# PROJECTIVE SHAPES
## TOPOLOGY AND MEANS

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## Abstract

The projective shape of an object consists of the geometric information that is invariant under different camera views. When describing an object as a configuration of $k$ points or "landmarks" in real projective space $\mathbf{RP}^d$, then the set of projective shapes can be defined as the set $\left(\mathbf{RP}^d\right)^k / \mathbf{PGL}(d)$ of equivalence classes of configurations under the component-wise action of projective transformations. Equipped with the quotient topology, the space of projective shapes is topologically ill-behaved just like in the cases of similarity and affine shapes. In particular, it is neither a manifold nor metrizable. In this thesis the topological structure of projective shape space is analysed in detail in quest for a reasonable topological subspace which is convenient enough for the application of mathematical tools. Further, it is shown that the topological subspace of Tyler regular shapes introduced by Kent and Mardia fulfills all required properties except for some number of landmarks $k$ and dimensions $d$. Then using Tyler standardization, Procrustes distances and Riemannian structures can be defined on the subspace of Tyler regular shapes. For one of these Procrustes distances, a projective mean shape is defined by using the more general concept of Fréchet means. Since the computation of the corresponding sample mean is rather intricate, a new mean is introduced and discussed.

## Abstract (german)

Die projektive Form eines Objektes ist die geometrische Information, die invariant unter projektiven Transformationen ist. Sie tritt natürlicherweise bei der Rekonstruktion von Objekten anhand Fotos unkalibrierter Kameras auf. Wenn ein Objekt als Punktmenge oder Konfiguration von Landmarken im $d$-dimensionalen reell-projektiven Raum $\mathbf{RP}^d$ beschrieben wird, so ist die Menge der projektiven Formen der Quotientenraum $\left(\mathbf{RP}^d\right)^k / \mathbf{PGL}(d)$ und damit kanonisch mit der Quotiententopologie versehen. Auf diesem topologischen Raum der projektiven Formen lassen sich jedoch aus topologischen Gründen viele mathematische Werkzeuge nicht anwenden, ein Phänomen, welches in ähnlicher Form auch bei den Räumen der Ähnlichkeits- bzw. affinen Formen auftritt. In der vorliegenden Arbeit wird die Topologie des projektiven Formenraumes gründlich untersucht, in Hinblick auf die Suche nach einem vernünftigen topologischen Unterraum, der hinreichende Eigenschaften für die Anwendung statistischer Methoden besitzt. Ein Beispiel für einen dieser gutartigen Unterräume ist der Raum der Tyler regulären Formen, der bereits durch Kent und Mardia betrachtet wurde. Deren Ergebnisse werden in dieser Arbeit noch erweitert. Dieser Unterraum ist zwar für einige Dimensionen $d$ und Anzahlen an Landmarken $k$ nicht optimal gewählt, jedoch liefert die so-genannte Tyler-Standardisierung dieser Formen einem sowohl Einbettungen in metrische Räume als auch eine Riemannsche Metrik auf diesem Unterraum. Für eine dieser Einbettungen werden die dazugehörige Fréchet-Erwartungs- sowie Mittelwerte definiert. Während die Konsistenz dieses Mittelwertes leicht zu zeigen ist, ist die Berechnung des extrinsischen Mittelwertes numerisch anspruchsvoll. Als Ersatz wird ein weiterer Erwartungs- bzw. Mittelwert definiert, dessen Berechnung diese Probleme umgeht.

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# Contents

# List of Symbols

| | |
|---|---|
| $\dot\cup, \coprod$ | disjoint union |
| $\wedge, \vee$ | section resp. join |
| $\lceil \cdot \rceil$ | ceiling function |
| $[\cdot, \cdot]$ | Lie bracket |
| $\langle \cdot, \cdot \rangle_F$ | Frobenius inner product |
| $\Vert \cdot \Vert$ | Euclidean norm |
| $\Vert \cdot \Vert_F$ | Frobenius norm |
| $\Vert \cdot \Vert_\infty$ | supremum norm |
| $\Vert \cdot \Vert_{\max}$ | max norm |
| $\mathscr{a}_d^k, \mathscr{A}_d^k$ | space of all projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ |
| $\mathbf{A}^d$ | $d$-dimensional real affine space |
| $\mathbf{Aff}(d)$ | affine group |
| $\mathscr{b}_d^k, \mathscr{B}_d^k$ | space of projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ with projective frame in first $d+2$ landmarks |
| $\mathbf{Bl}(p)$ | blur of an element $p$ |
| $\mathbf{C}_2^k$ | group of $k \times k$-dimensional sign matrices |
| $C(p), C^*(p)$ | collection of (non-trivial) projective subspace constraints fulfilled by $p$ |
| $\mathbf{Cl}(X)$ | closure of a subset $X$ |
| $\mathscr{d}_d^k, \mathscr{D}_d^k$ | space of decomposable projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ |
| $\mathbf{Diag}^*(k)$ | group of $k \times k$-dimensional, non-singular, diagonal matrices |
| $\mathscr{f}_d^k, \mathscr{F}_d^k$ | space of projective shapes resp. configuration of $k$ landmarks in $\mathbf{RP}^d$ with trivial isotropy group |
| $\mathscr{g}_d^k, \mathscr{G}_d^k$ | space of projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ in general position |
| $G(P)$ | graph of a configuration matrix $P$ |
| $\mathbf{GL}(d+1)$ | general linear group of $(d+1) \times (d+1)$-dimensional, non-singular matrices |
| $\mathbf{Gr}(d+1, r)$ | Grassmannian manifold of $r$-dimensional vector subspace of $\mathbf{R}^{d+1}$ |
| $\mathbf{I}_k$ | $k \times k$-dimensional identity matrix |
| $\mathscr{k}_d^k, \mathscr{K}_d^k$ | Kent's shape space, space of Tyler regular or Tyler semi-regular projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ |
| $\mathcal{L}(P)$ | linear hull, column space of a matrix $P$ |
| $\mathbf{N}$ | natural numbers (zero not included) |
| $\mathscr{n}_d^k(n), \mathscr{N}_d^k(n)$ | space of projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ bounded by projective subspace numbers $n$ |
| $\mathbf{O}(k)$ | orthogonal group of $k \times k$-dimensional, orthogonal matrices |
| $\mathfrak{o}(k)$ | Lie algebra to $\mathbf{O}(k)$, space of anti-symmetric, $k \times k$-dimensional matrices |
| $\mathscr{p}_d^k, \mathscr{P}_d^k$ | space of projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ with projective frame |
| $\mathbf{PGL}(d)$ | projective linear group, group of projective transformations |
| $\mathbf{R}$ | real numbers |
| $\mathbf{R}^*$ | multiplicative subgroup $\mathbf{R} \backslash \{0\}$ of $\mathbf{R}$ |
| $\mathbf{rk}\, p, \mathbf{rk}\, P$ | rank of a configuration resp. of a matrix |

| | |
|---|---|
| $\mathbf{RP}^d$ | $d$-dimensional real projective space |
| $r_d^k, \mathcal{R}_d^k$ | space of projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ of rank $d+1$ |
| $\mathbf{S}^d$ | $d$-dimensional unit sphere in $\mathbf{R}^{d+1}$ |
| $\mathbf{Sim}(d)$ | group of similarity transformations |
| $\mathbf{St}(d+1,r)$ | non-compact Stiefel manifold of $(d+1)\times r$-dimensional matrices of full rank |
| $\mathbf{St}^{\mathrm{O}}(d+1,r)$ | orthogonal Stiefel manifold of $(d+1)\times r$-dimensional matrices of full rank with orthogonal columns |
| $\mathbf{sym}(k)$ | space of equivalence classes of $k\times k$-dimensional, symmetric matrices under action of $\mathbf{C}_2^k$ |
| $\mathbf{Sym}(k)$ | space of $k\times k$-dimensional, symmetric matrices |
| $t_d^k, \mathcal{T}_d^k$ | space of Tyler regular projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ |
| $ter_d^k, \mathcal{T}er_d^k$ | space of Tyler extended-regular projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ |
| $tir_d^k, \mathcal{T}ir_d^k$ | space of Tyler irregular projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ |
| $\mathrm{tr}(A)$ | trace of a matrix $A$ |
| $\mathcal{TSC}(\mathcal{K}_d^k)$ | space of Tyler standardized matrix configurations of projective shape in $\mathcal{K}_d^k$ |
| $\mathbf{tsp}(\mathcal{K}_d^k)$ | space of equivalence classes of Tyler standardized projection matrices under action of $\mathbf{C}_2^k$ corresponding to projective shapes in $\mathcal{K}_d^k$ |
| $\mathbf{TSP}(\mathcal{K}_d^k)$ | space of Tyler standardized projection matrices corresponding to projective shapes in $\mathcal{K}_d^k$ |
| $tsr_d^k, \mathcal{T}sr_d^k$ | space of Tyler semi-regular projective shapes resp. configurations of $k$ landmarks in $\mathbf{RP}^d$ |

# Chapter 1

# Introduction

Consider taking $d$-dimensional images of a scene comprising of $k$ ordered points or landmarks in a $d$-dimensional hyperplane of $(d+1)$-dimensional space such that all $k$ landmarks are visible in the images. An important result from computer vision is that these images differ only by a projective transformation between themselves and from the original scene, even if the images are taken with different cameras. In particular, if the calibrations of the cameras are unknown, i.e., if there is no information available on the camera parameters such as focal length, angle between scene and film hyperplane, location of the camera, etc., then an image relays only information about the scene which is invariant under projective transformations (Hartley and Zisserman; 2003). The collection of this information is known as the scene's projective shape.

Projective shapes arise similarly in the problem of reconstruction of a 3-D scene from multiple camera views: a scene can be reconstructed from a set of uncalibrated 2-D images at best up to a projective transformation, so again, one retrieves only the scene's projective shape.

In both cases, one gains more information about the original object if one has more information about the camera(s), leading to other types of shape such as similarity shape or affine shape. Of course, if everything is known about the camera(s), then the original scene is completely reconstructable from its images.

Mathematically, an object or scene is described by a configuration $p = (p_1, \ldots, p_k) \in \left(\mathbf{RP}^d\right)^k$, i.e. a finite, ordered set of points or landmarks in real projective space $\mathbf{RP}^d$, while the shape $[p]$ of this configuration $p$ is its orbit or equivalence class

$$[p] = \left\{\alpha p = (\alpha p_1, \ldots, \alpha p_k) : \alpha \in \mathbf{PGL}(d)\right\}$$

under the component-wise action of the projective linear group $\mathbf{PGL}(d)$. The set of projective shapes is then the set of orbits

$$\boldsymbol{a}_d^k = \left(\mathbf{RP}^d\right)^k / \mathbf{PGL}(d).$$

This topological quotient is naturally equipped with the quotient topology, thus rendering $\boldsymbol{a}_d^k$ a topological space.

It is quite unpleasant to work with this abstract notion of projective shape, whence homogeneous coordinates will be used in this manuscript to describe configurations and projective shapes. In homogeneous coordinates, a configuration is given as a $k \times (d+1)$-dimensional matrix

$$P = \begin{pmatrix} P_{1\cdot} \\ \vdots \\ P_{k\cdot} \end{pmatrix} \in \mathbf{R}^{k \times (d+1)}$$

with its non-vanishing rows $P_{i\cdot} \in \mathbf{R}^{d+1} \backslash \{0\}$ representing the landmarks. Left-multiplication of $P$ with a non-singular, $k \times k$-dimensional matrix $D \in \mathbf{Diag}^*(k)$ corresponds then to the same configuration in $\mathbf{RP}^d$. Projective transformations act on such matrix configurations as right-multiplication with non-singular, $(d+1) \times (d+1)$-dimensional matrices $B \in \mathbf{GL}(d+1)$. Hence, the projective shape of a configuration matrix $P$ is the orbit

$$[P] = \{DPB : D \in \mathbf{Diag}^*(k),\, B \in \mathbf{GL}(d+1)\}.$$

Many mathematical applications, e.g. statistics, numerics, etc., require quantitative comparisons on the underlying space, i.e. a metric, or the space to be at least locally Euclidean, i.e., to be a topological manifold. Unfortunately, the topological space $\mathfrak{a}_d^k$ of projective shapes is neither metrizable nor a topological manifold, analogously to the situation with similarity and affine shape spaces. As in those cases, the solution to this problem is to find a topological subspace of $\mathfrak{a}_d^k$ which is fulfilling the requirements needed for the desired application. This turns out to be more complicated for projective shapes than for the cases of similarity and affine shapes in which the topological subspace of shapes corresponding to the configurations with trivial isotropy group is a differentiable Hausdorff manifold. Reasonable metrics have been defined and discussed on these subspaces of similarity resp. affine shape space (Dryden and Mardia; 1998; Groisser and Tagare; 2009).

The purpose of this thesis is to establish requirements for a topological subspace of projective shape space such that a multitude of mathematical tools can conveniently be applied on the topological subspace. However, the topological subspaces in question shall also be geometrically and topologically sensible. Of course, the objective is to determine conditions for topological subspaces for which these requirements are met, too.

This discussion of projective shape spaces is started with a clear application in mind: statistics and, in particular, the computation of a sample mean shape. For the classical definition of the sample mean as the arithmetic mean, a vector space structure is needed, but there is none to find in $\mathfrak{a}_d^k$, whence another definition of a mean has to be used. In Euclidean spaces, the sample mean is the minimizer of the sum of squared distances to the sample. This fact can be generalized to metric spaces, leading to the definition of the Fréchet sample mean as the set of minimizers of the Fréchet function

$$\tfrac{1}{n}\sum_{i=1}^{n} \mathbf{d}^2(\,\cdot\,, X_i)$$

for a sample $X_1, \ldots, X_n$ and a metric $\mathbf{d}$ (Fréchet; 1948). Hence, a metrizable space is what we aim for as a topological subspace of projective shape space. Recall from differential geometry that differentiable Hausdorff manifolds are metrizable since they allow both the definition of a Riemannian metric and an embedding into a Euclidean space. Hence, it is reasonable to require the topological subspace to be a differentiable Hausdorff manifold. Both the idea of embedding a *differentiable Hausdorff manifold* into a Euclidean space and the idea of endowing it with a Riemannian metric have been widely discussed before in a statistical context. The former idea leads to what is called extrinsic statistics, the latter to intrinsic statistics, see e.g. (Bhattacharya and Bhattacharya; 2012). In addition to being metrizable, differentiable Hausdorff manifolds possess of course the advantage of being locally Euclidean, i.e., mathematical tools on Euclidean spaces can be locally performed on manifolds.

Besides requiring the structure of a differentiable Hausdorff manifold, it is natural to ask that all landmarks play the same role in the chosen topological subspace of $\mathfrak{a}_d^k$; mathematically speaking, the subspace shall be *closed under permutations* of the landmarks.

One of the first observations to make when working with projective transformations is that they map lines to lines, planes to planes, etc. Hence, if some landmarks of a configuration lie in a projective subspace of $\mathbf{RP}^d$, then they will lie in a projective subspace of the same dimension under any projective transformation. So, the information of landmarks in projective subspaces is invariant under projective transformations and an attribute of the corresponding projective shape. We will say that a configuration resp. shape fulfills *projective subspace constraints.* As the third requirement, we request the chosen topological subspace of $\mathfrak{a}_d^k$ to include with a shape all shapes with the same or less projective subspace constraints, as well. In particular, if there is a small distortion on the landmarks of a configuration corresponding to a shape in a chosen topological subspace, then the distorted configuration fulfills less projective subspace constraints and its projective shape shall again be an element of the chosen subspace. This requirement will be called *respecting the hierarchy of projective subspace constraints*.

Finally, we seek topological subspaces that are *maximal* in the sense that further inclusion of shapes leads to infringement of at least one other requirement.

Of course, there have been some prior attempts to find a convenient topological subspace of projective shape space: Mardia and Patrangenaru (2005) used projective frames to define a topological subspace of projective shape space via common registration, just like Bookstein (1986) has done for similarity shapes: if one fixes a shape's first $d + 2$ landmarks to a predetermined projective frame, then the projective shape is uniquely determined by the coordinates of the remaining $k - d - 2$ landmarks. This procedure is only possible for the shapes which include a frame in its first $d+2$ landmarks. While this topological subspace has the drawback of not being closed under permutations, it respects the hierarchy of projective subspace constraints and is homeomorphic to the differentiable Hausdorff manifold $\left(\mathbf{RP}^d\right)^{k-d-2}$. Of course, the definition of a Riemannian metric resp. of an embedding into Euclidean space has been done before for $\mathbf{RP}^d$, as has the computation of sample means, cf. e.g. (Hotz and Kelma; 2016) for a discussion of the latter in the context of projective shapes.

Another topological subspace of $\mathfrak{a}_d^k$ was introduced and discussed by Kent and Mardia (2012): the subspace of Tyler regular shapes. They show that under some mild regularity conditions on a shape's projective subspace constraints a shape possesses a configuration of certain type which is unique up to the action of a discrete group and of the orthogonal group. With this so-called Tyler standardization of shapes, Procrustes distances can be defined on the topological subspace of shapes whose projective subspace constraints fulfill the regularity conditions. However, this topological subspace has been introduced through the existence of this standardization without giving it a topological or geometrical justification. It has so far been unclear if this subspace fulfills any of the requirements above, besides that the existence of metrics render this space a Hausdorff space, and that the regularity conditions imply that the space respects the hierarchy of projective subspace constraints. As it turns out, the topological subspace of Tyler regular shapes fulfills all of our requirements unless $k$ and $d + 1$ have a common divisor larger than 2 when maximality is not given. This subspace has been used in a statistical context to determine if the projective shapes of two buildings are distinguishable (Kent and Mardia; 2012, Sect. 9).

Using Tyler standardization and one of the metrics introduced by Kent and Mardia (2012), we will define the corresponding extrinsic mean shape on this space. The computation of such a sample mean turns out to be rather difficult, whence a new mean is introduced and discussed as well.

## Overview

First and foremost, this work presents a thorough discussion of the topology of projective shape space in search of topological subspaces which fulfill reasonable geometric and topological constraints for the application of mathematical tools. As it is demonstrated using direct techniques, the topology is ruled by geometrical and algebraic properties. In particular, we discuss which topological subspaces fulfill the separation axiom T1, which are Hausdorff, and which are differentiable manifolds. For the latter the notion of a projective frame is generalized to obtain charts on the topological subspace of free shapes. For a reasonable class of topological subspaces we give simple conditions for which the aforementioned requirements are met.

As an explicit example the topological subspace of Tyler regular shapes is discussed. Using Tyler standardization, we present the definition and computation of a new mean on this subspace and compare this mean with an extrinsic mean.

Chapter 2 recalls useful notions of projective geometry. Further, projective shape space is introduced and the notation is fixed for the remainder of the thesis. Additionally, the notion of projective subspace constraints for configurations is introduced. It is shown that these geometric entities are invariant under projective transformations, hence attributes of the projective shape of a configuration. The chapter also includes a short discussion of the occurrences of projective shape in computer vision.

In Chapter 3 a reasonable list of requirements for a useful topological subspace of projective shape space is presented. With this list in mind the topology of projective shape space is analyzed in detail, and criteria are determined for which topological subspaces fulfill these requirements. The main results of this chapter have been published in (Hotz et al.; 2016).

In Chapter 4 the topological subspace of Tyler regular shapes is discussed. This subspace has been introduced by Kent and Mardia (2012), and it allows the reduction of the group action through partial standardization. A new geometric reasoning for this so-called Tyler standardization is introduced (published in (Hotz et al.; 2016), too), and Kent and Mardia's results are expanded by proving which shapes—besides the Tyler regular ones—can be Tyler standardized. Tyler standardization also leads to embeddings of the subspace of Tyler regular shapes into metric spaces, as well as the definition of Riemannian metrics.

In Chapter 5 these embeddings are used to define an extrinsic population and sample mean shape for which consistency is proven. Unfortunately, the computation of the sample mean is rather hard since the projection on to the shape space can only be approximated by a gradient descent algorithm. As a remedy, the notion of a Tyler mean shape is introduced. The Tyler sample mean shape is easier to compute while being a strongly consistent estimator of the Tyler population mean shape. These means are compared in elementary examples.

Finally, the thesis concludes with a short discussion of the results and an outlook for future research.

## Related works

Concerning projective geometry, there are many textbooks available which include all of the theory needed for this discussion, cf. e.g. (Berger; 1987) or (Onishchik and Sulanke; 2006). An overview of projective geometry in the context of image analysis can be found in (Faugeras and Luong; 2001) and (Hartley and Zisserman; 2003).

Both (Faugeras and Luong; 2001) and (Hartley and Zisserman; 2003) additionally are standard textbooks for computer vision which discuss the occurrences of the different types of shape. In particular, they include the key observations that an image of a flat scene is a projective transformation of the scene, as well as that a scene can only be reconstructed up to a projective transformation from multiple uncalibrated camera views. Another recommendable book about computer vision is (Ma et al.; 2004).

The topologies of affine and similarity shape spaces have been discussed in a similar fashion as this thesis does for projective shape space.

Patrangenaru and Mardia (2003) noted in a short conference article that affine shape space is stratified into real Grassmannian manifolds; in particular, the top stratum of affine shape space is a real Grassmannian manifold and comprises of the shapes of configurations whose isotropy group is trivial. It is the largest sensible differentiable Hausdorff manifold in affine shape space. A detailed discussion of these statements was provided by Groisser and Tagare (2009). Additionally, Groisser and Tagare discuss a reasonable Riemannian metric for the top stratum.

Some key results in the discussion of similarity shapes were presented by Bookstein and Kendall. While Bookstein (1986) discussed—as already mentioned—the topological subspace given by common registration, the approach by Kendall (1984) introduced the idea of pre-shapes, i.e., to discuss only those shapes for which the group action can be reduced to a compact group action by choosing standardized representatives. We refer the reader to (Dryden and Mardia;

1998) and (Kendall et al.; 1999) for a detailed description of similarity shape space and further references.

As already mentioned, two topological subspaces of projective shape space have already been discussed. The approach through common registration by Mardia and Patrangenaru (2005) uses an earlier idea of Horadam (1970) and Goodall and Mardia (1999). Here, the first $d+2$ landmarks of a projective shape are fixed consuming all the degrees of freedom in the transformation group. Consequently, the projective shape is then given by the location of the remaining $k-d-2$ landmarks. However, this idea works only for those projective shapes whose first $d+2$ landmarks are in general position.

The approach by Kent and Mardia (2012) uses the idea of a projective pre-shape analogously to Kendall's approach to similarity shapes: under some conditions on its projective subspace constraints, a projective shape possesses a configuration which is unique up to the action of a compact group. Using this so-called Tyler standardization of projective shapes, Kent and Mardia defined and discussed Procrustes metrics on this topological subspace. We note that this topological subspace has also been discussed in the literature of geometric invariant theory (Mumford et al.; 1994, Ch. 3).

The definition of a mean on non-Euclidean spaces was introduced by Fréchet (1948) as the minimizer of the expected squared distance. Consistency results for these so-called Fréchet means were found by Ziezold (1977) and later by Bhattacharya and Patrangenaru (2003), while asymptotic behavior was discussed by Hendriks and Landsman (1998) and Bhattacharya and Patrangenaru (2005).

Consistency results for means of similarity shapes were presented by Kent and Mardia (1997) and Le (1998). Additionally, Bhattacharya and Bhattacharya (2012) introduced and discussed nonparametric statistical methods on manifolds and, in particular, on similarity, affine, and projective shape spaces. Further results in the context of inference on shape spaces were presented by Patrangenaru and Ellingson (2016).

The statistical results for projective shape space in Bhattacharya and Bhattacharya (2012) and Patrangenaru and Ellingson (2016) use the approach through common registration by Mardia and Patrangenaru (2005). For this topological subspace of projective shapes, an extrinsic sample mean and parametric tests have been discussed in the context of face recognition (Mardia and Patrangenaru; 2005), while Mardia et al. (2003) discussed the same extrinsic sample mean in the context of reconstruction of a planar scene from multiple images. Universal, non-asymptotic confidence sets for this extrinsic mean have been constructed by T. Hotz and the author of this thesis (Hotz and Kelma; 2016). To our knowledge, the approach through Tyler standardization has only been used by (Kent and Mardia; 2012) in a statistical context as we have noted above.

## Unpublished contributions of this thesis

The notion of projective geometry, projective shape, and its occurences in computer vision presented in Chapter 2 are, of course, well-known. The notion of projective subspace constraints was introduced by Kent et al. (2011) as "linear subspace constraints". New are the calculation rules for projective subspace constraints (Lemma 2.5), the partial order, the notion of "total decomposition" as well as the canonical block matrix structure of projective shapes (Proposition 2.7).

The main results of Chapter 3 have been published in (Hotz et al.; 2016) for which I consider myself the main author. Many remarks and examples have been added to the discussion, though. The results about a stratification of projective shape space (Proposition 3.2) as well as the exact computation of the blur of a projective shape (Proposition 3.10) are unpublished. Additionally, the characterization of Hausdorff spaces respecting the hierarchy of projective subspace constraints (Corollary 3.14) is new.

In Chapter 4 the geometric motivation of Tyler standardization was already published in (Hotz et al.; 2016). The results regarding Tyler standardization itself were introduced by Kent et al. (2011) for which now a comprehensive proof is given. Further, it was shown that Tyler standardization is differentiable (Remark 4.8). The thorough discussion of Tyler semi-regular shapes and Kent's shape space is new, as is the connection to the notion of finite unit norm tight frames in Hilbert spaces (Remark 4.6) resulting in a homeomorphism for shape spaces of different dimensions (Lemma 4.11). The discussion of the embedding of Tyler standardized projection matrices as well as the construction of a Riemannian metric in Section 4.2 are comprehensive extensions of published results; see (Kent and Mardia; 2012) resp. (Hotz et al.; 2016).

The construction of mean shapes in Chapter 5 is completely unpublished. While the statistical approach has been suggested by my supervisor Thomas Hotz, the thorough derivation of the presented results and examples is my contribution.

# Chapter 2

# Projective shape space

The importance of projective geometry is visible in image analysis: when taking an image of parallel lines in real world, e.g. railroad tracks, they do meet at the horizon which is usually modeled to be infinitely distant. In Euclidean or affine geometry parallel lines have no intersection point, and there are no points at infinity, whence these geometries are not the right framework for image analysis. In projective spaces there are points at infinity, and, in 2-D, lines always intersect with parallel lines intersecting at infinity. Hence, projective geometry is the natural geometry to work with in image analysis. In particular, central projections, and hence taking pictures with pinhole cameras, can conveniently be described in this framework.

There are two distinct approaches to projective geometry: the *synthetic* approach is the classical one, and it relates geometrical object (points, lines, planes, etc.) axiomatically. The *analytical* approach uses—contrary to its name—concepts from linear algebra, and will be used in this thesis since the representation in notation of linear algebra is very useful for our purposes.

The projective shape of an object comprising of a finite, ordered set of points or landmarks in $d$-dimensional real projective space $\mathbf{RP}^d$ is the information that remains if the information about the coordinate system on $\mathbf{RP}^d$ is removed. This kind of information arises naturally in computer vision. The coordinate transformation group of $\mathbf{RP}^d$ is the so-called projective linear group $\mathbf{PGL}(d)$, so the projective shape of an object is the orbit of the object under the component-wise action of $\mathbf{PGL}(d)$. The set of projective shapes of objects with $k$ landmarks in $d$-dimensional real projective space $\mathbf{RP}^d$ can then be described as the set of equivalence classes

$$\boldsymbol{a}_d^k = \left(\mathbf{RP}^d\right)^k / \, \mathbf{PGL}(d).$$

Equipped with the quotient topology, $\boldsymbol{a}_d^k$ is a topological space.

The main objective of this chapter is to fix the notation for the remainder of this thesis: in Section 2.1 we remind the reader of projective geometry, including real projective spaces and Grassmannians. In Section 2.2 it is shown how projective geometry is used to describe cameras in computer vision. In particular, the occurrences of projective shapes in single- and multiple-view settings are discussed. In Section 2.3 projective shapes and projective shape space are introduced thoroughly. Additionally, important invariants of the group action as well as canonical representations of configurations and shapes are discussed.

## 2.1 Real projective space and Grassmannians

The *d-dimensional real projective space* $\mathbf{RP}^d$ derived from $\mathbf{R}^{d+1}$ is defined to be the quotient space of $\mathbf{R}^{d+1}\backslash\{0\}$ modulo the component-wise action of the multiplicative group $\mathbf{R}^* = \mathbf{R}\backslash\{0\}$, i.e. modulo the equivalence relation

$$x \sim y \qquad \iff \qquad x = \lambda y \quad \text{for some } \lambda \in \mathbf{R}^*$$

for $x, y \in \mathbf{R}^{d+1}$. The quotient map is denoted by $\pi_p : \mathbf{R}^{d+1}\backslash\{0\} \to \mathbf{RP}^d$. Note that $\pi_p$ is a continuous, open mapping.

While $d$-dimensional real projective space $\mathbf{RP}^d$ can be understood as the space of one-dimensional vector subspaces of $\mathbf{R}^{d+1}$, i.e. of lines through the origin, an $i$-dimensional *projective subspace* of $\mathbf{RP}^d$, $1 \leqslant i \leqslant d$, is the image of an $(i+1)$-dimensional vector subspace $V$ of $\mathbf{R}^{d+1}$ under $\pi_p$, i.e. the set of one-dimensional vector subspaces of $V$. Hence, any $i$-dimensional projective subspace of $\mathbf{RP}^d$ is homeomorphic to $\mathbf{RP}^i$. One- and two-dimensional projective subspaces of $\mathbf{RP}^d$ are called (projective) lines respectively planes, while the elements of $\mathbf{RP}^d$ are called points. A (projective) hyperplane of $\mathbf{RP}^d$ is the image of a vector hyperplane.

A set of $k$ points in $\mathbf{RP}^d$ is said to be *projectively independent* if $k \leqslant d+1$ and there is no $(k-2)$-dimensional projective subspace of $\mathbf{RP}^d$ containing them; it is said to be *in general position* if any subset of at most $d+1$ points is projectively independent.

