Locally finite extensions and Gesztesy-Šeba realizations for the Dirac operator on a metric graph

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Chapter 1

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Abstract

We study extensions of direct sums of symmetric operators $S = \oplus_{n\in\mathbb{N}}S_n$. In general there is no natural boundary triplet for S^* even if there is one for every S_n^*, $n \in \mathbb{N}$. We consider a subclass of extensions of S which can be described in terms of the boundary triplets of S_n^* and investigate the self-adjointness, the semi-boundedness from below and the discreteness of the spectrum. Sufficient conditions for these properties are obtained from recent results on weighted discrete Laplacians. The results are applied to Dirac operators on metric graphs with point interactions at the vertices. In particular, we allow graphs with arbitrarily small edge length.

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1.1 Introduction

We consider direct sum operators $S = \bigoplus_{n\in\mathbb{N}} S_n$ in a direct sum Hilbert space $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ associated to a family of closed densely defined symmetric operators $\{S_n\}_{n\in\mathbb{N}}$, where S_n

is defined in the Hilbert space $\mathcal{H}_n$. It is easy to see that S is closed and symmetric. Furthermore, if S_n has self-adjoint extensions for all $n \in \mathbb{N}$, then also S has self-adjoint extensions.

The direct sum operator S can be viewed as an diagonal operator matrix with infinitely many entries. Its self-adjoint extensions are no longer diagonal. Here we are interested in the spectrum and related properties. Setting $\mathcal{H}_n = \{0\}$ for all but for two or three entries we end up with a 2×2 (3×3, respectively) operator matrix, see the books [30] and [14].

For the description of the extensions of closed symmetric operators and their spectral properties we use boundary triplets and their associated Weyl functions, see [8, 9, 13, 18]. A boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ consists of a Hilbert space $\mathcal{G}$ and a surjection $(\Gamma_0, \Gamma_1)^T :$ $\operatorname{dom} S^* \to \mathcal{G} \times \mathcal{G}$ that satisfies an abstract Green identity, cf. (1.3.1) below. Here the closed extensions of S correspond one to one to the closed linear subspaces $\Theta \subseteq \mathcal{G} \times \mathcal{G}$ and the extension of S is given by

$$S_\Theta := \{f \in \operatorname{dom} S^* \mid (\Gamma_0 f, \Gamma_1 f) \in \Theta\}. \tag{1.1.1}$$

In order to apply this approach to quantum graphs, we will write the extension (1.1.1) of S in a different, more suitable way: given a closed subspace $\mathcal{G}_{\mathrm{op}}$ of $\mathcal{G}$ and a closed operator L with $\operatorname{dom} L \subseteq \mathcal{G}_{\mathrm{op}}$ then a specific closed extension of S is given by

$$S_L = \{f \in \operatorname{dom} S^* \mid L\Gamma_0 f = P_{\mathcal{G}_{\mathrm{op}}} \Gamma_1 f, \ \Gamma_0 f \in \mathcal{G}_{\mathrm{op}}\}. \tag{1.1.2}$$

To illustrate the above abstract concept, we will briefly show how (1.1.2) looks like for a δ-type point interaction on a graph G with countable sets of vertices V and edges E and with the edge length function $\ell : E \to (0, \infty)$. Consider $\mathcal{H} = L^2(G) = \bigoplus_{e \in E} L^2(0, \ell(e))$ with the operator

$$S = \bigoplus_{e \in E} S_e, \quad \operatorname{dom} S_e = W_0^{2,2}(0, \ell(e)), \quad S_e := -\frac{d^2}{dx_e^2}, \tag{1.1.3}$$

where S_e is the minimal operator on the edge e associated with the differential expression above and $W_0^{2,2}(0, \ell(e))$ denotes the usual second order Sobolev space with boundary values equal to zero. The operator S in (1.1.3) is symmetric with the adjoint S^* defined on $W^{2,2}(G) := \bigoplus_{e \in E} W^{2,2}(0, \ell(e))$.

A point interaction of δ-type on a graph is an extension H_α of S. It is introduced for finite graphs in [3, 4] and for infinite graphs in [10]. The domain of H_α can be specified with a real-valued sequence $(\alpha(v))_{v \in V}$ by

$$\operatorname{dom} H_\alpha := \left\{ (\psi_e)_{e \in E} \in W^{2,2}(G) \cap \mathcal{C}(G) \ \Big| \ \sum_{(e,t) \in I_v} \operatorname{sgn}(e, t) \psi_e'(t\ell(e)) = \alpha(v)\psi(v), \ v \in V \right\}, \tag{1.1.4}$$

where $\mathcal{C}(G)$ is the set of continuous functions on G viewed as a metric space, $\psi(v)$ is the evaluation of ψ at the vertex v and I_v is the set of pairs (e, t) with $e \in E$, $t = 0, 1$. We

have $(e, 0) \in I_v$ if v is an initial vertex of the directed edge e and in this case we set $\operatorname{sgn}(e, 0) := 1$. Furthermore, we have $(e, 1) \in I_v$ if v is a terminal vertex of the directed edge e and we set $\operatorname{sgn}(e, 1) := -1$.

We show how (1.1.4) can be written in the form (1.1.2). First, we need a boundary triplet for S^*. It is well known [28, Example 15.3] that a boundary triplet $\{\mathcal{G}_e, \Gamma_0^e, \Gamma_1^e\}$ for S_e^* is given by

$$
\mathcal{G}_e := \mathbb{C}^2, \quad \Gamma_0^{(e)}\psi_e := \begin{pmatrix} \psi_e(0+) \\ \psi_e(\ell(e)-) \end{pmatrix}, \quad \Gamma_1^{(e)}\psi_e := \begin{pmatrix} \psi_e'(0+) \\ -\psi_e'(\ell(e)-) \end{pmatrix}. \tag{1.1.5}
$$

If $0 < \inf_{e \in E} \ell(e) < \sup_{e \in E} \ell(e) < \infty$, then it follows from [19] that a boundary triplet for S^* is given by the direct sum of the triplets (1.1.5),

$$
\{\mathcal{G}, \Gamma_0, \Gamma_1\} := \Big\{ \bigoplus_{e \in E} \mathcal{G}_e, \bigoplus_{e \in E} \Gamma_0^{(e)}, \bigoplus_{e \in E} \Gamma_1^{(e)} \Big\}. \tag{1.1.6}
$$

Each entry of an element of $\mathcal{G}$ corresponds to a vertex of the decoupled graph, i.e. the elements of $\mathcal{G}$ are sequences $(x_{(e,t)})_{(e,t) \in I}$ with $I := E \times \{0, 1\}$. For $\psi \in W^{2,2}(G)$ we write

$$
\Gamma_0 \psi := (\Gamma_0^{(e,t)}\psi)_{(e,t) \in I} = (\psi_e(t\ell(e)))_{(e,t) \in I},
$$
$$
\Gamma_1 \psi := (\Gamma_1^{(e,t)}\psi)_{(e,t) \in I} = (\operatorname{sgn}(e,t)\psi_e'(t\ell(e)))_{(e,t) \in I}.
$$

Using this boundary triplet, the condition $\psi \in C(G)$ in (1.1.4) is equivalent to

$$
(\Gamma_0^{(e,t)}\psi_e)_{(e,t) \in I_v} \in \mathcal{G}_v := \operatorname{span}\{1_v\}, \quad 1_v := (1, \dots, 1) \in \mathbb{C}^{|I_v|},
$$

for all $v \in V$. Let $\deg v := |I_v|$ be the *degree* of $v \in V$. Here and in the following we make the (crucial) assumption, that the graphs are *locally finite*, i.e.

$$
\deg v < \infty \quad \text{for all } v \in V.
$$

The expressions in the equality in (1.1.4) are equivalent to

$$
P_{\mathcal{G}_v}(\Gamma_1^{(e,t)}\psi_e)_{(e,t) \in I_v} = \frac{1}{\|1_v\|^2}((\Gamma_1^{(e,t)}\psi_e)_{(e,t) \in I_v}, 1_v)_{\mathbb{C}^{\deg v}} 1_v
$$
$$
= \frac{1}{\deg v} \sum_{(e,t) \in I_v} \operatorname{sgn}(e,t)\psi_e'(t\ell(e)) \cdot 1_v
$$
$$
= \frac{\alpha(v)}{\deg v}(\Gamma_0^{(e,t)}\psi_e)_{(e,t) \in I_v}
$$

Let ι_v be the natural embedding of elements of $\mathcal{G}_v$ in the sequence space $\mathcal{G}$. For the operator

$$
L := \oplus_{v \in V} L_v \quad \text{with } L_v \iota_v 1_v := \frac{\alpha(v)}{\deg v} \iota_v 1_v, \ \operatorname{dom} L_v = \iota_v \operatorname{span}\{1_v\}
$$

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on $\mathcal{G}_\mathcal{V} := \oplus_{v \in V} \iota_v \mathcal{G}_v$ we have

$$S_L = H_\alpha$$

in the case $0 < \inf_{e \in E} \ell(e) < \sup_{e \in E} \ell(e) < \infty$.

In Proposition 1.3.1, we show that the extension S_L of S is self-adjoint, semi-bounded from below and has discrete spectrum if and only if the operator L has this property. In our point interaction example the operator L is just an infinite diagonal matrix, therefore the above mentioned spectral properties translate easily to H_α, see [10].

If $\inf_{e \in E} \ell(e) = 0$, then there is no natural candidate for a boundary triplet associated to S^* since the operators in (1.1.6) are in general not defined on $\operatorname{dom} S^*$. However, it was shown in [19] that the triplet (1.1.6) is a so called *boundary relation* in the sense of [7]. To obtain a boundary triplet for S^* from (1.1.6) a regularization technique has been applied in [5, 10, 19, 23, 24]. Here we apply in Theorem 1.4.1 below the technique from [5] for operators where there exists $\lambda_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \in \bigcap_{n=0}^\infty \rho\big(S_n^*|_{\ker \Gamma_0^{(n)}}\big)$. Then a (regularized) boundary triplet $\left\{ \widetilde{\mathcal{G}}, \bigoplus_{n=0}^\infty \widetilde{\Gamma}_0^{(n)}, \bigoplus_{n=0}^\infty \widetilde{\Gamma}_1^{(n)} \right\}$ is given by

$$\widetilde{\mathcal{G}} := \mathcal{G}, \quad \widetilde{\Gamma}_0^{(n)} := \sqrt{\|M_n'(\lambda_0)\|}\,\Gamma_0^{(n)}, \quad \widetilde{\Gamma}_1^{(n)} := \frac{\Gamma_1^{(n)} - M_n(\lambda_0)\Gamma_0^{(n)}}{\sqrt{\|M_n'(\lambda_0)\|}}, \tag{1.1.7}$$

where M_n is the Weyl function of the boundary triplet $\{\mathcal{G}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$. Again, one can represent extensions of S in terms of an operator $\widetilde{L}$ (now with respect to the regularized triplet $\{\mathcal{G}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ from (1.1.7)) in the form of (1.1.2),

$$S_{\widetilde{L}} = \{f \in \operatorname{dom} S^* \mid \widetilde{L}\widetilde{\Gamma}_0 f = P_{\widetilde{\mathcal{G}}_{\mathrm{op}}} \widetilde{\Gamma}_1 f, \ \widetilde{\Gamma}_0 f \in \widetilde{\mathcal{G}}_{\mathrm{op}} \}, \tag{1.1.8}$$

where $\widetilde{L}$ is defined on some subspace $\widetilde{\mathcal{G}}_{op}$ of $\mathcal{G}$. Whereas in the example above the operator L is just an (infinite) diagonal operator, now, in general, the operator $\widetilde{L}$ has a more complex structure.

