Mathematical model of the laminated frame for a dome

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1 Introduction

For the International Colonial Exhibition of 1931 in Vincennes (Paris), the Belgian architect Henry Lacoste (°1885 Tournai, †1968 Brussels) was in charge of the construction of the pavilion for the Belgian Congo colony. The building encompassed three domes each reflecting the structure of a typical Central African cabin, as can be seen in the picture below (Figure 1). For the construction of these domes Lacoste used an at that time known industrial technique based on a wooden laminated frame, approximating the desired shape (system owned by the Europäische Zollbau Syndicat A.G.).

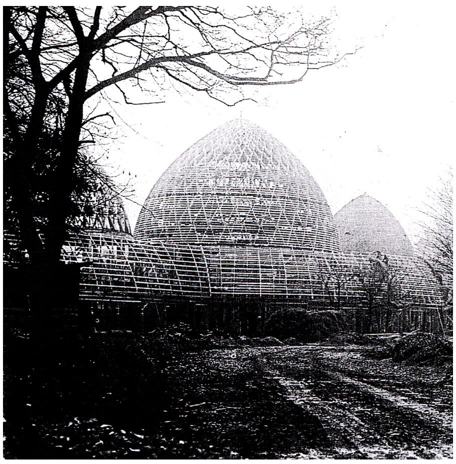

Figure 1: The Belgian Congo pavilion under construction (Paris, 1931)

A mathematical model of the wooden skeleton of the dome is developed, with a twofold aim: first, to reconstruct the original pavilion and next, to gain insight in the building technique such that it could be applied to constructions of other shapes.
In this contribution, we are reporting about the computational geometrical model for the Lacoste dome.

2 Material available

As almost all pavilions of the exhibition, including the one of Belgian Congo, have disappeared completely, the only remaining material at our disposal for the case study of this particular construction were a few photographs, a schematic plan by the architect and two papers reporting on technical details of the construction system, [1], [2].
All parameters appearing in the model have been estimated as accurately as possible on the basis of this available material.

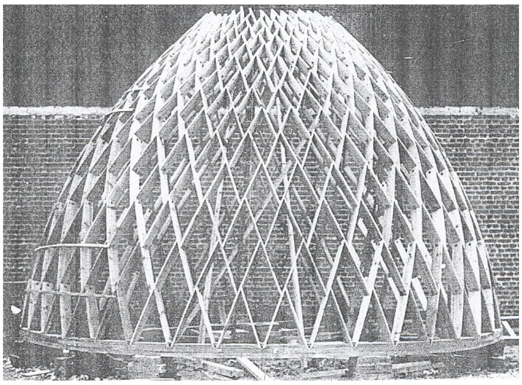

Figure 2: Photograph of the wooden laminated frame

3 Differential geometry aspects: a first approximation

From the architecture plan, it is seen that the dome is a "bullet" shaped surface of revolution, originating from the rotation of a part of a circle around a chord which is not a centre line (see Figure 3).

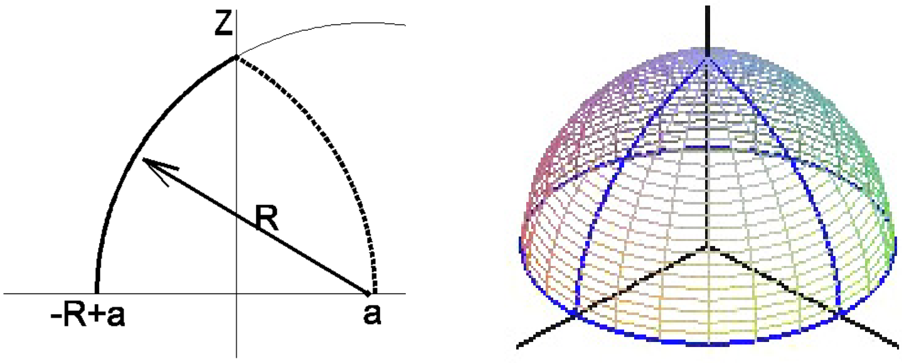

Figure 3: Generating the bullet shaped surface

If R is the radius of this circular arc and a is the distance of its centre to the axis of revolution, which is taken to be the Z-axis of a cartesian co-ordinate system, then the parametric equations of the dome are given by

$$P(u,v) = [R(\cos(u) - \cos(u^*))\cos(v), R(\cos(u) - \cos(u^*))\sin(v), R\sin(u)],$$
$$u \in [0, u^*],\ v \in [0, 2\pi],$$

where we have put

$$\cos(u^*) = \frac{a}{R}, \quad 0 < u^* < \frac{\pi}{2}.$$

The parameter u represents the latitude with respect to the XY-plane, while v stands for the longitude with respect to the XZ-plane.