Let $\{x_1, \ldots, x_{d+1}\}$ be a basis of $\mathbf{R}^{d+1}$. Then, any point $p \in \mathbf{RP}^d$ in $d$-dimensional real projective space has a representation as a vector $(p^1, \ldots, p^{d+1})^t \in \mathbf{R}^{d+1}$ in the considered basis with $\pi_p\big((p^1, \ldots, p^{d+1})^t\big) = p$. While this so-called *homogeneous coordinate vector* is only unique up to rescaling, it allows to describe projective space $\mathbf{RP}^d$ and its morphisms in the convenient notation of matrix calculus. Of course, one could require a homogeneous coordinate vector $(p^1, \ldots, p^{d+1})^t \in \mathbf{R}^{d+1}$ to be of norm 1, i.e., $(p^1, \ldots, p^{d+1})^t \in \mathbf{S}^d = \big\{p \in \mathbf{R}^{d+1} : \|p\|_2 = 1\big\}$. Then, the equivalence relation becomes

$$x \sim y \qquad \Longleftrightarrow \qquad x = \pm y$$

for $x, y \in \mathbf{S}^d$, and one easily obtains $\mathbf{RP}^d \cong \mathbf{S}^d \big/ \{\pm 1\}$.

Note that the $d+1$ points $p_i = \pi_p(x_i) \in \mathbf{RP}^d$, $i \in \{1, \ldots, d+1\}$, are not sufficient to determine the homogeneous coordinates of some other point $q \in \mathbf{RP}^d$ since any other basis of the form $\lambda_i x_i$, $\lambda_i \in \mathbf{R}^*$, $i \in \{1, \ldots, d+1\}$, would give the same points $p_i$. To resolve this uncertainty, another point $p_{d+2} \in \mathbf{RP}^d$ in general position is needed, e.g.

$$p_{d+2} = \pi_p(x_1 + \cdots + x_{d+1})$$

with homogeneous coordinates $(1, \ldots, 1)$, whereby only two proportional bases $\{x_1, \ldots, x_{d+1}\}$, $\{y_1, \ldots, y_{d+1}\}$ with $x_i = \lambda y_i$, $\lambda \in \mathbf{R}^*$, give the same points in $\mathbf{RP}^d$. As it turns out, there exists a unique set of $d+2$ points in general position to any homogeneous coordinate system, and vice versa. Therefore, such a sequence of $d+2$ points in general position is called *projective frame* or *projective basis*. In a (projective) frame, the first $d+1$ points are called *base points*, and the $(d+2)$-nd point is the *unit point*.

The transformation group for vector coordinates of $\mathbf{R}^{d+1}$ is the general linear group $\mathbf{GL}(d+1)$ acting transitively from left on $\mathbf{R}^{d+1}$. Such a change of basis transforms the homogeneous coordinate system. Since only uniform scalar multiplication of all basis elements does not change the homogeneous coordinates, the kernel of this action is given by scalar multiples of the identity matrix $\mathbf{I}_{d+1}$. Hence, the transformation group for homogeneous coordinate systems is the *projective linear group*

$$\mathbf{PGL}(d) = \mathbf{GL}(d+1) \big/ \mathbf{R}^* \mathbf{I}_{d+1} \tag{2.1}$$

acting simply transitively on the set of homogeneous coordinate systems, and thus on the set of frames as stated in the First Main Theorem of Projective Geometry, cf. e.g. (Berger; 1987, Prop. 4.5.10):

**Theorem 2.1.** *Let $(p_1, \ldots, p_{d+2})$, $(q_1, \ldots, q_{d+2})$ be two projective frames of $\mathbf{RP}^d$. There exists a unique projective transformation $\varphi \in \mathbf{PGL}(d)$ such that $q_i = \varphi(p_i)$ for all $i \in \{1, \ldots, d+2\}$.*

Alternatively, one can define local coordinate systems, and thus proving that $\mathbf{RP}^d$ is a $d$-dimensional manifold: again, let $\{x_1, \ldots, x_{d+1}\}$ be a basis of $\mathbf{R}^{d+1}$, and let

$$H_i = \big\{\pi_p\big((p^1, \ldots, p^{d+1})^t\big) \in \mathbf{RP}^d : p^i = 0\big\}$$

for $i \in \{1, \ldots, d+1\}$ be the hyperplane of $\mathbf{RP}^d$ comprising of those points whose $i$-th entry in homogeneous coordinates to the chosen basis is zero. Now, $\mathbf{RP}^d \backslash H_i$ is homeomorphic to $\mathbf{R}^d$, and a homeomorphism is e.g. given by the map

$$\begin{array}{rccc} \varphi_i : & \mathbf{RP}^d \backslash H_i & \longrightarrow & \mathbf{R}^d \\ & \pi_p\big((p^1, \ldots, p^{d+1})^t\big) & \longmapsto & \left(\frac{p^1}{p^i}, \ldots, \frac{p^{i-1}}{p^i}, \frac{p^{i+1}}{p^i}, \ldots, \frac{p^{d+1}}{p^i}\right)^t. \end{array} \tag{2.2}$$

These charts $\varphi_i$, $i \in \{1, \ldots, d+1\}$, are compatible and their domains cover $\mathbf{RP}^d$, rendering $\mathbf{RP}^d$ a $d$-dimensional differentiable manifold. The maps $\varphi_i$, $i \in \{1, \ldots, d+1\}$, are usually called *inhomogeneous coordinates*.

While $\mathbf{RP}^d$ is the manifold of one-dimensional vector subspaces of $\mathbf{R}^{d+1}$, the real *Grassmannian manifold* (short: Grassmannian) $\mathbf{Gr}(d+1, r)$ is the manifold of $r$-dimensional vector subspaces of $\mathbf{R}^{d+1}$, $r \in \{1, \ldots, d+1\}$, and thus generalizes real projective space. The Grassmannian $\mathbf{Gr}(d+1, r)$ is defined as the quotient space

$$\mathbf{Gr}(d+1, r) = \mathbf{St}(d+1, r) \,/\, \mathbf{GL}(r) \tag{2.3}$$

with $\mathbf{St}(d+1, r)$ being the space of $(d+1) \times r$-dimensional, real matrices of full rank and right-action of $\mathbf{GL}(r)$ on it. Here, the columns of the full rank matrices correspond to a basis of a vector subspace of $\mathbf{R}^{d+1}$ and the $\mathbf{GL}(r)$-action is the change of basis. The topological space $\mathbf{St}(d+1, r)$ is commonly known as the *non-compact Stiefel manifold*.

Equivalently, the Grassmannian can be defined via orthonormal bases, i.e.,

$$\mathbf{Gr}(d+1, r) = \mathbf{St}^{\mathrm{O}}(d+1, r) \,/\, \mathbf{O}(r) \tag{2.4}$$

with $\mathbf{St}^{\mathrm{O}}(d+1, r)$ being the space of $(d+1) \times r$-dimensional full rank matrices with orthonormal columns which is commonly known as the *orthogonal Stiefel manifold*. Note that $\mathbf{Gr}(d+1, 1) = \mathbf{RP}^d$ since $\mathbf{St}(d+1, 1) = \mathbf{R}^{d+1} \backslash \{0\}$ and $\mathbf{GL}(1) = \mathbf{R}^*$, respectively $\mathbf{St}^{\mathrm{O}}(d+1, 1) = \mathbf{S}^d$ and $\mathbf{O}(1) = \mathbf{C}_2 = \{\pm 1\}$.

The Grassmannian $\mathbf{Gr}(d+1, r)$ is a $r(d+1-r)$-dimensional, compact, differentiable Hausdorff manifold. It can be smoothly embedded into the Euclidean space $\mathbf{Sym}(d+1)$ by choosing a representative $X \in \mathbf{St}(d+1, r)$ to each element of $\mathbf{Gr}(d+1, r)$ and mapping $X$ to the orthogonal projection matrix $M_X = X(X^tX)^{-1}X^t$ which projects $\mathbf{R}^{d+1}$ orthogonally to the column space $\mathcal{L}(X)$ of $X$. This mapping

$$\begin{array}{rccc} \iota : & \mathbf{Gr}(d+1, r) & \longrightarrow & \mathbf{Sym}(d+1) \\ & \mathcal{L}(X) & \longmapsto & M_X \end{array} \tag{2.5}$$

is called *Veronese-Whitney embedding*. The Euclidean vector product on $\mathbf{Sym}(d+1)$ is given by the *Frobenius inner product* $\langle A, B \rangle_F = \mathrm{tr}(AB)$ for $A, B \in \mathbf{Sym}(d+1)$, while the corresponding norm $\|A\|_F = \sqrt{\mathrm{tr}(AA)}$ is called *Frobenius norm*. Note that $\mathbf{Gr}(d+1, r)$ is mapped to symmetric matrices of rank $r$ and norm $\sqrt{r}$.

The Veronese-Whitney embedding naturally gives a homeomorphism $T$ between $\mathbf{Gr}(d+1, r)$ and $\mathbf{Gr}(d+1, d+1-r)$ by mapping a vector subspace of $\mathbf{R}^{d+1}$ to its orthogonal complement, i.e.,

$$\begin{array}{rccc} T : & \mathbf{Gr}(d+1, r) & \longrightarrow & \mathbf{Gr}(d+1, d+1-r) \\ & \mathcal{L}(X) & \longmapsto & \iota^{-1}\big(\mathbf{I}_{d+1} - \iota(X)\big). \end{array} \tag{2.6}$$

The union $\mathfrak{P}^d = \bigcup_{r=1}^{d+1} \mathbf{Gr}(d+1, r) \cup \{\mathbf{o}\}$ of Grassmannians together with the trivial vector subspace $\mathbf{o} = \{0\} \subset \mathbf{R}^{d+1}$ is called *$d$-dimensional projective geometry over* $\mathbf{R}$, while its elements are called *projective subspaces*. The *projective dimension* of a projective subspace is given by the dimension of the corresponding vector space diminished by 1, or equivalently by the rank $\mathbf{rk}\,X$ of a representative $X \in \mathbf{St}(d+1, r)$ to the projective subspace minus 1.

The trivial vector subspace $\mathbf{o}$ is added to the geometry for mathematical reasons, whence two products can be defined on $\mathfrak{P}^d$: the *section* $U \wedge V$ (also called meet or intersection) of two projective subspaces $U, V \in \mathfrak{P}^d$ is just its intersection as sets, i.e.,

$$U \wedge V = U \cap V,$$

while the *join* of $U, V \in \mathfrak{P}^d$ is the smallest projective subspace in $\mathfrak{P}^d$ which contains both $U$ and $V$, i.e.,

$$U \vee V = \mathcal{L}\big(\iota(U) \cup \iota(V)\big).$$

The join is very useful to describe projective independence. A set $\{p_1, \ldots, p_k\}$ of points in real projective space is *projectively independent* if and only if the projective dimension of

$$p_1 \vee \cdots \vee p_k$$

equals $k - 1$. More general, a set $\{U_1, \ldots, U_k\}$ of projective subspaces of $\mathbf{RP}^d$ is called projectively independent if and only if the dimension of

$$U_1 \vee \cdots \vee U_k$$

as a vector subspace of $\mathbf{R}^{d+1}$ equals the sum of the dimensions of the vector subspaces $U_i$ of $\mathbf{R}^{d+1}$, $1 \leqslant i \leqslant k$, or equivalently, equals the sum of projective dimensions of the $U_i$, $1 \leqslant i \leqslant k$, plus $k - 1$.

In $\mathbf{RP}^d$ there exist sets of $d+1$ projectively independent points, e.g. the points corresponding to any basis of $\mathbf{R}^{d+1}$. Any set of $k > d + 1$ points is projectively dependent.

The morphisms in the category of projective geometries stem from morphisms on their corresponding vector spaces, i.e. from linear maps between them. A linear map $A : \mathbf{R}^{d+1} \to \mathbf{R}^{e+1}$, $d, e \in \mathbf{N}$, i.e. a matrix $A \in \mathbf{R}^{(e+1)\times(d+1)}$, naturally defines a map

$$\alpha : \quad \mathfrak{P}^d \quad \longrightarrow \quad \mathfrak{P}^e$$

between the corresponding projective geometries $\mathfrak{P}^d$ and $\mathfrak{P}^e$ by mapping a vector subspace $U$ of $\mathbf{R}^{d+1}$ to its image $AU$ under $A$. Such a morphism $\alpha$ preserves the operation of both the section and the join, i.e.,

$$\alpha(U \wedge V) = \alpha(U) \wedge \alpha(V) \qquad \text{and} \qquad \alpha(U \vee V) = \alpha(U) \vee \alpha(V)$$

for all projective subspaces $U, V \in \mathfrak{P}^d$. Two morphism $\alpha, \beta$ induced by linear maps $A, B$ are identical if and only if $A$ and $B$ are identical up to a scalar, i.e., $A = \lambda B$ for some $\lambda \in \mathbf{R}^*$, whence the set of morphisms is the projective space to the vector space of linear maps between the corresponding vector spaces.

Note that $\alpha$ can be reconstructed from its restriction

$$\alpha|_{\mathbf{RP}^d \cup \{\mathbf{o}\}} : \quad \mathbf{RP}^d \cup \{\mathbf{o}\} \quad \to \quad \mathbf{RP}^e \cup \{\mathbf{o}\}$$

since elements of Grassmannians are projective subspaces of the corresponding projective space.

Similarly, $A$ defines a map $\pi_p(A)$ between the corresponding projective spaces

$$\begin{array}{rrcl} \pi_p(A) : & \mathbf{RP}^d \setminus \pi_p\big(\ker(A)\backslash\{0\}\big) & \longrightarrow & \mathbf{RP}^e \\ & \pi_p(x) & \longmapsto & \pi_p\big(Ax\big) \end{array} \tag{2.7}$$

by passing on to the quotient spaces. The map $\pi_p(A)$ is well-defined since $A(\lambda x) = \lambda Ax$. Then, $\alpha$ or $\pi(A)$ is an isomorphism if $A$ is an isomorphism, i.e., if $A \in \mathbf{GL}(d+1)$, in which case $\pi_p(A)$ is defined on the whole of $\mathbf{RP}^d$. Such an isomorphism is called *projective transformation* or *homography*.

The automorphisms of a $d$-dimensional geometry form a group under the usual composition of maps which is again the projective linear group

$$\mathbf{PGL}(d) = \mathbf{GL}(d+1) \,/\, \mathbf{R}^*\mathbf{I}_{d+1}. \tag{2.8}$$

Note that projective transformations are homeomorphisms on $\mathbf{RP}^d$.

Projective geometry is in some sense a generalization of Euclidean and affine geometry. In particular, Euclidean and affine space are subspaces of projective space, and the corresponding automorphism groups are subgroups of the projective linear group.

Regarding affine geometry, let $H \subset \mathbf{RP}^n$ be a hyperplane in projective space. Then, $\mathbf{A}^d = \mathbf{RP}^d \backslash H$ is a $d$-dimensional affine space with $H$ being called the *hyperplane at infinity*. Parallelism, which is the property separating affine from projective geometry, is defined as follows: let $A, B \subseteq \mathbf{RP}^d$ be projective subspaces not lying in $H$, and $A' = A \cap \mathbf{A}^d$ resp. $B' = B \cap \mathbf{A}^d$ its affine counterparts. The affine subspaces $A'$ and $B'$ are said to be parallel if they only meet at infinity, i.e.,

$$A \wedge H \subseteq B \wedge H \quad \text{or} \quad B \wedge H \subseteq A \wedge H.$$

Affine transformations are thus projective transformations which preserve parallelism, i.e., the affine group $\mathbf{Aff}(d)$ is given by those elements of $\mathbf{PGL}(d)$ which map $H$ bijectively to itself:

$$\mathbf{Aff}(d) = \big\{ f \in \mathbf{PGL}(d) : f(H) = H \big\}.$$

When speaking of the hyperplane at infinity, most geometers think of the hyperplane $H_{d+1} = \big\{ \pi_p\big((p^1, \ldots, p^{d+1})^t\big) : p^{d+1} = 0 \big\}$ in homogeneous coordinates to the standard basis of $\mathbf{R}^{d+1}$. In these homogeneous coordinates, affine space $\mathbf{A}^d = \mathbf{RP}^d \backslash H$ is homeomorphic to $\mathbf{R}^d$ by inhomogeneous coordinates

$$\begin{array}{cccc} \varphi_{d+1}^{-1} : & \mathbf{A}^d \cong \mathbf{R}^d & \longrightarrow & \mathbf{RP}^d \backslash H \\ & (p^1, \ldots, p^d)^t & \longmapsto & \pi_p\big((p^1, \ldots, p^d, 1)^t\big), \end{array} \tag{2.9}$$

and affine transformations are given by matrices

$$\begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix}$$

with $A \in \mathbf{GL}(d)$ and $c \in \mathbf{R}^d$ acting from the left on homogeneous coordinate vectors. In particular, $\mathbf{Aff}(d) = \mathbf{R}^d \rtimes \mathbf{GL}(d)$.

Similarly, the similarity transformation group

$$\mathbf{Sim}(d) = \mathbf{R}^d \rtimes \big(\mathbf{R}^+ \times \mathbf{O}(d)\big)$$

generated by translations, rescaling and rotations/reflections forms a subgroup of $\mathbf{Aff}(d)$ and $\mathbf{PGL}(d)$. With respect to the embedding $\varphi_{d+1}^{-1}$, similarity transformations on $\mathbf{R}^d$ are given by matrices

$$\begin{pmatrix} sR & c \\ 0 & 1 \end{pmatrix}$$

with $s \in \mathbf{R}^+ = \{x \in \mathbf{R} : x > 0\}$, $R \in \mathbf{O}(d)$, and $t \in \mathbf{R}^d$. Of course, Euclidean transformations are similarity transformations with $s = 1$.

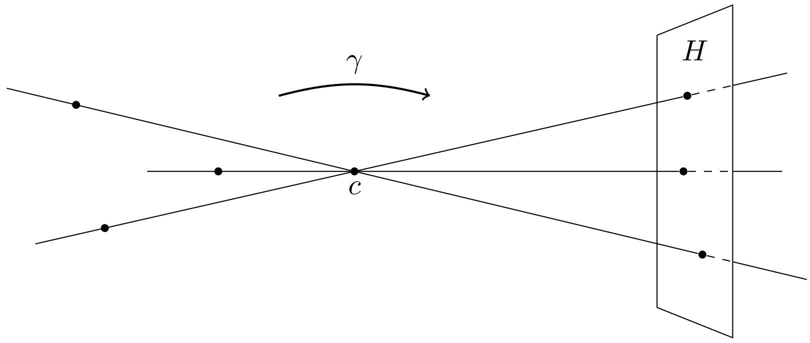



**Figure 2.1:** A central projection $\gamma$ mapping points in $\mathbf{RP}^3\backslash\{c\}$ to the hyperplane $H$ by using the unique projective lines through the projection center $c$.

## 2.2 Projective shapes in computer vision

Central projections can be easily described in the framework of projective geometry with the notion of join and section: let $H$ be a projective hyperplane in $\mathbf{RP}^{d+1}$, and let $c \in \mathbf{RP}^{d+1}$ be a point not incident with $H$, i.e., $c \notin H$.

For any $p \in \mathbf{RP}^{d+1}\backslash\{c\}$, there is a unique line connecting $p$ and $c$, namely $p \vee c$. This line intersects $H$ in the unique point $(p \vee c) \wedge H \in \mathbf{RP}^{d+1}$, defining a map

$$\begin{array}{rccc} \gamma : & \mathbf{RP}^{d+1}\backslash\{c\} & \longrightarrow & H, \\ & p & \longmapsto & (p \vee c) \wedge H \end{array} \tag{2.10}$$

as depicted in Figure 2.1 for $d = 2$. This so-called *central projection* $\gamma$ from $\mathbf{RP}^{d+1}\backslash\{c\}$ to $H$ with *projection center* (or *optical center*) $c$ is a linear map in homogeneous coordinates.

**Lemma 2.2.** *There is a linear map $C : \mathbf{R}^{d+2} \to \pi_p^{-1}(H) \cong \mathbf{R}^{d+1}$ such that $\gamma = \pi_p(C)$.*

Proof. Let $U$ be the hyperplane of $\mathbf{R}^{d+2}$ such that $\pi_p(U) = H$ and $V = \pi_p^{-1}(c)$. Then, $\mathbf{R}^{d+2} = U \oplus V$, and $\gamma$ is induced by the linear projection onto $U$. □

The $(d+1) \times (d+2)$-dimensional matrix $C$ corresponding to $\gamma$ is called *perspective projection matrix.* Of course, it is only unique up to rescaling and depends on the coordinate systems given on $\mathbf{RP}^{d+1}$ respectively $H \cong \mathbf{RP}^d$.

For $d + 1 = 3$ this map $\gamma$ describes the working mechanism of a pinhole camera or camera obscura. The projection center $c$ corresponds to the pinhole, while $H \cong \mathbf{RP}^2$ is the *image plane* of the camera. The matrix $C$ encodes the internal camera parameters and the camera's position and orientation in the surrounding space $\mathbf{RP}^3$. For a reasonable camera, $C$ should be of rank 3. The projection center is then the unique point $c \in \mathbf{RP}^3$ which satisfies $Cc = 0$ in homogeneous coordinates. Even though modern cameras have a focus and a lens to increase illuminance, which leads to distortion, the pinhole camera model is a good approximation for photography. Of course, one can define "pinhole cameras" for general dimensions $d \geqslant 1$.

In this thesis objects in space are modeled as finite configurations $p$ of landmarks, i.e. as elements of

$$\mathcal{A}_d^k = \left(\mathbf{RP}^d\right)^k, \tag{2.11}$$

resp. in homogeneous coordinates as $k \times (d+1)$-dimensional matrices $P \in \mathbf{R}^{k\times(d+1)}$ with the non-trivial rows of $P$ giving the homogeneous coordinates of the landmarks in $\mathbf{RP}^d$. In abuse of notation, we will write both $p \in \mathcal{A}_d^k$ and $P \in \mathcal{A}_d^k$ with the lower case letter $p$ always denoting a configuration in $\mathbf{RP}^d$, the corresponding upper case letter $P$ always denoting a configuration matrix.

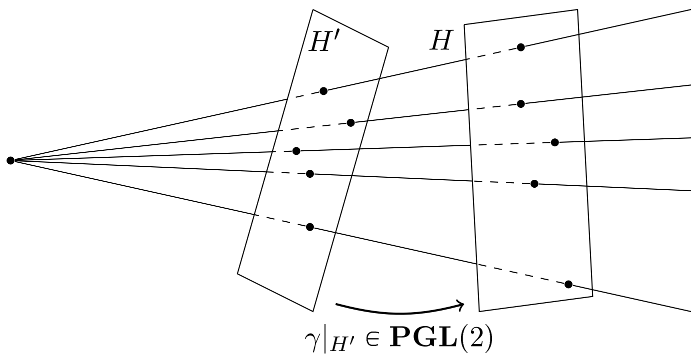



**Figure 2.2:** The image of a configuration in a hyperplane is a projective transformation of the configuration.

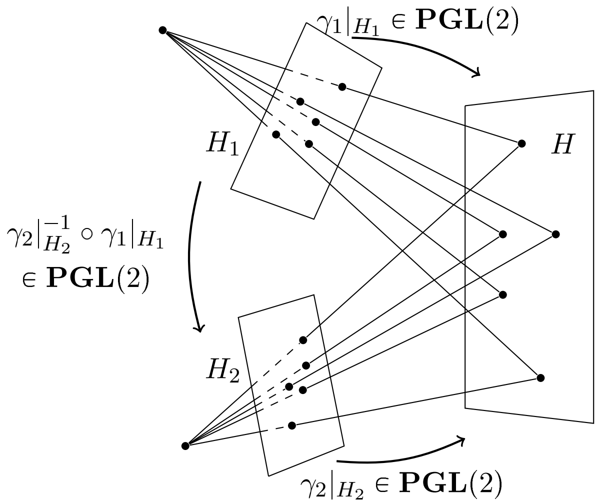



**Figure 2.3:** Two images of a configuration in a hyperplane are equivalent under $\mathbf{PGL}(d)$.

The image of an object $p = (p_1, \dots, p_k) \in \mathcal{A}_{d+1}^k$ under a central projection $\gamma$ is then the component-wise image $\gamma(p) = \big(\gamma(p_1), \dots, \gamma(p_k)\big) \in \mathcal{A}_d^k$. This is only well-defined if no point of $p$ coincides with the projection center $c$. In homogeneous coordinates, the image configuration is given by $PC^t$ with $P$ being a configuration matrix and $C$ the perspective projection matrix.

If the object $p$ itself lies in a hyperplane $H' \subset \mathbf{RP}^{d+1}$ disjoint with the projection center $c$, then the restriction of $\gamma$ to $H' \cong \mathbf{RP}^d$ is a projective transformation, i.e., $\gamma|_{H'} \in \mathbf{PGL}(d)$. In particular, the original configuration $p \in \mathcal{A}_d^k$ differs from the image $\gamma(p) \in \mathcal{A}_d^k$ only by a projective transformation. Therefore, an object $p$ cannot be completely reconstructed from an image if the camera's calibration is unknown. It can only be reconstructed up to a projective transformation and the information one retrieves is called the *projective shape* of $p$. By the same line of thought, two images of the same hyperplanar object are related by a projective transformation, see Figure 2.2 and Figure 2.3. This ambiguity decreases to affine or similarity transformations if more information about the camera parameters, i.e. about the matrix $C$, is given.

Of course, the cases $d = 1$ and $d = 2$ are the critical ones in reality.

Another topic in computer vision is the reconstruction of an object in real world from several images of it.

Two images $X = PC_1^t \in \mathcal{A}_2^k$ and $Y = PC_2^t \in \mathcal{A}_2^k$ of an unknown object $P \in \mathcal{A}_3^k$ taken with two cameras $C_1, C_2$ are considered. The objective is to reconstruct the object $P$ from the landmark correspondences $X_{i\cdot} \longleftrightarrow Y_{i\cdot}$. If the cameras are uncalibrated, i.e., if the camera matrices $C_1, C_2$ are unknown, then $P$ can be at best recovered up to a projective transformation

$\alpha \in \mathbf{PGL}(3)$ since the application of $A \in \mathbf{GL}(4)$ with $\alpha = \pi_p(A)$ to the scene and the cameras does not change the image:

$$X = PC_i^t = PA(A^{-1}C_i^t).$$

It has been shown that this projective ambiguity is also the worst case if there are sufficiently many well-distributed landmark correspondences, cf. (Hartley and Zisserman; 2003, Ch. 10 et seq.). Hence, the projective shape of the object $P$ is all the information that can be retrieved from multiple images by uncalibrated cameras.

All results from this section can be found in (Faugeras and Luong; 2001), (Hartley and Zisserman; 2003), and (Ma et al.; 2004).

## 2.3 Projective shape space

Geometrically, objects are described as configurations of landmarks, i.e. as a finite, ordered set of landmarks in space, while the shape of a configuration is the information that remains when removing the coordinate system the configuration is described in. As we have seen before, the set of coordinate systems may be described as the transformation group corresponding to the geometry. Then, the shape of an object is the orbit of the corresponding configuration under the component-wise group action, while the shape space is the topological quotient of the configuration space modulo the group action.

In the setup of projective geometry, an object is then of course described as a configuration $p = (p_1, \ldots, p_k)$ of $k \geqslant 1$ landmarks in $d$-dimensional[1] real projective space $\mathbf{RP}^d$, $d \geqslant 0$, i.e. as an element of

$$\mathcal{A}_d^k = \left(\mathbf{RP}^d\right)^k, \tag{2.12}$$

and the *projective shape* $[p]$ of such a configuration is the information about the object that is invariant under the component-wise action of $\mathbf{PGL}(d)$, i.e. the equivalence class or orbit

$$[p] = \left\{\alpha p = \big(\alpha(p_1), \ldots, \alpha(p_k)\big) : \alpha \in \mathbf{PGL}(d)\right\}. \tag{2.13}$$

The set[2]

$$a_d^k = \mathcal{A}_d^k \,/\, \mathbf{PGL}(d) = \left\{[p] : p \in \mathcal{A}_d^k\right\} \tag{2.14}$$

of projective shapes, i.e. the set of orbits in $\mathcal{A}_d^k$, is naturally equipped with the quotient topology. This topological space is thus called *projective shape space*. Note that the critical dimensions in real world are $d = 1, 2, 3$, while the number of landmarks is often quite large, in particular $k \geqslant d + 2$.

Recall that the quotient topology is the finest topology on $a_d^k$ making the projection

$$\pi : \quad \mathcal{A}_d^k \quad \longrightarrow \quad a_d^k \tag{2.15}$$

a continuous map. Here, the projection map $\pi$ is also an open map: since projective transformations are homeomorphisms on $\mathbf{RP}^d$, the preimage of the image of an open set $U \in \mathcal{A}_d^k$ is

$$\pi^{-1}\big(\pi(U)\big) = \bigcup_{\alpha \in \mathbf{PGL}(d)} \alpha U,$$

which is—as a union of open sets—open in $\mathcal{A}_d^k$. Thus, $\pi(U)$ is open.

It is common to describe configurations in homogeneous coordinates as $k \times (d+1)$-dimensional matrices $P$ with the non-trivial rows of $P$ giving the homogeneous coordinates of the landmarks

[1]The case $d = 0$ is rather boring: $\mathbf{RP}^0$ is a singleton and $\mathbf{PGL}(0)$ the trivial group, whence $\mathcal{A}_0^k$ and $a_0^k$ are singletons for any $k$, too. However, this case is added to the discussion to describe some results more elegantly.

[2]Note that configuration spaces are always denoted by upper case letters, the corresponding projective shape spaces by lower case letters.

in $\mathbf{RP}^d$. Since homogeneous coordinates are only unique up to rescaling, left-multiplication with non-singular, diagonal $k \times k$-dimensional matrices $D \in \mathbf{Diag}^*(k) \cong (\mathbf{R}^*)^k$ does not change the configuration in $\mathbf{RP}^d$. The group $\mathbf{PGL}(d)$ acts on $P$, contrary to Section 2.1, now as right-multiplication of non-singular matrices $B \in \mathbf{GL}(d+1)$ since landmarks are represented in homogeneous coordinates as row vectors in this matrix notation. Then, the (projective) shape of a matrix configuration $P$ under $\mathbf{PGL}(d)$ is the orbit

$$[P] = \big\{DPB : D \in \mathbf{Diag}^*(k),\, B \in \mathbf{GL}(d+1)\big\}. \tag{2.16}$$

Note that the joint action of $\mathbf{Diag}^*(k)$ and $\mathbf{GL}(d+1)$ is not effective on matrix configurations since

$$DPB = \big(\lambda D\big)\, P\, \big(\lambda^{-1} B\big)$$

for all $\lambda \in \mathbf{R}^*$, $D \in \mathbf{Diag}^*(k)$, $B \in \mathbf{GL}(d+1)$ and any $P \in \mathcal{A}_d^k$. In particular, any matrix configuration is preserved by the simultaneous left-action of $\lambda \mathbf{I}_k \in \mathbf{Diag}^*(k)$ and right-action of $\lambda^{-1}\mathbf{I}_{d+1} \in \mathbf{GL}(d+1)$. This ineffectiveness can be removed by fixing the scaling of one of the matrices, e.g. by requiring[3] $\det(B) = 1$.

