Nonlinear Interactions of Gravitational Waves

DISSERTATION

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Abstract

This work presents the foundations of a new solution technique for the characteristic initial value problem of colliding plane gravitational waves. It has extensive similarities to the approach of Alekseev and Griffiths in 2001, but uses the inverse scattering method with a Riemann-Hilbert problem. This allows for a further transformation to a continuous Riemann-Hilbert problem with a solution given in terms of an integral equation for a regular unknown function. Ambiguities in the solution of the initial Riemann-Hilbert problem lead to the construction of a whole family of exact spacetimes generalising the proper solution of the initial value problem. Therefore the described technique also serves as an interesting solution generating method. The procedure is exemplified by extending the Szekeres class of colliding wave spacetimes with two additional real parameters. The obtained solution features a limiting case of a new type of impulsive waves, which are circularly polarised. A semi-analytic approximation scheme for the solution to the general initial value problem of colliding plane waves is introduced.

Kurzfassung

Diese Arbeit präsentiert die Grundlagen einer neuen Lösungstechnik für das charakteristische Anfangswertproblem kollidierender ebener Gravitationswellen. Sie weist deutliche Ähnlichkeiten zu einem 2001 von Alekseev und Griffiths beschriebenen Verfahren auf, verwendet jedoch die inverse Streumethode mit einem Riemann-Hilbert-Problem. Dies erlaubt eine weitere Transformation zu einem stetigen Riemann-Hilbert-Problem, dessen Lösung in Form einer Integralgleichung für eine reguläre unbekannte Funktion gegeben ist. Mehrdeutigkeiten in der Lösung des anfänglichen Riemann-Hilbert-Problems führen zur Konstruktion einer ganzen Familie exakter Raumzeiten, welche die eigentliche Lösung des Anfangswertproblems verallgemeinert. Deshalb kann die vorgestellte Technik auch als eine interessante Methode zur Lösungserzeugung dienen. Das Verfahren wird anhand der Erweiterung der Szekeres-Klasse von Wellenkollisionsraumzeiten demonstriert. Die dadurch erhaltene Lösung beinhaltet als Grenzfall einen neuen Typ impulsiver Gravitationswellen mit zirkularer Polarisation. Ein halbanalytisches Approximationsschema für die Lösung des charakteristischen Anfangswertproblems kollidierender ebener Gravitationswellen wird vorgestellt.

Contents

1 Introduction

Wege entstehen dadurch, dass man sie geht.

Franz Kafka (1883 – 1924)

History and character of gravitational waves

In 1915 Albert Einstein's general theory of relativity (GR) replaced Newton's concept of gravity as an instantaneous force by a dynamical description of the four-dimensional spacetime [1]. This enables gravitation to be radiated in the form of distortions in the fabric of spacetime, as Einstein discovered shortly afterwards using a weak field approximation [2]. In his article he also presented the famous quadrupole formula describing the energy loss of a system caused by gravitational waves (GWs) to be proportional to the second time derivative of its quadrupole moment. Ironically, he stated that the value of this energy loss is practically zero in all conceivable cases. He has been proven wrong in the second half of the last century not simply due to his underestimation of technological advance, but also because he could not think of such compact and fast orbiting objects like binary neutron stars.

However, questions about the properties or even the existence of GWs were far from being settled at that time. After discussions with Willem de Sitter and Gunnar Nordström, Einstein had to correct his GW calculations in 1918 [3]. He derived three different types of GWs that were subject to further investigation until Arthur Eddington in 1922 showed that only the purely transversal GWs travel with the speed of light, whereas he proved the other two types to be mere artefacts of choosing an inapropriate coordinate system [4]. Some time later, in 1936, Einstein himself together with his student Nathan Rosen temporarily came to the conclusion that GWs do not exist. Einstein altered his opinion after an error was pointed out to him. Subsequently, the question arose if GWs can indeed carry energy, which was only resolved by Richard Feynmans 'sticky bead' argument: A GW causes test bodies to slide along a rigid mount and finally heat due to friction, thus energy from the GW must have been deposited into the system. Ultimately, Peter Bergmann described the transverse and quadrupolar nature of gravitational radiation by discussing the motion of test bodies [5], cf. figure 1.1.

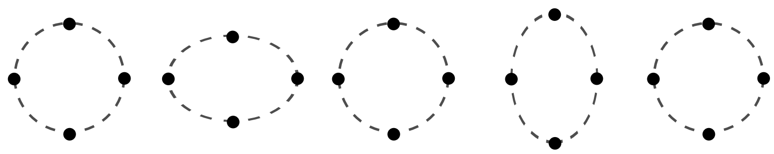

Figure 1.1: Sketch of one period of oscillation in a system of four test masses caused by a GW travelling orthogonal to their plane.

In the 1960s the first GW detectors were proposed. Joseph Weber built resonance detectors consisting of heavy aluminium cylinders and claimed to have measured several GW signals, but his findings could not be reproduced and were seriously questioned by theoretical arguments. Joseph Taylor and Alan Hulse discovered a binary pulsar in 1974 that featured an energy loss matching the quadrupole formula with astonishing precision [6], which is regarded as a first indirect measurement of GWs. Recently, after a long and labourious quest, *direct* observations of GWs emitted by black hole mergers were finally achieved by the interferometers of the LIGO cooperation [7], which simultaneously confirm the existence of GWs and of black hole binaries with around 30 solar masses. This discovery marks the beginning of a new era of GW astronomy, wherein the electromagnetic picture of our universe will be complemented by GWs observations.

Colliding plane gravitational waves

For the discussion of GW observations a linearised version of GR is sufficient. However, these waves are generated in regions with extraordinary strong gravity where the full nonlinear theory has to be applied. In this regime the interacting GWs behave very differently from the better known case of classical (vacuum) electromagnetic waves, whose undisturbed penetration through each other is a part of everyday experience. For example, interacting GWs tend to focus each other astigmatically.

This work aims to develop a deeper understanding of the nonlinear effects within the interaction of GWs. The detection of such effects seems to be completely out of reach of current instrumentation, but this is precisely the same situation as for Einstein's first considerations of GWs. A first step in this venture is surely the model of two colliding plane GWs, which is the simplest setting to study nonlinear wave interactions analytically. Therefore, many features of nonlinearity as well as conceptual issues like focussing properties and arising singularities have so far been discussed on the basis of colliding plane waves [8–12]. A lot of exact solutions have been described along with generating techniques constructing solutions in the interaction region and deriving the shape of the incoming waves afterwards (cf. the overview of Jerry Griffiths [9] or [13]). Isidore Hauser and Frederick Ernst pioneered the search for a method to address the characteristic initial value problem (IVP) of colliding plane GWs [14–17], which is the more natural task to calculate the interaction of arbitrarily prescribed initial waves. They also proved the existence and uniqueness of its solution [18]. George Alekseev and Griffiths [19, 20] described a more practical procedure for both colliding gravitational and electromagnetic waves leading to integral equations for singular unknown functions. The present work introduces a treatment of the characteristic IVP featuring many similarities to this approach, but allowing for an additional transformation to integral equations for a regular unknown function. It is expected to be better suited for approximations using spectral methods, but it is still too early to clearly compare the performance of the two approaches related to this goal.

Analogies to axially symmetric and stationary spacetimes

The major motivation for this work's new perspective on colliding plane GWs stems from a strong formal analogy to axially symmetric and stationary spacetimes (ASSS). This provides the opportunity to benefit from the highly evolved mathematical methods for ASSS in another, less developed field of research.

ASSS are defined by the existence of one spacelike and one timelike Killing vector field which commute, whereas a colliding plane GWs spacetime features two commuting spacelike Killing vector fields. In both cases the gravitational field equations essentially reduce to an Ernst equation, which has elliptic character for ASSS and hyperbolic character in case of colliding plane GWs. The Ernst equation is a nonlinear partial differential equation that fulfils the requirements to be treated with the so-called 'inverse scattering method' (ISM)[1], cf. [21] for a general introduction and [22] for ASSS.

In the course of the ISM, the analogies between ASSS and colliding plane GWs remain but steadily decrease. Firstly, for ASSS the boundary value problem of the elliptic Ernst equation can be reformulated as a 'linear problem' (LP) in the Neugebauer form containing a spectral parameter λ subtly depending on an actually independent spectral parameter k, cf. [22]. A formally analogue LP will be used in this work with some adoptions for the IVP of colliding plane GWs. The ISM proceeds by expressing

[1] The name stems from the application to the Korteweg-de Vries equation, where it implies the task of finding a potential fulfilling the Schroedinger equation for given 'scattering data'.

the LP solution as a solution of a Riemann-Hilbert problem (RHP), which is still similar in both cases, but at this level the first considerable differences occur. Most significant is the incidence of discontinuities in the jump matrix of the RHP for colliding plane GWs, which were not present in the case of ASSS. Finally, the solution of the RHP is written in terms of integral equations and a solution of the Ernst equation is inferred.

Dual objectives of the inverse scattering method

In chapter 2 the characteristic IVP of colliding plane GWs is introduced. This work is devoted to applying the ISM on that IVP, which is the immediate purpose of the chapters 3 and 4. Adapting a general method of Nikolaĭ Vekua [23], the ISM will be extended by a further transformation from the discontinuous RHP to a continuous RHP (CRHP) developed in chapter 5. In doing so, two principally different objectives arise that will be equivalently pursued: The search for analytic solutions for specific colliding wave spacetimes and an approximation scheme for the general characteristic IVP.

Due to ambiguities in the RHP caused by its discontinuities, the ISM will indeed turn out to serve as a tool to generalise existing exact colliding wave spacetimes, which will be clarified in chapter 6 and exemplified in chapter 7.

Finally a semi-analytic spectral expansion procedure to the integral equations of the CRHP is designed in chapter 8 in order to solve the general characteristic IVP of colliding plane GWs for arbitrary initial waves. Regarding this task, the ISM approach is complementary to the more common finite differencing schemes (cf. e.g. [24]) in the sense that the solution at a specified point can be calculated with high accuracy independent of its environment, especially without accumulating errors.

This work is based on the publication [25] written by the author under supervision of Reinhard Meinel.

2 The characteristic initial value problem for colliding plane waves

This work considers purely gravitational plane waves with distinct wavefronts and arbitrary polarisation colliding in a Minkowski background within the framework of standard GR.

For an introduction as well as for clarity of sign conventions, the foundations of Einstein's theory shall be stated. The focus will be laid on the Newman-Penrose components of the Weyl tensor describing GWs. Furthermore, the Szekeres line element describing colliding GWs is introduced and its interpretation for single waves is discussed, whereby a convenient formula for the 'wave profile' $\hat{\Psi}_4$ in Brinkmann coordinates is derived. The chapter is concluded by considering the partitioning of a colliding wave spacetime and the formulation of the IVP, where 'colliding wave conditions' have to be fulfilled by the initial values.

2.1 General relativity and gravitational waves

The Riemann curvature tensor is defined in terms of the Christoffel symbols Γ^a_{bc} as (confer e.g. [26])

$$R^a{}_{bcd} := \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \tag{2.1}$$

The purely covariant curvature tensor R_{abcd} possesses 20 degrees of freedom. Ten of them can be recovered in the Ricci tensor $R_{ab} := R^c{}_{acb}$, which is directly linked to the stress energy tensor T_{ab} describing the matter content of spacetime via the Einstein equations

$$G_{ab} := R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab}. \tag{2.2}$$

Therein g_{ab} is the metric tensor, $R := R^a_a$ is the Ricci scalar and G_{ab} is called Einstein tensor. Since this work considers only purely gravitational waves, (2.2) is reduced to its vacuum version with $T_{ab} = 0$. Hence the GWs must be described by the remaining

10 degrees of freedom from the curvature tensor, which are contained in its trace-free part, the Weyl tensor

$$C_{abcd} = R_{abcd} + g_{b[c}R_{d]a} - g_{a[c}R_{d]b} + \tfrac{1}{3}Rg_{a[c}g_{d]b}. \tag{2.3}$$

Herein the square brackets denote the antisymmetric part. The Weyl tensor can be nicely reexpressed by scalars in the framework of the Newman-Penrose formalism [27] (a good introduction can be found in [28], another one using spinors[1] in [10]). It operates with a so-called 'null tetrad' consisting of two real null vectors l^a, n^a and one spacelike complex null vector m^a as well as its conjugate. They are arranged in a way that all their scalar products vanish with the exception of

$$l^a n_a = 1, \qquad m^a \bar{m}_a = -1, \tag{2.4}$$

where $\bar{m}$ denotes the complex conjugate of m. In the Newman-Penrose formalism complete tensorial equations are decomposed with respect to this null tetrad, the details can be studied for instance in the books of Griffiths [9] or Sibgatullin [11]. Thereby the 10 degrees of freedom of the Weyl tensor are converted into the 5 complex scalars

$$\begin{aligned} &\Psi_0 := -C_{abcd}l^a m^b l^c m^d, \qquad \Psi_1 := -C_{abcd}l^a n^b l^c m^d, \qquad \Psi_2 := -C_{abcd}l^a m^b \bar{m}^c n^d, \\ &\Psi_3 := -C_{abcd}n^a l^b n^c \bar{m}^d, \qquad \Psi_4 := -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d. \end{aligned} \tag{2.5}$$

These quantities may be called the Newman-Penrose components of the Weyl tensor, but for brevity this work will refer to them as 'Weyl tensor components'. They are commonly employed in the relevant literature, since they can be physically interpreted as [9]

Ψ_0 : transverse wave component in the direction of n^a,

Ψ_1 : longitudinal wave component in the direction of n^a,

Ψ_2 : 'Coulomb component',

Ψ_3 : longitudinal wave component in the direction of l^a,

Ψ_4 : transverse wave component in the direction of l^a.

Another useful notion for the discussion of GWs is the Petrov classification, which is based on the concept of so-called 'repeated principal null directions'. The most com-

[1] Spinors are elements of an underlying complex two dimensional symplectic vector space, which is considered to be more fundamental than the tangent space over a Lorentzian manifold.

prehensive approach to them is via spinors [10, 29], where the principal null directions are simply four vectors associated with the unique spinor representation of the Weyl tensor. Spinors will not be treated in detail in this work and so for completeness the quite inconvenient definition of repeated principal null directions from [9] is given:

A null vector k^a is a principal null direction of multiplicity 1, 2, 3 or 4 respectively, if it obeys the equation

$$k_{[a}C_{b]cd[e}k_{f]}k^c k^d = 0, \quad C_{bcd[e}k_{f]}k^c k^d = 0, \quad C_{bcd[e}k_{f]}k^d = 0 \quad \text{or} \quad C_{bcde}k^d = 0. \tag{2.6}$$

A spacetime with four different principal null vectors is said to be algebraically general or of Petov type I. Other spacetimes are algebraically special and called Petrov type II, III or N respectively, if they feature a principal null direction with multiplicity 2, 3 or 4. A spacetime with two different repeated principal null directions with multiplicity 2 is denoted as Petrov type D. Finally, a conformally flat spacetime with vanishing Weyl tensor is said to be of Petrov type 0.

If the tetrad vector l^a is aligned with a principal null direction with multiplicity k, then the first k Newman-Penrose components of the Weyl tensor vanish. The same holds for an alignment of n^a with a principal null direction with multiplicity k for the last k components of the Weyl tensor [9].

Another interesting feature of the Weyl tensor components Ψ_i is that under reasonable assumptions in the far field of an otherwise arbitrary wave travelling in the direction l^a they fall off in the sequence of their indices [27]. This is referred to as 'peeling theorem', indicating that the more complicated components peel off one after another until only a plane wave is left described solely by Ψ_4. Only this Weyl tensor component is present in the linearised theory and causes on test bodies the effects depicted in figure 1.1 and measured by GW interferometers.

Finally, the only two second order scalar invariants of the Weyl tensor are [11]

$$\mathcal{I}_1 := C^{abcd}C_{abcd}, \qquad \mathcal{I}_2 := \tfrac{1}{2}\epsilon_{abcd}C^{abef}C^{cd}_{ef} \tag{2.7}$$

where ϵ_{abcd} is the totally antisymmetric Levi-Civita pseudotensor[2]. In vacuum, the Riemann tensor and the Weyl tensor are identical and hence $\mathcal{I}_1$ coincides with the Kretschmann scalar $R^{abcd}R_{abcd}$ as well as $\mathcal{I}_2$ with the Chern–Pontryagin scalar $\frac{1}{2}\epsilon_{abcd}R^{abef}R^{cd}_{ef}$. A complex combination of these scalars is easily expressed in terms of the Weyl tensor

[2]The reader may be reminded of $\epsilon_{abcd} = \sqrt{-\det(g_{ab})}e_{abcd}$, $e_{abcd} = \pm 1$ if (a, b, c, d) is an even/odd permutation of $(1, 2, 3, 4)$ and $e_{abcd} = 0$ else.

components via [30]

$$\mathcal{I} := \mathcal{I}_1 - \mathrm{i}\mathcal{I}_2 = 16(3\Psi_2^2 + \Psi_0\Psi_4 - 4\Psi_1\Psi_3). \tag{2.8}$$

2.2 The Szekeres metric for colliding plane waves

A general colliding plane wave spacetime features two commuting spacelike Killing vectors ∂_x and ∂_y [21]. The associated coordinates x and y parametrise the planes of symmetry and are complemented by two lightlike coordinates u and v. The derivation of Griffiths [8, 9] using the Newman-Penrose formalism shows that for the case of colliding plane waves the metric can be assumed to be orthogonal transitive, i.e. the metric tensor g_{ab} can be assumed to have blockdiagonal shape. Like all two-dimensional metrics the (u, v)-block of g_{ab} is conformally flat, which in the coordinates (u, v, x, y) leads to the Szekeres line element [31]

$$\mathrm{d}s^2 = 2\mathrm{e}^{-M}\mathrm{d}u\mathrm{d}v - \mathrm{e}^{-U}\left(\mathrm{e}^{V}\cosh W\mathrm{d}x^2 - 2\sinh W\mathrm{d}x\mathrm{d}y + \mathrm{e}^{-V}\cosh W\mathrm{d}y^2\right). \tag{2.9}$$

It contains the four real functions $M(u, v)$, $U(u, v)$, $V(u, v)$ and $W(u, v)$ only depending on u and v. The ISM will make extensive use of the complex Ernst potential[3]

$$E(u, v) := \mathrm{e}^{-V}\left(\operatorname{sech} W + \mathrm{i}\tanh W\right), \tag{2.10}$$

therefore the line element is rewritten depending directly on the Ernst potential as

$$\mathrm{d}s^2 = 2\mathrm{e}^{-M}\mathrm{d}u\mathrm{d}v - \frac{\mathrm{e}^{-U}}{\Re(E)}|\mathrm{d}x + \mathrm{i}E\mathrm{d}y|^2. \tag{2.11}$$

Herein $\Re(E)$ denotes the real part of E; $\Im(E)$ will later be used for the imaginary part. In the Szekeres coordinates (u, v, x, y) the tetrad is shown by [9] to be representable in the form

$$\begin{aligned} l_a &= \mathrm{e}^{-\frac{1}{2}M}\delta^u_a, & n_a &= \mathrm{e}^{-\frac{1}{2}M}\delta^v_a, \\ l^a &= \mathrm{e}^{\frac{1}{2}M}\delta^a_v, & n^a &= \mathrm{e}^{\frac{1}{2}M}\delta^a_u, \end{aligned} \qquad m^a = (0, 0, \xi^3, \xi^4) \tag{2.12}$$

where δ^a_b is the Kronecker symbol and the components of m^a can be chosen as

$$\xi^3 = \tfrac{1}{\sqrt{2}}\mathrm{e}^{\frac{1}{2}(U-V)}\left(\cosh\tfrac{W}{2} + \mathrm{i}\sinh\tfrac{W}{2}\right), \qquad \xi^4 = \tfrac{1}{\sqrt{2}}\mathrm{e}^{\frac{1}{2}(U+V)}\left(\sinh\tfrac{W}{2} + \mathrm{i}\cosh\tfrac{W}{2}\right).$$

[3] The reader may be reminded of $\operatorname{sech} x = \cosh^{-1} x$.

An alternative but less convenient representation of the components of m^a in terms of the Ernst potential is given by

$$\xi^3 = \tfrac{1}{\sqrt{2}} \mathrm{e}^{\frac{1}{2}U} \sqrt{|E| E \left[\Re(E)\right]^{-1}}, \qquad \xi^4 = \mathrm{i}E^{-1}\xi^3. \tag{2.13}$$

The ambiguity introduced by the square root in (2.13) does not affect the Newman-Penrose components of the Weyl tensor Ψ_i since they are quadratic in m^a and $\bar{m}^a$. They evaluate[4] to [9]

$$\Psi_0 = \frac{\mathrm{e}^M |E|}{2[\Re(E)]^2 E} \left[(E_{vv} - U_v E_v + M_v E_v)\Re(E) - E_v^2\right], \tag{2.14}$$

$$\Psi_2 = \frac{\mathrm{e}^M}{4} \left[\frac{\bar{E}_u E_v}{[\Re(E)]^2} - U_u U_v\right], \tag{2.15}$$

$$\bar{\Psi}_4 = \frac{\mathrm{e}^M |E|}{2[\Re(E)]^2 E} \left[(E_{uu} - U_u E_u + M_u E_u)\Re(E) - E_u^2\right], \tag{2.16}$$

whereby coordinates (in this case u and v) deployed as lower indices shall denote partial derivatives throughout this work[5]. As indicated by equations (2.14)-(2.16), the interaction region is usually algebraically general (Petrov type I) [9], but in some special cases it is of Petrov type D, e.g. in the solution of Subrahmanyan Chandrasekhar and Basilis Xanthopoulos [32].

2.3 Single wave metrics

A (part of a) spacetime with only a single wave present can be described by a line element of the form (2.9) or respectively (2.11), where all metric functions depend only on one of the lightlike coordinates u or v. In section 2.6 the incident waves will be treated with this approach. Using the coordinate transformation

$$u(u') = \int_0^{u'} \mathrm{e}^{M(\tilde{u})} \mathrm{d}\tilde{u}, \qquad \mathrm{d}u = \mathrm{e}^M \mathrm{d}u' \tag{2.17}$$

[4] In odrer to obtain the presented form of (2.15) it is necessary to look ahead to the field equation $U_{uv} = U_u U_v$ in (2.30).

[5] The single exception are Kronecker symbols like δ^a_u.

such a single wave line element is converted to the Rosen form[6]

$$\mathrm{d}s^2 = 2\mathrm{d}u'\mathrm{d}v - \mathrm{e}^{-U(u')}\Big(\mathrm{e}^{V(u')}\cosh W(u')\mathrm{d}x^2 - 2\sinh W(u')\mathrm{d}x\mathrm{d}y + \mathrm{e}^{-V(u')}\cosh W(u')\mathrm{d}y^2\Big). \quad (2.18)$$

Hermann Bondi, Felix Pirani and Ivor Robinson discussed that the metric (2.18) indeed contains a single plane wave [34]. In particular they demonstrated that this metric is invariant under the same 5-parameter group of motions as an electromagnetic plane wave travelling in the same direction. Thus a single GW propagates undisturbedly with the speed of light just like an electromagnetic wave.

A compellingly simple way to express a single plane wave is the Brinkmann form (given here in the parametrisation of [9] but going back to considerations of H. W. Brinkmann [35]) of the line element[7],

$$\mathrm{d}s^2 = \Big[(X^2 - Y^2)h_{11}(u') + 2XYh_{12}(u')\Big]\,\mathrm{d}u'^2 + 2\mathrm{d}u'\mathrm{d}r - \mathrm{d}X^2 - \mathrm{d}Y^2. \quad (2.19)$$

Employing a tetrad adapted to the Brinkmann coordinates (u', r, X, Y),

$$\hat{l}_a = \delta_a^{u'}, \quad \hat{n}_a = \Big([\tfrac{1}{2}(X^2 - Y^2)h_{11} + XYh_{12}], 1, 0, 0\Big), \quad \hat{m}_a = (0, 0, \tfrac{1}{\sqrt{2}}, -\tfrac{\mathrm{i}}{\sqrt{2}}), \quad (2.20)$$

the adapted Weyl tensor components are given by

$$\hat{\Psi}_0 = \hat{\Psi}_1 = \hat{\Psi}_2 = \hat{\Psi}_3 = 0, \qquad \hat{\Psi}_4 = (h_{11} + \mathrm{i}h_{12}). \quad (2.21)$$

The component $\hat{\Psi}_4$ is directly linked to the non-trivial part of the line element (2.19) and therefore usually referred to as 'the wave profile'. The Brinkmann metric is equipped with several interesting features:

- For impulsive waves the distribution valued part of $\hat{\Psi}_4$ directly enters the Brinkmann form of the metric, which nevertheless gives rise to a reasonable curvature tensor.
- Since the first four Weyl tensor components vanish, a spacetime containing only a single plane wave is always of Petrov type N.
- The vacuum Einstein equations are automatically fulfilled for an arbitrary wave

[6] The diagonal case of this form of the metric was first considered in an article by Albert Einstein and Nathan Rosen [33].

[7] In the form (2.19) the metric is already specified to purely gravitational waves where the coefficients in front of X^2 and Y^2 in g_{11} sum up to 0.

profile $\hat{\Psi}_4$. Therefore two GWs travelling in exactly the same direction can be simply superposed by adding the wave profiles.

2.4 Transformation from Rosen to Brinkmann metric

Implementing a recipe from the appendix of [36] (alternative descriptions can be found in [9] and [37]), a transition from the Rosen metric $g^{(\mathrm{R})}_{ab}$ to the Brinkmann form can be achieved via the transformation

$$v = r + \tfrac{1}{2}\, g^{(\mathrm{R})}_{AB}\, (Q^A_C)_{u'}\, Q^B_D\, X^C\, X^D, \qquad x^A = Q^A_B\, X^B \tag{2.22}$$

with $A, B, C, D \in \{3, 4\}$ as well as $x^3 := x$, $x^4 := y$, $X^3 := X$ and $X^4 := Y$. The matrix Q^A_B has to fulfil the conditions

$$g^{(\mathrm{R})}_{AB}\, Q^A_C\, Q^B_D = \delta_{AB} \tag{2.23}$$

$$g^{(\mathrm{R})}_{AB}\, (Q^A_C)_{u'}\, Q^B_D = g^{(\mathrm{R})}_{AB}\, Q^A_C\, (Q^B_D)_{u'} \tag{2.24}$$

which is achieved by

$$\begin{aligned} Q^3_3 &= \mathrm{e}^{\frac{1}{2}(U-V)}\sqrt{\cosh W}\cos P, & Q^4_3 &= \mathrm{e}^{\frac{1}{2}(U+V)}\sqrt{\operatorname{sech} W}(\cos P \sinh W - \sin P),\\ Q^3_4 &= -\mathrm{e}^{\frac{1}{2}(U-V)}\sqrt{\cosh W}\sin P, & Q^4_4 &= -\mathrm{e}^{\frac{1}{2}(U+V)}\sqrt{\operatorname{sech} W}(\cos P + \sin P \sinh W). \end{aligned} \tag{2.25}$$

This solution contains the function

$$\begin{aligned} P(u') &:= \frac{1}{2}\int_0^{u'} V_{\tilde{u}}(\tilde{u}) \sinh W(\tilde{u}) + W_{\tilde{u}}(\tilde{u}) \operatorname{sech} W(\tilde{u})\, \mathrm{d}\tilde{u}\\ &= \frac{1}{2}\int_0^{u'} \frac{|E(\tilde{u})|^2}{\Re(E(\tilde{u}))} \left(\frac{\Im(E(\tilde{u}))}{|E(\tilde{u})|^2}\right)_{\tilde{u}} \mathrm{d}\tilde{u} \end{aligned} \tag{2.26}$$

defined by (2.23) and (2.24) only up to an additive constant. According to [36,38], Q^A_B is unique up to u'-independent orthogonal transformations, which is not obvious from the above formula. Via the transformation (2.22), h_{11} and h_{12} can be represented in terms of the metric functions of the Rosen line element:

$$\begin{aligned} h_{11} &= h_1 h_3 - h_2 h_4, & h_{12} &= h_1 h_4 + h_2 h_3,\\ h_1 &:= \tfrac{1}{2}\left[\sin(2P)\sinh W + \cos(2P)\right], & h_2 &:= \tfrac{1}{2}\left[\cos(2P)\sinh W - \sin(2P)\right],\\ h_3 &:= U_{u'}V_{u'} - 2\tanh W V_{u'}W_{u'} - V_{u'u'}, & h_4 &:= \sinh W V_{u'}^2 - \operatorname{sech} W(V_{u'u'} - U_{u'}W_{u'}). \end{aligned} \tag{2.27}$$

Finally, the adapted Weyl tensor component related to the Brinkmann tetrad can be expressed in terms of the Weyl tensor component related to the Rosen tetrad:

$$\hat{\Psi}_4 = \mathrm{e}^{M-2\mathrm{i}P}(\operatorname{sech} W + \mathrm{i}\tanh W)\Psi_4 = \mathrm{e}^{M-2\mathrm{i}P}\operatorname{sign}(E)\Psi_4 \tag{2.28}$$

Therein $\operatorname{sign}(E) := E/|E| = \mathrm{e}^{i\arg(E)}$ has to be understood as the complex generalisation of the sign function. The precise transformation formulae (2.25)-(2.28) seem to be unprecedented in the literature. Amplitude and phase of the wave profile $\hat{\Psi}_4$ are of course constant along the (infinitely extended) wavefronts $u = \text{const}$, but differ from Ψ_4. However, the case of (constant) linear polarisation

$$E \in \mathbb{R} \quad \Leftrightarrow \quad W = 0 \quad \Rightarrow \quad \Psi_4 \in \mathbb{R}, \;\; P = 0, \;\; \hat{\Psi}_4 = \mathrm{e}^{M}\Psi_4 \in \mathbb{R}, \tag{2.29}$$

has an invariant meaning since the imaginary parts of both $\hat{\Psi}_4$ and Ψ_4 vanish simultaneously.

2.5 The Ernst equation

In this section a novel derivation of the Ernst equation is given using only the Ernst potential E contained in the metric (2.11). In vacuum, Einstein's field equation (2.2) equates the Einstein tensor G_{ab} to zero. A sophisticated combination of its components yields

$$\begin{aligned} 0 &= \tfrac{1}{3}\mathrm{e}^{U-M}\Re(E)^{-1}\left[G_{44} + |E|^2 G_{33} + 2\Im(E)G_{34} + 4\Re(E)\mathrm{e}^{M-U}G_{12}\right] \\ &= U_u U_v - U_{uv}. \end{aligned} \tag{2.30}$$

Employing this leads to the progenitor of the Ernst equation:

$$0 = 2\mathrm{i}\mathrm{e}^{U-M}\frac{\Re(E)^2}{\Im(E)}\left[G_{44} + E^2 G_{33} + 4E\mathrm{e}^{M-U}G_{12}\right] \tag{2.31}$$

$$= 2E_u E_v + \Re(E)(U_u E_v + U_v E_u - 2E_{uv}). \tag{2.32}$$

Using (2.30) and (2.32) to replace U_{uv} and E_{uv} respectively, $G_{11} = 0$ and $G_{22} = 0$ can be easily converted into

$$\Re(E)^2(2U_{uu} - U_u^2 + 2M_u U_u) = E_u \bar{E}_u, \tag{2.33}$$

$$\Re(E)^2(2U_{vv} - U_v^2 + 2M_v U_v) = E_v \bar{E}_v, \tag{2.34}$$

In addition, $G_{12} = 0$ yields with (2.30)

$$2\Re(E)^2(2M_{uv} + U_u U_v) = E_u \bar{E}_v + E_v \bar{E}_u. \tag{2.35}$$

On the other hand, the derivative of (2.33) with respect to v and the derivative of (2.34) with respect to u in consideration of (2.30) and (2.32) both yield also equation (2.35). Therefore (2.35) is redundant and the integrability condition of (2.33) and (2.34) is automatically fulfilled. All in all the six real equations for the non-vanishing components G_{11}, G_{12}, G_{22}, G_{33}, G_{34} and G_{44} of the Einstein tensor are converted into the real equation (2.30), the complex Ernst equation (2.32) and the integrable system (2.33) and (2.34). Their decoupling suggests a solution procedure in the sense that first U can be determined from (2.30), secondly E can be determined from (2.32) and finally M is given by (2.33) and (2.34).
Equation (2.30) has the general solution

$$\mathrm{e}^{-U} = f(u) + g(v) \tag{2.36}$$

containing two arbitrary functions $f(u)$ and $g(v)$. In order for the spacetime to contain single wave regions where U depends only on a single lightlike coordinate, the choice

$$f = \tfrac{1}{2} \text{ for } u \leq 0, \quad g = \tfrac{1}{2} \text{ for } v \leq 0, \quad f'(0) = 0 = g'(0), \tag{2.37}$$

is made in accordance with Griffiths [9]. From rewriting (2.33) as

$$2\mathrm{e}^{-M}(\mathrm{e}^{M} U_u)_u = U_u^2 + \frac{|E_u|^2}{\Re(E)^2} > 0 \tag{2.38}$$

it follows that $U(u, v_0)$ for constant v_0 is monotonically increasing and hence f monotonically decreasing for $u > 0$. Similarly g is monotonically decreasing for $v > 0$. Using f and g as coordinates for $(u > 0, v > 0)$, with the help of

$$U_u = -\frac{f_u}{f+g}, \quad U_v = -\frac{g_v}{f+g} \tag{2.39}$$

equation (2.32) becomes the hyperbolic Ernst equation

$$\Re(E)\left(2E_{fg} + \frac{E_f + E_g}{f+g}\right) = 2E_f E_g. \tag{2.40}$$

This is a nonlinear partial differential equation with two variables. Moreover, as will be shown in section 3.2, a linear problem exists whose integrability condition is equivalent

to (2.40) and thus the ISM is applicable.
Having determined E and U, the function M can be obtained afterwards by integration of the field equations (2.33) and (2.34). If the coordinate rescaling freedom $u \to u'(u)$ is eliminated by making a distinct choice for the shape of $f(u)$ (as will be done in (2.45)), the whole metric is specified through the knowledge of E. Together with E also the function $E' = aE + \mathrm{i}b$ ($a, b \in \mathbb{R}$) is a solution to the Ernst equation (2.40). As the next section will show, it is appropriate to fix this freedom by demanding as connection to the Minkowski background the normalisation

$$E(\tfrac{1}{2}, \tfrac{1}{2}) = 1. \tag{2.41}$$

2.6 Spacetime regions and boundaries

As anticipated in section 2.3, it is appropriate to divide a colliding wave spacetime into four regions [9, 39] with the following coordinate dependencies of the metric functions:

$$\begin{aligned}
I: &\quad u < 0, v < 0: &\quad E = 1, &\quad M = 0, &\quad \mathrm{e}^{-U} &= 1,\\
II: &\quad u \geq 0, v < 0: &\quad E(u,0), &\quad M(u,0) =: M_{II}(u), &\quad \mathrm{e}^{-U} &= \tfrac{1}{2} + f(u),\\
III: &\quad u < 0, v \geq 0: &\quad E(0,v), &\quad M(0,v) =: M_{III}(v), &\quad \mathrm{e}^{-U} &= \tfrac{1}{2} + g(v),\\
IV: &\quad u \geq 0, v \geq 0: &\quad E(u,v), &\quad M(u,v), &\quad \mathrm{e}^{-U} &= f(u) + g(v).
\end{aligned} \tag{2.42}$$

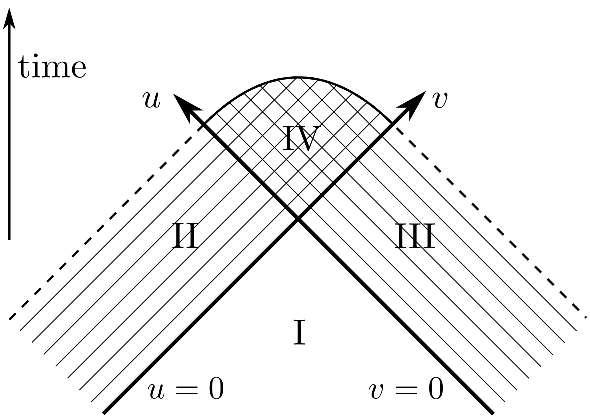

Figure 2.1: Identification of the four spacetime regions of colliding GWs adapted from Griffiths [9], see also [31, 39, 40].

This partition is illustrated in figure 2.1. Its physical interpretation is that on a Minkowski background (I) two plane waves propagate undisturbedly in opposite directions (II and III) until their collision and nonlinear interaction (IV). In this setup the Minkowski space (I) is bounded by the lightlike fronts $u = 0$ and $v = 0$ of the two different waves. The intersection of these wave fronts is spanned by two space-like vectors, which can be made orthogonal to the time direction via a Lorentz boost. Therefore it is sufficient to consider only head-on collisions as described by the metric (2.11).

As the general IVP shall be solved in this work, it is necessary to prescribe arbitrary initial waves and calculate the solution of the Ernst potential in the interaction region IV. Using the functions f and g as coordinates in region IV, the 'left wave' in region II is specified by the value of $E(f, \frac{1}{2})$ on the boundary $g = \frac{1}{2}$ (i.e. $v = 0, u > 0$) to region IV. This implies automatically knowledge of the derivative $E_f(f, \frac{1}{2})$. If in addition arbitrary initial data for the other partial derivative $E_g(f, \frac{1}{2})$ were prescribed, this would similarly determine a second derivative $E_{fg}(f, \frac{1}{2})$ which in general would not fulfil the Ernst equation (2.40). This phenomenon is a special property of the curve $g = \frac{1}{2}$ (and analogously also of $f = \frac{1}{2}$), which is therefore called a 'characteristic curve' of the corresponding second order partial differential equation, cf. [41]. Impulses and shocks can be shown to generically propagate along characteristic curves. In fact, as Roger Penrose showed [42], it is a well posed problem to find a solution E to the Ernst equation (2.40) with given initial values $E(f, \frac{1}{2})$ and $E(\frac{1}{2}, g)$. This means that a unique solution exists at least in a neighbourhood of the boundaries where the initial data is defined [9].

According to [43] curvature singularities can be classified either as being of scalar or of non-scalar nature. The existence of a scalar curvature singularity is established if some scalar curvature invariant (e.g. the Ricci scalar R or the Kretschmann scalar $R^{abcd}R_{abcd}$) is shown to become singular on the boundary of a spacetime. In contrast, there is a non-scalar curvature singularity if all of the (infinitely many) scalar curvature invariants are regular but at least one component of the Riemann tensor R_{abcd} with respect to an orthonormal basis is badly behaved. Throughout each of these curvature singularities no continuation of the metric is possible.

