ON A QUATERNIONIC FORM OF THE MAXWELL EQUATIONS FOR THE TIME-DEPENDENT ELECTROMAGNETIC FIELDS IN CHIRAL MEDIA

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1 Maxwell's equations for chiral media

In [2] a quaternionic reformulation of the time-harmonic Maxwell equations for chiral media was proposed and used in [4] in order to construct complete systems of quaternionic fundamental solutions convenient for numerical analysis of scattering boundary value problems. In the present contribution we give a quaternionic reformulation of time-dependent Maxwell's equations for chiral media. The Maxwell system is written as a single quaternionic equation. We obtain a fundamental solution of this equation and use it for solving Maxwell's system with sources.

Consider time-dependent Maxwell's equations with sources

$$\operatorname{rot} \vec{E}(t,x) = -\partial_t \vec{B}(t,x);$$

$$\operatorname{rot} \vec{H}(t,x) = \partial_t \vec{D}(t,x) + \vec{j}(t,x),$$

$$\operatorname{div} \vec{E}(t,x) = \frac{\rho(t,x)}{\varepsilon}, \qquad \operatorname{div} \vec{H}(t,x) = 0 \tag{1}$$

and the constitutive relations of Drude-Born-Fedorov corresponding to the chiral media (see, e.g., [7], [8], [9])

$$\vec{B}(t,x) = \mu(\vec{H}(t,x) + \beta \operatorname{rot} \vec{H}(t,x)),$$

$$\vec{D}(t,x) = \varepsilon(\vec{E}(t,x) + \beta \operatorname{rot} \vec{E}(t,x)), \tag{2}$$

here β is the chirality measure of the medium. β, ε, μ are real constants.

We use also the Maxwell system with incorporated constitutive relations (2)

$$\operatorname{rot} \vec{H}(t,x) = \varepsilon(\partial_t \vec{E}(t,x) + \beta \partial_t \operatorname{rot} \vec{E}(t,x)) + \vec{j}(t,x), \tag{3}$$

$$\operatorname{rot} \vec{E}(t,x) = -\mu(\partial_t \vec{H}(t,x) + \beta \partial_t \operatorname{rot} \vec{H}(t,x)). \tag{4}$$

Separating $\vec{E}$ and $\vec{H}$ we obtain the equations which represent analogues of the wave equations for non-chiral media

$$\operatorname{rot}\operatorname{rot} \vec{E}(x) + \varepsilon\mu \partial_t^2 \vec{E}(x) + 2\beta\varepsilon\mu \partial_t^2 \operatorname{rot} \vec{E}(x) + \beta^2 \varepsilon\mu \partial_t^2 \operatorname{rot}\operatorname{rot} \vec{E}(x)$$
$$= -\mu \partial_t \vec{j}(x) - \beta\mu \partial_t \operatorname{rot} \vec{j}(x),$$

$$\operatorname{rot}\operatorname{rot} \vec{H}(x) + \varepsilon\mu \partial_t^2 \vec{H}(x) + 2\beta\varepsilon\mu \partial_t^2 \operatorname{rot} \vec{H}(x) + \beta^2 \varepsilon\mu \partial_t^2 \operatorname{rot}\operatorname{rot} \vec{H}(x) = \operatorname{rot} \vec{j}(x). \tag{5}$$

It should be noted that when $\beta = 0$, (5) reduce to the wave equations for non-chiral media.

2 Some notations from quaternionic analysis

We will consider biquaternion-valued functions defined in some domain $\Omega \subset \mathbb{R}^3$. On the set of continuously differentiable such functions the well known Moisil-Teodoresco operator is defined by the expression $D := i_1 \frac{\partial}{\partial x_1} + i_2 \frac{\partial}{\partial x_2} + i_3 \frac{\partial}{\partial x_3}$ (see, e.g., [1]). Denote $D_\alpha := D + \alpha$, where $\alpha \in \mathbb{C}$. The fundamental solution for this operator is known [5] (see also [6]):

$$K_\alpha(x) = -\operatorname{grad}\Theta_\alpha(x) + \alpha\Theta_\alpha(x) = \left(\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|}\right)\Theta_\alpha(x), \tag{6}$$

$x = \sum_{k=1}^{3} x_k i_k$. We assume that $\operatorname{Im}\alpha \geq 0$, and the fundamental solution $\Theta_\alpha(x)$ of the Helmholtz operator is chosen as follows

$$\Theta_\alpha(x) = -\frac{e^{i\alpha|x|}}{4\pi|x|}.$$

3 Field equations in quaternionic form

In this section we rewrite the field equations from Section 1 in quaternionic form.

