Finite Cell-Elements of Higher Order

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Summary

The method of the finite elements is an adaptable numerical procedure for interpolation as well as for the numerical approximation of solutions of partial differential equations. The basis of these procedure is the formulation of suitable finite elements and element decompositions of the solution space. Classical finite elements are based on triangles or quadrangles in the two-dimensional space and tetrahedron or hexahedron in the three-dimensional space. The use of arbitrary-dimensional convex and non-convex polyhedrons as the geometrical basis of finite elements increases the flexibility of generating finite element decompositions substantially and is sometimes the only way to get a clear decomposition.

Introducing a local coordinate system for arbitrary convex polyhedrons permits a uniform formulation of an element within numeric methods. Based on the natural neighbor interpolation [6] it is possible to define natural element coordinates on arbitrary convex polyhedrons. The extension of the natural element coordinates on non-convex polyhedrons is presented.

Convex and non-convex polyhedrons in combination with a simple local coordinate system, the natural element coordinates, permit a uniform element formulation of interpolation functions and test functions. With these polyhedrons and interpolation functions it is also possible to formulate parametric finite cell elements for a better geometry approximation.

The accuracy of the finite element interpolation or approximation can be controlled by either applying the *h*-version or by utilizing the *p*-version. The *h*-version uses a local or global refinement of the decomposition. The use of polyhedrons and parametric cells simplifies the algorithms. The basic idea of the *p*-version is to keep the finite element mesh fixed and to increase the polynomial degree of the interpolation functions. In this paper we present construction methods for Lagrangian as well as for hierarchic interpolation functions based on natural element coordinates.

1 Introduction

Since the middle of the 60's the finite elements method (FEM) has set up as a standard tool for obtaining numerical solutions for partial differential equations which arise in many different engineering disciplines.

With the finite element method the investigation area is divided into small partial areas on which the solution of the problem is approximated by using easy functions that generally are polynomials. An improvement of the approximated solution can be reached by refinement (*h*-method) of the decomposition or by using higher order polynomials (*p*-method).

The *h*-method uses the fact that the number of the nodes or degrees of freedom is responsible for the exactness of the approximation. It is to be expected a better approach of the solution with a finer decomposition under use the same interpolation functions.

With the *p*-method the decomposition of the investigation area is unchanged. In order to attain a better approach to the exact solution, the degree of the used polynomials is increased. The increase of the degree of the polynomials provides advantages compared with a refinement of the decomposition. Increasing of the degree of the polynomials of the *p*-method leads to an exponential decrease of the global error, while the conventional refinement of the decomposition only leads to the algebraic convergence. With the increases of the degree of the polynomials the exactness of the element also increases, nevertheless, requires a higher geometrical exactness with the modeling (edge curve adaptation). Due to the blending function method, curved boundaries can easily be considered without increasing the number of elements.

2 Geometric Basis

The geometric basis of the finite elements is formed by a compact non-empty set in the Euclidean space $\mathbb{R}^n$. Typical representatives of compact sets in the Euclidean space are such, which are finite and bounded. The simplest compact sets are bounded intervals in the $\mathbb{R}^1$, triangles and squares in the $\mathbb{R}^2$ as well as tetrahedron or hexahedron in $\mathbb{R}^3$. Convex polyhedrons are generalizations of the described geometrical elements.

2.1 Convex and non-convex Polyhedron

Convex polyhedrons Z in the n-dimensional Euclidean space $\mathbb{R}^n$ can be described in different equivalent kinds. The definition of a convex polyhedron over the convex hull of given reference points, called vertices $E=\{e^1, e^2, \dots e^N\}$, frequently forms the basis for suitable construction methods and/or algorithms. In the context of the formulation of finite elements the description of convex polyhedrons over the Minkowsky product

$$Z := \{p : p = \lambda_1 e^1 + \lambda_2 e^2 + \dots + \lambda_N e^N, \lambda_i \geqslant 0 \wedge \sum_i \lambda_i = 1\} \quad (1)$$

appears more suitable.