As anticipated by the sum $f + g$ in the denominator within (2.40) (and discussed in detail in [9]), the colliding wave spacetime features a generic scalar curvature singularity at $f+g = 0$. This is indicated by the solid curved line in figure 2.1). The occurrence of a singularity can be understood considering the mutual focussing properties of waves in GR: A gravitational plane wave induces convergence and shear into a congruence of orthogonal null geodesics [9]. Alternatively, the singularity can be regarded as an artefact of the unphysical idealisation of perfectly plain wavefronts.

For a large variety of exceptional cases the singularity at $f + g = 0$ is replaced by a Killing-Cauchy horizon [9]. Then the metric is extendible throughout $f + g = 0$, but the extension is not uniquely defined by initial values in the past of the Killing-Cauchy horizon. These horizons are conjectured to be unstable [12] and for collinearly polarised waves this instability has been rigorously proven [44].

The regions II and III are confined by coordinate degeneracies on lightlike hypersurfaces, which are depicted by dashed lines in figure 2.1. They can be identified with the points $-f = g = \frac{1}{2}$ and $f = -g = \frac{1}{2}$ and inherit their singular character [9]. Nevertheless, for the vacuum case considered here the scalar invariants of the Weyl tensor starting with (2.8) are the only candidates for diverging scalar curvature invariants. However, no scalar invariant can be a function of Ψ_4 or respectively Ψ_0 alone, since this would imply Ψ_4 or Ψ_0 to be a coordinate invariant scalar itself. But since only Ψ_4 or respectively Ψ_0 is non-vanishing in the regions II and III, this implies that all invariants of the Weyl tensor vanish in these incoming wave regions. Therefore the outer boundaries of the regions II and III cannot be scalar curvature singularities on their own. On the other hand Griffiths [9] shows that only a zero set of timelike geodesics actually ends in the outer boundaries of the regions II and III, whereas they enter the interaction region and reach $f + g = 0$ in the general case. Thus the term 'fold singularity' has been established to indicate the special topological character of these boundaries, which is not visible in the two-dimensional section depicted in figure 2.1.

2.7 Colliding wave conditions

In the last section the spacetime was divided into four different regions and separate expressions were introduced that describe the metric in these individual parts. In order to give rise to a reasonable spacetime, the metric functions have to fulfil appropriate junction conditions on the borders between the regions.

How these junction conditions shall be formulated for GWs was the topic of a long lasting debate. André Lichnerowicz [45] demanded for general hypersurfaces that coordinates should exist, in which the metric is C^1 (continuously differentiable) and piecewise C^2 (two times continuously differentiable with at most a finite number of exceptional hypersurfaces). This condition works well for non-null hypersurfaces, but it excludes impulsive waves like the Khan-Penrose solution [40].

Werner Israel [46, 47] reformulated the Lichnerowicz condition by claiming that the second fundamental form should have at most a finite jump. Such a finite jump leads to a surface stress energy density. However, for null hypersurfaces as $u = 0$ and $v = 0$ linking the regions (2.42), the normal vector has zero norm and so the concept of the second fundamental form breaks down. Considering only null hyperfurfaces, E. H. Robson [48] showed that it is appropriate to use the relaxed junction conditions of Stephen O'Brien and John Synge [49]. For the lightlike hypersurface $u = 0$ these conditions read [9]

$$g_{ab},\ g^{ij}\partial_u g_{ij},\ g^{iu}\partial_u g_{ij} \in C^0, \qquad i,j = 1,2,3. \tag{2.43}$$

Applied to the metric in (2.11), condition (2.43) and its analogue for $v = 0$ lead to the demands

$$u = 0 \wedge v = 0: \qquad E, M \in C^0, \quad U \in C^1. \tag{2.44}$$

Thus it is possible to perform C^1-transformations $u \to u'(u)$ and $v \to v'(v)$ to arrange

$$f = \tfrac{1}{2} - (c_1 u)^{n_1}\Theta(u), \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2}\Theta(v), \tag{2.45}$$

where $\Theta(\cdot)$ is the Heaviside step function and $c_{1/2}$ can be interpreted as magnitudes of the waves. Alternatively, such C^1-transformations could be used to achieve $M_{II}(u) = 0$ and $M_{III}(v) = 0$ in the incoming wave regions. Then $f(u)$ and $g(v)$ were determined by the field equations (2.33), (2.34) and the junction conditions (2.37) with the Minkowski background. Also in this case the exponents $n_{1/2}$ from (2.45) would describe the first order behaviour of $f(u)$ and $g(v)$ because they cannot be changed by C^1-transformations. This work operates with the first alternative, requiring (2.45).

A procedure intending to determine the metric function M from a solution of the Ernst equation has to assure that the result is indeed continuous across $u = 0$ and $v = 0$. A condition resulting from the continuity of M at $(u = 0+, v \leq 0)$ can be derived by inserting (2.45) into (2.38) and evaluating each term in leading order for $u \to 0$, which yields

$$|E_u|^2 = 2n_1(n_1 - 1)c_1^{n_1}u^{n_1-2} + 2n_1 c_1^{n_1}u^{n_1-1}M_u - n_1^2 c_1^{2n_1}u^{2n_1-2}. \tag{2.46}$$

Herein already $\Re(E) = 1$ was inserted since $E(0-, v \leq 0) = 1$ and $E \in C^0$. To fulfil $U \in C^1$ at least $n_1 > 1$ is necessary in (2.45). The continuity of M now requires the first term on the right hand side of (2.46) to be the dominant one because $M_u(0+, v \leq 0) \sim u^{-1}$ would imply a violation of $M \in C^0$. Therefore (2.46) should be read as a description of the Ernst potential's derivative near the wave front. In doing so, five different cases can be distinguished depending on the value of n_1:

- For $1 < n_1 < 2$ the divergence of u^{n_1-2} leads to an unbounded derivative of the Ernst potential $E_u(0+, v \leq 0)$ and so the second derivative $E_{uu}(0+, v \leq 0)$, which contributes to $\Psi_4(0+, v \leq 0)$ in (2.16), cannot even be described via a

δ-distribution. Therefore the case $1 < n_1 < 2$ is usually excluded by demanding[8]

$$n_{1/2} \geq 2, \tag{2.47}$$

where already the analogue statement for n_2 is incorporated. The index '1/2' is in this work supposed to denote a statement holding both for index 1 and for index 2 inserted throughout the entire expression.

- For $n_1 = 2$ the u^{n_1-2}-term in (2.46) is finite at $u = 0+$ which yields a finite value for $E_u(0+, v \leq 0)$. Since $E_u(0-, v \leq 0) = 0$, the second derivative contains a distributional part $E_{uu} \sim \delta(u)$ and so does the Weyl tensor component $\Psi_4(0+, v \leq 0)$. In this case the wave is said to have an impulsive wavefront.

- For $2 < n_1 < 4$ the derivative of the Ernst potential $E_u(0+, v \leq 0)$ is zero at $u = 0$, but $E_{uu}(0+, v \leq 0)$ and hence $\Psi_4(0+, v \leq 0)$ is still unbounded.

- For $n_1 = 4$ the second derivative $E_{uu}(0+, v \leq 0)$ is finite but different from $E_{uu}(0-, v \leq 0) = 0$. The corresponding jump in $\Psi_4(u, v \leq 0)$ describes a step wave.

- For $n_1 > 4$ finally $E_{uu}(u, v \leq 0)$ and $\Psi_4(u, v \leq 0)$ are continuous.

All these properties are directly inherited by the 'wave profile' $\hat{\Psi}_4$ via the transformation (2.28).
Since the characteristic IVP will be treated in the coordinates f and g, equation (2.46) needs to be rewritten replacing u by f. An analogue expression then exists for E_g. Keeping only the dominant term on the right hand side, the results are the so-called 'colliding wave conditions' first formulated by Hauser and Ernst [14], which within the normalisation (2.41) have the form

$$\lim_{(f,g)\to(\frac{1}{2},\frac{1}{2})} \left[(\tfrac{1}{2} - f) E_f \bar{E}_f \right] = 2k_1, \qquad \lim_{(f,g)\to(\frac{1}{2},\frac{1}{2})} \left[(\tfrac{1}{2} - g) E_g \bar{E}_g \right] = 2k_2, \tag{2.48}$$

with

$$k_{1/2} := 1 - \frac{1}{n_{1/2}}. \tag{2.49}$$

[8] In a spacetime with matter $1 < n_1 < 2$ is linked with an impulsive component of the Ricci tensor R_{ab}, cf. [9].

Relation (2.47) translates to a quite narrow range for $k_{1/2}$:

$$\frac{1}{2} \leq k_{1/2} < 1. \tag{2.50}$$

Equations (2.48) are conditions concerning only the limits of E_f and E_g in the Minkowski point $f = g = \frac{1}{2}$ implying the continuity of M at $(u = 0, v \leq 0)$ and $(u \leq 0, v = 0)$, but they are also sufficient to assure the validity of the boundary conditions (2.44) throughout $u = 0$ and $v = 0$, cf. [15]. For example the values of $M_v(0\pm, v \geq 0)$ are determined via (2.34) by v-derivatives of E and U which are continuous across $u = 0$ and hence M_v is also continuous at the boundary between region III and IV. Since the continuity of M at $u = v = 0$ is assured by (2.48), this implies that M is also continuous across $u = 0$ for $v \geq 0$.

It may be noticed that the colliding wave conditions (2.48) actually demand the derivatives of E at the Minkowski point $f = g = \frac{1}{2}$ to feature the inverse root-like divergences $|E_f| \sim (\frac{1}{2} - f)^{-\frac{1}{2}}$ and $|E_g| \sim (\frac{1}{2} - g)^{-\frac{1}{2}}$. Therefore the initial values of the Ernst potential must have an appropriate root-like behaviour near $f = g = \frac{1}{2}$ with an offset given by $E(\frac{1}{2}, \frac{1}{2}) = 1$. In the context of the characteristic IVP the colliding wave conditions are a matter of choosing suitable initial values for E featuring divergent derivatives at $(f = \frac{1}{2}, g = \frac{1}{2})$.

3 Inverse scattering method for collinear polarisation

The main purpose of this work is to investigate the characteristic IVP described in chapter 2 via the ISM, which will be explained in detail below. The introduction of the severely restricted case of collinearly polarised colliding GWs will provide a nice testbed for this approach. Using the ISM, the general solution for collinear polarisation given by Hauser and Ernst [14] in terms of integrals via generalized Abel transformations will be reproduced.

3.1 Scheme of inverse scattering

In the course of the ISM, the nonlinear Ernst equation is expressed as the integrability condition of a system of linear partial differential equations, the so-called 'linear problem' (LP). Its solution depends in addition to the variables f and g also on a complex spectral parameter λ. Furthermore, an appropriate Riemann-Hilbert problem (RHP) is constructed that is supposed to have the same solution as the LP. However this approach retains a heuristic element at that point, since the relation between the solution spaces of the LP and the RHP is not fully clarified within this work.

A Riemann-Hilbert problem can be described as the task to find a complex function with a prescribed *multiplicative* jump on a given contour in the extended complex plane[1] of the spectral parameter. For arbitrary polarised GWs, the LP and the RHP consist of matrix valued equations, but they reduce to scalar ones in the collinear case. In correspondence with the boundaries of the IVP, the contour of the RHP has to consist of two specific parts of the real axis (cf. figure 3.4). Using its *additive* jump on the contour, the solution of the RHP can be uniquely represented via a Cauchy type integral. The multiplicative jump equation defining the RHP can then be translated into integral equations for this additive jump.

All in all the ISM consists of the LP and the RHP and effectively transforms the

[1] i.e. the Riemann sphere $\mathbb{C} \cup \infty$.

IVP of a nonlinear partial differential equation into linear integral equations, which is schematically depicted in figure 3.1. In the case of colliding plane waves however, due to the colliding wave conditions, the latter have the severe disadvantage of dealing with a singular unknown function, which can be understood as follows: The mathematical literature on Riemann-Hilbert problems operates entirely with closed contours. In order to match with that picture, it is possible to construct a single closed contour for the RHP by connecting the original contour parts, whereby the jump matrix has to be set to $\mathbb{1}$ on the added parts. Regrettably, the singularity in the derivatives of the Ernst potential demanded by the colliding wave conditions (2.48) leads to a jump matrix which tends to a finite value different from $\mathbb{1}$ at the ends of the contour. In that sense, the jump matrix of the RHP is discontinuous and this in general implies the RHP solution to diverge at the endpoints of the original contour. Therefore also its additive jump function diverges and the integral equations determining it are harder to handle because of the unboundedness of their unknown function.

An interesting possibility to circumvent the issues connected to integral equations for a *singular* function is the transformation to a continuous RHP (CRHP), which will be achieved in this work in chapter 5. The additive jump function determining the CRHP solution is then given in terms of an integral equation for a *regular* unknown function, cf. figure 3.1.

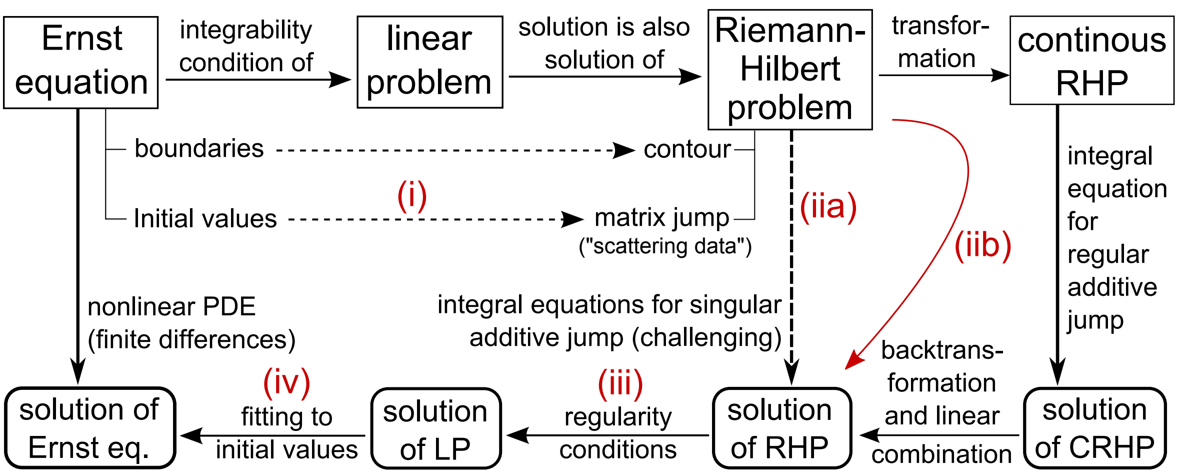

Figure 3.1: Scheme of the ISM with additional transformation to a continuous RHP.

The singularities in the RHP solution do not only complicate the RHP integral equations but also introduce an ambiguity into the RHP solution. Since the singularities can have two different appearances at each contour part, there are overall four different basic solutions of the RHP, which can be linearly combined. The transformation to the CRHP allows for a representation of these basic solutions in terms of the non-singular

CRHP solutions. Moreover, holomorphicity conditions on the linear combination of the basic solutions are imposed in order to assure that it solves the LP. By reduction of these holomorphicity conditions to simple algebraic relations it is shown that they can in general be fulfilled. Therefore the RHP solutions indeed generate LP solutions, which in turn imply solutions to the Ernst equation. However, it is not proven that the 'right' solution to the initially posed IVP is among them. Thus at the end of the described procedure it has to be checked if possible remaining degrees of freedom can be specified to yield the solution to the original IVP.
In that sense, to presumably solve a specific IVP with given initial data $E(f, \frac{1}{2})$ and $E(\frac{1}{2}, g)$ via the ISM, the following four steps indicated in figure 3.1 have to be carried out:

(i) Translating the initial data into the jump matrix by solving a system of ordinary differential equations (ODEs). For special cases an analytical treatment is possible.

(ii) Solving the Riemann-Hilbert problem, in the general case by expansion of an additive jump function in Chebyshev polynomials

   a) via integral equations for a singular additive jump, which is more challenging

   b) via transformation to the CRHP and its integral equation for a regular additive jump with better numerical properties

(iii) Evaluating holomorphicity conditions, which assure that the RHP solution fulfils the LP. These are purely algebraic equations to determine the linear combination coefficients of the RHP's four basic solutions.

(iv) Fixing the remaining degrees of freedom to adapt the solution to its initial data, if possible

The whole process of the ISM shall be illustrated by examining the case of collinearly polarised GWs. The contour of the collinear case will be directly transferred to the RHP for GWs with arbitrary polarisation.

3.2 Linear problem for collinear polarisation

As described in section 2.6, within the Newman-Penrose formalism the singular waves in the spacetime regions II and III are described by the complex Weyl tensor components $\Psi_0(v)$ and $\Psi_4(u)$, respectively. For linearly polarized initial waves the phases of these components are constant in region II and III. If these constant phases are even identical, then the polarisation of the waves is aligned and the metric can be diagonalised in all four regions simultaneously containing only a real Ernst potential. This very special setup is called the collision of collinearly polarised GWs.

The LP for this collinear case is to find the function $\Phi^{\|\mathrm{LP}}(f,g;\lambda)$ satisfying

$$\Phi_f^{\|\mathrm{LP}} = (1+\lambda)A\Phi^{\|\mathrm{LP}}, \qquad \lambda(f,g;k) = \sqrt{\frac{k-g}{k+f}}, \tag{3.1}$$
$$\Phi_g^{\|\mathrm{LP}} = (1+\lambda^{-1})B\Phi^{\|\mathrm{LP}},$$

where A and B are real functions of f and g whereas k is a complex 'independent spectral parameter', which enters the equations through the complex 'spectral parameter' λ depending on f, g and k. The partial derivatives ∂_f and ∂_g are taken with constant k rather than constant λ. The spectral parameter λ is defined on the Riemann sphere $\mathbb{C}_\lambda := \mathbb{C}\cup\{\infty\}$. The solution $\Phi^{\|\mathrm{LP}}$ can be thought of either as a function on $\mathbb{C}_\lambda$ or as a function of k defined on a two-sheeted Riemann surface $\mathbb{C}_k$ with branch cut along the segment $[-f,g]$ of the $\Re(k)$-axis, a twofold covering of the Riemann sphere illustrated in figure 3.2. Like for $\mathbb{C}_\lambda$, the (two different) points ∞ are both considered to be part of these sheets. Moreover, the sheet with $\lambda = 1$ for $k = \infty$ shall be called the upper one and the sheet with $\lambda = -1$ for $k = \infty$ the lower one.

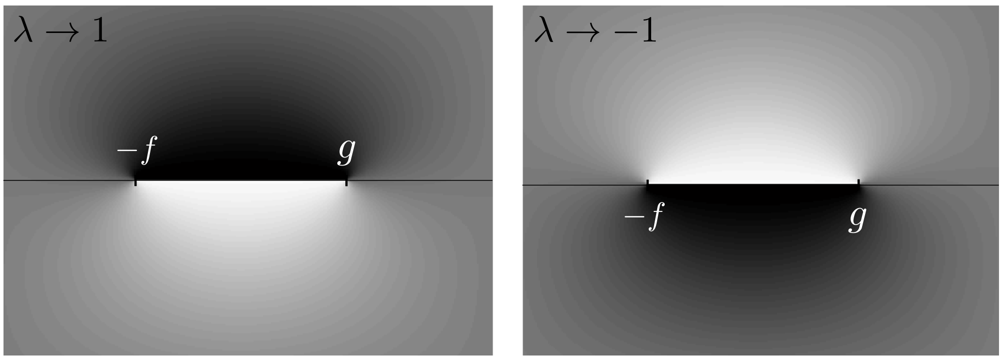

Figure 3.2: Illustration of the two-sheeted Riemann surface $\mathbb{C}_k$ consisting of an upper (left) and a lower (right) sheet. At the branch cut $[-f,g]$ bright area is connected to bright area and dark area to dark area.

Representing the derivatives of $\lambda(f,g;k)$ as

$$\lambda_f = \frac{\lambda}{2(f+g)}(\lambda^2-1), \qquad \lambda_g = \frac{1}{2\lambda(f+g)}(\lambda^2-1), \tag{3.2}$$

the integrability condition $\Phi^{\|\mathrm{LP}}_{fg} = \Phi^{\|\mathrm{LP}}_{gf}$ of the LP (3.1) evaluates to

$$(1+\lambda)A_g + \frac{\lambda^2-1}{2\lambda(f+g)}A = (1+\lambda^{-1})B_f - \frac{\lambda^2-1}{2\lambda(f+g)}B. \tag{3.3}$$

This equation has to be fulfilled for all values of k, but since only λ is affected by a change of k it can simultaneously be conceived as valid for all values of λ. Therefore the coefficients of each power of λ in (3.3) can be compared separately. In doing so the terms independent of λ simply yield $A_g = B_f$, which assures the existence of a potential $\psi(f,g)$ with

$$\psi_f = A, \qquad \psi_g = B. \tag{3.4}$$

The only other independent equation inferred from (3.3) can then be written as

$$\psi_{fg} + \frac{\psi_f + \psi_g}{2(f+g)} = 0. \tag{3.5}$$

This is indeed the Euler-Poisson-Darboux equation, a linearised version of the Ernst equation in case of real E, which can be derived from (2.40) by setting

$$\psi = \tfrac{1}{2}\ln E. \tag{3.6}$$

By (3.1) the LP solution $\Phi^{\|\mathrm{LP}}$ is only defined up to multiplication with a function of k. This freedom shall be fixed by demanding the normalisation

$$\Phi^{\|\mathrm{LP}}\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 1 \qquad \forall k, \tag{3.7}$$

whereby this work uses the following convention for an arbitrary function $F(f,g;\lambda)$ depending on f, g and λ: where F is displayed with 2 arguments as in (3.7), these should be understood as the values of f and g, but where F is displayed with a single argument as in (3.8), this should be taken as the value of λ.

It may be observed that the points $\lambda \to \pm 1$ are solely approached by $k \to \infty$ in the corresponding sheet independent of the values of (f,g). Consequently, derivatives λ_f and λ_g from (3.2) vanish in $\lambda = \pm 1$ and thus these points are predestined to gain insights into the LP and also into the RHP later. Indeed, evaluating the LP (3.1) at $\lambda = -1$ yields $\Phi^{\|\mathrm{LP}}_f(-1) = 0 = \Phi^{\|\mathrm{LP}}_g(-1)$, i.e. $\Phi^{\|\mathrm{LP}}(-1) = \mathrm{const}\ \forall (f,g)$. Since the value of $\Phi^{\|\mathrm{LP}}(\frac{1}{2},\frac{1}{2};-1)$ is fixed to 1 by (3.7) this means

$$\Phi^{\|\mathrm{LP}}(-1) = 1 \qquad \forall(f,g). \tag{3.8}$$

A corresponding relation will be used to normalise the RHP. On the other hand, evaluating (3.1) at $\lambda = 1$ leads to

$$[\ln \Phi^{\|\mathrm{LP}}(-1)]_f = 2A = 2\psi_f, \qquad [\ln \Phi^{\|\mathrm{LP}}(-1)]_g = 2B = 2\psi_g. \tag{3.9}$$

These relations imply together

$$\Phi^{\|\mathrm{LP}}(1) = \mathrm{e}^{-2\psi(\frac{1}{2},\frac{1}{2})}\mathrm{e}^{2\psi}, \tag{3.10}$$

where the integration constant has already been determined by (3.7). Considering (3.6), the LP solution normalised by (3.7) is seen to be linked to an Ernst potential with the normalisation (2.41) via

$$E = \Phi^{\|\mathrm{LP}}(1). \tag{3.11}$$

Hence the problem of solving the Ernst equation is transformed into the LP in the sense that if a solution to the normalised LP is obtained, the corresponding solution of the normalised Ernst equation can be read off from (3.11). On the other hand, the solution $E(f,g)$ to the IVP is uniquely determined by the initial values $E(f,\frac{1}{2})$, $E(\frac{1}{2},g)$, and the functions A and B determining the unique LP solution are given unambiguously in terms of the Ernst potential by (3.4). Therefore the initial values also induce a unique solution of the normalised LP, i.e. there is a 1:1 correspondence between IVP and normalised LP.

3.3 Riemann-Hilbert problem for collinear polarisation

The RHP connected to the LP (3.1) is to find a function $\Phi^{\|}(f,g;\lambda)$ which is analytic everywhere in the complex Riemann k-surface except on the contour $\Gamma^{(k)}$, where it has a jump described by the equation

$$k \in \Gamma^{(k)}: \qquad \Phi^{\|}_{+} = \alpha(k)\Phi^{\|}_{-}. \tag{3.12}$$

Herein $\Phi^{\|}_{+}$ is the inner (left to the contour) and $\Phi^{\|}_{-}$ the outer (right to the contour) limit of $\Phi^{\|}$. In addition, the freedom of multiplying $\Phi^{\|}$ with a function of f and g is fixed by demanding the normalisation

$$\Phi^{\|}(-1) = 1 \qquad \forall f, g. \tag{3.13}$$

The contour $\Gamma^{(k)}$ in the k-surface is chosen to consist of a first part $\Gamma_1^{(k)}$ directed from $k = -\frac{1}{2}$ in the upper sheet through the branch point $k = -f$ to $k = -\frac{1}{2}$ in the lower sheet and a second part $\Gamma_2^{(k)}$ directed from $k = \frac{1}{2}$ in the lower sheet through the branch point $k = g$ to $k = \frac{1}{2}$ in the upper sheet, cf. figure 3.3. By setting $k = \pm\frac{1}{2}$ the contour

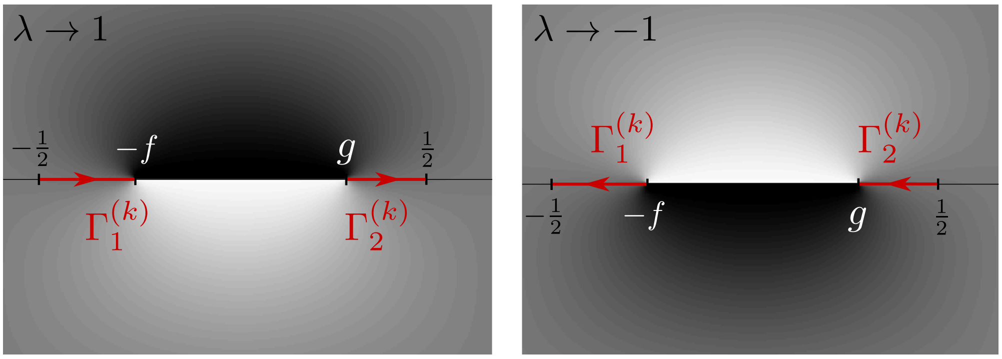

Figure 3.3: The two parts $\Gamma_1^{(k)}$ and $\Gamma_2^{(k)}$ of the contour $\Gamma^{(k)}$ in the upper (left) and lower (right) sheet of the two-sheeted Riemann k-surface.

endpoints on the λ-sphere are obtained:

$$\lambda_1 := \sqrt{\frac{\frac{1}{2} + g}{\frac{1}{2} - f}}, \qquad \lambda_2 := \sqrt{\frac{\frac{1}{2} - g}{\frac{1}{2} + f}}. \tag{3.14}$$

They lie on the real axis and satisfy $0 < \lambda_2 < 1 < \lambda_1$. The contour Γ on the λ-sphere is divided into Γ_1 corresponding to $\Gamma_1^{(k)}$ and Γ_2 corresponding to $\Gamma_2^{(k)}$. The first part Γ_1 is directed from λ_1 through $\lambda = \infty$ to $-\lambda_1$ and the second part Γ_2 is directed from $-\lambda_2$ through $\lambda = 0$ to λ_2, cf. figure 3.4.

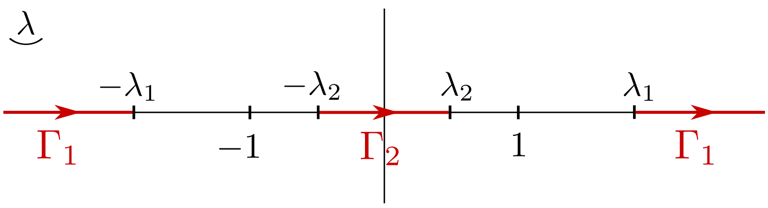

Figure 3.4: The two parts Γ_1 and Γ_2 of the contour Γ in the extended λ-plane.

The contour vanishes for $f = \frac{1}{2} = g$, which suggests $\Phi^{\|}(\frac{1}{2}, \frac{1}{2}) = \text{const}$ in this limit. This conjecture will be discussed in detail in section 6.7. With that constant set to 1 by (3.13) the LP normalisation $\Phi^{\|}(\frac{1}{2}, \frac{1}{2}) = 1$ is reproduced.

The inner and outer limit $\Phi_+^{\|}$ and $\Phi_-^{\|}$ as well as their reciprocals can be expanded at $\lambda = 0$ above or respectively below the contour into the power series

$$\Phi_\pm^{\|} = a_{0\pm} + a_{1\pm}\lambda + a_{2\pm}\lambda^2 + \cdots, \qquad (\Phi_\pm^{\|})^{-1} = b_{0\pm} + b_{1\pm}\lambda + b_{2\pm}\lambda^2 + \cdots \,. \tag{3.15}$$

Therein the coefficients $a_{i\pm}$ and $b_{i\pm}$ are functions of f and g. Recalling (3.2), the derivative of $\Phi_\pm^{\|}$ with respect to g is then given by

$$(\Phi_\pm^{\|})_g = (a_{0\pm})_g + (a_{1\pm})_g\lambda + \cdots + \frac{(\lambda^2-1)}{2\lambda(f+g)}\left(a_{1\pm} + 2a_{2\pm}\lambda + \cdots\right). \tag{3.16}$$

Since the multiplicative jump α in (3.12) is a function depending only on the independent spectral parameter k, (3.12) together with its derivative with respect to g yields

$$(\Phi_+^{\|})_g(\Phi_+^{\|})^{-1} = \alpha(k)(\Phi_-^{\|})_g(\Phi_+^{\|})^{-1} = (\Phi_-^{\|})_g(\Phi_-^{\|})^{-1}. \tag{3.17}$$

It follows that the terms $\Phi_g^{\|}(\Phi^{\|})^{-1}$ and similarly $\Phi_f^{\|}(\Phi^{\|})^{-1}$ exhibit no jump on the contour $\Gamma^{(k)}$. Assuming the expansions (3.15) and (3.16) to be valid also on Γ, their coincidence on Γ due to (3.17) determines the unique series for $\Phi_g^{\|}(\Phi^{\|})^{-1}$ in the neighbourhood of $\lambda = 0$ to be

$$\Phi_g^{\|}(\Phi^{\|})^{-1} = c_{-1}\lambda^{-1} + c_0 + c_1\lambda + \ldots \,. \tag{3.18}$$

Within the treatment of the CRHP it will be even shown that there exist solutions of the RHP fulfilling the holomorphicity conditions:

$$\begin{aligned} &\Phi_f^{\|}(\Phi^{\|})^{-1} \text{ is holomorphic in } \mathbb{C}_\lambda\backslash\{\infty\},\\ &\Phi_g^{\|}(\Phi^{\|})^{-1} \text{ is holomorphic in } \mathbb{C}_\lambda\backslash\{0\}. \end{aligned} \tag{3.19}$$

If (3.19) holds, then (3.18) is indeed the proper Laurent series of $\Phi_g^{\|}(\Phi^{\|})^{-1}$ valid throughout $\mathbb{C}_\lambda$. This immediately implies that $\Phi_g^{\|}(\Phi^{\|})^{-1} - c_{-1}\lambda^{-1}$ is holomorphic everywhere on the complex sphere and thus a constant due to Liouville's theorem, i.e. $c_i(f,g) = 0\,\forall i \geq 1$. Consideration of (3.13) finally leads to $c_{-1} = c_0$ and hence allows the representation

$$\Phi_g^{\|}(\Phi^{\|})^{-1} = (1+\lambda^{-1})B, \tag{3.20}$$

which is equivalent to the second line of the LP in (3.1). An analogous expansion of $\Phi^{\|}_g(\Phi^{\|})^{-1}$ at $\lambda = \infty$ yields the first line of the LP if the holomorphicity conditions hold. On the other hand, the LP solution due to $\Phi^{\|\mathrm{LP}}_f(\Phi^{\|\mathrm{LP}})^{-1} = U$ as well as $\Phi^{\|\mathrm{LP}}_g(\Phi^{\|\mathrm{LP}})^{-1} = V$ clearly obeys the holomorphicity conditions and so they are both necessary and sufficient for the RHP solution to also solve the LP. Together with the consideration of normalisation above, it follows that a solution $\Phi^{\|}$ of the RHP normalised by (3.13) and fulfilling the holomorphicity conditions (3.19) is simultaneously a solution $\Phi^{\|\mathrm{LP}}_f$ of a LP of the form (3.1) normalised by (3.7). According to (3.11) a solution to the Ernst equation is then given by $E = \Phi^{\|\mathrm{LP}}(1) = \Phi^{\|}(1)$.

Within the treatment of the CRHP it will become clear that the RHP in the form (3.12) with a multiplicative jump features a wider class of solutions which are not all solutions of the LP. Therefore the holomorphicity conditions are a nontrivial restriction. On the other hand, this work will not proof that each LP solution which is induced by an IVP solution is indeed the solution of a RHP. The possibility of this proof will be discussed in the outlook in section 9.1, but for the time being the attempt to represent the LP solution via a RHP has to be regarded as a heuristic approach.

3.4 General solution for collinear polarisation

To complete the ISM, the initial values $E(f, \frac{1}{2})$ and $E(\frac{1}{2}, g)$ have to be related to the jump function α of the RHP. In the collinear case a more direct approach is to link $\psi(f, \frac{1}{2}) = \frac{1}{2}\ln E(f, \frac{1}{2})$ and $\psi(\frac{1}{2}, g) = \frac{1}{2}\ln E(\frac{1}{2}, g)$ with the additive jump $\mathrm{i}\mu^{\|}$ of the logarithm of the LP solution. In order to derive this additive jump of $\ln \Phi^{\|\mathrm{LP}}$ at a given point k on the contour $\Gamma_2^{(k)}$, the second line of the LP (3.1) at $f = \frac{1}{2}$ is written in the form

$$\left(\ln \Phi^{\|\mathrm{LP}}\left(\tfrac{1}{2}, g\right)\right)_g = (1 + \lambda^{-1})\psi_g\left(\tfrac{1}{2}, g\right). \tag{3.21}$$

With the normalisation $\Phi^{\|\mathrm{LP}}(\frac{1}{2}, \frac{1}{2}) = 1$ the integration from $g' = \frac{1}{2}$ to $g' = g$ yields

$$\ln \Phi^{\|\mathrm{LP}}(\tfrac{1}{2}, g) = \psi(\tfrac{1}{2}, g) - \int_g^{\frac{1}{2}} \sqrt{\frac{k + \frac{1}{2}}{k - g'}}\psi_{g'}\mathrm{d}g'. \tag{3.22}$$

Therein the root under the integral still has to be calculated with the sign of $[\lambda(\frac{1}{2}, g'; k)]^{-1}$. Using $k = \kappa + \mathrm{i}\epsilon$ it can be written as

$$[\lambda(\tfrac{1}{2}, g'; \kappa + \mathrm{i}\epsilon)]^{-1} = \sqrt{\frac{\kappa + \mathrm{i}\epsilon + \frac{1}{2}}{\kappa + \mathrm{i}\epsilon - g'}} = \sqrt{\frac{(\kappa + \frac{1}{2})(\kappa - g') + \epsilon^2 - \mathrm{i}\epsilon(\frac{1}{2} + g')}{(\kappa - g')^2 + \epsilon^2}}.$$

In the upper sheet the continuous extension from $\epsilon \to +\infty$ with $\lambda = 1$ to $\epsilon \searrow 0$ (inner side of the contour) yields $(\kappa + \frac{1}{2})/(\kappa - g')$ for $g' < \kappa$ and $-\mathrm{i}(\kappa + \frac{1}{2})/(g' - \kappa)$ for $g' > \kappa$. The continuous extension from $\epsilon \to -\infty$ with $\lambda = 1$ to $\epsilon \nearrow 0$ (outer side of the contour) gives the conjugate limits. Therefore $\ln \Phi^{\|\mathrm{LP}}(\frac{1}{2}, g)$ has the jump

$$\mathrm{i}\mu_2^{\|} := \ln \Phi_+^{\|\mathrm{LP}} - \ln \Phi_-^{\|\mathrm{LP}} = 2\mathrm{i}\sqrt{\tfrac{1}{2} + k} \int_k^{\frac{1}{2}} \mathrm{d}g' \frac{\psi_{g'}(\frac{1}{2}, g')}{\sqrt{g' - k}} \tag{3.23}$$

along the segment $[g, \frac{1}{2}]$, which is the half of $\Gamma_2^{(k)}$ lying in the upper sheet.

Switching to the lower sheet alters the sign of λ as well as the direction of integration so that the same jump function is obtained. Analogously the additive jump on $\Gamma_1^{(k)}$ is given by

$$\mathrm{i}\mu_1^{\|} := \ln \Phi_+^{\|\mathrm{LP}} - \ln \Phi_-^{\|\mathrm{LP}} = -2\mathrm{i}\sqrt{\tfrac{1}{2} - k} \int_{-k}^{\frac{1}{2}} \mathrm{d}f' \frac{\psi_{f'}(f', \frac{1}{2})}{\sqrt{k + f'}} \tag{3.24}$$

in both sheets. Taking the exponential of (3.23) and (3.24) yields

$$\Phi_+^{\|\mathrm{LP}} = \mathrm{e}^{\mathrm{i}\mu^{\|}} \Phi_-^{\|\mathrm{LP}}, \qquad \mu^{\|} := \begin{cases} \mu_1^{\|}, & \lambda \in \Gamma_1; \\ \mu_2^{\|}, & \lambda \in \Gamma_2. \end{cases} \tag{3.25}$$

Since the LP solution obeys this jump equation with multiplicative jump $\mathrm{e}^{\mathrm{i}\mu^{\|}}$, the RHP must necessarily feature the jump

$$\alpha := \mathrm{e}^{\mathrm{i}\mu^{\|}} \tag{3.26}$$

to possibly give rise to LP solutions. This is the desired relation between the initial values and the RHP jump. By considering equations (3.23)-(3.25) this jump $\mathrm{e}^{\mathrm{i}\mu^{\|}}$ is seen to be indeed independent of the coordinates f and g. It was crucial to choose the contour to be effectively independent of the coordinates f and g to arrive at this result[2].