On the basis of the material available (see section 2), more specifically the photograph of the wooden structure under construction (see Figure 2), it occurred to us that, identifying in a first approximation the joints of the successive laminae with a geometric point, the curve connecting those points should be a rhumb line or loxodrome, enclosing a constant angle with the meridians $v =$ constant. .

If $v = f(u)$ is the equation of the right handed loxodrome cutting the meridians under the constant angle ψ, then f satisfies the differential equation:

$$f'(u) = \frac{\tan(\psi)}{\cos(u) - \cos(u^*)},$$

which yields

$$v = f(u) = \frac{\tan(\psi)}{\sin(u^*)} \ln \frac{1 - \cos(u^*) + \sin(u^*)\tan(\frac{u}{2})}{1 - \cos(u^*) - \sin(u^*)\tan(\frac{u}{2})},$$

or

$$u = f^{-1}(v) = 2\arctan\left[(1 - \cos(u^*))\frac{\exp(\frac{\sin(u^*)}{\tan(\psi)}v) - 1}{\exp(\frac{\sin(u^*)}{\tan(\psi)}v) + 1}\right]$$

where we have chosen the constant of integration such that

$$v_0 = f(u_0), \quad u_0 = 0, \quad v_0 = 0.$$

Clearly, the corresponding left handed loxodrome issuing from the same point $(0, 0)$ is then given by $v = -f(u)$.

Plotting the dome surface and several of those loxodromes (right and left handed) yields a result resembling quite convincingly the available photographs, especially the picture of the ceiling (see Figure 4) showing the "spirals" as the orthogonal projection of the loxodromes on a horizontal plane.

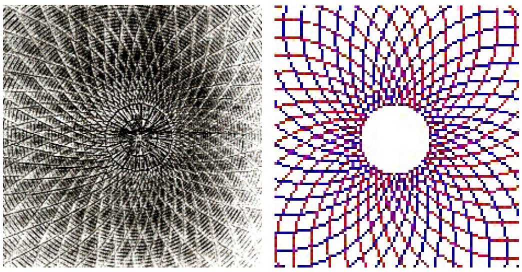

Figure 4: Photograph of the ceiling versus projection of the loxodromes

4 A second approximation

The wooden construction is set up in 15 layers along parallel circles, the top of the dome being cut off and replaced by an ad hoc construction of the "Congo Star". Note that the first layer starts at the parallel circle corresponding to the latitude $u = u_1 \neq 0$, and not at the equator $u = 0$. The basic circle is divided into 40 segments of 9° each, with longitudes v_i, $i = 1, \ldots, 40$, where $v_{i+1} - v_i = 9°$ or $v_{i+1} = v_1 + i.9°$.

Except for half of the planks in the first layer, all planks span two layers, their lengths diminishing with the height of the layer. However the joints appear to keep their size, quite obvious from a manufacturing point of view. In a second approximation we take this non-trivial joints into account by identifying each wooden strip as a part of a loxodrome. In this way the endpoints of the laminae appear as the intersection of a right handed and a left handed loxodrome.

We denote by $r_{i,0}$ and $l_{i,0}$ ($i = 1, \ldots, 40$) the right handed, respectively left handed loxodrome issuing from the point (u_1, v_i) on the lowest parallel circle, and by $r_{i,j}$ and $l_{i,j}$ ($i = 1, \ldots, 40$; $j = 1, \ldots, 6$) the loxodromes obtained by rotating $r_{i,0}$ and $l_{i,0}$ over an angle $j\Delta v$, respectively $-j\Delta v$, the angle Δv being determined by the dimensions of the joints construction. The equations of those loxodromes are given by

$$\begin{array}{ll} \text{for } r_{i,j}: & v - v_i - j\Delta v = f(u) - v_1 \text{ or } v - (i-1).9° - j\Delta v = f(u) \\ \text{for } l_{i,j}: & v - v_i + j\Delta v = -f(u) + v_1 \text{ or } v - (i-1).9° + j\Delta v = -f(u) + 2v_1 \end{array}$$

with $f(u_1) = v_1$.