Let us introduce the following quaternionic operator

$$A := \beta\sqrt{\varepsilon\mu}\,\partial_t D + \sqrt{\varepsilon\mu}\,\partial_t - iD \tag{7}$$

and consider the purely vectorial biquaternionic function

$$V(t,x) = \vec{E}(t,x) - i\sqrt{\frac{\mu}{\varepsilon}}\vec{H}(t,x).$$

The quaternionic equation

$$AV(t,x) = \sqrt{\frac{\mu}{\varepsilon}}\vec{j}(t,x) + i\frac{\rho(t,x)}{\varepsilon} \tag{8}$$

has the scalar and the vector parts in the form:

$$-\beta\sqrt{\varepsilon\mu}\,\partial_t \operatorname{div}\vec{E}(t,x) + \sqrt{\frac{\mu}{\varepsilon}}\operatorname{div}\vec{H}(t,x) +$$

$$i(\operatorname{div}\vec{E}(t,x) + \beta\mu\partial_t \operatorname{div}\vec{H}(t,x)) = i\frac{\rho(t,x)}{\varepsilon}; \tag{9}$$

$$\sqrt{\varepsilon\mu}\,\partial_t \operatorname{rot}\vec{E}(t,x) + \sqrt{\varepsilon\mu}\,\partial_t\vec{E}(t,x) - \sqrt{\frac{\mu}{\varepsilon}}\operatorname{rot}\vec{H}(t,x) -$$

$$-i(\operatorname{rot}\vec{E}(t,x) + \beta\mu\partial_t \operatorname{rot}\vec{H}(t,x) + \mu\partial_t\vec{H}(t,x)) = \sqrt{\frac{\mu}{\varepsilon}}\vec{j}(t,x). \tag{10}$$

The real part of (10) coincides with (3) and the imaginary part coincides with (4). Applying divergence to the equations (3) and (4) gives us

$$\partial_t \operatorname{div} \vec{H}(t;x) = 0 \quad \text{and} \quad \partial_t \operatorname{div} \vec{E}(t;x) = 0.$$

Taking into account the last two equalities we obtain from (9) that the vectors $\vec{E}$ and $\vec{H}$ satisfy the equation (1).

Thus the quaternionic equation (8) is equivalent to the Maxwell system (1), (3) and (4).

It should be noted that for $\beta = 0$ from (7) we obtain the operator which was studied in [3] with the aid of the factorization of the wave operator for non-chiral media

$$\varepsilon\mu\partial_t^2 - \Delta_x = (\sqrt{\varepsilon\mu}\,\partial_t + iD)(\sqrt{\varepsilon\mu}\,\partial_t - iD).$$

In the case under consideration we obtain a similar result. Let us denote by A^* the complex conjugate operator of A

$$A^* := -\beta\sqrt{\varepsilon\mu}\,\partial_t D + \sqrt{\varepsilon\mu}\,\partial_t + iD$$

For simplicity we consider now a sourceless situation. In this case the equations (5) are homogeneous and can be represented as follows

$$AA^*\vec{U}(t;x) = 0,$$

where $\vec{U}$ stands for $\vec{E}$ or for $\vec{H}$.

4 The fundamental solution for the operator A

We construct the fundamental solution for the operator A using the results of the previous section and well known facts from quaternionic analysis. Consider the following equation

$$Af(t;x) = (-\beta\sqrt{\varepsilon\mu}\,\partial_t D + \sqrt{\varepsilon\mu}\,\partial_t - iD)f(t;x) = \delta(t)\delta(x).$$

Applying the Fourier transform F with respect to the time-variable t we arrive at

$$(-\beta\sqrt{\varepsilon\mu}\,i\omega D + \sqrt{\varepsilon\mu}\,i\omega - iD)F(\omega;x) = \delta(x),$$

where $F(w,x)=\mathcal{F}\{f(t,x)\}$. The last equation can be rewritten as follows

$$(D+\alpha)(\beta\sqrt{\varepsilon\mu}\,\omega-1)iF(w,x)=\delta(x),$$

where $\alpha=\frac{\sqrt{\varepsilon\mu}\,\omega}{\beta\sqrt{\varepsilon\mu}\,\omega-1}$. The fundamental solution of D_α is given by (6), so we have

$$(\beta\sqrt{\varepsilon\mu}\,\omega-1)iF(w,x)=-\operatorname{grad}\Theta_\alpha(x)+\alpha\Theta_\alpha(x)=(\alpha+\frac{x}{|x|^2}-i\alpha\frac{x}{|x|})\Theta_\alpha(x),$$

from where

$$F(w,x)=\left[\left(\frac{-\sqrt{\varepsilon\mu}\,\omega}{(\beta\sqrt{\varepsilon\mu}\,\omega-1)^2}\right)\left(1-\frac{ix}{|x|}\right)+\frac{ix}{|x|^2}\frac{1}{\beta\sqrt{\varepsilon\mu}\,\omega-1}\right]\frac{e^{i|x|\frac{\sqrt{\varepsilon\mu}\,\omega}{\beta\sqrt{\varepsilon\mu}\,\omega-1}}}{4\pi|x|}$$

Applying the inverse Fourier transform we obtain the fundamental solution of the operator A.

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