However, the relation (3.26) does not readily prove the LP solution to be indeed representable by a RHP solution, since the holomorphicity of $\Phi^{\|\mathrm{LP}}$ outside the contour is not assured. On the other hand, also the solution of the RHP with α given by (3.26) is not proven to be a LP solution because the holomorphicity conditions (3.19) are not

[2]E.g. the dependence of $\Gamma_1^{(k)}$ on f can be seen to be marginal by replacing it with the contour $\Gamma_{1'}^{(k)}$ on the $\Re(k)$-axis directed from $k = -\frac{1}{2}$ in the upper sheet through $k = -f$ to $k = \frac{1}{2}$ (passing the branch cut on any side), turning around and going back (passing the branch cut on the same side) through $k = -f$ to $k = -\frac{1}{2}$ in the lower sheet.

shown to be fulfilled. It is actually easy to recognise the class of RHP solutions to be wider than the class of LP solutions by transforming the jump equation equivalently into

$$
\begin{aligned}
k \in \Gamma_1^{(k)}: &\qquad \ln \Phi_+^{\|} - \ln \Phi_-^{\|} = \mathrm{i}\mu_1^{\|} + 2\pi m_1 \mathrm{i}, &\qquad m_1 \in \mathbb{Z}, \\
k \in \Gamma_2^{(k)}: &\qquad \ln \Phi_+^{\|} - \ln \Phi_-^{\|} = \mathrm{i}\mu_2^{\|} + 2\pi m_2 \mathrm{i}, &\qquad m_2 \in \mathbb{Z}.
\end{aligned}
\tag{3.27}
$$

In principle every choice of m_1 and m_2 leads to a different RHP solution, but as discussed in section 4.5 it is reasonable to restrict this freedom to the domain $|\mu_{1/2}^{\|} + 2\pi m_{1/2}| < 2\pi$, yielding in general two possible values for each $\mu_{1/2}^{\|}$. The linearity of the RHP allows also for linear combinations of them. Thus in the collinear case the ambiguity of the RHP solution is a direct consequence of the ambiguity of the complex logarithm.

For arbitrary polarised GWs only *multiplicative* jump data can be inferred from the IVP, leading to similar ambiguities as described above. A special aspect of the ISM in the collinear case is that an *additive* jump function connected with the LP solution is given by (3.23) and (3.24). Therefore a unique RHP solution $\Phi_\mu^{\|}$ defined by

$$
k \in \Gamma^{(k)}: \qquad \ln \Phi_{\mu+}^{\|} - \ln \Phi_{\mu-}^{\|} = \mu^{\|}. \tag{3.28}
$$

can be associated with the LP solution: either $\Phi_\mu^{\|} = \Phi^{\|\mathrm{LP}}$ holds, or no RHP solution solves the LP. Using this specified additive jump, $\ln \Phi_\mu^{\|}$ can be expressed in terms of a Cauchy type integral on the λ-sphere,

$$
\ln \Phi_\mu^{\|} = \frac{1}{2\pi} \int_\Gamma \left(\frac{1}{\lambda' - \lambda} - \frac{1}{\lambda' + 1} \right) \mu^{\|}(k') \mathrm{d}\lambda'. \tag{3.29}
$$

The second term under the integral assures the normalisation $\Phi_\mu^{\|}(-1) = 1$. The effect of the first term under the integral can be understood by thinking of each partial contour $\Gamma_{1/2}$ to be closed by some line segment lying in a domain where $\mu_{1/2}^{\|}$ can be analytically continued. Due to Cauchy's residue theorem, the value of the integral in (3.29) taken over these closed curves $\Gamma_{1/2}^{\circ}$ will increase by $2\pi\mathrm{i}\mu^{\|}$ if λ moves across $\Gamma_{1/2}^{\circ}$ from its exterior to its interior, which is equivalent to (3.28).

With the help of

$$
\left(\frac{1}{\lambda' - 1} - \frac{1}{\lambda' + 1} \right) = -\frac{2(k' + f)}{f + g} \quad \text{and} \quad \frac{d\lambda'}{dk'} = \frac{1}{2\lambda'} \frac{f + g}{(k' + f)^2} \tag{3.30}
$$

the integral in (3.29) evaluated at $\lambda = 1$ can be transferred into the k-surface yielding

$$\frac{1}{2}\ln\Phi^{\|}_{\mu}(1) = \frac{1}{2\pi}\int_{-\frac{1}{2}}^{-f}\frac{\mu_1^{\|}(k')\mathrm{d}k'}{\sqrt{(k'-g)(k'+f)}} - \frac{1}{2\pi}\int_{g}^{\frac{1}{2}}\frac{\mu_2^{\|}(k')\mathrm{d}k'}{\sqrt{(k'-g)(k'+f)}}. \tag{3.31}$$

Note that on the contour λ is always real with $\lambda' > 0$ in the upper sheet and $\lambda' < 0$ in the lower one. Hence the integrals taken in opposite directions in the two sheets add up. Finally, if $\Phi^{\|}_{\mu}$ fulfilled the holomorphicity conditions (3.19), this would imply $\frac{1}{2}\ln\Phi^{\|}_{\mu}(1) = \frac{1}{2}\ln\Phi^{\|\mathrm{LP}}(1) = \psi$ and thus

$$\psi = \frac{1}{2\pi}\int_{-\frac{1}{2}}^{-f}\frac{\mu_1^{\|}(k')\mathrm{d}k'}{\sqrt{(k'-g)(k'+f)}} - \frac{1}{2\pi}\int_{g}^{\frac{1}{2}}\frac{\mu_2^{\|}(k')\mathrm{d}k'}{\sqrt{(k'-g)(k'+f)}}. \tag{3.32}$$

However, the validity of the holomorphicity conditions shall not be investigated in further detail here. Instead it is sufficient to note that the solution proposed by the ISM, which consists of the equations (3.23), (3.24) and (3.32), is exactly the unique IVP solution that Isidore Hauser and Frederick Ernst derived via generalised Abel transformations in [14]. In this sense the heuristic ISM approach indeed leads to the right IVP solution $\frac{1}{2}\ln\Phi^{\|}_{\mu}(1) = \psi$ although not even this directly proves $\Phi^{\|\mathrm{LP}} = \Phi^{\|}_{\mu}$.
For $g = \frac{1}{2}$ the second integral in (3.32) vanishes and (3.32) together with (3.24) has indeed the shape of an Abel transformation between ψ and $\mu_1^{\|}$. However, since the colliding wave conditions (2.48) require the divergent behaviour $\psi_f(f, \frac{1}{2}) \sim (\frac{1}{2} - f)^{-\frac{1}{2}}$, generalising considerations were needed. Hauser and Ernst showed that $\mu_1^{\|}$ calculated via (3.24) exists but does not vanish at the contour endpoint $k = -\frac{1}{2}$.
In the same article [14] Hauser and Ernst presented a RHP similar to (3.12), where the independent spectral parameter lies in a simple complex plane. Its solution Φ_H is related to $\Phi^{\|}_{\mu}$ by

$$\ln\Phi^{\|}_{\mu}(k) = -(k+f)\lambda\Phi_H(2k) + \psi \tag{3.33}$$

and it uses the jump functions

$$g_3(\sigma) = \frac{\mu_1^{\|}(\sigma/2)}{2\sqrt{1-\sigma}}, \qquad g_2(\sigma) = \frac{\mu_2^{\|}(\sigma/2)}{2\sqrt{1+\sigma}}. \tag{3.34}$$

Note that the jump functions $\mu_{1/2}^{\|}$ from (3.23) and (3.24) both are defined on the interval $[-\frac{1}{2}, \frac{1}{2}]$, but for given f and g only the values of $\mu_1^{\|}$ on $\Gamma_1^{(k)}$ and the values of $\mu_2^{\|}$ on $\Gamma_2^{(k)}$ appear in (3.32). Obviously real initial values $\psi_f(f, \frac{1}{2})$ and $\psi_g(\frac{1}{2}, g)$ lead to real $\mu_{1/2}^{\|}$ and via (3.32) to a real solution ψ.

4 Inverse scattering method for arbitrary polarisation

A derivation similar to the axially symmetric and stationary case in [50] leads for arbitrary polarisation to the general versions of the LP and the RHP. These are formally very similar to the collinear case, but matrix valued. From the initial values in general only the calculation of multiplicative jump data is possible and this is accomplished by solving an ODE. Explicit formulae are derived for the boundary values of the multiplicative jump matrix, which determine the singular behaviour of the solution and indicate the character of its ambiguities. At the end of the chapter the integral equations for the singular additive jump data of the RHP are introduced and discussed.

4.1 Linear problem for arbitrary polarisation

The LP for colliding plane waves of arbitrary polarisation is to find the matrix $\Phi^{\mathrm{LP}}(f, g; \lambda)$ satisfying[1]

$$\begin{aligned} \Phi^{\mathrm{LP}}_f &= U\Phi^{\mathrm{LP}}, \\ \Phi^{\mathrm{LP}}_g &= V\Phi^{\mathrm{LP}}, \end{aligned} \qquad U = \begin{pmatrix} A & \lambda A \\ \lambda\bar{A} & \bar{A} \end{pmatrix}, \quad V = \begin{pmatrix} B & \lambda^{-1}B \\ \lambda^{-1}\bar{B} & \bar{B} \end{pmatrix}. \tag{4.1}$$

Herein A and B are complex functions of f and g and the spectral parameter λ is defined as before in (3.1). In addition, the freedom of right multiplying a matrix function of k to the solution, $\Phi^{\mathrm{LP}} \to \Phi^{\mathrm{LP}}C(k)$, is fixed by demanding the normalisation

$$\Phi^{\mathrm{LP}}(\tfrac{1}{2}, \tfrac{1}{2}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \forall k. \tag{4.2}$$

The integrability condition $\Phi^{\mathrm{LP}}_{fg} = \Phi^{\mathrm{LP}}_{gf}$ of (4.1) leads via comparison of coefficients of

[1]The matrix U should not be confused with the metric function denoted by the same symbol.

λ-powers to the three independent equations

$$\begin{aligned} B_f - A_g =& A\bar{B} - B\bar{A}, && (4.3)\\ A_g =& - A\bar{B} + AB - \frac{A}{2(f+g)} - \frac{B}{2(f+g)}, && (4.4)\\ -B_f =& B\bar{A} - AB + \frac{A}{2(f+g)} + \frac{B}{2(f+g)}. && (4.5) \end{aligned}$$

Addition of (4.4) and (4.5) as well as their complex conjugates yields

$$\bar{A}_g - \bar{B}_f + A_g - B_f = 0, \tag{4.6}$$

which assures the existence of a real potential h with

$$\mathrm{d}h = (A+\bar{A})\mathrm{d}f + (B+\bar{B})\mathrm{d}g. \tag{4.7}$$

This in turn grants the existence of a complex potential E with

$$\mathrm{d}E = A\mathrm{e}^h df + B\mathrm{e}^h dg, \tag{4.8}$$

because the corresponding integrability condition is exactly (4.3). It follows that $\mathrm{d}(E+\bar{E}) = \mathrm{d}\mathrm{e}^h$ and the additive constant in h may be chosen to allow for

$$E + \bar{E} = \mathrm{e}^h. \tag{4.9}$$

Combining (4.8) and (4.9) the functions A and B can be represented in terms of E by

$$A = \frac{E_f}{E+\bar{E}}, \qquad B = \frac{E_g}{E+\bar{E}}. \tag{4.10}$$

Inserting these relations in (4.4) or (4.5), respectively, leads to the Ernst equation (2.40) in both cases. Since the Ernst equation with (4.10) also implies (4.3), the Ernst equation accompanied by (4.10) is equivalent to the system (4.3)-(4.5) derived from the integrability condition.

Using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.11}$$

the following relations between the matrices $U(\lambda)$ and $V(\lambda)$ and their values at $-\lambda$

and $\bar{\lambda}$ respectively can be stated:

$$\sigma_3 W(-\lambda)\sigma_3 = W(\lambda) = \sigma_1 \bar{W}(\bar{\lambda})\sigma_1 \quad \text{with} \qquad W = U, V. \tag{4.12}$$

Therefore, from a given column vector $v(\lambda)$ solving the LP the new solution $\sigma_3 v(-\lambda)$ can be derived because of

$$\partial_f \left(\sigma_3 v(-\lambda)\right) = \sigma_3 U(-\lambda) v(-\lambda) = U(\lambda) \left(\sigma_3 v(-\lambda)\right). \tag{4.13}$$

Analogously, $\sigma_1 \bar{v}(\bar{\lambda})$ is another column vector solving the LP. Denoting the components of v by v_1 and v_2, the sum $\Phi_1 := v(\lambda) + \sigma_1 \bar{v}(\bar{\lambda})$ can be represented by the single scalar function φ^{LP} via

$$\Phi_1 = \begin{pmatrix} v_1 + \bar{v}_2(\bar{\lambda}) \\ v_2 + \bar{v}_1(\bar{\lambda}) \end{pmatrix} =: \begin{pmatrix} \varphi^{\mathrm{LP}}(\lambda) \\ \bar{\varphi}^{\mathrm{LP}}(\bar{\lambda}) \end{pmatrix}. \tag{4.14}$$

This vector can be complemented by the (in general linearly independent) vector $\Phi_2 := \sigma_3 \Phi_1(-\lambda)$ to form the system

$$\Phi^{\mathrm{LP}} = \begin{pmatrix} \varphi^{\mathrm{LP}}(\lambda) & -\varphi^{\mathrm{LP}}(-\lambda) \\ \bar{\varphi}^{\mathrm{LP}}(\bar{\lambda}) & \bar{\varphi}^{\mathrm{LP}}(-\bar{\lambda}) \end{pmatrix}. \tag{4.15}$$

It depends only on a single scalar function φ^{LP} which shall be called 'scalar solution of the LP'. The representation (4.15) is consistent with the normalisation (4.2) providing that

$$\varphi^{\mathrm{LP}}(\tfrac{1}{2}, \tfrac{1}{2}) = 1 \quad \forall k, \tag{4.16}$$

and so the matrix solution of the LP and the RHP later on will be assumed to have this structure (4.15), which will be abbreviated by stating 'Φ^{LP} is in normal form with the scalar function φ^{LP}'.

With the identity $(\ln \det M)_x = \mathrm{Tr}(M_x M^{-1})$, holding for an arbitrary square matrix M, it is possible to write

$$(\ln \det \Phi^{\mathrm{LP}})_f = \mathrm{Tr}(\Phi^{\mathrm{LP}}_f (\Phi^{\mathrm{LP}})^{-1}) = \mathrm{Tr}(U) = A + \bar{A} = \frac{E_f + \bar{E}_f}{E + \bar{E}} = (\ln(E + \bar{E}))_f$$

and hence

$$\det \Phi^{\mathrm{LP}} = (E + \bar{E}) F_1(g, k). \tag{4.17}$$

Including the analogous statement derived from the calculation of $(\ln \det \Phi^{\text{LP}})_g$ yields

$$\det \Phi^{\text{LP}} = (E + \bar{E})F(k). \tag{4.18}$$

On the other hand, the normal form representation (4.15) implies

$$\det \Phi^{\text{LP}} = \bar{\varphi}^{\text{LP}}(\bar{\lambda})\varphi^{\text{LP}}(-\lambda) + \text{c.c.}, \tag{4.19}$$

whereby an adapted version of the '+c.c.'-symbol defined by $a(\lambda)+\text{c.c.} := a(\lambda)+\bar{a}(\bar{\lambda})$ is applied throughout this work. Using the normalisation (4.16), relation (4.19) leads to $\det \Phi^{\text{LP}}(\frac{1}{2}, \frac{1}{2}) = 2\ \forall k$. Finally, a comparison with (4.18), bearing in mind $E(\frac{1}{2}, \frac{1}{2}) = 1$, gives

$$\det \Phi^{\text{LP}} = E + \bar{E} \qquad \forall f, g, k. \tag{4.20}$$

In particular (4.20) states that the determinant of the LP solution is a function depending only on the coordinates f and g.

From (4.20) follows, that a degeneracy of the LP solution in the form (4.15) can only occur for $\Re(E) = 0$, which is supposed to be connected with a singularity of the metric (2.11). Otherwise the normal form (4.15) is a fundamental system.

4.2 Motivation of the Riemann-Hilbert problem

In order to motivate the design of the RHP, some additional properties of the LP shall be derived, whereby this work proceeds analogously to [22] and [50].

The $(1,1)$-elements of the LP equations (4.1) yield for $\lambda = 1$ with (4.10):

$$\varphi_f^{\text{LP}}(1) = \frac{E_f}{E + \bar{E}}(\varphi^{\text{LP}}(1) + \bar{\varphi}^{\text{LP}}(1)), \tag{4.21}$$

$$\varphi_g^{\text{LP}}(1) = \frac{E_g}{E + \bar{E}}(\varphi^{\text{LP}}(1) + \bar{\varphi}^{\text{LP}}(1)). \tag{4.22}$$

Summation of (4.21) and its complex conjugate leads to $\left(\ln \Re[\varphi^{\text{LP}}(1)]\right)_f = [\ln \Re(E)]_f$, which integrates to

$$\Re[\varphi^{\text{LP}}(1)] = a(g)\Re(E), \qquad a(g) \in \mathbb{R}. \tag{4.23}$$

Thereby (4.21) can be written as $\varphi_f^{\text{LP}}(1) = a(g)E_f$, which integrates with the help of (4.23) to $\varphi^{\text{LP}}(1) = a(g)E + ib(g)$ with f-independent functions $a(g), b(g) \in \mathbb{R}$. Starting the analogue procedure with (4.22) reveals that a and b also do not depend on g,

$$\varphi^{\mathrm{LP}}(1) = aE + ib, \qquad a, b \in \mathbb{R}. \tag{4.24}$$

Interestingly, the free parameters a and b in the connection of $\varphi^{\mathrm{LP}}(1)$ and E are the same that transform one solution of the Ernst equation (2.40) into another. The Ernst potential with normalisation $E(\frac{1}{2}, \frac{1}{2}) = 1$ and the LP solution with normalisation $\varphi^{\mathrm{LP}}(\frac{1}{2}, \frac{1}{2}) = 1$ are connected by

$$\varphi^{\mathrm{LP}}(1) = E \qquad \forall f, g. \tag{4.25}$$

Evaluating the $(2,1)$-elements of the LP at $\lambda = -1$ leads to equations for $\mathrm{i}\varphi^{\mathrm{LP}}(-1)$ that are formally equivalent to (4.21) and (4.22) for $\varphi^{\mathrm{LP}}(1)$. In analogy to (4.24) follows

$$\mathrm{i}\varphi^{\mathrm{LP}}(-1) = \tilde{a}E + i\tilde{b}, \qquad \tilde{a}, \tilde{b} \in \mathbb{R}. \tag{4.26}$$

Inserting the normalisations $E(\frac{1}{2}, \frac{1}{2}) = 1$ and $\varphi^{\mathrm{LP}}(\frac{1}{2}, \frac{1}{2}) = 1$ yields $\tilde{a} = 0$ and $\tilde{b} = 1$ and hence the anticipation of the RHP normalisation

$$\varphi^{\mathrm{LP}}(-1) = 1 \qquad \forall f, g. \tag{4.27}$$

The fact that the matrices U and V have no jump on the contour Γ defined in section 3.3 can be expressed as

$$\Phi^{\mathrm{LP}}_{f+}\left[\Phi^{\mathrm{LP}}_{+}\right]^{-1} = \Phi^{\mathrm{LP}}_{f-}\left[\Phi^{\mathrm{LP}}_{-}\right]^{-1}, \qquad \Phi^{\mathrm{LP}}_{g+}\left[\Phi^{\mathrm{LP}}_{+}\right]^{-1} = \Phi^{\mathrm{LP}}_{g-}\left[\Phi^{\mathrm{LP}}_{-}\right]^{-1}. \tag{4.28}$$

Defining the jump matrix $J^{\mathrm{LP}} := \left[\Phi^{\mathrm{LP}}_{-}\right]^{-1}\Phi^{\mathrm{LP}}_{+}$ yields with (4.28) the two equations

$$\Phi^{\mathrm{LP}}_{+} = \Phi^{\mathrm{LP}}_{-}J^{\mathrm{LP}}, \qquad \Phi^{\mathrm{LP}}_{f+} = \Phi^{\mathrm{LP}}_{f-}J^{\mathrm{LP}} \tag{4.29}$$

holding simultaneously, which implies $J^{\mathrm{LP}}_{f} = 0$. Together with the analogously deduced relation $J^{\mathrm{LP}}_{g} = 0$ it follows that the jump matrix J^{LP} is solely depending on the independent spectral parameter k.

Moreover, the LP solution Φ^{LP} in the normal form (4.15) obeys the identities

$$\Phi^{\mathrm{LP}}\sigma_1\left[\Phi^{\mathrm{LP}}(-\lambda)\right]^{-1} = -\sigma_3, \qquad \Phi^{\mathrm{LP}}\sigma_3\left[\bar{\Phi}^{\mathrm{LP}}(\bar{\lambda})\right]^{-1} = \sigma_1. \tag{4.30}$$

Note that since Γ is a subset of the real axis, reading off the values of $\Phi^{\mathrm{LP}}(-\lambda)$ and

$\Phi^{\text{LP}}(\bar{\lambda})$ for $\lambda \in \Gamma$ implies changing the side of the contour:

$$\lambda \in \Gamma: \qquad \left[\Phi^{\text{LP}}(-\lambda)\right]_{\pm} = \Phi^{\text{LP}}_{\mp}(-\lambda), \qquad \left[\Phi^{\text{LP}}(\bar{\lambda})\right]_{\pm} = \Phi^{\text{LP}}_{\mp}(\bar{\lambda}) = \Phi^{\text{LP}}_{\mp}(\lambda). \tag{4.31}$$

Since the expressions in (4.30) are constants, they in particular have no jump on Γ, which can be expressed as

$$\Phi^{\text{LP}}_{+}\sigma_1\left[\Phi^{\text{LP}}_{-}(-\lambda)\right]^{-1} = \Phi^{\text{LP}}_{-}\sigma_1\left[\Phi^{\text{LP}}_{+}(-\lambda)\right]^{-1}, \qquad \Phi^{\text{LP}}_{+}\sigma_3\left[\bar{\Phi}^{\text{LP}}_{-}(\bar{\lambda})\right]^{-1} = \Phi^{\text{LP}}_{-}\sigma_3\left[\bar{\Phi}^{\text{LP}}_{+}(\bar{\lambda})\right]^{-1}.$$

Inserting $J^{\text{LP}} = \left[\Phi^{\text{LP}}_{-}\right]^{-1}\Phi^{\text{LP}}_{+}$, these relations easily lead to the identities

$$\mathbb{1} = \sigma_1 J^{\text{LP}}(-\lambda)\sigma_1 J^{\text{LP}}, \qquad \mathbb{1} = \sigma_3 \bar{J}^{\text{LP}}\sigma_3 J^{\text{LP}}. \tag{4.32}$$

With the additional assumption $J^{\text{LP}}(-\lambda) = J^{\text{LP}}(\lambda)$ they imply a shape of J^{LP} analogous to (4.34).

4.3 Riemann-Hilbert problem for arbitrary polarisation

Although the previous section completely anticipated the configuration of the RHP, it did not fully prove the LP solution Φ^{LP} to be representable by a RHP. The missing link is the holomorphicity of Φ^{LP} in $\mathbb{C}_\lambda\backslash\Gamma$. Therefore the RHP will be presented as a definition in this section and the matrix version of the holomorphicity conditions (3.19) will be shown to be necessary and sufficient for the RHP solution to also solve the LP.

In this sense, the RHP for arbitrary polarisation shall be defined as the task to find the matrix $\Phi(f, g; \lambda)$ analytic in $\mathbb{C}_\lambda\backslash\Gamma$ and satisfying on Γ the jump equation

$$\Phi_+ = \Phi_- J(k). \tag{4.33}$$

The jump matrix $J(k)$ has the form

$$J(k) = \begin{pmatrix} \alpha(k) & \beta(k) \\ -\beta(k) & \bar{\alpha}(k) \end{pmatrix}, \qquad \beta \in \mathbb{R},\ \bar{\alpha}\alpha + \beta^2 = 1, \tag{4.34}$$

exhibiting only one complex degree of freedom α. It is sufficient to consider the jump matrix to be identical in both sheets of the k-surface, which is equivalent to

$$J(-\lambda) = J(\lambda). \tag{4.35}$$

On Γ the chosen form (4.34) with (4.35) fulfils the identities[2]

$$\mathbb{1} = \sigma_1 J(-\lambda)\sigma_1 J, \qquad \mathbb{1} = \sigma_3 \bar{J}\sigma_3 J. \tag{4.36}$$

Insertion of $J(-\lambda) = \Phi_-^{-1}(-\lambda)\Phi_+(-\lambda)$ leads to

$$\Phi_-(-\lambda)\sigma_1 = \Phi_+(-\lambda)\sigma_1 J, \tag{4.37}$$

which is the RHP jump equation (4.33) for the matrix $\Phi(-\lambda)\sigma_1$. From (4.37) and the analogous relation derived from the second equation in (4.36) it is obvious that $\Phi(-\lambda)\sigma_1$ and $\bar{\Phi}(\bar{\lambda})\sigma_3$ obey the same jump equation (4.33) as Φ. This statement holds already for row vectors. In the same way as in section 4.1 using column vectors, it can be deduced that starting from an arbitrary row vector w solving (4.33) a matrix solution

$$\Phi = \begin{pmatrix} w - w(-\lambda)\sigma_1 \\ [\bar{w}(-\bar{\lambda}) - \bar{w}(-\bar{\lambda})\sigma_1]\sigma_3 \end{pmatrix} = \begin{pmatrix} \varphi(\lambda) & -\varphi(-\lambda) \\ \bar{\varphi}(\bar{\lambda}) & \bar{\varphi}(-\bar{\lambda}) \end{pmatrix} \tag{4.38}$$

to (4.33) can be constructed. This matrix Φ is in normal form with the scalar function φ as introduced in section 4.1 and only such solutions of the RHP shall be considered from now on. Within the representation (4.15) the jump equation (3.12) is equivalent to the single scalar jump equation

$$\varphi_+ = \alpha\varphi_- + \beta\varphi_+(-\lambda). \tag{4.39}$$

This is a special case of the analogue statement for the CRHP which will be proven at the end of section 5.7.

A general solution of (4.33) is only determined up to left multiplication with an arbitrary matrix $M(f,g)$. Within the class of solutions representable by the normal form (4.38) this matrix is confined to the diagonal form $M = \mathrm{diag}(m, \bar{m})$, which reduces the freedom in the solution to multiplication of φ with $m(f,g)$. Therefore the RHP (4.33) with a solution in the normal form (4.38) can be normalised by demanding

$$\varphi(-1) = 1 \qquad \forall f, g. \tag{4.40}$$

For $f = \frac{1}{2} = g$ the contour Γ vanishes and suggests $\Phi = \mathrm{const}$, confer the discussion in section 6.7. Considering (4.40) this reproduces the LP normalisation $\varphi(\frac{1}{2}, \frac{1}{2}) = 1$.

[2]Herein $J(-\lambda)$ is not replaced by J for a better illustration of the transition to the corresponding CRHP relations in (5.46).

Since the jump matrix J is only depending on the independent spectral parameter k, the expressions $\Phi_f\Phi^{-1}$ and $\Phi_g\Phi^{-1}$ feature no jump on Γ. As the investigation of the CRHP will show, there exist solutions Φ of the RHP in normal form (4.38) with a scalar function φ and fulfilling even the holomorphicity conditions

$$\begin{aligned} &\Phi_f\Phi^{-1} \text{ is holomorphic in } \mathbb{C}_\lambda\backslash\{\infty\},\\ &\Phi_g\Phi^{-1} \text{ is holomorphic in } \mathbb{C}_\lambda\backslash\{0\}. \end{aligned} \tag{4.41}$$

Also in the arbitrary polarised case a power series expansion of $\Phi_g\Phi^{-1}$ in λ analogue to (3.18) can be derived, but with matrix valued coefficients. If the holomorphicity conditions hold, this series has to be cut off after the constant member yielding

$$\Phi_g\Phi^{-1} = M_{-1}(f,g)\lambda^{-1} + M_0(f,g). \tag{4.42}$$

With the help of

$$\Phi(-1) = \begin{pmatrix} 1 & -\varphi(1) \\ 1 & \bar{\varphi}(1) \end{pmatrix}, \qquad \Phi(1) = \begin{pmatrix} \varphi(1) & -1 \\ \bar{\varphi}(1) & 1 \end{pmatrix}$$

the evaluation of (4.42) at $\lambda = \pm 1$ yields

$$M_0 - M_{-1} = \Phi_g(-1)\Phi^{-1}(-1) = \frac{1}{\varphi(1)+\bar{\varphi}(1)} \begin{pmatrix} \varphi_g(1) & -\varphi_g(1) \\ -\bar{\varphi}_g(1) & \bar{\varphi}_g(1) \end{pmatrix}, \tag{4.43}$$

$$M_0 + M_{-1} = \Phi_g(1)\Phi^{-1}(1) = \frac{1}{\varphi(1)+\bar{\varphi}(1)} \begin{pmatrix} \varphi_g(1) & \varphi_g(1) \\ \bar{\varphi}_g(1) & \bar{\varphi}_g(1) \end{pmatrix}. \tag{4.44}$$

Solving this system for M_0 and M_{-1} leads with (4.42) to

$$\Phi_g\Phi^{-1} = \frac{1}{\varphi(1)+\bar{\varphi}(1)} \begin{pmatrix} \varphi_g(1) & \lambda^{-1}\varphi_g(1) \\ \lambda^{-1}\bar{\varphi}_g(1) & \bar{\varphi}_g(1) \end{pmatrix}, \tag{4.45}$$

which is the second line of the LP (4.1) with $B = \varphi_g(1)/[\varphi(1)+\bar{\varphi}(1)]$. The first line follows analogously with $A = \varphi_f(1)/[\varphi(1)+\bar{\varphi}(1)]$. From the representation (4.10) of A and B in terms of E the relation $\varphi(1) = aE + ib$ with undetermined $a, b \in \mathbb{R}$ can be derived as in section 4.2. Therefore the scalar RHP solution φ with $\varphi(\frac{1}{2},\frac{1}{2}) = 1$ is connected to an Ernst potential with normalisation $E(\frac{1}{2},\frac{1}{2}) = 1$ by

$$\varphi(1) = E \qquad \forall f, g. \tag{4.46}$$

In conclusion, a RHP solution Φ in normal form (4.38) with the scalar solution φ, normalised by $\varphi(-1) = 1$ and fulfilling the holomorphicity conditions (4.41) is simultaneously also a solution Φ^{LP} of the LP (4.1) normalised by (4.16). A solution to the normalised Ernst equation is then given by $E = \varphi(1) = \varphi^{\mathrm{LP}}(1)$. In the next section the ISM is completed by relating the initial values $E(f, g = \frac{1}{2})$ and $E(f = \frac{1}{2}, g)$ to the jump functions α and β in a way that this Ernst potential inferred by (4.46) indeed matches the initial data.

For $\beta = 0$ the RHP (4.33) reduces to the collinearly polarised case with $\mu^{\parallel} := -\mathrm{i}\ln\alpha$, $\Phi^{\parallel} := \varphi = \bar{\varphi}(\bar{\lambda})$ and $\psi := \frac{1}{2}\ln E$.

4.4 Calculation of the jump matrix from initial data

Since the ISM aims at constructing a RHP which has the same solution as the LP connected with the IVP, it is necessary to require the RHP's jump matrix to be $J = J^{\mathrm{LP}} = \left[\Phi_-^{\mathrm{LP}}\right]^{-1}\Phi_+^{\mathrm{LP}}$. In matrix form this demand reads

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \varphi_-^{\mathrm{LP}}(\lambda) & -\varphi_+^{\mathrm{LP}}(-\lambda) \\ \bar{\varphi}_+^{\mathrm{LP}}(\bar{\lambda}) & \bar{\varphi}_-^{\mathrm{LP}}(-\bar{\lambda}) \end{pmatrix}^{-1} \begin{pmatrix} \varphi_+^{\mathrm{LP}}(\lambda) & -\varphi_-^{\mathrm{LP}}(-\lambda) \\ \bar{\varphi}_-^{\mathrm{LP}}(\bar{\lambda}) & \bar{\varphi}_+^{\mathrm{LP}}(-\bar{\lambda}) \end{pmatrix}. \tag{4.47}$$

and remembering $J(-\lambda) = J(\lambda)$ this can be solved for the jump functions:

$$\alpha = \frac{\bar{\varphi}_-^{\mathrm{LP}}(\lambda)\varphi_+^{\mathrm{LP}}(-\lambda) + \varphi_+^{\mathrm{LP}}(\lambda)\bar{\varphi}_-^{\mathrm{LP}}(-\lambda)}{\bar{\varphi}_+^{\mathrm{LP}}(\lambda)\varphi_+^{\mathrm{LP}}(-\lambda) + \bar{\varphi}_-^{\mathrm{LP}}(-\lambda)\varphi_-^{\mathrm{LP}}(\lambda)}, \quad \beta = \frac{\bar{\varphi}_+^{\mathrm{LP}}(\lambda)\varphi_+^{\mathrm{LP}}(\lambda) - \varphi_-^{\mathrm{LP}}(\lambda)\bar{\varphi}_-^{\mathrm{LP}}(\lambda)}{\bar{\varphi}_+^{\mathrm{LP}}(\lambda)\varphi_+^{\mathrm{LP}}(-\lambda) + \bar{\varphi}_-^{\mathrm{LP}}(-\lambda)\varphi_-^{\mathrm{LP}}(\lambda)}. \tag{4.48}$$

These relations simplify in the points $\lambda = \infty$ and $\lambda = 0$ where all arguments of φ^{LP} coincide. For a given $k \in [-\frac{1}{2}, \frac{1}{2}]$ it is possible to arrange $\lambda = \infty$ by $f = -k$ and $\lambda = 0$ by $g = k$. At this stage the additional denominations $\alpha_{1/2} := \alpha|_{\Gamma_{1/2}^{(k)}}$ and $\beta_{1/2} := \beta|_{\Gamma_{1/2}^{(k)}}$ for the elements of $J_{1/2} := J|_{\Gamma_{1/2}^{(k)}}$ as well as $\chi^{\mathrm{LP}} := \bar{\varphi}^{\mathrm{LP}}(\bar{\lambda})$ shall be introduced. Since $\infty \in \Gamma_1$ and $0 \in \Gamma_2$, the evaluation of (4.48) at $\lambda = \infty$ and $\lambda = 0$ then yields

$$\alpha_1 = \frac{2\chi_+^{\mathrm{LP}}(\infty)\varphi_+^{\mathrm{LP}}(\infty)}{|\varphi_+^{\mathrm{LP}}(\infty)|^2 + |\chi_+^{\mathrm{LP}}(\infty)|^2}, \qquad \beta_1 = \frac{|\varphi_+^{\mathrm{LP}}(\infty)|^2 - |\chi_+^{\mathrm{LP}}(\infty)|^2}{|\varphi_+^{\mathrm{LP}}(\infty)|^2 + |\chi_+^{\mathrm{LP}}(\infty)|^2}, \tag{4.49}$$

$$\alpha_2 = \frac{2\chi_+^{\mathrm{LP}}(0)\varphi_+^{\mathrm{LP}}(0)}{|\varphi_+^{\mathrm{LP}}(0)|^2 + |\chi_+^{\mathrm{LP}}(0)|^2}, \qquad \beta_2 = \frac{|\varphi_+^{\mathrm{LP}}(0)|^2 - |\chi_+^{\mathrm{LP}}(0)|^2}{|\varphi_+^{\mathrm{LP}}(0)|^2 + |\chi_+^{\mathrm{LP}}(0)|^2}. \tag{4.50}$$

In the next step, for a given $k \in [-\frac{1}{2}, \frac{1}{2}]$ the values of χ_+^{LP} and φ_+^{LP} at $\lambda = 0$ and $\lambda = \infty$ shall be calculated by integration of the LP for the first column $(\varphi^{\mathrm{LP}}, \chi^{\mathrm{LP}})^T$ of

the matrix solution Φ^{LP} in the (f,g)-plane. The starting point of this integration is conveniently chosen as $(\frac{1}{2},\frac{1}{2})$, where the normalisation (4.16) defines $\varphi^{\mathrm{LP}}=1=\chi^{\mathrm{LP}}$. As stated above, $\lambda=\infty$ is achieved at $(-k,\frac{1}{2})$ and $\lambda=0$ at $(\frac{1}{2},k)$. Furthermore, choosing the integration path along $g=\frac{1}{2}$ and $f=\frac{1}{2}$ respectively leads to two major simplifications. First of all, the LP can be reduced to a single ODE in both cases, and secondly only values on the boundaries of the IVP are used. Therefore $\alpha_{1/2}$ and $\beta_{1/2}$ can indeed be calculated from the initial values alone.
For the integration to result in the inner limit at $f\to -k$ or respectively $g\to k$, the point k has to be conceived lying infinitesimally above the real axis in the upper sheet of the k-surface or, equivalently, infinitesimally below the real axis in the lower sheet (black region in figure 3.3). There the value of λ is purely imaginary, and its imaginary part is positive. Therefore it is appropriate to define the positive quantities

$$\lambda^f := -\mathrm{i}\lambda = \sqrt{\frac{\frac{1}{2}-k}{f+k}} > 0, \qquad \lambda^g := \mathrm{i}/\lambda = \sqrt{\frac{\frac{1}{2}+k}{g-k}} > 0. \tag{4.51}$$

Hence the ODEs for the integration procedure described above read

$$g=\tfrac{1}{2}: \qquad \begin{pmatrix}\varphi_+^{\mathrm{LP}}\\ \chi_+^{\mathrm{LP}}\end{pmatrix}_{,f} = \begin{pmatrix}A & \mathrm{i}\lambda^f A\\ \mathrm{i}\lambda^f\bar{A} & \bar{A}\end{pmatrix}\begin{pmatrix}\varphi_+^{\mathrm{LP}}\\ \chi_+^{\mathrm{LP}}\end{pmatrix} \tag{4.52}$$

$$f=\tfrac{1}{2}: \qquad \begin{pmatrix}\varphi_+^{\mathrm{LP}}\\ \chi_+^{\mathrm{LP}}\end{pmatrix}_{,g} = \begin{pmatrix}B & -\mathrm{i}\lambda^g B\\ -\mathrm{i}\lambda^g\bar{B} & \bar{B}\end{pmatrix}\begin{pmatrix}\varphi_+^{\mathrm{LP}}\\ \chi_+^{\mathrm{LP}}\end{pmatrix}, \tag{4.53}$$

where (4.52) shall be integrated from $f=\frac{1}{2}$ to $f=-k$ and (4.53) shall be integrated from $g=\frac{1}{2}$ to $g=k$. The starting value is in both cases $(1,1)^T$ and A for $g=\frac{1}{2}$ as well as B for $f=\frac{1}{2}$ are given in terms of the initial values by (4.10). Subsequently $\alpha_{1/2}$ and $\beta_{1/2}$ are calculated via (4.49) and (4.50), which corresponds to step (i) of the ISM scheme in figure 3.1.

