Fractal Truss Structure and Automatic Form Generation Using Iterated Function System

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Summary

This paper describes a couple of new truss structures based on fractal geometry. One is the famous Sierpinski Gasket and another is a fractal triangle derived by means of applying a process forming leaves of a cedar tree using M. F. Barnsley's contraction mapping theory. Therefore a pair of x-y coordinates of an arbitrary nodal point on the structures are generated easily if IFS(Iterated Function System) codes and a scale of them are specified. Structural members are defined similarly. Thus data for frame analysis can be generated automatically, which is significant if the objective structure has complex configuration. Next analytical results under vertical and wind loadings in Japanese Building Code are shown. Here members are assumed to be timber and to have cross section of 15cm×15cm. Finally authors conclude that geometrically new truss structures were developed and automatic data generation for frame analysis was attained using IFS. Analytical results show they contribute to saving material when compared it with King-post truss.

1 Introduction

The modern structure is huge assemblage of various parts described by Euclidean geometry. It has been contributing to give mechanical rationality and safety to the modern architecture. However, in nature, there exists another form system, which can be described by the fractal geometry (cf. Mandelbrot [1], Barnsley [2]). The complex configuration such as botanies, blood vessels, lungs of creature, crystals and clouds looks so far from the modern structure but may lead to an innovative one because our technology has been learning so much from nature. Therefore authors developed a layered arch (cf. Mae and Asayama [3]) based on a form system a fractal tree has and showed it to be effective to resist wind force (cf. Mae and Asayama [4]). In this paper they present fractal truss structures that can be generated by applying a process forming leaves of a cedar tree and describe automatic data generation for frame analysis and show analytical results.

2 Geometry of Fractal Truss

Truss structures with fractal geometric form shown in Figure 1 are generated by means of contraction mapping repeatedly. The figures show convergent sequence of them set in 2-dimensional space. Each triangle generated repeatedly through the process in the figure is named a fractal truss in this paper, defined mathematically as follows :

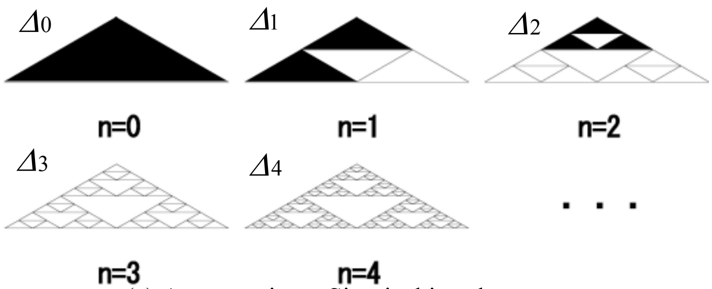
$$\Delta_n = f_1(\Delta_{n-1}) \cup f_2(\Delta_{n-1}) \cup f_3(\Delta_{n-1}) \quad (1)$$

$$\Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_n \supset \ldots \quad (2)$$

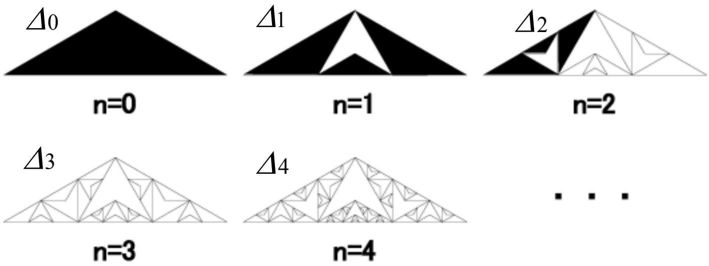
and

$$\Delta = \bigcap_{i=1}^{\infty} \Delta i \quad (3)$$

, which forms a perfect self-similar set.



(a) A truss using a Sierpinski gasket



(b) A fractal truss

Figure 1 Convergent sequence of fractal trusses.