The operator $\widetilde{L}$ from above, that describes the extensions with respect to the regularized boundary mappings, is studied in [5, 10, 19]. In [19] Schrödinger operators with point interactions on the real line are considered. In this case, roughly speaking, the operator L in (1.1.8) for a point interaction if (1.1.6) is a boundary triplet, is a diagonal operator, whereas the operator $\widetilde{L}$ is a Jacobi operator and therefore a correspondence of extensions describing such interactions and Jacobi operators is made in [19]. In particular, criteria for self-adjointness, semi-boundedness from below and discreteness of the spectrum are obtained from corresponding criteria for Jacobi operators. Later, in [5] the ideas of [19] were extended to the case of Dirac operators with point interactions on the real line, so called Gesztesy-Šeba realzations, see [12]. Recently, in [10] the regularization is applied to quantum graphs and Laplacians with point interactions are studied. In this case, the operator $\widetilde{L}$ in (1.1.8) is a discrete Laplacians on a weighted ℓ^2-space, see [16, 17, 11] and the references therein.

Here we consider a more general class of extensions of symmetric direct sum operators $S = \bigoplus_{n=0}^{\infty} S_n$: *locally finite extensions* S_L^{loc}. It turns out that the operator $\widetilde{L}$ from above is also a weighted discrete Laplacian. The locally finite extensions of S are such that they extend the quantum graph examples to more general structures. In particular, the symmetric operators S_n may have an arbitrary but finite defect indices

We study properties of the extensions S_L^{loc} like self-adjointness, semi-boundedness from below and discreteness of the spectrum in terms of the associated weighted discrete Laplacian $\widetilde{L}$ to the extension S_L^{loc}. We show that self-adjointness, semi-boundedness from below and discreteness of the spectrum of $\widetilde{L}$ implies the same property for S_L^{loc}. Sufficient conditions for such properties for S_L^{loc} are obtained recently in [11, 17].

In the case where (1.1.6) is not a boundary triplet, some recent approaches [26, 29] without using the regularization technique, lead to a parametrization of the self-adjoint extensions of S, but without explicit criteria for the above mentioned properties (like (self-adjointness, semi-boundedness from and discreteness of the spectrum).

Moreover, the boundary triplet approach to quantum graphs was previously applied in numerous works, see e.g. [2, 10, 20, 21, 25, 27]. In [2, 20, 21] finite graphs are considered. Graphs with an infinite number of edges but with finite vertex degree were considered in [25], under the assumption that $\ell(e) = 1$ for all $e \in E$, and assuming that $\inf_{e \in E} \ell(e) > 0$ in [27]. The study of the operators S_L was carried out in [2] for star-graphs and for quantum graphs satisfying $\inf_{e \in E} \ell(e) > 0$ in [22].

The paper is organized as follows: First, we recall linear relations in Hilbert space and boundary triplets. From the boundary triplet theory, we collect some results on the properties of the extension S_L given by (1.1.2) which can be described terms of the operator L and the Weyl function of an underlying boundary triplet for S^*. In Section 1.4 we introduce locally finite extension S_L^{loc} and construct an associated discrete Laplacian D_L such that roughly speaking $S_L^{\mathrm{loc}} = S_{D_L}$ holds in the sense of (1.1.8) with $\widetilde{L} = D_L$. From this relation, we obtain conditions for the self-adjointness, lower semi-boundedness and discreteness of the spectrum of S_L^{loc}. These conditions only depend on the matrices L_v, the subspaces $\mathcal{G}_v$ and the decoupled Weyl functions M_n. Finally, in Section 1.5 we apply our results to Dirac operators with point interactions on infinite graphs.

1.2 Linear relations in Hilbert spaces

Let $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$ be a separable Hilbert space. A *(closed) linear relation in* $\mathcal{H}$ is a (closed) subspace of $\mathcal{H} \times \mathcal{H}$ and the set of all closed linear relations in $\mathcal{H}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{H})$. For a linear operator T defined in $\mathcal{H}$ with values in $\mathcal{H}$, the graph of T is a linear relation in $\mathcal{H}$. The set of all closed linear operators in $\mathcal{H}$ is denoted by $\mathcal{C}(\mathcal{H})$. For the subspace of *bounded* linear operators defined on $\mathcal{H}$ we write $\mathcal{L}(\mathcal{H})$.

The *domain*, the *range*, the *kernel*, the *multivalued part* and the *inverse* of a linear

relation Θ in $\mathcal{H}$ are given by

$$
\begin{aligned}
\operatorname{dom}\Theta &:= \{f \in \mathcal{H} \mid (f, f') \in \Theta \text{ for some } f' \in \mathcal{H}\}, \\
\operatorname{ran}\Theta &:= \{f' \in \mathcal{H} \mid (f, f') \in \Theta \text{ for some } f \in \mathcal{H}\}, \\
\ker\Theta &:= \{f \in \mathcal{H} \mid (f, 0) \in \Theta\}, \\
\operatorname{mul}\Theta &:= \{f' \in \mathcal{H} \mid (0, f') \in \Theta\}, \\
\Theta^{-1} &:= \{(f', f) \in \mathcal{H}^2 \mid (f, f') \in \Theta\}.
\end{aligned}
$$

Recall that the (operator-like) *sum* of two linear relations Θ_1 and Θ_2 in $\mathcal{H}$ is given by

$$
\Theta_1 + \Theta_2 := \{(f, f_1' + f_2') \in \mathcal{H} \times \mathcal{H} \mid (f, f_1') \in \Theta_1, \ (f, f_2') \in \Theta_2\}.
$$

Let Θ be a closed linear relation in $\mathcal{H}$. The set of all $\lambda \in \mathbb{C}$ such that $(\Theta - \lambda)^{-1}$ is the graph of an operator from $\mathcal{L}(\mathcal{H})$ is called *resolvent set* $\rho(\Theta)$ of Θ. The complement of $\rho(\Theta)$ in $\mathbb{C}$ is the *spectrum* $\sigma(\Theta)$ of Θ. The *adjoint* Θ^* of a linear relation Θ in $\mathcal{H}$ is defined as

$$
\Theta^* := \{(g, g') \in \mathcal{H}^2 \mid (f', g)_{\mathcal{H}} = (f, g')_{\mathcal{H}} \text{ for all } (f, f') \in \Theta\}.
$$

A linear relation is called *symmetric (self-adjoint)* if $\Theta \subseteq \Theta^*$ (resp. $\Theta = \Theta^*$). For $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ we have

$$
\operatorname{mul}\Theta = (\operatorname{dom}\Theta^*)^{\perp}, \quad \operatorname{mul}\Theta^* = \operatorname{dom}\Theta^{\perp}.
$$

Given a self-adjoint linear relation Θ, we can associate a self-adjoint operator on the Hilbert space $\overline{\operatorname{dom}\Theta}$, see [1, Theorem 5.3]. Below, we present a somehow converse result.

Proposition 1.2.1 *Let $\mathcal{H}_{\mathrm{op}}$ be a closed subspace of a Hilbert space $\mathcal{H}$ and consider a densely defined operator L from $\mathcal{H}_{\mathrm{op}}$ to $\mathcal{H}_{\mathrm{op}}$. Define*

$$
\Theta_L := \{(f, Lf + g) \mid f \in \operatorname{dom}L, \ g \in \mathcal{H}_{\mathrm{op}}^{\perp}\} \subseteq \mathcal{H} \times \mathcal{H}. \tag{1.2.1}
$$

Then the following holds.

(a) *We have $\Theta_L^* = \Theta_{L^*}$. If L is closable, we have $\overline{\Theta_L} = \Theta_{\overline{L}}$.*

(b) *Θ_L is closed (symmetric, self-adjoint) if and only if L is closed (resp. symmetric, self-adjoint).*

(c) *If L is symmetric then all extensions $\widetilde{\Theta}$ with $\Theta_L \subseteq \widetilde{\Theta} \subseteq \Theta_L^*$ are of the form $\Theta_{\widetilde{L}}$, where $\widetilde{L}$ is an extension of L.*

(d) *If L is self-adjoint, then $\rho(L) = \rho(\Theta_L)$ and for all $\lambda \in \rho(L)$*

$$
(\Theta_L - \lambda)^{-1} = \begin{pmatrix} (L - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H}_{\mathrm{op}} \oplus \mathcal{H}_{\mathrm{op}}^{\perp}).
$$

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Proof. Let $(f, g) \in \Theta_L^*$. Then for all $(f', Lf' + g') \in \Theta_L$ with $f' \in \operatorname{dom} L$ and $g' \in \mathcal{H}_{\mathrm{op}}^\perp$ we have

$$(g, f') = (f, Lf' + g'). \tag{1.2.2}$$

Choosing $f' = 0$ we obtain $f \in \mathcal{H}_{\mathrm{op}}$. Therefore, we conclude from (1.2.2)

$$(g, f') = (f, Lf')$$

for all $f' \in \operatorname{dom} L$. This implies that $f \in \operatorname{dom} L^*$ and $P_{\mathcal{H}_{\mathrm{op}}} g = L^* f$. Hence,

$$(f, g) = (f, P_{\mathcal{H}_{\mathrm{op}}} g + P_{\mathcal{H}_{\mathrm{op}}^\perp} g) = (f, L^* f + P_{\mathcal{H}_{\mathrm{op}}^\perp} g) \in \Theta_{L^*}.$$

Assume conversely that $(f, L^* f + g) \in \Theta_{L^*}$ with $f \in \operatorname{dom} L^*$ and some $g \in \mathcal{H}_{\mathrm{op}}^\perp$. Then we have for all $(f', Lf' + g') \in \Theta_L$ with $f' \in \operatorname{dom} L$ and $g' \in \mathcal{H}_{\mathrm{op}}^\perp$

$$(f', L^* f + g) = (f', L^* f) = (Lf', f) = (Lf' + g', f)$$

and therefore $(f, L^* f + g) \in \Theta_L^*$. Thus we have seen that $\Theta_L^* = \Theta_{L^*}$.

Let $(f, \overline{L} f + g) \in \Theta_{\overline{L}}$ with $f \in \operatorname{dom} \overline{L}$ then there is a sequence $((f_n, Lf_n))_{n \in \mathbb{N}}$ which converges in $\mathcal{H}^2$ to $(f, \overline{L} f)$. But then $(f_n, Lf_n + g) \in \Theta_L$ converges in $\mathcal{H}^2$ to $(f, \overline{L} f + g)$, as $n \to \infty$, which implies $(f, \overline{L} f + g) \in \overline{\Theta_L}$.