4.5 The boundary values of the jump matrix

The boundary coefficients $A_\mathrm{b}, B_\mathrm{b} \in \mathbb{C}$ of the functions A and B as well as their amplitudes $\rho_{1/2} \in \mathbb{R}$ and phases $\phi_A, \phi_B \in \mathbb{R}$ shall be defined by

$$\begin{aligned} A_\mathrm{b} &:= \rho_1 \mathrm{e}^{\mathrm{i}\phi_A} := \lim_{(f,g)\to(\frac{1}{2},\frac{1}{2})} \left[\sqrt{\tfrac{1}{2} - f}A\right], \\ B_\mathrm{b} &:= \rho_2 \mathrm{e}^{\mathrm{i}\phi_B} := \lim_{(f,g)\to(\frac{1}{2},\frac{1}{2})} \left[\sqrt{\tfrac{1}{2} - g}B\right]. \end{aligned} \tag{4.54}$$

Considering (4.10), the colliding wave conditions (2.48) can be stated as

$$\rho_1 = \sqrt{\frac{k_1}{2}}, \qquad \rho_2 = \sqrt{\frac{k_2}{2}}. \tag{4.55}$$

From the domain (2.50) of $k_{1/2}$ follows

$$\tfrac{1}{2} \leq \rho_{1/2} < 2^{-\frac{1}{2}}. \tag{4.56}$$

In order to calculate the boundary values $J(\pm\lambda_1)$ of the jump matrix, the ODE system (4.52) is examined for $k = -\frac{1}{2} + \epsilon$. Substituting $f = \frac{1}{2} - \delta$, its leading order in δ is given by

$$\begin{pmatrix} \varphi_+^\mathrm{LP} \\ \chi_+^\mathrm{LP} \end{pmatrix}_\delta = -\delta^{-\frac{1}{2}} \begin{pmatrix} A_\mathrm{b} & \mathrm{i}\lambda^f A_\mathrm{b} \\ \mathrm{i}\lambda^f \bar{A}_\mathrm{b} & \bar{A}_\mathrm{b} \end{pmatrix} \begin{pmatrix} \varphi_+^\mathrm{LP} \\ \chi_+^\mathrm{LP} \end{pmatrix}, \qquad f = \frac{1}{2} - \delta,\ g = \frac{1}{2}, \tag{4.57}$$

which has to be integrated from $\delta = 0$, where $\chi_+^\mathrm{LP} = 1 = \varphi_+^\mathrm{LP}$ holds, to $\delta = \epsilon$. For $0 < \delta < \epsilon \ll 1$ it is $\lambda^f \gg 1$ and (4.57) reduces in leading order to

$$(\varphi_+^\mathrm{LP})_\delta = -\mathrm{i}[\delta(\epsilon-\delta)]^{-\frac{1}{2}} A_\mathrm{b} \chi_+^\mathrm{LP}, \qquad (\chi_+^\mathrm{LP})_\delta = -\mathrm{i}[\delta(\epsilon-\delta)]^{-\frac{1}{2}} \bar{A}_\mathrm{b} \varphi_+^\mathrm{LP}. \tag{4.58}$$

Substituting $s = 2\arcsin(\sqrt{\delta/\epsilon})$ yields

$$(\varphi_+^\mathrm{LP})_s = -\mathrm{i}A_\mathrm{b}\chi_+^\mathrm{LP}, \qquad (\chi_+^\mathrm{LP})_s = -\mathrm{i}\bar{A}_\mathrm{b}\varphi_+^\mathrm{LP}. \tag{4.59}$$

which has to be integrated from $s = 0$, where $\chi_+^\mathrm{LP} = 1 = \varphi_+^\mathrm{LP}$, to $s = \pi$. The solution matching these initial conditions is

$$\begin{pmatrix} \varphi_+^\mathrm{LP} \\ \chi_+^\mathrm{LP} \end{pmatrix} = \begin{pmatrix} \cos(|A_\mathrm{b}|s) - \frac{\mathrm{i}A_\mathrm{b}}{|A_\mathrm{b}|}\sin(|A_\mathrm{b}|s) \\ \cos(|A_\mathrm{b}|s) - \frac{\mathrm{i}|A_\mathrm{b}|}{A_\mathrm{b}}\sin(|A_\mathrm{b}|s) \end{pmatrix}. \tag{4.60}$$

Inserting $s = \pi$ yields $\varphi_+^{\text{LP}}(\infty)$ and $\chi_+^{\text{LP}}(\infty)$ with a trivial limit for $\epsilon \to 0$. With $|A_\text{b}|= \rho_1$ and $\arg A_\text{b} = \phi_A$ the boundary values of the jump functions at $\pm\lambda_1$ are then given by (4.49):

$$\alpha_{1\text{b}} := \alpha_1(k = -\tfrac{1}{2}) = \cos(2\pi\rho_1) - \mathrm{i}\cos(\phi_A)\sin(2\pi\rho_1), \tag{4.61}$$

$$\beta_{1\text{b}} := \beta_1(k = -\tfrac{1}{2}) = \sin(\phi_A)\sin(2\pi\rho_1). \tag{4.62}$$

Analogously, the investigation of the other ODE (4.53) for $k = \frac{1}{2} - \epsilon$ leads to

$$\alpha_{2\text{b}} := \alpha_2(k = \tfrac{1}{2}) = \cos(2\pi\rho_2) + \mathrm{i}\cos(\phi_B)\sin(2\pi\rho_2), \tag{4.63}$$

$$\beta_{2\text{b}} := \beta_2(k = \tfrac{1}{2}) = -\sin(\phi_B)\sin(2\pi\rho_2). \tag{4.64}$$

With relation (4.56) the range of $\Re(\alpha_{1\text{b}})$ and $\Re(\alpha_{2\text{b}})$ is given by

$$-1 \leq \Re(\alpha_{1/2\text{b}}) < \cos(\sqrt{2}\pi) < 0. \tag{4.65}$$

The equality in (4.65) is reached for impulsive waves:

$$\alpha_{1/2\text{b}} = -1 \quad \Leftrightarrow \quad \rho_{1/2} = \tfrac{1}{2} \quad \Leftrightarrow \quad k_{1/2} = \tfrac{1}{2} \quad \Leftrightarrow \quad n_{1/2} = 2. \tag{4.66}$$

A RHP with a Hölder continuous[3] jump matrix J has a unique solution under certain additional conditions [23]. But $\alpha_{1/2\text{b}} = 1$, which would be necessary for a continuous connection to the jump matrix $\mathbb{1}$ on a continued contour, is not consistent with the colliding wave conditions demanding (4.65). Therefore the RHP associated with the IVP for colliding plane GWs is discontinuous and thus its solution Φ features singularities, which will occur in different appearances. In anticipation of the rigorous treatment in chapter 5 the situation shall be illustrated using the toy model of a RHP with constant purely multiplicative jump on only one contour, $\alpha_2 = \alpha_{2\text{b}} = \mathrm{e}^{-2\pi\mathrm{i}\rho_2}$, $\alpha_1 = 1$, $\beta = 0$.
In a first step it is beneficial to define the fractions

$$L_1 := \frac{\lambda_1 + \lambda}{\lambda_1 - \lambda}, \qquad L_2 := \frac{\lambda + \lambda_2}{\lambda - \lambda_2}. \tag{4.67}$$

They obey the relations

$$\bar{L}_{1/2}(\bar{\lambda}) = L_{1/2}(\lambda) = L_{1/2}^{-1}(-\lambda) \tag{4.68}$$

[3]A Hölder continuous function f satisfies the Hölder condition $|f(x) - f(y)|\leq C|x - y|^\gamma$ for some $C \in \mathbb{R}, \gamma \in (0, 1]$ and all x, y. A matrix is Hölder continuous if all of its components are Hölder continuous. In particular the root-like function x^ρ with $0 < \rho < 1$ is Hölder continuous.

which will be used extensively in the remainder of this work. The terms $L_{1/2}^{\rho_{1/2}}$ and $L_{1/2}^{\rho_{1/2}-1}$ as well as their reciprocals feature a branch cut along $\Gamma_{1/2}$. They will be used as functions only in the λ-sheet with real value at $\lambda = 1$, where they shall be regarded as having a jump on the contour $\Gamma_{1/2}$. Their inner and outer limits at the contour are

$$\begin{array}{lll} \text{for } \lambda \in \Gamma_1, & (L_1^{\rho_1})_+ = \mathrm{e}^{\pi i \rho_1}|L_1^{\rho_1}|, & (L_1^{\rho_1})_- = \mathrm{e}^{-\pi i \rho_1}|L_1^{\rho_1}| \\ \text{for } \lambda \in \Gamma_2, & (L_2^{\rho_2})_+ = \mathrm{e}^{-\pi i \rho_2}|L_2^{\rho_2}|, & (L_2^{\rho_2})_- = \mathrm{e}^{\pi i \rho_2}|L_2^{\rho_2}| \end{array} \tag{4.69}$$

and analogous for $L_{1/2}^{\rho_{1/2}-1}$. This implies $L_{1/2}^{\rho_{1/2}}(-\lambda) = L_{1/2}^{-\rho_{1/2}}$. Now for the toy model RHP with $\alpha_2 = \alpha_{2\mathrm{b}} = \mathrm{e}^{-2\pi \mathrm{i} \rho_2}$, $\alpha_1 = 1$, $\beta = 0$ the scalar jump equation (4.39) decouples with respect to $\varphi(\lambda)$ and $\varphi(-\lambda)$. On Γ_2 it simply reads $\varphi_+ = \mathrm{e}^{-2\pi \mathrm{i} \rho_2}\varphi_-$ and features the set of unnormalised scalar solutions $\varphi_{2n} = L_2^{\rho_2 - n}$ with $n \in \mathbb{Z}$. For $n \leq 0$ they have at $\lambda = \lambda_2$ the singular behaviour $\varphi_{2n} \sim (\lambda - \lambda_2)^{-\rho_2 - n}$, which shall be expressed in the remainder of this work by saying that they diverge with divergence exponent $\rho_2 - n$.
The uniqueness of the solution to a CRHP stems from the requirement that it should not have any poles or zeros. But as the toy model shows, for the discontinuous RHP the divergences at the contour endpoints are generic. If the requirement of regularity at the contour endpoints is completely dropped, all φ_{2n} are solutions of the toy model RHP. Also in the general case an infinite series of solutions (not in normal form) can be generated by multiplying powers of L_1 and L_2 to a given solution Φ. However, it seems reasonable to restrict the search to the 'most regular' solutions with a divergence exponent lower than one since only in these cases the scalar solution is representable via a Cauchy integral over its additive jump.
For the toy model the restriction to divergence exponents lower than one yields the two independent scalar solutions $\varphi_{20} = L_2^{\rho_2}$ and $\varphi_{21} = L_2^{\rho_2 - 1}$. The first solution has the singular behaviour $\varphi_{20} \sim (\lambda - \lambda_2)^{-\rho_2}$ at $\lambda = \lambda_2$ and the second one has the singular behaviour $\varphi_{21} \sim (\lambda + \lambda_2)^{\rho_2 - 1}$ at $\lambda = -\lambda_2$.
Keeping the restriction of divergence exponents lower than one, also for a more general Lipschitz continuous[4] jump matrix diagonal at $\pm\lambda_2$ the scalar jump equation in a neighbourhood of λ_2 can only be solved by a function having in λ_2 either a branch point with divergence exponent ρ_2 or a zero. Analogously, at $-\lambda_2$ the scalar solution φ must have either a branch point with divergence exponent $1 - \rho_2$ or a zero. For a Lipschitz continuous jump matrix diagonal at the endpoints $\pm\lambda_1$ of the other contour part, the scalar solution φ has at λ_1 and $-\lambda_1$ branch points with divergence exponent

[4] A Lipschitz continuous function f satisfies the Lipschitz condition $|f(x) - f(y)| \leq C|x - y|$ for some $C \in \mathbb{R}$ and all x, y. A matrix is Lipschitz continuous if all of its components are Lipschitz continuous. Each Lipschitz continuous function is also Hölder continuous. In particular the root-like function x^ρ with $0 < \rho < 1$ is not Lipschitz continuous.

ρ_1 or $1 - \rho_1$ respectively or zeros.

In the general case $\beta_{1/2} \neq 0$, the scalar jump equations (4.39) for the contour endpoints $\pm\lambda_{1/2}$ become coupled and φ is expected to diverge at both endpoints of $\Gamma_{1/2}$ with the same divergence exponent $\rho_{1/2}$ or $1-\rho_{1/2}$. In a linear combination of these solutions the greater divergence exponent $\rho_{1/2}$ will dominate the behaviour at the contour endpoints.

4.6 RHP integral equations for a singular additive jump

Using its additive jump function $\mu^\varphi(\lambda')$, φ can be expressed as the Cauchy type integral

$$\varphi(\lambda) = 1 + \frac{1}{2\pi\mathrm{i}} \int_\Gamma \left(\frac{1}{\lambda' - \lambda} - \frac{1}{\lambda' + 1} \right) \mu^\varphi(\lambda')\mathrm{d}\lambda' \tag{4.70}$$

with the correct normalisation $\varphi(-1) = 1$. With the Cauchy principal value $⨍$ the inner and outer limit of the integral

$$I(\lambda) = \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mu(\lambda')\mathrm{d}\lambda'}{\lambda' - \lambda} \tag{4.71}$$

over a contour C through λ can, according to the Sokhotski-Plemelji formula (confer e.g. [23] or [51]), be represented as

$$I_+(\lambda) = \frac{1}{2\pi\mathrm{i}} ⨍_C \frac{\mu(\lambda')\mathrm{d}\lambda'}{\lambda' - \lambda} + \frac{1}{2}\mu(\lambda), \qquad I_-(\lambda) = \frac{1}{2\pi\mathrm{i}} ⨍_C \frac{\mu(\lambda')\mathrm{d}\lambda'}{\lambda' - \lambda} - \frac{1}{2}\mu(\lambda). \tag{4.72}$$

Roughly speaking, the jump $\mu(\lambda)$ of the integral $I(\lambda)$ is equally partitioned to both sides of the contour if C runs straight through the point λ. Insertion of (4.72) into (4.39) with the kernel $F(\lambda, \lambda') := (\lambda' - \lambda)^{-1} - (\lambda' + 1)^{-1}$ from (4.70) as well as $\mu^\varphi_{1/2} := \mu^\varphi|_{\Gamma_{1/2}}$ yields the integral equation

$$\begin{aligned}
1 + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_1} F(\lambda, \lambda')\mu^\varphi_1(\lambda')\mathrm{d}\lambda' + \frac{1}{2\pi\mathrm{i}} ⨍_{\Gamma_2} F(\lambda, \lambda')\mu^\varphi_2(\lambda')\mathrm{d}\lambda' + \tfrac{1}{2}\mu^\varphi_2(\lambda) = \\
\alpha \left[1 + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_1} F(\lambda, \lambda')\mu^\varphi_1(\lambda')\mathrm{d}\lambda' + \frac{1}{2\pi\mathrm{i}} ⨍_{\Gamma_2} F(\lambda, \lambda')\mu^\varphi_2(\lambda')\mathrm{d}\lambda' - \tfrac{1}{2}\mu^\varphi_2(\lambda) \right] \\
+ \beta \left[1 + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_1} F(-\lambda, \lambda')\mu^\varphi_1(\lambda')\mathrm{d}\lambda' + \frac{1}{2\pi\mathrm{i}} ⨍_{\Gamma_2} F(-\lambda, \lambda')\mu^\varphi_2(\lambda')\mathrm{d}\lambda' + \tfrac{1}{2}\mu^\varphi_2(-\lambda) \right]
\end{aligned} \tag{4.73}$$

for $\lambda \in \Gamma_2$. A similar relation is obtained for $\lambda \in \Gamma_1$. For each set of fixed coordinate values (f, g) these are coupled inhomogeneous linear singular integral equations of the second kind[5]. For given jump functions α and β they can in principle be used to

[5] A singular integral equation contains an improper integral like a principal value; an integral equation of the second kind contains the unknown function both inside and outside the integral. An

determine the additive jump μ^φ which via the Cauchy integral (4.70) yields the scalar RHP solution φ. This corresponds to step (iia) in figure 3.1.

These integral equations may be solved analytically for some special cases, in general they can be evaluated by an expansion in Chebyshev polynomials. But due to the expected inverse root-like divergences of the scalar solution φ at the contour endpoints discussed in section 4.5, also its additive jumps μ^φ have to diverge, e.g. at $\lambda = \pm\lambda_2$ like $\mu_2^\varphi \sim (\lambda \mp \lambda_2)^{-\rho_2}$. Therefore the integral equations of the RHP have to be evaluated for singular unknown functions.

The integral over Γ_2 is caused by the inverse root-like behaviour of μ_2^φ to diverge for $\lambda \to \pm\lambda_2$, which makes it difficult to evaluate the integral equation at the contour endpoints. A naive ansatz for an expansion of μ_2^φ may include the terms $a(\lambda-\lambda_2)^{-\rho_2}$ and $b(\lambda+\lambda_2)^{-\rho_2}$, but a precise approximation of the coefficients a and b seems challenging. Inaccuracies in a and b may on the other hand have a strong influence on the scalar solution φ. Moreover, the consideration of the toy model in section 4.5 showed that the solution of the integral equation is not unique, which might complicate the task of approximately solving (4.73).

Nevertheless, Shahmorad and Ahdiaghdam presented an expansion scheme for singular integral equations with divergent expansion functions built up of Chebyshev polynomials of several kinds with astonishing accuracy [52], see also [53]. An application of such methods on the RHP's integral equation (4.73) is an interesting direction of further investigations, but they have to be considerably adapted to match the case of two different contours.

This work will proceed with a transformation to the CRHP, which is necessary to prove the existence of RHP solutions fulfilling the holomorphcity conditions (4.41) and to fully understand their properties. This will lead to an integral equation for the continuous additive jump function of the scalar CRHP solution, which has to be solved in step (iib) of the solution scheme in figure 3.1. This integral equation appears to be better suited at least for a naive numerical approximation.

inhomogeneous integral equation has terms which do not contain the unknown function.

5 Transformation to a continuous Riemann-Hilbert problem

The transformation to a continuous Riemann-Hilbert problem is inspired by a recipe described by Vekua in [23], where a jump matrix discontinuity is removed through multiplication with an appropriate branch cut perpendicular to the contour. The RHP described in section 4.3 features four discontinuities at the endpoints of the partial contours $\Gamma_{1/2}$. Transformations are introduced which simultaneously remove the two discontinuities at the endpoints of a single partial contour in two different ways based on the two functions $L_{1/2}^{\rho_{1/2}}$ and $L_{1/2}^{\rho_{1/2}-1}$. Thereby iterated transformations lead to four different CRHP which are conjectured to completely classify the ambiguity of the RHP. In addition, two other types of transformation are needed to reach the CRHP. The three basic transformations will be first introduced in a generic manner and secondly their role in the sequence of transformations leading to the CRHP is illustrated. Subsequently the full transformation consisting of these basic transformations is assembled and its properties are studied.

5.1 The extended Riemann-Hilbert problem

In a first step, the RHP is generalised into the extended RHP (ERHP)

$$\Phi_+ = \Phi_- G \tag{5.1}$$

with the slightly modified 'generalised jump matrix'

$$G = \begin{pmatrix} \alpha & \gamma+\beta \\ \gamma-\beta & \bar{\alpha} \end{pmatrix}, \quad \gamma, \beta \in \mathbb{R}. \tag{5.2}$$

This matrix shall be defined on the whole real λ-axis, which serves as the contour of the ERHP denoted by $\Gamma_{\Re}$. This is indeed the most obvious way to connect Γ_1 and Γ_2 to a closed contour on the λ-sphere. The corresponding contour on the k-surface consists of the real axis outside the branch cut $[-f, g]$ in both sheets. Furthermore, the generalised jump matrix G shall be featuring the properties

$$\det G = \alpha\bar{\alpha} + \beta^2 - \gamma^2 = 1, \qquad \alpha(-\lambda) = \alpha, \beta(-\lambda) = \beta, \gamma(-\lambda) = -\gamma. \tag{5.3}$$

In contrast to the RHP jump matrix $J(k)$, the ERHP jump matrix G will acquire a dependence on the coordinates f and g in the course of the transformation to the CRHP. For $\gamma \neq 0$ the ERHP jump matrix G is neither unitary nor symmetric in λ any more. For clarity of notation, the ERHP is treated without normalisation. Out of the resulting scalar RHP solution φ the Ernst potential $E = \varphi(1)/\varphi(-1)$ with the right normalisation (2.41) can be easily derived afterwards. The ERHP jump matrix induced by the RHP jump matrix J is denoted as

$$G_J := \begin{cases} J, & \lambda \in \Gamma; \\ \mathbb{1} & \text{else.} \end{cases} \tag{5.4}$$

As in [23] it is necessary to demand the jump matrix J of the initial RHP to be Lipschitz continuous at the endpoints of Γ. Thus for later reference it can be stated

$$\lim_{\lambda \to \lambda_{1/2}} (\lambda - \lambda_{1/2})^x (J - J(\lambda_{1/2})) = 0 \quad \text{for} \quad |x| < 1. \tag{5.5}$$

Moreover, for technical reasons the derivation of the CRHP given in this work is restricted to non-impulsive waves by demanding

$$\tfrac{1}{2} < \rho_{1/2} < 2^{-\frac{1}{2}} \tag{5.6}$$

and hence excluding the case $\rho_1 = \frac{1}{2} \vee \rho_2 = \frac{1}{2}$, where $L_{1/2}^{\rho_{1/2}}$ and $L_{1/2}^{\rho_{1/2}-1}$ become the inverse of each other.

In the following sections transformations like $G \to G'$ will be described by expressing the new jump functions α', β' and γ' in terms of the old ones. It can be easily checked that the relations (5.3) remain valid in all cases. The effect of the transformations leading to the CRHP are illustrated by means of an example in figure 5.1.

5.2 Rotation transformation

A rotation transformation, which converts (5.1) to $\Phi'_+ = \Phi'_- G'$, shall be defined by

$$\Phi' = \Phi R_\delta, \quad G' = R_\delta^{-1} G R_\delta, \qquad R_\delta = \begin{pmatrix} \cos\delta & \mathrm{i}\sin\delta \\ \mathrm{i}\sin\delta & \cos\delta \end{pmatrix}. \tag{5.7}$$

The normal form of Φ is preserved and the scalar solution as well as the Ernst potential $E := \varphi(1)$ transform as

$$\varphi' = \cos\delta\varphi - \mathrm{i}\sin\delta\varphi(-\lambda), \qquad E' = \cos\delta E - \mathrm{i}\sin\delta. \tag{5.8}$$

Note that if φ was normalised by $\varphi(-1) = 1$, then a normalised scalar solution of the transformed ERHP and the corresponding Ernst potential are given by

$$\varphi'' = \frac{\cos\delta\varphi - \mathrm{i}\sin\delta\varphi(-\lambda)}{\cos\delta - \mathrm{i}\sin\delta E}, \qquad E'' = \frac{\cos\delta E - \mathrm{i}\sin\delta}{\cos\delta - \mathrm{i}\sin\delta E} \tag{5.9}$$

This is the Ernst potential contained in a metric of the form (2.11) after a clockwise rotation of the x-y-plane by an angle δ,

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5.10}$$

Secondly, E'' from (5.9) is exactly the result of a so-called 'Ehlers transformation' $E'' = E/(1 - \mathrm{i}\tan\delta E)$ with subsequent normalisation in virtue of (2.41), cf. e.g. [9]. The jump functions transform under (5.7) as

$$\begin{aligned} \Im(\alpha') &= \cos(2\delta)\Im(\alpha) + \sin(2\delta)\beta, & \Re(\alpha') &= \Re(\alpha), \\ \beta' &= -\sin(2\delta)\Im(\alpha) + \cos(2\delta)\beta, & \gamma' &= \gamma. \end{aligned} \tag{5.11}$$

Starting with the induced ERHP jump matrix G_J, the boundary values (4.61)-(4.64) of the RHP jump functions transform under the rotation transformation (5.7) to

$$\alpha'_{1\mathrm{b}} = \cos(2\pi\rho_1) - \mathrm{i}\cos(\phi_A + 2\delta)\sin(2\pi\rho_1), \quad \beta'_{1\mathrm{b}} = \sin(\phi_A + 2\delta)\sin(2\pi\rho_1), \tag{5.12}$$

$$\alpha'_{2\mathrm{b}} = \cos(2\pi\rho_2) + \mathrm{i}\cos(\phi_B + 2\delta)\sin(2\pi\rho_2), \quad \beta'_{2\mathrm{b}} = -\sin(\phi_B + 2\delta)\sin(2\pi\rho_2). \tag{5.13}$$

Thus the clockwise coordinate rotation in the x-y-plane by an angle δ corresponds to a counterclockwise rotation of A_b and B_b in the complex plane by an angle 2δ. If the initial values imply $\phi_A - \phi_B = n\pi, n \in \mathbb{R}$, then the RHP jump matrix can be diagonalized at all four contour endpoints simultaneously, which leads to tremendous simplifications in the transition to the CRHP. We will call this case 'initially collinearly polarised GWs'.

By convention the rotation matrices R_{δ_1} and R_{δ_2} for the diagonalisation of the matrices $J_{1/2\mathrm{b}} := J_1(\lambda_{1/2})$ shall be chosen with

$$\delta_1 := (\pi - \phi_A)/2, \qquad \delta_2 := (\pi - \phi_B)/2. \tag{5.14}$$

Applying these transformations yields respectively

$$G'|_{\pm(\lambda_1+0)} = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\rho_1} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\rho_1} \end{pmatrix} \quad \text{or} \quad G'|_{\pm(\lambda_2-0)} = \begin{pmatrix} \mathrm{e}^{-2\pi\mathrm{i}\rho_2} & 0 \\ 0 & \mathrm{e}^{2\pi\mathrm{i}\rho_2} \end{pmatrix}. \tag{5.15}$$

From now on the freedom of rotating the x-y-plane shall be used to choose coordinates in which the jump matrix is initially diagonal at $\pm\lambda_2$ (cf. figure 5.1 (a)), i.e.

$$G_J|_{\pm(\lambda_2-0)} = \mathrm{diag}(\mathrm{e}^{-2\pi\mathrm{i}\rho_2}, \mathrm{e}^{2\pi\mathrm{i}\rho_2}). \tag{5.16}$$

This is equivalent to setting $\phi_B = \pi$.

5.3 Singularity transformation

A singularity transformation, which converts (5.1) to $\tilde{\Phi}_+ = \tilde{\Phi}_-\tilde{G}$, shall be defined by

$$\tilde{\Phi} = \Phi S^K_{1/2}, \qquad \tilde{G}^K = (S^K_{1/2-})^{-1} G S^K_{1/2+}. \tag{5.17}$$

Herein K is an index which takes the values 'e' and 'o' designating the two possibilities of using either an 'even' or an 'odd' singularity transformation matrix,

$$S^{\mathrm{e}}_{1/2} := \begin{pmatrix} L_{1/2}^{1-\rho_{1/2}} & 0 \\ 0 & L_{1/2}^{\rho_{1/2}-1} \end{pmatrix} \quad \text{or} \quad S^{\mathrm{o}}_{1/2} := \begin{pmatrix} L_{1/2}^{1-\rho_{1/2}} & 0 \\ 0 & L_{1/2}^{\rho_{1/2}} \end{pmatrix}. \tag{5.18}$$

The even transformation preserves the normal form of the ERHP solution, whereas in the odd case $\Phi S^{\mathrm{o}}_{1/2}$ is not in normal form. This property can be restored in the following way: As will be explicated in section (5.7), also in the context of the ERHP $\Phi(-\lambda)\sigma_1$ solves the same jump equation as Φ. Therefore, using also the freedom of adding solutions and left multiplying matrices independent of λ, from $\Phi S^{\mathrm{o}}_{1/2}$ another solution

$$\tilde{\Phi}^{\mathrm{o}} := \Phi S^{\mathrm{o}}_{1/2} - \sigma_3 \Phi(-\lambda) S^{\mathrm{o}}_{1/2}(-\lambda)\sigma_1 \tag{5.19}$$

to the jump equation with jump matrix $\tilde{G}^{\mathrm{o}}$ can be constructed. It is indeed in normal form with the scalar function

$$\tilde{\varphi}^{\mathrm{o}} = L_{1/2}^{\rho_{1/2}}(1 + L_{1/2})\varphi. \tag{5.20}$$

Note that if the odd transformation had been defined by the alternative matrix

$$S_1^{\mathrm{o}*} := \mathrm{diag}(L_1^{-\rho_1}, L_1^{\rho_1 - 1}) = L_1^{-1} S_1^{\mathrm{o}} \tag{5.21}$$

instead of S_1^{o}, then the linear combination in (5.19) would lead to exactly the same solution $\tilde{\Phi}^{\mathrm{o}}$ in normal form with $\tilde{\varphi}^{\mathrm{o}}$ from (5.20).

Evaluation of the inner and outer limits of $S^K_{1/2}$ similar to (4.69) leads to the following transformation of the off-diagonal elements of the jump matrix:

$$\begin{aligned} \tilde{\gamma}^K + \tilde{\beta}^K &= \varepsilon^K_{1/2} |L_{1/2}|^{x^K_{1/2}} (\gamma + \beta), \\ \tilde{\gamma}^K - \tilde{\beta}^K &= \varepsilon^K_{1/2} |L_{1/2}|^{-x^K_{1/2}} (\gamma - \beta), \end{aligned} \tag{5.22}$$

Therein $\varepsilon^K_{1/2} := \mathrm{sign}(\det S^K_{1/2})$ is just a sign which is 1 in the even case, $\varepsilon^{\mathrm{e}}_{1/2} = 1$, whereas for the odd case holds $\varepsilon^{\mathrm{o}}_{1/2}|_{\Gamma_{1/2}} = -1$ and $\varepsilon^{\mathrm{o}}_{1/2}|_{\Gamma_{\Re}/\Gamma_{1/2}} = 1$.

The exponents

$$x^{\mathrm{e}}_{1/2} := 2\rho_{1/2} - 2, \quad \text{and} \quad x^{\mathrm{o}}_{1/2} := 2\rho_{1/2} - 1 \tag{5.23}$$

due to (5.6) lie in the ranges

$$-1 < x^{\mathrm{e}}_{1/2} < 0, \qquad 0 < x^{\mathrm{o}}_{1/2} < 1. \tag{5.24}$$

Solved for the individual jump functions, the singularity transformation reads

$$\begin{aligned} \tilde{\gamma}^K &= \varepsilon^K_{1/2} \frac{1}{2} \left[(|L_{1/2}|^{x^K_{1/2}} + |L_{1/2}|^{-x^K_{1/2}})\gamma + (|L_{1/2}|^{x^K_{1/2}} - |L_{1/2}|^{-x^K_{1/2}})\beta \right], \\ \tilde{\beta}^K &= \varepsilon^K_{1/2} \frac{1}{2} \left[(|L_{1/2}|^{x^K_{1/2}} - |L_{1/2}|^{-x^K_{1/2}})\gamma + (|L_{1/2}|^{x^K_{1/2}} + |L_{1/2}|^{-x^K_{1/2}})\beta \right], \\ \tilde{\alpha}^K &= \begin{cases} \mathrm{e}^{\mp 2\pi\mathrm{i}\rho_{1/2}}\alpha, & \lambda \in \Gamma_{1/2}; \\ \alpha & \text{else}; \end{cases} \quad \text{('-' associated with index '1').} \end{aligned} \tag{5.25}$$

Considering (5.5), due to $|x^K_{1/2}| < 1$ the application of the singularity transformation to a ERHP jump matrix G_J diagonal at $\pm\lambda_2$ yields (cf. figure 5.1 (b)):

$$\tilde{G}^K_J := (S^K_{2-})^{-1} G_J S^K_{2+}, \qquad \tilde{G}^K_J(\pm\lambda_2) = \mathbb{1}. \tag{5.26}$$

The jump matrix $\tilde{G}^K_J$ is Hölder continuous at $\pm\lambda_2$, but not necessarily Lipschitz continuous, whereas at $\pm\lambda_1$ the jump matrix is still Lipschitz continuous. However, since $\tilde{\gamma}^K \neq 0$ the transformed jump matrix $\tilde{G}^K_J$ is no longer unitary and so another type of transformation is necessary to restore the unitarity at least at the endpoints $\pm\lambda_1$ of the other contour. Thereafter the jump matrix may be diagonalised at $\pm\lambda_1$ by a rotation transformation and made continuous by a singularity transformation with S^K_1.
The matrices S^{e}_2 and S^{o}_2 can be generalised to the class

$$S^n_2 := \begin{pmatrix} L_2^{1-\rho_2} & 0 \\ 0 & L_2^{\rho_2 - n} \end{pmatrix}, \qquad n \in \mathbb{Z} \tag{5.27}$$

of transformation matrices which remove the jump of α in a jump matrix G_J diagonal at $\pm\lambda_2$. However, for the general transformation with S^n_2 the exponent x^K_2 in (5.25) has to be replaced by $x^n_2 := 2\rho_2 - 1 - n$. Furthermore $|x^n_2| < 1$, which is necessary to get at $\pm\lambda_2$ convergent functions $\tilde{\gamma}^K$ and $\tilde{\beta}^K$ from Lipschitz continuous initial jump functions γ and β, is equivalent to $n \in \{0, 1\}$. Hence there are no transformation matrices in the class (5.27) which are linearly independent of S^{e}_2 and S^{o}_2 and lead to $\tilde{G}^K_J(\pm\lambda_2) = \mathbb{1}$ for Lipschitz continuous initial jump matrices J.
Upon this result the matrices $S^{\mathrm{e}}_{1/2}$ and $S^{\mathrm{o}}_{1/2}$ are conjectured to be the only in principle different transformations removing the discontinuities at a pair of contour endpoints and leading to $\tilde{G}^K_J(\pm\lambda_2) = \mathbb{1}$ for Lipschitz continuous initial jump matrices J. The case is different for restricted conditions on the initial jump matrix. For example in the collinear case $\beta = 0 = \gamma$, transformations S^n_2 with arbitrary n lead to $\tilde{G}^K_J(\pm\lambda_2) = \mathbb{1}$. However, as discussed at the end of section 4.5, also in this case only solutions with a divergence exponents lower than one are desirable which seems to restrict the class of permitted transformations again to $S^{\mathrm{e}}_{1/2}$ and $S^{\mathrm{o}}_{1/2}$.

5.4 Unitarisation transformation

A unitarisation transformation converting (5.1) to $\hat{\Phi}_+ = \hat{\Phi}_-\hat{G}$ shall be defined by

$$\hat{\Phi} = \Phi U^K, \quad \hat{G} = (U_-^K)^{-1} G U_+^K, \qquad U^K := \begin{pmatrix} w^K\Lambda^K & 0 \\ 0 & (w^K\Lambda^K)^{-1} \end{pmatrix}, \tag{5.28}$$

where the constituents of the unitarisation matrix U^K are defined as[1]

$$w^K := \left(\mathrm{sign}\left[\Lambda^K(\lambda_1)\right]\right)^{-1}, \qquad \Lambda^K := \begin{cases} \frac{\lambda+\lambda_U^K}{\lambda-\bar{\lambda}_U^K}, & \Im(\lambda) > 0; \\ \frac{\lambda+\bar{\lambda}_U^K}{\lambda-\lambda_U^K}, & \Im(\lambda) < 0. \end{cases} \tag{5.29}$$

The phase factor w^K is constant in each half-sphere and compensates the phase of Λ^K in the contour endpoints $\pm\lambda_1$. Similar to $L_{1/2}$, the functions w^K and Λ^K obey

$$\bar{w}^K(\bar{\lambda}) = w^K(\lambda) = 1/w^K(-\lambda), \qquad \bar{\Lambda}^K(\bar{\lambda}) = \Lambda^K(\lambda) = 1/\Lambda^K(-\lambda). \tag{5.30}$$

Under the action of the unitarisation transformation the normal form of the solution matrix is maintained. The jump functions are mapped to

$$\begin{aligned} \hat{\gamma} &= \tfrac{1}{2}\left[(|\Lambda^K|^{-2}+|\Lambda^K|^2)\gamma + (|\Lambda^K|^{-2}-|\Lambda^K|^2)\beta\right], \\ \hat{\beta} &= \tfrac{1}{2}\left[(|\Lambda^K|^2-|\Lambda^K|^{-2})\gamma + (|\Lambda^K|^{-2}+|\Lambda^K|^2)\beta\right], \end{aligned} \qquad \hat{\alpha} = (w_+^K)^2\,\mathrm{sign}^2(\Lambda_+^K)\alpha. \tag{5.31}$$

The free parameter λ_U^K in (5.29) shall be chosen so that

$$|\Lambda^K(\lambda_1)|^2 = (L_2(\lambda_1))^{x_2^K} \tag{5.32}$$

holds, and so the unitarisation transformation applied after the singularity transformation with S_2^K exactly reverts the effect of S_2^K on the off-diagonal matrix elements displayed in (5.22) at the points $\pm\lambda_1$. Therefore the unitarisation transformation applied after removing the discontinuities of G_J at $\pm\lambda_2$ by $\tilde{G}_J^K = (S_{2-}^K)^{-1} G_J S_{2+}^K$ yields (cf. figure 5.1 (c)):

$$\hat{G}_J^K := (U_-^K)^{-1}\tilde{G}_J^K U_+^K, \qquad \hat{G}_J^K|_{\pm(\lambda_1-0)} = \mathbb{1}, \quad \hat{G}_J^K|_{\pm(\lambda_1+0)} = J(\pm\lambda_1). \tag{5.33}$$

Hence the unitarisation transformation reproduces the initial settings at $\pm\lambda_1$ with $\hat{G}_J^K$ still Lipschitz continuous at these points. Furthermore $\hat{G}_J^K \neq \mathbb{1}$ results almost

[1] The reader may be reminded that $\mathrm{sign}(a) := \mathrm{e}^{i\arg(a)}$ has to be understood as the complex generalisation of the sign function.

everywhere on $\Gamma_{\Re}$ and so the matrix solution $\hat{\Phi}$ is no longer described by a single expression for both sides of the contour. Finally by choosing $\Im(\lambda_U^K) > 0$ it can be assured that Λ^K has neither zeros nor poles. A possible choice for λ_U^K with $|\Lambda^K(\lambda_1)|^2 = (L_2(\lambda_1))^{x_2^K}$ and $\Im(\lambda_U^K) > 0$ is

$$\lambda_U^{\mathrm{e}} = -\lambda_2 + \mathrm{i}\sqrt{\frac{(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^{-x_2^{\mathrm{e}}} - (\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)^{-x_2^{\mathrm{e}}}}{(\lambda_1+\lambda_2)^{-x_2^{\mathrm{e}}} - (\lambda_1-\lambda_2)^{-x_2^{\mathrm{e}}}}}, \tag{5.34}$$

$$\lambda_U^{\mathrm{o}} = +\lambda_2 + \mathrm{i}\sqrt{\frac{(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^{x_2^{\mathrm{o}}} - (\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)^{x_2^{\mathrm{o}}}}{(\lambda_1+\lambda_2)^{x_2^{\mathrm{o}}} - (\lambda_1-\lambda_2)^{x_2^{\mathrm{o}}}}}. \tag{5.35}$$

5.5 The full transformation formula

The jump matrix $\hat{G}_J^K$ made unitary again at $\pm\lambda_1$ in (5.33) can now be diagonalised by the rotation transformation (cf. figure 5.1 (d))

$$G_J'^K := R_{\delta_1}^{-1}\hat{G}_J^K R_{\delta_1}. \qquad G_J'^K|_{\pm(\lambda_1-0)} = \mathbb{1}, \quad G_J'^K|_{\pm(\lambda_1+0)} = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\rho_1} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\rho_1} \end{pmatrix}. \tag{5.36}$$

Finally, the discontinuities at $\pm\lambda_1$ are removed by the singularity transformation

$$G_c^{IK} := (S_{2-}^I)^{-1} G_J'^K S_{2+}^I, \qquad G_c^{IK}(\pm\lambda_1) = \mathbb{1}, \quad G_{\mathrm{c}}^{IK} \text{ Hölder continuous on } \Gamma_{\Re} \tag{5.37}$$

analogous to the procedure at $\pm\lambda_2$, cf. figure 5.1 (e). In summary, all the subsequent transformations lead to the CRHP

$$\Omega_+^{IK} = \Omega_-^{IK} G_{\mathrm{c}}^{IK}, \tag{5.38}$$

which is obtained from the ERHP (5.1) by applying the combined transformation

$$\Omega^{IK} := \Phi S_2^K U^K R_{\delta_1} S_1^I, \tag{5.39}$$

$$G_{\mathrm{c}}^{IK} := (S_{1-}^I)^{-1} R_{\delta_1}^{-1} (U_-^K)^{-1} (S_{2-}^K)^{-1} G_J S_{2+}^K U_+^K R_{\delta_1} S_{1+}^I. \tag{5.40}$$

The jump matrix G_{c}^{IK} of the CRHP depends in contrast to G_J on the coordinates f and g. This is an interesting similarity to the treatment of Alekseev and Griffiths [20], where the non-analytic behaviour of the solution at the wavefronts is handled by 'dynamical' monodromy data and generalised integral evolution equations.