The rank of a configuration matrix $P$ is obviously invariant under both the left-action of $\mathbf{Diag}^*(k)$ and the right-action of $\mathbf{GL}(d+1)$, i.e., the rank $\mathbf{rk}\, p$ of a configuration $p \in \mathcal{A}_d^k$ is well-defined as the rank of one representing matrix configuration $P$, as is the rank $\mathbf{rk}[p]$ of a projective shape $[p] \in \mathfrak{a}_d^k$. Similarly, the group actions preserve the linear dependencies of the rows of $P$, or projectively speaking, projective transformations $\alpha \in \mathbf{PGL}(d)$ map projective subspaces of $\mathbf{RP}^d$ to projective subspaces of the same dimension, i.e. points to points, lines to lines, etc., as we have already seen in Section 2.1. Hence, if $j$ landmarks of a configuration $p$ lie in an $i$-dimensional projective subspace, then the same is true for any equivalent configuration $\alpha p$. So, this information is invariant under $\mathbf{PGL}(d)$, too, and a property of its projective shape $[p]$.

**Definition 2.3.** Let $j \in \{1,\dots,d\}$ and $I \subseteq \{1,\dots,k\}$ be a subset of size $|I| \geqslant j$. A configuration $p \in \mathcal{A}_d^k$ *fulfills the projective subspace constraint* $(I,j)$ if and only if the projective dimension of $\bigvee_{i\in I} p_i$ is at most $j-1$, or equivalently if and only if the landmarks $p_i$, $i \in I$, lie in a projective subspace of projective dimension $j-1$. In other words, $\mathbf{rk}\, p_I \leqslant j$ with $p_I$ denoting the restriction of $p$ to landmarks with index $i \in I$.

We denote the *collection of projective subspace constraints* fulfilled by a configuration $p \in \mathcal{A}_d^k$ by

$$C(p) = \big\{(I,j) : p \text{ fulfills } (I,j)\big\}. \tag{2.17}$$

A projective subspace constraint $(I,j) \in C(p)$ is said to be *non-trivial* if $I \subseteq \{1,\dots,k\}$ is of cardinality strictly larger than $j$. The *collection of non-trivial projective subspace constraints* fulfilled by a configuration $p \in \mathcal{A}_d^k$ is denoted by

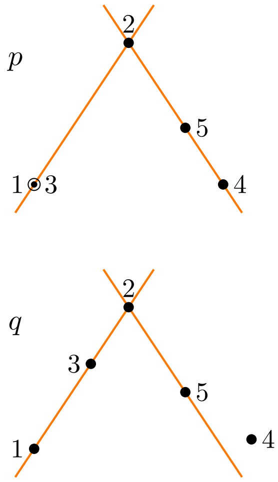
$$C^*(p) = \big\{(I,j) \in C(p) : (I,j) \text{ is non-trivial}\big\}. \tag{2.18}$$

Further, $(I,j) \in C(p)$ is called *decomposable* in $C(p)$ if there are projective subspace constraints $(I_1,j_1),(I_2,j_2) \in C(p)$ with disjoint, non-empty sets $I_1, I_2 \subset I$ and integers $j_1, j_2 \in \{1,\dots,d\}$ such that $I_1 \dot\cup I_2 = I$ and $j_1 + j_2 = j$. Else, $(I,j) \in C(p)$ is called *non-decomposable.* A configuration $p$ is said to be decomposable resp. non-decomposable if $\big(\{1,\dots,k\}, d+1\big)$ is decomposable resp. non-decomposable, slightly generalizing our notation. ◊

Note that any configuration $p \in \mathcal{A}_d^k$ fulfills the subspace constraints $\big(\{i\},1\big)$, $1 \leqslant i \leqslant k$. Further, $\mathbf{rk}\, p_I = j$ for non-decomposable projective subspace constraints $(I,j) \in C(p)$; otherwise, $(I,j)$ decomposes into $(I\backslash\{i\}, j-1), (\{i\},1) \in C(p)$ for any $i \in I$.

**Corollary 2.4.** *The collection of projective subspace constraints $C(p)$ from Definition 2.3 is well-defined for any projective shape $[p] \in \mathfrak{a}_d^k$.*

[3] For odd dimensions $d$ (even $d+1$), fixing the determinant of $B \in \mathbf{GL}(d+1)$ only lessens the ineffectiveness since multiplication of $B$ and $D$ by $-1$ has still no effect on $DPB$.




$$C^*(p) = \Big\{ \big(\{1,3\},1\big), \big(\{1,2,3\},2\big), \big(\{2,4,5\},2\big) \Big\}$$

$$C^*(q) = \Big\{ \big(\{1,2,3\},2\big) \Big\}$$

**Figure 2.4:** Two configurations $p, q \in \mathscr{A}_2^5$ and their respective collections of non-trivial projective subspace constraints. The configuration $p$ is decomposable since $\big(\{1,\dots,5\},3\big)$ decomposes into $\big(\{1,3\},1\big), \big(\{2,4,5\},2\big)$. Obviously, $C^*(p) \supset C^*(q)$, whence $C(p) \supset C(q)$, i.e., $p$ is less constrained than $q$, while $q$ is less regular than $p$. The lines are only added to visualize projective subspace constraints.

**Lemma 2.5.** *Let $p \in \mathscr{A}_d^k$ fulfill the projective subspace constraints $(I_1, j_1), (I_2, j_2) \in C(p)$. Then:*

*(i) $p$ fulfills the projective subspace constraint $(I, j_1)$ for all subsets $I \subseteq I_1$ with $|I| \geqslant j_1$;*

*(ii) $p$ fulfills the projective subspace constraint $(I_1 \dot\cup \{i\}, j_1 + 1)$ for all $i \in I_1^c = \{1, \dots, k\} \backslash I_1$;*

*(iii) $p$ fulfills the projective subspace constraint $(I_1 \cup I_2, j_1 + j_2)$ (if $|I_1 \cup I_2| \geqslant j_1 + j_2$).*

Note that (ii) is a special case of (iii) for $I_2 = \{i\} \nsubseteq I_1$ and $j_2 = 1$.

Proof. (i) Note that $\mathbf{rk}\, p_I \leqslant \mathbf{rk}\, p_{I_1} \leqslant j_1$ for all $I \subseteq I_1$.

(ii) The rank of $p_{I_1}$ increases at most by 1 by adding another landmark to the subconfiguration.

(iii) $\mathbf{rk}\, p_{I_1 \cup I_2} \leqslant \mathbf{rk}\, p_{I_1} + \mathbf{rk}\, p_{I_2} \leqslant j_1 + j_2$. □

On the set of collections of projective subspace constraints, we can naturally define the partial order of inclusion. A configuration $p$ resp. shape $[p]$ is said to be *less constrained* than $q$ resp. $[q]$ if $C(p) \subset C(q)$, and *less regular* if $C(p) \supset C(q)$, see Figure 2.4.

This partial order is apparent in the topology of $\boldsymbol{a}_d^k$.

**Lemma 2.6.** *Let $p \in \mathscr{A}_d^k$ and $[p] \in \boldsymbol{a}_d^k$. Then, the following holds:*

*(i) There is an open neighborhood $U \subseteq \mathscr{A}_d^k$ of $p$ such that all elements of $U$ are less or equally constrained than $p$, i.e., $C(q) \subseteq C(p)$ for all $q \in U$.*

*(ii) There is an open neighborhood $V \subseteq \boldsymbol{a}_d^k$ of $[p]$ such that all elements of $V$ are less or equally constrained than $[p]$, i.e., $C(q) \subseteq C(p)$ for all $[q] \in V$.*

*(iii) The subsets $\big\{q \in \mathscr{A}_d^k : C(q) \subseteq C(p)\big\}$ and $\big\{[q] \in \boldsymbol{a}_d^k : C(q) \subseteq C(p)\big\}$ are open in $\mathscr{A}_d^k$ resp. $\boldsymbol{a}_d^k$.*

Proof. The statements (i) and (ii) are special cases of statement (iii).

The set $\big\{Q \in \mathscr{A}_d^k : C(Q) \subseteq C(P)\big\}$ of less constrained matrix configurations is an open subset of $\mathbf{R}^{k \times (d+1)}$ since small distortions of the entries of a matrix $P$ do not produce "more" linear dependencies in the rows. Since the projection map $\pi_p$ is open, the set $\big\{q \in \mathscr{A}_d^k : C(q) \subseteq C(p)\big\}$ is open. Further, the set $\big\{[q] \in \boldsymbol{a}_d^k : C(q) \subseteq C(p)\big\}$ is open since the projection map $\pi : \mathscr{A}_d^k \to \boldsymbol{a}_d^k$ is open. □

We can now give a canonical matrix representation for a projective shape $[p] \in \mathfrak{a}_d^k$ which illustrates the decomposability of $[p]$. Please note that this representation is tremendously important for the remainder of this thesis.

**Proposition 2.7.** *Let $p \in \mathcal{A}_d^k$ be a configuration and $[p] \in \mathfrak{a}_d^k$ its projective shape. Then, the following holds:*

*(i) There is a unique subset*

$$\big\{(I_1, j_1), \ldots, (I_s, j_s)\big\} \subseteq C(p) \tag{2.19}$$

*of non-decomposable projective subspace constraints fulfilled by $p$ resp. $[p]$ with $\bigcup_{r=1}^s I_r = \{1, \ldots, k\}$ and $\sum_{r=1}^s j_r = \mathbf{rk}\, p$. This subset is called* total decomposition *of $p$ resp. $[p]$. The projective subspaces spanned by the landmarks $p_{I_r}$, $1 \leqslant r \leqslant s$, are projectively independent.*

*(ii) There is a permutation $\sigma \in S_k$ of the landmarks such that $[\sigma p]$ has a block matrix representation of the partitioned form*

$$\begin{pmatrix} P_1 & 0 & \ldots & 0 & 0 \\ 0 & P_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \ldots & 0 & P_s & 0 \end{pmatrix} \tag{2.20}$$

*with $P_r \in \mathcal{A}_{j_r-1}^{|I_r|} \subset \mathbf{R}^{|I_r| \times j_r}$, $1 \leqslant r \leqslant s$.*

Proof. (i) Note that a projective subspace constraint $(I, j)$ with $j = 1$ is necessarily non-decomposable. To obtain a total decomposition, start with the projective subspace constraint $\big(\{1, \ldots, k\}, \mathbf{rk}\, p\big) \in C(p)$. If it is non-decomposable, then there is nothing to prove. If it is decomposable, then it decomposes into two projective subspace constraints $(I_1, j_1), (I_2, j_2) \in C(p)$ with $j_1 + j_2 = j$, $I_1 \cup I_2 = I$, $I_1 \cap I_2 = \varnothing$. Check these projective subspace constraints for decomposability and iterate this procedure until all projective subspace constraints are non-decomposable. Since $j_1, j_2 < j$, this algorithm will terminate after finite iterations.

For a total decomposition $\big\{(I_1, j_1), \ldots, (I_s, j_s)\big\}$ of $p \in \mathcal{A}_d^k$, the projective subspaces spanned by the landmarks $p_{I_r}$, $1 \leqslant r \leqslant s$, are projectively independent since $\sum \mathbf{rk}\, p_{I_r} = \sum \mathbf{rk}\, p_{j_r} = \mathbf{rk}\, p$, cf. page 10.

To prove the uniqueness of the total decomposition, assume that there are two distinct total decompositions $\big\{(I_1, j_1), \ldots, (I_s, j_s)\big\} \neq \big\{(I'_1, j'_1), \ldots, (I'_t, j'_t)\big\} \subseteq C(p)$ of $\big(\{1, \ldots, k\}, \mathbf{rk}\, p\big)$ into non-decomposable projective subspace constraints. Let $p \in \mathcal{A}_d^k$ be a representative of $[p]$, and let $(I_r, j_r) \neq (I'_u, j'_u)$ be distinct projective subspace constraints of $p$ with $I_r \cap I'_u \neq \varnothing$. If $I_r = I'_u$, then $j_r \neq j'_u$, and consequently $\mathbf{rk}\, p_{I_r} = j_r \neq j'_u = \mathbf{rk}\, p_{I'_u}$, contradicting $I_r = I'_u$. Therefore, let $I_r \backslash I'_u \neq \varnothing$ (w.l.o.g.). Then, $(I_r, j_r)$ decomposes into $\big(I_r \backslash I'_u, \mathbf{rk}\, p_{I_r \backslash I'_u}\big)$ and $\big(I_r \cap I'_u, \mathbf{rk}\, p_{I_r \cap I'_u}\big)$ since the projective subspace spanned by landmarks of $p_{I_r \backslash I'_u}$ is projectively independent from the projective subspace spanned by the landmarks of $p_{I'_u}$ and $p_{I_r \cap I'_u}$. Hence, $(I_r, j_r)$ is decomposable in contradiction to the assumption, whence the total decompositions $\big\{(I_1, j_1), \ldots, (I_s, j_s)\big\}$ and $\big\{(I'_1, j'_1), \ldots, (I'_t, j'_t)\big\}$ are identical.

(ii) Let $\big\{(I_1, j_1), \ldots, (I_s, j_s)\big\} \subseteq C(p)$ be the unique subset of non-decomposable projective subspace constraint from (i). Further, let $\sigma \in S_k$ be a permutation such that

$$\begin{aligned} \sigma(I_1) &= \big\{1, \ldots, |I_1|\big\}, \\ \sigma(I_2) &= \big\{|I_1| + 1, \ldots, |I_1| + |I_2|\big\}, \\ &\vdots \\ \sigma(I_s) &= \left\{\sum_{r=1}^{s-1} |I_r| + 1, \ldots, k\right\}, \end{aligned}$$

and let $\Sigma$ be the permutation matrix permuting the standard basis vectors $e_1, \dots, e_k$ of $\mathbf{R}^k$ such that $\Sigma e_i = e_{\sigma(i)}$. Let $P$ be a matrix representing $[p]$. When permuting the rows of $P$ to $\Sigma P$, successive blocks of $|I_r|$ rows span the $j_r$-dimensional vector subspace $S_r$ to the corresponding projective subspace constraint $(I_r, j_r)$. Note that these vector subspaces $S_r$ are projectively independent since the projective subspace constraints are non-decomposable, and thus they only intersect mutually in the origin, i.e., $S_r \cap S_t = \{0\}$. Let $B \in \mathbf{GL}(d+1)$ be a non-singular matrix such that $S_1$ is mapped to the vector subspace spanned by the first $j_1$ standard basis vectors of $\mathbf{R}^{d+1}$, $S_2$ is mapped to the vector subspace spanned by the next $j_2$ standard basis vectors, and so forth. Then, the matrix $\hat{P} = \Sigma P B$ has the form described in Equation (2.20).□

**Remarks 2.8.** (i) The canonical matrix representation is not unique since neither the permutation $\sigma$ in Proposition 2.7(ii) nor the blocks $P_r$ are unique. The composition of $\sigma$ with any permutation of the blocks or within the blocks gives another canonical matrix representation. Meanwhile, the blocks $P_r$ are only unique up to left-multiplication with non-singular diagonal matrices $D \in \mathbf{Diag}^*(|I_r|)$ and right-multiplication with non-singular matrices $B \in \mathbf{GL}(j_r)$, so a decomposable shape "decomposes into non-decomposable shapes of lower dimension", cf. Proposition 3.2.

(ii) For a non-decomposable shape $[p]$, the total decomposition is $\{(\{1, \dots, k\}, d+1)\}$, and any representing matrix configuration is canonical. Note that $(\{1, \dots, k\}, d+1)$ is not an element of $C(p)$, so the total decomposition is, technically speaking, not a subset of $C(p)$.

(iii) Similar to Proposition 2.7, one can show that a shape $[p] \in \mathfrak{a}_d^k$ with $(I, j) \in C(p)$ (after permuting the rows) has a block matrix representation

$$\begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix} \tag{2.21}$$

for some matrices $P_{11} \in \mathbf{R}^{|I| \times j}$, $P_{21} \in \mathbf{R}^{|I^c| \times j}$ and $P_{22} \in \mathbf{R}^{|I^c| \times (d+1-j)}$. ◊

The projective subspace constraints of a configuration resp. shape can also be reconstructed from the so-called *volume cross ratios* which are invariants in the algebraic sense: let $p \in \mathcal{A}_d^{d+3}$ such that the projective subspaces $p_1 \vee \cdots \vee p_{d-1} \vee p_i$ are $(d-1)$-dimensional for all $i \in \{d, \dots, d+3\}$, and pairwise different for at least all but one pair of indexes. Let $P$ be a matrix representation of $p$, and denote the submatrix of $P$ comprising of the rows of $P$ with index $I \subset \{1, \dots, k\}$ by $P_I$. The value

$$\mathbf{cr}(p_1, \dots, p_{d+3}) = \frac{|P_{\{1,\dots,d-1,d,d+1\}}|}{|P_{\{1,\dots,d-1,d,d+2\}}|} \frac{|P_{\{1,\dots,d-1,d+2,d+3\}}|}{|P_{\{1,\dots,d-1,d+1,d+3\}}|} \in \mathbf{R} \cup \{\infty\} \tag{2.22}$$

with $|\cdot|$ denoting the determinant of the configuration in homogeneous coordinates is then invariant under the action of $\mathbf{PGL}(d)$ and is called *volume cross ratio*, cf. (Olver; 1999) or (Boutin and Kemper; 2005) for a discussion of the case $d = 2$. Note that we allow at most one of the determinant to take the value 0 for $p_1, \dots, p_{d+3}$, else the cross ratio is not defined for $p_1, \dots, p_{d+3}$.

This definition generalizes the usual notion of a cross ratio which is defined on the real projective line $\mathbf{RP}^1$ via homogeneous coordinates: let $\pi_p((p^1, 1)^t), \dots, \pi_p((p^4, 1)^t) \in \mathbf{RP}^1$ be four landmarks on the line with at most one pair coincidence $p^r = p^s$ for $r \neq s$. Then, the cross ratio is defined as the quotient

$$\mathbf{cr}\big(\pi_p((p^1, 1)^t), \dots, \pi_p((p^1, 1)^t)\big) = \frac{(p^1 - p^2)(p^3 - p^4)}{(p^1 - p^3)(p^2 - p^4)} \in \mathbf{R} \cup \{\infty\}.$$

If and only if a configuration resp. shape fulfills some non-trivial subspace constraints, then some volume cross ratios take values 0, 1, $\infty$ or are not defined since some determinants vanish in this case.

**Remark 2.9.** While the focus of this manuscript is on projective shapes, similarity and affine shapes can be described in the same manner: configurations in $\mathbf{R}^d$ can be described in homogeneous coordinates as matrix configurations $P$ with $P_{i,d+1} \neq 0$ for all $i \in \{1, \ldots, k\}$. The similarity and affine groups act by right-multiplication by matrices

$$B = \begin{pmatrix} A & 0 \\ c^t & 1 \end{pmatrix},$$

such that the similarity shape of $P$ is the orbit

$$\left\{ DP \left( \begin{smallmatrix} sR & 0 \\ c^t & 1 \end{smallmatrix} \right) : D \in \mathbf{Diag}^*(k), s \in \mathbf{R}^*, R \in \mathbf{O}(d), c \in \mathbf{R}^d \right\}$$

while the affine shape of $P$ is the orbit

$$\left\{ DP \left( \begin{smallmatrix} A & 0 \\ c^t & 1 \end{smallmatrix} \right) : D \in \mathbf{Diag}^*(k), A \in \mathbf{GL}(d), c \in \mathbf{R}^d \right\}. \qquad \diamond$$

# Chapter 3

# The topology of projective shape space

The main objective of this thesis is to find "good" topological subspaces $\mathscr{y}$ of projective shape space $\mathscr{a}_d^k$ on which well-known mathematical tools (e.g. statistics, optimization, etc.) can be applied. Here of course, the question arises what the meaning of "good topological subspace" shall be.

The topological subspaces we are looking for shall fulfill the following properties:

**1) differentiable Hausdorff manifold**
Many mathematical tools require the topological subspace to be a metric space such that projective shapes can be distinguished by a distance function. There are two main concepts to metricise a topological space: first, one might require the topological space to be a Riemannian manifold. In statistics, this would lead to intrinsic statistics which uses the metric of the Riemannian manifold. For mathematical convenience, the completeness of the Riemannian structure should be a property one would like to add. Alternatively, an embedding of the topological space into some metric space would also equip the topological space with a metric, namely the subspace metric. This would lead to extrinsic statistics. Either way, the topological space itself has to be Hausdorff for the respective structure to exist. If a topological subspace of $\mathscr{a}_d^k$ is a *differentiable Hausdorff manifold*, then the existence of both a Riemannian metric (Lee; 2013, Prop. 13.3) and of an embedding into Euclidean space (Lee; 2013, Thm. 6.15) is guaranteed, whence we will look for this structure. Additionally, manifolds allow the application of local formalisms, e.g. optimization, statistics, etc., using the local homeomorphy to Euclidean space.

**2) closure under permutations**
The first statistical approach to projective shape space via projective frames by Mardia and Patrangenaru (2005) is in some way analogous to Bookstein's approach for similarity shapes. Let $\mathscr{B}_d^k \subseteq \mathscr{A}_d^k$ be the set of configurations whose first $d+2$ landmarks form a projective frame. The corresponding shape space $\mathscr{b}_d^k$ is then homeomorphic to $k-d-2$ copies of $\mathbf{RP}^d$ by standardizing a shape's first $d+2$ landmarks to a fixed projective frame, see Lemma 3.15. This approach has the drawback that it was chosen to have the first $d+2$ landmarks form a frame, i.e., these landmarks play a special role in this approach without cause. A reasonable topological subspace of projective shapes should not have such a designation. Mathematically speaking, a reasonable topological subspace of $\mathscr{a}_d^k$ should be *closed under permutation* of the landmarks' order, i.e., the inclusion of the shape of $(p_1, \dots, p_k)$ shall induce the inclusion of $(p_{\sigma(1)}, \dots, p_{\sigma(k)})$ for all permutations $\sigma$ of $\{1, \dots, k\}$, see Figure 3.1 (a).

**3) respecting the hierarchy of projective subspace constraints**
As noted in Section 2.3, the geometry of projective shapes can be described by projective subspace constraints. With inclusion of a shape $[p]$, it is natural to ask for the inclusion of all less constrained shapes, i.e., the inclusion of all shapes $[q]$ with $C(q) \subseteq C(p)$, see Figure 3.1 (b). A

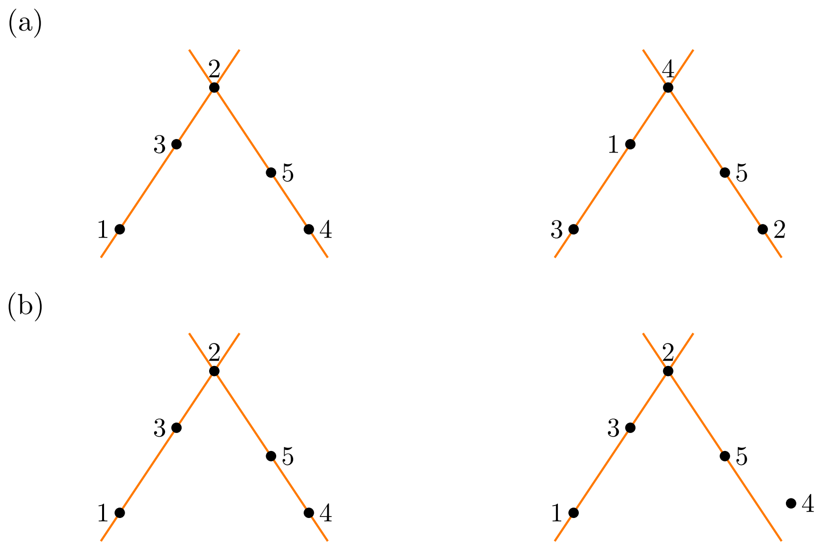



**Figure 3.1:** (a) Two configurations in $\mathcal{A}_2^5$. A topological subspace of $\mathscr{a}_d^k$ which is closed under permutations and includes the shape of one of the configurations includes the other shape, too. (b) Two configurations in $\mathcal{A}_2^5$. Let the $\mathcal{Y} \subseteq \mathcal{A}_2^5$ be a topological subspace including the left configuration. If the corresponding shape space $\mathscr{y}$ respects the hierarchy of projective subspace constraints, then the shape of the right configuration is in $\mathscr{y}$, too, since the right configuration is less constrained.

topological subspace of $\mathscr{a}_d^k$ fulfilling this property is said to "*respect the hierarchy of projective subspace constraints*".

**4) maximality**
Of course, one will want to choose the topological subspace as large as possible while fulfilling the above properties.

Please note that $\mathscr{a}_d^k$ is not Hausdorff for any $k > 1$ and $d \in \mathbf{N}$, i.e., $\mathscr{a}_d^k$ does not fulfill property 1). Indeed it is not even T1 (see Section 3.1), whence we indeed have to look for a true topological subspace of $\mathscr{a}_d^k$.

This chapter is structured as follows: first, a few topological subspaces of $\mathscr{a}_d^k$ will be distinguished which are of special interest for algebraic or geometric reasons. In Section 3.2 it is shown that the quotient topology on $\mathscr{a}_d^k$ and its topological subspaces inherits some properties from the configuration space $\mathcal{A}_d^k$ since the quotient group $\mathbf{PGL}(d)$ consists of homeomorphism. In Section 3.3 we will discuss which projective shapes can be separated from each other by open neighborhoods. It turns out that the largest T1 subset which respects the hierarchy of projective subspace constraints is both given by algebraic and geometric properties. Additionally, we will precisely state the criteria for which a topological subspace is Hausdorff. In Section 3.4 the notion of a projective frame is generalized to obtain charts on the shape space corresponding to the configurations with trivial isotropy group. Finally, the class of topological subspaces bounded by projective subspace numbers is introduced in Section 3.5, and it is shown in which cases these topological subspaces fulfill the properties 1) to 4).

## 3.1 Topological subspaces of special interest

The space $\mathscr{a}_d^k$ of all projective shapes is closed under permutations and respects the hierarchy of projective subspace constraints. It is, however, not a Hausdorff manifold (unless $k = 1$) as we show in Sections 3.3 and 3.4. There is also a quick argument for that: consider the trivial configurations $p \in \mathcal{A}_d^k$ in which all landmarks coincide, i.e., $p_i = p_j$ for all $i, j \in \{1, \ldots, k\}$. These trivial configurations are equivalent under $\mathbf{PGL}(d)$ since $\mathbf{PGL}(d)$ acts transitively on $\mathbf{RP}^d$. Furthermore, $\mathbf{PGL}(d)$ includes the action of rescaling of configurations (in homogeneous coordinates), i.e., any configuration $q \in \mathcal{A}_d^k$ has an equivalent configuration with its landmarks arbitrarily close together. Topologically speaking, any neighborhood of a trivial configuration $p$ contains a configuration which is equivalent to $q$; respectively in shape space, the only neighborhood of the trivial shape $[p]$ is $\mathfrak{a}_d^k$. Consequently, $\mathfrak{a}_d^k$ is not Hausdorff or even T1. This phenomenon also arises in similarity and affine shape space.

Therefore, we have to find a topological subspace of $\mathfrak{a}_d^k$ to fulfill the aforementioned requirements. In this matter, a few topological subspaces of $\mathcal{A}_d^k$ resp. $\mathfrak{a}_d^k$ deserve special attention due to algebraic, geometric or historic reasons:

- $\mathcal{G}_d^k$, which contains a configuration $p = (p_1, \ldots, p_k) \in \mathcal{A}_d^k$ if and only if its landmarks $p_1, \ldots, p_k \in \mathbf{RP}^d$ are in ***g**eneral position*, i.e., if and only if any subconfiguration $p_I$ of size $|I| \leqslant d+1$ is of rank $\mathbf{rk}\, p_I = |I|$. In particular, $p$ fulfills only trivial projective subspace constraints. The elements of $\mathcal{G}_d^{d+2}$ are projective frames.

- $\mathcal{B}_d^k$, which contains a configuration $p = (p_1, \ldots, p_k) \in \mathcal{A}_d^k$ if and only if the first $d+2$ landmarks in $p$ form a projective frame, i.e., if and only if $(p_1, \ldots, p_{d+2}) \in \mathcal{G}_d^{d+2}$. They allow to define the equivalent of ***B**ookstein coordinates* for similarity shapes (Mardia and Patrangenaru; 2005, p. 1672; $\mathcal{B}_d^k$ being called $G(k, d)$ there).

- $\mathcal{P}_d^k$, which contains a configuration $p \in \mathcal{A}_d^k$ if and only if arbitrary $d+2$ landmarks of $p$ form a projective frame, i.e., if and only if there exists a ***p**ermutation* $\sigma \in S_k$ such that $\sigma p \in \mathcal{B}_d^k$ (Mardia and Patrangenaru; 2005, Remark 2.1; $\mathcal{P}_d^k$ being called $\mathcal{FC}_d^k$ there).

- $\mathcal{F}_d^k$, which contains a configuration $p \in \mathcal{A}_d^k$ if and only if the isotropy group of $p$ is trivial, i.e., $\{\alpha \in \mathbf{PGL}(d) : \alpha p = p\} = \{e\}$. Such configurations are said to be ***f**ree* or *regular* under the group action of $\mathbf{PGL}(d)$. In homogeneous coordinates a matrix configuration is free if and only if $P = DPB$ is equivalent to $D = \lambda \mathbf{I}_k$ and $B = \lambda^{-1}\mathbf{I}_{d+1}$ being multiples of identity matrices for some $\lambda \in \mathbf{R}^k$.

- $\mathcal{D}_d^k$, which contains a configuration $p \in \mathcal{A}_d^k$ if and only if it is ***d**ecomposable*, i.e., there is a partition $\{I_1, I_2\}$ of $\{1, \ldots, k\}$ into disjoint, non-empty sets $I_1, I_2$ such that $\mathbf{rk}\, p_{I_1} + \mathbf{rk}\, p_{I_2} \leqslant d+1$, see Definition 2.3.

- $\mathcal{R}_d^k$, which contains a configuration $p \in \mathcal{A}_d^k$ if and only if $p$ is of **r**ank $d+1$, i.e., there is no projective subspace of dimension $m < d$ containing all landmarks. In particular, any corresponding configuration matrix $P$ is of rank $d+1$, and $(\{1, \ldots, k\}, d) \notin C(p)$.