Conversely, let $(f, g') \in \overline{\Theta_L}$ then there exists a sequence $(f_n, Lf_n + g_n) \in \Theta_L$ with $g_n \in \mathcal{H}_{\mathrm{op}}^\perp$ and $f_n \in \operatorname{dom} L$ which converges to (f, g'). As $\mathcal{H}^2 = (\mathcal{H}_{\mathrm{op}} \times \mathcal{H}_{\mathrm{op}}) \oplus (\mathcal{H}_{\mathrm{op}}^\perp \times \mathcal{H}_{\mathrm{op}}^\perp)$ we have $(f_n, Lf_n) \to (f, P_{\mathcal{H}_{\mathrm{op}}} g')$ and $(0, g_n) \to (0, P_{\mathcal{H}_{\mathrm{op}}^\perp} g')$, as $n \to \infty$. Therefore $f \in \operatorname{dom} \overline{L}$ with $\overline{L} f = P_{\mathcal{H}_{\mathrm{op}}} g'$. Hence

$$(f, g') = (f, \overline{L} f + P_{\mathcal{H}_{\mathrm{op}}^\perp} g') \in \Theta_{\overline{L}}.$$

The assertion (b) is a consequence of (a). We show (c). Let $\widetilde{\Theta}$ be an extension of Θ_L with $\Theta_L \subseteq \widetilde{\Theta} \subseteq \Theta_{L^*}$. Obviously,

$$\operatorname{dom} L = \operatorname{dom} \Theta_L \subseteq \operatorname{dom} \widetilde{\Theta} \subseteq \operatorname{dom} \Theta_{L^*} = \operatorname{dom} L^*.$$

Set $\widetilde{L} := L^* | \operatorname{dom} \widetilde{\Theta}$. Then $\widetilde{L}$ is an extension of L. As $\widetilde{\Theta} \subset \Theta_{L^*}$, every element $(f, g') \in \widetilde{\Theta}$ satisfies $f \in \operatorname{dom} \widetilde{\Theta} \subset \operatorname{dom} L^*$ and has a representation

$$(f, g') = (f, L^* f + g)$$

for some $g \in \mathcal{H}_{\mathrm{op}}^\perp$. As $L^* f = \widetilde{L} f$ for $f \in \operatorname{dom} \widetilde{\Theta}$, $(f, g') \in \Theta_{\widetilde{L}}$ follows. Hence, $\widetilde{\Theta} \subset \Theta_{\widetilde{L}}$. The converse inclusion is obvious and (c) is shown.

For the last statement observe that we have for all $\lambda \in \mathbb{C}$

$$(\Theta_L - \lambda)^{-1} = \left\{ ((L - \lambda)f + g, f) \mid f \in \operatorname{dom} L, \ g \in \mathcal{H}_{\mathrm{op}}^\perp \right\}.$$

From this (d) follows easily. Q.E.D.

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1.3 Extension theory of symmetric operators with boundary triplets

We review the boundary triplet theory following [8], see also [18].

Definition 1.3.1 *For a densely defined symmetric operator $A \in \mathcal{C}(\mathcal{H})$ in a Hilbert space $\mathcal{H}$ we say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* if $(\mathcal{G}, (\cdot, \cdot)_{\mathcal{G}})$ is a Hilbert space, $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} A^* \to \mathcal{G}^2$ is surjective and the following abstract Green identity holds*

$$(A^* f, g)_{\mathcal{H}} - (f, A^* g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}. \tag{1.3.1}$$

Boundary triplets are a standard tool to describe all closed extensions of a given symmetric operator. For a densely defined symmetric operator $A \in \mathcal{C}(\mathcal{H})$, we fix a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^*. The extension A_Θ of A corresponding to a parameter $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ is defined as

$$\operatorname{dom} A_\Theta := \{f \in \operatorname{dom} A^* \mid (\Gamma_0 f, \Gamma_1 f) \in \Theta\}, \quad A_\Theta f := A^* f.$$

The correspondence between the closed linear relations $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and the closed extensions A_Θ of A is bijective (see, e.g., [8]). The following two special self-adjoint extensions of A will play a prominent role:

$$A_0 := A_{\{0\} \times \mathcal{G}} = A^*|_{\ker \Gamma_0} \quad \text{and} \quad A_1 := A_{\mathcal{G} \times \{0\}} = A^*|_{\ker \Gamma_1}$$

In [8] a correspondence of properties between $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and $A_\Theta \in \mathcal{C}(\mathcal{G})$ was established using the concept of the γ-*field* and the *Weyl function*.

$$\gamma : \rho(A_0) \to \mathcal{L}(\mathcal{H}, \mathcal{G}), \ \gamma(\lambda) := (\Gamma_0|_{\mathcal{N}_\lambda})^{-1}, \quad \mathcal{N}_\lambda(A) := \{f \in \operatorname{dom} A^* \mid A^* f = \lambda f\},$$

$$M : \rho(A_0) \to \mathcal{L}(\mathcal{G}), \quad M(\lambda) := \Gamma_1 \gamma(\lambda).$$

Here we prefer the following description of the extensions. Let L be a densely defined operator on a subspace $\mathcal{G}_{\mathrm{op}}$ of $\mathcal{G}$ mapping into $\mathcal{G}_{\mathrm{op}}$. We consider the relation Θ_L from (1.2.1) and the associated extension $A_L := A_{\Theta_L}$ and therefore

$$\operatorname{dom} A_L = \{f \in \operatorname{dom} A^* \mid \Gamma_0 f \in \operatorname{dom} L, \ L\Gamma_0 f = P_{\overline{\operatorname{dom} L}} \Gamma_1 f\}, \tag{1.3.2}$$

where $P_{\overline{\operatorname{dom} L}}$ is the orthogonal projection onto $\mathcal{G}_{\mathrm{op}} = \overline{\operatorname{dom} L}$ since L is assumed to be densely defined in $\mathcal{G}_{\mathrm{op}}$. Proposition 1.2.1 and some well known results on the relationship between $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and $A_\Theta \in \mathcal{C}(\mathcal{H})$ from [8, 19] lead to the next statement. Here we use the notation $\mathfrak{S}_p(\mathcal{H})$ with $p \in (0, \infty]$ for the two sided *Schatten-von Neumann ideal* and we denote by $n_\pm(A) := \dim \mathcal{N}_{\pm i}(A)$ the *defect numbers* of a symmetric densely defined linear operator A.

Proposition 1.3.1 *Let A be a densely defined symmetric operator in $\mathcal{H}$ with boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^* and let L be a densely defined operator in a subspace $\mathcal{G}_{\mathrm{op}}$ of $\mathcal{G}$ then the following holds.*

(a) *A_L is self-adjoint (symmetric) if and only if L is self-adjoint (resp. symmetric).*

(b) *$A_{\overline{L}} = \overline{A_L}$, $A_{L^*} = A_L^*$ and $n_\pm(A_L) = n_\pm(L)$.*

(c) *If L is symmetric, then there is a bijective correspondence between the extensions of L and the extensions of A_L.*

(d) *For $\lambda \in \rho(A_0)$ we have $\lambda \in \rho(A_L)$ if and only if $0 \in \rho(\Theta_L - M(\lambda))$. In this case the Krein resolvent formula holds*

$$(A_L - \lambda)^{-1} - (A_0 - \lambda)^{-1} = \gamma(\lambda)(\Theta_L - M(\lambda))^{-1}\gamma(\overline{\lambda})^*.$$

(e) *Let $(A_0 - \lambda_0)^{-1} \in \mathfrak{S}_p(\mathcal{H})$ for some $\lambda_0 \in \rho(A_0)$ and $p \in [1, \infty]$. Then $(A_L - \lambda)^{-1} \in \mathfrak{S}_p(\mathcal{H})$ if and only if $(L - \lambda)^{-1} \in \mathfrak{S}_p(\mathcal{G})$ for $\lambda \in \rho(L)$.*

Let A be a densely defined symmetric operator which is semi-bounded from below, i.e. $A \geq \gamma$ for some $\gamma \in \mathbb{R}$. Then there is a distinguished, in some sense maximal, semi-bounded self-adjoint extension $A_F \geq \gamma$, which is called the *Friedrichs extension* of A, see e.g. [28, Section 10.4].

Given boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ of A^* with Weyl function M such that A_0 equals the Friedrichs extension A_F, then we use the notation $M(\lambda) \rightrightarrows -\infty$ for $\lambda \to -\infty$ to indicate that for any $\gamma > 0$ there exists λ_γ with $-M(\lambda_\gamma) \geq \gamma$.

We collect some results on nonnegative extensions from [8, 9], see also [28].

Proposition 1.3.2 *Given a densely defined symmetric operator $A \in \mathcal{C}(\mathcal{H})$, a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^* with $A_0 = A_F \geq \gamma$ for $\gamma > 0$ and a self-adjoint operator $L \in \widetilde{\mathcal{C}}(\mathcal{G}_{\mathrm{op}})$ on a subspace $\mathcal{G}_{\mathrm{op}}$ of $\mathcal{G}$. Then the following holds.*

(a) *$L - P_{\mathcal{G}_{\mathrm{op}}} M(\lambda_0)|_{\mathcal{G}_{\mathrm{op}}} \geq 0$ for $\lambda_0 < \gamma$ implies $A_L \geq \lambda_0$.*

(b) *If $M(\lambda) \rightrightarrows -\infty$ for $\lambda \to -\infty$ then A_L is semi-bounded from below if and only if L is semi-bounded from below.*

In the lemma below, we decribe the change of a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ under unitary transformations of the space $\mathcal{G}$.

Lemma 1.3.1 *Let $A \in \mathcal{C}(\mathcal{H})$ be a densely defined symmetric operator with a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^* and a unitary operator $U : \mathcal{G} \to \widehat{\mathcal{G}}$ then $\{\widehat{\mathcal{G}}, U\Gamma_0, U\Gamma_1\}$ is a boundary triplet for A^* with Weyl function $\lambda \mapsto UM(\lambda)U^*$ on $\rho(A^*|_{\ker \Gamma_0})$. Furthermore the extension A_L given by (1.3.2) can be written with $\widehat{L} := ULU^*$ as*

$$\mathrm{dom}\, A_L = \{f \in \mathrm{dom}\, A^* \mid U\Gamma_0 f \in \mathrm{dom}\, \widehat{L}, \quad \widehat{L} U\Gamma_0 f = P_{\overline{\mathrm{dom}\, \widehat{L}}} U\Gamma_1 f\}.$$

Proof. Since U is unitary, the mapping $f \mapsto (U\Gamma_0 f, U\Gamma_1 f)$ from $\operatorname{dom} A^*$ into $\widehat{\mathcal{G}}^2$ is onto and the abstract Green identity (1.3.1) holds. Hence $\{\widehat{\mathcal{G}}, U\Gamma_0, U\Gamma_1\}$ is a boundary triplet for A^* with $A^*|_{\ker \Gamma_0} = A^*|_{\ker U\Gamma_0}$ and Weyl function $\lambda \mapsto UM(\lambda)U^*$ which is defined for all $\lambda \in \rho(A^*|_{\ker \Gamma_0})$. Given that $f \in \operatorname{dom} A_L$ then we have $\Gamma_0 f \in \operatorname{dom} L$ and $L\Gamma_0 f = P_{\overline{\operatorname{dom} L}}\Gamma_1 f$ which is equivalent to

$$U\Gamma_0 f \in U\operatorname{dom} L, \quad ULU^*U\Gamma_0 f = UP_{\overline{\operatorname{dom} L}}U^*U\Gamma_1 f. \tag{1.3.3}$$

Furthermore, it is easy to see that

$$\operatorname{dom}\widehat{L} = \operatorname{dom} ULU^* = \operatorname{dom} LU^* = U\operatorname{dom} L. \tag{1.3.4}$$

Moreover $UP_{\overline{\operatorname{dom} L}}U^*$ is an orthogonal projection, satisfying

$$UP_{\overline{\operatorname{dom} L}}U^* = P_{U\overline{\operatorname{dom} L}} = P_{\overline{U\operatorname{dom} L}} = P_{\overline{\operatorname{dom}\widehat{L}}}. \tag{1.3.5}$$

Rewriting (1.3.3) with (1.3.4) and (1.3.5) completes the proof of the lemma. Q.E.D.