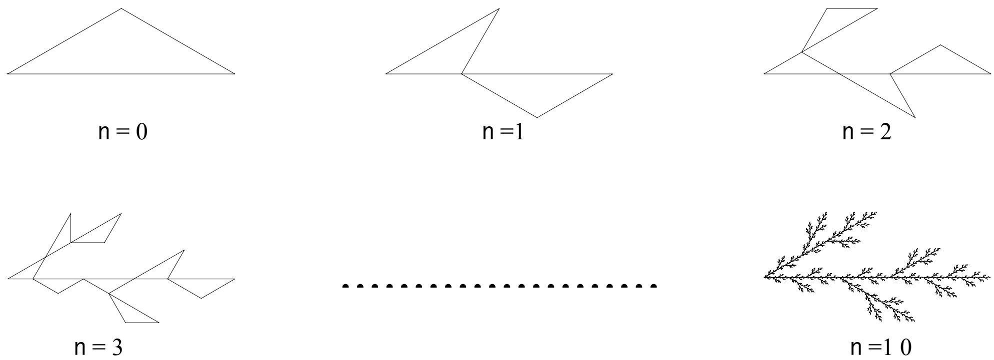


Figure 2 Convergent sequence of leaves of a cedar tree.

Figure 1(a) is the famous Sierpinski Gasket and (b) is developed by applying a process forming leaves of a cedar tree shown in Figure 2. Obviously they are truss structures but look somewhat different from those engineers are familiar with because of their self-similarity. Hausdorff dimension of the gasket is constant (log3 / log2) . However the value of another truss depends on base angles, given by the following equation (4) under the condition that the mathematical fractals are defined by contraction mapping of sets not overlapping each other.

$$\sum_{i=1}^{m} \lambda_i^D = 1 \tag{4}$$

Here λ_i (i = 1,2,3) denotes contractivity. Figure 3 shows a detail of the fractal truss that has unsymmetric base angles, θ_1 and θ_2. It has contractivities λ_1, λ_2 and λ_3 as follows :

$$\lambda_1 = \frac{\tan\theta_2}{\cos\theta_1(\tan\theta_1 + \tan\theta_2)} \tag{5}$$

$$\lambda_2 = \frac{\tan\theta_1}{\cos\theta_2(\tan\theta_1 + \tan\theta_2)} \tag{6}$$

$$\lambda_3 = 1 - (\lambda_1^2 + \lambda_2^2). \tag{7}$$

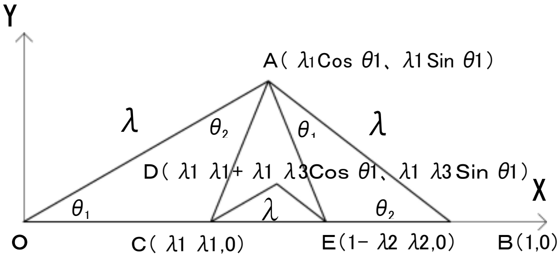


Figure 3 Contractivity and each vertex of a fractal truss.

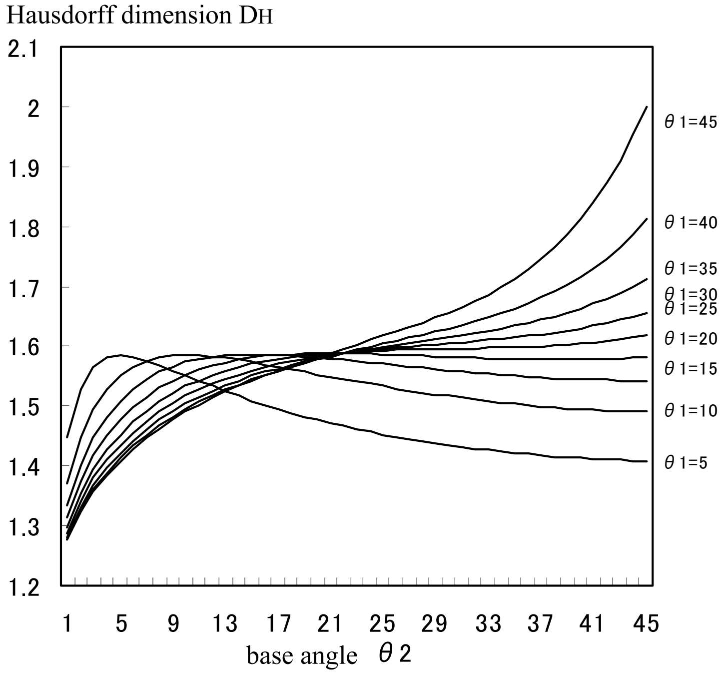


Figure 4 Variation of Hausdorff dimension of a fractal truss to base angle.