The largest number of m, for which there are *(m+1)* linear independent points in the set of the vertices E, is called the dimension of the convex polyhedron Z. An m-dimensional convex polyhedron has a finite number of sides and each side is again a convex polyhedron. The $(m-1)$-dimensional sides of Z are called facets, the one-dimensional sides edges and the zero-dimensional sides are the vertices.

The class of the considered geometry can be extended to special non-convex polyhedrons [5], if these can be constructed over regularized set operations $\cap^*$, $\cup^*$, $\setminus^*$ [1] from the convex hull of the vertices and convex sub-polyhedrons. These convex sub-polyhedrons describe the points of the convex hull, which are not contained in the non-convex polyhedron.

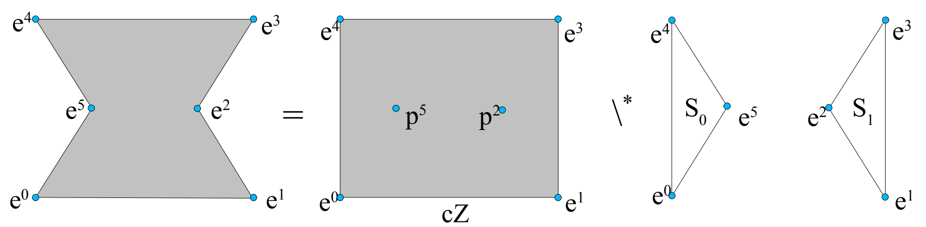


Figure 1: Construction of non-convex polyhedron

For the use of polyhedrons as geometrical basis of finite elements it is appropriate to represent the points of a polyhedron by means of local coordinates. A local coordinate system must be applicable both to convex and to non-convex polyhedrons.

2.2 Natural element coordinates

The formulation of a local coordinate system permits an uniform element formulation in the method of the finite elements. The description of the convex polyhedron Z by the Minkowsky product (1) of its vertices E suggests to use the factors λ_i of the linear combination as element coordinates. If an m-dimensional convex polyhedron has $m+1$ linear independent vertices, the factors are unique and called barycentric coordinates. If a convex polyhedron consists of more than $m+1$ vertices, the factors are not unique. If the natural neighborhood coordinates introduced by Sibson [6] are restricted to the convex polyhedron, one receives unique natural element coordinates, which are related to the vertices of the convex polyhedron.

2.2.1 Convex Polyhedron

The determination of the natural element coordinates of a point x concerning the convex polyhedron Z is based on the computation of the Voronoi diagram of second order concerning the vertices and the point x.

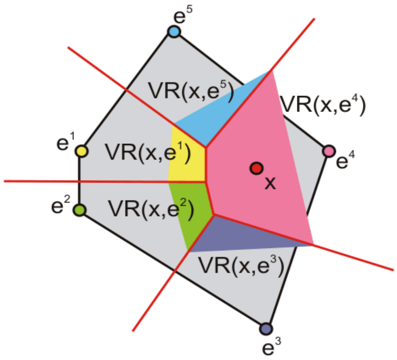


Figure 2: Voronoi decomposition of the convex polyhedron using sub-regions

Firstly, the Voronoi decomposition of first order of a convex polyhedron is determined by its vertices e^i. Each vertex of the convex polyhedron has its own Voronoi region. The Voronoi region of a vertex e^i is the set of all points p which has a smaller or equal distance to the vertex e^i as their distance to the remaining vertices e^j:

$$VR(e^i) := \{p \in \mathbb{R}^n : d(p, e^i) \leq d(p, e^j) \forall\, j \neq i\} \quad . \tag{2}$$