1.4 Locally finite extensions of direct sums of symmetric operators

In this section, we introduce direct sum operators and their locally finite extensions. Throughout this section we consider a family of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ with inner product $(\cdot,\cdot)_{\mathcal{H}_n}$ and densely defined symmetric operators $S_n \in \mathcal{C}(\mathcal{H}_n)$ with boundary triplets $\{\mathcal{G}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for S_n^* such that $\dim\mathcal{G}_n < \infty$, $n \in \mathbb{N}$. We introduce the direct sum Hilbert space $\mathcal{H}$,

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n := \{x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathcal{H}_n, (x,x)_{\mathcal{H}} < \infty\}$$

with inner product

$$((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})_{\mathcal{H}} := \sum_{n=0}^{\infty}(x_n, y_n)_{\mathcal{H}_n}.$$

Acting on $\mathcal{H}$ we introduce the direct sum operator $S := \bigoplus_{n=0}^{\infty} S_n$ via

$$\operatorname{dom} S := \left\{(f_n)_{n\in\mathbb{N}} \,\Big|\, f_n \in \operatorname{dom} S_n, \sum_{n=0}^{\infty}\|S_n f_n\|_{\mathcal{H}_n}^2 < \infty\right\}, \quad S(f_n)_{n\in\mathbb{N}} := (S_n f_n)_{n\in\mathbb{N}}.$$

The case of a finite dimensional direct sum Hilbert space $\mathcal{H}$ is obtained by setting $\mathcal{H}_n := \{0\}$ and $S_n := 0$ for all $n \geq N$ and some $N \in \mathbb{N}$. It is easy to see that S is densely defined, closed with the adjoint

$$\left(\bigoplus_{n=0}^{\infty} S_n\right)^* = \bigoplus_{n=0}^{\infty} S_n^*.$$

Since $S_n \subseteq S_n^*$ for all $n \in \mathbb{N}$ it is easy to see that S is symmetric with $n_\pm(S) = \sum_{n=0}^\infty n_\pm(S_n)$. To describe the extensions of S, the natural candidate for a boundary triplet for S^* is given by $\mathcal{G} := \bigoplus_{n=0}^\infty \mathcal{G}_n$ with the boundary mappings Γ_i, $i = 1, 2$,

$$\operatorname{dom}\Gamma_i := \left\{ (f_n)_{n\in\mathbb{N}} : f_n \in \operatorname{dom}\Gamma_i^{(n)}, \ \sum_{n=0}^\infty \|\Gamma_i^{(n)} f_n\|^2 < \infty \right\}, \quad \Gamma_i(f_n)_{n\in\mathbb{N}} := \left(\Gamma_i^{(n)} f_n \right)_{n\in\mathbb{N}}$$

which can also be written in the form

$$\mathcal{G} := \bigoplus_{n=0}^\infty \mathcal{G}_n, \quad \Gamma_0 := \bigoplus_{n=0}^\infty \Gamma_0^{(n)}, \quad \Gamma_1 := \bigoplus_{n=0}^\infty \Gamma_1^{(n)}. \tag{1.4.1}$$

In general, the operators Γ_0 and Γ_1 are only defined on a subspace of $\operatorname{dom} S^*$ such that (1.4.1) is not a boundary triplet for S^*. However, it was shown in [19] that the triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ given by (1.4.1) forms a single valued boundary relation in the sense of [7].

We use a particular regularization from [5] for the direct sum triplet (1.4.1) for operators with a common real point in the resolvent set, i.e. we assume that for $S_{n0} := S_n^*|_{\ker\Gamma_0^{(n)}}$ there exist $\lambda_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subseteq \bigcap_{n\in\mathbb{N}} \rho(S_{n0})$. In the theorem below we use [5, Theorem 2.12], to provide a boundary triplet for the direct sum operator S^*.

Theorem 1.4.1 *Let $\{S_n\}_{n\in\mathbb{N}}$ be a family of densely defined symmetric linear operators $S_n \in \mathcal{C}(\mathcal{H}_n)$ with boundary triplets $\{\mathcal{G}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for S_n^* and Weyl functions M_n and $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subseteq \bigcap_{n=0}^\infty \rho(S_{n0})$ for some $\varepsilon > 0$ and $\lambda_0 \in \mathbb{R}$. Then $\left\{ \bigoplus_{n=0}^\infty \mathcal{G}_n, \bigoplus_{n=0}^\infty \widetilde{\Gamma}_0^{(n)}, \bigoplus_{n=0}^\infty \widetilde{\Gamma}_1^{(n)} \right\}$ with*

$$\widetilde{\Gamma}_0^{(n)} := \sqrt{\|M_n'(\lambda_0)\|}\,\Gamma_0^{(n)}, \quad \widetilde{\Gamma}_1^{(n)} := \sqrt{\|M_n'(\lambda_0)\|}^{-1} \left(\Gamma_1^{(n)} - M_n(\lambda_0)\Gamma_0^{(n)} \right) \tag{1.4.2}$$

is a boundary triplet for $S^* = \bigoplus_{n=0}^\infty S_n^*$. The Weyl function $\widetilde{M}$ of this triplet is given by

$$\widetilde{M} : \rho(S^*|_{\ker\Gamma_0}) \to \mathcal{L}(\mathcal{G}), \ \lambda \mapsto \bigoplus_{n=0}^\infty \widetilde{M}_n(\lambda) \ with$$

$$\widetilde{M}_n := \frac{1}{\|M_n'(\lambda_0)\|}(M_n - M_n(\lambda_0)). \tag{1.4.3}$$

The construction of this regularization implies that $S_n^*|_{\ker\Gamma_0^{(n)}} = S_n^*|_{\ker\widetilde{\Gamma}_0^{(n)}}$ and therefore

$$S^*|_{\ker\widetilde{\Gamma}_0} = \bigoplus_{n=0}^\infty S_{n0}. \tag{1.4.4}$$

The remainder of this section is devoted to locally finite extensions. We assume that the Hilbert space $\mathcal{G}$ is given as the direct sum Hilbert space

$$\mathcal{G} = \bigoplus_{n=0}^\infty \mathcal{G}_n = \bigoplus_{n=0}^\infty \mathbb{C}^{d_n} \quad \text{with } d_n < \infty \text{ and } \mathcal{G}_n = \mathbb{C}^{d_n}.$$

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The elements of $\mathcal{G}$ are sequences of the form $x = (x_i)_{i \in I}$ where

$$I := \{(n, d) \mid n \in \mathbb{N}, \ d = 1, \ldots, d_n\}.$$

In the following we will consider a partition of I into subsets I_v where v is an element of a countable index set V such that the following conditions are fulfilled:

(i) $|I_v| < \infty$,

(ii) $I_v \cap I_w = \emptyset$ for all $v, w \in V$ with $v \neq w$,

(iii) $\bigcup_{v \in V} I_v = I$.

Since $\Gamma_i^{(n)} f_n \in \mathbb{C}^{d_n}$ the sequence $(\Gamma_i^{(n)} f_n)_{n \in \mathbb{N}}$, $i = 0, 1$ is an element of $\times_{n=0}^{\infty} \mathbb{C}^{d_n}$, but not necessarily of the Hilbert space $\bigoplus_{n=0}^{\infty} \mathbb{C}^{d_n}$. Thus, we can view it as sequence $(\Gamma_i^{(n,d)} f_n)_{(n,d) \in I}$ where

$$\Gamma_i^{(n,d)} f_n := (\Gamma_i^{(n)} f_n)_d, \quad 1 \leq d \leq d_n, \quad i = 0, 1$$

is the d-th entry of $\Gamma_i^{(n)} f_n$. With this we introduce for $f \in \operatorname{dom} S^*$

$$\Gamma_i^v f := (\Gamma_i^{(n,t)} f_n)_{(n,t) \in I_v}, \quad i = 0, 1.$$

Before we continue with the definition of locally finite extensions, we illustrate the definitions from above with the quantum graph example from the introduction.

Example 1.4.1 *Consider the densely defined symmetric operators $S_1, \ldots, S_N$ with domains* $\operatorname{dom} S_n := W_0^{2,2}(0, \ell(e_n))$, $n = 1, \ldots, N$ *with* $S_n \psi_n := -\psi_n''$. *Then a boundary triplet for* S_n^* *with* $n = 1, \ldots, N$ *is given by*

$$\{\mathbb{C}^2, (\psi_n(0+), \psi_n(\ell(e_n)-))^\top, (\psi_n'(0+), -\psi_n'(\ell(e_n)-))^\top\},$$

Hence $d_n = 2$ for all n and therefore

$$I = \{1, \ldots, N\} \times \{1, 2\}.$$

Consider the index set $V = \{v_1, \ldots, v_{N+1}\}$. We introduce $I_{v_i} := \{(i, 1)\}$ for $i = 1, \ldots, N$ and $I_{v_{N+1}} := \{(i, 2) : i = 1, \ldots, N\}$. It is easy to see that the conditions (i)-(iii) from above are satisfied. For each index $i = 1, \ldots, N+1$ there is an edge associated with it and the sets I_{v_i} describe which edges are glued together at the vertex v_i which leads to a graph. In this simple example all vertices v_i, $i = 1, \ldots, N$ corresponds so singleton sets I_{v_i}, i.e., only one vertex leads to v_i, whereas in v_{N+1} we have N vertices. Hence the underlying graph is a star graph with $N + 1$ vertices and N edges. Furthermore, we have

$$\Gamma_0^{v_n}(\psi_j)_{j=1}^N = \psi_n(0+), \quad \Gamma_1^{v_n}(\psi_j)_{j=1}^N = \psi_n'(0+), \quad n = 1, \ldots, N,$$

$$\Gamma_0^{v_{N+1}}(\psi_j)_{j=1}^N = (\psi_1(\ell(e_1)-), \psi_2(\ell(e_2)-), \ldots, \psi_N(\ell(e_N)-))^\top,$$

$$\Gamma_1^{v_{N+1}}(\psi_j)_{j=1}^N = (-\psi_1'(\ell(e_1)-), -\psi_2'(\ell(e_2)-), \ldots, -\psi_N'(\ell(e_N)-))^\top.$$

Obviously, one easily can construct examples with infinitely many vertices and edges. Observe, as we only consider locally finite extensions, that always $|I_v| < \infty$ holds, which means, in the cases of graph-like constructions, that in each edge there are only finitely many vertices.