The fractal dimension is calculated by means of substituting equation (5), (6) and (7) into (4). Figure 4 shows relationship between Hausdorff dimension and base angle. The value ranges from 1.3 to 2.0 and increases when both of base angles are larger than 20 degrees.

3 IFS Codes for Fractal Truss and Automatic Data Generation for Frame Analysis

Iterated Function System (IFS) defining configurations of the fractal structures can be written as

$$W\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} e \\ f \end{Bmatrix} \quad (8)$$

,where α_{11}, α_{12}, α_{21}, α_{22}, e and f are IFS codes as shown Table 1 and 2. The later was derived by the authors using M. F. Barnsley's contraction mapping theory(Barnsley [2]). A pair of x-y coordinates of an arbitrary point on an original structure Δ_0 shown previously in Figure

Table 1 IFS code for unsymetric Sierpinski gasket.

w	a_{11}	a_{12}	a_{21}	a_{22}	e	f
1	1/2	0	0	1/2	0	0
2	1/2	0	0	1/2	$\tan\theta_2/(\tan\theta_1+\tan\theta_2)$	$\tan\theta_1\tan\theta_2/(\tan\theta_1+\tan\theta_2$
3	1/2	0	0	1/2	1/2	0

Table 2 IFS code for an unsymetric fractal truss.

w	a_{11}	a_{12}	a_{21}	a_{22}	e	f
1	$\lambda_1\cos\theta_1$	$\lambda_1\sin\theta_1$	$\lambda_1\sin\theta_1$	$-\lambda_1\cos\theta_1$	0	0
2	λ_3	0	0	λ_3	λ_1^2	0
3	$1-\lambda_1\cos\theta_1$	$\{(1-\lambda_1\cos\theta_1)^2-\lambda^2\}/\lambda_1\sin\theta_1$	$-\lambda_1\sin\theta_1$	$-1+\lambda_1\cos\theta_1$	$\lambda_1\cos\theta_1$	$\lambda_1\sin\theta$