The Voronoi region of second order of a convex polyhedron is determined concerning its vertices e^i and a point x of the convex polyhedron. A Voronoi region of second order is the set of points p, whose distance to the point x is smaller or equal their distance to a vertex e^i, if its distance to this vertex is smaller or equal their distance to the remaining vertices e^j:

$$VR(x, e^i) := \{p \in \mathbb{R}^n : d(p, x) \leq d(p, e^i) \leq d(p, e^j) \forall\, j \neq i\} \quad . \tag{3}$$

The natural element coordinates of the point x concerning the vertex e^i are determined over the Voronoi regions of second order (see Figure 3). Each Voronoi region of first or second order assigns itself a Lebesgue measure $\mu(VR(e^i))$ or $\mu(VR(x, e^i))$ (see [2]). This measure corresponds to the common surface area in the 2-dimensional Euclidean space. The ratio between the measure of the Voronoi region of second order of a vertex and the point x to measure of the Voronoi regions of first order of the point x concerning all vertices of the convex polyhedron is called the unique natural element coordinates

$$\lambda_i(x, e_i) := \frac{\mu(VR(x, e^i))}{\mu(VR(x))} \quad . \tag{4}$$

If the considered point x lies outside of the convex polyhedron, then no representation in natural element coordinates exists. If the point x is accurately on a facet of the polyhedron, the resulting Voronoi regions of second order have infinite measures. It can be shown by limit value calculation [2] that the calculation of the natural element coordinates depends only on vertices of the facet and thus the calculation is limited to the convex polyhedron of the facet.

The natural element coordinates are equivalent to the barycentric coordinates of the simplex and have a bilinear characteristic inside the squares. The natural element coordinates contain the well-known local coordinate systems of classical finite element theory.

2.2.2 Non-convex Polyhedron

Firstly, the construction of the natural element coordinates is limited to convex polyhedrons. The natural element coordinates of convex polyhedrons is transferable to non-convex polyhedrons only by an adjustment of the construction. The natural element coordinates of convex polyhedrons are based on the Voronoi decomposition of the vertices. The neighbors between the vertices, resulting from the Voronoi decomposition, consider the space of the convex hull with respect to the vertices. When the construction of the natural element coordinates of the convex polyhedron is directly transferred to that of non-convex polyhedrons, too many neighbors are considered.

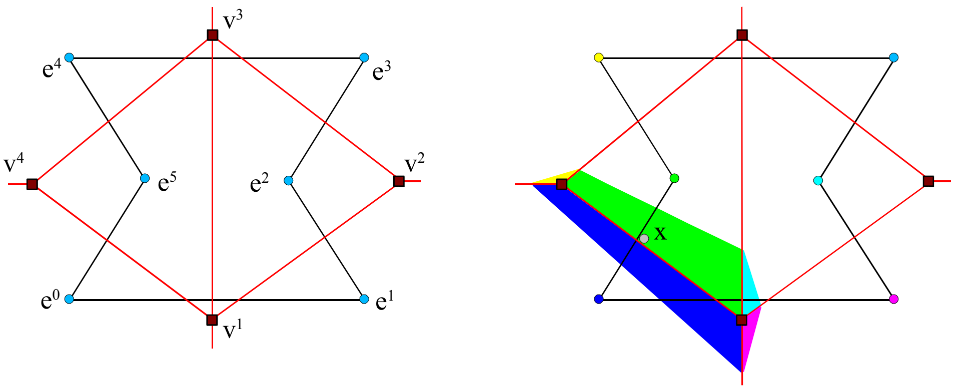


Figure 3: Voronoi-Decomposition of a non convex polyhedron

For example, in the non-convex polyhedron of figure 3 the vertices e^0 and e^4 are neighbors due to the Voronoi decomposition. The point x which can be represented in natural element coordinates falls into the direct spheres of influence of both vertices. The sphere of influence is given by the Voronoi circumcircle, in whose construction both vertices are involved. The center of this circumcircle is the Voronoi vertex v^4. The neighborhood relationship between the vertices e^0 and e^4 , which is represented by the Voronoi vertex v^4, must not be considered in the non-convex polyhedron.

