3$$

has a solution for any right part $\mu \in F(\Omega)$ and the solution u belongs to $G(\Omega)$. Then any solution $f \in F(\Omega)$ of (3) has the form $f = (D - M^{\vec{\alpha}})g$, where $g \in \mathrm{im}\,(D + M^{\vec{\alpha}})(G(\Omega))$ and g_k satisfy the equations $(-\Delta + v_k)g_k = 0$ in Ω.

Finally, let u_k be a fundamental solution of the operator $-\Delta + v_k$, $k = 0, 1, 2, 3$. Then the integral operator $T_k\varphi(x) = \int_\Omega u_k(x - y)\varphi(y)dy$ under some natural conditions is a right inverse operator for the operator $-\Delta + v_k$. Denote

$$T_{\vec{\alpha}} f = (D - M^{\vec{\alpha}})(\sum_{k=0}^{3} (T_k f_k e_k)).$$

It can be verified that $T_{\vec{\alpha}}$ is a right inverse operator for the operator $D + M^{\vec{\alpha}}$.

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