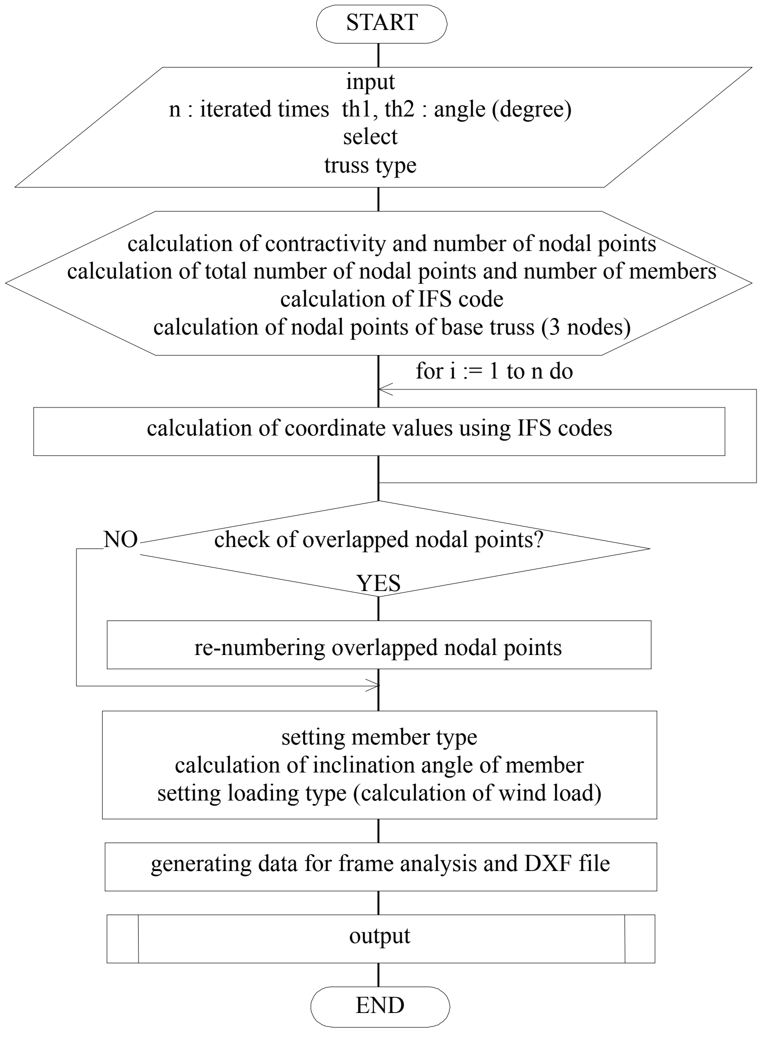


Figure 4 Flow chart of automatic data generation for frame analysis.

1 is transformed into one on the sub-structures by equation (8). Figure4 shows a flow chart of computing. Here some nodal points, which are normally vertexes of a small triangle generated by IFS, are overlapped when giving them node numbers under the above algorithm because each triangular element comes in contact with adjacent one at the vertexes. Therefore all nodal points are compared each other, examining values of the x-y coordinates and if they have the

same values in x and y directions simultaneously, a large node number is replaced by a small one. Then the nodal points are re-numbered. Total numbers of them and structural members on the fractal truss generated by contraction mapping repeated n times can be written as follows:

$$(3^{n+1} + 3)/2 \quad (8)$$

and

$$3^{n}. \quad (9)$$

Thus data for frame analysis are generated easily if IFS codes, a scale of the original structure and loading are specified. Figure 5 shows a user interface of the program described above.

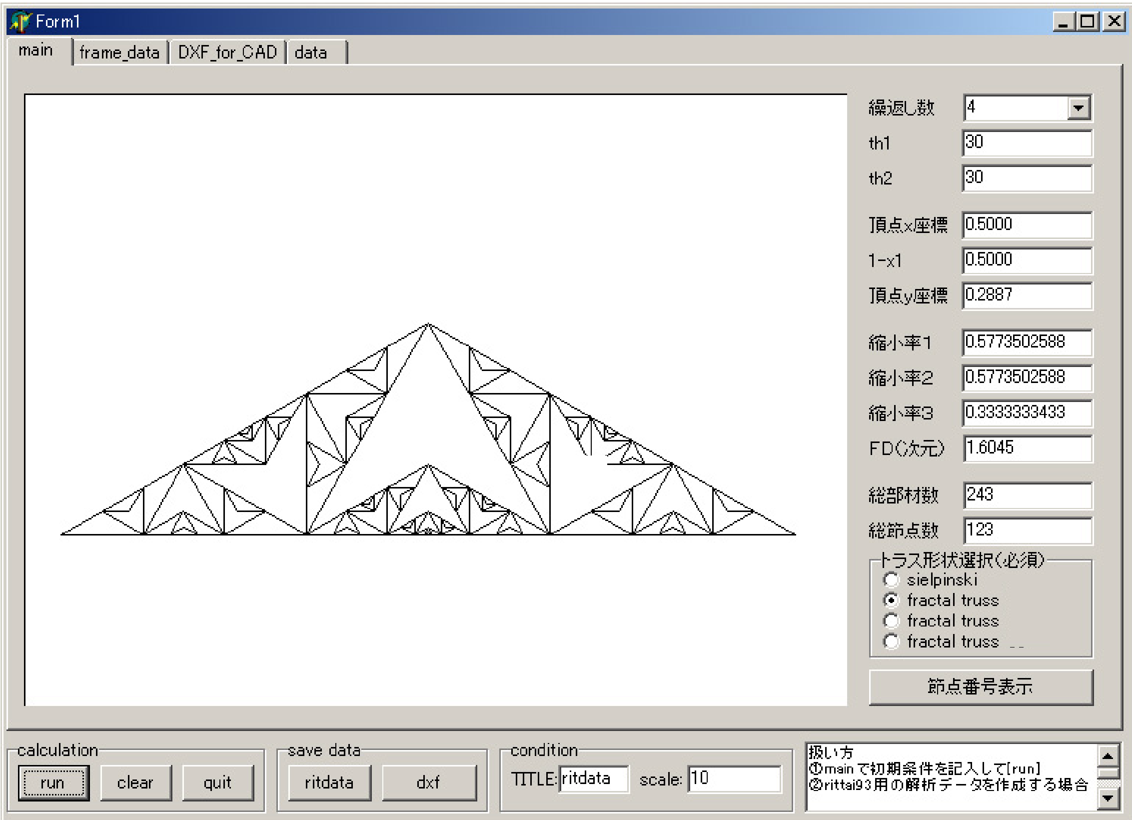
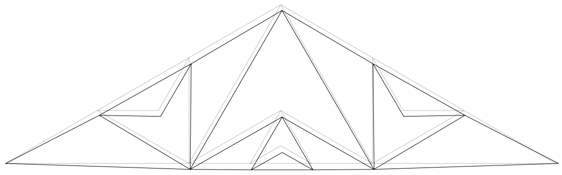