In fact (5.39) and (5.40) describe four different CRHPs for the index values $IK \in \{\mathrm{ee}, \mathrm{eo}, \mathrm{oe}, \mathrm{oo}\}$ which give rise to four independent RHP solutions. Furthermore,

in the solution Ω^{IK} the normal form is temporarily lost. It will be newly established by constructing a fundamental matrix Θ^{IK} of the CRHP in normal form in section 5.7. Out of Θ^{IK} the four independent RHP solutions will be constructed in section 5.8.
To illustrate the subsequent transformations of the jump matrix from G_J to G_{c}^{IK}, an example including two even singularity transformations is given in figure 5.1 starting with the arbitrary choice

$$\alpha_J = \begin{cases} \frac{1}{2}(\mathrm{e}^{\frac{9\mathrm{i}\pi}{10}} - \frac{\mathrm{i}}{5})[1-\cos(\pi\lambda_1/\lambda)], & \lambda \in \Gamma_1; \\ \frac{1}{5}\mathrm{e}^{\frac{4\mathrm{i}\pi}{5}}[4-\cos(\pi\lambda/\lambda_2)], & \lambda \in \Gamma_2; \\ 1 & \text{else}; \end{cases} \tag{5.41}$$

$$\beta_J = 1 - \alpha_J \bar{\alpha}_J, \tag{5.42}$$

$$\gamma_J = 0. \tag{5.43}$$

It is adapted to the demand that the initial jump matrix G_J is diagonal at $\pm\lambda_2$ (cf. figure 5.1 (a)), which is in general achieved by the choice of the coordinates x and y. This example demonstrates that the full transformation in (5.40) indeed leads to a continuous jump matrix. On the other hand it also shows that the price to pay for this continuity is a root-like behaviour of $\Im(\alpha)$, β and γ near the contour endpoints $\pm\lambda_{1/2}$ with infinitely steep slopes.

5.6 The degree of the solution row vectors

In order to discuss the properties of the CRHP solution Ω^{IK}, a point λ_p shall be fixed on the imaginary axis of the λ-sphere (one may think of $\lambda_p = \mathrm{i}$). Further a λ_p-regular function shall be defined as a function which is only allowed to have poles or respectively zeros in λ_p and the degree of a λ_p-regular function $f(\lambda)$ shall be defined as

$$\text{degree of } f(\lambda) \quad := \begin{cases} n, & f(\lambda) \text{ has pole of order } n \text{ in } \lambda_p; \\ 0, & 0 \neq f(\lambda_p) \neq \infty; \\ -n, & f(\lambda) \text{ has zero of order } n \text{ in } \lambda_p. \end{cases}$$

The degree of a Matrix shall be defined as the maximum of the degrees of its elements. According to [23] (where a finite closed contour with $\lambda_p = \infty$ is discussed), for a two-dimensional CRHP there exists a fundamental matrix $\Omega^{IK} = (\Omega_1^{IK}; \Omega_2^{IK})$ characterized by the λ_p-regular and linearly independent solution row vectors Ω_1^{IK} and Ω_2^{IK} having minimal degree $\varkappa_1$ and $\varkappa_2$, respectively. From this fundamental matrix all

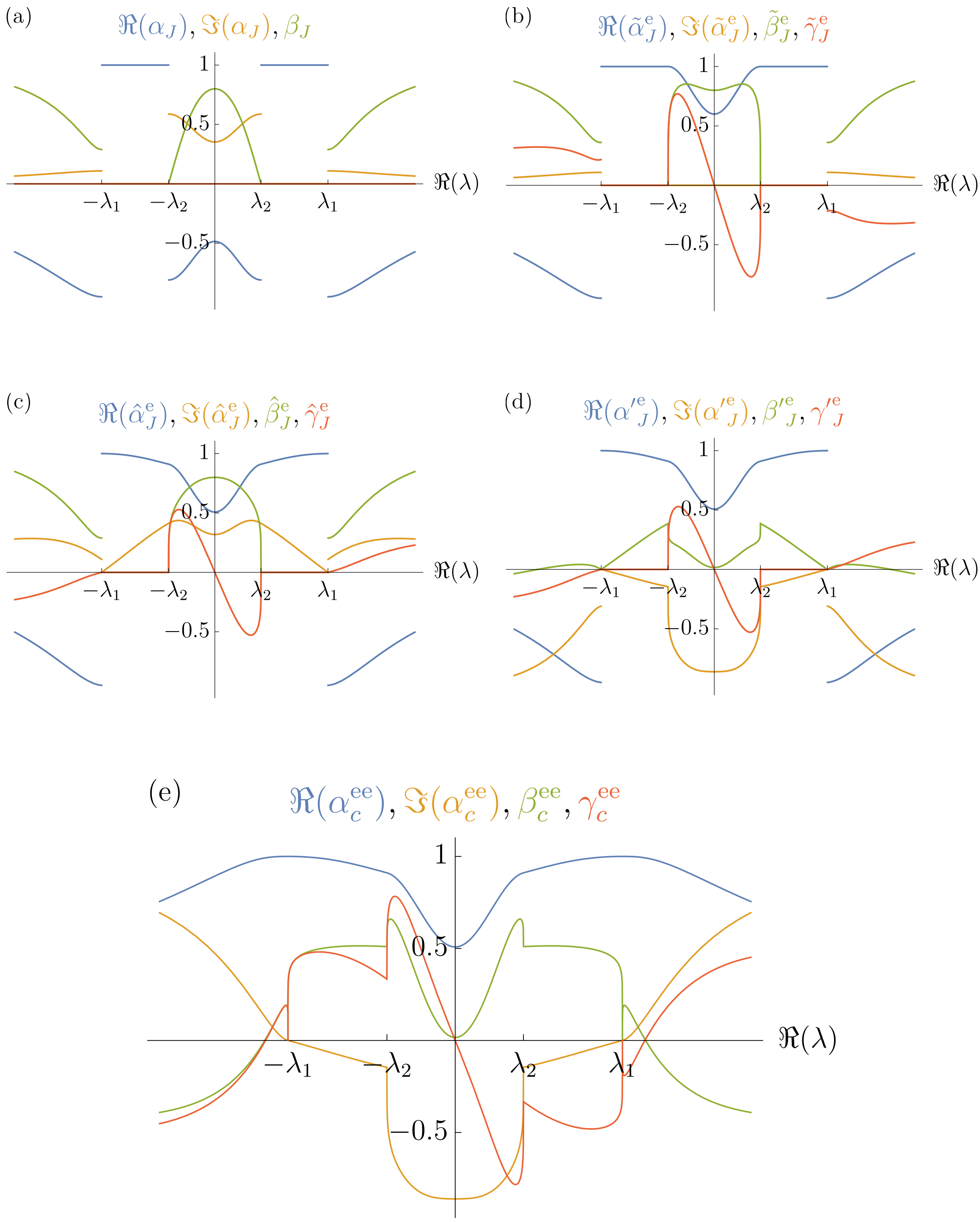

Figure 5.1: Visualisation of the transformation of the jump functions $\Re(\alpha)$ (blue), $\Im(\alpha)$ (orange), β (green) and γ (red) contained in the jump matrices G_J (a), $\tilde{G}^{\mathrm{e}}_J = (S^e_{2-})^{-1} G_J S^e_{2+}$ (b), $\hat{G}^{\mathrm{e}}_J = (U^e_-)^{-1} \tilde{G}^{\mathrm{e}}_J U^e_+$ (c), $G'^{e}_J = R^{-1}_{\delta_1} \hat{G}^{\mathrm{e}}_J R_{\delta_1}$ (d) and $G^{\mathrm{ee}}_c = (S^e_{1-})^{-1} G'^{e}_J S^e_{1+}$ (e). The final continuous jump matrix G^{ee}_c features a rather steep root-like behaviour at the contour endpoints $\pm\lambda_{1/2}$.

solution vectors can be constructed as linear combinations (as the case may be with coefficients depending on λ) of Ω_1^{IK} and Ω_2^{IK}. It is shown in [23] to have the following two properties:

$$\det \Omega^{IK} \neq 0 \; \forall \lambda \neq \lambda_p, \tag{5.44}$$

$$0 < (\lambda - \lambda_p)^{\varkappa_1 + \varkappa_2} \det \Omega^{IK}(\lambda_p) < \infty. \tag{5.45}$$

Thus in this work's nomenclature $\det \Omega^{IK}$ is λ_p-regular with degree $\varkappa_D = \varkappa_1 + \varkappa_2$. Furthermore, due to $\det G_c^{IK} = \mathbb{1}$ the determinant of the CRHP solution, $\det \Omega^{IK}$, is continuous across the contour $\Gamma_{\Re}$. This can be recognised to imply the holomorphicity of $\det \Omega^{IK}$ near $\Gamma_{\Re}$ by the following consideration:

Lemma 1. *Be R a simply connected open subset of the complex sphere $\mathbb{C}_\lambda$ which is divided by a contour Γ_R into two open parts R^+ and R^- not containing Γ_R. If a function ω is holomorphic in R^+ and R^- as well as continuous in $R^+ \cup \Gamma_R$ and $R^- \cup \Gamma_R$ whereby its values on Γ_R from each side coincide, then ω is holomorphic in R.*

Proof. Holomorphicity is equivalent to the statement that every contractable line integral vanishes. The contour C of each integral $I = \int_C \omega(\lambda) \mathrm{d}\lambda$ across Γ_R can be partitioned into a contour C^+ lying in R^+ and a contour C^- lying in R^- by simultaneously adding and subtracting the section of Γ_R enclosed by C. The contours C^+ and C^- are in turn limits of sequences of curves C_a^+ or respectively C_a^- which lie entirely in R^+ or respectively R^-. Because of the holomorphicity of ω, the integrals over C_a^+ and C_a^- vanish and due to the continuity of ω this also holds for the integrals over C^+ and C^- as well as finally for I. □

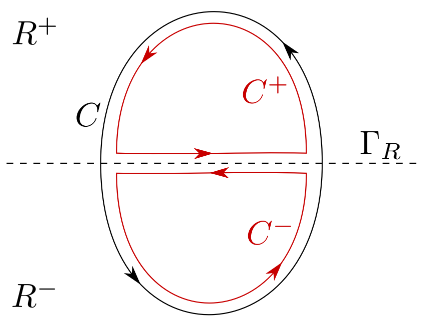

Figure 5.2: Partitioning of the curve C.

Therefore $\det \Omega^{IK}$ is holomorphic in $\mathbb{C}_\lambda \setminus \{\lambda_p\}$, due to (5.44) without even having zeros in that domain. Either $\det \Omega^{IK}$ or $(\det \Omega^{IK})^{-1}$ is then holomorphic on the entire λ-sphere $\mathbb{C}_\lambda$ and thus constant due to Liouville's theorem (and so in fact both $\det \Omega^{IK}$ and $(\det \Omega^{IK})^{-1}$ are constant). In consequence $\varkappa_D = 0 = \varkappa_1 + \varkappa_2$ holds for the degree of the determinant.

5.7 The fundamental matrix of the CRHP

Using (5.2) and (5.3) it follows that the CRHP jump matrix $G_{\rm c}^{IK}$ fulfils the same identities on $\Gamma_{\Re}$ as the RHP jump matrix J on Γ in (4.36):

$$\mathbb{1} = \sigma_1 G_{\rm c}^{IK}(-\lambda)\sigma_1 G_{\rm c}^{IK}, \qquad \mathbb{1} = \sigma_3 \bar{G}_{\rm c}^{IK}\sigma_3 G_{\rm c}^{IK}. \tag{5.46}$$

Inserting the jump equation (5.38) in the forms $G_{\rm c}^{IK}(-\lambda) = (\Omega_-^{IK})^{-1}(-\lambda)\Omega_+^{IK}(-\lambda)$ and $\bar{G}_{\rm c}^{IK} = (\bar{\Omega}_-^{IK})^{-1}(\bar{\lambda})\bar{\Omega}_+^{IK}(\bar{\lambda})$ shows that $\Omega^{IK}(-\lambda)\sigma_1$ and $\bar{\Omega}^{IK}(\bar{\lambda})\sigma_3$ fulfil the same jump equation (5.38) as Ω^{IK}, but the zeros or respectively poles of these new solutions lie in $-\lambda_p = \bar{\lambda}_p$. Again this statement holds already for row vectors. Involving the fraction

$$L_p := \frac{\lambda_p + \lambda}{\lambda_p - \lambda}, \qquad \bar{L}_p(\bar{\lambda}) = L_p^{-1}(\lambda) = L_p(-\lambda) \tag{5.47}$$

it is possible to construct the λ_p-regular solutions $L_p^{\varkappa_1} w(-\lambda)\sigma_1$ and $L_p^{\varkappa_1}\bar{w}(\bar{\lambda})\sigma_1$ with the same degree from a given solution row vector w. Assuming $w = (w_1, w_2)$ to have the minimal degree $\varkappa_1$, another λ_p-regular solution row vector $\Theta_1^{IK} := w - L_p^{\varkappa_1} w(-\lambda)\sigma_1$ with degree $\varkappa_1$ can be constructed. Within the subtraction no new zeros can arise because w has already minimal degree. The components of Θ_1^{IK} can be represented via a scalar function ϑ^{IK} defined by

$$\Theta_1^{IK} = \Big(w_1 - L_p^{\varkappa_1} w_2(-\lambda), w_2 - L_p^{\varkappa_1} w_1(-\lambda)\Big) =: \Big(\vartheta^{IK}, -L_p^{\varkappa_1}\vartheta^{IK}(-\lambda)\Big) \tag{5.48}$$

From defining the matrix

$$\Theta^{IK} := \begin{pmatrix} \Theta_1^{IK} \\ L_p^{\varkappa_1}\bar{\Theta}_1^{IK}(\bar{\lambda})\sigma_3 \end{pmatrix} = \begin{pmatrix} \vartheta^{IK} & -L_p^{\varkappa_1}\vartheta^{IK}(-\lambda) \\ L_p^{\varkappa_1}\bar{\vartheta}^{IK}(\bar{\lambda}) & \bar{\vartheta}^{IK}(-\bar{\lambda}) \end{pmatrix} \tag{5.49}$$

and calculating

$$\det \Theta^{IK}(0) = |\vartheta_+^{IK}(0)|^2 + |\vartheta_-^{IK}(0)|^2 \neq 0 \tag{5.50}$$

it can be seen that Θ_1^{IK} and $L_p^{\varkappa_1}\bar{\Theta}_1^{IK}(\bar{\lambda})\sigma_3$ are both linearly independent λ_p-regular solution row vectors with degree $\varkappa_1$. Thus $\varkappa_1 = \varkappa_2$ holds, but since $\varkappa_1 + \varkappa_2 = 0$ this results in $\varkappa_1 = \varkappa_2 = 0$, which means that the row vectors constituting the fundamental matrix feature neither zeros nor poles. However, the scalar function ϑ^{IK} still may have zeros in points different from 0 and ∞. One such fundamental matrix of the CRHP is indeed Θ^{IK} and it can be written in normal form with the scalar solution ϑ^{IK}:

$$\Theta^{IK} = \begin{pmatrix} \vartheta^{IK} & -\vartheta^{IK}(-\lambda) \\ \bar{\vartheta}^{IK}(\bar{\lambda}) & \bar{\vartheta}^{IK}(-\bar{\lambda}) \end{pmatrix}. \tag{5.51}$$

According to [23], the solution to a RHP with Hölder continuous jump matrix is continuous in both of the closed domains separated by the contour, which are considered to include the contour itself. Furthermore, all solutions to the CRHP can be represented as linear combinations of the columns of the fundamental matrix [23], so the solution (5.51) to the CRHP is unique up to left multiplication of a matrix independent of λ or respectively the multiplication of ϑ^{IK} with a function independent of λ if the normal form shall be conserved.

For the normal form matrix Θ^{IK} the jump equation $\Theta^{IK}_{+} = \Theta^{IK}_{-} G^{IK}_{\mathrm{c}}$ is equivalent to the single scalar jump equation

$$\vartheta^{IK}_{+} = \alpha_c \vartheta^{IK}_{-} + (\beta_c - \gamma_c)\vartheta^{IK}_{+}(-\lambda). \tag{5.52}$$

resulting from its $(1,1)$-element. Recalling (5.3) the other matrix elements

$$\begin{aligned}
(1,2):& \qquad -\vartheta^{IK}_{-}(-\lambda) = \bar{\alpha}_c \vartheta^{IK}_{+}(-\lambda) + (\gamma_c + \beta_c)\vartheta^{IK}_{-}, \\
(2,1):& \qquad \bar{\vartheta}^{IK}_{-}(\bar{\lambda}) = \alpha_c \bar{\vartheta}^{IK}_{+}(\bar{\lambda}) + (\gamma_c - \beta_c)\bar{\vartheta}^{IK}_{-}(-\bar{\lambda}), \\
(2,2):& \qquad \bar{\vartheta}^{IK}_{+}(-\bar{\lambda}) = \bar{\alpha}_c \bar{\vartheta}^{IK}_{-}(-\bar{\lambda}) + (\gamma_c + \beta_c)\bar{\vartheta}^{IK}_{+}(\bar{\lambda}),
\end{aligned}$$

reduce to (5.52) via the identities

$$\begin{aligned}
(1,1) = \overline{(2,2)}(-\bar{\lambda}) &= \left[(1,2)(-\lambda) + (\gamma_c - \beta_c)(1,2)\right]/\bar{\alpha} \\
&= \left[\overline{(2,1)}(\bar{\lambda}) + (\gamma_c - \beta_c)\overline{(2,1)}(-\bar{\lambda})\right]/\bar{\alpha}.
\end{aligned}$$

By setting $\gamma_c = 0$ this consideration yields also the equivalence of the original RHP jump equation (4.33) to the scalar jump equation (4.39).

5.8 The normal form solutions of the RHP

Now the transformation (5.39) to the CRHP shall be gradually reverted, where the normal form of the matrix solution must be ensured in each step. In the ‘even’ case the first partial inverse transformation $\Theta'^{\mathrm{e}K} := \Theta^{\mathrm{e}K}(S^{\mathrm{e}}_1)^{-1}$ directly yields a matrix in normal form with

$$\vartheta'^{\mathrm{e}K} = L_1^{\rho_1 - 1}\vartheta^{\mathrm{e}K}. \tag{5.53}$$

In the ‘odd’ case, analogous to the discussion in section 5.3, the linear combination $\Theta'^{\mathrm{o}K} := \Theta^{\mathrm{o}K}(S_1^{\mathrm{o}})^{-1} - \sigma_3\Theta^{\mathrm{o}K}(-\lambda)(S_1^{\mathrm{o}})^{-1}(-\lambda)\sigma_1$ is in normal form with the scalar function

$$\vartheta'^{\mathrm{o}K} = L_1^{\rho_1-1}(1+L_1)\vartheta^{\mathrm{o}K} \tag{5.54}$$

The second partial inverse transformation $\hat{\Theta}^{IK} = \Theta'^{IK}R_{\delta_1}^{-1}$ yields directly a matrix $\hat{\Theta}^{IK}$ in normal form with

$$\hat{\vartheta}^{IK} = \cos\delta_1\vartheta'^{IK} + \mathrm{i}\sin\delta_1\vartheta'^{IK}(-\lambda). \tag{5.55}$$

Likewise the inverse transformation $\tilde{\Theta}^{IK} = \hat{\Theta}^{IK}(U^K)^{-1}$ yields a matrix $\tilde{\Theta}^{IK}$ in normal form with

$$\tilde{\vartheta}^{IK} = (w^K\Lambda^K)^{-1}\hat{\vartheta}^{IK}. \tag{5.56}$$

At last, the inverse transformation with $(S_2^K)^{-1}$ is treated analogously to the inverse transformation with $(S_1^I)^{-1}$ above. A solution Φ^{IK} to the initial RHP is obtained which is in normal form with one of the scalar functions

$$\varphi^{I\mathrm{e}} = L_2^{\rho_2-1}\tilde{\vartheta}^{I\mathrm{e}} \quad \text{or} \quad \varphi^{I\mathrm{o}} = L_2^{\rho_2-1}(1+L_2)\tilde{\vartheta}^{I\mathrm{o}}. \tag{5.57}$$

In summary, via the CRHP the four matrix solutions Φ^{ee}, Φ^{oe}, Φ^{eo} and Φ^{oo} are obtained in normal form with the scalar functions given in terms of the CRHP solutions ϑ^{IK} as

$$\begin{aligned}
\varphi^{\mathrm{ee}} &= L_2^{\rho_2-1}(w^{\mathrm{e}}\Lambda^{\mathrm{e}})^{-1}\left[\cos\delta_1 L_1^{\rho_1-1}\vartheta^{\mathrm{ee}} + \mathrm{i}\sin\delta_1 L_1^{1-\rho_1}\vartheta^{\mathrm{ee}}(-\lambda)\right],\\
\varphi^{\mathrm{oe}} &= L_2^{\rho_2-1}(w^{\mathrm{e}}\Lambda^{\mathrm{e}})^{-1}\left[\cos\delta_1 L_1^{\rho_1-1}(1+L_1)\vartheta^{\mathrm{oe}} + \mathrm{i}\sin\delta_1 L_1^{1-\rho_1}(1+L_1^{-1})\vartheta^{\mathrm{oe}}(-\lambda)\right],\\
\varphi^{\mathrm{eo}} &= L_2^{\rho_2-1}(1+L_2)(w^{\mathrm{o}}\Lambda^{\mathrm{o}})^{-1}\left[\cos\delta_1 L_1^{\rho_1-1}\vartheta^{\mathrm{eo}} + \mathrm{i}\sin\delta_1 L_1^{1-\rho_1}\vartheta^{\mathrm{eo}}(-\lambda)\right],\\
\varphi^{\mathrm{oo}} &= L_2^{\rho_2-1}(1+L_2)(w^{\mathrm{o}}\Lambda^{\mathrm{o}})^{-1}\\
&\qquad\cdot\left[\cos\delta_1 L_1^{\rho_1-1}(1+L_1)\vartheta^{\mathrm{oo}} + \mathrm{i}\sin\delta_1 L_1^{1-\rho_1}(1+L_1^{-1})\vartheta^{\mathrm{oo}}(-\lambda)\right].
\end{aligned} \tag{5.58}$$

The solution of the four scalar jump equations (5.52) and the construction of these RHP solutions is subsumed in step (iib) of the solution scheme in figure 3.1.

6 Holomorphicity conditions for solutions of the linear problem

In this chapter a LP solution Φ^{LP} is constructed out of the four independent RHP solutions in (5.58). Via a rather technical investigation of the terms in $\Phi^{\mathrm{LP}}_f(\Phi^{\mathrm{LP}})^{-1}$ and $\Phi^{\mathrm{LP}}_g(\Phi^{\mathrm{LP}})^{-1}$ its holomorphicity conditions are shown to reduce to two purely algebraic relations. They can in general be fulfilled and even seem to be not fully determining for Φ^{LP}, which may lead to the generation of a class of solutions. At the end of the chapter partially rather heuristic considerations of the limits $f \to \frac{1}{2}$ and $g \to \frac{1}{2}$ are presented including interesting conjectures for the evaluation of the colliding wave conditions for the Ernst potential inferred from Φ^{LP}.

6.1 Construction of the solution to the linear problem

As investigated in section 4.3, the remaining freedom in an unnormalised RHP solution in normal form with φ consists in the multiplication of φ with a function of f and g. Therefore the most general ansatz for a LP solution is the linear combination of the four RHP solutions (5.58), which can be expressed on matrix and scalar level as

$$\Phi^{\mathrm{LP}} = \Phi^{\mathrm{oo}} + \mathrm{diag}(p, \bar{p})\Phi^{\mathrm{eo}} + \mathrm{diag}(q, \bar{q})\Phi^{\mathrm{oe}} + \mathrm{diag}(r, \bar{r})\Phi^{\mathrm{ee}}, \tag{6.1}$$

$$\varphi^{\mathrm{LP}} = \varphi^{\mathrm{oo}} + p\varphi^{\mathrm{eo}} + q\varphi^{\mathrm{oe}} + r\varphi^{\mathrm{ee}}. \tag{6.2}$$

A fourth coefficient would be redundant since the solution will be subject to a subsequent normalisation. With this solution the LP matrices can be represented as

$$U = \Phi^{\mathrm{LP}}_f(\Phi^{\mathrm{LP}})^{-1}, \qquad V = \Phi^{\mathrm{LP}}_g(\Phi^{\mathrm{LP}})^{-1}. \tag{6.3}$$

The φ-coefficients p, q and r are functions of the coordinates f and g and have to be arranged to make U holomorphic in $\mathbb{C}_\lambda \backslash \{\infty\}$ and V holomorphic in $\mathbb{C}_\lambda \backslash \{0\}$. These are the generic holomorphicity conditions (4.41), which will be specified now for the ansatz (6.1). A preliminary investigation of $\det \Phi^{\mathrm{LP}}$ will be useful before the direct consideration of U and V.

In section 4.1 the LP solution was shown to feature $\det \Phi^{\mathrm{LP}} = E + \bar{E}$ and so the determinant has to be independent of λ. On the other hand, the determinant $\det \Phi^{\mathrm{LP}}$ is composed of the four independent RHP solutions φ^{IK} in (5.58) and thus, due to the absence of poles in the four CRHP solutions ϑ^{IK}, the determinant $\det \Phi^{\mathrm{LP}}$ can only inherit the poles at $\pm\lambda_{1/2}$ from $L_{1/2}^{\pm 1}$. Since $\det G_J = \mathbb{1}$, the determinant $\det \Phi^{\mathrm{LP}}$ has no jump on the contour $\Gamma_{\Re}$ and is hence, according to Lemma 1, holomorphic up to the contour endpoints $\pm\lambda_{1/2}$. Since $\det \Phi^{\mathrm{LP}}(-\lambda) = \det \Phi^{\mathrm{LP}}$ because of the normal form (cf. (4.19)), for $\det \Phi^{\mathrm{LP}}$ to be independent of λ it is sufficient to assure that $\det \Phi^{\mathrm{LP}}$ has no isolated singularities at $\lambda_{1/2}$.

6.2 Holomorphicity condition for $\det \Phi^{\mathrm{LP}}$ at λ_2

In order to derive a first necessary condition for the φ-coefficients from the λ-independence of $\det \Phi^{\mathrm{LP}}$, after insertion of (5.58) the constituents of φ^{LP} regular in λ_2 are collected in the two functions

$$\begin{aligned}\psi_2^{\mathrm{e}} := (w^{\mathrm{e}}\Lambda^{\mathrm{e}})^{-1}\{&\cos\delta_1 L_1^{\rho_1-1}\left[(1+L_1)\vartheta^{\mathrm{oe}} + q^{-1} r \vartheta^{\mathrm{ee}}\right]\\ &+ \mathrm{i}\sin\delta_1 L_1^{1-\rho_1}\left[(1+L_1^{-1})\vartheta^{\mathrm{oe}}(-\lambda) + q^{-1} r\vartheta^{\mathrm{ee}}(-\lambda)\right]\}, \end{aligned} \tag{6.4}$$

$$\begin{aligned}\psi_2^{\mathrm{o}} := (w^{\mathrm{o}}\Lambda^{\mathrm{o}})^{-1}\{&\cos\delta_1 L_1^{\rho_1-1}\left[(1+L_1)\vartheta^{\mathrm{oo}} + p\vartheta^{\mathrm{eo}}\right]\\ &+ \mathrm{i}\sin\delta_1 L_1^{1-\rho_1}\left[(1+L_1^{-1})\vartheta^{\mathrm{oo}}(-\lambda) + p\vartheta^{\mathrm{eo}}(-\lambda)\right]\}. \end{aligned} \tag{6.5}$$

Using these expressions the scalar LP solution can be represented as

$$\varphi^{\mathrm{LP}}(\lambda) = L_2^{\rho_2-1}\left[(1+L_2)\psi_2^{\mathrm{o}} + q\psi_2^{\mathrm{e}}\right]. \tag{6.6}$$

Considering (5.53) and (5.54), the functions ψ_2^K can be regarded as scalar solutions of the ERHP with the jump matrix $\tilde{G}_J^K := (S_{2-}^K)^{-1} G_J S_{2+}^K$ obtained after the first singularity transformation which removed the discontinuity at $\pm\lambda_2$. Because of $\tilde{G}_J^K(\pm\lambda_2) = \mathbb{1}$ these scalar solutions have no jump at $\pm\lambda_2$:

$$\psi_{2+}^K(\pm\lambda_2) = \psi_{2-}^K(\pm\lambda_2). \tag{6.7}$$

Inserting (6.6) and remembering the properties (4.68) of $L_{1/2}$, the part of $\det\Phi^{\mathrm{LP}}$ proportional to L_2 can be identified by[1]

$$\det\Phi^{\mathrm{LP}} = \bar{\varphi}^{\mathrm{LP}}(\bar{\lambda})\varphi^{\mathrm{LP}}(-\lambda) + \text{c.c.} = \left(\bar{\psi}_2^{\mathrm{o}}(\bar{\lambda})\left[\psi_2^{\mathrm{o}}(-\lambda) + q\psi_2^{\mathrm{e}}(-\lambda)\right] + \text{c.c.}\right) L_2 + R, \quad (6.8)$$

where R is the remainder consisting of a term proportional to L_2^{-1} and a term independent of L_2. Due to (6.7) the prefactor in front of L_2 in (6.8) has a unique value at $\lambda = \lambda_2$ and its vanishing at this point is equivalent to

$$(\kappa_2 + \bar{\kappa}_2)|\psi_2^{\mathrm{o}}(\lambda_2)|^2 = 0, \qquad \kappa_2 := (\psi_2^{\mathrm{o}}(\lambda_2))^{-1}\left[\psi_2^{\mathrm{o}}(-\lambda_2) + q\psi_2^{\mathrm{e}}(-\lambda_2)\right]. \qquad (6.9)$$

If (6.9) holds, then $(\lambda - \lambda_2)\det\Phi^{\mathrm{LP}}$ is zero at $\lambda = \lambda_2$, which excludes a pole of $\det\Phi^{\mathrm{LP}}$ at λ_2 as well as an essential singularity. Therefore $\det\Phi^{\mathrm{LP}}$ is regular at λ_2. If on the other hand (6.9) does not hold, then $\det\Phi^{\mathrm{LP}}$ has a pole of first order in λ_2 and because of its symmetry also in $-\lambda_2$. Hence (6.9) is equivalent to the holomorphicity of $\det\Phi^{\mathrm{LP}}$ at $\pm\lambda_2$ and a necessary condition for the coefficients p, q and r to obtain a LP solution from the linear combination (6.1).

6.3 Holomorphicity condition for $\det\Phi^{\mathrm{LP}}$ at λ_1

The derivation of a holomorphicity condition for $\det\Phi^{\mathrm{LP}}$ at λ_1 is similar to the procedure above, though a bit more involved since the dependency of $\det\Phi^{\mathrm{LP}}$ on L_1 is more complicated than its dependency on L_2. At first it is beneficial to introduce for the prefactors of the scalar solutions φ^{IK} the notation

$$H^{\mathrm{e}} := L_2^{\rho_2-1}(w^{\mathrm{e}}\Lambda^{\mathrm{e}})^{-1}, \qquad H^{\mathrm{o}} := L_2^{\rho_2-1}(1+L_2)(w^{\mathrm{o}}\Lambda^{\mathrm{o}})^{-1}. \qquad (6.10)$$

Due to the definitions and the properties of w^I and Λ^I described in section 5.4, $w^K(\lambda_1)\Lambda^K(\lambda_1)$ has the following values:

$$\begin{aligned} w^{\mathrm{e}}(\lambda_1)\Lambda^{\mathrm{e}}(\lambda_1) &= L_{12}^{\rho_2-1}, \\ w^{\mathrm{o}}(\lambda_1)\Lambda^{\mathrm{o}}(\lambda_1) &= L_{12}^{\rho_2-\frac{1}{2}}, \end{aligned} \qquad \text{with} \quad L_{12} := L_1(\lambda_2) = L_2(\lambda_1) = \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}. \qquad (6.11)$$

This yields for H^{e} and H^{o} at $\pm\lambda_1$:

$$H^{\mathrm{e}}(-\lambda_1) = 1 = H^{\mathrm{e}}(\lambda_1), \qquad H^{\mathrm{o}}(-\lambda_1) = L_{12}^{-1/2} + L_{12}^{1/2} = H^{\mathrm{o}}(\lambda_1). \qquad (6.12)$$

[1] The reader may be reminded of this work's convention $a(\lambda) + \text{c.c.} := a(\lambda) + \bar{a}(\bar{\lambda})$.