When constructing the Voronoi-decomposition of the non-convex polyhedron, only neighbors are considered, which are element of both convex and non-convex polyhedron.

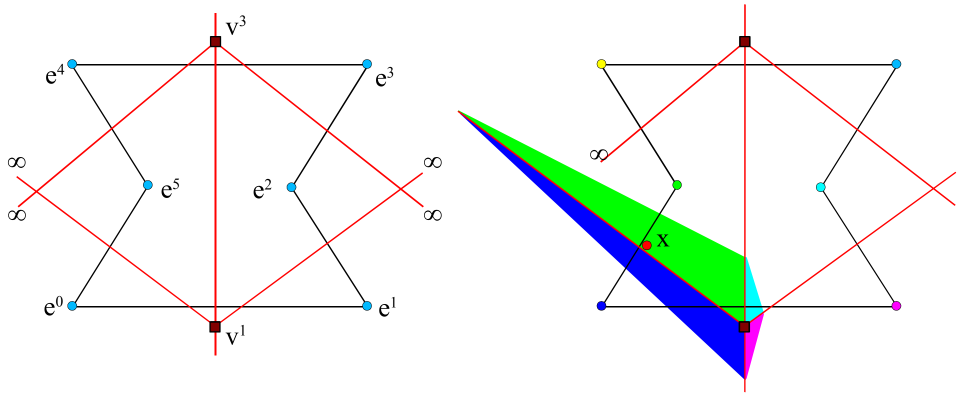


Figure 4: “Non-convex” Voronoi-decomposition of a non-convex cell

For the non-convex polyhedron in figure 4, the neighbors between the vertices e^0 and e^4 as well as the vertices e^1 and e^3 are part of the convex polyhedron but not of the non-convex polyhedron. These neighbors are represented by the Voronoi-vertices v^1 and v^4 (Figure 3). If the Voronoi vertices are not considered, the “non-convex” Voronoi decomposition of the vertices arises (Figure 4).
The natural element coordinates which result form the “non-convex” Voronoi decomposition, possess the same characteristics as that of a convex polyhedron.

2.3 Parametric cells

The geometry of polyhedral elements is planar bounded. Investigation areas which are describable as polyhedrons can be decomposed accurately. In order to be able to describe investigation areas with curved boundaries parametric cells are used in addition as geometrical support of the finite elements.
The parametric cells that are used here are special topological cells [2]. Parametric cells are described by convex or non-convex polyhedrons and a map F (a homeomorphism). The map is described by form functions for the facets and is formulated in the natural element coordinates of the polyhedron

$$x = \sum_{edges} N_{ij}(\lambda(r)) + \sum_{sides} N_{ij\ldots k}(\lambda(r)) + \ldots \quad (6)$$

Functions of the following kind are used as form functions on the edges:

$$N_{ij} := \left|\lambda_i(r) + \lambda_j(r)\right|^l \cdot f_{ij}\left|s(r)\right| \quad (7)$$

with

$$s_m := \frac{\lambda_m(r)}{\lambda_i(r) + \lambda_j(r)}, \quad m \in (i, j) \quad . \quad (8)$$

The effect of these form functions decreases over the cell uniformly. The decay of the form function with the power l can be controlled inside the cell.

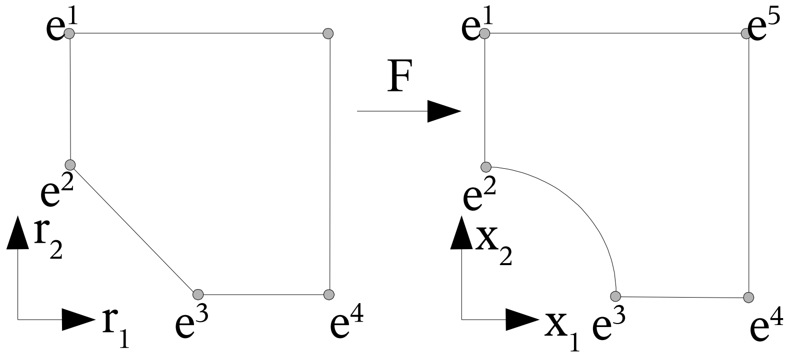


Figure 5: Parametric cell

Exemplary the formulation of map $f(s_i,s_j)$ for the edge e^2 and e^3 in figure 5 is to be represented here. The edge e^2 and e^3 can be understood as a chord of a circle sector and the function f has the following form