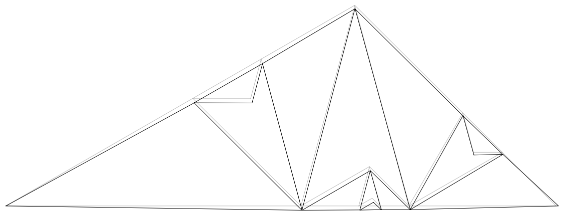
Figure 5 A user interface of a program generating data for frame analysis.

4 Analytical Result

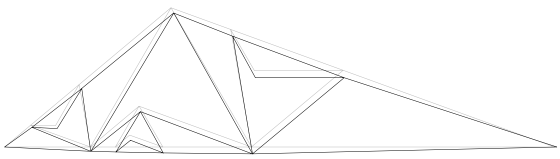
Analytical results under vertical loading and wind loading in Japanese Building Code are shown. Members are assumed to be timber and to have cross section of 15cm×15cm. Young's modulus is 784kN/cm^2(80 ton/cm^2). Figure 6 shows deflections of the fractal truss to vertical loading, concentrating distributed unit loads to each nodal point on the roof. Deflection at the center of the span seems to be small in an unsymmetric model with a pair of base angle 45 and 30 degrees in Figure 6 (d). Figure 7 shows deflection of a symmetric fractal truss with base angle of 30 degrees to wind loading in Japanese Building Code. Similarly Figure 8 shows one of an unsymmetric model with base angle of 30 and 45 degrees. The deflection is also small in unsymmetric one. This can be thought the effect of difference between roof heights arising from giving a base angle of 45 degrees to the model.