In particular H^{e} and H^{o} have no jump at $\pm\lambda_1$. Moreover, these functions obey $\bar{H}^{\mathrm{K}}(\bar{\lambda}) = H^{\mathrm{K}}$ because the corresponding relations hold already for w^{K}, Λ^{K} and L_2. The constituents of φ^{LP} regular in $\pm\lambda_1$ (with indices p and m abbreviating 'plus' and 'minus') can be collected in the four functions[2]

$$\begin{aligned} \psi^{\mathrm{e}}_{1\mathrm{p}} &:= H^{\mathrm{o}}\vartheta^{\mathrm{eo}} + \frac{r}{p}H^{\mathrm{e}}\vartheta^{\mathrm{ee}}, & \psi^{\mathrm{e}}_{1\mathrm{m}} &:= H^{\mathrm{o}}\vartheta^{\mathrm{eo}}(-\lambda) + \frac{r}{p}H^{\mathrm{e}}\vartheta^{\mathrm{ee}}(-\lambda), \\ \psi^{\mathrm{o}}_{1\mathrm{p}} &:= H^{\mathrm{o}}\vartheta^{\mathrm{oo}} + qH^{\mathrm{e}}\vartheta^{\mathrm{oe}}, & \psi^{\mathrm{o}}_{1\mathrm{m}} &:= H^{\mathrm{o}}\vartheta^{\mathrm{oo}}(-\lambda) + qH^{\mathrm{e}}\vartheta^{\mathrm{oe}}(-\lambda). \end{aligned} \tag{6.13}$$

With these functions the scalar LP solution φ^{LP} from (6.2) can be expressed as

$$\begin{aligned} \varphi^{\mathrm{LP}}(\lambda) = \cos\delta_1 L_1^{\rho_1-1}(1+L_1)\psi^{\mathrm{o}}_{1\mathrm{p}} + \mathrm{i}\sin\delta_1 L_1^{1-\rho_1}(1+L_1^{-1})\psi^{\mathrm{o}}_{1\mathrm{m}} \\ + p\left[\cos\delta_1 L_1^{\rho_1-1}\psi^{\mathrm{e}}_{1\mathrm{p}} + \mathrm{i}\sin\delta_1 L_1^{1-\rho_1}\psi^{\mathrm{e}}_{1\mathrm{m}}\right]. \end{aligned} \tag{6.14}$$

Because of $G^{IK}_{\mathrm{c}}(\pm\lambda_1) = \mathbb{1}$ the scalar CRHP solution ϑ^{IK} has no jump in $\pm\lambda_1$ and thus evaluating (6.13) at $\pm\lambda_1$ using (6.12) leads to

$$\psi^{K}_{1\mathrm{p}+}(\pm\lambda_1) = \psi^{K}_{1\mathrm{p}-}(\pm\lambda_1) = \psi^{K}_{1\mathrm{m}+}(\mp\lambda_1) = \psi^{K}_{1\mathrm{m}-}(\mp\lambda_1). \tag{6.15}$$

Unlike the situation at λ_2, during the calculation of $\det\Phi^{\mathrm{LP}}$ out of (6.14) terms proportional to various powers of L_1 occur, but only the terms with $L_1^{2\rho_1}$ and L_1 could potentially lead to an isolated singularity because they have an exponent greater or equal to one. The prefactor $(\mathrm{i}\sin\delta_1\cos\delta_1\bar{\psi}^{\mathrm{o}}_{1\mathrm{p}}(\bar{\lambda})\psi^{\mathrm{o}}_{1\mathrm{m}}(-\lambda) + \mathrm{c.c.})$ in front of $L_1^{2\rho_1}$ in $\det\Phi^{\mathrm{LP}}$ can be converted in the following way:

$$\begin{aligned} &\mathrm{i}\sin\delta_1\cos\delta_1\bar{\psi}^{\mathrm{o}}_{1\mathrm{p}}(\bar{\lambda})\psi^{\mathrm{o}}_{1\mathrm{m}}(-\lambda) + \mathrm{c.c.} \\ &= \mathrm{i}\sin\delta_1\cos\delta_1\left[H^{\mathrm{o}}\bar{\vartheta}^{\mathrm{oo}}(\bar{\lambda}) + \bar{q}H^{\mathrm{e}}\bar{\vartheta}^{\mathrm{oe}}(\bar{\lambda})\right]\left[H^{\mathrm{o}}(-\lambda)\vartheta^{\mathrm{oo}} + qH^{\mathrm{e}}(-\lambda)\vartheta^{\mathrm{oe}}\right] + \mathrm{c.c.} \\ &= \mathrm{i}\sin\delta_1\cos\delta_1\left[qH^{\mathrm{e}}(-\lambda)H^{\mathrm{o}}\vartheta^{\mathrm{oe}}\bar{\vartheta}^{\mathrm{oo}}(\bar{\lambda}) + \bar{q}H^{\mathrm{e}}H^{\mathrm{o}}(-\lambda)\bar{\vartheta}^{\mathrm{oe}}(\bar{\lambda})\vartheta^{\mathrm{oo}}\right] + \mathrm{c.c.} \\ &= \mathrm{i}\sin\delta_1\cos\delta_1\left[H^{\mathrm{e}}(-\lambda)H^{\mathrm{o}} - H^{\mathrm{e}}H^{\mathrm{o}}(-\lambda)\right]\left[q\vartheta^{\mathrm{oe}}\bar{\vartheta}^{\mathrm{oo}}(\bar{\lambda}) + \mathrm{c.c.}\right] \end{aligned} \tag{6.16}$$

In the last line the term

$$\left[H^{\mathrm{e}}(-\lambda)H^{\mathrm{o}} - H^{\mathrm{e}}H^{\mathrm{o}}(-\lambda)\right] = (1+L_2)w^{\mathrm{e}}\Lambda^{\mathrm{e}}(w^{\mathrm{o}}\Lambda^{\mathrm{o}})^{-1} + (1+L_2^{-1})(w^{\mathrm{e}}\Lambda^{\mathrm{e}})^{-1}w^{\mathrm{o}}\Lambda^{\mathrm{o}}$$

is zero at λ_1 because of (6.12). Due to the regular definitions of w^{K}, Λ^{K} and L_2 at λ_1 this is at least a (linear) simple zero at both sides of the contour (i.e. from both sides of the contour an analytical continuation in a neighbourhood of λ_1 exists and these

[2]Note that due to the λ-dependence of H^K the relation $\psi^K_{1\mathrm{p}}(\lambda) = \psi^K_{1\mathrm{m}}(-\lambda)$ does not hold in general.

analytical continuations have at least simple zeros in λ_1). Therefore the $L_1^{2\rho_1}$-term does not contribute to $\left[(\lambda-\lambda_1)\det\Phi^{\mathrm{LP}}\right]_{\lambda=\lambda_1}$.
The coefficient in front of L_1 reads:

$$\cos^2\delta_1\bar{\psi}^{\mathrm{o}}_{1\mathrm{p}}(\bar{\lambda})\left(\psi^{\mathrm{o}}_{1\mathrm{p}}(-\lambda)+p\psi^{\mathrm{e}}_{1\mathrm{p}}(-\lambda)\right)+\sin^2\delta_1\psi^{\mathrm{o}}_{1\mathrm{m}}(-\lambda)\left(\bar{\psi}^{\mathrm{o}}_{1\mathrm{m}}(\bar{\lambda})+\bar{p}\bar{\psi}^{\mathrm{e}}_{1\mathrm{m}}(\bar{\lambda})\right)+\text{c.c.}$$

Considering (6.15) it has a definite value at the point λ_1 whose vanishing is equivalent to

$$(\kappa_1+\bar{\kappa}_1)|\psi^{\mathrm{o}}_{1\mathrm{p}}(\lambda_1)|^2=0,\qquad \kappa_1:=(\psi^{\mathrm{o}}_{1\mathrm{p}}(\lambda_1))^{-1}\left[\psi^{\mathrm{o}}_{1\mathrm{p}}(-\lambda_1)+p\psi^{\mathrm{e}}_{1\mathrm{p}}(-\lambda_1)\right].\tag{6.17}$$

If and only if this relation holds, then $(\lambda-\lambda_1)\det\Phi^{\mathrm{LP}}$ is zero at λ_1. Using the same argument than above at λ_2, equation (6.17) is equivalent to the holomorphicity of $\det\Phi^{\mathrm{LP}}$ at $\pm\lambda_1$ and a necessary condition for the coefficients p, q and r to obtain a LP solution from the linear combination (6.1). Both (6.9) and (6.17) together are equivalent to the holomorphicity of $\det\Phi^{\mathrm{LP}}$ throughout $\mathbb{C}_\lambda$ and hence to the statement that $\det\Phi^{\mathrm{LP}}$ is independent of λ.

6.4 Construction of the LP matrices U and V

Assuming that (6.9) and (6.17) are fulfilled and hence $\det\Phi^{\mathrm{LP}}$ is independent of λ, it is useful to express the LP matrix U defined by (6.3) through

$$\begin{aligned}\det\Phi^{\mathrm{LP}}U&=\begin{pmatrix}\varphi_f^{\mathrm{LP}} & -\varphi_f^{\mathrm{LP}}(-\lambda)\\ \bar{\varphi}_f^{\mathrm{LP}}(\bar{\lambda}) & \bar{\varphi}_f^{\mathrm{LP}}(-\bar{\lambda})\end{pmatrix}\begin{pmatrix}\bar{\varphi}^{\mathrm{LP}}(-\bar{\lambda}) & \varphi^{\mathrm{LP}}(-\lambda)\\ -\bar{\varphi}^{\mathrm{LP}}(\bar{\lambda}) & \varphi^{\mathrm{LP}}\end{pmatrix}\\ &=\begin{pmatrix}\varphi_f^{\mathrm{LP}}\bar{\varphi}^{\mathrm{LP}}(-\bar{\lambda})+\varphi_f^{\mathrm{LP}}(-\lambda)\bar{\varphi}^{\mathrm{LP}}(\bar{\lambda}) & \varphi_f^{\mathrm{LP}}\varphi^{\mathrm{LP}}(-\lambda)-\varphi_f^{\mathrm{LP}}(-\lambda)\varphi^{\mathrm{LP}}\\ \bar{\varphi}_f^{\mathrm{LP}}(\bar{\lambda})\bar{\varphi}^{\mathrm{LP}}(-\bar{\lambda})-\bar{\varphi}_f^{\mathrm{LP}}(-\bar{\lambda})\bar{\varphi}^{\mathrm{LP}}(\bar{\lambda}) & \bar{\varphi}_f^{\mathrm{LP}}(\bar{\lambda})\varphi^{\mathrm{LP}}(-\lambda)+\bar{\varphi}_f^{\mathrm{LP}}(-\bar{\lambda})\varphi^{\mathrm{LP}}\end{pmatrix}\end{aligned}\tag{6.18}$$

and V through an analogous expression. As discussed in section 4.3, since G_J is only a function of k the LP matrices U and V calculated via (6.3) have no jump on $\Gamma_{\Re}$. Like the determinant $\det\Phi^{\mathrm{LP}}$ they can only inherit poles of $L_{1/2}^{\pm1}$ at $\pm\lambda_{1/2}$. In addition, considering the formulae for λ_f and λ_g in (3.2), a pole in U can arise at $\lambda=\infty$ and a pole in V can arise at $\lambda=0$. In consequence, U and V are holomorphic in $\mathbb{C}_\lambda\backslash\{\infty\}$ and $\mathbb{C}_\lambda\backslash\{0\}$ respectively, as required by the holomorphicity conditions (4.41), if and only if the φ^{LP}-coefficients p, q and r can be arranged so that isolated singularities at $\pm\lambda_{1/2}$ are prevented.
Due to the symmetries of (6.18) it is again sufficient to investigate only the points $\lambda_{1/2}$.

Furthermore, thanks to $U_{22} = \bar{U}_{11}(\bar{\lambda})$ and $U_{21} = \bar{U}_{12}(\bar{\lambda})$ only the matrix elements U_{11} and U_{12} as well as analogously V_{11} and V_{12} have to be considered. Another very useful circumstance is that using the formulae for λ_f and λ_g in (3.2) the derivatives of $L_{1/2}$ can be expressed by

$$L_{1f} = -\frac{\lambda\lambda_1}{f+g}L_1, \qquad L_{2f} = -\frac{\lambda\lambda_2}{f+g}L_2, \tag{6.19}$$

$$L_{1g} = -\frac{1}{\lambda\lambda_1(f+g)}L_1, \qquad L_{2g} = -\frac{1}{\lambda\lambda_2(f+g)}L_2. \tag{6.20}$$

Therefore at $\lambda_{1/2}$ the exponents of $L_{1/2}$ and hence also the divergent behaviour of powers of $L_{1/2}$ is preserved under the coordinate derivatives, although the positions $\pm\lambda_{1/2}$ of these singularities do depend on f and g. This is an artefact of the discussion on the λ-sphere, whereas on the k-surface the positions $k = \pm\frac{1}{2}$ are fixed.

6.5 Holomorphicity condition for the LP matrices at λ_2

Since the restriction to $\lambda = \lambda_2$ is solely enforced by setting $k = \frac{1}{2}$, inserting $\lambda = \lambda_2$ and taking coordinate derivatives commutes, e.g. $\psi^K_{2f}|_{\lambda=\pm\lambda_2} = [\psi^K_2(\pm\lambda_2)]_f =: \psi^K_{2f}(\pm\lambda_2)$. Hence from (6.7) follow the analogous relations for the derivatives of $\psi^K_2(\pm\lambda_2)$:

$$\psi^K_{2f+}(\pm\lambda_2) = \psi^K_{2f-}(\pm\lambda_2), \qquad \psi^K_{2g+}(\pm\lambda_2) = \psi^K_{2g-}(\pm\lambda_2). \tag{6.21}$$

The prefactor in front of L_2 in $(\det\Phi^{\mathrm{LP}})U_{12} = \varphi^{\mathrm{LP}}_f\varphi^{\mathrm{LP}}(-\lambda) - \varphi^{\mathrm{LP}}_f(-\lambda)\varphi^{\mathrm{LP}}$ reads

$$\left(\psi^{\mathrm{o}}_{2f} - \rho_2\frac{\lambda\lambda_2}{f+g}\psi^{\mathrm{o}}_2\right)\left[\psi^{\mathrm{o}}_2(-\lambda) + q\psi^{\mathrm{e}}_2(-\lambda)\right] \\ - \psi^{\mathrm{o}}_2\left(\left[\psi^{\mathrm{o}}_2(-\lambda) + q\psi^{\mathrm{e}}_2(-\lambda)\right]_f - (1-\rho_2)\frac{\lambda\lambda_2}{f+g}\left[\psi^{\mathrm{o}}_2(-\lambda) + q\psi^{\mathrm{e}}_2(-\lambda)\right]\right)$$

Using the definition (6.9) of κ_2, the vanishing of this expression at λ_2 is easily seen to be equivalent to

$$\left(\kappa_{2f} + (2\rho_2 - 1)\frac{\lambda_2^2}{f+g}\kappa_2\right)(\psi^{\mathrm{o}}_2(\lambda_2))^2 = 0. \tag{6.22}$$

For $\psi^{\mathrm{o}}_2(\lambda_2) \neq 0$ this leads together with the analogous calculation for $(\det\Phi^{\mathrm{LP}})V_{12}$ to the following two differential equations holding for $k = \frac{1}{2}$ and arbitrary (f, g):

$$(\ln\kappa_2)_f = (1-2\rho_2)\frac{\lambda_2^2}{f+g}, \qquad (\ln\kappa_2)_g = (1-2\rho_2)\frac{\lambda_2^{-2}}{f+g}. \tag{6.23}$$

The system is integrable and has the solution

$$\kappa_2 = \mathrm{i}C_2 \left(\frac{(\frac{1}{2}+f)(\frac{1}{2}-g)}{f+g} \right)^{2\rho_2 - 1}, \quad C_2 \in \mathbb{C} \tag{6.24}$$

where (6.9) yields even $C_2 \in \mathbb{R}$. The exceptional solution $\psi_2^{\mathrm{o}}(\lambda_2) = 0\,\forall(f,g)$ to (6.22) will be discussed in section 6.7.

The prefactor in front of L_2 in $(\det \Phi^{\mathrm{LP}})U_{11} = \varphi_f^{\mathrm{LP}}\bar{\varphi}^{\mathrm{LP}}(-\bar{\lambda}) + \varphi_f^{\mathrm{LP}}(-\lambda)\bar{\varphi}^{\mathrm{LP}}(\bar{\lambda})$ is

$$\left(\psi_{2f}^{\mathrm{o}} - \rho_2 \frac{\lambda\lambda_2}{f+g}\psi_2^{\mathrm{o}}\right)\left[\bar{\psi}_2^{\mathrm{o}}(-\bar{\lambda}) + \bar{q}\bar{\psi}_2^{\mathrm{e}}(-\bar{\lambda})\right] \\ + \bar{\psi}_2^{\mathrm{o}}(\bar{\lambda})\left(\left[\psi_2^{\mathrm{o}}(-\lambda) + q\psi_2^{\mathrm{e}}(-\lambda)\right]_f - (1-\rho_2)\frac{\lambda\lambda_2}{f+g}\left[\psi_2^{\mathrm{o}}(-\lambda) + q\psi_2^{\mathrm{e}}(-\lambda)\right]\right)$$

and its vanishing at λ_2 is equivalent to

$$\left[\kappa_{2f} + (2\rho_2 - 1)\frac{\lambda_2^2}{f+g}\kappa_2 + (\kappa_2 + \bar{\kappa}_2)\left(\frac{\psi_{2f}^{\mathrm{o}}(\lambda_2)}{\psi_2^{\mathrm{o}}(\lambda_2)} - \rho_2\frac{\lambda_2^2}{f+g}\right)\right]|\psi_2^{\mathrm{o}}(\lambda_2)|^2 = 0. \tag{6.25}$$

This equation is automatically fulfilled if (6.9) and (6.22) hold. The same applies to the prefactor in front of L_2 in $(\det \Phi^{\mathrm{LP}})V_{11}$.

6.6 Holomorphicity condition for the LP matrices at λ_1

Due to (6.19) and (6.20), the expressions $(\det \Phi^{\mathrm{LP}})U_{11}$ and $(\det \Phi^{\mathrm{LP}})U_{12}$ feature powers of L_1 with the same variety of exponents as $\det \Phi^{\mathrm{LP}}$. Again only the terms containing $L_1^{2\rho_1}$ and L_1 could lead to isolated singularities at λ_1. It will be shown that due to the regular behaviour of the functions H^K defined in (6.10) with $H^K(-\lambda_1) = H^K(\lambda_1)$ and its derivatives, the coefficients of $L_1^{2\rho_1}$ all contain at least simple zeros on both sides of the contour. Since under this condition such terms are bounded by $C_{\lambda_1}(\lambda - \lambda_1)^{2\rho_1 - 1}$ at $\lambda = \lambda_1$ for some C_{λ_1}, it can be concluded that they cannot lead to isolated singularities at λ_1. At first the prefactor in front of $L_1^{2\rho_1}$ in $(\det \Phi^{\mathrm{LP}})U_{11}$ shall be considered, which takes the form

$$\frac{\sin 2\delta_1}{2\mathrm{i}}\left[\psi_{1\mathrm{p}f}^{\mathrm{o}}\bar{\psi}_{1\mathrm{m}}^{\mathrm{o}}(-\bar{\lambda}) - \psi_{1\mathrm{m}f}^{\mathrm{o}}(-\lambda)\bar{\psi}_{1\mathrm{p}}^{\mathrm{o}}(\bar{\lambda}) - \rho_2\frac{\lambda\lambda_2}{f+g}\left(\bar{\psi}_{1\mathrm{p}}^{\mathrm{o}}(\bar{\lambda})\psi_{1\mathrm{m}}^{\mathrm{o}}(-\lambda) - \mathrm{c.c.}\right)\right].$$

Herein the last term inside the square brackets is proportional to (6.16) and can therefore not lead to isolated singularities as discussed above. Inserting (6.13) the first and the second term together admit of the following conversion:

$$\begin{aligned}
&\tfrac{1}{2\mathrm{i}}\sin 2\delta_1 \left[\psi^{\mathrm{o}}_{1\mathrm{p}f}\bar{\psi}^{\mathrm{o}}_{1\mathrm{m}}(-\bar{\lambda}) - \psi^{\mathrm{o}}_{1\mathrm{m}f}(-\lambda)\bar{\psi}^{\mathrm{o}}_{1\mathrm{p}}(\bar{\lambda})\right] \\
&= \tfrac{1}{2\mathrm{i}}\sin 2\delta_1 \Big[\left(H^{\mathrm{o}}\vartheta^{\mathrm{oo}} + qH^{\mathrm{e}}\vartheta^{\mathrm{oe}}\right)_f \left(H^{\mathrm{o}}(-\lambda)\bar{\vartheta}^{\mathrm{oo}}(\bar{\lambda}) + \bar{q}H^{\mathrm{e}}(-\lambda)\bar{\vartheta}^{\mathrm{oe}}(\bar{\lambda})\right) \\
&\qquad\qquad - \left(H^{\mathrm{o}}(-\lambda)\vartheta^{\mathrm{oo}} + qH^{\mathrm{e}}(-\lambda)\vartheta^{\mathrm{oe}}\right)_f \left(H^{\mathrm{o}}\bar{\vartheta}^{\mathrm{oo}}(\bar{\lambda}) + \bar{q}H^{\mathrm{e}}\bar{\vartheta}^{\mathrm{oe}}(\bar{\lambda})\right)\Big] \\
&= \tfrac{1}{2\mathrm{i}}\sin 2\delta_1 \Big[\left(H^{\mathrm{o}}H^{\mathrm{e}}(-\lambda) - H^{\mathrm{o}}(-\lambda)H^{\mathrm{e}}\right)\left(\vartheta^{\mathrm{oo}}_f\bar{q}\bar{\vartheta}^{\mathrm{oe}}(\bar{\lambda}) - (q\vartheta^{\mathrm{oe}})_f\bar{\vartheta}^{\mathrm{oo}}(\bar{\lambda})\right) \\
&\quad + \left(H^{\mathrm{o}}_f H^{\mathrm{o}}(-\lambda) - H^{\mathrm{o}}_f(-\lambda)H^{\mathrm{o}}\right)\vartheta^{\mathrm{oo}}\bar{\vartheta}^{\mathrm{oo}}(\bar{\lambda}) + \left(H^{\mathrm{e}}_f H^{\mathrm{e}}(-\lambda) - H^{\mathrm{e}}_f(-\lambda)H^{\mathrm{e}}\right)|q|^2\vartheta^{\mathrm{oe}}\bar{\vartheta}^{\mathrm{oe}}(\bar{\lambda}) \\
&\quad + \left(H^{\mathrm{o}}_f H^{\mathrm{e}}(-\lambda) - H^{\mathrm{o}}_f(-\lambda)H^{\mathrm{e}}\right)\vartheta^{\mathrm{oo}}\bar{q}\bar{\vartheta}^{\mathrm{oe}}(\bar{\lambda}) + \left(H^{\mathrm{e}}_f H^{\mathrm{o}}(-\lambda) - H^{\mathrm{e}}_f(-\lambda)H^{\mathrm{o}}\right)q\vartheta^{\mathrm{oe}}\bar{\vartheta}^{\mathrm{oo}}(\bar{\lambda})\Big]
\end{aligned} \tag{6.26}$$

A thorough study of the definitions of the quantities contained in the functions H^K and H^K_f reveals that for $f < \frac{1}{2}$ they can be analytically continued in a neighbourhood of $k = -\frac{1}{2}$ from both sides of the contour. Furthermore, from $H^K(\lambda_1) = H^K(-\lambda_1)\ \forall(f,g)$ derived in section 6.3 follows $H^K_f(\lambda_1) = H^K_f(-\lambda_1)$. Therefore, taking also into account the regularity of H^K and H^K_f at $\pm\lambda_1$, all the brackets containing $H^K(\pm\lambda)$ and $H^K_f(\pm\lambda)$ on the right hand side of (6.26) have at least a simple zero at λ_1 from both sides of the contour. Thus the terms in (6.26) do not yield isolated singularities at λ_1.

An analogous treatment of the prefactor in front of $L_1^{2\rho_1}$ in $(\det\Phi^{\mathrm{LP}})U_{12}$ results in the expression

$$\begin{aligned}
&\tfrac{\mathrm{i}}{2}\sin 2\delta_1 \Big[\left(H^{\mathrm{o}}H^{\mathrm{e}}(-\lambda) - H^{\mathrm{o}}(-\lambda)H^{\mathrm{e}}\right)\left(\vartheta^{\mathrm{oo}}_f q\vartheta^{\mathrm{oe}} - (q\vartheta^{\mathrm{oe}})\vartheta^{\mathrm{oo}}\right) \\
&\quad + \left(H^{\mathrm{o}}_f H^{\mathrm{o}}(-\lambda) - H^{\mathrm{o}}_f(-\lambda)H^{\mathrm{o}}\right)\vartheta^{\mathrm{oo}}\vartheta^{\mathrm{oo}} + \left(H^{\mathrm{e}}_f H^{\mathrm{e}}(-\lambda) - H^{\mathrm{e}}_f(-\lambda)H^{\mathrm{e}}\right)|q|^2\vartheta^{\mathrm{oe}}\vartheta^{\mathrm{oe}} \\
&\quad + \left(H^{\mathrm{o}}_f H^{\mathrm{e}}(-\lambda) - H^{\mathrm{o}}_f(-\lambda)H^{\mathrm{e}} + H^{\mathrm{e}}_f H^{\mathrm{o}}(-\lambda) - H^{\mathrm{e}}_f(-\lambda)H^{\mathrm{o}}\right)q\vartheta^{\mathrm{oe}}\vartheta^{\mathrm{oo}}\Big].
\end{aligned} \tag{6.27}$$

Again all the brackets containing $H^K(\pm\lambda)$ and $H^K(\pm\lambda)_f$ in (6.27) have at least a simple zero at λ_1 from both sides of the contour and therefore these terms do not yield isolated singularities at λ_1 for $f < \frac{1}{2}$. In summary, the terms containing $L_1^{2\rho_1}$ in $(\det\Phi^{\mathrm{LP}})U_{12}$ and $(\det\Phi^{\mathrm{LP}})U_{11}$ do not give rise to isolated singularities in case of $f < \frac{1}{2}$ and the same result is obtained for $(\det\Phi^{\mathrm{LP}})V_{12}$ and $(\det\Phi^{\mathrm{LP}})V_{11}$ in case of $g < \frac{1}{2}$. The study of the limits $f \to \frac{1}{2}$ and $g \to \frac{1}{2}$ is more involved and will be discussed in section 6.7.

The evaluation of the prefactors in front of L_1 in the LP matrix elements is similar to the investigation in section 6.5, but additionally the relation (6.15) between $\psi^K_{1\mathrm{p}}(\pm\lambda_1)$ and $\psi^K_{1\mathrm{m}}(\pm\lambda_1)$ as well as its derivatives have to be used. This yields the analogous holomorphicity conditions for $(\det\Phi^{\mathrm{LP}})U_{12}$ and $(\det\Phi^{\mathrm{LP}})U_{11}$,

$$\left(\kappa_{1f}+(2\rho_1-1)\frac{\lambda_1^2}{f+g}\kappa_1\right)(\psi^{\mathrm{o}}_{1\mathrm{p}}(\lambda_1))^2=0, \quad (6.28)$$

$$\left[\kappa_{1f}+(2\rho_1-1)\frac{\lambda_1^2}{f+g}\kappa_1+(\kappa_1+\bar\kappa_1)\left(\frac{\psi^{\mathrm{o}}_{1\mathrm{p}f}(\lambda_1)}{\psi^{\mathrm{o}}_{1\mathrm{p}}(\lambda_1)}-\rho_1\frac{\lambda_1^2}{f+g}\right)\right]|\psi^{\mathrm{o}}_{1\mathrm{p}}(\lambda_1)|^2=0 \quad (6.29)$$

and corresponding relations for $(\det\Phi^{\mathrm{LP}})V_{12}$ and $(\det\Phi^{\mathrm{LP}})V_{11}$. Again, (6.29) is automatically fulfilled if (6.17) and (6.28) hold.

Equation (6.28) has a solution analogous to (6.24), wherein also the condition (6.17) inferred from the holomorphicity of the determinant confines κ_1 to be purely imaginary. With real constants C_1 and C_2, the full system of holomorphicity conditions can be stated as

$$\begin{aligned}\kappa_2&=\mathrm{i}C_2\left(\frac{(\frac12+f)(\frac12-g)}{f+g}\right)^{2\rho_2-1}=:\mathrm{i}C_2L_{02}^{2\rho_2-1}, && C_2\in\mathbb{R},\\ \kappa_1&=\mathrm{i}C_1\left(\frac{(\frac12-f)(\frac12+g)}{f+g}\right)^{2\rho_1-1}=:\mathrm{i}C_1L_{01}^{2\rho_1-1}, && C_1\in\mathbb{R}.\end{aligned} \quad (6.30)$$

These relations assure that the combined holomorphicity conditions at λ_2, (6.9), (6.22) and (6.25) as well as their V-counterparts, and the combined holomorphicity conditions at λ_1, (6.17), (6.28) and (6.29) as well as their V-counterparts respectively are fulfilled. However, at this stage there are also the exceptional solutions $\psi^{\mathrm{o}}_2(\lambda_2)=0$ for the combined holomorphicity conditions at λ_2 and $\psi^{\mathrm{o}}_{1\mathrm{p}}(\lambda_1)=0$ for the combined holomorphicity conditions at λ_1. They will be excluded on a heuristic level by an argument in section 6.7.

In order to determine the LP solution from the linear combination of the RHP solutions, the equations (6.30) have to be solved, which represents step (iii) of the solution scheme in figure 3.1. The convenience of these two purely algebraic relations is essential for the performance of the whole ISM: it would not have been very beneficial for solving the IVP of the Ernst equation if the ISM had led to an equally challenging system of holomorphicity conditions.

Another aspect of the solution procedure becoming evident in (6.30) is that these are two equations for the three φ^{LP}-coefficients p, q and r. Therefore a function of f and g may be left free to choose in the scalar LP solution φ^{LP} and thus the ISM may serve as a solution generating technique, which will be confirmed in chapter 7. However, from the above discussion it is not clear that the 'right' LP solution uniquely associated with the IVP definitely lies in the class of solutions generated via the ISM.

6.7 Behaviour of the RHP solution for $g \to \frac{1}{2}$ and $f \to \frac{1}{2}$

The last section is devoted to a study of the RHP and its solutions near the wave fronts, i.e. in the limits $f \to \frac{1}{2}$, which implies $\lambda_1 \to \infty$ and $L_1 \to 1$, as well as $g \to \frac{1}{2}$, which implies $\lambda_2 \to 0$ and $L_2 \to 1$. In particular it collects interesting considerations concerning these limits, which cannot be formulated as rigorous statements within the scope of this work. Again, the discussion benefits a lot from consulting the transformation to the CRHP.

For $g \to \frac{1}{2}$ the contour part Γ_2 contracts symmetrically around the point $\lambda = 0$. At first it shall be investigated if the limit of the ERHP solution for $g \to \frac{1}{2}$ is indeed the solution for the ERHP with

$$G_{J_1} := \begin{cases} J, & \lambda \in \Gamma_1; \\ \mathbb{1} & \text{else.} \end{cases} \tag{6.31}$$

i.e. for the ERHP formulated as if the contour part Γ_2 never had existed.

Note that the ERHP jump matrix G_J converges pointwise to G_{J_1} only for $\lambda \neq 0$, whereas its limit at $\lambda = 0$ is the boundary value of the jump matrix,

$$\lim_{g \to \frac{1}{2}} G_J(0) = J_{2\text{b}} = \operatorname{diag}(\mathrm{e}^{-2\pi \mathrm{i} \rho_2}, \mathrm{e}^{2\pi \mathrm{i} \rho_2}). \tag{6.32}$$

On the other hand, the singularity transformation matrices S_2^K defined in (5.18) converge pointwise to $\mathbb{1}$ for $\lambda \neq 0$ and at $\lambda = 0$ their (f, g)-independent values read

$$S_{2+}^K(0) = \operatorname{diag}(\mathrm{e}^{\pi \mathrm{i} \rho_2}, \mathrm{e}^{-\pi \mathrm{i} \rho_2}), \qquad S_{2-}^K(0) = \operatorname{diag}(\mathrm{e}^{-\pi \mathrm{i} \rho_2}, \mathrm{e}^{\pi \mathrm{i} \rho_2}). \tag{6.33}$$

Therefore the ERHP jump matrix $\tilde{G}_J^K := (S_{2-}^K)^{-1} G_J S_{2+}^K$ after the first singularity transformation converges pointwise to G_{J_1}. Because of $\lim_{g \to \frac{1}{2}} U^K = \mathbb{1}$ throughout $\Gamma_{\Re}$, the corresponding CRHP matrix converges also to the CRHP matrix obtained from G_{J_1} via rotation transformation and the second singularity transformation,

$$\lim_{g \to \frac{1}{2}} G_c^{IK} = (S_{1-}^I)^{-1} R_{\delta_1}^{-1} G_{J_1} R_{\delta_1} S_{1+}^I. \tag{6.34}$$

Similarly, also the pointwise convergence

$$\lim_{f \to \frac{1}{2}} G_c^{IK} = R_{\delta_1}^{-1} (S_{2-}^K)^{-1} G_{J_2} S_{2+}^K R_{\delta_1}, \qquad G_{J_2} := \begin{cases} J, & \lambda \in \Gamma_2; \\ \mathbb{1} & \text{else.} \end{cases} \tag{6.35}$$

can be established. In particular these limiting values are independent of the values of K and I respectively, which means that the corresponding pairs of solutions are reunited for $g = \frac{1}{2}$ and $f = \frac{1}{2}$ respectively. A numerical analysis suggests even the uniform convergence of G_c^{IK} for $f \to \frac{1}{2}$ and for $g \to \frac{1}{2}$.

In the combined limit $f, g \to \frac{1}{2}$, the CRHP jump matrix converges pointwise to $\mathbb{1}$, which trivially corresponds to the ERHP with $G_J = \mathbb{1}$. The unique scalar solution without zeros and poles to this ERHP is $\varphi(\frac{1}{2}, \frac{1}{2}) = \text{const}$, where the constant is set to 1 in case of a RHP normalised by $\varphi(-1) = 1$. This coincides with the normalisation $\varphi^{\text{LP}}(\frac{1}{2}, \frac{1}{2}) = 1$ of the LP. If it could be shown that the pointwise convergence of the CRHP jump matrix to a Hölder continuous limit G_c^{IK*} implies the pointwise convergence of the CRHP solution to the solution of the CRHP with G_c^{IK*}, then the right normalisation of LP solutions obtained from RHP solutions would be proven. Regrettably, to the knowledge of the author no such statement exists.

However, other evidence for the regularity of the RHP solution at $f \to \frac{1}{2}$ and $g \to \frac{1}{2}$ stems from an approximate solution for $f \approx \frac{1}{2}$ and $g \approx \frac{1}{2}$, which is motivated in the following way:

For $g = \frac{1}{2} - \epsilon$, $\epsilon \ll 1$ the contour in the k-surface contracts to the twofold covering of $[\frac{1}{2} - \epsilon, \frac{1}{2}]$. On this interval the difference[3] $||J(k) - J_{2\text{b}}||_\infty$ between the jump matrix and its constant boundary value is bounded due to the Lipschitz continuity of $J(k)$. In the limit $g \to \frac{1}{2}$ the deviation $||J(k) - J_{2\text{b}}||_\infty$ goes to zero.

On the other hand, exact solutions of the ERHP with the constant jump matrix $G_J = J_{2b}$ on Γ_2 and vanished contour Γ_1 (i.e. $f = \frac{1}{2}$ and $G_J|_{\Gamma_\Re \setminus \Gamma_2} = \mathbb{1}$) can be obtained. The two independent solutions with a divergence exponent lower than one have the (unnormalised) scalar solutions

$$\varphi_{20} = L_2^{\rho_2}, \qquad \varphi_{21} = L_2^{\rho_2 - 1} \tag{6.36}$$

introduced in section 4.5. This fits perfectly to the representation (6.6) of the linear combination φ^{LP}, which also contains only terms proportional to $L_2^{\rho_2}$ and $L_2^{\rho_2-1}$.

Similarly, the ERHP with constant jump matrix $G_J = J_{1b}$ on the other contour Γ_1 and vanished Γ_2 (i.e. $g = \frac{1}{2}$ and $G_J|_{\Gamma_\Re \setminus \Gamma_1} = \mathbb{1}$) is exactly solved by the scalar solutions $\varphi'_{10} = L_1^{\rho_1}$ and $\varphi'_{11} = L_1^{\rho_1 - 1}$ in an x-y-frame where J_{1b} is diagonal. For this work's standard frame where J_{2b} is diagonal, a rotation transformation of φ'_{10} and φ'_{11} with

[3] The expression $||J(k) - J_{2\text{b}}||_\infty := \sup_{k \in [\frac{1}{2} - \epsilon, \frac{1}{2}]} |J(k) - J_{2\text{b}}|$ denotes the uniform norm on the interval $[\frac{1}{2} - \epsilon, \frac{1}{2}]$.

$R_{-\delta_1}$ yields the (unnormalised) solutions

$$\varphi_{10} = L_1^{\rho_1} + \mathrm{i}\tan\delta_1 L_1^{-\rho_1}, \qquad \varphi_{11} = L_1^{\rho_1-1} + \mathrm{i}\tan\delta_1 L_1^{1-\rho_1}. \tag{6.37}$$

In a further step it can be observed that the value of the linear combination $\varphi_2 := \varphi_{20} + a_2(f,g)\varphi_{21}$ solving the scalar jump equation for constant jump functions on Γ_2 goes to the λ-independent value $1 + a_2(f,g)$ on the other contour part Γ_1 in both of the limits $f \to \frac{1}{2}$ or $g \to \frac{1}{2}$. A corresponding statement holds for $\varphi_1 := \varphi_{10} + a_1(f,g)\varphi_{11}$. Hence the scalar solution of a RHP with constant jump matrices on both contour parts $\Gamma_{1/2}$ is expected to be approximated by the product $\varphi_1\varphi_2$ for $f \approx \frac{1}{2}$ and $g \approx \frac{1}{2}$. In conclusion, the scalar solution of a *general* RHP is conjectured to behave for $f \approx \frac{1}{2}$ and $g \approx \frac{1}{2}$ like the approximate solution

$$\varphi^* = \varphi_1\varphi_2 = (L_2^{\rho_2} + a_2 L_2^{\rho_2-1})\left[(L_1^{\rho_1} + a_1 L_1^{1-\rho_1}) + \mathrm{i}\tan\delta_1(L_1^{-\rho_1} + a_1 L_1^{\rho_1-1})\right] \tag{6.38}$$

based on the two assumptions:

- solutions are expected to converge to constant jump solutions for $f, g \to \frac{1}{2}$,
- scalar solutions are expected to factorise into two solutions solving the scalar jump equation on each partial contour separately for $f, g \to \frac{1}{2}$.

In the double limit $f, g \to \frac{1}{2}$ the approximation φ^* has indeed the λ-independent value $(1 + a_2)(1 + a_1)(1 + \mathrm{i}\tan\delta_1)$ and thus an approximate LP solution with the right normalisation is obtained by $\varphi^{*\mathrm{LP}} = \varphi^*/\varphi^*(-1)$.

Finally, it is especially instructive to evaluate the colliding wave conditions (2.48) for φ^*. Using the Ernst potential $E^* = \varphi^*(1)/\varphi^*(-1)$ corresponding to the normalised RHP solution, the first condition yields

$$\frac{1}{2}\lim_{g\to\frac{1}{2}}\left[\sqrt{\tfrac{1}{2} - g}\,E_g^*(\tfrac{1}{2}, g)\right] = -\rho_2 + \frac{a_2(\frac{1}{2},\frac{1}{2})}{1 + a_2(\frac{1}{2},\frac{1}{2})}. \tag{6.39}$$

Remembering $\phi_B = \pi$ arranged by the choice of x and y made at the end of section (5.2), this matches the definition (4.54) of ρ_2 only for $a_2(\frac{1}{2},\frac{1}{2}) = 0$. Evaluation of the second colliding wave condition is most easily done in coordinates diagonalising $J_{1\mathrm{b}}$ and leads similarly to $a_1(\frac{1}{2},\frac{1}{2}) = 0$. This suggests that for scalar solutions giving rise to Ernst potentials matching the colliding wave conditions, the approximation (6.38) can be refined to

$$\varphi(f \approx \tfrac{1}{2}, g \approx \tfrac{1}{2}) \approx \varphi^{**} = L_2^{\rho_2}(L_1^{\rho_1} + \mathrm{i}\tan\delta_1 L_1^{-\rho_1}). \tag{6.40}$$

In this sense the colliding wave conditions seem to require the terms with divergence exponents $\rho_{1/2}$ in φ to be dominant in the limit $f, g \to \frac{1}{2}$. Considering the representation (6.6) of the scalar LP solution, φ^{LP} approaches φ^{**} for $[\psi_2^{\mathrm{o}} + q\psi_2^{\mathrm{e}}]_{f=\frac{1}{2},g=\frac{1}{2}} = 0$. This can be seen to be equivalent to

$$\left.\frac{q\vartheta^{\mathrm{oe}} + r\vartheta^{\mathrm{ee}}}{\vartheta^{\mathrm{oo}} + p\vartheta^{\mathrm{eo}}}\right|_{f=\frac{1}{2},g=\frac{1}{2}} = 1. \tag{6.41}$$

Analogously the colliding wave condition for $E_f(\frac{1}{2}, \frac{1}{2})$ seems to imply

$$\left.\frac{p\vartheta^{\mathrm{eo}} + r\vartheta^{\mathrm{ee}}}{\vartheta^{\mathrm{oo}} + q\vartheta^{\mathrm{oe}}}\right|_{f=\frac{1}{2},g=\frac{1}{2}} = 1. \tag{6.42}$$

Equations (6.41) and (6.42) are supposed to be conditions for the remnant degrees of freedom in the coefficients p, q and r which assure that the colliding wave conditions hold and thus a proper colliding wave spacetime is generated. These are only two (constant) conditions, whereas the discussion at the end of the last section implied that a functional degree of freedom could be left undetermined by the holomorphicity conditions (6.30). Therefore, a whole class of spacetimes may result from the ISM and the Ernst potential matching the initial data may have to be identified within this class, which is the last step (iv) of the solution scheme in figure 3.1.
Moreover, the exceptional solution $\psi_{1/2}^{\mathrm{o}}(\lambda_{1/2}) = 0$ to the holomorphicity conditions would eliminate the terms with divergence exponent $\rho_{1/2}$ in φ^{LP} for all f and g and hence should be excluded on the basis of this discussion.