Example 1.4.2 *Here we give an example for a star graph with finite edges and vertices but with infinite edge length. Consider the densely defined symmetric operators $S_1, \ldots, S_N$ from Example 1.4.1 and, in addition, $\mathrm{dom}\, S_{N+1} := W_0^{2,2}(0, \infty)$ with $S_{N+1} \psi_{N+1} := -\psi''_{N+1}$. Then a boundary triplet for S_n^* with $n = 1, \ldots, N$ is given as in Example 1.4.1 and a triplet for S_{N+1}^* is given by $\{\mathbb{C}, \psi_{N+1}(0+), \psi'_{N+1}(0+)\}$, see e.g. [28, Example 15.5]. Hence $d_n = 2$ for all $n = 1, \ldots, N$ but $d_{N+1} = 1$ and therefore*

$$I = (\{1, \ldots, N\} \times \{1, 2\}) \cup \{(N+1, 1)\}.$$

Consider the index set $V = \{v_1, \ldots, v_{N+1}\}$. We introduce $I_{v_i} := \{(i, 1)\}$ for $i = 1, \ldots, N$, $I_{v_{N+1}} := \{(i, 2) : i = 1, \ldots, N\} \cup \{(N+1, 1)\}$. As above we have a star graph, but with one vertex less, as the edge corresponding to $N+1$ is a semi-axis,

$$\Gamma_0^{v_n}(\psi_j)_{j=1}^{N+1} = \psi_n(0+), \ \Gamma_1^{v_n}(\psi_j)_{j=1}^{N+1} = \psi'_n(0+), \quad n = 1, \ldots, N,$$

$$\Gamma_0^{v_{N+1}}(\psi_j)_{j=1}^{N+1} = (\psi_1(\ell(e_1)-), \psi_2(\ell(e_2)-), \ldots, \psi_N(\ell(e_N)-), \psi_{N+1}(0+))^\top,$$

$$\Gamma_1^{v_{N+1}}(\psi_j)_{j=1}^{N+1} = (-\psi'_1(\ell(e_1)-), -\psi'_2(\ell(e_2)-), \ldots, -\psi'_N(\ell(e_N)-), \psi'_{N+1}(0+))^\top.$$

Similarly, one can construct graphs with infinitely many vertices and edges. Moreover, we stress that we are able to allow $d_n > 2$ with leads to structures which do no longer allow an interpretation as a graph.

Now let $\mathcal{G}_v$ be a subspace of $\mathbb{C}^{|I_v|}$ and consider the Hermitian matrix $L_v : \mathcal{G}_v \to \mathcal{G}_v$. We introduce the locally finite extension S_L^{loc} of S

$$\mathrm{dom}\, S_L^{\mathrm{loc}} := \left\{ f \in \mathrm{dom}\, S^* \mid L_v \Gamma_0^v f = P_{\mathcal{G}_v} \Gamma_1^v f, \ \Gamma_0^v f \in \mathcal{G}_v, \ v \in V \right\},$$
$$S_L^{\mathrm{loc}} f := S^* f.$$

It is shown in Proposition 1.4.1 below that S_L^{loc} is the adjoint of the operator $S_L^{\mathrm{min}} \subseteq S^*$ with

$$\mathrm{dom}\, S_L^{\mathrm{min}} := \left\{ f \in \mathrm{dom}\, S_L^{\mathrm{loc}} \mid \mathrm{supp}\, (\Gamma_0^v f)_{v \in V}, \mathrm{supp}\, (P_{\mathcal{G}_v} \Gamma_1^v f)_{v \in V} \text{ finite} \right\}$$

where we used the *support* of a sequence $x = (x_i)_{i \in I} \in \mathbb{C}^I$ given by

$$\mathrm{supp}\, x := \{i \in I \mid x_i \neq 0\}.$$

For its proof we need a variant of the abstract Green identity (1.3.1).

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Lemma 1.4.1 *Let $f, g \in \operatorname{dom} S^*$ then*

$$(S^* f, g) - (f, S^* g) = \sum_{v \in V} (\Gamma_1^v f, \Gamma_0^v g) - (\Gamma_0^v f, \Gamma_1^v g). \qquad (1.4.5)$$

Furthermore, given $v \in V$, $y_0 \in \mathcal{G}_v$ and $y_1 \in \mathcal{G}_v^{\perp}$ there exists $g = (g_n)_{n \in \mathbb{N}} \in \operatorname{dom} S_L^{\min}$ with finite support such that the following equations hold

$$\begin{aligned}
\Gamma_0^v g &= y_0, \quad \Gamma_1^v g = y_1 + L_v y_0, \\
\Gamma_0^w g &= \Gamma_1^w g = 0, \quad \text{for all } w \in V \setminus \{v\}.
\end{aligned} \qquad (1.4.6)$$

Proof. First, we show that for all $f = (f_n)_{n \in \mathbb{N}}, g = (g_n)_{n \in \mathbb{N}} \in \operatorname{dom} S^*$, the sum $\sum_{n=0}^{\infty} (S_n^* f_n, g_n)$ converges absolutely. From Cauchy-Bunjakowski and Hölder inequality, we have

$$\sum_{n=0}^{\infty} |(S_n^* f_n, g_n)| \le \sum_{n=0}^{\infty} \|S_n^* f_n\| \|g_n\| \le \|S^* f\| \|g\| < \infty.$$

Next, using the abstract Green identity (1.3.1) for the operators S_n^* and changing the order of summation leads to

$$\begin{aligned}
(S^* f, g) - (f, S^* g) &= \sum_{n=0}^{\infty} (S_n^* f_n, g_n) - (f_n, S_n^* g_n) \\
&= \sum_{n=0}^{\infty} (\Gamma_1^{(n)} f_n, \Gamma_0^{(n)} g_n) - (\Gamma_0^{(n)} f_n, \Gamma_1^{(n)} g_n) \\
&= \sum_{v \in V} (\Gamma_1^v f, \Gamma_0^v g) - (\Gamma_0^v f, \Gamma_1^v g),
\end{aligned}$$

where the last equality follows from $\bigcup_{v \in V} I_v = I$.

For the proof of the second assertion we construct $g = (g_n)_{n \in \mathbb{N}} \in \operatorname{dom} S_L^{\min}$ satisfying the equations (1.4.6). Consider $n \in \mathbb{N}$ and the set I_v. Given that $(n, d) \notin I_v$ for all $d = 1, \ldots, d_n$ then we set $g_n := 0$. For $(n, d) \in I_v$, for some $d = 1, \ldots, d_n$, the surjectivity of $(\Gamma_0^{(n)}, \Gamma_1^{(n)})^{\top} : \operatorname{dom} S_n^* \to \mathcal{G}_n \times \mathcal{G}_n$ for all $n \in \mathbb{N}$ implies that we can choose g_n such that the first and second equation in (1.4.6) hold. From the construction we also have the lower system of equations in (1.4.6) hold. Q.E.D.

Next, we show that S_L^{loc} is the adjoint of $S_L^{\min}$.

Proposition 1.4.1 *We have $S_L^{\mathrm{loc}} = (S_L^{\min})^*$, in particular S_L^{loc} is closed.*

Proof. Let $f \in (S_L^{\min})^*$ then we have from (1.4.5) for all $g \in \operatorname{dom} S_L^{\min}$

$$0 = (S^* f, g) - (f, S^* g) = \sum_{v \in V} (\Gamma_1^v f, \Gamma_0^v g) - (\Gamma_0^v f, \Gamma_1^v g). \qquad (1.4.7)$$

For this equation we use (1.4.6) from Lemma 1.4.1 with $y_0 = 0$ and $y_1 \in \mathcal{G}_v^\perp$ which leads to $(\Gamma_0^v f, y_1) = 0$. Since y_1 was arbitrary, we conclude that $\Gamma_0^v f \in \mathcal{G}_v$ for all $v \in V$. Choose $g \in \text{dom } S_L^{\min}$ that solves (1.4.6) for $y_1 = 0$ and arbitrary $y_0 \in \mathcal{G}_v$. With (1.4.7) this leads to

$$0 = (\Gamma_1^v f, y_0) - (\Gamma_0^v f, L_v y_0) = (P_{\mathcal{G}_v} \Gamma_1^v f - L_v \Gamma_0^v f, y_0).$$

Since $y_0 \in \mathcal{G}_v$ was arbitrary, we see $P_{\mathcal{G}_v} \Gamma_1^v f = L_v \Gamma_0^v$ for all $v \in V$ this proves $f \in \text{dom } S_L^{\text{loc}}$.

Assume conversely that $f \in \text{dom } S_L^{\text{loc}}$. For all $g \in \text{dom } S_L^{\min}$ we have

$$\sum_{v \in V} (\Gamma_1^v f, \Gamma_0^v g) - (\Gamma_0^v f, \Gamma_1^v g) = \sum_{v \in V} (P_{\mathcal{G}_v} \Gamma_1^v f, \Gamma_0^v g) - (\Gamma_0^v f, P_{\mathcal{G}_v} \Gamma_1^v g)$$

$$= \sum_{v \in V} (L_v \Gamma_0^v f, \Gamma_0^v g) - (\Gamma_0^v f, L_v \Gamma_0^v g)$$

$$= 0$$

which implies with (1.4.5), $f \in \text{dom } (S_L^{\min})^*$. Q.E.D.

We prove the main theorem of this section that allows us to describe the extension S_L^{loc} with operators on $\ell^2(\widehat{V})$ for a countable index set $\widehat{V}$. For this we use the notation

$$C(\widehat{V}) := \{(f_v)_{v \in \widehat{V}} \in \ell^2(\widehat{V}) \mid \text{supp } f \text{ finite}\}.$$

Furthermore, for the subspaces $\mathcal{G}_v$ of $\mathbb{C}^{|I_v|}$ we use the canonical embedding

$$\iota_v : \mathcal{G}_v \to \oplus_{n \in \mathbb{N}} \mathbb{C}^{d_n}, \quad (x_{(n,d)})_{(n,d) \in I_v} \mapsto (y_{(n,d)})_{(n,d) \in I},$$

$$y_{(n,d)} := \begin{cases} x_{(n,d)}, & \text{if } (n,d) \in I_v, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\iota_v(\mathcal{G}_v)$ is a subspace of $\mathcal{G}$ and we have an orthogonal sum decomposition

$$\mathcal{G}_{\mathcal{V}} := \bigoplus_{v \in V} \iota_v \mathcal{G}_v. \tag{1.4.8}$$

In the following, we consider an orthogonal basis $\{b_w\}_{w \in \widehat{V}}$ of the subspace $\mathcal{G}_{\mathcal{V}}$, which has the property that each b_w is an element of an orthogonal basis for some $\mathcal{G}_v$ and $\widehat{V}$ is a countable set of indices. In the theorem below we will make use of the unitary operator $U : \mathcal{G}_{\mathcal{V}} \to \ell^2(\widehat{V})$ given by $b_w \mapsto \|b_w\| e_w$.

Theorem 1.4.2 *Let $\{S_n\}_{n \in \mathbb{N}}$ be a family of densely defined symmetric linear operators $S_n \in \mathcal{C}(\mathcal{H}_n)$ with boundary triplets $\{\mathcal{G}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for S_n^* and Weyl functions M_n and $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subseteq \bigcap_{n=0}^\infty \rho(S_{n0})$ for some $\varepsilon > 0$ and $\lambda_0 \in \mathbb{R}$. Consider S_L^{loc} with Hermitian matrices L_v, subspaces $\mathcal{G}_v$, $\mathcal{G}_{\mathcal{V}}$ given by (1.4.8) with orthogonal basis $\{b_w\}_{w \in \widehat{V}}$ and the operator $L = \oplus_{v \in V} L_v$ on $\mathcal{G}_{\mathcal{V}}$. Then the following holds.*

(a) *The operator $L_{\min}$ in $\ell^2(\widehat{V})$ with $\operatorname{dom} L_{\min} = C(\widehat{V})$ given as an infinite matrix operator,*

$$L_{\min} := \left(\frac{((L - \bigoplus_{n \in \mathbb{N}} M_n(\lambda_0))b_v, b_w)}{\|Rb_v\| \|Rb_w\|} \right)_{v,w \in \widehat{V}}, \quad R := \bigoplus_{n=0}^{\infty} \sqrt{\|M_n'(\lambda_0)\|} I_{\mathbb{C}^{d_n}},$$

with $\operatorname{dom} R := U^{-1}C(\widehat{V})$, satisfies $S_L^{\min} = S_{L_{\min}}$ and $S_L^{\mathrm{loc}} = S_{L_{\min}^*}$.