The endpoints of all laminae are then given by the following intersections:

$$\begin{aligned} A_{2k,i} &= r_{i,k-1} \cap l_{i+2k-1,k-1} \\ B_{2k,i} &= r_{i,k-1} \cap l_{i+2k-1,k} \end{aligned}$$

for $k = 1, \ldots, 8$, and

$$\begin{aligned} A_{2k+1,i} &= r_{i,k-1} \cap l_{i+2k,k} \\ B_{2k+1,i} &= r_{i,k} \cap l_{i+2k,k} \end{aligned}$$

for $k = 1, \ldots, 7$, yielding the co-ordinates:

$$\begin{aligned} A_{2k,i} &= \begin{cases} v = v_1 + (2i + 2k - 3).4,5° \\ u = f^{-1}(v_1 - (k-1)\Delta v + (2k-1).4,5°) \end{cases} \\ B_{2k,i} &= \begin{cases} v = v_1 - \Delta v + (2i + 2k - 3).4,5° \\ u = f^{-1}(v_1 - k\Delta v + (2k-1).4,5°) \end{cases} \\ A_{2k+1,i} &= \begin{cases} v = v_1 - \Delta v + (i + k - 1).9° \\ u = f^{-1}(v_1 - k\Delta v + k.9°) \end{cases} \\ B_{2k+1,i} &= \begin{cases} v = v_1 + (i + k - 1).9° \\ u = f^{-1}(v_1 - k\Delta v + k.9°) \end{cases} \end{aligned}$$

After having determined, from the dimensions of the architecture plan, that $u_1 = 3,60°$, and from the photograph in Figure 3 that the longitude increment $\Delta v = 1,5°$, the co-ordinates of all endpoints are calculated. The resulting computer image (Figure 5) shows an astonishing resemblance with the photographs.

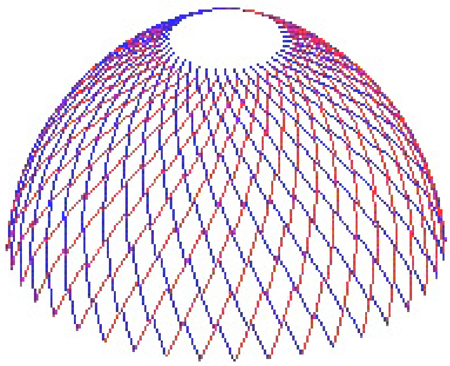

Figure 5: Computer image of the second approximation

5 Construction of the dome: first ideas

For a real wooden reconstruction of the Lacoste dome there is still a major problem to solve: to determine the shape and the dimensions of the laminae.
There are 15 types of laminae, each type occurring 40 times due to the $\frac{\pi}{20}$-periodicity. For the laminae inclined to the right the basic types are:

$$B_{1,1}A_{3,1};\ B_{3,1}A_{5,1};\ B_{5,1}A_{7,1};\ B_{7,1}A_{9,1};\ B_{9,1}A_{11,1};\ B_{11,1}A_{13,1};\ B_{13,1}A_{15,1}.$$

For the ones inclined to the left the basic types are

$$B_{1,16}A_{2,15};\ B_{2,15}A_{4,13};\ B_{4,13}A_{6,11};\ B_{6,11}A_{8,9};\ B_{8,9}A_{10,7};\ B_{10,7}A_{12,5};\ B_{12,5}A_{14,3};\ B_{14,3}A_{16,1}.$$