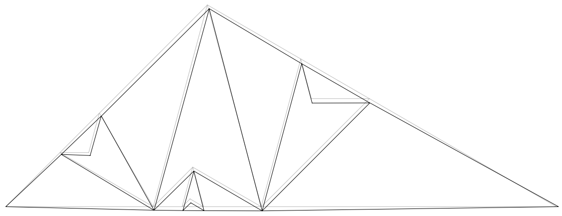
(a) base angle：30° 30°



(b) base angle：30° 45°

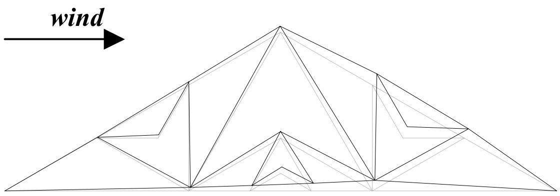


(c) base angle：40° 20°

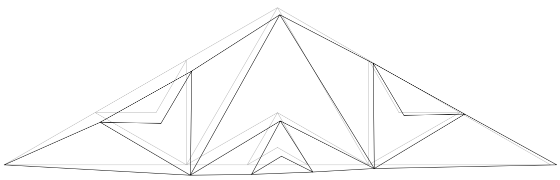


(d) base angle：45° 30°

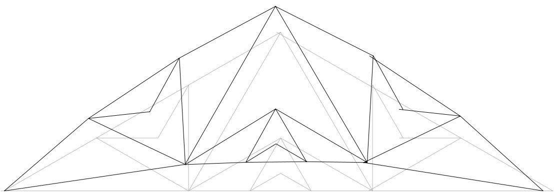
Figure 6 Deflection of a fractal truss to vertical loading.



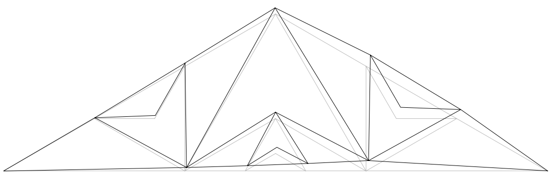
(a) positive pressure : internal pressure(0,0)



(b) positive pressure : internal pressure(0,0.2)

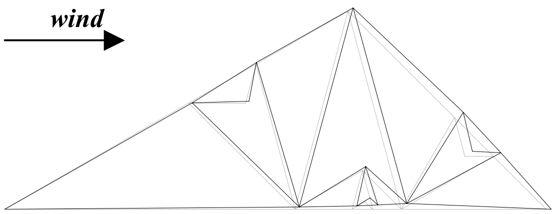


(c) negative pressure : internal pressure(0,0)

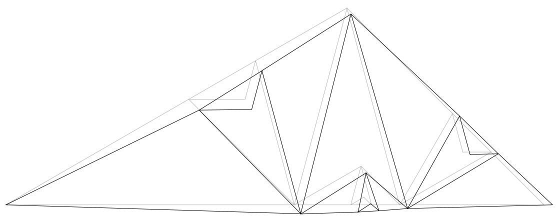


(d) negative pressure : internal pressure(0,0.2)

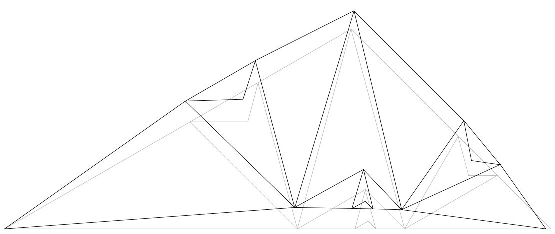
Figure 7 Deflection of a fractal truss to wind loading.



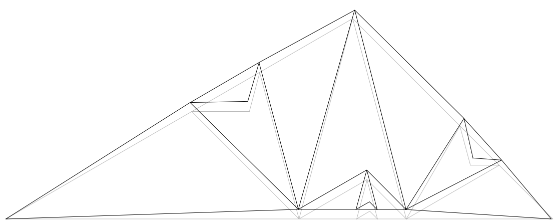
(a) positive pressure : internal pressure(0,0)



(b) positive pressure : internal pressure(0,0.2)



(c) negative pressure : internal pressure(0,0)



(d) negative pressure : internal pressure(0,0.2)

Figure 8 Deflection of a fractal truss to wind loading.