- $\mathcal{N}_d^k(n)$ for $n = (n_1, \ldots, n_d) \in \mathbf{N}^d$ with $n_1 < \cdots < n_d$, which contains a configuration $p \in \mathcal{A}_d^k$ if and only if any projective subspace constraint $(I, j) \in C(p)$ fulfills $|I| \leqslant n_j$. The topological subspace $\mathcal{N}_d^k(n)$ is said to be *bounded by projective subspace **n**umbers $n$*.

- $\mathcal{T}_d^k$, which contains a configuration if and only $|I| < k\frac{j}{d+1}$ for any projective subspace constraint $(I, j) \in C(p)$. These configurations are called ***T**yler regular* by Kent and Mardia (2012).

Recall that a topological subspace $\mathcal{Y}$ of configuration space $\mathcal{A}_d^k$ is always denoted by an upper case letter, the corresponding topological subspace $\mathfrak{y} \subseteq \mathfrak{a}_d^k$ of projective shapes by a lower case letter, for example $\mathcal{A}_d^k, \mathcal{B}_d^k$, etc. for the configuration spaces, $\mathfrak{a}_d^k, \mathfrak{b}_d^k$, etc. for the corresponding shape spaces.

Of course, some of the topological subspaces defined above include another or are mutual complements in $\mathcal{A}_d^k$.

**Proposition 3.1.** *The following holds for all $d, k \geqslant 1$:*

*(i)* $\mathcal{P}_d^k \subseteq \mathcal{F}_d^k \subset \mathcal{R}_d^k$;

*(ii)* $\mathcal{A}_d^k = \mathcal{D}_d^k \mathbin{\dot\cup} \mathcal{F}_d^k$ *for all* $d, k$;

*(iii)* $\mathcal{A}_d^k = \mathcal{D}_d^k$ *resp.* $\mathcal{F}_d^k = \varnothing$ *if and only if* $k \leqslant d+1$;

*(iv)* $\mathcal{G}_d^{d+2} = \mathcal{B}_d^{d+2} = \mathcal{P}_d^{d+2} = \mathcal{F}_d^{d+2}$ *is a singleton;*

*(v)* $\mathcal{G}_d^k \subseteq \mathcal{N}_d^k(n)$ *for any* $n \in \mathbf{N}^d$ *with equality if and only if* $n_j = j$ *for all* $j \leqslant \min\{d, k\}$;

*(vi)* $\mathcal{N}_d^k(n) = \mathcal{A}_d^k$ *if and only if* $n_1 \geqslant k$.

*For* $k > d+2$*:*

*(vii)* $\mathcal{G}_d^k \subset \mathcal{B}_d^k \subset \mathcal{P}_d^k$*;*

*(viii)* $\mathcal{P}_d^k \subseteq \mathcal{F}_d^k$ *with equality if and only if* $d = 1, 2$ *or* $k = d+3$;

Proof. (i) Let $P \notin \mathcal{R}_d^k$ be not of full rank, i.e., $\mathbf{rk}\, P < d+1$. Then, there is a basis $\{x_1, \dots, x_{d+1}\}$ of $\mathbf{R}^{d+1}$ such that the rows $P_{i\cdot}$ of $P$ are in the space spanned by $x_1, \dots, x_d$, i.e., $P_{i\cdot} \in \mathcal{L}(\{x_1, \dots, x_d\})$ for all $i \in \{1, \dots, k\}$. Let $\{x_1, \dots, x_d, x'_{d+1}\}$ be another basis with $x_{d+1} \neq x'_{d+1}$. The basis transformation matrix $B$ is then no scalar multiple of the identity matrix $\mathbf{I}_{d+1}$, but it leaves $P$ unchanged, i.e., $P = PB$. Therefore, $P$ is not free, whence free configurations are of full rank, i.e., $\mathcal{F}_d^k \subseteq \mathcal{R}_d^k$. Theorem 2.1 states that $\mathbf{PGL}(d)$ acts freely on the set $\mathcal{P}_d^{d+2}$ of frames. Hence, $\mathbf{PGL}(d)$ acts freely on $\mathcal{P}_d^k$ for any $k \geqslant 1$, and $\mathcal{P}_d^k \subseteq \mathcal{F}_d^k \subseteq \mathcal{R}_d^k$.

However, shapes comprising of only $d+1$ distinct landmarks are always decomposable, and thus not free due to (ii). Hence, $\mathcal{P}_d^k \subseteq \mathcal{F}_d^k \subset \mathcal{R}_d^k$.

(ii) We will show that decomposable implies not free, and vice versa. Let $P \in \mathcal{A}_d^k$ be a matrix configuration. If $P$ is not of full rank, then $P$ is decomposable since $\big(\{1, \dots, k\}, d+1\big)$ decomposes into $\big(\{1, \dots, k-1\}, d\big), \big(\{k\}, 1\big) \in C(P)$. From (i) we conclude that $P$ is also not free.

Now, assume that $P$ is of rank $\mathbf{rk}\, P = d+1$ and decomposable, i.e., there are projective subspace constraints $(I, j)$, $(I^c, d+1-j) \in C(P)$. By Proposition 2.7, there is a permutation matrix $\Sigma$ of the vertices and a matrix $A \in \mathbf{GL}(d+1)$ such that $\Sigma P A$ is in canonical block structure $\left(\begin{smallmatrix} \hat P_I & 0 \\ 0 & \hat P_{I^c} \end{smallmatrix}\right)$. Then,

$$\begin{aligned}\begin{pmatrix} \hat P_I & 0 \\ 0 & \hat P_{I^c} \end{pmatrix} &= \underbrace{\begin{pmatrix} \lambda \mathbf{I}_{|I|} & 0 \\ 0 & \mathbf{I}_{|I^c|} \end{pmatrix}}_{D} \begin{pmatrix} \hat P_I & 0 \\ 0 & \hat P_{I^c} \end{pmatrix} \underbrace{\begin{pmatrix} \lambda^{-1} \mathbf{I}_j & 0 \\ 0 & \mathbf{I}_{d+1-j} \end{pmatrix}}_{B} \\ &= D \Sigma P A B,\end{aligned}$$

whence $\Sigma P A$ is not free. Therefore, $\Sigma P$ is not free, and neither is $P$ since $P = \Sigma^{-1} D \Sigma P A B$ with $\Sigma^{-1} D \Sigma$ being a diagonal matrix and $AB \in \mathbf{GL}(d+1)$.

For the opposite direction, assume $P$ is not free, i.e., there are a non-singular, diagonal matrix $D \in \mathbf{Diag}^*(k)$ and a non-singular matrix $B \in \mathbf{GL}(d+1)$, $B \neq \lambda \mathbf{I}_{d+1}$, $\lambda \in \mathbf{R}^*$, such that $P = DPB$. Then, the rows of $P$ are left eigenvectors of $B$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_k$. There are at least two distinct values among the $\lambda_i$, $i \in \{1, \dots, k\}$, otherwise $B = \lambda_1 \mathbf{I}_{d+1}$ in contradiction to the assumption. Then, the rows of $P$ divide into classes of corresponding eigenvalues, and $P$ fulfills the projective subspace constraints $\big(I, \mathbf{rk}\, P_I\big), \big(I^c, \mathbf{rk}\, P_{I^c}\big) \in C(P)$ with $I = \{i : \lambda_i = \lambda_1\}$, while $\mathbf{rk}\, P_I + \mathbf{rk}\, P_{I^c} = d+1$, whence $P$ is decomposable.

(iii) If $k < d+1$, there are no configurations of full rank and thus no free configurations due to (i). For $k = d+1$, the configurations of full rank, i.e. those in general position, are decomposable since the trivial projective subspace constraints $\big(\{1\}, 1\big), \big(\{2\}, 1\big), \dots, \big(\{k\}, 1\big)$ give a decomposition. Hence, there are no free configurations if $k \leqslant d+1$.

(iv) Let $p \in \mathcal{A}_d^{d+2}$ be a configuration which fulfills a non-trivial projective subspace constraint $(I, j) \in C^*(p)$. Then, $|I| \geqslant j+1 > j$, and $(I^c, \mathbf{rk}\, p_{I^c}) \in C(p)$ is a trivial projective subspace constraint fulfilled by $p$ with $\mathbf{rk}\, p_{I^c} = |I^c| = d+2-|I| \leqslant d+1-j$, i.e., $p \in \mathcal{A}_d^{d+2}$ is decomposable if and only it fulfills a non-trivial projective subspace constraint. After reminding the reader that a frame consists of $d+2$ landmarks in general position, the equalities are obvious. Since all frames are equivalent under the action of $\mathbf{PGL}(d)$ since $\mathbf{PGL}(d)$ acts transitively on them, see Theorem 2.1, $\mathcal{p}_d^{d+2}$ comprises of one element.

(v) By definition $n_j \geqslant j$ for all feasible vectors $n = (n_1, \dots, n_d) \in \mathbf{N}^d$, so configurations fulfilling only trivial projective subspace constraints are always allowed, i.e., $\mathcal{G}_d^k \subseteq \mathcal{N}_d^k(n)$. There are only configurations with trivial projective subspace constraints in $\mathcal{N}_d^k(n)$ if $n = (1, \dots, d)$ or if $k = 1$.

(vi) $\mathcal{N}_d^k(n) = \mathcal{A}_d^k$ if and only if the projective subspace numbers $n = (n_1, \dots, n_d)$ allow for all possible projective subspace constraints, i.e., even $\big(\{1, \dots, k\}, 1\big)$, so $n_1 \geqslant k$ and, consequently, $n_d > \dots > n_1 \geqslant k$.

(vii) Recall that a frame consists of $d+2$ landmarks in general position. Then, there is nothing left to prove.

(viii) The statement $\mathcal{P}_d^k \subseteq \mathcal{F}_d^k$ follows directly from Theorem 2.1. For the statement regarding equality we refer to Section 3.4, in particular to page 42 for the cases $d = 1, 2$. □

From Proposition 2.7 and Proposition 3.1, we conclude that $\mathcal{a}_d^k$ can be decomposed into disjoint subsets with fixed total decomposition.

**Proposition 3.2.** *Let $k \geqslant 1$ and $d \geqslant 1$. Then,*

$$\begin{aligned} \mathcal{a}_d^k &\cong \coprod_{r=1}^{d+1} \coprod_{s=1}^{r} \coprod_{\substack{\{(I_1,j_1),\dots,(I_s,j_s)\} \\ \text{is total decomposition} \\ \text{with } \sum_{n=1}^s j_n = r}} \mathcal{f}_{j_1-1}^{|I_1|} \times \dots \times \mathcal{f}_{j_s-1}^{|I_s|} \\ &\cong \coprod_{s=1}^{d+1} \coprod_{\substack{\{(I_1,j_1),\dots,(I_s,j_s)\} \\ \text{is total decomposition}}} \mathcal{f}_{j_1-1}^{|I_1|} \times \dots \times \mathcal{f}_{j_s-1}^{|I_s|}. \end{aligned}$$

Note that the stratification of Proposition 3.2 is only a set-theoretic one, but not a topological one. However, Proposition 3.10 states how the strata, which turn out to be manifolds of different dimensions (see Theorem 3.24), are glued together. Analogous stratifications have been proven for similarity shape space (Kendall et al.; 1999, Sect. 2.6) as well as affine shape space (Groisser and Tagare; 2009, Thm. 4.2). In these cases the strata can be ordered in terms of matrix ranks. This is not possible for the stratification of projective shape space given above since projective shapes of full rank are not necessarily free. However, a partial order can be given by the dimensions of the strata.

## 3.2 Properties of all topological subspaces

The quotient topology on $\mathcal{a}_d^k$ inherits some properties from the topology on $\mathcal{A}_d^k$. First of all, recall that the projection map $\pi : \mathcal{A}_d^k \to \mathcal{a}_d^k$ is continuous by the definition of the quotient topology, and open since projective transformations are homeomorphisms on $\mathcal{A}_d^k$. It easily follows that the quotient topology on $\mathcal{a}_d^k$ fulfills the first and second axiom of countability, i.e., there is a countable set of open subsets of $\mathcal{a}_d^k$ such that any open subset of $\mathcal{a}_d^k$ is a union of some of these distinguished open subsets.

**Lemma 3.3.** *The topology on $\mathcal{a}_d^k$ fulfills the first and second axiom of countability.*

Proof. $\mathcal{A}_d^k = \left(\mathbf{RP}^d\right)^k$ fulfills the second axiom of countability since it is a finite product of second-countable spaces. Hence, there is a countable base $(U_n)_{n\in\mathbf{N}}$ of the topology. Any open subset $V \subseteq \mathfrak{a}_d^k$ is of the form $V = \pi(U)$ for an open $U \in \mathcal{A}_d^k$, i.e., $V = \pi(U) = \pi\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} \pi(U_i)$ for some $I \subseteq \mathbf{N}$. Since $\pi$ is open, $\left(\pi(U_n)\right)_{n\in\mathbf{N}}$ is a base of the topology on $\mathfrak{a}_d^k$. Thus, $\mathfrak{a}_d^k$ is second-countable, and consequently first-countable, too. □

A classical result of general topology states that the topology of a first-countable space is determined by sequences (Kelley; 1955, Ch. 2, Thm. 8). In particular, the topology of $\mathfrak{a}_d^k$ is determined by sequences.

**Corollary 3.4.** *A subset $\mathcal{y} \subseteq \mathfrak{a}_d^k$ is open if and only if for any $[p] \in \mathcal{y}$ and any sequence $\left([p_n]\right)_{n\in\mathbf{N}}$ with limit $[p]$ there is an $N \in \mathbf{N}$ such that $[p_n] \in \mathcal{y}$ for all $n \geqslant N$.*

Furthermore, when thinking about a converging sequence in shape space $\mathfrak{a}_d^k$, one may always think about a converging sequence in configuration space $\mathcal{A}_d^k$.

**Corollary 3.5.** *To any sequence $\left([p_n]\right)_{n\in\mathbf{N}}$ in projective shape space $\mathfrak{a}_d^k$ with limit $[p]$ and any configuration $q \in \pi^{-1}\left([p]\right)$, there is a sequence $(q_n)_{n\in\mathbf{N}}$ in configuration space $\mathcal{A}_d^k$ with limit $q$ such that $[p_n] = \pi(q_n)$.*

Proof. Let $\left([p_n]\right)_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{a}_d^k$ with limit $[p]$ and $q \in \mathcal{A}_d^k$ with $[p] = \pi(q)$. Since $\mathcal{A}_d^k$ fulfills the first axiom of countability, there is a countable base $\left(U_m\right)_{m\in\mathbf{N}}$ of neighborhoods at $q$. W.l.o.g. $U_m \supseteq U_k$ for all $k > m$, otherwise, consider the countable base $\left(U'_m\right)_{m\in\mathbf{N}}$ with $U'_m = \bigcap_{k\leqslant m} U_k$. For all $m \in \mathbf{N}$ there exists an $N_m \in \mathbf{N}$ such that $[p_n] \in \pi(U_m)$ for all $n > N_m$ since $\left([p_n]\right)_{n\in\mathbf{N}}$ has limit $[p]$ and $\pi(U_m)$ is a neighborhood of $[p]$. Now, choose the sequence $(q_n)_{n\in\mathbf{N}}$ such that $\pi(q_n) = [p_n]$ and $q_n \in U_m$ for all $n > N_m$ for all $m \in \mathbf{N}$. □

Note that Lemma 3.3, Corollary 3.4 and Corollary 3.5 also hold for topological subspaces of $\mathfrak{a}_d^k$ since they inherit the property of the axioms of countability through the subspace topology.

Additionally, dense topological subspaces of $\mathcal{A}_d^k$ are again dense under $\pi$.

**Lemma 3.6.** *Let $\mathcal{Y} \subseteq \mathcal{A}_d^k$ be dense in $\mathcal{A}_d^k$. Then, $\mathcal{y} = \pi(\mathcal{Y})$ is dense in $\mathfrak{a}_d^k$. In particular, the topological subspace $\mathfrak{g}_d^k$ of shapes with all landmarks in general position is dense in $\mathfrak{a}_d^k$.*

Proof. Let $U \subseteq \mathfrak{a}_d^k$ be a neighborhood of $[p] \in \mathfrak{a}_d^k$. Then, $\pi^{-1}(U)$ is a neighborhood of $p \in \mathcal{A}_d^k$ and thus contains an element $q \in \mathcal{Y}$ since $\mathcal{Y}$ is dense in $\mathcal{A}_d^k$. Hence, $[q] \in \pi\left(\pi^{-1}(U)\right) \cap \mathcal{y}$, so $\mathcal{y}$ is dense in $\mathfrak{a}_d^k$.

For the second statement, let $p \in \mathcal{A}_d^k$ be an arbitrary configuration. Then, one can resolve all non-trivial projective subspace constraints of $p$ by arbitrary small perturbations on the landmarks of $p$, whence a configuration $q \in \mathcal{G}_d^k$ in general position can be found in any neighborhood of $p$. Thus, $\mathcal{G}_d^k$ is dense in $\mathcal{A}_d^k$, as is $\mathfrak{g}_d^k$ in $\mathfrak{a}_d^k$. □

Further, topological subspaces respecting the hierarchy of projective subspace constraints are open in $\mathfrak{a}_d^k$.

**Proposition 3.7.** *A topological subspace $\mathcal{y} \subseteq \mathfrak{a}_d^k$ which respects the hierarchy of projective subspace constraints is open. In particular, the topological subspace $\mathfrak{g}_d^k$ of shapes with all landmarks in general position is open.*

Proof. Both statements are direct consequences of Lemma 2.6. □

## 3.3 T1 and Hausdorff subspaces

As mentioned in requirement 1), we are interested in Hausdorff subspaces of $\mathfrak{a}_d^k$. For the construction of these, it is important to understand which shapes $[q] \in \mathfrak{a}_d^k$ can be separated from a fixed shape $[p] \in \mathfrak{a}_d^k$ by an open neighborhood and which cannot.

To describe the degree of separation, topologists introduced separation axioms. Besides Hausdorffness, two more notions of separation will be discussed here.

A topological space $\mathcal{M}$ is said to be

**T0** if for any two elements $p, q \in \mathcal{M}$ there is an open neighborhood of $p$ or $q$ not containing the other element;

**T1** if for any two elements $p, q \in \mathcal{M}$ there are open neighborhoods $U_p$ and $U_q$ of $p$ resp. $q$ not containing the other element, i.e., $q \notin U_p$ and $p \notin U_q$;

**Hausdorff or T2** if for any two elements $p, q \in \mathcal{M}$ there are disjoint open neighborhoods of $p$ and $q$.

Obviously, a Hausdorff space is T1, too, while a T1 space is also T0.

To understand the separation properties of a topological space $\mathcal{M}$, it is very useful to compute the intersection of all open neighborhoods to an element $p \in \mathcal{M}$. Groisser and Tagare (2009) have considered this set in their discussion of affine shape space, and it was called the *blur* $\mathbf{Bl}(p)$ of $p$ in $\mathcal{M}$ there. An element $p \in \mathcal{M}$ is said to be *blurry* in the case that its blur is a strict superset of $\{p\}$, and *unblurry* if $\mathbf{Bl}(p) = \{p\}$.

Note that the blur of an element $p \in \mathcal{M}$ depends heavily on the topological space in which the blur is considered. However, if $\mathcal{U}$ is any topological subspace of $\mathcal{M}$, then the blur of an element $p$ in $\mathcal{U}$ is a subset of the blur of $p$ in $\mathcal{M}$ with equality if $\mathcal{U}$ is open.

The blur can also be defined via sequences:

**Lemma 3.8.** *Let $\mathcal{M}$ be a topological space and $p, q \in \mathcal{M}$ elements in $\mathcal{M}$. Then, $q \in \mathbf{Bl}(p)$ if and only if $p$ is a limit point of the constant sequence $(q)_{n\in\mathbf{N}}$.*

PROOF. The sequence $(q)_{n\in\mathbf{N}}$ converges to $p$ if and only if $(q)_{n\in\mathbf{N}}$ is ultimately in every neighborhood of $p$, i.e., if and only if $q$ is in every neighborhood of $p$. By definition, $q$ is in every neighborhood of $p$ if and only if $q \in \mathbf{Bl}(p)$. □

The more familiar concept of the *closure* $\mathbf{Cl}(U)$ of a set $U$ is similarly defined as the intersection of all *closed* supersets of $U$. Groisser and Tagare (2009) have pointed out that the concepts of the blur $\mathbf{Bl}(p)$ and the closure $\mathbf{Cl}(p)$ of an element $p \in \mathcal{M}$ are basically interchangeable.

**Lemma 3.9 (Groisser and Tagare (2009), Lemma 5.2).** *Let $\mathcal{M}$ be a topological space and $p, q \in \mathcal{M}$, and let $\mathbf{Cl}(p)$ denote the closure of $\{p\}$ in $\mathcal{M}$. Then, $q \in \mathbf{Cl}(p)$ if and only if $p \in \mathbf{Bl}(q)$. In particular, every element of $\mathcal{M}$ is closed, i.e., $\mathbf{Cl}(p) = \{p\}$, if and only if every element is unblurry.*

Recall that a topological space is T1 if and only if all of its elements are closed (Arkhangel'skiĭ and Fedorchuk; 1990, Sect. 2.6, Prop. 13). Consequently, the topological subspace of $\mathfrak{a}_d^k$ comprising of unblurry shapes is T1.

From now on, only the case $k \geqslant d + 3$ will be discussed, the case $k \leqslant d + 2$ is less relevant and less interesting as Proposition 3.1 shows. The blurry shapes in $\mathfrak{a}_d^k$ are then characterized by decomposability.

**Proposition 3.10.** *A shape $[p] \in \mathfrak{a}_d^k$ is blurry in $\mathfrak{a}_d^k$ if and only if $[p]$ is decomposable.*

*A shape $[q] \in \mathfrak{a}_d^k$ is in the blur of $[p]$ in $\mathfrak{a}_d^k$ if and only if there is a permutation $\sigma$ of the landmarks such that*

$$\sigma[p] = \left[\begin{pmatrix} P_1 & 0 & \cdots & 0 & 0 \\ 0 & P_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & P_s & 0 \end{pmatrix}\right] \tag{3.1}$$

*has a block representation, and*

$$\sigma[q] = \left[\begin{pmatrix} P_1 & P_{12} & \cdots & P_{1s} & P_{10} \\ 0 & P_2 & \cdots & P_{2s} & P_{20} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & P_s & P_{s0} \end{pmatrix}\right] \tag{3.2}$$

*is upper block "triangular" for some $P_{ij}$ of suitable dimension. In particular, the blur $\mathbf{Bl}([p])$ of $[p]$ consists only of less constrained shapes, i.e., $C(q) \subseteq C(p)$ for all $[q] \in \mathbf{Bl}([p])$.*

Before proving Proposition 3.10, let us give a simple example to show the concept of the proof: let $[P] \in \mathfrak{a}_d^k$ be of rank $d+1$ with $(\{1,\ldots,i\},j), (\{i+1,\ldots,k\}, d+1-j) \in C(P)$ with $i \in \{1,\ldots,k\}$, and let $P$ be a representing matrix configuration in block diagonal form, i.e.,

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

for some matrices $P_1 \in \mathcal{A}_{j-1}^i$ and $P_2 \in \mathcal{A}_{d-j}^{k-i}$. The sequence $(Q_n)_{n\in\mathbf{N}}$ given by

$$Q_n = \begin{pmatrix} P_1 & \frac{1}{n}Y \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_i & 0 \\ 0 & n\mathbf{I}_{k-i} \end{pmatrix} \begin{pmatrix} P_1 & Y \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} \mathbf{I}_j & 0 \\ 0 & n^{-1}\mathbf{I}_{d+1-j} \end{pmatrix} \tag{3.3}$$

has limit $P$ in $\mathcal{A}_d^k$ for any $Y \in \mathbf{R}^{i\times(d+1-j)}$, while the corresponding sequence in projective shape space $\mathfrak{a}_d^k$ is constant since Equation (3.3) shows

$$[Q_n] = \left[\begin{pmatrix} P_1 & \frac{1}{n}Y \\ 0 & P_2 \end{pmatrix}\right] = \left[\begin{pmatrix} P_1 & Y \\ 0 & P_2 \end{pmatrix}\right].$$

Due to Lemma 3.8,

$$\left[\begin{pmatrix} P_1 & Y \\ 0 & P_2 \end{pmatrix}\right] \in \mathbf{Bl}([P]), \quad \text{and analogously} \quad \left[\begin{pmatrix} P_1 & 0 \\ Z & P_2 \end{pmatrix}\right] \in \mathbf{Bl}([P])$$

for any $Y \in \mathbf{R}^{i\times(d+1-j)}$, $Z \in \mathbf{R}^{(k-i)\times j}$. Now, there is a $Y \in \mathbf{R}^{i\times(d+1-j)}$ or $Z \in \mathbf{R}^{(k-i)\times j}$ which breaks a projective subspace constraint of $[P]$, whence $\mathbf{Bl}([P]) \neq \{[P]\}$.

This idea of "different speeds of convergence" employed in Equation (3.3) will be used a few more times in this thesis, e.g. in a more evolved way in the following proof of Proposition 3.10. It has been introduced before in (Kent et al.; 2011, Sect. 3.4), albeit in a less general way.

Proof (Proposition 3.10). It suffices to show the second statement that the blur comprises of shapes as in Equation (3.2). The other statements then follow immediately.

Let $P \in \mathcal{D}_d^k$ be a decomposable matrix configuration of shape $[p]$ with total decomposition $\{(I_1,j_1),\ldots,(I_s,j_s)\} \subseteq C(p)$ and define $j_0 = d+1-\mathbf{rk}\,P$. By Proposition 2.7, there is a permutation matrix $\Sigma$ and a non-singular matrix $B \in \mathbf{GL}(d+1)$ such that the matrix $\hat{P} = \Sigma P B$ is a block "diagonal" matrix

$$\begin{pmatrix} \hat{P}_1 & 0 & \cdots & 0 & 0 \\ 0 & \hat{P}_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \hat{P}_s & 0 \end{pmatrix}$$

with matrices $\hat{P}_r \in \mathbf{R}^{|I_r| \times j_r}$, $1 \leqslant r \leqslant s$. By using different speeds of convergence, the sequence $\big(D_n Q B_n\big)_{n \in \mathbf{N}}$ with

$$Q = \begin{pmatrix} \hat{P}_1 & \hat{P}_{12} & \cdots & \hat{P}_{1s} & \hat{P}_{10} \\ 0 & \hat{P}_2 & \cdots & \hat{P}_{2s} & \hat{P}_{20} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \hat{P}_s & \hat{P}_{s0} \end{pmatrix},$$

$$D_n = \begin{pmatrix} \mathbf{I}_{|I_1|} & 0 & \cdots & 0 \\ 0 & n\mathbf{I}_{|I_2|} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n^{s-1}\mathbf{I}_{|I_s|} \end{pmatrix} \in \mathbf{Diag}^*(k),$$

$$B_n = \begin{pmatrix} \mathbf{I}_{j_1} & 0 & \cdots & 0 & 0 \\ 0 & n^{-1}\mathbf{I}_{j_2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & n^{-s+1}\mathbf{I}_{j_s} & 0 \\ 0 & \cdots & \cdots & 0 & n^{-s}\mathbf{I}_{j_0} \end{pmatrix} \in \mathbf{GL}(d+1)$$

has limit $\hat{P}$ for any matrices $\hat{P}_{rt} \in \mathbf{R}^{|I_r| \times j_t}$ while being constant in shape space. Hence, $[Q] \in \mathbf{Bl}\big([\hat{P}]\big)$ for any matrices $\hat{P}_{rt} \in \mathbf{R}^{|I_r| \times j_t}$ by Lemma 3.8, as has been discussed similarly by (Kent et al.; 2011, Sect. 3.4). Analogously, the sequence $\big(\Sigma^{-1} D_n \Sigma \Sigma^{-1} Q B_n B\big)_{n \in \mathbf{N}}$ has limit $P$, and $\big[\Sigma^{-1} Q\big] \in \mathbf{Bl}\big([p]\big)$. As mentioned in Remarks 2.8, there is more than one permutation to obtain a block structure as in Equation (3.1), i.e., all of the shapes as in Equation (3.2) are indeed included in the blur.

Further, we have to show that these shapes actually comprise the blur of $[p]$, i.e., that it suffices to think in such block structures: denote the indices in $P$ of the block $P_r$ by $I_r \times J_r$. The non-decomposable projective subspace constraints of $P$ are then given by $\big(I_r, |J_r|\big)$, $1 \leqslant r \leqslant s$. Let $[Q]$ be a shape with $[Q] \in \mathbf{Bl}\big([P]\big)$, i.e., the constant sequence $\big([Q]\big)_{n \in \mathbf{N}}$ has limit $[P]$ by Lemma 3.8, and let $Q$ be a representing configuration of $[Q]$. Then, there is a sequence $(P_n)_{n \in \mathbf{N}}$ of matrix configurations with limit $P$ and $[P_n] = [Q]$ for all $n \in \mathbf{N}$ (Corollary 3.5). In particular, there are non-singular diagonal matrices $D_n$ and non-singular matrices $B_n \in \mathbf{GL}(d+1)$ such that

$$D_n P_n = Q B_n \tag{3.4}$$

for all $n \in \mathbf{N}$. Without loss of generality the following can be assumed:

- $B_n$ *is diagonal for all* $n \in \mathbf{N}$: using a singular value decomposition for $B_n$, one obtains the existence of diagonal matrices $D_n, E_n$ and orthogonal matrices $U_n, V_n \in \mathbf{O}(d+1)$ such that $D_n P_n = Q V_n E_n U_n^t$ or equivalently $D_n P_n U_n = Q V_n E_n$. The sequences $(U_n)_{n \in \mathbf{N}}$ and $(V_n)_{n \in \mathbf{N}}$ have common converging subsequences since $\mathbf{O}(d+1)$ is compact, whence we can assume $U_n \to U$, $V_n \to V$, and consequently $P_n U_n \to PU$ and $Q V_n \to QV$ without restriction. Since right-multiplication by an orthogonal matrix does not change the projective shape of $P_n$ resp. $Q$, we can choose $P_n, Q$ such that the corresponding $B_n$ is diagonal.

- $\|B_n\|_\infty = 1$ *for all* $n \in \mathbf{N}$: if otherwise, consider $\|B_n\|_\infty^{-1} D_n$ and $\|B_n\|_\infty^{-1} B_n$ instead of $D_n$ and $B_n$.