We thus identify the outer edge of each lamina with a straight line segment $B_{*,*}A_{*,*}$, where $B_{*,*}$ is the lower end and $A_{*,*}$ the upper end of the lamina considered, the cartesian co-ordinates of which can be easily calculated. The length of all outer edges and the angle between two cutting outer edges are then calculated by means of the inner product of the corresponding vectors.
To fix the ideas consider the plank with outer edge $B_{3,2}A_{5,2}$ (see Figure 6) for which

$$\cos(\beta_{3,2}) = \frac{\overrightarrow{B_{3,2}A_{5,2}} \cdot \overrightarrow{B_{2,3}A_{4,1}}}{\|\overrightarrow{B_{3,2}A_{5,2}}\|\ \|\overrightarrow{B_{2,3}A_{4,1}}\|}$$

and

$$\cos(\alpha_{5,2}) = \frac{\overrightarrow{B_{3,2}A_{5,2}} \cdot \overrightarrow{B_{4,3}A_{6,1}}}{\|\overrightarrow{B_{3,2}A_{5,2}}\|\ \|\overrightarrow{B_{4,3}A_{6,1}}\|}$$

For the calculation of the angles $\gamma_{5,2}$, $\delta_{3,2}$, $\varepsilon_{5,2}$ and $\eta_{3,2}$ we assume the edges $A_{5,2}A'_{5,2}$ and $B_{3,2}B'_{3,2}$ to lay along the normals $n_{A,5,2}$ and $n_{B,3,2}$ to the dome surface at the points $A_{5,2}$ and $B_{3,2}$ respectively.
If the normal $n_{A,5,2}$ intersects the axis of revolution at the point $O_{A,5}$ then

$$\cos(\gamma_{5,2}) = \frac{\overrightarrow{O_{A,5}A_{5,2}} \cdot \overrightarrow{B_{3,2}A_{5,2}}}{\|\overrightarrow{O_{A,5}A_{5,2}}\|\ \|\overrightarrow{B_{3,2}A_{5,2}}\|}$$

$$\cos(\varepsilon_{5,2}) = \frac{\overrightarrow{O_{A,5}A_{5,2}} \cdot \overrightarrow{A_{6,1}B_{4,3}}}{\|\overrightarrow{O_{A,5}A_{5,2}}\|\ \|\overrightarrow{A_{6,1}B_{4,3}}\|}$$

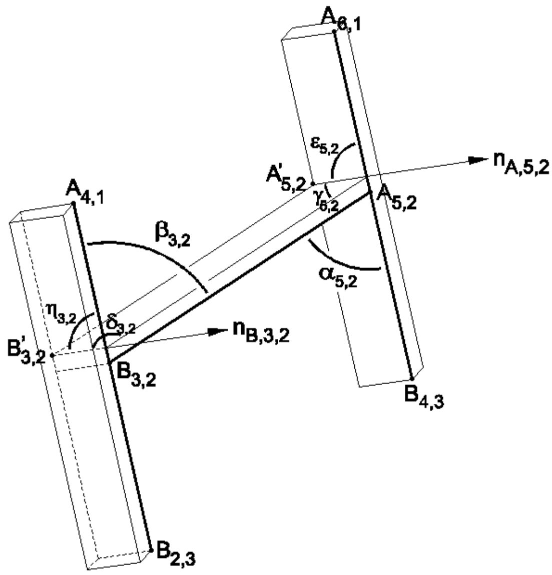

Figure 6: Example of the characterising angles between the laminae

and similarly

$$\cos(\delta_{3,2}) = \frac{\overrightarrow{O_{B,3}B_{3,2}} \cdot \overrightarrow{A_{5,2}B_{3,2}}}{\|\overrightarrow{O_{B,3}B_{3,2}}\| \ \|\overrightarrow{A_{5,2}B_{3,2}}\|}$$

$$\cos(\eta_{3,2}) = \frac{\overrightarrow{O_{B,3}B_{3,2}} \cdot \overrightarrow{A_{4,1}B_{2,3}}}{\|\overrightarrow{O_{B,3}B_{3,2}}\| \ \|\overrightarrow{A_{4,1}A_{2,3}}\|}$$

In this way, the shape and dimensions of all laminae are fully determined.

References

[1] L'architecture d'aujourd'hui, no.11 (1938): Le bois et ses nouvelles applications dans la construction.

[2] Architettura e Arti Decorative, Vol.5-6 (1931): Gaetano Minnucci, Notiziario tecnico: coperture a lamelle, pp.408-412.