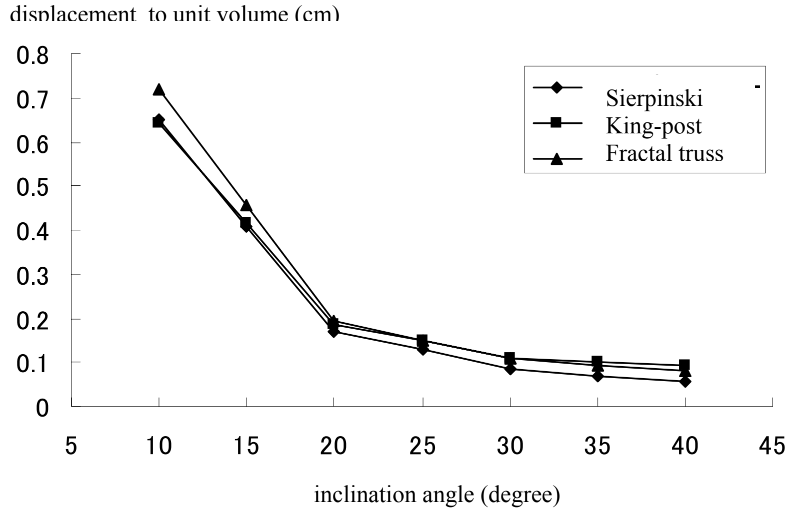


Figure 9 Relationship between displacement to vertical loading and inclination angle.

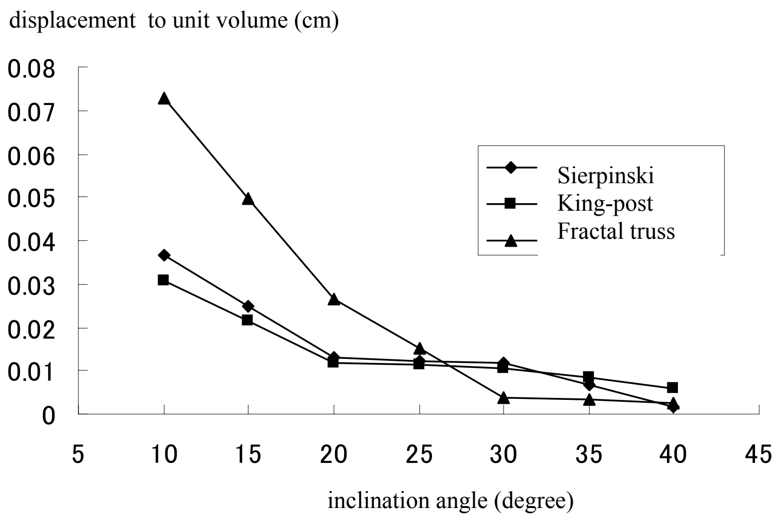


Figure 10 Relationship between displacement to wind loading and inclination angle.

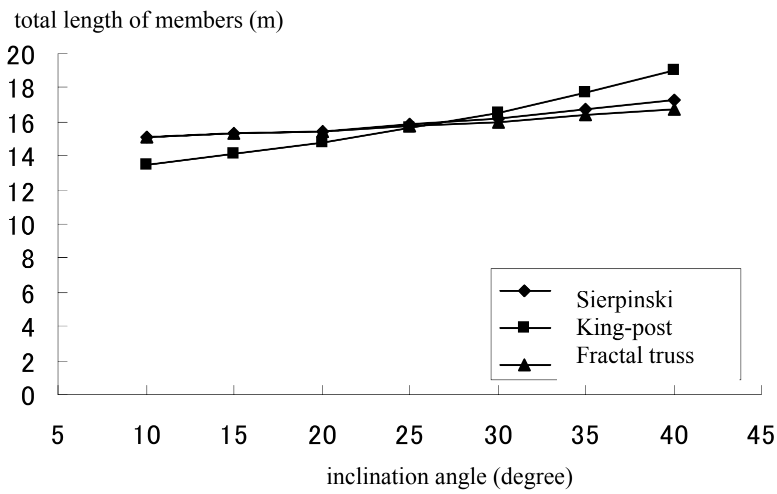


Figure 11 Relationship between total length of members and inclination angle.

Figure 9 shows variation of displacement at the center of the span to base angle in symmetric models generated one time contraction mapping when acting vertical load on them as described before. Values of displacements are not so different each other. However those to unit volume of a fractal truss become smaller than King-post at the range from 25 to 40 degrees of inclination angle. The reason can be thought that total length of members of King-post is larger than other trusses at the same range shown in Figure 11.

5 Conclusion

The authors conclude that geometric characteristics of the new truss structures including fractal dimension are shown and Automatic data generation for frame analysis was attained using IFS. Analytical results show they contribute to saving material when compared with King-post truss under wind loading.