- $(B_n)_{n \in \mathbf{N}}$ *converges to some limit* $B$ *with* $\|B\|_\infty = 1$: $(B_n)_{n \in \mathbf{N}}$ is w.l.o.g. bounded in the supremum norm, hence possesses at least a converging subsequence. Consequently, we can assume $Q B_n \to QB$, too.

- $(D_n)_{n \in \mathbf{N}}$ *converges to some limit* $D$ with $\|D_n\|_\infty \leqslant \rho$, $\rho > 0$, for all $n \in \mathbf{N}$; else a row of $P$ would be the null vector since $D_n P_n \to QB$ and $P_n \to P$, which is impossible.

Now, if $\left(\frac{(D_n)_{ii}}{(B_n)_{jj}}\right)_{n\in\mathbf{N}}$ diverges, then $(P_n)_{ij}$ has to converge to 0 due to Equation (3.4), i.e., $P_{ij} = 0$; if $\left(\frac{(D_n)_{ii}}{(B_n)_{jj}}\right)_{n\in\mathbf{N}}$ converges to 0, then $Q_{ij} = 0$ and thus also $(P_n)_{ij} = P_{ij} = 0$. Consequently, there is a $j \in J_r$ to any $i \in I_r$, $1 \leqslant r \leqslant s$, such that $\frac{(D_n)_{ii}}{(B_n)_{jj}} \to c \neq 0$ as $n$ goes to infinity since $P$ has non-trivial rows. If there were $i \in I_r$ and $j \in J_r$ such that $\left(\frac{(D_n)_{ii}}{(B_n)_{jj}}\right)_{n\in\mathbf{N}}$ diverges or $\frac{(D_n)_{ii}}{(B_n)_{jj}} \to 0$ as $n$ goes to infinity, then one obtains a decomposition of $(I_r, j_r)$ by merging rows and columns of same speed of convergence, and $(I_r, j_r)$ is decomposable in contradiction to the assumption. Hence, there are blocks of different speeds of convergence corresponding to the blocks of $P$. When ordering these speeds in a decreasing order, one obtains the proposed block structure of $Q$. Note that the elements of $\{1, \ldots, k\} \setminus \bigcup_{r=1}^{s} I_r$ belong to the trivial columns of $P$.

To see that the blur $\mathbf{Bl}([p])$ of a shape $[p]$ contains only less constrained shapes, recall that the topological subspace

$$\{[q] \in \mathfrak{a}_d^k : C(q) \subseteq C(p)\}$$

is open by Lemma 2.6. Thus, $\mathbf{Bl}([p]) \subseteq \{[q] \in \mathfrak{a}_d^k : C(q) \subseteq C(p)\}$ by the definition of the blur.□

Proposition 3.10 states that open topological subspaces of $\mathfrak{a}_d^k$, e.g. subspaces respecting the hierarchy of projective subspace constraints, cannot be T1 and even less Hausdorff if a decomposable shape is included. However, $\mathfrak{a}_d^k$ is T0, so are all of its topological subspaces since less regular shapes can be separated from less constrained shapes by an open neighborhood of the latter, while equally constrained shapes can even be separated in the T1 sense, which can be concluded from the later Theorem 3.24. By Proposition 3.1, the largest T1 subspace of $\mathfrak{a}_d^k$ respecting the hierarchy of projective subspace constraints is therefore the topological subspace $\mathfrak{f}_d^k = \mathfrak{a}_d^k \setminus \mathfrak{d}_d^k$ of free shapes.

**Example 3.11.** In $\mathfrak{a}_1^k$ decomposable shapes are either of rank 1 (the trivial shape where all landmarks coincide) or their total decomposition is $\{(I, 1), (I^c, 1)\}$ for a non-trivial subset $I \subset \{1, \ldots, k\}$ (shapes comprising of exactly two distinct landmarks). Therefore, the topological subspace $\mathfrak{f}_1^k$ of free shapes comprises of those with at least three distinct landmarks, i.e. a frame, so $\mathfrak{f}_1^k = \mathfrak{p}_1^k$.

For $k = 4$, the blur in $\mathfrak{a}_1^4$ of the shape $[p] \in \mathfrak{d}_1^4$ with $p_1 = p_2$ and $p_3 = p_4$, but $p_1 \neq p_3$ (double pair coincidence) comprises of $[p]$ and the single pair coincidences $[q]$ with $q_3 = q_4$ and $[r]$ with $r_1 = r_2$ since

$$[p] = \left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}\right], \quad [q] = \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\right], \quad [r] = \left[\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

Here, the representative of $[q]$ and the first representative of $[p]$ are already in the structures of Proposition 3.10, while for $[r]$ the permutation interchanging the first two landmarks with the latter two has to be applied on the representative of $[r]$ and the second representative of $[p]$. The blur of the trivial shape is $\mathfrak{a}_1^4$, and the blur of the shape

$$[s] = \left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \in \mathfrak{d}_1^4$$

with the triple coincidence $s_1 = s_2 = s_3 \neq s_4$ comprises of all less constrained shapes. ◊

Of course, we are looking for Hausdorff spaces, and the topological subspace $\mathfrak{f}_d^k$ of the free is not Hausdorff for any $d \geqslant 1$ and $k \geqslant d + 3$ as we will see shortly.

Consider a shape $[p] \in \mathscr{r}_d^k$ which fulfills the projective subspace constraint $\big(\{1,\ldots,i\}, j\big)$, i.e., $[p]$ has a block matrix representation

$$P = \begin{pmatrix} P_1 & 0 \\ Z & P_2 \end{pmatrix}$$

for some matrices $P_1 \in \mathscr{A}_{j-1}^{i}$, $P_2 \in \mathscr{A}_{d-j}^{k-i}$, and $Z \in \mathbf{R}^{(k-i)\times j}$, see Remarks 2.8. Then, the sequence $\big([P_n]\big)_{n\in\mathbf{N}}$ with

$$P_n = \begin{pmatrix} P_1 & \frac{1}{n}Y \\ Z & P_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_i & 0 \\ 0 & n\mathbf{I}_{k-i} \end{pmatrix} \begin{pmatrix} P_1 & Y \\ \frac{1}{n}Z & P_2 \end{pmatrix} \begin{pmatrix} \mathbf{I}_j & 0 \\ 0 & \frac{1}{n}\mathbf{I}_{d+1-j} \end{pmatrix}$$

has limit points $[P]$ and

$$[Q] = \left[\begin{pmatrix} P_1 & Y \\ 0 & P_2 \end{pmatrix}\right]$$

with $\big(\{i+1,\ldots,k\}, d+1-j\big) \in C(Q)$. Meanwhile sequences in first-countable, Hausdorff spaces have at most one limit point (Kelley; 1955, Ch. 2, Thm. 3), whence a topological subspace of $\mathscr{a}_d^k$ containing $[P]$, $[Q]$ and $[P_n]$ for all $n \in \mathbf{N}$ simultaneously is not Hausdorff.

A generalization of this observation gives us a criterion to determine if a topological subspace of $\mathscr{a}_d^k$ is Hausdorff.

**Proposition 3.12.** *Let $\mathscr{y} \subseteq \mathscr{r}_d^k$ be a topological subspace containing all shapes in general position, i.e., $\mathscr{g}_d^k \subseteq \mathscr{y}$. The subspace $\mathscr{y}$ is not Hausdorff if and only if there are two distinct shapes $[p],[q] \in \mathscr{y}$ which after simultaneous reordering of rows by some permutation $\sigma$ have the block structure*

$$\sigma[p] = \left[\begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ 0 & \vdots & \ddots & \vdots \\ \vdots & P_{l-1,2} & \cdots & P_{l-1,m} \\ 0 & \cdots & 0 & P_{lm} \end{pmatrix}\right] \tag{3.5}$$

*and*

$$\sigma[q] = \left[\begin{pmatrix} D_1P_{11}B_1 & 0 & \cdots & 0 \\ Q_{21} & \cdots & Q_{2,m-1} & \vdots \\ \vdots & \ddots & \vdots & 0 \\ Q_{l1} & \cdots & Q_{l,m-1} & D_lP_{lm}B_m \end{pmatrix}\right], \tag{3.6}$$

*with $P_{rs}, Q_{rs}$ being matrices of the same dimensions for all $r, s$, and*

*(i) $l, m > 1$ since $[p] \neq [q]$,*

*(ii) if $P_{rs}, Q_{rs} \neq 0$, then $Q_{rs} = D_rP_{rs}B_s$ with $D_r$ diagonal and non-singular, $B_s$ non-singular,*

*(iii) $P_{rs} = 0$ if there is a pair $(a,b) \neq (r,s)$ with $a \leqslant r$, $b \geqslant s$ and $Q_{ab} \neq 0$,*

*(iv) $Q_{rs} = 0$ if there is a pair $(a,b) \neq (r,s)$ with $a \geqslant r$, $b \leqslant s$ and $P_{ab} \neq 0$.*

For an illustration of the form of the configurations $P = \big(P_{rs}\big)$ and $Q = \big(Q_{rs}\big)$ see Figure 3.2.

Proof. The strategy of the proof is to use the definition of Hausdorff spaces via sequences in first-countable spaces: if two distinct elements $p, q \in \mathcal{M}$ in a first-countable topological space $\mathcal{M}$ do not possess disjoint open neighborhoods, then there is a sequence with limit points $p$ and $q$ (Kelley; 1955, Ch. 2, Thm. 3). Equivalently, this means that sequences in Hausdorff spaces

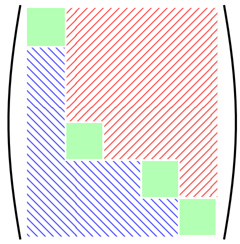


**Figure 3.2:** The form of the configurations $P$ and $Q$ in Proposition 3.12. The configuration $P$ is zero in the blue, hatched area ( ) due to (iii), while $Q$ is zero in the red, hatched area ( ) due to (iv). In the green area ( ), the corresponding matrices are equivalent due to (ii).

possess at most one limit point. In shape space $\mathscr{y}$, this gives us the equation $D_nP_n = Q_nB_n$ for all $n \in \mathbf{N}$, and sequences $\big([P_n]\big)_{n\in\mathbf{N}}$, $\big([Q_n]\big)_{n\in\mathbf{N}}$ with distinct but non-separable limit points $[P]$ resp. $[Q]$. As in the proof of Proposition 3.10 we can w.l.o.g. assume that $B_n$ is diagonal for all $n \in \mathbf{N}$, and that the sequences $(B_n)_{n\in\mathbf{N}}, (D_n)_{n\in\mathbf{N}}$ converge to singular matrices. By using the method of different speeds of convergence, we will then obtain the described form of the configurations $P$ and $Q$.

For the other direction the idea of different speeds of convergence will be used to construct a shape in any neighborhood of some $[P], [Q] \in \mathscr{y}$ of the described form.

Recall that $\mathscr{a}_d^k$ and all of its topological subspaces are first-countable (Lemma 3.3). Let $[p], [q] \in \mathscr{y}$ be distinct shapes which cannot be separated in the Hausdorff sense, i.e., there are no disjoint open neighborhoods of $[p]$ and $[q]$. Then, there is a sequence $\big([r_n]\big)_{n\in\mathbf{N}}$ in $\mathscr{y}$ with limits $[p], [q]$. We can assume $[r_n] \in \mathscr{g}_d^k$ to be in general position for all $n \in \mathbf{N}$ since $\mathscr{g}_d^k$ is dense in $\mathscr{y} \subseteq \mathscr{a}_d^k$ by Lemma 3.6. By Corollary 3.5, there are sequences $(P_n)_{n\in\mathbf{N}}$ with limit $P$ and $(Q_n)_{n\in\mathbf{N}}$ with limit $Q$ in the configuration space $\mathscr{A}_d^k$ such that $\pi(P_n) = \pi(Q_n) = [r_n]$ for all $n \in \mathbf{N}$ and $\pi(P) = [p]$, $\pi(Q) = [q]$. Further, there are matrices $D_n \in \mathbf{Diag}^*(k)$ and $B_n \in \mathbf{GL}(d+1)$ such that

$$D_nP_n = Q_nB_n$$

for all $n \in \mathbf{N}$ since $P_n$ and $Q_n$ are of the same projective shape.

Without loss of generality the following can be assumed:

- *$B_n$ is diagonal for all $n \in \mathbf{N}$*: in fact, using a singular value decomposition for $B_n$, one obtains the existence of diagonal matrices $D_n \in \mathbf{Diag}^*(k)$, $E_n \in \mathbf{Diag}^*(d+1)$ and orthogonal matrices $U_n, V_n \in \mathbf{O}(d+1)$ such that $D_nP_n = Q_nV_nE_nU_n^t$ or equivalently $D_nP_nU_n = Q_nV_nE_n$. The sequences $(U_n)_{n\in\mathbf{N}}$ and $(V_n)_{n\in\mathbf{N}}$ have common converging subsequences since $\mathbf{O}(d+1)$ is compact, so w.l.o.g. $U_n \to U$, $V_n \to V$, $P_nU_n \to PU$ and $Q_nV_n \to QV$. Since right-multiplication by an orthogonal matrix does not change the projective shape of $P_n$ resp. $Q_n$, we can choose $P_n, Q_n$ such that the corresponding $B_n$ is diagonal.

- *$\|B_n\|_\infty = 1$ for all $n \in \mathbf{N}$*; otherwise, consider the matrices $\|B_n\|_\infty^{-1}D_n$ and $\|B_n\|_\infty^{-1}B_n$ instead of $D_n$ and $B_n$.

- *$(B_n)_{n\in\mathbf{N}}$ converges to some limit $B$ with $\|B\|_\infty = 1$*: the sequence $(B_n)_{n\in\mathbf{N}}$ is w.l.o.g. bounded in the supremum norm (see above), whence it possesses a converging subsequence. Thus, we can assume $Q_nB_n \to QB$ without restriction, too.

- *$(D_n)_{n\in\mathbf{N}}$ converges to some limit $D$*, hence $\|D_n\|_\infty \leqslant \rho$, $\rho > 0$, for all $n \in \mathbf{N}$; else, since $D_nP_n \to QB$ and $P_n \to P$, a row of $P$ would be the null vector which is impossible.

- *$B$ and $D$ are singular, but non-trivial matrices, i.e., $B, D \neq 0$*: if $B$ was non-singular, so would be $D$; else $QB$ and thus $Q$ would have a vanishing row which is impossible. If $D$ was non-singular, so would be $B$; otherwise, $P$ would be of rank less than $d+1$ in contradiction to the assumption $\mathcal{y} \subseteq r_d^k$. If both are non-singular, then $P = D^{-1}QB$ in contradiction to $[p] \neq [q]$. Hence, both $B$ and $D$ are singular. Further, $B$ is non-trivial since $\|B\|_\infty = 1$, while $D$ is non-trivial since $B$ is non-trivial and $P$ and $Q$ are of full rank.

Recall that neither $P$ nor $Q$ may have trivial rows or columns by assumption. By reordering of rows and columns, one may assume that $\left(\frac{(D_n)_{ii}}{(D_n)_{jj}}\right)_{n\in\mathbf{N}}$ and $\left(\frac{(B_n)_{ii}}{(B_n)_{jj}}\right)_{n\in\mathbf{N}}$ converge to a finite limit for all $i < j$, so $(D_n)_{ii}$ does not grow faster than $(D_n)_{jj}$ for all $i < j$. The merger of columns respectively rows of equal speed of convergence leads then to the proposed block structure of $P$ and $Q$. If the sequence $\left(\frac{(D_n)_{ii}}{(B_n)_{jj}}\right)_{n\in\mathbf{N}}$ converges to a non-zero value for some $i, j$, then the corresponding block is of type (ii). If the sequence $\left(\frac{(D_n)_{ii}}{(B_n)_{jj}}\right)_{n\in\mathbf{N}}$ converges to 0, then $Q_{ij} = 0$ which explains type (iv). Concerning blocks of type (iii), consider the equalities $P_nF_n = G_nQ_n$ with $F_n = B_n^{-1}/\|B_n^{-1}\|_\infty$ and $G_n = D_n^{-1}/\|B_n^{-1}\|_\infty$ for all $n \in \mathbf{N}$. If the sequence $\left(\frac{(D_n)_{ii}}{(B_n)_{jj}}\right)_{n\in\mathbf{N}}$ diverges, or equivalently, the sequence $\left(\frac{(B_n)_{jj}}{(D_n)_{ii}}\right)_{n\in\mathbf{N}} = \left(\frac{\|B_n\|_\infty^{-1}(D_n^{-1})_{ii}}{\|B_n\|_\infty^{-1}(B_n^{-1})_{jj}}\right)_{n\in\mathbf{N}} = \left(\frac{(G_n)_{ii}}{(F_n)_{jj}}\right)_{n\in\mathbf{N}}$ converges to 0, then $P_{ij} = 0$ which explains type (iii).

Finally, we have to show that the upper left and bottom right blocks are of type (ii): since every row of $Q$ is non-trivial, $\left(\frac{(D_n)_{kk}}{(B_n)_{d+1,d+1}}\right)_{n\in\mathbf{N}}$ does not converge to 0. The corresponding sequence of inverses $\left(\frac{(B_n)_{d+1,d+1}}{(D_n)_{kk}}\right)_{n\in\mathbf{N}}$ does not converge to 0 since $P$ is of full rank. Consequently, these sequences converge to a non-zero number, i.e., to blocks of type (ii). Analogously, $\left(\frac{(B_n)_{11}}{(D_n)_{11}}\right)_{n\in\mathbf{N}}$ converges to a non-zero number since $P$ has no trivial row and $Q$ is of full rank. This finishes the proof that $P$ and $Q$ are of the described form.

Conversely, assume there exist $[P], [Q] \in \mathcal{y}$ with $P, Q$ in the described form. Let $U_{[p]}$ and $U_{[q]}$ be arbitrary open neighborhoods of $[p]$ resp. $[q]$ in $\mathcal{y}$. Further, let $\mathcal{Y} = \pi^{-1}(\mathcal{y})$, and let $B_r(P)$ be the open ball with radius $r$ and center $P$ in the space $\mathcal{Y}$ of matrix configurations equipped with the max norm. Then, there is a $\delta > 0$ such that $B_\delta(P) \subseteq \pi^{-1}\big(U_{[p]}\big)$ and $B_\delta(Q) \subseteq \pi^{-1}\big(U_{[q]}\big)$. We will construct a configuration $\tilde{A} \in \mathcal{G}_d^k \subseteq \mathcal{Y}$ such that its shape $[\tilde{A}]$ is an element of both $\pi\big(B_\delta(P)\big)$ and $\pi\big(B_\delta(Q)\big)$. For $n \in \mathbf{N}$ consider block diagonal matrices

$$\tilde{D}_n = \begin{pmatrix} n^{d_1}\tilde{D}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & n^{d_l}\tilde{D}_l \end{pmatrix} \in \mathbf{Diag}^*(k)$$

and

$$\tilde{B}_n = \begin{pmatrix} n^{-b_1}\tilde{B}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & n^{-b_m}\tilde{B}_m \end{pmatrix} \in \mathbf{GL}(d+1)$$

with non-singular diagonal matrices $\tilde{D}_r$, non-singular matrices $\tilde{B}_s$, and speeds of convergence $d_r, b_s \in \mathbf{N}_0$ such that

- $b_r > b_s$, $d_r > d_s$ for all $r > s$;
- $b_s = d_r$ and $\tilde{D}_r = D_r, \tilde{B}_s = B_s$ for pairs $(r, s)$ with $P_{rs}, Q_{rs} \neq 0$, and thus $Q_{rs} = D_rP_{rs}B_s$ (blocks of type (ii));
- otherwise, $b_s \neq d_r$ and $\tilde{D}_r = \mathbf{I}, \tilde{B}_s = \mathbf{I}$; more precisely, let $b_s > d_r$ for all $(r, s)$ with $P_{rs} \neq 0$ (blocks of type (iv)), while $b_s < d_r$ for all $(r, s)$ with $Q_{rs} \neq 0$ (blocks of type (iii)).

Then,

$$\max\big\{n^{b_s-d_r} : (r, s) \text{ with } Q_{rs} \neq 0, P_{rs} = 0\big\} \leqslant n^{-1}$$

and

$$\max\{n^{d_r-b_s} : (r,s) \text{ with } P_{rs} \neq 0, Q_{rs} = 0\} \leqslant n^{-1}.$$

Further, define a configuration $A = (A_{rs})$ with the same block structure as $P$ and $Q$ and entries

$$A_{rs} = \begin{cases} P_{rs} & \text{if } P_{rs} \neq 0, \\ n^{b_s-d_r}\tilde{D}_r^{-1}Q_{rs}\tilde{B}_s^{-1} & \text{if } P_{rs} = 0. \end{cases}$$

The equivalent configuration $\tilde{D}A\tilde{B}$ is then given by

$$(\tilde{D}A\tilde{B})_{rs} = \begin{cases} Q_{rs} & \text{if } P_{rs} = 0, \\ n^{d_r-b_s}\tilde{D}_r P_{rs}\tilde{B}_s & \text{if } P_{rs} \neq 0. \end{cases}$$

Now, choose $n$ large enough such that

$$n^{-1} \cdot \max_{(r,s)}\Big\{\|\tilde{D}_r P_{rs}\tilde{B}_s\|_{\max}, \|\tilde{D}_r^{-1}Q_{rs}\tilde{B}_s^{-1}\|_{\max}\Big\} < \delta,$$

whence both $\|A - P\|_{\max} < \delta$ and $\|\tilde{D}A\tilde{B} - Q\|_{\max} < \delta$, i.e., $[A] \in \pi(B_\delta(P)) \cap \pi(B_\delta(Q))$ as subsets of $\mathcal{a}_d^k$. Since $\mathcal{g}_d^k$ is dense in $\mathcal{a}_d^k$ by Lemma 3.6 and $\pi(B_\delta(P)) \cap \pi(B_\delta(Q)) \neq \varnothing$ is open, there is a shape $[\tilde{A}] \in \mathcal{g}_d^k$ such that

$$[\tilde{A}] \in \pi(B_\delta(P)) \cap \pi(B_\delta(Q)) \cap \mathcal{g}_d^k \subseteq U_{[p]} \cap U_{[q]}.$$

Consequently, $U_{[p]}$ and $U_{[q]}$ are not disjoint, whence $\mathcal{y}$ is not Hausdorff. □

Proposition 3.12 shows that neither the topological subspace $\mathcal{f}_d^k$ of free shapes nor the topological subspace $\mathcal{p}_d^k$ of shapes with a frame is Hausdorff for any $k > d+2$: the configurations

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

are elements of $\mathcal{P}_d^{d+3} \subseteq \mathcal{F}_d^{d+3}$ since the first resp. last $d+2$ landmarks form a frame. Thus, $\mathcal{p}_d^{d+3}$ is not Hausdorff since $[P]$ and $[Q]$ are of the described form of Proposition 3.12 (see also Figure 3.2) and $\mathcal{g}_d^k \subset \mathcal{p}_d^k \subseteq \mathcal{f}_d^k$. For $k > d+3$, some of the landmarks may be repeated.

**Example 3.13.** In the case $d = 1$ and $k = 4$, the discussion above shows that e.g. the topological subspace $\mathcal{y}$ comprising of the shapes in general position and the single pair coincidences $[p], [q]$ with three distinct landmarks $p_1, p_2, p_3$ resp. $q_2, q_3, q_4$, but $p_3 = p_4$ resp. $q_1 = q_2$, is not Hausdorff. In fact, then

$$[p] = \left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\right] \quad \text{and} \quad [q] = \left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\right]$$

with $l = m = 2$. Recall that $\mathcal{y}$ is T1 since all shapes of $\mathcal{y}$ are free. ◊

Proposition 3.12 simplifies for topological subspaces respecting the hierarchy of projective subspace constraints.

**Corollary 3.14.** *Let $\mathscr{y} \subseteq \mathscr{r}_d^k$ with $\mathscr{g}_d^k \subseteq \mathscr{y}$. If $\mathscr{y}$ is not Hausdorff, then there are two distinct shapes $[p], [q] \in \mathscr{y}$, $[p] \neq [q]$ such that $(I, j) \in C(p)$ and $(I^c, d+1-j) \in C(q)$. If $\mathscr{y}$ additionally respects the hierarchy of projective subspace constraints, the converse statement is also true.*

Proof. The first statement follows immediately from the block structure of $[p]$ and $[q]$ in Proposition 3.12, simply let $I$ contain the rows below the upper left block of $[p]$ as in Equation (3.5), and let $j$ be the number of columns to the right of the upper left block.

For the converse statement in case that $\mathscr{y}$ respects the hierarchy of projective subspace constraints, let $[p], [q] \in \mathscr{y}$, $[p] \neq [q]$, be distinct shapes with $(I, j) \in C(p)$ and $(I^c, d+1-j) \in C(q)$. Then, there are representatives such that after simultaneous reordering of the rows

$$[p] = \left[\begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix}\right] \quad \text{and} \quad [q] = \left[\begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix}\right]$$

with $P_{11}, Q_{11} \in \mathbf{R}^{|I| \times j}$, etc. Consider the matrices

$$[r] = \left[\begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}\right] \quad \text{and} \quad [s] = \left[\begin{pmatrix} R_{11} & S_{12} \\ 0 & R_{22} \end{pmatrix}\right]$$

in the same block structure with $R_{11} \in \mathcal{G}_{j-1}^{|I|}$, $R_{22} \in \mathcal{G}_{d-j}^{|I^c|}$, as well as $R_{21} \in \mathcal{G}_{j-1}^{|I|^c}$ and $S_{12} \in \mathcal{G}_{d-j}^{|I|}$ such that the subconfigurations

$$\begin{pmatrix} R_{21} & R_{22} \end{pmatrix} \in \mathcal{G}_d^{|I^c|} \quad \text{and} \quad \begin{pmatrix} R_{11} & S_{12} \end{pmatrix} \in \mathcal{G}_d^{|I|},$$

i.e., $[r_{I^c}] \in \mathscr{g}_d^{|I^c|}$ and $[s_I] \in \mathscr{g}_d^{|I|}$ are in general position. Then, $[r]$ and $[s]$ are less or equally constrained than $[p]$ resp. $[q]$ and, since $\mathscr{y}$ respects the hierarchy of projective subspace constraints, contained in $\mathscr{y}$. Since $[r]$ and $[s]$ are of the block structure as in Proposition 3.12, $\mathscr{y}$ is not Hausdorff. □

## 3.4 Topological subspaces with manifold structure

In the requirements of a good topological subspace of $\mathscr{a}_d^k$, we listed "differentiable Hausdorff manifold." We already know from the previous section how to check the Hausdorff property, but we still do not know which topological subspaces of $\mathscr{a}_d^k$ are topological manifolds and can be given a differentiable structure.

Before we begin constructing charts, recall that a *topological manifold* $\mathcal{M}$ of dimension $n$ is a second-countable topological space for which every element $p \in \mathcal{M}$ has a neighborhood $U$ that is homeomorphic to an open subset $V$ of $\mathbf{R}^n$. Such a homeomorphism

$$\varphi : \ U \longrightarrow V$$

is called *(coordinate) chart*, and a family of charts whose domains cover $\mathcal{M}$ is called an *atlas*. If $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are two charts with $U_1 \cap U_2 \neq \emptyset$, the composite map $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$ is called the *transition map* between these two charts. Two charts $\varphi_1$ and $\varphi_2$ are *compatible* if either $U_1 \cap U_2 = \emptyset$ or its transition map is a diffeomorphism. A *differentiable manifold* $\mathcal{M}$ of dimension $n$ is a topological manifold together with a *differentiable structure*, i.e. a maximal atlas of compatible charts.

While we do not require a manifold to be Hausdorff, a manifold is T1 by our definition: let $p, q \in \mathcal{M}$ be two elements of the manifold $\mathcal{M}$. Then, there are charts $\varphi_p : U_p \to V_p$ and $\varphi_q : U_q \to V_q$ with $p \in U_p$ and $q \in U_q$. If $p \notin U_q$ and $q \notin U_p$, then $U_p$ and $U_q$ are the requested open neighborhoods of $p, q$. If $p, q$ are in the domain of the same chart, i.e., $p \in U_q$ or $q \in U_p$, then $p$ and $q$ can be separated in $U_q$ resp. $U_p$ and $\mathcal{M}$ even in the Hausdorff sense since $U_q$ resp. $U_p$ is open in $\mathcal{M}$ and homeomorphic to the Hausdorff space $V_q \subseteq \mathbf{R}^n$ resp. $V_p \subseteq \mathbf{R}^n$.

From Proposition 3.10 we concluded that, for $k \geqslant d+2$, the space $\mathfrak{f}_d^k$ of the free shapes is the largest T1 subspace of $\mathfrak{a}_d^k$ which respects the hierarchy of projective subspace constraints. Indeed, $\mathfrak{f}_d^k$ will turn out to be a differentiable manifold (Theorem 3.24), as it is the case for similarity and affine shape space. This is true even for $k \leqslant d+2$ since, by Proposition 3.1, $\mathfrak{f}_d^{d+2}$ is—as a singleton—a zero-dimensional manifold, while $\mathfrak{f}_d^k = \varnothing$ for $k < d+2$. However, we will continue to consider just the case $k > d+2$.

Before constructing compatible charts on $\mathfrak{f}_d^k$, we will shortly recall the approach to projective shape space by Mardia and Patrangenaru (2005) via a topological subspace defined through projective frames: since there is a unique projective transformation mapping a frame $(p_1, \ldots, p_{d+2})$ of $d+2$ points in general position to another frame $(q_1, \ldots, q_{d+2})$, see Theorem 2.1, the group action of $\mathbf{PGL}(d)$ on $\mathcal{A}_d^k$ can be removed from a projective shape $[p]$ by choosing the representation $q$, $\pi(q) = [p]$, with a fixed frame. Of course, this standardization is only possible if the shape contains a frame, and one quickly obtains that the topological subspace of shapes with a frame in a fixed subset of $d+2$ landmarks is a differentiable Hausdorff manifold.

**Lemma 3.15 (Mardia and Patrangenaru (2005), Prop. 2.3).** *Let $\mathcal{B}_d^k \subseteq \mathcal{A}_d^k$ the topological subspace of configurations with a frame in its first $d+2$ landmarks. The corresponding shape space $\mathfrak{b}_d^k$ is then homeomorphic to the $d(k-d-2)$-dimensional differentiable Hausdorff manifold*

$$\mathfrak{b}_d^k \cong \left(\mathbf{RP}^d\right)^{k-d-2} \tag{3.7}$$

*and respects the hierarchy of projective subspace constraints.*

Note that this statement is the projective analogon to Bookstein coordinates in similarity shape space, cf. Bookstein (1986). It has first been discovered in the case $d = 2$ by Goodall and Mardia (1999) and by Horadam (1970).