(b) *We have $n_\pm(S_L^{\min}) = n_\pm(L_{\min})$ and there is a bijective correspondence between the self-adjoint extensions of $L_{\min}$ and the self-adjoint extensions of $S_L^{\min}$.*

(c) *Assume that $\bigoplus_{n=0}^{\infty} S_{n0} = S_F \geq \gamma$ with $\gamma > 0$ and that $\widetilde{M}$ given by (1.4.3) satisfies $\widetilde{M}(\lambda) \rightrightarrows -\infty$ for $\lambda \to -\infty$. Let $\widetilde{L}$ be a self-adjoint extension of $L_{\min}$ which is semi-bounded from below then $S_{\widetilde{L}}$ is semi-bounded from below.*

Proof. For the proof of (a), we use the regularized boundary triplet $\{\mathcal{G}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ defined in (1.4.2) for $f = (f_n)_{n \in \mathbb{N}} \in \operatorname{dom} S^*$ as

$$\widetilde{\Gamma}_0 f = (\sqrt{\|M_n'(\lambda_0)\|}\Gamma_0^{(n)} f_n)_{n \in \mathbb{N}},$$
$$\widetilde{\Gamma}_1 f = (\|M_n'(\lambda_0)\|^{-1/2}(\Gamma_1^{(n)} - M_n(\lambda_0)\Gamma_0^{(n)}) f_n)_{n \in \mathbb{N}}.$$

Consider $M := \bigoplus_{n=0}^{\infty} M_n(\lambda_0)$ with $\operatorname{dom} M := U^{-1}C(\widehat{V})$ and let $\widetilde{L}_{\min}$ be given by

$$\operatorname{dom} \widetilde{L}_{\min} := RU^{-1}C(\widehat{V}), \quad \widetilde{L}_{\min} f := P_{\overline{\operatorname{ran} R}} R^{-1}(L - M)R^{-1} f.$$

We show that

$$\operatorname{dom} S_L^{\min} = \{f \in \operatorname{dom} S^* | \widetilde{L}_{\min} \widetilde{\Gamma}_0 f = P_{\overline{\operatorname{ran} R}} \widetilde{\Gamma}_1 f, \ \widetilde{\Gamma}_0 f \in RU^{-1}C(\widehat{V}), \ \operatorname{supp} (P_{\mathcal{G}_v} \Gamma_1^v)_{v \in V} \text{ finite}\}. \tag{1.4.9}$$

Let $f \in \operatorname{dom} S^*$ be in the set on the right hand side of (1.4.9). Obviously $\operatorname{supp} \Gamma_0 f$ is finite and rewriting the conditions on the right hand side of (1.4.9) we obtain

$$P_{\overline{\operatorname{ran} R}} R^{-1}(L - M)R^{-1} R(\Gamma_0^{(n)} f_n)_{n \in \mathbb{N}} = \widetilde{L}_{\min} \widetilde{\Gamma}_0 f$$
$$= P_{\overline{\operatorname{ran} R}} \widetilde{\Gamma}_1 f$$
$$= P_{\overline{\operatorname{ran} R}} (\|M_n(\lambda_0)\|^{-1/2}(\Gamma_1^{(n)} - M_n(\lambda_0)\Gamma_0^{(n)}) f_n)_{n \in \mathbb{N}}$$

and therefore

$$P_{\overline{\operatorname{ran} R}} R^{-1} L (\Gamma_0^{(n)} f_n)_{n \in \mathbb{N}} = P_{\overline{\operatorname{ran} R}} (\|M_n(\lambda_0)\|^{-1/2} \Gamma_1^{(n)} f_n)_{n \in \mathbb{N}}. \tag{1.4.10}$$

Note that $(\|M_n(\lambda_0)\|^{-1/2}\Gamma_1^{(n)}f_n)_{n\in\mathbb{N}} \in \mathcal{G}$, since $\widetilde{\Gamma}_1 f \in \mathcal{G}$ and $\operatorname{supp}\Gamma_0 f$ is finite. The definition of R implies that $\{Rb_w\}_{w\in\widehat{V}}$ is an orthogonal basis of $\operatorname{ran}R$. Furthermore, we have from (1.4.10) that for all $w \in \widehat{V}$

$$(P_{\overline{\operatorname{ran}R}}R^{-1}L(\Gamma_0^{(n)}f_n)_{n\in\mathbb{N}}, Rb_w) = (P_{\overline{\operatorname{ran}R}}(\|M_n(\lambda_0)\|^{-1/2}\Gamma_1^{(n)}f_n)_{n\in\mathbb{N}}, Rb_w)$$

which is equivalent to

$$((L_v\Gamma_0^v f)_{v\in V}, b_w) = (L(\Gamma_0^{(n)}f_n)_{n\in\mathbb{N}}, b_w) = ((\Gamma_1^{(n)}f_n)_{n\in\mathbb{N}}, b_w) = (P_{\mathcal{G}_v}(\Gamma_1^v f)_{v\in V}, b_w)$$

for all $w \in \widehat{V}$. Note that $(\Gamma_1^{(n)}f_n)_{n\in\mathbb{N}}$ and $L(\Gamma_0^{(n)}f_n)_{n\in\mathbb{N}}$ are in general not in $\mathcal{G}$ but the formal scalar product of these sequences with b_v exists, because the support of b_v is finite. Since for each $v \in V$ there exists a subset of $\{b_w\}_{w\in\widehat{V}}$ which is an orthogonal basis for $\mathcal{G}_v$, we see that

$$L_v\Gamma_0^v f = P_{\mathcal{G}_v}\Gamma_1^v f$$

for all $v \in V$ and all f in the set of the right hand side of (1.4.9). Moreover, $\widetilde{\Gamma}_0 f \in RU^{-1}C(\widehat{V})$, hence $\Gamma_0 f \in U^{-1}C(\widehat{V})$ and, by construction, $\Gamma_0^v f \in \mathcal{G}_v$ follows. Thus we have proven that $f \in \operatorname{dom}S_L^{\min}$.

Assume conversely that $f \in \operatorname{dom}S_L^{\min}$ then we have that for finitely many $v \in V$ that

$$L_v\Gamma_0^v f = P_{\mathcal{G}_v}\Gamma_1^v, \quad \Gamma_0^v f \in \mathcal{G}_v$$

and $\Gamma_0^v f = P_{\mathcal{G}_v}\Gamma_1^v f = 0$ otherwise. Obviously $\widetilde{\Gamma}_0 f \in RU^{-1}C(\widehat{V})$ and $\operatorname{supp}(P_{\mathcal{G}_v}\Gamma_1^v f)_{v\in V}$ is finite. Furthermore, it is also clear from the calculations in the first part of the proof, that for all $w \in \widehat{V}$

$$(P_{\overline{\operatorname{ran}R}}R^{-1}L(\Gamma_0^{(n)}f_n)_{n\in\mathbb{N}}, Rb_w) = (P_{\overline{\operatorname{ran}R}}(\|M_n(\lambda_0)\|^{-1/2}\Gamma_1^{(n)}f_n)_{n\in\mathbb{N}}, Rb_w)$$

holds. Since $\operatorname{span}\{Rb_w\}_{w\in\widehat{V}}$ is dense in $\overline{\operatorname{ran}R}$ we have

$$\widetilde{L}_{min}\widetilde{\Gamma}_0 f = P_{\overline{\operatorname{ran}R}}\widetilde{\Gamma}_1 f.$$

Thus the identity (1.4.9) holds.

We apply Lemma 1.3.1 to obtain a different representation of $S_{\widetilde{L}_{\min}}$ in terms of the boundary triplet $\{\hat{U}\mathcal{G}, \hat{U}\Gamma_0, \hat{U}\Gamma_1\}$ where $\hat{U} : \operatorname{ran}R \to \ell^2(\widehat{V})$ is given by $Rb_w \mapsto \|Rb_w\|e_w$. and with the operator $L_{\min} = \hat{U}\widetilde{L}_{\min}\hat{U}^*$ which is given by

$$\frac{(\widetilde{L}_{\min}Rb_v, Rb_w)}{\|Rb_v\|\|Rb_w\|} = \frac{(P_{\overline{\operatorname{ran}R}}R^{-1}(L-M)R^{-1}Rb_v, Rb_w)}{\|Rb_v\|\|Rb_w\|}$$

$$= \frac{(R^{-1}(L-M)b_v, P_{\overline{\operatorname{ran}R}}Rb_w)}{\|Rb_v\|\|Rb_w\|}$$

$$= \frac{((L - \bigoplus_{n\in\mathbb{N}} M_n(\lambda_0))b_v, b_w)}{\|Rb_v\|\|Rb_w\|} = (L_{\min})_{v,w}$$

The assertion (b) follows immediately from (a) and Proposition 1.3.1. An application of Proposition 1.3.2 (b) yields (c). Q.E.D.

Under the assumption that the direct sum triplet (1.4.1) is a boundary triplet for S^*, we have that $\bigoplus_{n=0}^{\infty} M_n(\lambda_0)$, R and R^{-1} are bounded and we obtain the following special case of Theorem 1.4.2. For quantum graphs with edge length bounded from below, this result was also obtained in [22].

Corollary 1.4.1 *Assume that the triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ given by (1.4.1) is a boundary triplet for S^*, then S_L^{loc} has the following properties:*

(a) *S_L^{loc} is self-adjoint.*

(b) *Assume that $S^*|_{\ker \Gamma_0} = S_F \geq \gamma$ with $\gamma > 0$ and that $\bigoplus_{n=0}^{\infty} M_n(\lambda) \rightrightarrows -\infty$, as $\lambda \to -\infty$, then S_L^{loc} is semi-bounded from below if and only if there exists $C > -\infty$ with $(L_v x, x) \geq C \|x\|^2$ for all $x \in \mathcal{G}_v$ and all $v \in V$.*

Proof. Since S_L^{loc} is closed, it remains to show by Theorem 1.4.2 that $L_{\min}$ is essentially self-adjoint. Every L_v is unitarily equivalent to a diagonal matrix and therefore the operator $\left(\frac{(Lb_v, b_w)}{\|Rb_v\| \|Rb_w\|} \right)_{v,w \in \widehat{V}}$ is unitarily equivalent to a densely defined multiplication operator on $\ell^2(\widehat{V})$, and hence essentially self-adjoint. Since $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet, [5, Theorem 2.12] implies that the operators R, R^{-1} and $\bigoplus_{n=0}^{\infty} M_n(\lambda_0)$ are bounded. Therefore $L_{\min}$ is just a bounded and symmetric perturbation of an essentially self-adjoint operator and hence essentially self-adjoint according to the Kato-Rellich theorem [15, Theorem V.4.4]. Assertion (b) is a consequence of the boundedness of R, R^{-1} and of $\bigoplus_{n=0}^{\infty} M_n(\lambda_0)$ and follows from Theorem 1.4.2 (a). Q.E.D.