7 Example: Generalisation of the Szekeres class of solutions

In order to exemplify the solution generation technique embedded in the described inverse scattering procedure, the generalisation of the Szekeres class of collinearly polarised colliding wave spacetimes [31] is studied. This class is a unification of the first exact colliding plane wave solutions including the one of K. A. Khan and Roger Penrose [40] and a step wave solution found even earlier by George Szekeres [39].

The treatment of the Szekeres class of solutions with the ISM leads to a generalised family of colliding wave spacetimes. The initial waves can be regarded as having a specific chirality and the metric functions as well as derived quantities feature interesting structures depending on the relation of the initial waves' chiralities. Moreover, this class of solutions features a limit of what could be called 'circularly polarised impulsive waves'. Finally, the singularity at $f + g = 0$ is shown to be a scalar curvature singularity.

7.1 General solution of the linear problem

Remarkably, the Szekeres class of collinearly polarised vacuum solutions also corresponds to a very easy solution in terms of the ISM. Using the abbreviation

$$L_{1/2\mathrm{p}} := L_{1/2}(1) = L_{1/2}^{-1}(-1), \tag{7.1}$$

the Ernst potential of the Szekeres class reads $E_{\mathrm{Sz}} = L_{1\mathrm{p}}^{2\rho_1} L_{2\mathrm{p}}^{2\rho_2}$ with the exponents $\rho_{1/2}$ varying in the range (4.56) prescribed by the colliding wave conditions. This E_{Sz} can be regarded as the associated Ernst potential of a RHP with scalar solution

$$\varphi_{\mathrm{Sz}} = L_1^{\rho_1} L_2^{\rho_2}. \tag{7.2}$$

The piecewise constant jump matrix of the associated matrix solution Φ_{Sz} is given by

$$J = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \quad \text{with} \quad \alpha = \begin{cases} \mathrm{e}^{-2\pi i \rho_2} & \text{on } \Gamma_2, \\ \mathrm{e}^{2\pi i \rho_1} & \text{on } \Gamma_1, \\ 1 & \text{else.} \end{cases} \tag{7.3}$$

Since it is diagonal everywhere on $\Gamma_{\Re}$, the corresponding spacetime in particular lies in the class of initially collinearly polarised GWs and it follows that $R_{\delta_1} = \mathbb{1}$. Furthermore, there is actually no need for the unitarisation transformation with U^K. However, in order to illustrate the ISM procedure, the derivation presented here will make use of the full transformation formula (5.40) maintaining also the transformation with U^K. This leads to the CRHP jump matrix

$$G_c^{IK} = \mathrm{diag}(\alpha^K, \bar{\alpha}^K), \qquad \alpha^K = (w_+^K)^2 \,\mathrm{sign}^2(\Lambda_+^K) \tag{7.4}$$

and the scalar CRHP solutions $\vartheta^{IK} = w^K \Lambda^K$. Via inverse transformation the four different scalar RHP solutions

$$\varphi^{\mathrm{ee}} = L_1^{\rho_1 - 1} L_2^{\rho_2 - 1}, \qquad \varphi^{\mathrm{oe}} = L_1^{\rho_1 - 1} L_2^{\rho_2 - 1}(1 + L_1), \tag{7.5}$$

$$\varphi^{\mathrm{eo}} = L_1^{\rho_1 - 1} L_2^{\rho_2 - 1}(1 + L_2), \qquad \varphi^{\mathrm{oo}} = L_1^{\rho_1 - 1} L_2^{\rho_2 - 1}(1 + L_1)(1 + L_2). \tag{7.6}$$

are obtained. The holomorphicity conditions imposed on the linear combination $\varphi^{\mathrm{LP}} = \varphi^{\mathrm{oo}} + p\varphi^{\mathrm{eo}} + q\varphi^{\mathrm{oe}} + r\varphi^{\mathrm{ee}}$ to satisfy the corresponding LP read

$$\kappa_2 = (\psi_2^{\mathrm{o}}(\lambda_2))^{-1} \left[\psi_2^{\mathrm{o}}(-\lambda_2) + q\psi_2^{\mathrm{e}}(-\lambda_2)\right] \overset{!}{=} \mathrm{i}C_2 L_{02}^{2\rho_2 - 1}, \tag{7.7}$$

$$\kappa_1 = (\psi_{1\mathrm{p}}^{\mathrm{o}}(\lambda_1))^{-1} \left[\psi_{1\mathrm{p}}^{\mathrm{o}}(-\lambda_1) + p\psi_{1\mathrm{p}}^{\mathrm{e}}(-\lambda_1)\right] \overset{!}{=} \mathrm{i}C_1 L_{01}^{2\rho_1 - 1} \tag{7.8}$$

with $C_{1/2} \in \mathbb{R}$, $L_{01/2}$ defined in (6.30) and

$$\psi_2^{\mathrm{e}} = L_1^{\rho_1 - 1}\left[(1 + L_1) + r/q\right], \qquad \psi_2^{\mathrm{o}} = L_1^{\rho_1 - 1}\left[(1 + L_1) + p\right], \tag{7.9}$$

$$\psi_{1\mathrm{p}}^{\mathrm{e}} = L_2^{\rho_2 - 1}\left[(1 + L_2) + r/p\right], \qquad \psi_{1\mathrm{p}}^{\mathrm{o}} = L_2^{\rho_2 - 1}\left[(1 + L_2) + q\right]. \tag{7.10}$$

Using the identity $1 + L_1 L_2 - L_{12}(L_1 + L_2) = 0$, they can be solved for p and q to give the LP solution

$$\varphi^{\mathrm{LP}} = L_1^{\rho_1} L_2^{\rho_2}\left[1 - L_{12}^2 + L_{12}^2\left(1 + \mathrm{i}C_1 L_{01}^{2\rho_1 - 1} L_{12}^{2\rho_2 - 2} L_1^{-1}\right)\left(1 + \mathrm{i}C_2 L_{02}^{2\rho_2 - 1} L_{12}^{2\rho_1 - 2} L_2^{-1}\right)\right]. \tag{7.11}$$

The corresponding Ernst potential $E = \varphi^{\mathrm{LP}}(1)/\varphi^{\mathrm{LP}}(-1)$ obeys the Ernst equation and the colliding wave conditions without further restrictions. Since $L_{01/2} \to 0$ and $L_{12} \to 1$ in the double limit $f, g \to \frac{1}{2}$, this can be regarded as corresponding to the fact that φ^{LP} has the right limiting behaviour $\varphi^{\mathrm{LP}}(f \approx \frac{1}{2}, g \approx \frac{1}{2}) = L_1^{\rho_1} L_2^{\rho_2}$ suggested by (6.40). For this class of solutions the third functional degree of freedom not determined by the two holomorphicity conditions turns out to be an overall factor in φ, which has already been omitted in (7.11) due to its insignificance for the physical Ernst potential. Nevertheless, the two scalar real parameters $C_{1/2}$ remain, in terms of which (7.11) is a generalisation of the Szekeres class (7.2). The scalar RHP solution $\varphi_{\mathrm{Sz}} = L_1^{\rho_1} L_2^{\rho_2}$ corresponding to the original Szekeres class is reproduced for $C_1 = 0 = C_2$. The special cases $C_2 = 0$ and $C_1 = 0$ correspond to the 'unilateral' generalised Szekeres classes

$$\begin{aligned} \varphi_1^{\mathrm{LP}} &= L_1^{\rho_1 - 1} L_2^{\rho_2} (L_1 + \mathrm{i} C_1 L_{01}^{2\rho_1 - 1} L_{12}^{2\rho_2}), \\ \varphi_2^{\mathrm{LP}} &= L_1^{\rho_1} L_2^{\rho_2 - 1} (L_2 + \mathrm{i} C_2 L_{02}^{2\rho_2 - 1} L_{12}^{2\rho_1}). \end{aligned} \tag{7.12}$$

Note that also for the limiting case $\rho_{1/2} = \frac{1}{2}$ of impulsive waves, which had been excluded in the derivation of the CRHP, the expression (7.11) leads to a solution of the Ernst equation fulfilling the colliding wave conditions.

7.2 Metric functions

From (7.11) the Ernst potential $E = \varphi^{\mathrm{LP}}(1)/\varphi^{\mathrm{LP}}(-1)$ of the generalised Szekeres class is determined to be

$$E = L_{1\mathrm{p}}^{2\rho_1} L_{2\mathrm{p}}^{2\rho_2} \frac{1 - L_{12}^2 + L_{12}^2 \left(1 + \mathrm{i} C_1 L_{01}^{2\rho_1 - 1} L_{12}^{2\rho_2 - 2} L_{1\mathrm{p}}^{-1}\right) \left(1 + \mathrm{i} C_2 L_{02}^{2\rho_2 - 1} L_{12}^{2\rho_1 - 2} L_{2\mathrm{p}}^{-1}\right)}{1 - L_{12}^2 + L_{12}^2 \left(1 + \mathrm{i} C_1 L_{01}^{2\rho_1 - 1} L_{12}^{2\rho_2 - 2} L_{1\mathrm{p}}\right) \left(1 + \mathrm{i} C_2 L_{02}^{2\rho_2 - 1} L_{12}^{2\rho_1 - 2} L_{2\mathrm{p}}\right)}.$$

For $C_{1/2} \neq 0$ the Ernst potential is complex and hence the Szekeres class of collinearly polarised colliding waves is expanded to general polarisation. Via the field equations (2.33) and (2.34) the last metric function e^{-M} can be determined as

$$\begin{aligned} \mathrm{e}^{-M} &= \frac{f_u g_v}{c_1 c_2 n_1 n_2 \sqrt{f+g}} L_{01}^{-2\rho_1^2} L_{02}^{-2\rho_2^2} L_{12}^{-4\rho_1\rho_2} \left[\left(1 + C_1 C_2 L_{01}^{2\rho_1 - 1} L_{02}^{2\rho_2 - 1} L_{12}^{2\rho_1 + 2\rho_2 - 2}\right)^2 \right. \\ &\qquad \left. + \left(C_2 L_{02}^{2\rho_2 - 1} L_{12}^{2\rho_1} - C_1 L_{01}^{2\rho_1 - 1} L_{12}^{2\rho_2}\right)^2 \right] \\ &= \frac{f_u g_v}{c_1 c_2 n_1 n_2 \sqrt{f+g}} L_{01}^{-2\rho_1^2} L_{02}^{-2\rho_2^2} L_{12}^{-4\rho_1\rho_2} |\varphi_r(0)|^2 \end{aligned} \tag{7.13}$$

with the prefactor-reduced LP solution $\varphi_r := L_1^{-\rho_1} L_2^{-\rho_2} \varphi^{\mathrm{LP}}$.

Of considerable interest is the limit of the metric functions at the singularity $f+g=0$, because it is rather oppositional to their behaviour in the special case of the Szekeres class. Setting $\epsilon := f+g > 0$ yields for $\epsilon \to 0$ the limiting behaviours $L_{1/2\mathrm{p}} \sim \epsilon^{-1}$, $L_{01/2} \sim \epsilon^{-1}$ and $L_{12} \sim \epsilon^{-1}$. Using the definitions (2.45) of $f(u)$ and $g(v)$, the limiting behaviour of the metric functions for $\epsilon \to 0$ can be expressed by

$$
\begin{aligned}
E &\sim -\mathrm{i}2^{-2}\mathcal{S}, \qquad \mathcal{S} := 2^{4\rho_1}(\tfrac{1}{2}-f)^{3-2\rho_1-2\rho_2} C_1^{-1} + 2^{4\rho_2}(\tfrac{1}{2}+f)^{3-2\rho_1-2\rho_2} C_2^{-1}, \\
e^{-M} &\sim 2^{-8\rho_1\rho_2}(\tfrac{1}{2}-f)^{-2\rho_1(\rho_1+2\rho_2-2)}(\tfrac{1}{2}+f)^{-2\rho_2(\rho_2+2\rho_1-2)}\mathcal{D}^2 \epsilon^{(\rho_1+\rho_2-\frac{1}{2})(2\rho_1+2\rho_2-3)}, \\
&\qquad \mathcal{D} := 2^{4\rho_2}(\tfrac{1}{2}-f)^{2(\rho_1+\rho_2-1)} C_1 - 2^{4\rho_1}(\tfrac{1}{2}+f)^{2(\rho_1+\rho_2-1)} C_2.
\end{aligned}
\tag{7.14}
$$

In contrast to the divergence $E_{\mathrm{Sz}} = L_{1\mathrm{p}}^{2\rho_1} L_{2\mathrm{p}}^{2\rho_2} \sim \epsilon^{-2(\rho_1+\rho_2)}$ of the Szekeres class Ernst potential, in the general case $C_{1/2} \neq 0$ the Ernst potential converges for all values of $\rho_{1/2}$ to a purely imaginary value at the singularity $f+g=0$. However, this also leads to a coordinate degeneracy in the metric (2.11). For opposite signs of C_1 and C_2 this limit $E|_{\epsilon=0}$ has a zero at $\mathcal{S}=0$, which can be explained by an investigation of the initial waves described by the Ernst potential's boundary values

$$
E(f,\tfrac{1}{2}) = L_{1\mathrm{p}}^{2\rho_1} \frac{1+\mathrm{i}C_1 L_{01}^{2\rho_1-1} L_{1\mathrm{p}}^{-1}}{1+\mathrm{i}C_1 L_{01}^{2\rho_1-1} L_{1\mathrm{p}}}, \qquad E(\tfrac{1}{2},g) = L_{2\mathrm{p}}^{2\rho_2} \frac{1+\mathrm{i}C_2 L_{02}^{2\rho_2-1} L_{2\mathrm{p}}^{-1}}{1+\mathrm{i}C_2 L_{02}^{2\rho_2-1} L_{2\mathrm{p}}}, \tag{7.15}
$$

which are illustrated in figure 7.1. They feature the root-like behaviour at $f=g=\frac{1}{2}$ demanded by the colliding wave conditions (2.48). In addition their monotonic phase change of $\pm\frac{\pi}{2}$ from the real value at $f=g=\frac{1}{2}$ to the purely imaginary value at $\epsilon=0$ indicates a considerable change of the polarisation angle along the incident waves. The examination of the wave profiles in section 7.4 confirms that the initial waves of the generalised Szekeres class can be regarded as having a definite chirality given by the sign of C_1 or C_2, respectively.

In (7.15), the expression for $E(f,\frac{1}{2})$ depends only on C_1 as well as $E(\frac{1}{2},g)$ depends only on C_2. Thus the unilateral generalisations (7.12) of the Szekeres class describe the collision of an initial wave from the original Szekeres class and a generalised one. Moreover a change in the sign of C_1 or C_2 is equivalent to the complex conjugation of $E(f,\frac{1}{2})$ or $E(\frac{1}{2},g)$, respectively. These two properties are also inherited by the incident wave profiles $\hat{\Psi}_4$ and $\hat{\Psi}_0$ constructed out of the boundary values of E via (2.14) and (2.16) as well as (2.28) and an equivalent transformation formula for $\hat{\Psi}_0$. Therefore the sign of $C_{1/2}$ defines the chirality of the corresponding initial wave. Taking into account the initial waves' opposite directions of propagation, the case of opposite signs of C_1

and C_2 depicted in figure 7.1 can be referred to as 'equal chirality' of the initial waves. The zero in $E|_{\epsilon=0}$ is then seen to occur only for this case of incoming waves with equal chirality since this case also implies opposite signs of $\Im[E(f,\frac{1}{2})]$ and $\Im[E(\frac{1}{2},g)]$.

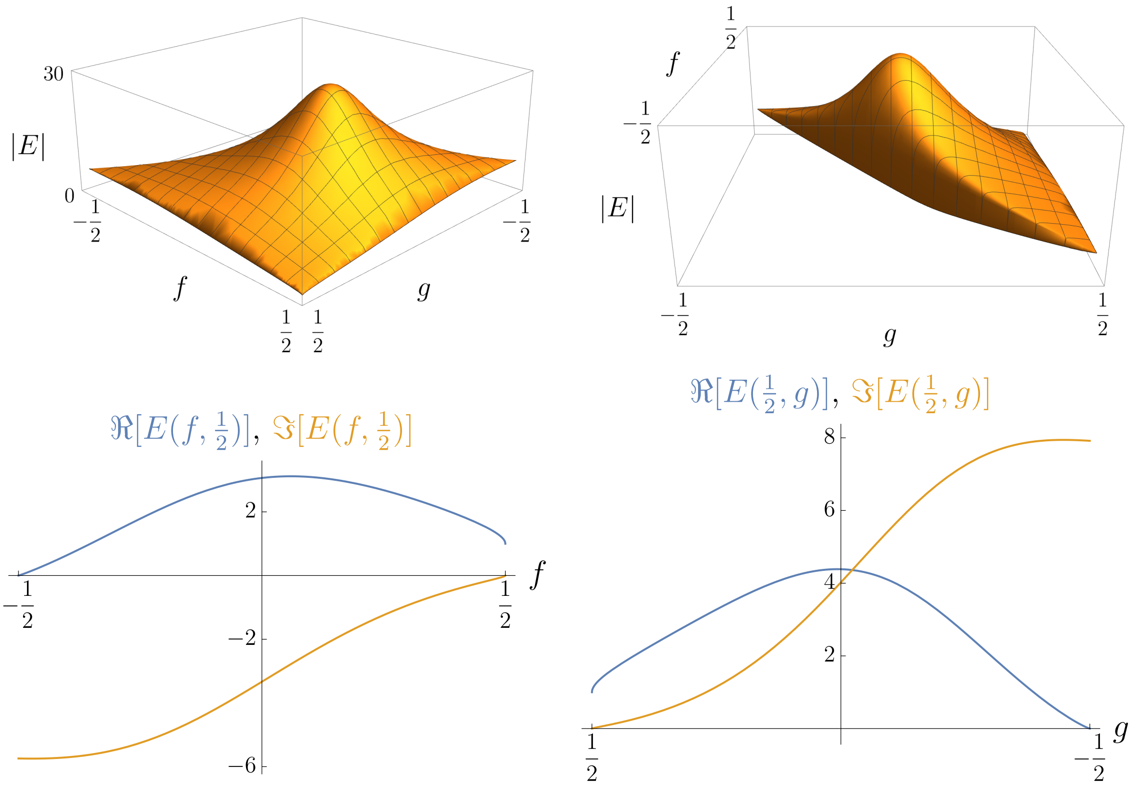

Figure 7.1: *Top*: The absolute value of the Ernst potential E in region IV viewed from two perspectives for $c_{1/2}=1$, $\rho_1=\frac{55}{100}$, $\rho_2=\frac{6}{10}$, $C_1=\frac{1}{5}$, $C_2=-\frac{1}{6}$ featuring a zero at $\mathcal{S}=0$ on $f+g=0$ and a bump inside region IV. *Bottom*: The boundary values of the Ernst potential describing initial waves with equal chirality.

Figure 7.1 also shows a bump of the Ernst potential inside region IV for initial waves with equal chirality, which does not occur for the case $\text{sign}(C_1C_2)=1$ of incoming waves with opposite chirality.

In the limit $\epsilon\to 0$ in (7.14), the metric function e^{-M} behaves like $\epsilon^{(\rho_1+\rho_2-\frac{1}{2})(2\rho_1+2\rho_2-3)}$. Therein the first factor of the exponent of ϵ is always positive in the allowed range $\frac{1}{2}\leq\rho_{1/2}<2^{-\frac{1}{2}}$, whereas for the second factor the relation $-1\leq 2\rho_1+2\rho_2-3<0$ holds. Therefore e^{-M} diverges at $f+g=0$ in the general case, though for the Szekeres class it features the limiting behaviour

$$\mathrm{e}^{-M_{\mathrm{Sz}}}\sim 2^{-8\rho_1\rho_2}\left(\tfrac{1}{2}-f\right)^{-2\rho_1(\rho_1+2\rho_2)}\left(\tfrac{1}{2}+f\right)^{-2\rho_2(\rho_2+2\rho_1)}\epsilon^{-\frac{1}{2}+2(\rho_1+\rho_2)^2}$$

and thus vanishes at $f+g=0$. In the general case $C_{1/2}\neq 0$ the reciprocal e^{M} goes to zero at $f+g=0$ with the exception of a pole at $\mathcal{D}=0$ for initial waves with *opposite* chirality, as can be studied in figure 7.2. This pole occurs for all values of $\rho_{1/2}$ permitted by the range (4.56).

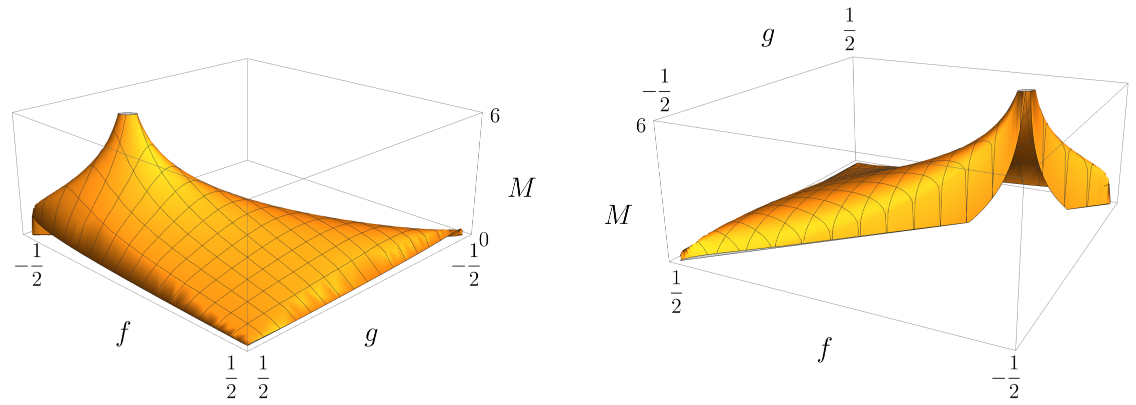

Figure 7.2: The metric function M in region IV viewed from two perspectives for $c_{1/2}=1$, $n_1=5$, $n_2=6$, $C_1=\frac{1}{5}$, $C_2=\frac{2}{5}$ featuring a pole at $\mathcal{D}=0$ on $f+g=0$.

7.3 Weyl tensor components

Evaluating the formulae (2.14)-(2.16), the Weyl tensor components with respect to the Szekeres tetrad (2.12) can be represented as

$$\Psi_0=\mathrm{e}^{M}g_v^2\frac{\operatorname{sign}\left[\varphi_r^{-1}(1)\varphi_r^{-1}(-1)\varphi_r^2(0)\right]}{2(f+g)^2\bar{\varphi}_r(0)}P_0, \tag{7.16}$$

$$\Psi_2=\mathrm{e}^{M}f_ug_v\frac{\operatorname{sign}\left[\varphi_r^2(0)\right]}{4(f+g)^2\bar{\varphi}_r(0)}P_2, \tag{7.17}$$

$$\bar{\Psi}_4=\mathrm{e}^{M}f_u^2\frac{\operatorname{sign}\left[\varphi_r^{-1}(1)\varphi_r^{-1}(-1)\varphi_r^{-2}(0)\right]}{2(f+g)^2\varphi_r(0)}\bar{P}_4, \tag{7.18}$$

using the expressions

$$\begin{aligned}P_0:=\;&F_0(\rho_1,\rho_2)+C_1C_2L_{01}^{2\rho_1-1}L_{02}^{2\rho_2-1}L_{12}^{2\rho_1+2\rho_2-2}F_0(\rho_1-1,\rho_2-1)\\&-\mathrm{i}C_1L_{01}^{2\rho_1-1}L_{12}^{2\rho_2}F_0(\rho_1-1,\rho_2)+\mathrm{i}C_2L_{02}^{2\rho_2-1}L_{12}^{2\rho_1}F_0(\rho_1,\rho_2-1),\end{aligned} \tag{7.19}$$

$$\begin{aligned}P_4:=\;&F_4(\rho_1,\rho_2)+C_1C_2L_{01}^{2\rho_1-1}L_{02}^{2\rho_2-1}L_{12}^{2\rho_1+2\rho_2-2}F_4(\rho_1-1,\rho_2-1)\\&-\mathrm{i}C_1L_{01}^{2\rho_1-1}L_{12}^{2\rho_2}F_4(\rho_1-1,\rho_2)+\mathrm{i}C_2L_{02}^{2\rho_2-1}L_{12}^{2\rho_1}F_4(\rho_1,\rho_2-1),\end{aligned} \tag{7.20}$$

$$\begin{aligned}P_2:=\;&F_2(\rho_1,\rho_2)+C_1C_2L_{01}^{2\rho_1-1}L_{02}^{2\rho_2-1}L_{12}^{2\rho_1+2\rho_2-2}F_2(\rho_1-1,\rho_2-1)\\&+\mathrm{i}C_1L_{01}^{2\rho_1-1}L_{12}^{2\rho_2}F_2(\rho_1-1,\rho_2)-\mathrm{i}C_2L_{02}^{2\rho_2-1}L_{12}^{2\rho_1}F_2(\rho_1,\rho_2-1),\end{aligned} \tag{7.21}$$

$$F_0(a,b) := 4(a\lambda_1^{-1} + b\lambda_2^{-1})^3 - a\lambda_1^{-3} - b\lambda_2^{-3}, \tag{7.22}$$

$$F_4(a,b) := 4(a\lambda_1 + b\lambda_2)^3 - a\lambda_1^3 - b\lambda_2^3, \tag{7.23}$$

$$F_2(a,b) := 4ab\lambda_1^{-1}\lambda_2^{-1}(\lambda_1 - \lambda_2)^2 + 4(a+b)^2 - 1. \tag{7.24}$$

Only Ψ_4 is non-vanishing for the left initial wave in region II and Ψ_0 is the only non-vanishing Weyl tensor component for the right wave in region III. The occurrence of the Coulomb component Ψ_2 in the interaction region is a direct manifestation of the nonlinearity of the wave interaction. With the abbreviations

$$R_1 := \mathrm{i}C_1 \left(\frac{1-2f}{1+2f}\right)^{2\rho_1 - 1}, \qquad R_2 := \mathrm{i}C_2 \left(\frac{1-2g}{1+2g}\right)^{2\rho_2 - 1} \tag{7.25}$$

as well as the boundary values of e^{-M},

$$\mathrm{e}^{-M}\Big|_{f=\frac{1}{2}} = (\tfrac{1}{2} + f)^{2\rho_1^2 - \frac{1}{2}}(1 - R_1^2), \qquad \mathrm{e}^{-M}\Big|_{g=\frac{1}{2}} = (\tfrac{1}{2} + g)^{2\rho_2^2 - \frac{1}{2}}(1 - R_2^2), \tag{7.26}$$

the incoming waves can be described by

$$\Psi_0\big|_{f=\frac{1}{2}} = \frac{1}{2}c_2^2 n_2^2 \frac{(\frac{1}{2} - g)^{4\rho_2^2 - 3/2}(2\rho_2 - 1)\left[\rho_2(1+2\rho_2) + R_2(3 - 5\rho_2 + 2\rho_2^2)\right]}{(\frac{1}{2} + g)^{2\rho_2^2 + 3/2}(1 + R_2)^3 \,\mathrm{sign}\left[(1+2g)(1-R_2)^2 + 8R_2\right]}, \tag{7.27}$$

$$\bar{\Psi}_4\Big|_{g=\frac{1}{2}} = \frac{1}{2}c_1^2 n_1^2 \frac{(\frac{1}{2} - f)^{4\rho_1^2 - 3/2}(2\rho_1 - 1)\left[\rho_1(1+2\rho_1) + R_1(3 - 5\rho_1 + 2\rho_1^2)\right]}{(\frac{1}{2} + f)^{2\rho_1^2 + 3/2}(1 + R_1)^3 \,\mathrm{sign}\left[(1+2f)(1-R_1)^2 + 8R_1\right]}. \tag{7.28}$$

From the boundary values of the Ernst potential these expressions inherit the properties to depend only on one of the constants $C_{1/2}$ and to become their conjugates via change of the sign of this constant.

Near the wave front $f = \frac{1}{2}$ of the left wave holds for the boundary value $\Psi_4|_{g=\frac{1}{2}}$ of the Weyl tensor component

$$f \approx \tfrac{1}{2}: \qquad \Psi_4\big|_{g=\frac{1}{2}} \sim \tfrac{1}{2}c_1^2 n_1^2 \rho_1(4\rho_1^2 - 1)(\tfrac{1}{2} - f)^{-3/2 + 4\rho_1^2}, \tag{7.29}$$

which is the same asymptotical behaviour as for the Szekeres class. As anticipated by the discussion at the end of section 2.7, for $\frac{1}{2} \le \rho_1 < \sqrt{3/8}$, i.e. $2 \le n_1 < 4$, the incoming Weyl tensor component $\Psi_4|_{g=\frac{1}{2}}$ is unbounded at the wave front $f = \frac{1}{2}$; for $\sqrt{3/8} \le \rho_1 < \sqrt{1/2}$ (i.e. $4 \le n_1$) it is bounded.

At the fold singularity $f = -\frac{1}{2}$ the Weyl tensor component diverges as

$$f \approx -\tfrac{1}{2}: \qquad \Psi_4\big|_{g=\frac{1}{2}} \sim \mathrm{i}c_1^2 n_1^2 \frac{3 - 11\rho_1 + 12\rho_1^2 - 4\rho_1^3}{2C_1^2 \,\mathrm{sign}(C_1)}(\tfrac{1}{2} + f)^{-3/2 - 2(1-\rho_1)^2}. \tag{7.30}$$

with purely imaginary coefficient, whereas the Szekeres class has the stronger divergence behaviour

$$f \approx -\tfrac{1}{2}: \qquad \Psi_4\big|_{g=\frac{1}{2}} \sim \tfrac{1}{2} c_1^2 n_1^2 \rho_1 (4\rho_1^2 - 1)(\tfrac{1}{2} + f)^{-3/2 - 2\rho_1^2}. \tag{7.31}$$

That divergence of Ψ_4 is a strong hint for the existence of a non-scalar curvature singularity at the boundary $f = -\frac{1}{2}, v < 0$ of region II. This singularity character has already been confirmed for the Szekeres class [54].

7.4 Wave profiles and the limit of circularly polarised impulsive waves

The wave profile, i.e. the Weyl tensor component $\hat{\Psi}_4$ with respect to the Brinkmann tetrad (2.20), can be calculated via the relation (2.28). This transformation's effect is illustrated by plotting the factor $\hat{\Psi}_4/\Psi_4 = \mathrm{e}^{M-2\mathrm{i}P}\,\mathrm{sign}(E)$ in figure 7.3 for a bounded wavefront with $n_1 = 5$. The absolute value of this factor is simply e^M given by the reciprocal of (7.26), whereas the phase depending on P is only given in terms of the integral (2.26). As figure 7.3 shows, this phase change within the transformation to $\hat{\Psi}_4$ is only significant near the fold singularity $f = -\frac{1}{2}$ and only for medium values of C_1.

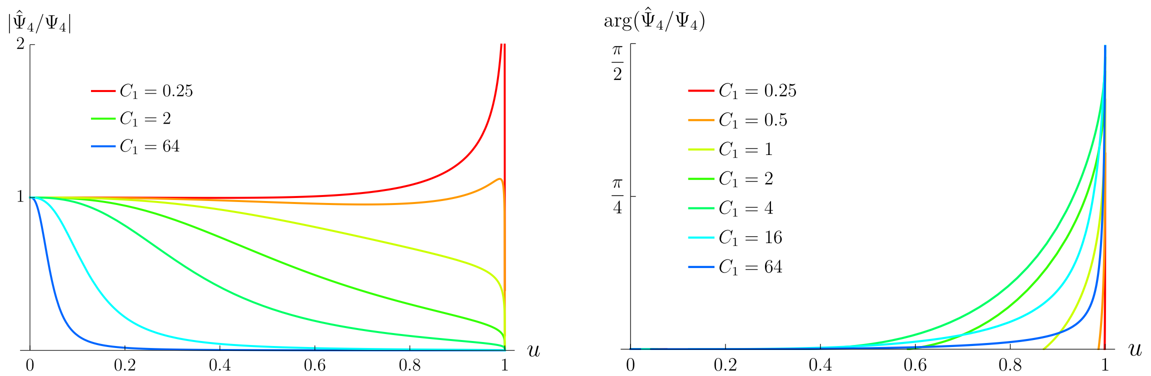

Figure 7.3: Amplitude $|\hat{\Psi}_4/\Psi_4|$ and phase $\arg(\hat{\Psi}_4/\Psi_4)$ of the transformation factor between Ψ_4 and the wave profile $\hat{\Psi}_4$ for $n_1 = 5$ and $c_1 = 1$.

As anticipated in section (7.2), the wave profiles displayed in figure 7.4 feature a monotonic change of their phase angle from 0 at the wave front $f = \frac{1}{2}$ (i.e. $u = 0$) to 2π at the fold singularity $f = -\frac{1}{2}$ (i.e. $u = 1$ for $c_1 = 1$). Therefore a distinct chirality can indeed be assigned to them.

A particular interesting feature of the wave profiles of the generalised Szekeres class is their shape for large values of $C_{1/2}$. As illustrated in figure 7.4, $\Psi_4\big|_{g=\frac{1}{2}}$ compactifies into a pulse at the wave front $f = \frac{1}{2}$ (i.e. $u = 0$) for increasing C_1 featuring a full revolution of the polarisation angle during that pulse. Note that the 'wave strength' c_1 (cf. (2.45)) has been fixed to 1 in figure 7.4, but can be easily modified to adjust the height of the pulse. In consequence, this generalised class of solution can provide analytical formulae for a new type of impulsive waves, which shall be called 'circularly polarised impulsive waves'. However, some attention may have to be paid to the remnant divergence of $\Psi_4|_{g=\frac{1}{2}}$ at the fold singularity $f = -\frac{1}{2}$ (i.e. $u = 1$ for $c_1 = 1$). Qualitatively, the Weyl tensor component Ψ_4 has the same limit as $\hat{\Psi}_4$ for large values of $C_{1/2}$, but the convergence to impulsive waves is slower.

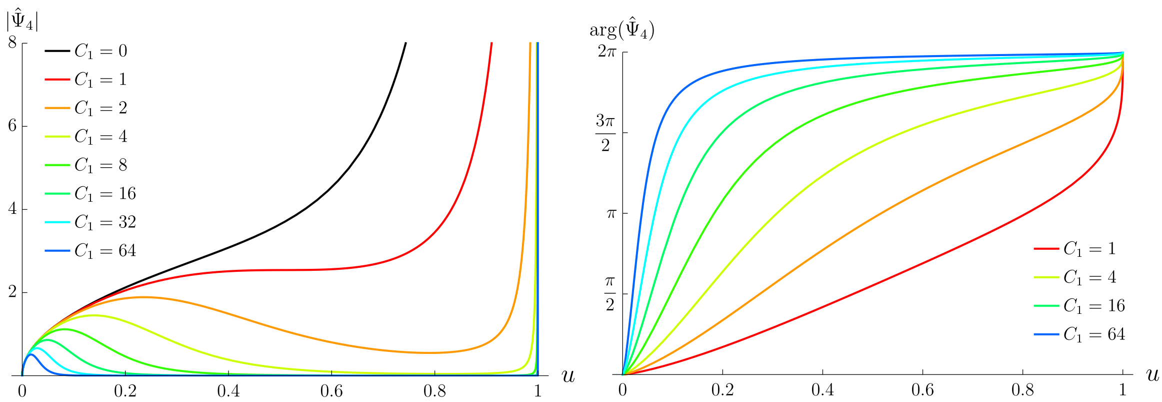

Figure 7.4: Absolute value and polarisation angle of the initial wave profile $\Psi_4|_{g=\frac{1}{2}}$ for $n_1 = 5$ and $c_1 = 1$ approximating a circularly polarised pulsed wave for increasing C_1.

Due to $M(\frac{1}{2}, \frac{1}{2}) = 0$, the behaviour of the wave profile $\hat{\Psi}_4$ at the wave front $f = \frac{1}{2}$ is identical to that of Ψ_4 in (7.29). Despite its small value for medium values of u in the limit of circularly polarised impulsive waves, at the fold singularity $f = -\frac{1}{2}$ the wave profile still diverges as

$$f \approx -\tfrac{1}{2}: \qquad \hat{\Psi}_4\Big|_{g=\frac{1}{2}} \sim c_1^2 n_1^2 \frac{3 - 11\rho_1 + 12\rho_1^2 - 4\rho_1^3}{2C_1^4 \operatorname{sign}(C_1)} (\tfrac{1}{2} + f)^{-1-4(1-\rho_1)^2}. \qquad (7.32)$$

The Szekeres class exhibits the stronger divergence behaviour

$$f \approx -\tfrac{1}{2}: \qquad \hat{\Psi}_4\Big|_{g=\frac{1}{2}} \sim \tfrac{1}{2} c_1^2 n_1^2 \rho_1 (4\rho_1^2 - 1)(\tfrac{1}{2} + f)^{-1-4\rho_1^2}. \qquad (7.33)$$

7.5 The character of the singularity at $f + g = 0$

To clarify the character of the singularity at $f + g = 0$, the complex scalar curvature invariant (2.8) of the Weyl tensor for the generalised Szekeres class is computed to

$$\mathcal{I} = 16[3\Psi_2^2 + \Psi_0\Psi_4] = \frac{c_1^2 c_2^2 n_1^2 n_2^2 L_{01}^{4\rho_1^2} L_{02}^{4\rho_2^2} L_{12}^{8\rho_1\rho_2}}{(f+g)^3 \varphi_r^6(\infty)} (3P_2^2 + 4P_0P_4). \tag{7.34}$$

At $\epsilon = f + g \to 0$ it diverges for $C_{1/2} \neq 0$ as

$$\mathcal{I} \sim c_1^2 c_2^2 n_1^2 n_2^2 2^{16\rho_1\rho_2} (\tfrac{1}{2} - f)^{8\rho_1(\rho_1+\rho_2-1)} (\tfrac{1}{2} + f)^{8\rho_2(\rho_1+\rho_2-1)} \mathcal{D}^{-4} P_{\mathcal{I}} \epsilon^{-3-4(\rho_1+\rho_2-1)^2}, \tag{7.35}$$

$$P_{\mathcal{I}} := 3\left[1 - 4(\rho_1+\rho_2-1)^2\right]^2 + 4\left[4(\rho_1+\rho_2-1)^3 - \rho_1 - \rho_2 + 1\right]^2. \tag{7.36}$$

Therefore the boundary $f + g = 0$ of region IV is a scalar curvature singularity as it is for the Szekeres class. Nevertheless, the divergence is weaker than for the Szekeres class where the curvature invariant behaves like

$$\mathcal{I}_{\mathrm{Sz}} \sim c_1^2 c_2^2 n_1^2 n_2^2 2^{16\rho_1\rho_2} (\tfrac{1}{2} - f)^{8\rho_1(\rho_1+\rho_2)} (\tfrac{1}{2} + f)^{8\rho_2(\rho_1+\rho_2)} P_{\mathcal{I}_{\mathrm{Sz}}} \epsilon^{-3-4(\rho_1+\rho_2)^2}, \tag{7.37}$$

$$P_{\mathcal{I}_{\mathrm{Sz}}} := \left[1 - 4(\rho_1+\rho_2)^2\right]^2 \left[3 + 4(\rho_1+\rho_2)^2\right].$$

In case of initial waves with opposite chirality, there is a complicated pole structure at $\mathcal{D} = 0$ on top of the divergence behaviour at the boundary $f + g = 0$. From the study of $\Re(\mathcal{I})$ and $\Im(\mathcal{I})$ in figure 7.5 this structure can be seen to resemble a higher order pole of a complex function. The exact position of that pole is determined by the ratio of C_1 and C_2.