Proof. $\mathbf{PGL}(d)$ acts transitively and effectively on frames (see Theorem 2.1), hence there is a unique configuration $q$ representing the shape $[p]$ such that $q_i = \pi_p(e_i)$ for all $1 \leqslant i \leqslant d+1$ and $q_{d+2} = \pi_p(e_1 + \cdots + e_{d+1})$ with $e_i$ denoting the $i$-th canonical basis vector of $\mathbf{R}^{d+1}$. Mapping $[p]$ to $q_{\{d+3,\ldots,k\}}$ then gives a homeomorphism to $(\mathbf{RP}^d)^{k-d-2}$. In matrix notation, there is a representation $Q$ of the form

$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} & & \mathbf{I}_{d+1} & & \\ 1 & 1 & \cdots & 1 & 1 \\ & & Q_{\{d+3,\ldots,k\}} & & \end{pmatrix} \tag{3.8}$$

for any $[p] \in \mathfrak{b}_d^k$ which is unique up to rescaling of the last $k-d-2$ rows. □

**Example 3.16.** By Lemma 3.15, $\mathfrak{b}_2^5$ is homeomorphic to $\mathbf{RP}^2$, and projective shapes in $\mathfrak{b}_2^5$ can be visualized as elements in $\mathbf{R}^2 \subset \mathbf{RP}^2$ using inhomogeneous coordinates, see Equation (2.9): any shape $[p] \in \mathfrak{b}_2^5$ possesses a representation $P$ such that the projective frame in the shape's first 4 landmarks is standardized to a square in $\mathbf{R}^2$, i.e. e.g.

$$P = \begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ P_{51} & P_{52} & P_{53} \end{pmatrix}.$$

In this representation, $P_{53}$ is either 0 (if the fifth landmark is at infinity) or can be chosen to be 1. In the latter case, $\big(P_{51}, P_{52}\big)$ are the coordinates of the fifth landmark in $\mathbf{R}^2$. $\Diamond$

While $\mathfrak{b}_d^k$ obviously respects the hierarchy of projective subspace constraints, it is not closed under relabeling for $k \geqslant d+3$ since $\mathcal{B}_d^k$ then includes the configuration with $p_{\{1,\ldots,d+2\}}$ being a frame and $p_i = p_{d+2}$ for all $i \in \{d+3, \ldots, k\}$, but not all of its permutations.

The largest topological subspace of $\mathfrak{b}_d^k$ which is closed under relabeling is the subspace $\mathfrak{g}_d^k$ of shapes in general position.

**Corollary 3.17.** *The topological subspace $\mathfrak{g}_d^k$ of shapes in general position is homeomorphic to a $d(k-d-2)$-dimensional differentiable Hausdorff manifold, closed under permutations and respects the hierarchy of projective subspace constraints.*

Proof. $\mathfrak{g}_d^k$ is open in $\mathfrak{b}_d^k$ due to Proposition 3.7 since it respects the hierarchy of projective subspace constraints. An open topological subspace of a manifold is a manifold of the same dimension itself which can be seen by restriction of the corresponding charts. $\square$

Unfortunately, $\mathfrak{g}_d^k$ has the drawback of not being maximal unless $k = 4$ and $d = 1$, as we will see in Section 3.5.

The closure of $\mathfrak{b}_d^k$ under permutations is by definition the topological subspace $\mathfrak{p}_d^k$ of shapes containing a frame in arbitrary $d+2$ landmarks. While $\mathfrak{p}_d^k$ is a differentiable manifold, it is not Hausdorff for any $d \geqslant 1$, $k \geqslant d+3$, see Proposition 3.12.

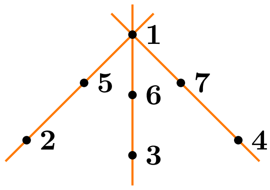
**Corollary 3.18.** *The topological space $\mathfrak{p}_d^k$ of shapes with a frame is homeomorphic to a differentiable T1 manifold of dimension $d(k-d-2)$.*

Proof. From Lemma 3.15 we obtain homeomorphisms from the topological subspaces of shapes with a frame in a fixed subset of $d+2$ landmarks to $(\mathbf{RP}^d)^{k-d-2}$. These subspaces of shapes with a frame respect the hierarchy of projective subspace constraints and are thus open in $\mathfrak{a}_d^k$ and $\mathfrak{p}_d^k$ due to Proposition 3.7. Therefore, these homeomorphisms can be considered as "manifold-valued" charts on $\mathfrak{p}_d^k$. To obtain ordinary charts on $\mathfrak{p}_d^k$, one can compose the manifold-valued charts with charts on the manifold $(\mathbf{RP}^d)^{k-d-2}$, e.g. inhomogeneous coordinates. The transition maps are then just multiplications with non-singular diagonal and non-singular matrices as well as division by non-vanishing parameters depending smoothly on the representation matrix, whence we obtain compatible charts on $\mathfrak{p}_d^k$ rendering $\mathfrak{p}_d^k$ a differentiable T1 manifold of dimension $d(k-d-2)$. See the more general setup presented in the later Example 3.22 for a detailed construction of these charts.

At the end of Section 3.3 (page 34), it was shown that $\mathfrak{p}_d^k$ is not Hausdorff. $\square$

Unfortunately, for $d \geqslant 3$ there are free shapes that do not include a frame: e.g. for $d = 3$, take three projective lines in $\mathbf{RP}^3$ with their join being $\mathbf{RP}^3$ and their section being a point, i.e., three non-coplanar lines with a common intersection point. Put two distinct landmarks on each line and another on the intersection point (see Figure 3.3). The resulting configuration of seven landmarks is free since it is not decomposable; if it was decomposable, then the seven landmarks would either decompose into a set of landmarks lying in a projective hyperplane and a set of coinciding landmarks (decomposition $(I, 3), (I^c, 1)$) or into two sets of landmarks lying in projective lines (decomposition $(I, 2), (I^c, 2)$), which is not the case. However, this configuration does not contain a frame since there are no $d+2 = 5$ landmarks in general position. The same argument works for $k = 6$ when the landmark on the intersection point is removed. For $k \geqslant 8$ free shapes without a frame are constructed by multiplication of landmarks. Free shapes without a frame can be constructed analogously for any $d > 3$ by using $d$ non-coplanar projective lines with a common intersection point.

Hence, a free shape does not necessarily contain a frame, and thus frames only form an atlas on $\mathfrak{p}_d^k \subseteq \mathfrak{f}_d^k$, but not on $\mathfrak{f}_d^k$ for $d \geqslant 3$.




$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

**Figure 3.3:** A free configuration $P \in \mathcal{F}_3^7$ in $\mathbf{RP}^3$ without a frame. All landmarks are distinct and lie on three non-coplanar lines with common intersection point; landmark 1 lies on the intersection point.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$G(P)$

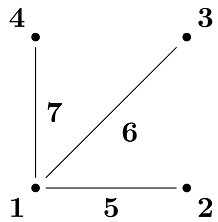



**Figure 3.4:** The free configuration $P \in \mathcal{A}_3^7$ without a frame from Figure 3.3 and its graph corresponding to the base points $P_{\{1,2,3,4\}}$.

From Proposition 3.1, we know that free shapes are non-decomposable, and vice versa. In particular, a free shape $[p]$ is of full rank, i.e., there are at least $d+1$ landmarks in general position in $[p]$, say, its first $d+1$ landmarks. Let $P = \binom{P_0}{P_1}$ be a matrix configuration with $[P] = [p]$ and $P_0$ the submatrix consisting of the first $d+1$ rows of $P$. Then, $P_0 \in \mathbf{GL}(d+1)$, whence

$$\tilde{P} = PP_0^{-1} = \begin{pmatrix} \mathbf{I}_{d+1} \\ \tilde{P}_1 \end{pmatrix} \tag{3.9}$$

is also a matrix configuration of shape $[p]$ with $\tilde{P}_1 = P_1 P_0^{-1}$ consisting of non-trivial rows. To such a configuration $P$, define an edge-colored, undirected *graph* $G(P) = \big(V(P), E(P)\big)$ by taking the columns of $\tilde{P}$ as vertices, i.e., $V(P) = \{1, \ldots, d+1\}$. Let there an edge labeled with "$l$" between two distinct vertices $i$ and $j$ if both $\tilde{P}_{li} \neq 0$ and $\tilde{P}_{lj} \neq 0$ for $l \in \{d+2, \ldots, k\}$ (see Figure 3.4 as an example). The set of edges $E(P) = \bigcup_{l=d+2}^{k} E_l$ has a partition into sets of edges $E_l$ labeled with "color" $l \in \{d+2, \ldots, k\}$. Note that multiple edges between two vertices which are labeled differently are allowed. Loops are not allowed, though.

Note that this definition of the graph $G(P)$ of a configuration $P$ with its first $d+1$ landmarks in general position is well-defined and invariant under $\mathbf{PGL}(d)$: for an equivalent configuration $Q = DPB$ let $D_0 \in \mathbf{Diag}^*(d+1)$ be the upper left square block of $D$ with $d+1$ rows, $D_1 \in \mathbf{Diag}^*(k-d-1)$ be the lower right square block of $D$ with $k-d-1$ rows, $Q_0 \in \mathbf{GL}(d+1)$ be the first $d+1$ landmarks (here: rows) of $Q$, and $Q_1$ be the last $k-d-1$ landmarks of $Q$, i.e.,

$$\begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} = \begin{pmatrix} D_0 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \end{pmatrix} B,$$

and in particular $Q_i = D_i P_i B$, $i = 0, 1$. Then, $\tilde{Q}_1$ in $\tilde{Q} = QQ_0^{-1}$ is given by

$$Q_1 Q_0^{-1} = D_1 P_1 B \big(D_0 P_0 B\big)^{-1} = D_1 P_1 P_0^{-1} D_0^{-1},$$

whence $\tilde{P}_1$ is only unique up to left- and right-multiplication by non-singular diagonal matrices. However, these actions do not affect the graph $G(P)$ since they only rescale rows and columns.

Of course, this definition of the graph is only well-defined if the configuration's resp. shape's first $d+1$ landmarks are in general position. It can, however, easily be extended to any configuration resp. shape with fixed $d+1$ landmarks in general position by mapping the corresponding submatrix to the identity matrix. These distinguished landmarks are said to be the *base points.* Note that the ordering of the chosen landmarks is not critical for the graph since a permutation of the landmarks will only permute the vertices. Without restriction, we will assume the base points to be in ascending order.

It turns out that the graph of a shape encodes algebraic information. In fact, the graph is connected if and only if the shape is free.

**Proposition 3.19.** *Let $P \in \mathscr{A}_d^k$ be a configuration with its first $d+1$ landmarks in general position. Then, $G(P)$ is connected if and only if $P$ is free.*

PROOF. If $G(P)$ is not connected, then the vertices of $G(P)$, i.e. the columns of $\tilde{P}$ in Equation (3.9), split into two or more connected components $J_1, \ldots, J_s$, $s \geqslant 2$, with $J_r \neq \varnothing$ for all $r \in \{1, \ldots, s\}$ and $\sum_{r=1}^{s} |J_r| = d+1$. Define

$$I_r = \left\{ i \in \{1, \ldots, k\} : p_i \in \bigvee_{j \in J_r} p_j \right\}, \quad r = 1, \ldots, s,$$

as the set of rows of $\tilde{P}$ which are in the projective subspace spanned by the base points numbered by elements of $J_r$. Then, $\big(I_r, |J_r|\big) \in C(P)$ for all $r \in \{1, \ldots, s\}$. The sets $I_r$ are pairwise disjoint since the $J_r$ are pairwise disjoint and the first $d+1$ landmarks of $P$ are in general position. Further, $\bigcup_{r=1}^{s} I_r = \{1, \ldots, k\}$. Hence,

$$\big\{ \big(I_r, |J_r|\big) : r = 1, \ldots, s \big\}$$

is the total decomposition of $P$, i.e., $P$ is decomposable, and thus—according to Proposition 3.1—not free.

Conversely, suppose that $G(P)$ is connected. Without restriction we can assume that $P = \tilde{P}$, i.e., $P_{\{1,\ldots,d+1\}} = \mathbf{I}_{d+1}$. Suppose that there are matrices $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_k) \in \mathbf{Diag}^*(k)$ and $B \in \mathbf{GL}(d+1)$ leaving $P$ unchanged, i.e., $DPB = P$. Then, Equation (3.9) implies

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_{d+1}) \mathbf{I}_{d+1} B = \mathbf{I}_{d+1}$$

for the first $d+1$ rows of $P$, and consequently $B = \operatorname{diag}(\lambda_1^{-1}, \ldots, \lambda_{d+1}^{-1})$. If two columns $i, j$ are connected by an edge in $G(P)$, then there is a row $P_{l\cdot}$ such that both $P_{li} \neq 0$ and $P_{lj} \neq 0$ by the definition of the graph. From the identities

$$P_{li} = \lambda_l \big(PB\big)_{li} = \lambda_l P_{li} \lambda_i^{-1} \qquad \text{and} \qquad P_{lj} = \lambda_l \big(PB\big)_{lj} = \lambda_l P_{lj} \lambda_j^{-1},$$

we then conclude

$$\lambda_i = \lambda_j = \lambda_l,$$

and thus $\lambda_1 = \ldots = \lambda_{d+1}$ since $G(P)$ is connected. Consequently, the isotropy group of $P$ comprises of multiples of the identity matrix $D = \lambda_1 \mathbf{I}_k$ and $B = \lambda_1^{-1} \mathbf{I}_{d+1}$, i.e., $P$ is free. □

In the following the $d+1$ base points together with a connected *tree* $G$ with edges labeled with the remaining landmarks will be called a *pseudo-frame*. Note that a tree contains no circles and gets disconnected if an edge is removed; hence, it is a minimal substructure of a connected graph. A shape $[p]$ is said to contain the pseudo-frame $\big(\{i_1, \ldots, i_{d+1}\}, G\big)$ if its landmarks with indices $\{i_1, \ldots, i_{d+1}\}$ are in general position and the corresponding graph of $[p]$ has $G$ as a subgraph. From Proposition 3.19 we immediately conclude that $f_d^k$ comprises of the shapes which contain a pseudo-frame.

**Corollary 3.20.** *A shape is free if and only if it contains a pseudo-frame.*

Proof. If a shape is free, then its graph to some ordered set of $d+1$ landmarks in general position is connected by Proposition 3.19. A spanning tree of this graph together with the $d+1$ landmarks gives a pseudo-frame.

Vice versa, if a shape contains a pseudo-frame, then the graph corresponding to the base points is connected, and the shape is free. $\square$

The shapes including a fixed pseudo-frame form a differentiable Hausdorff manifold, generalizing Lemma 3.15.

**Proposition 3.21.** *Let $\big(\{i_1,\dots,i_{d+1}\},G\big)$ be a pseudo-frame with tree $G=\big(\{1,\dots,d+1\},E\big)$. Denote the number of edges in $G$ labeled with the landmark $l$ by $|E_l|$, and let $\#E=\big|\{l : E_l\neq\varnothing\}\big|$ be the number of colors in $G$. The topological subspace of all shapes containing the pseudo-frame $\big(\{i_1,\dots,i_{d+1}\},G\big)$ is then homeomorphic to the $d(k-d-2)$-dimensional differentiable Hausdorff manifold*

$$\big(\mathbf{RP}^d\big)^{k-d-1-\#E}\times\bigtimes_{\substack{l=d+2:\\ E_l\neq\varnothing}}^{k}\mathbf{R}^{d-|E_l|}. \tag{3.10}$$

Proof. Note that $\sum_{l=d+2}^{k}|E_l|=d$ is the number of edges in the tree $G$ with $d+1$ vertices. Then, the dimension of the final factor of the product is

$$\sum_{\substack{l=d+2:\\ E_l\neq\varnothing}}^{k} d-|E_l| = \#E\cdot d-\sum_{l=d+2}^{k}|E_l| = d\big(\#E-1\big),$$

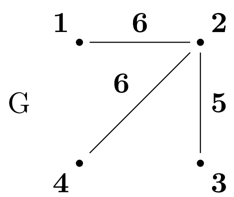
whence the dimension of the product is $d(k-d-2)$.

To construct a homeomorphism, the idea is to give a standardized matrix configuration to any shape with pseudo-frame $\big(\{i_1,\dots,i_{d+1}\},G\big)$. Let $[p]\in\mathfrak{a}_d^k$ be a shape with this pseudo-frame, and let $P$ be any matrix configuration of this shape. Then, there is a unique matrix $A\in\mathbf{GL}(d+1)$ such that $P_{\{i_1,\dots,i_{d+1}\}}A=\mathbf{I}_{d+1}$, namely $A=P^{-1}_{\{i_1,\dots,i_{d+1}\}}$. Additionally, there are non-singular matrices $D\in\mathbf{Diag}^*(k)$ and $B\in\mathbf{Diag}^*(d{+}1)\subset\mathbf{GL}(d{+}1)$ such that $(DPAB)_{li}=1$ for columns $i\in\{1,\dots,d+1\}$ with an adjacent edge labeled with $l$ while still $(DPAB)_0=\mathbf{I}_{d+1}$. Note that $B$ is unique[1] while $D_{ii}$ is only unique for $i\in\{1,\dots,d+1\}\cup\{l : E_l\neq\varnothing\}$. Then, a homeomorphism mapping $[p]$ to

$$\big(\mathbf{RP}^d\big)^{k-d-1-\#E}\times\bigtimes_{\substack{l=d+2:\\ E_l\neq\varnothing}}^{k}\mathbf{R}^{d-|E_l|}$$

is defined in the following way: the rows of $DPAB$ with numbers not in $E$ are only well-defined up to rescaling, i.e., they are elements of $\mathbf{RP}^d$, and they will be mapped to $\big(\mathbf{RP}^d\big)^{k-d-1-\#E}$. The rows of $DPAB$ with numbers in $\{l : E_l\neq\varnothing\}$ will be mapped to $\mathbf{R}^{d-|E_l|}$ by omitting the entries fixed to 1; see the following Example 3.22. $\square$

**Example 3.22.** Consider a shape $[p]\in\mathfrak{a}_3^k$ with pseudo-frame $\big((1,2,3,4),G\big)$ with $G$ being the following edge-coloured tree:




[1] Uniqueness is—of course—only given up to the usual scalar multiplication of $D$ and $B$

Here, $\#E = 2$, $|E_5| = 1$ and $|E_6| = 2$. Let $P$ be a corresponding matrix configuration. The standardization described in the proof of Proposition 3.21 brings $P$ then to a matrix of form

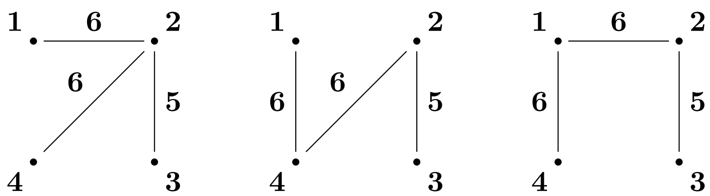
$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ Q_{51} & 1 & 1 & Q_{54} \\ 1 & 1 & Q_{63} & 1 \\ & \multicolumn{2}{c}{Q_{\{7,\dots,k\}}} & \end{pmatrix} \tag{3.11}$$

for some $Q_{51}, Q_{54}, Q_{63} \in \mathbf{R}$ and some $Q_{\{7,\dots,k\}} \in \mathcal{A}_3^{k-6}$. The configuration $Q$ is, for $\tilde{P} = PP_{\{1,2,3,4\}}^{-1}$, given by the matrix multiplication

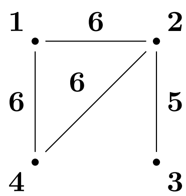
$$Q = \underbrace{\operatorname{diag}\left(\tilde{P}_{61}, \tilde{P}_{62}, \tilde{P}_{53}, \tilde{P}_{64}, \tilde{P}_{62}\tilde{P}_{52}^{-1}, 1, D_{77}, \dots\right)}_{D} \cdot P \cdot \underbrace{P_{\{1,2,3,4\}}^{-1}}_{A} \cdot \underbrace{\operatorname{diag}\left(\tilde{P}_{61}^{-1}, \tilde{P}_{62}^{-1}, \tilde{P}_{53}^{-1}, \tilde{P}_{64}^{-1}\right)}_{B} . \tag{3.12}$$

Since the entries $D_{77}, \dots, D_{kk} \in \mathbf{R}^*$ are arbitrary, $Q_{\{7,\dots,k\}}$ is only unique up to left-multiplication with non-singular diagonal matrices, i.e. a configuration in $\mathcal{A}_3^{k-6} = \left(\mathbf{RP}^3\right)^{k-6}$. In contrast, the entries $Q_{51}, Q_{54}, Q_{63} \in \mathbf{R}$ are uniquely given by Equation (3.12). Hence, we obtain that the topological subspace of shapes with this pseudo-frame is homeomorphic to $\left(\mathbf{RP}^3\right)^{k-6} \times \mathbf{R}^3$, as it was proposed by Proposition 3.21. Note that the standardization $Q$ depends smoothly on the entries of $P$. ◊

**Remarks 3.23.** (a) In Example 3.22 the following three edge-colored trees give the same standardization, and consequently the same homeomorphisms:




And this edge-colored graph does give it, too:




In general, the construction of the homeomorphism in the proof of Proposition 3.21 does not change if in the definition of a pseudo-frame one allows $G$ to be an edge-colored tree, but with its unicolored subgraphs completed. These kind of graphs are called trees of cliques.

(b) Considering only the distinguished landmarks, a frame gives rise to a complete graph with only one color (see Figure 3.5). A frame is thus a pseudo-frame with a unicolored tree $G$, i.e., $\#E = 1$. In particular, Proposition 3.21 is a generalization of Lemma 3.15, as are the correspondent standardizations. For $d = 1$, the notion of pseudo-frame and frame are identical since there is only one tree with two vertices which is automatically unicolored since it only has one edge. ◊

Now, similar to the situation with $\mathfrak{b}_d^k$ and $\mathfrak{p}_d^k$ (see Lemma 3.15 and Corollary 3.18), Proposition 3.21 gives us finitely many, manifold-valued charts for the topological subspace $\mathfrak{f}_d^k$ of free shapes, whence we can now prove that $\mathfrak{f}_d^k$ is a differentiable manifold.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$G(P)$

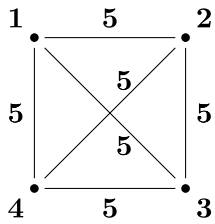



**Figure 3.5:** A projective frame $P \in \mathcal{G}_3^5$ and its complete graph $G(P)$ corresponding to the base points $P_{\{1,2,3,4\}}$. All spanning trees of $G(P)$ give a pseudo-frame.

**Theorem 3.24.** *The topological space $\mathscr{f}_d^k$ of free shapes is a $d(k-d-2)$-dimensional differentiable manifold.*

PROOF. $\mathscr{f}_d^k$ is the topological subspace of shapes which contain a pseudo-frame. Proposition 3.21 gives us homeomorphisms from open subsets of $\mathscr{f}_d^k$ to differentiable manifolds, i.e. manifold-valued charts. As in the proof of Corollary 3.18, charts on $\mathscr{f}_d^k$ are then obtained by composition of these manifold-valued charts with charts of the differentiable manifolds. In the matrix notation as in Example 3.22, the corresponding transition maps are then just multiplications with non-singular diagonal and non-singular matrices depending smoothly on the representation matrix. Consequently, $\mathscr{f}_d^k$ is a differentiable manifold. □

We would like to point out that, for $d = 1$ and $d = 2$, any shape with a pseudo-frame already contains a frame, i.e., $\mathscr{f}_d^k = \mathscr{p}_d^k$ for $d = 1, 2$. While pseudo-frames are already frames in the case $d = 1$, the critical shape to consider in the case $d = 2$ is (in the form of Equation (3.9))

$$[p] = \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ u & v & w \\ x & y & z \\ & \vdots & \end{pmatrix} \right].$$

Let there be a pseudo-frame, say, in the first five rows of $[p]$. The shape $[p]$ contains a frame if either all of $u, v, w \neq 0$ or all of $x, y, z \neq 0$. So, assume that there is a vanishing value in both of the rows. Since there is a pseudo-frame in the first five rows of $[p]$, i.e., since the three columns are connected by the rows 4 and 5, there is at most one vanishing value in these rows, and it cannot be in the same column. For the sake of argument, let $u, y = 0$ be the vanishing values. Then, $p_{\{1,2,4,5\}}$ is a frame. Consequently, the differentiable manifolds $\mathscr{f}_1^k$ and $\mathscr{f}_2^k$ are already covered by the charts associated to frames. On $\mathscr{f}_2^k$ pseudo-frames give a larger atlas.

Open subsets of $\mathscr{f}_d^k$ are differentiable manifolds. For topological subspaces respecting the hierarchy of projective subspace constraints, we obtain the following result.

**Corollary 3.25.** *Let $\mathscr{y} \subseteq \mathscr{a}_d^k$ be a Hausdorff subspace respecting the hierarchy of projective subspace constraints. Then, $\mathscr{y} \subseteq \mathscr{f}_d^k$ is a differentiable submanifold.*

PROOF. If $\mathscr{y}$ respects the hierarchy of projective subspace constraints, then it is an open subset of $\mathscr{a}_d^k$ due to Lemma 2.6.

Further, $\mathscr{y}$ is Hausdorff if and only if there are no shapes $[p], [q] \in \mathscr{y}$ with $(I, j) \in C(p)$ and $(I^c, d+1-j) \in C(q)$ by Corollary 3.14. Consequently, there is no decomposable shape $[r] \in \mathscr{y}$ with $(I, j), (I^c, d+1-j) \in C(r)$ since $\mathscr{y}$ respects the hierarchy of projective subspace constraints. Hence, $\mathscr{y}$ is an open subset of $\mathscr{f}_d^k$ and thus a differentiable manifold. □

Proposition 3.2 states that $\mathscr{a}_d^k$ is a stratified space with its strata being products of spaces of free shapes. Using Theorem 3.24 we conclude that the strata are differentiable manifolds with its top stratum being $\mathscr{f}_d^k$. Recall that Proposition 3.10 states how the strata are glued together.

## 3.5 Topological subspaces bounded by projective subspace numbers

One class of topological subspaces being closed under permutations and respecting the hierarchy of projective subspace constraints is the class of *topological subspaces bounded by projective subspace numbers*: to a vector $n = (n_1, \dots, n_d) \in \mathbf{N}^d$ with $1 < n_1 < n_2 < \dots < n_d$ define the topological subspace

$$\mathcal{N}_d^k(n) = \{p \in \mathcal{A}_d^k : |I| \leqslant n_j \text{ for all } (I, j) \in C(p)\}, \tag{3.13}$$

comprising of those configurations $p$ for which there are at most $n_j$ landmarks in any $(j-1)$-dimensional projective subspace of $\mathbf{RP}^d$, cf. Section 3.1. Recall from Proposition 3.1 that $\mathcal{N}_d^k(n)$ contains $\mathcal{G}_d^k$ for all feasible $n \in \mathbf{N}^d$ since $n_j \geqslant j$ for all $1 \leqslant j \leqslant d$, while $\mathcal{N}_d^k(n) = \mathcal{G}_d^k$ if and only if $n_j = j$ for all $1 \leqslant j \leqslant d$. Further, $\mathcal{N}_d^k(n) = \mathcal{A}_d^k$ if and only if $n_j \geqslant k$ for all $1 \leqslant j \leqslant d$.

**Remark 3.26.** It is not a restriction to require $n = (n_1, \dots, n_d) \in \mathbf{N}^d$ to be strictly increasing: let $(n_1, \dots, n_d) \in \mathbf{N}^d$ be an arbitrary, not necessarily increasing vector of projective subspace numbers. Since $(I, j_2) \in C(p)$ follows from $(I, j_1) \in C(p)$ for $j_1 < j_2$ and any $p \in \mathcal{A}_d^k$ (see Lemma 2.5), the vectors $(n_1, \dots, n_d)$ and

$$\big(\min(n_1, \dots, n_d),\ \min(n_2, \dots, n_d), \dots, \min(n_{d-1}, n_d),\ n_d\big)$$

give rise to the very same topological subspace, so $(n_1, \dots, n_d)$ is w.l.o.g. increasing. Additionally, if $n_m < k$ for some $2 \leqslant m \leqslant d$, then all projective subspace constraints $(I, j) \in C(p)$ with smaller dimension $j < m$ contain less landmarks, i.e., $|I| < n_m$; if there were a configuration $p \in \mathcal{N}_d^k(n)$ with $(I, m-1) \in C(p)$ for some $I$ with $|I| = n_m$, then $\big(I \dot\cup \{i\}, m\big) \notin C(p)$ with $i \in I^c$ in contradiction to Lemma 2.5. Hence, w.l.o.g. $n_j < n_m$ for all $j < m$. $\Diamond$

Of course, we are interested in projective subspace numbers $n$ which give rise to differentiable Hausdorff manifolds $\mathfrak{n}_d^k(n) = \mathcal{N}_d^k(n) \,/\, \mathbf{PGL}(d)$. Conditions for such feasible $n \in \mathbf{N}^d$ can be deduced from Corollary 3.14 and Corollary 3.25.

**Theorem 3.27.** *Let $n = (n_1, \dots, n_d)$ be a vector of projective subspace numbers. Then, the following statements are equivalent:*

*(i)* $\mathfrak{n}_d^k(n)$ *is Hausdorff;*

*(ii)* $\mathfrak{n}_d^k(n) \subseteq \mathfrak{f}_d^k$*;*

*(iii)* $\mathfrak{n}_d^k(n)$ *is an open Hausdorff submanifold of* $\mathfrak{f}_d^k$*;*

*(iv)* $n_j + n_{d+1-j} < k$ *for all* $1 \leqslant j \leqslant d$.

Proof. The implications (i)⇒(ii) and (i)⇒(iii) hold due to Corollary 3.25. (iii)⇒(i) is obvious. Regarding (ii)⇒(i), recall that in non-Hausdorff subspaces there are shapes $[p]$ and $[q]$ with $(I, j) \in C(p)$ and $(I^c, d+1-j) \in C(q)$ for some $I \subseteq \{1, \dots, k\}$ and $j \in \{1, \dots, d\}$ (see Corollary 3.14). Thus, if $\mathfrak{n}_d^k(n)$ is not Hausdorff, then it contains, by construction, also a shape $[r]$ fulfilling both projective subspace constraints $(I, j), (I^c, d+1-j)$, i.e. a decomposable shape, whence $\mathfrak{n}_d^k(n) \nsubseteq \mathfrak{f}_d^k$. Hence, subspaces bounded by projective subspace numbers consisting only of free shapes are Hausdorff.