6 References

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Mae, T and Asayama, S. (2004). *A Study on Statistic Characteristics of a Layered Arch with Fractal Geometric Form Against Wind Loading.* Journal of Environmental Eng.(Transaction of AIJ). No.576.(in Japanese) .

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Earnest cooperation of Ryuichi KOBAYASHI and Yoshifumi IKEDA, graduate students of Tokyo Denki University is deeply acknowledged.

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Final Abstract

1.Preface

The modern structure is huge assemblage of various parts described by Euclidean geometry. It has been contributing to give mechanical rationality and safety to the modern architecture. However, in nature, there exists another form system, which can be described by the fractal geometry (cf. Mandelbrot, Barnsley). The complex configuration such as botanies, blood vessels, lungs of creature, crystals and clouds looks so far from the modern structure but may lead to an innovative one because our technology has been learning so much from nature. Therefore authors developed a layered arch (cf. Mae and Asayama) and showed it to be effective to resist wind force (cf. Mae and Asayama). In this paper they present fractal truss structures and describe automatic data generation for frame analysis and show analytical results.

2. Geometry of Fractal Truss

Truss structures with fractal geometric form are generated by means of contraction mapping repeatedly as follows:

$$\Delta_n = f_1(\Delta_{n-1}) \cup f_2(\Delta_{n-1}) \cup f_3(\Delta_{n-1}) \tag{1}$$

$$\Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_n \supset \ldots \tag{2}$$

and

$$\Delta = \bigcap_{i=1}^{\infty} \Delta i \tag{3}$$

, which forms a perfect self-similar set. They are named fractal trusses in this paper. Hausdorff dimensions D=$\dim_H(\Delta)$ are given by the following equation (4) under the condition that the mathematical fractals are defined by contraction mapping of sets not overlapping each other.

$$\sum_{i=1}^{m} \lambda_i^D = 1 \tag{4}$$

Here λ_i (i = 1,2,3) denotes contractivity.

The fractal truss that has unsymmetric base angles, θ_1 and θ_2. It has contractivities λ_1, λ_2 and λ_3 as follows :

$$\lambda_1 = \frac{\tan\theta_2}{\cos\theta_1(\tan\theta_1 + \tan\theta_2)} \tag{5}$$

$$\lambda_2 = \frac{\tan\theta_1}{\cos\theta_2(\tan\theta_1 + \tan\theta_2)} \tag{6}$$

and

$$\lambda_3 = 1 - (\lambda_1^2 + \lambda_2^2). \tag{7}$$

The fractal dimension is calculated by means ofsubstituting equation (5), (6) and (7) into (4). Hausdorff dimension depends on the base angles. The value ranges from 1.3 to 2.0.

3. IFS Codes for Fractal Truss and Automatic Data Generation for Frame Analysis

Iterated Function System (IFS) defining configurations of the fractal structures in two dimensional space can be written as

$$W\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} e \\ f \end{Bmatrix} \tag{8}$$

,where a_{11}, a_{12}, a_{21}, a_{22}, e and f are IFS codes. They are derived using M. F. Barnsley's contraction mapping theory. A pair of x-y coordinates of an arbitrary point on an original structure Δ_0 decribed previously is transformed into one on the sub-structures by equation (8). Therefore those of all nodal points on the structure are generated easily if IFS codes and a scale of the original structure are specified. Structural members are defined similarly.

4. Analytical Result

Analytical results under vertical loading and wind loading in Japanese Building Code are presented. Members are assumed to be timber and to have a cross section of 15cm × 15cm. Young's modulous is 784kN/cm^2(80 ton/cm^2) and span of analyticl model is 10 meters.

Displacements per unit volume under vertical loadings are not so different eah other. However those of a fractal trass under wind loadings become smaller than King-post at the range from 25 to 40 degrees of inclination angle. The reason can be thought that total length of members of King-post is larger than other trusses at the same range of the base angle.

5. Concluding Remarks

The authors conclude that geometric characteristics of the new truss structures including fractal dimension are shown and Automatic data generation for frame analysis was attained using IFS. Analytical results show they contribute to saving material when compared with King-post truss under wind loading.