Since (1.4.1) is in general not a boundary triplet, we use the results of [11, 17] to provide conditions on the self-adjointness of S_L^{loc} and the discreteness of the spectrum of all self-adjoint extensions in the theorem below. For this we associate with S_L^{loc} the formal discrete Laplacian D_L on the weighted space

$$\ell^2(\widehat{V}, m) := \left\{ (x_v)_{v \in \widehat{V}} \in \mathbb{C}^{\widehat{V}} \mid \sum_{v \in \widehat{V}} m(v) |x_v|^2 < \infty \right\}$$

with $m(v) := \|Rb_v\|^2$, where b_v is an element of an orthogonal basis of the subspace $\mathcal{G}_{\mathcal{V}}$ defined in (1.4.8) and the scalar product in $\ell^2(\widehat{V}, m)$ is given by

$$(x, y)_m := \sum_{v \in \widehat{V}} m(v) x_v \overline{y_v}.$$

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We define an operator D_L with domain $\operatorname{dom} D_L := C(\widehat{V})$ via

$$(D_L f)_v := \frac{1}{\|Rb_v\|^2}\left(\sum_{w\in\widehat{V}} b(v,w)(f_v - f_w) + c(v)f_v\right),$$

$$b(v,w) := \left(\left(\bigoplus_{n=0}^{\infty} M_n(\lambda_0) - L\right)b_v, b_w\right), \quad v\neq w, \quad b(v,v) := 0, \tag{1.4.11}$$

$$c(v) := \left(\left(L - \bigoplus_{n=0}^{\infty} M_n(\lambda_0)\right)b_v, b_v\right) - \sum_{w\in\widehat{V}} b(v,w).$$

The elements of $\{b_v\}_{v\in\widehat{V}}$ have finite support, and if b_{v_1} and b_{v_2} are elements of a basis for $\mathcal{G}_{w_1}$ and $\mathcal{G}_{w_2}$ with $w_1 \neq w_2$ then $\operatorname{supp} b_{v_1} \cap \operatorname{supp} b_{v_2} = \emptyset$. Also, the support of $\left(\bigoplus_{n=0}^{\infty} M_n(\lambda_0) - L\right)b_v$ (considered as a sequence) is finite. Hence, for fixed $w \in \widehat{V}$ we have $b(v,w) \neq 0$ for only finitely many $v \in \widehat{V}$. As in [10, 16] we consider the *weighted degree*

$$\operatorname{Deg} : \widehat{V} \to (0,\infty), \quad v \mapsto \frac{1}{\|Rb_v\|^2}\sum_{w\in\widehat{V}} b(v,w). \tag{1.4.12}$$

Theorem 1.4.3 *Consider the operator S_L^{loc} and the associated discrete Laplacian* (1.4.11). *Assume that $b(v,w) \geq 0$ holds for all $v, w \in \widehat{V}$. Then the following holds.*

(a) *The operator S_L^{loc} is self-adjoint if one of the following conditions holds.*

   (i) *Assume that $\inf_{v\in\widehat{V}} \frac{c(v)}{\|Rb_v\|^2} > -\infty$ and that for all sequences $\{v_n\}_{n\in\mathbb{N}}$ in $\widehat{V}$ with $b(v_n, v_{n+1}) > 0$ for all $n \in \mathbb{N}$ we have $\sum_{n=1}^{\infty} \|Rb_{v_n}\|^2 = \infty$.*

   (ii) *The weighted degree Deg is bounded.*

(b) *All self-adjoint extensions of $S_L^{\min}$ are in one-to-one correspondence with the self-adjoint extensions of D_L.*

(c) *All self-adjoint extensions $\widehat{S}$ of $S_L^{\min}$ satisfy $(\widehat{S} - \lambda)^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for some $\lambda \in \rho(\widehat{S})$ if the following conditions hold.*

   (i) *For all $v, w \in \widehat{V}$ there exists $k \in \mathbb{N}$ and $v_0, \ldots, v_k$ such that $v_0 = v$, $v_k = w$ and $b(v_i, v_{i+1}) > 0$ for all $i = 0, \ldots, k-1$.*

   (ii) *Let $\left(\bigoplus_{n\in\mathbb{N}} S_{n0} - \lambda\right)^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $\lambda \in \rho\left(\bigoplus_{n\in\mathbb{N}} S_{n0}\right)$.*

   (iii) *Let $\sum_{v,w\in V, b(v,w)\neq 0} b(v,w)^{-1} < \infty$, $\sum_{v\in\widehat{V}} \|Rb_v\|^2 < \infty$, $\inf_{v\in\widehat{V}} \frac{c(v)}{\|Rb_v\|^2} > -\infty$.*

(d) *Assume that $\bigoplus_{n\in\mathbb{N}} S_{n0} = S_F \geq \gamma$ with $\gamma > 0$ and that $\widetilde{M}(\lambda) \rightrightarrows -\infty$ for $\lambda \to -\infty$ and $\inf_{v\in\widehat{V}} \frac{c(v)}{\|Rb_v\|^2} > -\infty$ then all self-adjoint extensions of $S_L^{\min}$ are semi-bounded from below.*

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Proof. First, we prove the results for $c(v) \geq 0$. To prove (a), we use that by Proposition 1.4.1 the operator S_L^{loc} is closed. It remains by Theorem 1.4.2 (b) and Proposition 1.3.1 (a) to show that the operator given by

$$(L_{\min})_{v,w} := \frac{((L - \bigoplus_{n \in \mathbb{N}} M_n(\lambda_0))b_v, b_w)}{\|Rb_v\|\|Rb_w\|}$$

is essentially self-adjoint on $C(\widehat{V})$. A straight forward calculation shows that $L_{\min}$ is unitary equivalent via $U : \ell^2(\widehat{V}, m) \to \ell^2(\widehat{V})$, $(x_v)_{v \in \widehat{V}} \mapsto (\|Rb_v\|x_v)_{v \in \widehat{V}}$ to the operator D_L. The assumption in (i) on the sequences $(v_n)_{n \in \mathbb{N}}$ in V and the invariance $D_L C(\widehat{V}) \subseteq C(\widehat{V})$ allows us to apply [17, Theorem 6] which yields the essential self-adjointness of D_L on $C(\widehat{V})$. This shows the essential self-adjointness of $S_L^{\min} = S_{L_{\min}}$ by Proposition 1.3.1 (a). The assumption (ii) implies by [16, Theorem 11] that D_0 given by D_L with $c(v) = 0$ for all $v \in \widehat{V}$ is bounded. Therefore D_L on $C(\widehat{V})$ is the bounded and symmetric perturbation of the essentially self-adjoint multiplication operator $(x_v)_{v \in \widehat{V}} \mapsto (\frac{c(v)}{\|Rb_v\|^2}x_v)_{v \in \widehat{V}}$ on $C(\widehat{V})$ hence essentially self-adjoint because of the Kato-Rellich theorem [15, Theorem V.4.4]. The correspondence in (b) is a consequence of Theorem 1.4.2 (b).

The assertion (c) follows from [11, Theorem 5.1] applied to D_L which shows that all self-adjoint extensions of D_L have resolvents in $\mathfrak{S}_1(\ell^2(\widehat{V}, m))$. Note that the assumptions of this Theorem 5.1 are satisfied because of $\sum_{v \in \widehat{V}} m(v) = \sum_{v \in \widehat{V}} \|Rb_v\|^2 < \infty$ and (i) and (iii), see also [11, Example 4.6]. The assumption (ii) that $\left(\bigoplus_{n \in \mathbb{N}} S_{n0} - \lambda\right)^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $\lambda \in \rho\left(\bigoplus_{n \in \mathbb{N}} S_{n0}\right)$ together with Proposition 1.3.1 (e) imply that $(\widehat{S} - \lambda)^{-1} \in \mathfrak{S}_1(\mathcal{H})$. This proves (c).

Let $\widehat{S}_L$ be an extension of $S_L^{\min}$ and $\widehat{D}_L$ be an extension of D_L on $C(\widehat{V})$ with $\widehat{S}_L = S_{\widehat{D}_L}$. It was shown in [17, p. 206] that $\widehat{D}_L$ has the same action as D_L. For $f \in \operatorname{dom} \widehat{D}_L$ with $(f, f)_m = 1$ we see from $b(v, w) \geq 0$ that

$$(\widehat{D}_L f, f)_m = \sum_{v \in \widehat{V}} m(v)(\widehat{D}_L f)_v \overline{f_v}$$

$$= \frac{1}{2} \sum_{v,w \in \widehat{V}} b(v, w)|f_v - f_w|^2 + \sum_{v \in \widehat{V}} c(v)|f_v|^2$$

$$\geq \sum_{v \in \widehat{V}} c(v)|f_v|^2 \geq \inf_{v \in \widehat{V}} \frac{c(v)}{\|Rb_v^2\|}(f, f)_m = \inf_{v \in \widehat{V}} \frac{c(v)}{\|Rb_v\|^2}.$$

Proposition 1.3.2 (a) applied to the regularized boundary triplet $\{\mathcal{G}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ from Theorem 1.4.1 yields that $S_{\widehat{D}_L}$ is semi-bounded from below. Here we used that due to (1.4.4) we have $S^*|_{\ker \widetilde{\Gamma}_0} = S_F$.

Assume now that $\inf_{v \in \widehat{V}} \frac{c(v)}{\|Rb_v\|^2} > -\infty$ holds. Then the operator $\widehat{D}_L$ is the bounded perturbation of an operator $\widehat{D}_L^+$ where we replace $c(v)$ with its positive part $c(v)^+ :=$

$\max\{c(v), 0\}$. Therefore we can apply the previous arguments to $\widehat{D}_L^+$. By assumption, $\widehat{D}_L$ is a bounded perturbation of $\widehat{D}_L^+$ again the Kato-Rellich theorem shows that self-adjointness is preserved which proves (a) and (c). Furthermore, (d) follows from Proposition 1.3.2 (b). Q.E.D.

1.5 Gesztesy-Šeba realizations of Dirac operators on metric graphs

In this section, we define the Gestezy-Šeba realization of Dirac operators on a locally finite graphs given by a set of vertices V and a set of edges E. On each edge $e \in E$ with finte length $\ell(e)$ we consider the Dirac operator

$$D_e := \begin{pmatrix} c^2/2 & -ic\frac{d}{dx_e} \\ -ic\frac{d}{dx_e} & -c^2/2 \end{pmatrix}, \quad \operatorname{dom} D_e := H_0^1(0, \ell(e)) \otimes \mathbb{C}^2,$$

where c denotes the speed of light. It was shown in [5, Lemma 3.1] that a boundary triplet for D_e^* is given by

$$\mathcal{G}_e := \mathbb{C}^2, \quad \hat{\Gamma}_0^{(e)} \begin{pmatrix} \psi_{e,1} \\ \psi_{e,2} \end{pmatrix} := \begin{pmatrix} \psi_{e,1}(0+) \\ ic\psi_{e,2}(\ell(e)-) \end{pmatrix}, \quad \hat{\Gamma}_1^{(e)} \begin{pmatrix} \psi_{e,1} \\ \psi_{e,2} \end{pmatrix} := \begin{pmatrix} ic\psi_{e,2}(0+) \\ \psi_{e,1}(\ell(e)-) \end{pmatrix}$$

with the Weyl function for $\lambda \in \rho(D_e^*|_{\ker \hat{\Gamma}_0^{(e)}})$

$$\hat{M}_e(\lambda) := \frac{1}{\cos(\ell(e)k(\lambda))} \begin{pmatrix} ck_1(\lambda)\sin(\ell(e)k(\lambda)) & 1 \\ 1 & (ck_1(\lambda))^{-1}\sin(\ell(e)k(\lambda)) \end{pmatrix},$$

where we abbreviate

$$k(\lambda) := c^{-1}\sqrt{\lambda^2 - (c^2/2)^2}, \quad k_1(\lambda) := \frac{ck(\lambda)}{\lambda + c^2/2} = \sqrt{\frac{\lambda - c^2/2}{\lambda + c^2/2}}$$

with $\sqrt{\cdot}$ such that $k(x) > 0$ for $x > \frac{c^2}{2}$. Under the assumption that $\sup_{e \in E} \ell(e) < \infty$, it was shown in [5, Equation (3.56)] that for some $\varepsilon > 0$ we have $(\frac{c^2}{2} - \varepsilon, \frac{c^2}{2} + \varepsilon) \subseteq \bigcap_{e \in E} \rho(D_e^*|_{\ker \hat{\Gamma}_0^{(e)}})$ and

$$\hat{M}_e\left(\frac{c^2}{2}\right) = \begin{pmatrix} 0 & 1 \\ 1 & \ell(e) \end{pmatrix}, \quad \hat{M}_e'\left(\frac{c^2}{2}\right) = \begin{pmatrix} \ell(e) & \frac{\ell(e)^2}{2} \\ \frac{\ell(e)^2}{2} & \frac{\ell(e)}{c^2} + \frac{\ell(e)^3}{3} \end{pmatrix}. \tag{1.5.1}$$