To proof the equivalence of (i) and (iv), note that there is no pair of shapes $[p], [q] \in \mathfrak{n}_d^k(n)$ with $(I, j) \in C(p)$ and $(I^c, d+1-j) \in C(q)$ for some $I \subseteq \{1, \dots, k\}$ and $j \in \{1, \dots, d\}$ if and only if $n_j + n_{d+1-j} < k$ for all $1 \leqslant j \leqslant d$. This is the case if and only if $n_j + n_{d+1-j} < k$ for all $1 \leqslant j \leqslant d$. $\square$

The set $\mathbf{N}^d$ may be equipped with the partial order induced by the component-wise total order on $\mathbf{N}$. Then, a vector $n \in \mathbf{N}^d$ is said to be *maximal* if $\mathfrak{n}_d^k(n)$ is a differentiable Hausdorff manifold and there is no $m > n$ such that $\mathfrak{n}_d^k(m)$ is a differentiable Hausdorff manifold, too. This notion of maximality accords with requirement 4) of the introduction of this chapter since the addition of further projective shapes would automatically lead to the violation of requirement 1). While there might be more than one maximal vector $n \in \mathbf{N}^d$ for given $d$ and $k$, there is a simple criterion for the maximality of $n$ resp. the corresponding topological subspace $\mathfrak{n}_d^k(n)$.

**Corollary 3.28.** *Let $n = (n_1, \ldots, n_d) \in \mathbf{N}^d$ be a vector of projective subspace numbers. The topological subspace $\mathfrak{n}_d^k(n)$ is then maximal if and only if*

$$n_j + n_{d+1-j} = k - 1 \tag{3.14}$$

*for all $j \in \{1, \ldots, d\}$, respectively $2 \cdot n_{(d+1)/2} = k - 2$ in the critical case for odd $d$ and even $k$.*

The topological subspace $\mathfrak{g}_d^k$ of shapes in general position is bounded by projective subspace numbers $n_j = j$ for $j \in \{1, \ldots, d\}$, whence $\mathfrak{g}_d^k$ is a differentiable Hausdorff manifold for $k \geqslant d+3$ by Theorem 3.27. In the case $d = 1$ and $k = 4$, $\mathfrak{g}_1^4$ is maximal since $2 \cdot n_1 = d + 1 = k - 2$. Otherwise however, $\mathfrak{g}_d^k$ is not maximal; then, $n_1 + n_d = d + 1 < k - 1$, so $n_d$ can be increased by 1 without violating the bound in point (iv) of Theorem 3.27 if $d > 1$, or $n_1$ and $n_d$ if $k > d + 3$.

Another example for a topological subspace of $\mathfrak{a}_d^k$ bounded by projective subspace numbers is the space $\mathfrak{t}_d^k$ of Tyler regular shapes introduced by Kent and Mardia (2012). Here, $\mathfrak{t}_d^k = \mathfrak{n}_d^k(t)$ is the topological subspace bounded by projective subspace numbers $t = (t_1, \ldots, t_d) \in \mathbf{N}^d$ with

$$t_j = \left\lceil j \frac{k}{d+1} \right\rceil - 1, \quad j \in \{1, \ldots, d\}, \tag{3.15}$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Then, the corresponding configuration space $\mathcal{T}_d^k$ comprises of those configurations $p$ which fulfill

$$|I| < k \tfrac{\mathbf{rk}\, p_I}{d+1}$$

for all $I \subset \{1, \ldots, k\}$ (see also Section 3.1).

By Theorem 3.27, $\mathfrak{t}_d^k$ is a differentiable Hausdorff manifold since

$$\begin{aligned} t_j + t_{d+1-j} &= \left\lceil j \frac{k}{d+1} \right\rceil - 1 + \left\lceil (d+1-j) \frac{k}{d+1} \right\rceil - 1 \\ &< j \frac{k}{d+1} + (d+1-j) \frac{k}{d+1} \\ &= k \end{aligned}$$

for all $1 \leqslant j \leqslant d$. In general, charts given by projective frames do not suffice to cover $\mathfrak{t}_d^k$ for $d \geqslant 3$ since there are Tyler regular shapes which do not contain a frame; as an example see the shape discussed in Figure 3.3 on page 38. Of course, charts given by pseudo-frames are sufficient since $\mathfrak{t}_d^k \subset \mathfrak{f}_d^k$.

Unfortunately, $\mathfrak{t}_d^k$ is not maximal for some $d$ and $k$.

**Proposition 3.29.** *The topological subspace $\mathfrak{t}_d^k$ is a maximal choice in the class of subspaces bounded by subspace numbers if and only if the greatest common divisor of $k$ and $d+1$ is either 1 or 2. In particular, $\mathfrak{t}_d^k$ is maximal for*

*(i) relatively prime $k$ and $d+1$,*

*(ii) $d = 1$ and arbitrary $k \geqslant d+3$, as well as*

*(iii) arbitrary $d$ and $k = d+3$.*

Proof. Recall that $k$ and $d+1$ are relatively prime if and only if their greatest common divisor $c = \gcd(k, d+1)$ is 1. If the quotient $\frac{jk}{d+1}$ is not integral for some $j \in \{1, \dots, d\}$, then $\frac{(d+1-j)k}{d+1}$ is not integral, and $t_j + t_{d+1-j} = k - 1$ due to rounding.

The quotient $\frac{jk}{d+1}$ is integral if and only if $j < d+1$ is a multiple of $\frac{d+1}{c}$. However, for $j = \frac{d+1}{c}$

$$\begin{aligned} t_{(d+1)/c} + t_{d+1-(d+1)/c} &= \left\lceil \frac{d+1}{c} \frac{k}{d+1} \right\rceil - 1 + \left\lceil (c-1) \frac{d+1}{c} \frac{k}{d+1} \right\rceil - 1 \\ &= \frac{k}{c} + (c-1)\frac{k}{c} - 2 \\ &= k - 2, \end{aligned}$$

whence $t_{d+1-(d+1)/c}$ can be increased by 1 by Corollary 3.28 unless $t_{d+1-(d+1)/c} = t_{(d+1)/c}$ in the case $c = 2$.

The cases (i)–(iii) follow easily: (i) is obvious. If $d = 1$ as in (ii), then $d + 1 = 2$, whence the greatest common divisor of $d + 1$ and $k$ is either 1 or 2. For the case $k = d + 3$ as in (iii), recall that the the greatest common divisor $c = \gcd(k, d + 1)$ of $k$ and $d + 1$ is also a divisor of $k - d - 1 = 2$. Then, $c$ is 1 if both $d + 1$ and $k$ are odd, and 2 if both $d + 1$ and $k$ are even. □

While the space $\boldsymbol{t}_d^k$ of Tyler regular shapes is not maximal in some cases, it has other properties which prove helpful for the definition of embeddings resp. Riemannian metrics as will be discussed in Chapter 4.

Of course, there are other ways to construct topological subspaces of $\boldsymbol{a}_d^k$ which are closed under permutations and respect the hierarchy of projective subspace constraints, e.g. by taking the closure under permutations of $\{[q] \in \boldsymbol{a}_d^k : C(q) \subseteq C(p)\}$ for a chosen $[p] \in \boldsymbol{a}_d^k$, i.e.,

$$\boldsymbol{c}_d^k(p) = \{[q] \in \boldsymbol{a}_d^k : \exists \sigma \in S_k \text{ s.t. } C(\sigma q) \subseteq C(p)\}. \tag{3.16}$$

Again, one can easily check if such a space is a differentiable Hausdorff manifold with Corollary 3.14 and Corollary 3.25. The topological subspace $\boldsymbol{c}_d^k(p)$ is e.g. Hausdorff if there are no projective subspace constraints $(I, j), (I', d+1-j) \in C(p)$ such that $|I| + |I'| = k$ with $I, I'$ not necessarily disjoint.

# Chapter 4

# Tyler regular shapes

The notion of Tyler regularity of configurations respectively shapes was introduced by Kent and Mardia (2012). The main motivation for a discussion of Tyler regular shapes was the observation that they possess a Tyler standardization, i.e., to any Tyler regular shape $[p] \in t_d^k$ there is a representing matrix configuration $P$ such that

$$P_{i\cdot}P_{i\cdot}^t = \tfrac{d+1}{k} \quad \text{for all } 1 \leqslant i \leqslant k \tag{4.1}$$

and

$$P^tP = \sum_{i=1}^{k} P_{i\cdot}^t P_{i\cdot} = \mathbf{I}_{d+1}. \tag{4.2}$$

As it is shown in Section 4.1, Tyler regular shapes are the only free shapes that are Tyler standardizable. For some $k$ and $d$ however, there are decomposable shapes which allow Tyler standardization, too. Additionally, a geometric reasoning for Tyler standardization will be presented.

Using the corresponding Tyler standardized projection matrices $M_P = PP^t$, the topological subspace of Tyler standardizable shapes can be embedded into a metric space, see Section 4.2. A Riemannian metric can only be defined for Tyler regular shapes through Tyler standardization, though.

## 4.1 Tyler standardization

Via the matrix representation of projective shape, one can obtain another noteworthy approach to projective shape. Let $[P] \in r_d^k$ be a shape of full rank and $P$ be a matrix representation of $[P]$. By definition, $P$ is only unique up to left-multiplication with non-singular diagonal $k \times k$-dimensional matrices $D \in \mathbf{Diag}^*(k)$ and right-multiplication of non-singular $(d+1)\times(d+1)$-dimensional matrices $B \in \mathbf{GL}(d+1)$. Instead of considering the $k \times (d+1)$-dimensional matrix $P$ as an aggregation of rows representing the landmarks, one can also consider the matrix $P$ as an aggregation of columns forming a basis of the $(d+1)$-dimensional column space $\mathcal{L}(P) \subset \mathbf{R}^k$, i.e., $P$ is an element of the non-compact Stiefel manifold $\mathbf{St}(k, d+1)$. In this latter approach, the right-action of $\mathbf{GL}(d+1)$ on $P$ is then the change of basis vectors of $\mathcal{L}(P)$. In particular, an orthonormal basis of the column space $\mathcal{L}(P)$ can be chosen as a representation, i.e. a matrix $P \in \mathbf{St}^{\mathrm{O}}(k, d+1)$ with orthonormal columns. Then,

$$P^tP = \mathbf{I}_{d+1}$$

with $P$ being unique up to the action of the orthogonal group $\mathbf{O}(d+1)$ from the right.

In this approach, the left-action of $\mathbf{Diag}^*(k)$ on configurations $P$ can be considered as an action on the Grassmannian manifold $\mathbf{Gr}(k, d+1)$ of $(d+1)$-dimensional subspaces of $\mathbf{R}^k$, see Section 2.1.

Using the Veronese-Whitney embedding $\iota$ of $\mathbf{Gr}(k, d+1)$ into $\mathbf{Sym}(k)$, cf. page 9, elements of the Grassmannian $\mathbf{Gr}(k, d+1)$ can be represented by the corresponding orthogonal projection matrices

$$M_P = P\big(P^tP\big)^{-1}P^t \in \mathbf{Sym}(k) \tag{4.3}$$

mapping elements of $\mathbf{R}^k$ orthogonally onto the column space of $P$. This, of course, simplifies to $M_P = PP^t$ if $P^tP = \mathbf{I}_{d+1}$, i.e., if $P$ fulfills Equation (4.2). The symmetric matrix $M_P$ is then—as an orthogonal projection—a $k \times k$-dimensional matrix of rank and trace $d+1$. In this representation, the group $\mathbf{Diag}^*(k)$ of non-singular diagonal matrices acts infinitesimally on the Grassmannian $\mathbf{Gr}(k, d+1)$ by conjugation as follows[1]: for a non-singular diagonal matrix $D = \operatorname{diag}(D_i, i = 1, \ldots, k)$ in a sufficiently small neighborhood of $\mathbf{I}_k$ the inverse of $P^tD^2P$ can be represented as a Neumann series (Shalit; 2017, Prop. 8.3.9):

$$\big(P^tD^2P\big)^{-1} = \big(\mathbf{I}_{d+1} - (\mathbf{I}_{d+1} - P^tD^2P)\big)^{-1} = \sum_{n=0}^{\infty} \big(\mathbf{I}_{d+1} - P^tD^2P\big)^n.$$

Further, $\frac{\partial}{\partial D_i} D = e_ie_i^t$ with $e_i$ denoting the $i$-th canonical basis vector of $\mathbf{R}^k$. Then,

$$\begin{aligned}
\tfrac{\partial}{\partial D_i} M_{DP} &= \tfrac{\partial}{\partial D_i}\big[DP\big(P^tD^2P\big)^{-1}P^tD\big] \\
&= e_ie_i^tP\big(P^tD^2P\big)^{-1}P^tD + DP\big(P^tD^2P\big)^{-1}P^te_ie_i^t \\
&\quad + DP\left[\sum_{n=1}^{\infty}\sum_{l=1}^{n} \big(\mathbf{I}_{d+1} - P^tD^2P\big)^{l-1} \cdot \big(-2D_iP^te_ie_i^tP\big)\big(\mathbf{I}_{d+1} - P^tD^2P\big)^{n-l}\right]P^tD.
\end{aligned}$$

For the derivative $\frac{\partial}{\partial D_i} M_{DP}$ at $D = \mathbf{I}_k$, $P^tP = \mathbf{I}_{d+1}$, and consequently for $P^tD^2P = \mathbf{I}_{d+1}$, $D_i = 1$, and $M_P = PP^t$, we conclude

$$\begin{aligned}
\tfrac{\partial}{\partial D_i} M_{DP} &= e_ie_i^tPP^t + PP^te_ie_i^t - 2PP^te_ie_i^tPP^t \\
&= \big(\underbrace{e_ie_i^tM_P - M_Pe_ie_i^t}_{\text{antisymmetric}}\big)M_P + M_P\big(\underbrace{M_Pe_ie_i^t - e_ie_i^tM_P}_{\text{antisymmetric}}\big).
\end{aligned}$$

Meanwhile, the infinitesimal action of the orthogonal group $\mathbf{O}(k)$ acting by conjugation on $M_P \in \mathbf{Gr}(k, d+1)$ is given by

$$\tfrac{\partial}{\partial t}\big|_{t=0} O(t)M_PO(t)^t = \dot O(0)M_P + M_P\dot O^t(0) = \dot O(0)M_P - M_P\dot O(0)$$

for a differentiable curve $\mathbf{R} \ni t \mapsto O(t) \in \mathbf{O}(k)$ with $O(0) = \mathbf{I}_k$ and antisymmetric $\dot O(0) \in \mathfrak{o}(k) = \big\{M \in \mathbf{R}^{k\times k} : M = -M^t\big\}$. Hence, $\frac{\partial}{\partial D_i} M_{DP}$ is an infinitesimal rotation in the plane spanned by $M_Pe_i$ and $e_i$. Therefore, fixing the angle

$$\begin{aligned}
\big\langle e_i, M_Pe_i\big\rangle = e_i^tM_Pe_i &= e_i^tM_PM_Pe_i = \big\|M_Pe_i\big\|^2 \\
&= e_i^tPP^te_i = P_{i\cdot}P_{i\cdot}^t = \|P_{i\cdot}\|^2
\end{aligned}$$

in $\mathbf{R}^k$ for all $1 \leqslant i \leqslant k$ fixes the remaining action of $\mathbf{Diag}^*(k)$ on $M_P$, and thus standardizes the projection matrix $M_P$ resp. the configuration $P$. Since we still require invariance under permutations, all directions $e_i$ resp. landmarks $P_{i\cdot}$ have to be treated equally, i.e., there is a constant $C \in \mathbf{R}$ such that

$$P_{i\cdot}P_{i\cdot}^t = C \quad \text{for all } 1 \leqslant i \leqslant k.$$

The values $P_{i\cdot}P_{i\cdot}^t$ are the diagonal elements of the orthogonal projection matrix $M_P$, whence we conclude $C = \frac{d+1}{k}$ since $M_P$ has trace $d+1$. Of course, fixing the norm of the rows $P_{i\cdot}$ of $P$ does not completely remove the action of the diagonal group $\mathbf{Diag}^*(k)$ since multiplication with $\pm 1$ is still allowed.

[1]The following ideas on this page are property of T. Hotz.

This standardization of a projective shape respectively a configuration fulfilling Equations (4.1) and (4.2) is called *Tyler standardization* and was first introduced by Kent and Mardia (2006). Unfortunately, Tyler standardization does only remove the action of $\mathbf{PGL}(d)$ up to a compact group since the right-action of $\mathbf{O}(d+1)$ and left-action of the group

$$\mathbf{C}_2^k = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \in \mathbf{Diag}^*(k) : \lambda_i \in \{\pm 1\} = \mathbf{C}_2 \text{ for all } i \in \{1, \ldots, k\} \right\} \tag{4.4}$$

of *sign matrices* remain. Further, Tyler standardization is not possible for all shapes $[p] \in \mathfrak{a}_d^k$, but only for Tyler regular shapes and certain decomposable ones.

The following topological subspaces of $\mathcal{A}_d^k$ have been first discussed by Kent et al. (2011) in an unpublished article:

$\mathcal{T}_d^k$ is, as in Section 3.1, the topological space of ***Tyler regular*** configurations, i.e., $p \in \mathcal{T}_d^k$ if and only if $|I| < k\frac{j}{d+1}$ for any $(I, j) \in C(p)$.

$\mathcal{T}sr_d^k$, which contains a configuration $p$ if and only if $p$ is ***Tyler semi-regular***, i.e., $p \in \mathcal{D}_d^k$ with $|I| = k\frac{j}{d+1}$ and $|I^c| = k\frac{d+1-j}{d+1}$ for all pairs $(I, j), (I^c, d+1-j) \in C(p)$, while $|I| < k\frac{j}{d+1}$ for all other projective subspace constraints $(I, j) \in C(p)$.

$\mathcal{T}er_d^k$, which contains a configuration $p$ if and only if $p$ is ***Tyler extended-regular***, i.e., $p$ is neither Tyler regular nor Tyler semi-regular, but $|I| \leqslant k\frac{j}{d+1}$ for all $(I, j) \in C(p)$.

$\mathcal{T}ir_d^k$, which contains a configuration $p$ if and only if $p$ is ***Tyler irregular***, i.e., $p$ fulfills a projective subspace constraint $(I, j) \in C(p)$ such that $|I| > k\frac{j}{d+1}$.

For relatively prime $d+1$ and $k$, however, there are no Tyler extended- and semi-regular configurations and shapes.

**Proposition 4.1.** *$\mathcal{T}sr_d^k, \mathcal{T}er_d^k = \varnothing$ if and only if $d+1$ and $k$ are relatively prime. In particular, there are Tyler extended- and semi-regular shapes if $\mathfrak{t}_d^k$ is not maximal, but also in the case that the greatest common divisor of $k$ and $d+1$ is 2.*

Proof. There are Tyler extended- and semi-regular configuration resp. shapes if and only if the quotient $\frac{jk}{d+1}$ is integer for some $j \in \{1, \ldots, d\}$. The latter is the case if and only if $d+1$ and $k$ have a common divisor $c \geqslant 2$.

The second statement holds due to Proposition 3.29. □

Tyler regular shapes are free (Section 3.5), while Tyler semi-regular shapes are decomposable by definition. There can be Tyler extended- and irregular shapes of both kinds. Tyler extended- and semi-regular shapes are necessarily of full rank since $t_d < k\frac{d}{d+1} < k$ for any $k$and $d$.

Let $c$ be the greatest common divisor of $k$ and $d+1$, and let $k' = k/c$ and $d' = (d+1)/c$. Further, let $\mathcal{P}(k, k')$ be the set of partitions of $\{1, \ldots, k\}$ into subsets whose cardinalities are multiples of $k'$. To a partition $\theta = \{I_1, \ldots, I_s\} \in \mathcal{P}(k, k')$ with $|I_r| = m_r k' = \frac{j_r k}{d+1} = \frac{j_r k'}{d'}$ define $\mathfrak{t}_d^k(\theta)$ to be the space of Tyler semi-regular shapes with total decomposition $\{(I_1, j_1), \ldots, (I_s, j_s)\}$ resp. the space of Tyler regular shapes $\mathfrak{t}_d^k$ in the case that $\theta = \{\{1, \ldots, k\}\}$. Then, the space $\mathfrak{k}_d^k = \mathfrak{t}_d^k \cup \mathfrak{tsr}_d^k$ of Tyler standardizable shapes is, as a set, the disjoint union of the topological subspaces $\mathfrak{t}_d^k(\theta)$, i.e.,

$$\mathfrak{k}_d^k = \coprod_{\theta \in \mathcal{P}(k,k')} \mathfrak{t}_d^k(\theta),$$

while

$$\mathfrak{t}_d^k(\theta) \cong \bigtimes_{r=1}^{l} \mathfrak{t}_{m_r d'}^{m_r k'}$$

as one can easily conclude from the canonical block matrix structure for decomposable shapes, see Proposition 2.7 and Proposition 3.2.

The blur of a Tyler semi-regular shape $[p]$ is disjoint from $\boldsymbol{t}_d^k$ and comprises of Tyler extended-regular shapes besides one Tyler semi-regular shape in the blur of $[p]$—namely $[p]$ itself—since $\mathscr{k}_d^k = \boldsymbol{t}_d^k \cup \boldsymbol{tsr}_d^k$ will turn out to be a topological manifold for any $d$ and $k$. Of course, the blur of a Tyler semi-regular shape can also be discussed using Proposition 3.10.

The manifold structure of $\mathscr{k}_d^k = \boldsymbol{t}_d^k \cup \boldsymbol{tsr}_d^k$ can be obtained through the use of pseudo-frames. Of course, pseudo-frames cover $\boldsymbol{t}_d^k \subset \boldsymbol{f}_d^k$, but not $\boldsymbol{tsr}_d^k \subset \boldsymbol{d}_d^k$; however, one can substitute any Tyler semi-regular shape with a free one from its blur which is then covered by a pseudo-frame. This accords with the topology of $\mathscr{k}_d^k$, and one obtains actually differentiable manifolds by this procedure. Unfortunately, it is unclear if the resulting differentiable structures are independent of the choice of the free shapes in the blur of Tyler semi-regular shapes.

The topological subspace $\mathscr{k}_d^k$ has been suggested to me as the topological subspace of choice by John T. Kent through personal communication, and will thus be called "**K**ent's shape space". Of course, $\mathscr{k}_d^k = \boldsymbol{t}_d^k$ if and only if $d+1$ and $k$ are relatively prime due to Proposition 4.1, otherwise $\mathscr{k}_d^k$ has the drawback of not respecting the hierarchy of projective subspace constraints. Thus, $\mathscr{k}_d^k$ fulfills all requirements from the introduction of Chapter 3 if and only if $\gcd(k, d+1) = 1$.

**Example 4.2 (Kent et al. (2011), Sect. 7.2).** In the case $d = 1$ and $k = 4$, a shape $[p]$ is Tyler regular if $|I| < 4 \cdot \frac{1}{2} = 2$ for any $(I, 1) \in C(p)$, i.e., Tyler regular shapes are already in general position and consist of four distinct landmarks. The Tyler semi-regular shapes are the three shapes with double pair coincidences; the Tyler extended-regular shapes are the six shapes with a single pair coincidence, while Tyler irregular shapes have at least three coinciding landmarks. $\diamondsuit$

Tyler standardization is only possible for Tyler regular and Tyler semi-regular shapes.

**Theorem 4.3 (Kent et al. (2011), Thm. 3).** *There is a matrix configuration $P$ of shape $[p]$ such that the rows of $P$ are of norm $\sqrt{\frac{d+1}{k}}$, i.e.,*

$$P_{i\cdot}P_{i\cdot}^t = \tfrac{d+1}{k} \quad \textit{for all } i \in \{1, \ldots, k\}, \tag{4.1}$$

*and the columns of $P$ are orthonormal, i.e.,*

$$P^t P = \sum_{i=1}^{k} P_{i\cdot}^t P_{i\cdot} = \mathbf{I}_{d+1}, \tag{4.2}$$

*if and only if $[p] \in \mathscr{k}_d^k = \boldsymbol{t}_d^k \cup \boldsymbol{tsr}_d^k$ is Tyler regular or Tyler semi-regular. The matrix $P$ is unique up to left-multiplication with sign matrices $D \in \mathbf{C}_2^k$ and up to right-multiplication with orthogonal matrices $B \in \mathbf{O}(d+1)$. A matrix configuration $P$ fulfilling Equations* (4.1) *and* (4.2) *is said to be* Tyler standardized.

For the proof of Theorem 4.3 we need the following result:

**Proposition 4.4.** *Let $X \in \mathscr{A}_d^k$ be a configuration matrix. There is a Tyler standardized matrix $P$ of shape $[X]$ if and only if*

$$A = \tfrac{d+1}{k} \sum_{i=1}^{k} \frac{X_{i\cdot}^t X_{i\cdot}}{X_{i\cdot} A^{-1} X_{i\cdot}^t} \tag{4.5}$$

*has a positive definite, symmetric solution $A \in \mathbf{GL}(d+1)$.*

Proof. First, let $P = DXB$ be Tyler standardized, i.e.,

$$\mathbf{I}_{d+1} = P^t P = B^t X^t D^2 X B = B^t \left( \sum_{i=1}^{k} X_{i\cdot}^t D_{ii}^2 X_{i\cdot} \right) B$$

and

$$\tfrac{d+1}{k} = P_{i\cdot}P_{i\cdot}^t = D_{ii}^2 X_{i\cdot} BB^t X_{i\cdot}^t.$$

Then,

$$\tfrac{d+1}{k}\sum_{i=1}^{k}\frac{X_{i\cdot}^t X_{i\cdot}}{X_{i\cdot}^t BB^t X_{i\cdot}} = \sum_{i=1}^{k}\frac{X_{i\cdot}^t X_{i\cdot}}{D_{ii}^{-2}} = \sum_{i=1}^{k} X_{i\cdot}^t D_{ii}^2 X_{i\cdot} = \left(B^t\right)^{-1}B^{-1} = \left(BB^t\right)^{-1}.$$

with $\left(BB^t\right)^{-1} \in \mathbf{Sym}(k)$ being positive definite.

The other direction has already been shown by Kent and Mardia (2012): let $A \in \mathbf{GL}(d+1)$ be a positive definite, symmetric solution of Equation (4.5). This solution is at most unique up to scale, so w.l.o.g. $\det(A) = 1$. Let $B$ be the unique positive definite, symmetric square root of $A^{-1}$, and let $D$ be the diagonal matrix with entries $D_{ii} = \left(\tfrac{k}{d+1}X_{i\cdot}A^{-1}X_{i\cdot}\right)^{-1/2}$. Then, $P = DXB$ is Tyler standardized. □

Finally, we are able to prove Theorem 4.3:

Proof (Theorem 4.3). The existence of a Tyler standardization has already been shown for Tyler regular shapes by Kent and Mardia (2012). The proof given here follows their line of thought. In the unpublished manuscript (Kent et al.; 2011) a sketch of a proof for the full statement can be found.

The proof of Theorem 4.3 is based on Proposition 4.4 and results of Kent and Tyler (1988) (and earlier work of Tyler (1987a,b)) about the existence of a solution to Equation (4.5).

For a Tyler regular configurations $X \in \mathcal{T}_d^k$, a solution to Equation (4.5) exists (Kent and Tyler; 1988, Thm. 1) and is unique up to a scalar multiple (Kent and Tyler; 1988, Thm. 2). Further, if $X \in \mathcal{Tir}_d^k$ is Tyler irregular, then there is no solution (Kent and Tyler; 1988, Thm. 3), and thus no Tyler standardization due to Proposition 4.4.

Regarding Tyler semi- resp. extended-regular configurations, the corresponding results about the existence of a solution to Equation (4.5) can be found in (Auderset et al.; 2005). However, the statements can be proven directly with a little more insight as well: Tyler semi-regular shapes can be understood as direct products of Tyler regular ones, see page 49. In particular, if $[p] \in tsr_d^k$ is Tyler semi-regular, then it has a representation $X$ in canonical block structure with Tyler regular blocks on the diagonal with dimension ratio equal to $\frac{k}{d+1}$, i.e., after reordering of the rows

$$X = \begin{pmatrix} X_{11} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_{ss} \end{pmatrix}$$

for some $X_{rr} \in \mathcal{T}_{d_r}^{k_r}$ with $\frac{k_r}{d_r+1} = \frac{k}{d+1}$, $1 \leqslant r \leqslant s$. After Tyler standardization of these diagonal blocks, the full matrix is Tyler standardized.

Concerning the Tyler extended-regular case $[p] \in ter_d^k$, suppose there were a Tyler standardization $P$ of shape $[p]$. Since $[p]$ is Tyler-extended regular, $[p]$ fulfills a projective subspace constraint $(I, j) \in C(p)$ such that $|I| = \frac{jk}{d+1}$ and $(I^c, d+1-j) \notin C(p)$. Then, there is a rotation matrix $B \in \mathbf{O}(d+1)$ such that (again after reordering the rows)

$$PB = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix}$$

for some $X_{11} \in \mathcal{T}_{j-1}^{|I|}$, $X_{21} \in \mathbf{R}^{|I^c|\times j}$ and $X_{22} \in \mathcal{R}_{j-1}^{|I^c|}$. Since $P$ is Tyler standardized, so is $PB$. In particular, the columns of $PB$ are orthogonal, whence $X_{21} = 0$ and $(I^c, d+1-j) \in C(p)$ in contradiction to the assumption. Therefore, there is no Tyler standardization to Tyler extended-regular shapes, finishing the proof. □

The existence of Tyler standardization offers topological advantages, in particular with respect to the definition of a metric, see Section 4.2 for a discussion of the latter.

**Proposition 4.5.** *The topological space $\mathcal{TSC}(\mathcal{K}^k_d) \subseteq \mathbf{R}^{k\times(d+1)}$ of all Tyler standardized configuration matrices is compact. Further, the topological space $\mathcal{TSC}(\boldsymbol{t}^k_d)$ of Tyler standardized configuration matrices corresponding to Tyler regular shapes is a differentiable submanifold of $\mathbf{R}^{k\times(d+1)}$.*

Proof. The space $\mathcal{TSC}(\mathcal{K}^k_d)$ of Tyler standardized configurations is, of course, a subset of $\mathbf{R}^{k\times(d+1)}$. Even more, Equation (4.2) states that Tyler standardized configurations are elements of the orthogonal Stiefel manifold $\mathbf{St}^{\mathrm{O}}(k, d+1)$ which comprises of orthonormal bases of $(d+1)$-dimensional vector subspaces of $\mathbf{R}^k$. Equation (4.1) specifies these elements. As a pre-image of a closed set under a continuous function, $\mathcal{TSC}(\mathcal{K}^k_d)$ is itself closed. Further, it is also bounded in the Euclidean norm of $\mathbf{R}^{k\times(d+1)}$, whence $\mathcal{TSC}(\mathcal{K}^k_d)$ is compact.