$$f_{ij}=R\begin{bmatrix}\cos\left(\varphi_i s_i+\varphi_j s_j\right)-s_i\cos\left(\varphi_i\right)-s_j\cos\left(\varphi_j\right)\\ \sin\left(\varphi_i s_i+\varphi_j s_j\right)-s_i\sin\left(\varphi_i\right)-s_j\sin\left(\varphi_j\right)\end{bmatrix}$$

with R is the radius of the circle and φ_i and φ_j the angle between the axis r_1 and e^2 respectively e^3.

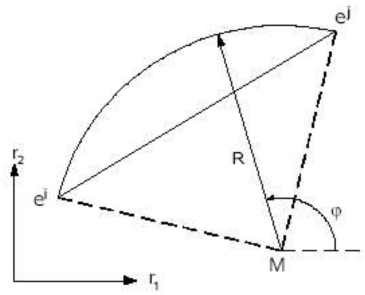


Figure 6: Map for a circle sector

In the following figure 7 the characteristic of the natural element coordinates on the parametric cell, changed by the transformation, is illustrated.

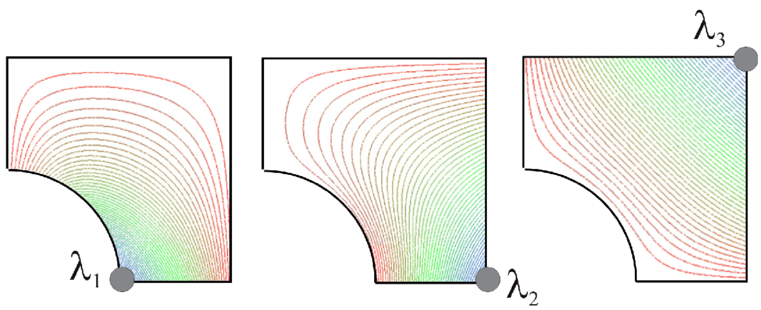


Figure 7: Transformed natural element coordinates

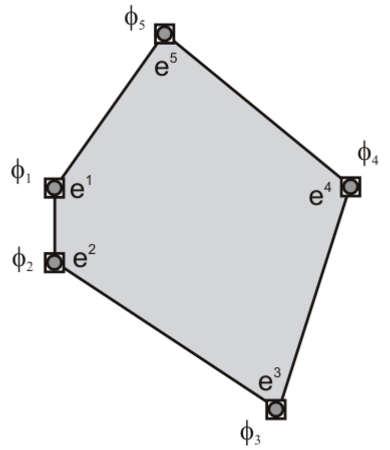
3 Interpolation Functions

The basic idea of the finite element method is the approximation of the unknown solution by using simple interpolation functions with still unknown parameters. The interpolation functions are defined on the geometric basis of the finite elements. The solution of the

partial differential equation will be transferred to the solution of a system of algebraic equations for the unknown parameters.
The natural element coordinates allow a generalized formulation of Lagrange-type as well as hierarchical interpolation functions on polyhedrons and/or parametric cells. Now, we concentrate on the definition of edge-oriented interpolation functions.