To describe a point interaction on a graph, we consider the boundary triplet for D_e^* given by a unitary transformation

$$\begin{pmatrix} \Gamma_0^{(e)} \\ \Gamma_1^{(e)} \end{pmatrix} := \begin{bmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{bmatrix} \begin{pmatrix} \hat{\Gamma}_0^{(e)} \\ \hat{\Gamma}_1^{(e)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{e,1}(0+) \\ ic\psi_{e,2}(\ell(e)-) \\ ic\psi_{e,2}(0+) \\ \psi_{e,1}(\ell(e)-) \end{pmatrix}$$

with $W_{00}, W_{01}, W_{10}, W_{11} \in \mathbb{C}^{2\times 2}$ and therefore

$$\Gamma_0^{(e)} \begin{pmatrix} \psi_{e,1} \\ \psi_{e,2} \end{pmatrix} = \begin{pmatrix} \psi_{e,1}(0+) \\ i\psi_{e,1}(\ell(e)-) \end{pmatrix}, \quad \Gamma_1^{(e)} \begin{pmatrix} \psi_{e,1} \\ \psi_{e,2} \end{pmatrix} := \begin{pmatrix} ic\psi_{e,2}(0+) \\ c\psi_{e,2}(\ell(e)-) \end{pmatrix}.$$

It was shown in [6] that such a unitary transformation leads to a boundary triplet with the Weyl function given by

$$M_e(\lambda) = (W_{10} + W_{11}\hat{M}_e(\lambda))(W_{00} + W_{01}\hat{M}_e(\lambda))^{-1}$$

$$= \frac{ck_1(\lambda)}{\sin(\ell(e)k(\lambda))} \begin{pmatrix} \cos(\ell(e)k(\lambda)) & -i \\ i & -\cos(\ell(e)k(\lambda)) \end{pmatrix} \tag{1.5.2}$$

for all $\lambda \in \rho(D_e^*|_{\ker \hat{\Gamma}_0^{(e)}}) \cap \rho(D_e^*|_{\ker \Gamma_0^{(e)}})$.

Introduce the set I_v with $(e, 0) \in I_v$ if $e \in E$ and e has v as initial vertex and $(e, 1) \in I_v$ if $e \in E$ and e has v as terminal vertex. The vectors $b_v \in \mathcal{G}$ are given by

$$(b_v)_{(e,t)} := \begin{cases} 1, & \text{if } (e, 0) \in I_v, \\ i, & \text{if } (e, 1) \in I_v, \\ 0, & \text{if } (e, t) \notin I_v. \end{cases}$$

Let $(\alpha(v))_{v\in V}$ be a real sequence. The operator GS_α is given by

$$\mathrm{dom}\,\mathrm{GS}_\alpha := \left\{ (\psi_1, \psi_2)^\top \in \bigoplus_{e\in E} D_e^* \ : \ \psi_1 \in \mathcal{C}(G), \ ic \sum_{(e,t)\in I_v} \mathrm{sgn}\,(e,t)\psi_{e,2}(t\ell(e)) = \alpha(v)\psi_1(v), \ v \in V \right\},$$

where $\mathcal{C}(G)$ is the set of continuous functions on G viewed as a metric space and $\psi_1(v)$ is the value of ψ_1 at the vertex v. We follow here [5] and call this operator *Gestesy-Šeba realization.*

If $\sup_{e\in E} \ell(e) < \infty$, it can easily be seen from (1.5.2) that for some $\varepsilon > 0$ we have $(\frac{c^2}{2} - \varepsilon, \frac{c^2}{2} + \varepsilon) \subseteq \bigcap_{e\in E} \rho(D_e^*|\ker \Gamma_0^{(e)})$ and that

$$M_e\left(\frac{c^2}{2}\right) = \frac{1}{\ell(e)} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

We also see from (1.5.2) and (1.5.1) with $T := \left(W_{00} + W_{01}\hat{M}_e\left(\frac{c^2}{2}\right)\right)^{-1}$ that

$$M'_e\left(\frac{c^2}{2}\right) = W_{11}\hat{M}'_e\left(\frac{c^2}{2}\right)T - \left(W_{10} + W_{11}\hat{M}_e\left(\frac{c^2}{2}\right)\right)TW_{01}\hat{M}'_e\left(\frac{c^2}{2}\right)T$$

$$= \begin{pmatrix} \frac{1}{\ell(e)c^2} + \frac{\ell(e)}{3} & -\frac{i\ell(e)}{2} + \frac{i}{\ell(e)c^2} + \frac{i\ell(e)}{3} \\ \frac{i\ell(e)}{2} - \frac{i}{\ell(e)c^2} - \frac{i\ell(e)}{3} & \frac{1}{\ell(e)c^2} + \frac{\ell(e)}{3} \end{pmatrix}$$

and this implies

$$\left\| M'_e\left(\frac{c^2}{2}\right) \right\| \geq (1,0)M'_e\left(\frac{c^2}{2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\ell(e)c^2} + \frac{\ell(e)}{3} \geq \frac{1}{\ell(e)c^2}. \tag{1.5.3}$$

Furthermore, we define

$$\mathcal{G}_v := \operatorname{span}\{1_v\}, \quad 1_v := ((b_v)_{(e,t)})_{(e,t)\in I_v} \quad \text{and} \quad L_v 1_v := \frac{\alpha(v)}{\deg v} 1_v.$$

We have according to (1.4.11) for $v \neq w$

$$\begin{aligned}
b(v,w) &:= \left(\left(\bigoplus_{e\in E} M_e\left(\frac{c^2}{2}\right) - L\right)b_v, b_w\right) = \left(\bigoplus_{e\in E} M_e\left(\frac{c^2}{2}\right)b_v, b_w\right) \\
&= \begin{cases} \ell(e)^{-1} & \text{if } e = vw \in E, \\ 0 & \text{if } e = vw \notin E, \end{cases}
\end{aligned}$$

and we see for $v \in V$

$$\begin{aligned}
c(v) &:= \left(\left(L - \bigoplus_{e\in E} M_e\left(\frac{c^2}{2}\right)\right)b_v, b_v\right) - \sum_{w\in V, w\neq v}\left(\bigoplus_{e\in E} M_e\left(\frac{c^2}{2}\right)b_v, b_w\right) \\
&= (L_v b_v, b_v) = \alpha(v).
\end{aligned}$$

As an application of Theorem 1.4.3, we have the following result on the self-adjointness of the Gesztesy-Šeba realizations.

Proposition 1.5.1 *Consider a locally finite graph with set of vertices V and set of edges E and let $\{\alpha(v)\}_{v\in V}$ be a real-valued sequence. Then the operator GS_α is a locally finite extension of $\bigoplus_{e\in E} D_e$ and if $\sup_{e\in E} \ell(e) < \infty$ then GS_α is self-adjoint.*

Proof. We show that GS_α is a locally finite extension of $\bigoplus_{e\in E} D_e$. Let $I := E \times \{0,1\}$, then

$$\Gamma_0 \psi = (\Gamma_0^{(e,t)}(\psi_{e,1}, \psi_{e,2})_{e\in E})_{(e,t)\in I} = (i^t \psi_{e,1}(t\ell(e)))_{(e,t)\in I},$$
$$\Gamma_1 \psi = (\Gamma_1^{(e,t)}(\psi_{e,1}, \psi_{e,2})_{e\in E})_{(e,t)\in I} = (ci^{1-t}\psi_{e,1}(t\ell(e)))_{(e,t)\in I}$$

and therefore

$$\Gamma_0^v(\psi_{e,1}, \psi_{e,2})_{e\in E} := (i^t \psi_{e,1}(t\ell(e)))_{(e,t)\in I_v}, \quad \Gamma_1^v(\psi_{e,1}, \psi_{e,2})_{e\in E} := (ci^{1-t}\psi_{e,2}(t\ell(e)))_{(e,t)\in I_v}.$$

Since $\mathcal{G}_v = \mathrm{span}\,\{1_v\}$ for all $v \in V$, we see that $\psi_1 \in \mathcal{C}(G)$ is equivalent to the condition $\Gamma_0^v(\psi_{e,1}, \psi_{e,2})_{e\in E} \in \mathcal{G}_v$ for all $v \in V$. Moreover, it is easy to see that the sum condition in the definition of $\mathrm{dom}\,GS_\alpha$ is equivalent to

$$P_{\mathcal{G}_v} \Gamma_1^v(\psi_{e,1}, \psi_{e,2})_{e\in E} = \frac{1}{\|1_v\|^2}(\Gamma_1^v(\psi_{e,1}, \psi_{e,2})_{e\in E}, 1_v)1_v$$

$$= \frac{1}{\deg v} \sum_{(e,t)\in I_v} \overline{(b_v)_{(e,t)}}ci^{1-t}\psi_{e,2}(t\ell(e))1_v$$

$$= \frac{ic}{\deg v} \sum_{(e,t)\in I_v} \mathrm{sgn}\,(e,t)\psi_{e,2}(t\ell(e))1_v$$

$$= \frac{\alpha(v)}{\deg v}\psi_1(v)1_v = L_v \Gamma_0^v(\psi_{e,1}, \psi_{e,2})_{e\in E}.$$

Thus, we have seen that GS_α is a locally finite extension of $\bigoplus_{e\in E} D_e$.

For $\sup_{e\in E} \ell(e) < \infty$, the assumptions of Theorem 1.4.2 are fulfilled. To see that GS_α is self-adjoint, we apply Theorem 1.4.3 (a). The estimate (1.5.3) implies that the weighted degree (1.4.12) satisfies

$$\mathrm{Deg}(v) = \frac{\sum_{w\in V} b(v,w)}{\|Rb_v\|^2} = \frac{\sum_{w\in V} b(v,w)}{\sum_{e=vw} \|M_e'(\frac{c^2}{2})\|} \le \frac{\sum_{e=vw} \ell(e)^{-1}}{\sum_{e=vw} \frac{1}{c^2 \ell(e)}} = c^2 < \infty$$

for all $v \in V$, where the summation $\sum_{e=vw}$ is taken over all edges e that contain v as a vertex. Hence, according to Theorem 1.4.3, GS_α is self-adjoint. Q.E.D.

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