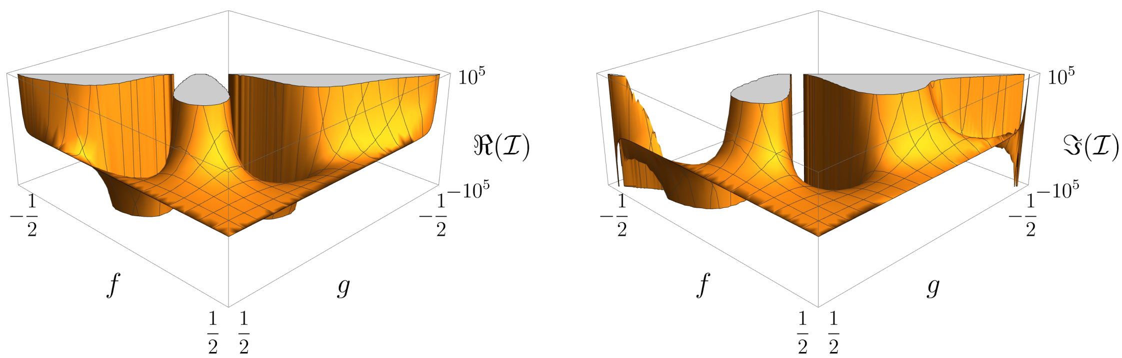

Figure 7.5: Real and imaginary part of the scalar invariant $\mathcal{I}$ of the Weyl tensor in region IV for $c_{1/2} = 1$, $n_1 = 5$, $n_2 = 6$, $C_1 = 1$, $C_2 = \frac{6}{5}$ featuring a structure resembling a higher order pole of a complex function at $\mathcal{D} = 0$ on $f + g = 0$.

8 Approximation scheme for the continuous RHP

The goal of this chapter is to provide the basic tools for a numerical implementation of the developed ISM solution procedure via spectral expansion of an integral form of the CRHP's scalar jump equation. A suitable set of functions to approximate the additive jump μ^{IK} of the scalar CRHP solution ϑ^{IK} is derived and their Cauchy integrals are given analytically. Therewith a proper system of algebraic equations is developed to determine the coefficients of the expansion functions.
The implementation of the full numerical approximation is beyond the scope of this work. For a proof of principle of a numerical solution along these lines, a simple approximation scheme using a collocation method is presented in this chapter and applied to the toy model of two instead of four regions with separate ansatz for μ^{IK}.

8.1 CRHP integral equation for a regular additive jump

The scalar CRHP solution ϑ^{IK} can be expressed by its additive jump function μ^{IK} via

$$\vartheta^{IK}(\lambda) = c_{\vartheta}^{IK} + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\Re}} \frac{\mu^{IK}(\lambda')\mathrm{d}\lambda'}{\lambda' - \lambda}. \tag{8.1}$$

Since the normalisation of ϑ^{IK} was not fixed, it is only determined up to multiplication with an overall factor. In the following, $\vartheta^{IK} = 1$ shall be chosen as a normalisation, but it may be noted that problems could arise for $c_{\vartheta}^{IK} \approx 0$.
Inserting (8.1) in the scalar jump equation (5.52) of the CRHP leads, similarly to the derivation in section 4.6, to the integral form of the jump equation

$$\begin{aligned}(\alpha_c + \beta_c - \gamma_c - 1)c_{\vartheta}^{IK} = (1-\alpha_c)\frac{1}{2\pi \mathrm{i}} ⨍_{\Gamma_{\Re}} \frac{\mu^{IK}(\lambda')\mathrm{d}\lambda'}{\lambda' - \lambda} + (\gamma_c - \beta_c)\frac{1}{2\pi \mathrm{i}} ⨍_{\Gamma_{\Re}} \frac{\mu^{IK}(\lambda')\mathrm{d}\lambda'}{\lambda' + \lambda} \\ + \tfrac{1}{2}(\alpha_c + 1)\mu^{IK} + \tfrac{1}{2}(\gamma_c - \beta_c)\mu^{IK}(-\lambda). \end{aligned} \tag{8.2}$$

The additive jump μ^{IK} is continuous because the scalar CRHP solution has a continuous extension on the contour $\Gamma_{\Re}$ from both sides. Therefore (8.2) is an inhomogeneous linear singular integral equation of the second kind for a continuous unknown function. However, μ^{IK} will in general inherit the root-like behaviour of the jump matrix components near $\pm\lambda_{1/2}$. Nevertheless, the Cauchy integrals in (8.2) over the root-like parts of μ^{IK} are still regular.

As described in section 5.3, the second singularity transformation $G_c^{IK} := (S_{2-}^I)^{-1} {G'_J}^K S_{2+}^I$ introduces terms $\sim |\lambda \pm \lambda_1|^{1-|x_1^I|}$ into the jump functions β_c and γ_c at both sides of the points $\pm\lambda_1$. Similarly the first singularity transformation $\tilde{G}_J^K := (S_{2-}^K)^{-1} G_J S_{2+}^K$ introduces terms $\sim |\lambda \pm \lambda_2|^{1-|x_2^K|}$ into the jump functions $\tilde{\beta}$ and $\tilde{\gamma}$ at the ends $\pm(\lambda_2 - 0)$ of the contour part Γ_2, but not on $\pm(\lambda_2 + 0)$ outside Γ_2 since $G_J[\pm(\lambda_2 + 0)] = \mathbb{1}$. The CRHP jump functions β_c and γ_c and, due to the rotation transformation ${G'_J}^K := R_{\delta_1}^{-1} \hat{G}_J^K R_{\delta_1}$ also $\Re(\alpha_c)$, inherit this root-like behaviour on $\pm(\lambda_2 - 0)$, as can be seen in the example in figure 5.1. On the other hand, a root-like behaviour of μ^{IK} at $\pm(\lambda_2 - 0)$ implies root-like behaviours of ϑ^{IK} at these contour ends. They are expected to have an analytic continuation in the punctured neighbourhood of $\pm\lambda_2$ which has also a root-like behaviour on $\pm(\lambda_2 + 0)$. Hence the ansatz for the additive jump function μ^{IK} has to contain terms accounting for the root-like behaviour at both sides of the points $\pm\lambda_{1/2}$. Depending on K, the values of the exponents $1 - |x_{1/2}^K|$ lie in the ranges

$$0 < 1 - |x_{1/2}^{\mathrm{e}}| = 2\varrho_{1/2} - 1 < \sqrt{2} - 1 < \tfrac{1}{2}, \tag{8.3}$$

$$\tfrac{1}{2} < 2 - \sqrt{2} < 1 - |x_{1/2}^{\mathrm{o}}| = 2 - 2\varrho_{1/2} < 1. \tag{8.4}$$

Moreover, different spectral expansions for the following four sections of the contour have to be used:

$$\mu^{IK} = \begin{cases} \mu_1^{IK} & \text{on } \Gamma_1, \\ \mu_2^{IK} & \text{on } \Gamma_2, \\ \mu_3^{IK} & \text{on } \Gamma_3 := [\lambda_2, \lambda_1], \\ \mu_4^{IK} & \text{on } \Gamma_4 := [-\lambda_1, -\lambda_2]. \end{cases} \tag{8.5}$$

In this sense (8.2) has to be regarded as a coupled system for the partial jump functions μ_i^{IK}. In the expansion of each μ_i^{IK}, N_i functions adapted to the corresponding interval shall be used. The details of the expansion are exemplified focussing on μ_2^{IK}, which shall be represented by the N_2 functions $(\lambda_2 - \lambda)^{1-|x_2^K|}$, $(\lambda_2 + \lambda)^{1-|x_2^K|}$ and λ^j, $j = 0 \ldots N_2 - 3$:

$$\mu_2^{IK} = a_2(\lambda_2 - \lambda)^{1-|x_2^K|} + b_2(\lambda_2 + \lambda)^{1-|x_2^K|} + \sum_{j=0}^{N_2-3} c_{2,j}\lambda^j. \tag{8.6}$$

For $\lambda \in \mathbb{R}$ the integrals of the components of μ_2^{IK} can be calculated as

$$\begin{aligned} I_{2j} &:= \quad \fint_{\Gamma_2} \frac{\lambda'^j}{\lambda' - \lambda} \mathrm{d}\lambda' = \sum_{k=1}^{j} \frac{\lambda_2 - (-\lambda_2)^k}{k} \lambda^{j-k} + \lambda^j \ln \left| \frac{\lambda_2 - \lambda}{\lambda_2 + \lambda} \right|, \\ I_{2m} &:= \fint_{\Gamma_2} \frac{(\lambda_2 - \lambda')^r}{\lambda' - \lambda} \mathrm{d}\lambda' = \Re \left[\frac{(2\lambda_2)^{r+1}}{(r+1)(\lambda_2 - \lambda)} \, {}_2F_1 \left(1, r+1; r+2; \frac{2\lambda_2}{\lambda_2 - \lambda} \right) \right], \\ I_{2\mathrm{p}} &:= \fint_{\Gamma_2} \frac{(\lambda_2 + \lambda')^r}{\lambda' - \lambda} \mathrm{d}\lambda' = -I_{2m}(-\lambda), \end{aligned} \tag{8.7}$$

where ${}_2F_1$ is the hypergeometric function and the appropriate value of I_{2m} is obtained by setting $r = 1 - |x_2^K|$. The integral I_{2m} has the limit $-(2\lambda_2)^j/j$ at λ_2 and behaves at $-\lambda_2$ like

$$\lambda \approx -\lambda_2 : \qquad I_{2m} \sim (2\lambda_2)^r \ln|\lambda + \lambda_2| + (2\lambda_2)^r [\ln(2\lambda_2) + H_r], \tag{8.8}$$

where H_r is the harmonic number with index r which can be defined as

$$H_r := -\sum_{k=1}^{\infty} \binom{r}{k} \frac{(-1)^k}{k}. \tag{8.9}$$

The expressions (8.7) shall be inserted into the integral equation (8.2) to turn it into an algebraic equation for the coefficients of the partial jumps μ_i^{IK}, which can be evaluated for selected values of λ. This treatment is called a ‘collocation method’ and leads to a linear system of equations for the coefficients of the μ_i^{IK}. The implementation of a more sophisticated method is beyond the scope of this work, but highly accurate results are expected following the methods proposed in e.g. [52, 53].

8.2 System of equations for the approximation

The integral I_{2j} diverges logarithmically at both contour endpoints $\pm\lambda_2$ and I_{2m} diverges logarithmically at $-\lambda_2$. This is however just an artefact of the reduction of the integral domain to Γ_2 in (8.7). If μ^{IK} is ensured to be exactly continuous, the Cauchy integrals in (8.2) will not diverge. This means that the logarithmically divergent part of the integral equation (8.2) is automatically fulfilled for continuous μ^{IK}. Hence it is appropriate at each of the four contour endpoints $\pm\lambda_{1/2}$ to require the two conditions:

1. continuity of μ^{IK},
2. validity of the non-divergent part of the integral jump equation (8.2).

Since the expansions of the partial jumps μ_i^{IK} consists of N_i functions, for each contour section Γ_i another $N_i - 2$ equations are needed to determine the N_i coefficients a_i, b_i, $c_{i,j}$, $(j = 0 \dots N_i - 3)$. Therefore additionally the integral jump equation is required to be exactly valid on the $N_i - 2$ inner points of a N_i-point Lobatto lattice[1] on Γ_i. For Γ_2 these points are

$$\lambda_{2,k} := \lambda_2 \cos\left(\frac{\pi k}{N_2 - 1}\right), \qquad k = 1 \dots N_2 - 2. \tag{8.10}$$

They are more dense towards the contour endpoints, making them well adapted for the polynomial expansion.

In summary, the approximation scheme uses $N := \sum_{i=1}^{4} N_i$ equations consisting of

- 4 continuity conditions of μ^{IK} at the contour endpoints $\pm\lambda_{1/2}$,
- 4 non-divergent parts of the integral jump equation (8.2) at $\pm\lambda_{1/2}$,
- $N - 8$ times the integral jump equation (8.2) evaluated at the inner points of the Lobatto lattices,

to determine the coefficients of the N functions:

- 8 root functions $\pm\lambda_{1/2}$ accounting for root-like behaviour at $\pm\lambda_{1/2}$
- $N - 8$ monomials adapted to the corresponding contour.

Since c_ϑ^{IK} was set to one, this linear system of equations in general has a unique solution. The monomials may be replaced by Chebychev polynomials to make the elements of its coefficient matrix have the same order of magnitude, but this also complicates the expressions for the analytic integrals.

The approximation scheme outlined above yields an approach to the solution of the CRHP for fixed coordinates (f, g) and a specific choice of I and K. To obtain the value of the Ernst potential on the surface $(f = f_0, g = g_0)$ in the spacetime, the additive jumps μ^{IK} of all four solutions ϑ^{IK} have to be calculated. Using the values $\vartheta^{IK}(\pm\lambda_{1/2})$ the holomorphicity conditions (6.30) have to be evaluated for the coefficients p, q and r determining the scalar LP solution φ^{LP}. Finally, the values of ϑ^{IK} taken at $\lambda = \pm 1$ together with these coefficients yield the Ernst potential $E = \varphi^{\mathrm{LP}}(1)/\varphi^{\mathrm{LP}}(-1)$. In particular, the Cauchy integral (8.1) for ϑ^{IK} has to be calculated only for the points $\pm\lambda_{1/2}$ and ± 1.

[1] An n-point Lobatto lattice contains the n extremal points of the Chebychev polynomial of order $n - 1$ on a given interval, which includes both endpoints of the interval.

8.3 Toy model with two contour sections

This chapter is concluded by demonstrating the numeric approximation scheme for a toy model with only the two contour sections Γ_2 and $\Gamma_\Re \backslash \Gamma_2$. For this purpose, the jump functions α_c^{ee}, β_c^{ee} and γ_c^{ee} on Γ_2 are taken from the example at the end of section 5.5 (cf. figure 5.1) and continued with $\alpha_c^{\mathrm{ee}} = \mathrm{const}$, $\gamma_c^{\mathrm{ee}} \sim 1/\lambda$ and $\beta_c^{\mathrm{ee}} = \sqrt{1 - \alpha_c^{\mathrm{ee}}\bar{\alpha}_c^{\mathrm{ee}} + (\gamma_c^{\mathrm{ee}})^2}$ on $\Gamma_\Re \setminus \Gamma_2$, cf. figure 8.1.

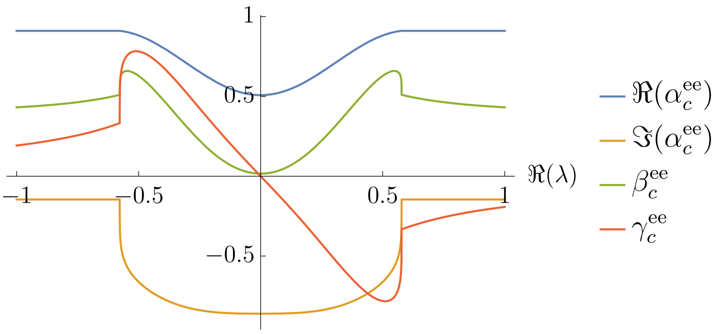

Figure 8.1: Jump functions of the toy model, which is identical with the example in section 5.5 on Γ_2.

In figure 8.2 the additive jump function on Γ_2 calculated as described in section 8.2 with $N_2 = 20$ is shown. Additionally, in the right panel of figure 8.2 the approximation is realised without the root-like terms so that μ^{ee} is composed only of monomials. The oscillations occurring in this case demonstrate the necessity of the root-like terms in the ansatz (8.6) for μ^{ee}. When the approximated jump function μ^{ee} is inserted into the integral jump equation (8.2), the error between the supporting points is of order 10^{-5} or respectively 10^{-2} without root-like terms.

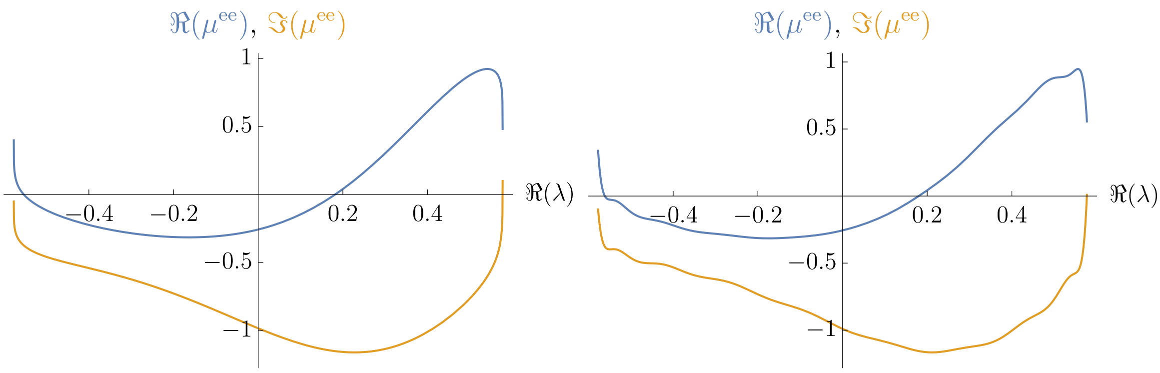

Figure 8.2: Additive jump function of the toy model on Γ_2 for $N_2 = 20$ approximated with the ansatz (8.6) (left) and without the root-like terms (right).

This toy model already covers all principal issues of the numeric approach and thus no further difficulties are expected in the treatment of the full CRHP.

9 Conclusions

Using the ISM and the subsequent transformation to a CRHP, within this work a solution procedure for the characteristic IVP of colliding plane waves was derived. Originally the intention of the ISM was the heuristic search for exact solutions to nonlinear differential equations, whose validity is checked afterwards. Also for axially symmetric and stationary spacetimes the ISM was mostly used in that sense to generate specific solutions without mathematical rigour. In a general solution procedure for the IVP of colliding waves, however, a deeper understanding of the involved mathematical problems is desirable. This work was at least partially able to meet that demand by establishing the following statements:

1. The IVP solution is uniquely associated with a LP solution.

2. A RHP whose solution is supposed to also solve the LP must necessarily have the jump matrix derived from the initial values of the Ernst potential via (4.52) and (4.53).

3. The linear combination of the four RHP solutions Φ^{ee}, Φ^{oe}, Φ^{eo} and Φ^{oo} describes the general solution of the RHP for a given jump matrix if the conjecture in section 5.3 holds. It stated that $S^{\mathrm{e}}_{1/2}$ and $S^{\mathrm{o}}_{1/2}$ are the only in principle different transformations removing the discontinuities at a pair of contour endpoints and leading to $\tilde{G}^K_J(\pm\lambda_2) = \mathbb{1}$ for Lipschitz continuous initial jump matrices J.

4. A RHP solution is also a LP solution if and only if it fulfils the holomorphicity conditions (4.41), which are in turn equivalent to the algebraic relations (6.30).

5. This form (6.30) shows that the holomorphicity conditions can in general be fulfilled, and thus the RHP gives indeed rise to LP solutions, which in turn lead to solutions of the Ernst equation. However, it is not rigorously proven that the right solution of the IVP is among them.

6. Based on the conjecture on the asymptotic form $\varphi(f \approx \frac{1}{2}, g \approx \frac{1}{2}) \approx \varphi^{**}$ given in (6.40), LP solutions represented by normalised RHP solutions are expected

to automatically have the right LP normalisation. If that conjecture holds, the colliding wave conditions are equivalent to the relations (6.41) and (6.42) for the coefficients p, q and r of φ^{LP}.

Deployed as an algorithm for a special IVP, the crucial problem within the presented ISM scheme consists of the solution of the integral form (8.2) of the scalar CRHP jump equation. In contrast to this, the derivation of the jump matrix from the initial data via the ODEs (4.52) and (4.53) is easily done with high numerical accuracy, if not analytically. Although the jump matrix can only be approximated in case of generic initial data, the transformation to the CRHP only depending on its exactly known boundary values $J_{1/2\mathrm{b}}$ is given analytically by (5.39) and (5.40). The holomorphicity conditions (6.30) adapting the RHP solution to the LP are just algebraic equations and finally remnant degrees of freedom have to be fixed by a simple comparison with the initial data.

In special cases where a fully analytic treatment is possible, the fourfold ambiguity contained in the solution to the discontinuous RHP and the possible remnant functional degree of freedom in the LP solution lead to the construction of families of exact solutions. In this sense the described procedure serves also as a solution generating technique which generalises existing colliding wave solutions and leads to insights into the structure of colliding plane waves. This was demonstrated by the generalisation of the collinearly polarised Szekeres class of colliding wave spacetimes to a class with general polarisation. A distinct notion of chirality could be associated to the initial waves and interesting features of the metric functions were associated to colliding waves of either equal or opposite chirality. A scalar curvature singularity in the interaction region has been identified for the generalised Szekeres class and evidence for a non-scalar curvature singularity at the 'fold singularity' has been given. Moreover, a transformation to Weyl tensor components with respect to a tetrad adapted to Brinkmann coordinates, the so-called wave profiles, was achieved. Via the study of their shapes an interesting limiting case with circularly polarised impulsive waves has been discovered.

Finally, a semi-analytic approximation scheme was designed implementing a collocation method. Based on approximation functions with analytically given Cauchy integrals, the integral form of the scalar CRHP jump equation was transformed into a system of algebraic equations. A toy model showed this approximation scheme to be functional and confirmed the considerations on the root-like behaviour of the additive jump function at the contour endpoints.

9.1 Outlook

This work's novel approach to colliding plane waves prepares the ground for various promising directions of further research. The most important ones are:

- **Implementing an approximation scheme for the full IVP**: The author's primary goal of adjacent investigations is to accomplish the spectral expansion of the CRHP outlined in chapter 8 and extend it into a full approximation scheme for the general characteristic IVP of colliding plane GWs. This enables a study of the properties of colliding waves under systematic variation of the initial data. The physical interpretation of the obtained spacetimes shall be considerably expanded. Of special methodical interest would also be the comparison with finite differencing schemes to reveal the complementarity of the two approaches. Possibly also other expansion functions will turn out to yield an even better performance in the approximation of the CRHP solution.

- **Investigating the limit of circularly polarised impulsive waves**: It seems to be highly worthwhile to explicitly calculate the limit of circularly polarised impulsive waves discovered in section 7.4. This should easily result in the exploration of the physical properties of this new concept of GWs.

- **Including impulsive waves in the ISM scheme**: The case $\rho_1 = \frac{1}{2}$ or $\rho_2 = \frac{1}{2}$ of impulsive waves was excluded in the derivation of the CRHP. It implies that the exponent $x^{\mathrm{e}}_{1/2} := 2\rho_{1/2} - 2$ associated with even transformations would take the value 1, which means that it is not ensured to yield a Hölder continuous jump matrix when applied to an initially Lipschitz continuous jump matrix.

  On the other hand, for an impulsive wave in region II or III the boundary value of the RHP jump matrix on the corresponding contour takes the value $J_{1\mathrm{b}} = -\mathbb{1}$ or $J_{2\mathrm{b}} = -\mathbb{1}$, respectively. This property is invariant under rotation transformation and unitarisation transformation and hence these types of transformation can be used to set the derivatives of the jump functions β and γ instead of their values to zero at $\pm\lambda_i$. After appropriate preparation, the discontinuities in the ERHP are expected to be removable by the alternative singularity transformations S^{e}_i and $(S^{\mathrm{e}}_i)^{-1}$ instead of S^{e}_i and S^{o}_i. The inverse transformation leads directly (i.e. without the linear combinations that were necessary to restore the normal form after odd singularity transformations) to the construction of four RHP solutions out of the CRHP solutions. Therewith the derivation of the holomorphicity conditions has to be reassessed carefully for a spacetime with at least one impulsive wave, but a simplification compared to the non-impulsive case is expected.

- **Generalising other known colliding wave solutions**: This work's approach allows for the generalisation of other known colliding wave solutions by simply recalculating them within the ISM scheme. Another promising candidate for gaining analytic results by this procedure is the Nutku-Halil spacetime describing the collision of impulsive waves with non-aligned polarisation. It is extensively discussed in [55] and can be represented by the Ernst potential

$$E_{\mathrm{NH}} = \frac{1 + \mathrm{e}^{-\mathrm{i}\alpha_{\mathrm{N}}}\sqrt{(\frac{1}{2}+f)(\frac{1}{2}-g)} + \mathrm{e}^{\mathrm{i}\alpha_{\mathrm{N}}}\sqrt{(\frac{1}{2}-f)(\frac{1}{2}+g)}}{1 - \mathrm{e}^{-\mathrm{i}\alpha_{\mathrm{N}}}\sqrt{(\frac{1}{2}+f)(\frac{1}{2}-g)} - \mathrm{e}^{\mathrm{i}\alpha_{\mathrm{N}}}\sqrt{(\frac{1}{2}-f)(\frac{1}{2}+g)}}. \tag{9.1}$$

Evaluation of the ODEs (4.52) and (4.53) yields the constant jump matrix $J = -\mathbb{1}$ on Γ independent of the phase parameter α_{N}. The corresponding RHP has the four independent scalar solutions $\varphi_1 = L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}$, $\varphi_2 = L_1^{\frac{1}{2}} L_2^{-\frac{1}{2}}$, $\varphi_3 = L_1^{-\frac{1}{2}} L_2^{\frac{1}{2}}$ and $\varphi_4 = L_1^{-\frac{1}{2}} L_2^{-\frac{1}{2}}$. A relation to the generalised Szekeres class evaluated for $\rho_{1/2} \to \frac{1}{2}$ is expected.

- **Increasing the rigorousness of the ISM**: In the case of the Korteweg-de Vries equation a rigorous derivation of the ISM solution procedure was possible, cf. [56]. It guaranteed that the solution given in terms of the Gelfand-Levitan-Marchenko integral equation solves every IVP. In the case of colliding plane GWs, a thorough investigation of the analyticity properties of the LP solution could lead to similar results. The author expects that it is possible to show the LP solution to be holomorphic up to the jump on Γ, which would imply that each solution of the LP is also a RHP solution. Together with the generality of the derived RHP solutions, depending on the validity of the conjecture on the uniqueness of the discontinuity removing transformations $S^{\mathrm{e}}_{1/2}$ and $S^{\mathrm{o}}_{1/2}$, this would imply that the LP solution associated with the IVP is among the RHP solutions.

- **Reconsidering the RHP integral equations**: Maybe a more compact version of the procedure can be derived reconsidering the singular integral equations for the initial RHP again via an ansatz for the scalar RHP solution's additive jumps μ^{φ} including terms with the right divergence exponents $\rho_{1/2}$ and $1 - \rho_{1/2}$. If an analytic evaluation of conditions for the LP's holomorphicity similar to those developed in chapter 6 can be used to exactly determine the singular part of μ^{φ}, then the singular integral equations in section 4.6 might be decomposed into an automatically fulfilled singular part and a regular part ready for a simple spectral approximation. On the other hand the implementation of approximation schemes adapted to unbounded unknown functions is also very promising.

- **Investigating initially collinearly polarised GWs**: Massive simplifications in the described solution procedure occur for the case of initially collinearly polarised GWs introduced in section 5.2, where the boundary values $J_{1/2\mathrm{b}}$ of the jump matrix can be diagonalised simultaneously. Then neither a unitarisation transformation nor a rotation transformation is necessary for the transition to the CRHP. Interestingly, each general colliding wave spacetime can be represented as limit of a family of initially collinearly polarised colliding wave spacetimes.

- **Including electromagnetic waves**: The starting point for the presented ISM scheme were the vacuum Einstein equations. A transition to the electrovacuum Einstein-Maxwell equations, which could also include electromagnetic contributions to the colliding waves, could build on extensive previous work for axially symmetric and stationary spacetimes as well as some results already obtained for the case of colliding waves. To reflect the Einstein-Maxwell equations, a second Ernst potential is introduced and the Ernst equation is replaced by a coupled system of nonlinear partial differential equations for these two Ernst potentials. The LP and the RHP are generalised by equations for 3×3 matrices with properties very similar to those discussed in this work, for example the RHP is expected to have the same contour.

In addition it would round off the ISM treatment of the IVP for colliding plane GWs to clarify the physical implications of the restriction to Lipschitz continuous RHP jump matrices.

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Bibliography

[1] A. Einstein (1915); Die Feldgleichungen der Gravitation; *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)*; 844–847.

[2] A. Einstein (1916); Näherungsweise Integration der Feldgleichungen der Gravitation; *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)*; 688–696.

[3] A. Einstein (1918); Über Gravitationswellen; *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)*; 154–167.

[4] A. S. Eddington (1922); The Propagation of Gravitational Waves; *Proceedings of the Royal Society of London Series A*; **102**: 268–282.

[5] P. Bergmann (1969); *The riddle of gravitation.*; Dover Publications.

[6] J. H. Taylor, L. A. Fowler and P. M. McCulloch (1979); Measurements of general relativistic effects in the binary pulsar PSR 1913+16; *Nature*; **277**: 437–440.

[7] B. P. Abbott, R. Abbott, T. D. Abbott, M. R. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. X. Adhikari *et al.* (2016); Observation of Gravitational Waves from a Binary Black Hole Merger; *Physical Review Letters*; **116**: 061102.

[8] J. B. Griffiths (1976); The collision of plane waves in general relativity.; *Annals of Physics*; **102**: 388–404.

[9] J. B. Griffiths (1991); *Colliding plane waves in general relativity.*; Oxford University Press.

[10] J. Steward (1991); *Advanced general relativity*; Cambridge University Press.

[11] N. R. Sibgatullin (1991); *Oscillations and Waves in Strong Gravitational and Electromagnetic Fields*; Springer-Verlag Berlin Heidelberg.

[12] U. Yurtsever (1989); Singularities and horizons in the collisions of gravitational waves; *Physical Review D*; **40**: 329–359.

[13] J. B. Griffiths and J. Podolský (2009); *Exact Space-Times in Einstein's General Relativity*; Cambridge University Press.

[14] I. Hauser and F. J. Ernst (1989); Initial value problem for colliding gravitational plane waves. I; *Journal of Mathematical Physics*; **30**: 872–887.

[15] I. Hauser and F. J. Ernst (1989); Initial value problem for colliding gravitational plane waves. II; *Journal of Mathematical Physics*; **30**: 2322–2336.

[16] I. Hauser and F. J. Ernst (1990); Initial value problem for colliding gravitational plane waves. III; *Journal of Mathematical Physics*; **31**: 871–881.

[17] I. Hauser and F. J. Ernst (1991); Initial value problem for colliding gravitational plane waves. IV; *Journal of Mathematical Physics*; **32**: 198–209.

[18] I. Hauser and F. J. Ernst (2001); Proof of a Generalized Geroch Conjecture for the Hyperbolic Ernst Equation; *General Relativity and Gravitation*; **33**: 195–293.

[19] G. A. Alekseev and J. B. Griffiths (2001); Solving the Characteristic Initial-Value Problem for Colliding Plane Gravitational and Electromagnetic Waves; *Physical Review Letters*; **87**: 221101.

[20] G. A. Alekseev and J. B. Griffiths (2004); Collision of plane gravitational and electromagnetic waves in a Minkowski background: solution of the characteristic initial value problem; *Classical and Quantum Gravity*; **21**: 5623–5654.

[21] V. Belinski and E. Verdaguer (2005); *Gravitational Solitons*; Cambridge University Press.

[22] G. Neugebauer and R. Meinel (2003); Progress in relativistic gravitational theory using the inverse scattering method; *Journal of Mathematical Physics*; **44**: 3407–3429.

[23] N. P. Vekua (1967); *Systems of singular integral equations*; P. Noordhoff, Groningen.

[24] J. Frauendiener and C. Stevens (2014); Numerical evolution of plane gravitational waves in the Friedrich-Nagy gauge; *Physical Review D*; **89**: 104026.

[25] S. Palenta and R. Meinel (2017); A continuous Riemann-Hilbert problem for colliding plane gravitational waves; *Classical and Quantum Gravity*; **34**: 195011.

[26] R. Meinel (2016); *Spezielle und allgemeine Relativitätstheorie für Bachelorstudenten*; Springer-Verlag Berlin Heidelberg.

[27] E. Newman and R. Penrose (1962); An Approach to Gravitational Radiation by a Method of Spin Coefficients; *Journal of Mathematical Physics*; **3**: 566–578.

[28] S. Chandrasekhar (1992); *The mathematical theory of black holes*; Oxford University Press.

[29] R. Penrose and W. Rindler (1988); *Spinors and Space-Time*; Cambridge University Press.

[30] C. Cherubini, D. Bini, S. Capozziello, R. Ruffini and L. Z. Fang (2002); Second Order Scalar Invariants of the Riemann Tensor; *International Journal of Modern Physics D*; **11**: 827–841.

[31] P. Szekeres (1972); Colliding plane gravitational waves.; *Journal of Mathematical Physics*; **13**: 286–294.

[32] S. Chandrasekhar and B. C. Xanthopoulos (1986); A new type of singularity created by colliding gravitational waves; *Proceedings of the Royal Society of London Series A*; **408**: 175–208.

[33] A. Einstein and N. Rosen (1937); On Gravitational Waves; *Journal of The Franklin Institute*; **223**: 43–54.

[34] H. Bondi, F. A. E. Pirani and I. Robinson (1959); Gravitational Waves in General Relativity. III. Exact Plane Waves; *Proceedings of the Royal Society of London Series A*; **251**: 519–533.

[35] H. W. Brinkmann (1923); On Riemann Spaces Conformal to Euclidean Space; *Proceedings of the National Academy of Science*; **9**: 1–3.

[36] M. Blau and M. O'Loughlin (2003); Homogeneous plane waves; *Nuclear Physics B*; **654**: 135–176.

[37] H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers and E. Herlt (2009); *Exact solutions of Einstein's field equations*; Cambridge University Press.

[38] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos (2002); Penrose limits, supergravity and brane dynamics; *Classical and Quantum Gravity*; **19**: 4753–4805.

[39] P. Szekeres (1970); Colliding Gravitational Waves; *Nature*; **228**: 1183–1184.

[40] K. A. Khan and R. Penrose (1971); Scattering of Two Impulsive Gravitational Plane Waves; *Nature*; **229**: 185–186.

[41] A. Sommerfeld (1958); *Partielle Differentialgleichungen der Physik*; Akademische Verlagsgesellschaft Geest und Portik KG, Leipzig.

[42] R. Penrose (1980); Null hypersurface initial data for classical fields of arbitrary spin and for general relativity; *General Relativity and Gravitation*; **12**: 225–264.

[43] G. F. R. Ellis and B. G. Schmidt (1977); Singular space-times; *General Relativity and Gravitation*; **8**: 915–953.

[44] J. B. Griffiths (2005); The stability of Killing Cauchy horizons in colliding plane wave space-times; *General Relativity and Gravitation*; **37**: 1119–1128.

[45] A. Lichnerowicz (1955); *Theories relativistes de la gravitation et de l'electromagnetisme. Relativite generale et theories unitaires*; Masson, Paris.

[46] W. Israel (1966); Singular hypersurfaces and thin shells in general relativity; *Nuovo Cimento B Serie*; **44**: 1–14.

[47] W. Israel (1967); [Erratum] Singular hypersurfaces and thin shells in general relativity; *Nuovo Cimento B Serie*; **48**: 463–463.

[48] E. H. Robson (1972); Junction conditions in general relativity theory; *Annales de l'Institut Henri Poincaré*; **16**: 41–50.

[49] S. O'Brien and J. L. Synge (1952); Jump Conditions at Discontinuities in General Relativity; *Communications of the Dublin Institute for Advanced Studies*; **9**: 1–20.

[50] R. Meinel, M. Ansorg, A. Kleinwächter, G. Neugebauer and D. Petroff (2012); *Relativistic Figures of Equilibrium*; Cambridge University Press.

[51] P. Blanchard and E. Brüning (2003); *Mathematical methods in physics: distributions, Hilbert space operators and variational methods*; Springer-Verlag Berlin Heidelberg.

[52] S. A. Sedaghat Shahmorad (2015); Approximate solution of a system of singular integral equations of the first kind by using Chebyshev polynomials; *arXiv:1508.01873*.

[53] M. Abdulkawi (2015); Solution of Cauchy type singular integral equations of the first kind by using differential transform method; *Applied Mathematical Modelling*; **39**: 2107 – 2118.

[54] D. A. Konkowski and T. M. Helliwell (1989); Singularities in colliding gravitational plane-wave spacetimes; *Classical and Quantum Gravity*; **6**: 1847–1854.

[55] S. Chandrasekhar and V. Ferrari (1984); On the Nutku-Halil solution for colliding impulsive gravitational waves; *Proceedings of the Royal Society of London Series A*; **396**: 55–74.

[56] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov (1984); *Theory of solitons: The inverse scattering method*; Consultants Bureau, New York.

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