3.1 Lagrangian Interpolation Function

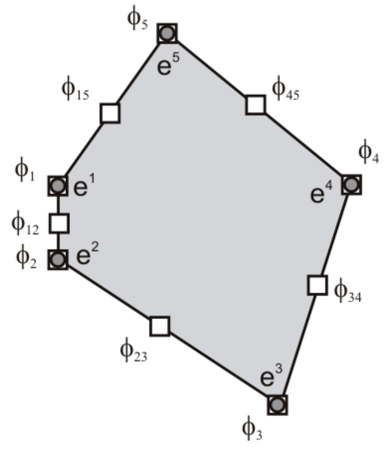
The Lagrangian interpolation functions ϕ_i can be used as interpolation functions in the finite element method. Now, we present the formulation in natural element coordinates. For edge-linear interpolations the interpolation functions consist of natural element coordinates of the associated vertices exclusively.



$$\phi_1(\lambda) := \lambda_1$$
$$\phi_2(\lambda) := \lambda_2$$
$$\phi_3(\lambda) := \lambda_3$$
$$\phi_4(\lambda) := \lambda_4$$
$$\phi_5(\lambda) := \lambda_5$$

Figure 8: Lagrangian interpolation functions - edge-linear

For edge-square interpolations the interpolation functions for the centric degree of freedom of a edge is the product of the two natural element coordinates of the associated vertices multiplied by a pre-factor. The interpolation function associated to a vertex depends on all natural element coordinates of the vertices of all outgoing edges.



$$\phi_1(\lambda) := \lambda_1 \cdot (1 - 2\cdot\lambda_2) \cdot (1 - 2\cdot\lambda_5)$$
$$\phi_2(\lambda) := \lambda_2 \cdot (1 - 2\cdot\lambda_1) \cdot (1 - 2\cdot\lambda_3)$$
$$\phi_3(\lambda) := \lambda_3 \cdot (1 - 2\cdot\lambda_2) \cdot (1 - 2\cdot\lambda_4)$$
$$\phi_4(\lambda) := \lambda_4 \cdot (1 - 2\cdot\lambda_3) \cdot (1 - 2\cdot\lambda_5)$$
$$\phi_5(\lambda) := \lambda_5 \cdot (1 - 2\cdot\lambda_4) \cdot (1 - 2\cdot\lambda_1)$$
$$\phi_{12}(\lambda) := 4\cdot\lambda_1\cdot\lambda_2$$
$$\phi_{23}(\lambda) := 4\cdot\lambda_2\cdot\lambda_3$$
$$\phi_{34}(\lambda) := 4\cdot\lambda_3\cdot\lambda_4$$
$$\phi_{45}(\lambda) := 4\cdot\lambda_4\cdot\lambda_5$$
$$\phi_{15}(\lambda) := 4\cdot\lambda_1\cdot\lambda_5$$

Figure 9: Lagrangian interpolation functions - edge-square

The presented interpolation functions permit a c^0-continuity interpolation on decompositions consisting of polyhedrons and parametric cells and form the basis for finite element approximations. Figure 10 shows the characteristic of third order interpolation functions on an edge of a parametric cell.

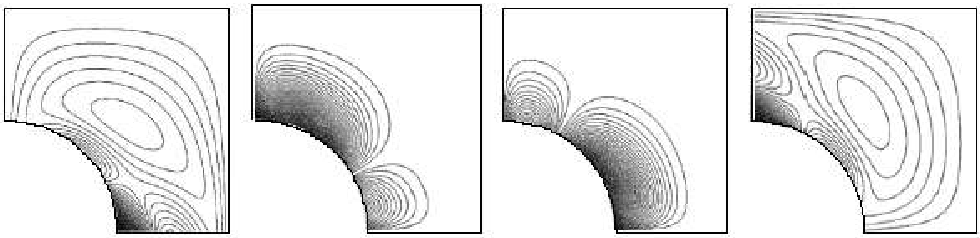


Figure 10: Set of edge cubic interpolation functions

3.2 Hierarchic Interpolation Functions

The formulation of a *p*-version on the basis of polyhedrons and parametric cells requires the definition of a family of interpolation functions, which possess a hierarchical layout. Differently to the Lagrangian interpolation functions, the interpolation functions of lower polynomial degree used first can be further used with an extension of the basis by additional interpolation functions of higher polynomial degree.

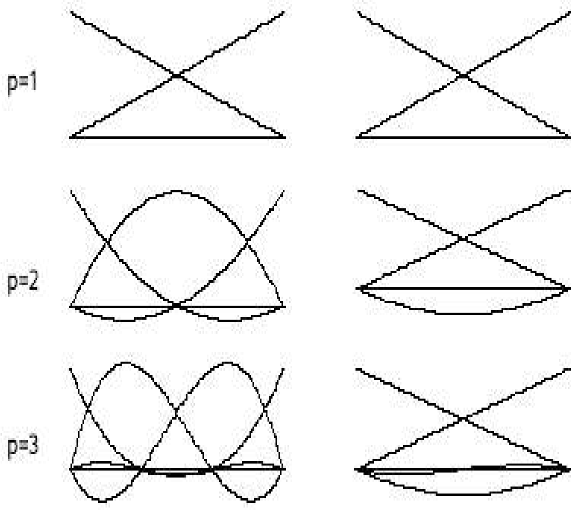


Figure 11: Set of one-dimensional Lagrange and hierarchic interpolation functions

A interpolation function of arbitrary degree can be won by a linear combination of the interpolation functions ϕ_0 and ϕ_1 with a series of polynomial functions. The polynomial functions in this case must meet the demand that their value at the ends of their definition range, thus at the vertices of the edge, is zero. The square approximation of a function *u* of a linear component results to

$$u = u_0 \phi_0 + u_1 \phi_1 + a_2 \phi_2$$

with

$$\phi_0 := \lambda_1$$
$$\phi_1 := \lambda_0$$
$$\phi_2 := \lambda_0 \cdot \lambda_1 \, .$$

The parameter a_2 is certain a freely selectable value and is determined in the context of the approximation.
In a similar way cubic interpolation functions on the component can be selected, extending the square interpolation function by a cubic portion.

$$u = u_0 \phi_0 + u_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3$$

On the basis of the representation in the linear case the development of the hierarchical interpolation functions can be transferred into the n-dimensional space.

$$\phi_n(\lambda_i, \lambda_j) = \prod_{k=1}^{n} (\lambda_i - \alpha_k(\lambda_i + \lambda_j)),\ n \geq 2 \qquad (17)$$

with $$\alpha_k = \frac{k-1}{n-1}.$$

The following figure shows the characteristic of the hierarchical interpolation function on an edge of a parametric cell.

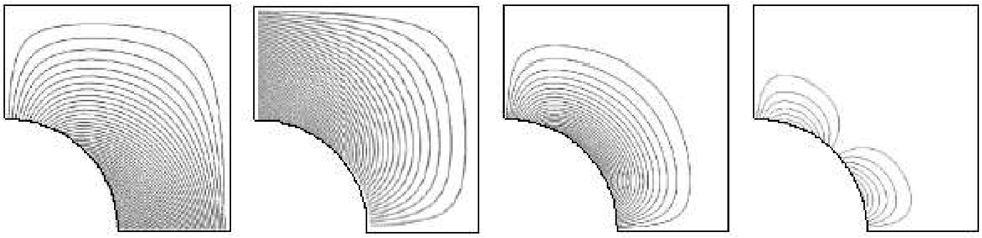


Figure 12: Set of hierarchical interpolation functions of order three

4 Conclusion

Generalized finite elements based on convex and non-convex polyhedrons are presented. Due to the blending function method parametric cells are introduced. Introducing the natural element coordinates on polyhedrons and parametric cells a generalized formulation of form functions and interpolation functions could be achieved. The possibilities for the formulation of higher interpolation functions in form of Lagrange and hierarchical edge oriented functions are presented. In order to be able to describe complete polynomials, the introduction of an interior degree of freedom in the element is additionally necessary.

5 References

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