The space $\mathcal{TSC}(\boldsymbol{t}^k_d)$ of Tyler standardized configurations corresponding to Tyler regular shapes is a differentiable submanifold of both $\mathbf{R}^{k\times(d+1)}$ and the orthogonal Stiefel manifold due to the regular value theorem (Dykema and Strawn; 2006, Thm. 4.3; $\mathcal{TSC}(\boldsymbol{t}^k_d)$ being called $\mathcal{F}^{\mathbf{R}}_{k,d+1}$ there). □

Any Tyler semi-regular shape possesses a Tyler standardized configuration in canonical block matrix structure, i.e., with its landmarks lying in orthogonal, complementary linear subspaces of $\mathbf{R}^{d+1}$. The isotropy group of a Tyler standardized configuration in canonical block matrix structure comprises of simultaneous multiplication of block sign matrices from the left and right. With this in mind note that the first statement of Proposition 4.5 is not true for $\mathcal{TSC}(\boldsymbol{t}^k_d)$ if $\boldsymbol{t}^k_d \neq \mathcal{K}^k_d$, i.e., if $d+1$ and $k$ are not relatively prime. Similarly, the second statement is not true for $\mathcal{TSC}(\mathcal{K}^k_d)$ in the case that $k$ and $d+1$ are not relatively prime. Here, the Tyler standardized configurations corresponding to Tyler semi-regular shapes are "points of higher dimension".

**Remark 4.6.** In a Hilbert space $\mathcal{H}$ there is the notion of a frame as a list of vectors $F = (f_i)_{i\in I}$ in $\mathcal{H}$ satisfying

$$A\|v\|^2 \leqslant \sum_{i\in I} |\langle v, f_i\rangle|^2 \leqslant B\|v\|^2 \quad \text{for all } v \in \mathcal{H}$$

for some constants $A, B \geqslant 0$. With this notion in mind, Tyler standardized configurations can be understood as frames of length $k$ in $\mathbf{R}^{d+1}$. Since the vectors lie, up to rescaling, on the unit sphere due to Equation (4.1), and since the frame bounds $A, B$ can be chosen to be equal to each other with value 1 (resp. $\frac{k}{d+1}$ after the aforementioned rescaling) due to Equation (4.2), we concern ourselves actually with what is called a finite unit norm tight frame (also: finite spherical tight frame, finite normalized tight frame), cf. e.g. (Dykema and Strawn; 2006). ◊

**Example 4.7 (Kent and Mardia (2012), Sect. 7).** In the case $d = 1$ and $k = 4$, Tyler standardized configurations are—up to rotation, reflections, and signs—those of the form

$$P = \sqrt{\tfrac{1}{2}} \begin{pmatrix} \cos\varphi & \sin\varphi \\ \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \tag{4.6}$$

for $\varphi \in [0, \pi)$ or a permutation thereof. The configuration $P$ is Tyler semi-regular for $\varphi \in \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$. Note that the cross ratio of $P$ is $-\tan^2 2\varphi$ for $\varphi \notin \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$. ◊

Unfortunately, a solution to Equation (4.5), and thus a Tyler standardization to a configuration resp. shape, can only be approximated numerically (Kent and Tyler; 1988).

**Remark 4.8.** The solution of Equation (4.5) (if existent) is the maximum likelihood estimator for the angular central Gaussian distribution on $\mathbf{RP}^d$, see (Tyler; 1987b). In particular, it is the minimizer of the negative log-likelihood function (up to a constant positive factor)

$$\rho_X(A) = \sum_{i=1}^{k} \log\left(\frac{X_{i\cdot}A^{-1}X_{i\cdot}^t}{X_{i\cdot}X_{i\cdot}^t}\right). \tag{4.7}$$

for symmetric, positive definite, $(d+1)\times(d+1)$-dimensional matrices $A$. Auderset et al. (2005) have shown that $\rho_X$ has a unique minimum in the space $\Theta_{d+1}$ of symmetric, positive definite, $(d+1)\times(d+1)$-dimensional matrices of determinant 1 if and only if the configuration $X$ is Tyler regular. For Tyler semi-regular configurations decomposing into $s$ Tyler regular parts, the minimizers of $\rho_X$ form a submanifold of $\Theta_{d+1}$ of dimension $s-1$. For Tyler extended- and irregular configurations, $\rho_X$ admits no minimum.

Note that the minimizer of $\rho_X$ depends differentiably on $X \in \mathcal{T}_d^k$: the set $\Theta_{d+1}$ is naturally equipped with a symmetric space structure (Auderset et al.; 2005, Appendix A); in particular, geodesic can be defined. Using the language of differentiable geometry, the function $\rho : (X, A) \mapsto \rho_X(A)$ is twice continuously differentiable with respect to $(X, A) \in \mathcal{T}_d^k \times \Theta_{d+1}$, and $\rho_X$ is (geodesically) strictly convex on $\Theta_{d+1}$ for any $X \in \mathcal{T}_d^k$ (Auderset et al.; 2005, Thm. 2). Let $A_0$ be the minimizer at $X_0 \in \mathcal{T}_d^k$, i.e., $\nabla_A \rho_{X_0}(A_0) = 0$. Then, the Jacobian of $\nabla_A \rho_{X_0}$ at $A_0$ is invertible since it equals the transpose of the Hessian of $\rho$ which is positive definite for strictly convex functions, and thus invertible. By the implicit function theorem there is an open neighborhood of $A_0$ and a continuously differentiable function $\alpha$ with

$$\begin{aligned} \alpha(X) &= A \quad \text{s.t.} \quad \nabla_A \rho_X(A) = 0 \\ &= \operatorname*{arg\,min}_{A \in \Theta_{d+1}} \rho_X(A). \end{aligned}$$

So, the minimizer of $\rho_X$ depends indeed differentiably on $X \in \mathcal{T}_d^k$. We immediately conclude that Tyler standardization of configurations $X \in \mathcal{T}_d^k$ is continuously differentiable, i.e., the map

$$\begin{array}{ccc} \mathcal{T}_d^k & \longrightarrow & \mathcal{TSC}(t_d^k) \\ X & \longmapsto & DXB \end{array} \tag{4.8}$$

with diagonal matrix $D$ with entries $D_{ii} = \left(\frac{k}{d+1} X_{i\cdot}\alpha(X)^{-1}X_{i\cdot}\right)^{-1/2}$ and $B \in \mathbf{GL}(d+1)$ being the unique positive definite square root of $\alpha(X)^{-1}$ is continuously differentiable. ◊

So, Tyler standardizable shapes $[p] \in \mathcal{K}_d^k$ can be mapped to Tyler standardized configuration matrices $P \in \mathbf{R}^{k\times(d+1)}$ uniquely up to the discrete group action of $\mathbf{C}_2^k = \{\pm 1\}^k \subset \mathbf{Diag}^*(k)$ from the left and the right-action of $\mathbf{O}(d+1)$. The latter ambiguity, however, can be removed by moving on to the corresponding orthogonal projection matrices $M_P = PP^t$.

Note that $M_P$ is of rank $d+1$ and Frobenius norm $\sqrt{d+1}$ and is the matrix comprising of the inner products of the rows of the Tyler standardized matrix $P$, i.e.,

$$\left(M_P\right)_{ij} = P_{i\cdot}P_{j\cdot}^t.$$

In particular, the diagonal of $M_P$ is constant $\frac{d+1}{k}$ since the rows of $P$ are of norm $\sqrt{\frac{d+1}{k}}$. Further, the rows and the columns of $M_P$ are of norm $\sqrt{\frac{d+1}{k}}$:

$$\begin{aligned} \sum_{j=1}^{k} \left(M_P\right)_{ij}^2 &= \sum_{j=1}^{k} \left(P_{i\cdot}P_{j\cdot}^t\right)^2 = \sum_{j=1}^{k} P_{i\cdot}P_{j\cdot}^t P_{j\cdot}P_{i\cdot}^t = \sum_{j=1}^{k} \operatorname{tr}\left(P_{i\cdot}^t P_{i\cdot} P_{j\cdot}^t P_{j\cdot}\right) \\ &= \operatorname{tr}\Big(P_{i\cdot}^t P_{i\cdot} \underbrace{\sum_{j=1}^{k} P_{j\cdot}^t P_{j\cdot}}_{P^tP = \mathbf{I}_{d+1}}\Big) = \operatorname{tr}\left(P_{i\cdot}^t P_{i\cdot}\right) = P_{i\cdot}P_{i\cdot}^t = \left(M_P\right)_{ii} = \tfrac{d+1}{k} \end{aligned} \tag{4.10}$$

for all $i \in \{1, \ldots, k\}$.

**Corollary 4.9.** *Let $[p] \in \mathfrak{a}_d^k$. If and only if $[p] \in \mathfrak{k}_d^k$ is Tyler regular or Tyler semi-regular, there is an orthogonal projection matrix $M \in \mathbf{Sym}(k)$ with constant diagonal equal to $\frac{d+1}{k}$ such that any $k \times (d+1)$-dimensional matrix with its columns comprising of a basis of the range of $M$ is of shape $[p]$. The matrix $M$ is unique up to conjugation with sign matrices $s \in \mathbf{C}_2^k$. An orthogonal projection matrix with constant diagonal is said to be* Tyler standardized*.*

Similarly to the situation with Tyler standardized configuration matrices (Proposition 4.5), the set $\mathbf{TSP}\big(\mathfrak{k}_d^k\big)$ of Tyler standardized projection matrices is naturally a subset of the space $\mathbf{Sym}(k) \subset \mathbf{R}^{k \times k}$ of symmetric matrices and of the set of orthogonal projection matrices $\iota\big(\mathbf{Gr}(k, d+1)\big)$. Note that $\mathbf{TSP}\big(\mathfrak{k}_d^k\big)$ is a closed subset of $\mathbf{Sym}(k)$ and $\iota\big(\mathbf{Gr}(k, d+1)\big)$ since it is a pre-image of a closed set under a continuous function. Further, Tyler standardized projection matrices are bounded in the Frobenius norm, whence $\mathbf{TSP}\big(\mathfrak{k}_d^k\big)$ is a compact set.

As in the situation of Tyler standardized configurations, $\mathbf{TSP}\big(\mathfrak{t}_d^k\big)$ is a submanifold of $\mathbf{Sym}(k)$ by the regular value theorem (see (Dykema and Strawn; 2006, Thm. 4.3; $\mathbf{TSP}\big(\mathfrak{t}_d^k\big)$ being called $\mathcal{G}_{k,d+1}^{\mathbf{R}}$ there)). The map

$$\begin{array}{cccc} \psi: & \mathbf{TSP}\big(\mathfrak{t}_d^k\big) & \longrightarrow & \mathfrak{t}_d^k \\ & M_P & \longmapsto & [P] \end{array} \tag{4.11}$$

is a differentiable covering map, i.e., differentiable, surjective, and each projective shape $[P] \in \mathfrak{t}_d^k$ has a neighborhood $U_{[P]}$ such that $\psi$ restricted to each connected component of $\psi^{-1}\big(U_{[P]}\big)$ is a diffeomorphism to $U_{[P]}$.

Meanwhile, $\mathbf{TSP}\big(\mathfrak{k}_d^k\big)$ is only a submanifold of $\mathbf{Sym}(k)$ if there are no Tyler semi-regular shapes and thus $\mathbf{TSP}\big(\mathfrak{k}_d^k\big) = \mathbf{TSP}\big(\mathfrak{t}_d^k\big)$, i.e., if $k$ and $d+1$ are relatively prime, see Example 4.10. Note that Tyler standardized projection matrices of Tyler semi-regular shapes will be block diagonal after a suitable permutation of rows and columns since Tyler semi-regular shapes possess a Tyler standardized matrix configuration in canonical block structure, see page 52.

**Example 4.10 (Kent and Mardia (2012), Sect. 7).** In the case $d = 1$ and $k = 4$, Tyler standardized projection matrices are of the form

$$M = \begin{pmatrix} \frac{1}{2} & a & b & d \\ a & \frac{1}{2} & c & e \\ b & c & \frac{1}{2} & f \\ d & e & f & \frac{1}{2} \end{pmatrix}. \tag{4.12}$$

From Equation (4.10) we conclude

$$\begin{aligned} a^2 + b^2 + d^2 + \left(\tfrac{1}{2}\right)^2 &= \tfrac{1}{2}, \\ a^2 + c^2 + e^2 + \left(\tfrac{1}{2}\right)^2 &= \tfrac{1}{2}, \\ b^2 + c^2 + f^2 + \left(\tfrac{1}{2}\right)^2 &= \tfrac{1}{2}, \end{aligned}$$

so $M$ is determined up to signs by $a, b, c \in \mathbf{R}$. By addition of these equations we obtain

$$a^2 + b^2 + c^2 = \tfrac{1}{2}\Big(\tfrac{3}{4} - \big(\underbrace{d^2 + e^2 + f^2}_{\frac{1}{4}}\big)\Big) = \tfrac{1}{4},$$

so $M$ is already determined by two of the three values. Indeed, by computation of the Tyler standardized projection matrices from the Tyler standardized configuration matrices in Example 4.7, we observe that there are only three kinds of Tyler standardized projection matrices:

$$\begin{pmatrix} \frac{1}{2} & x & y & 0 \\ x & \frac{1}{2} & 0 & -y \\ y & 0 & \frac{1}{2} & x \\ 0 & -y & x & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & x & 0 & y \\ x & \frac{1}{2} & -y & 0 \\ 0 & -y & \frac{1}{2} & x \\ y & 0 & x & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & 0 & x & y \\ 0 & \frac{1}{2} & -y & x \\ x & -y & \frac{1}{2} & 0 \\ y & x & 0 & \frac{1}{2} \end{pmatrix} \tag{4.13}$$

with $x = \frac{1}{2}(\cos^2\varphi - \sin^2\varphi) = \frac{1}{2}\cos 2\varphi \in [-\frac{1}{2}, \frac{1}{2}]$ and $y = \sin\varphi\cos\varphi = \frac{1}{2}\sin 2\varphi \in [-\frac{1}{2}, \frac{1}{2}]$. By mapping $M$ to $(a, b, c) \in \mathbf{R}^3$, we can think of $\mathbf{TSP}\big(\mathscr{k}_1^4\big)$ as three great circles on the sphere $\mathbf{S}^2 \subset \mathbf{R}^3$ with radius $\frac{1}{2}$ which intersect orthogonally. The intersection points (when $x = 0$ or $y = 0$) correspond to Tyler semi-regular shapes. $\Diamond$

Due to the homeomorphism

$$\begin{array}{rccc} T: & \mathbf{Gr}(k, d+1) & \longrightarrow & \mathbf{Gr}(k, k-d-1) \\ & V & \longmapsto & \iota^{-1}\big(\mathbf{I}_k - \iota(V)\big), \end{array}$$

see Equation (2.6), we immediately conclude that $\mathbf{TSP}\big(\mathscr{k}_d^k\big)$ is homeomorphic to $\mathbf{TSP}\big(\mathscr{k}_{k-d-2}^k\big)$.

By transition to the quotient spaces, we obtain that $\mathscr{k}_d^k$ and $\mathscr{k}_{k-d-2}^k$ are homeomorphic.

**Lemma 4.11 (Dykema and Strawn (2006); Cor. 2.7).** *$\mathscr{k}_d^k$ and $\mathscr{k}_{k-d-2}^k$ are homeomorphic.*

PROOF. The group $\mathbf{C}_2^k$ acts on $\mathbf{Gr}(k, d+1)$ and $\mathbf{Gr}(k, k-d-1)$ by conjugation on the corresponding $k \times k$-dimensional orthogonal projection matrices $M$. This action leaves the diagonal elements of $M$ untouched and is, in particular, trivial on the identity matrix $\mathbf{I}_k$. Therefore, the action commutes with the homeomorphism $T$, and we obtain a well-defined homeomorphism $\kappa_d^k : \mathscr{k}_d^k \to \mathscr{k}_{k-d-2}^k$ by restriction of $\iota \circ T \circ \iota^{-1}$ to Tyler standardized projection matrices. $\square$

**Example 4.12.** By Lemma 4.11 $\mathscr{k}_1^5 = \mathscr{t}_1^5 = \mathscr{n}_1^5(2)$ and $\mathscr{k}_2^5 = \mathscr{t}_2^5 = \mathscr{n}_2^5(1,3)$ are homeomorphic two-dimensional differentiable Hausdorff manifolds.

By Lemma 3.15 $\mathscr{b}_1^5$ is homeomorphic to the 2-torus $\mathbf{T}^2 \cong \mathbf{S}^1 \times \mathbf{S}^1 \cong \mathbf{RP}^1 \times \mathbf{RP}^1$. Some elements of $\mathscr{b}_1^5$ are not in $\mathscr{t}_1^5$, namely those shapes $[p] \in \mathscr{b}_1^5$ with a triple coincidence $\big(\{i, 4, 5\}, 1\big) \in C(p)$ for $i \in \{1, 2, 3\}$. However, those shapes cannot be separated in the Hausdorff sense in $\mathscr{a}_1^5$ from the shapes $[q] \in \mathscr{t}_1^5$ with the double coincidence $\big(\{1, 2, 3\}\backslash\{i\}, 1\big) \in C(q)$ by Proposition 3.12 as should be clear from the following two exemplary shapes in $\mathscr{a}_1^5$:

$$[p] = \left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\right] \qquad \text{and} \qquad [q] = \left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ x & 1 \\ y & 1 \\ z & 1 \end{pmatrix}\right]$$

The shape $[p]$ contains a frame in its first three coordinates, i.e., $[p] \in \mathscr{b}_1^5$. The shape $[q]$ is Tyler regular if $(x, y, z) \in \mathbf{R}^3\backslash\{(a, a, a) : a \in \mathbf{R}\}$ with rescaling of $(x, y, z)$ not changing the shape. Through right-multiplication with a suitable non-singular matrix and left-multiplication with a suitable diagonal, non-singular matrix, one can standardize $[q]$ even further such that $x = 0$ (w.l.o.g.). Then, $(y, z) \in \mathbf{R}^2\backslash\{0\}$ with rescaling not changing the shape, i.e., $[q]$ is determined by an element of $\mathbf{RP}^1$. The topological subspace of Tyler regular shapes which cannot be separated from the shapes $[p] \in \mathscr{b}_1^5$ with triple coincidence $\big(\{i, 4, 5\}, 1\big) \in C(p)$ for $i \in \{1, 2, 3\}$ is thus homeomorphic to $\mathbf{RP}^1$. Hence, to obtain $\mathscr{t}_1^5$, one has to insert a projective line at the shapes $[p] \in \mathscr{b}_1^5$ with a triple coincidence $\big(\{i, 4, 5\}, 1\big) \in C(p)$ for $i \in \{1, 2, 3\}$. Topologically, this is equivalent to forming the so-called connected sum of $\mathscr{b}_1^5 \cong \mathbf{T}^2$ and $\mathbf{RP}^2$, i.e., by cutting out an open subset homeomorhpic to the open disc $B_1(0) \in \mathbf{R}^2$ (or equivalently to homeomorphic to $\mathbf{R}^2$) in both topological spaces and identifying the resulting spaces by a homeomorphism of the arisen boundaries, cf. (Massey; 1991). The line at infinity of $\mathbf{RP}^2$ corresponds then to the inserted projective line. So, $\mathscr{t}_1^5$ is homeomorphic to

$$\mathscr{t}_1^5 \cong \mathbf{T}^2 \# \mathbf{RP}^2 \# \mathbf{RP}^2 \# \mathbf{RP}^2;$$

see Figure 4.1.

On the other hand, $\mathscr{b}_2^5$ is homeomorphic to $\mathbf{RP}^2$ by Lemma 3.15. However, the shapes $[p] \in \mathscr{b}_2^5$ with a single pair coincidence $\big(\{i, 5\}, 1\big) \in C(p)$ are not Tyler regular. In $\mathscr{a}_2^5$ they cannot be separated from the shapes $[q] \in \mathscr{t}_2^5$ with three landmarks on a projective line, i.e., $\big(\{1, 2, 3, 4\}\backslash\{i\}, 2\big) \in C(q)$ as the following two shapes illustrate:

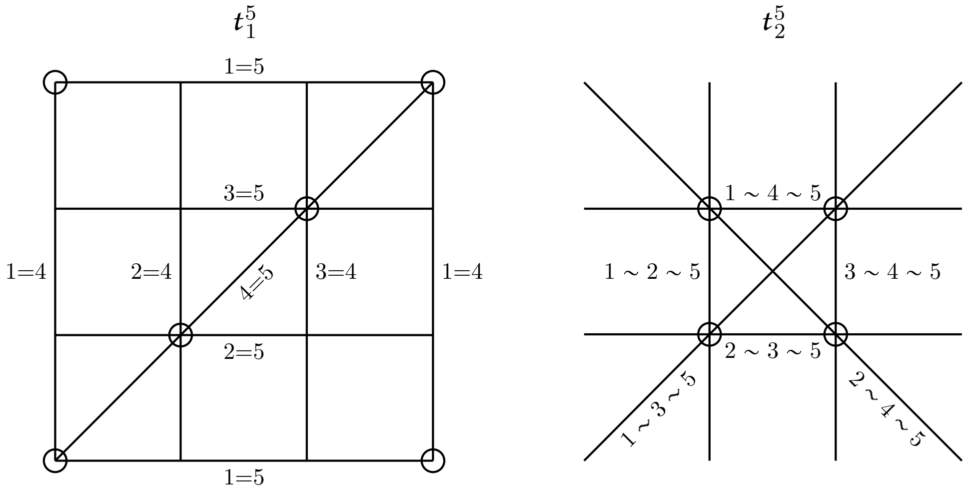



**Figure 4.1:** $t_1^5$ is $\mathscr{b}_1^5 \cong \mathbf{T}^2$ (here presented as a square with opposite edges identified) with the three triple coincidences replaced by $\mathbf{RP}^1$ (here presented as circles). The line denoted with "$1 = 4$" represents the topological subspace of shapes $[r] \in \mathscr{b}_1^5$ with $r_1 = r_4$, i.e., with $\big(\{1,4\},1\big) \in C(r)$), etc.
Similarly, $t_2^5$ is $\mathscr{b}_2^5 \cong \mathbf{RP}^2$ (here presented without its line at infinity as $\mathbf{R}^2$) with the four single pair coincidences replaced by $\mathbf{RP}^1$ (presented as circles). The line denoted with "$1 \sim 2 \sim 5$" represents the topological subspace of shapes $[r] \in \mathscr{b}_2^5$ with $\big(\{1,2,5\},2\big) \in C(r)$, etc.
These topological spaces are homeomorphic as was discussed in Example 4.12.

$$[p] = \left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right] \quad \text{and} \quad [q] = \left[\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ w & x & 1 \\ y & z & 1 \end{pmatrix}\right]$$

The shape $[p]$ contains a frame in its first four coordinates, i.e., $[p] \in \mathscr{b}_2^5$. The shape $[q]$ is Tyler regular if $(w,x),(y,z) \in \mathbf{R}^2$ are distinct, i.e., if $(w,x) \neq (y,z)$. Through right-multiplication with a suitable non-singular matrix and left-multiplication with a suitable diagonal, non-singular matrix, one can standardize $[q]$ such that $(w,x) = (0,0)$ (w.l.o.g.). Then, $(y,z) \in \mathbf{R}^2 \backslash \{(0,0)\}$ with rescaling not changing the shape. The topological subspaces of Tyler regular shapes which cannot be separated from the shapes $[p] \in \mathscr{b}_1^5$ with a single pair coincidence $\big(\{i,5\},1\big) \in C(p)$ are thus homeomorphic to $\mathbf{RP}^1$. Hence, $t_2^5$ is homeomorphic to a connected sum of five real projective planes, i.e.,

$$t_2^5 \cong \mathbf{RP}^2 \# \mathbf{RP}^2 \# \mathbf{RP}^2 \# \mathbf{RP}^2 \# \mathbf{RP}^2;$$

see Figure 4.1.

Additionally to Lemma 4.11, $t_1^5$ and $t_2^5$ are also homeomorphic by a result about two-dimensional manifolds (Massey; 1991, Sect. I.7, Lem. 7.1) which states that the connected sum of a 2-torus $\mathbf{T}^2$ and a real projective plane $\mathbf{RP}^2$ is homeomorphic to the connected sum of three real projective planes, i.e.,

$$\mathbf{T}^2 \# \mathbf{RP}^2 \cong \mathbf{RP}^2 \# \mathbf{RP}^2 \# \mathbf{RP}^2. \qquad \diamond$$

Of course, a Tyler standardized projection matrix of a Tyler standardizable shape is only unique up to conjugation with sign matrices. One may remove the ambiguity of the remaining action of $\mathbf{C}_2^k$ by squaring all entries of $M_P = \big(m_{ij}\big)$. The emerging matrix $N_P = \big(n_{ij}\big)$ with

$$n_{ij} = m_{ij}^2 = \big(P_{i\cdot} P_{j\cdot}^t\big)^2$$

does then, of course, not depend on the choice of the sign matrix.

The symmetric matrix $N_P$ is itself again an inner product matrix, namely the inner product matrix to a configuration in the Euclidean space $\mathbf{Sym}(d+1)$ of symmetric matrices equipped with the Frobenius inner product $\langle A, B\rangle_F = \operatorname{tr}(AB)$ since

$$\begin{aligned}\left(N_P\right)_{ij} &= \left(P_{i\cdot}P_{j\cdot}^t\right)^2 \\ &= P_{i\cdot}P_{j\cdot}^t P_{j\cdot}P_{i\cdot}^t \\ &= \operatorname{tr}\left(P_{i\cdot}P_{j\cdot}^t P_{j\cdot}P_{i\cdot}^t\right) \\ &= \operatorname{tr}\left(P_{i\cdot}^t P_{i\cdot}P_{j\cdot}^t P_{j\cdot}\right) \\ &= \left\langle P_{i\cdot}^t P_{i\cdot}, P_{j\cdot}^t P_{j\cdot}\right\rangle_F.\end{aligned}$$

A configuration $\iota(P) = \left(P_{1\cdot}^t P_{1\cdot}, \ldots, P_{k\cdot}^t P_{k\cdot}\right)^t \in \left(\mathbf{Sym}(d+1)\right)^k$ of rank 1 orthogonal projection matrices is then Tyler standardized if

$$\left\langle P_{i\cdot}^t P_{i\cdot}, \mathbf{I}_{d+1}\right\rangle_F = \operatorname{tr}\left(P_{i\cdot}^t P_{i\cdot}\mathbf{I}_{d+1}\right) = \operatorname{tr}\left(P_{i\cdot}P_{i\cdot}^t\right) = P_{i\cdot}P_{i\cdot}^t = \tfrac{d+1}{k} \tag{4.14}$$

for all $i \in \{1, \ldots, k\}$ and

$$\sum_{i=1}^{k} P_{i\cdot}^t P_{i\cdot} = P^t P = \mathbf{I}_{d+1}, \tag{4.15}$$

i.e., Tyler standardized configurations in $\mathbf{Sym}(d+1)$ are configurations of orthogonal projection matrices (up to the factor $\frac{d+1}{k}$) which have constant angle to the identity matrix and are centered if the negative identity matrix is added to the configuration. The augmentation of the negative identity matrix to $\iota(P)$ does therefore not give any extra information. The inner product matrix of this augmented configuration will be denoted with $N_P^1$. Note that the Tyler standardized configuration $\iota(P)$ in $\mathbf{Sym}(d+1)$ corresponding to a shape $[P] \in \mathcal{K}_d^k$ is only unique up to conjugation with orthogonal matrices $B \in \mathbf{O}(d+1)$ since the Tyler standardization $P$ is only unique up to right-multiplication with orthogonal matrices $B \in \mathbf{O}(d+1)$ and left-multiplication with sign matrices (the latter action is removed by the embedding $\iota$).

For $d = 1$, the matrices $M_P$ and $N_P$ contain the same information, as was pointed out by Kent and Mardia (2012), so the mapping

$$\begin{array}{rccc}\nu: & \mathcal{K}_1^k & \longrightarrow & \mathbf{Sym}(k) \\ & [p] & \longmapsto & N_P\end{array} \tag{4.16}$$

is a topological embedding, i.e., injective and homeomorphic onto its image. Unfortunately, this is not true for all $d > 1$ and $k$ as Example 4.13 shows[2].

Of course, one might try to remove the ambiguity of the $\mathbf{C}_2^k$-action by considering the matrix $\operatorname{abs}(M_P)$ comprising of the absolutes of the entries of the inner product matrix $M_P$, but $\operatorname{abs}(M_P)$ does obviously contain the same information as $N_P$, so there is no extra insight.

**Example 4.13.** For $k = 6$ and $d = 2$, consider the matrix configuration

$$P = \frac{1}{\sqrt{2}\sqrt{1+g^2}} \cdot \begin{pmatrix} 0 & 1 & g \\ 0 & 1 & -g \\ -1 & -g & 0 \\ -1 & g & 0 \\ g & 0 & 1 \\ g & 0 & -1 \end{pmatrix} \in \mathcal{G}_2^6$$

[2]This example was discovered by Thomas Hotz in (Blumenthal; 1970, Ch. IX, Sect. 80).

with $g = \frac{1+\sqrt{5}}{2}$ being the golden ratio. This matrix configuration $P$ is Tyler standardized, while the corresponding orthogonal projection matrix is

$$M_P = PP^t = \begin{pmatrix} \frac{1}{2} & h & h & -h & -h & h \\ h & \frac{1}{2} & h & -h & h & -h \\ h & h & \frac{1}{2} & h & h & h \\ -h & -h & h & \frac{1}{2} & h & h \\ -h & h & h & h & \frac{1}{2} & -h \\ h & -h & h & h & -h & \frac{1}{2} \end{pmatrix}$$

with $h = \frac{1-g^2}{2(1+g^2)} = \frac{-g}{2(1+g^2)} = \frac{1}{2\sqrt{5}}$, i.e., this configuration consists of 6 evenly distributed landmarks on $\mathbf{RP}^2$ and is the projection of the icosahedron (12 evenly distributed landmarks on the sphere $\mathbf{S}^2$) with opposite landmarks identified.

Now, there are permutations $\sigma$ of the landmarks such that $[\sigma P] \neq [P]$, e.g. the permutation fixing the first 4 landmarks (a frame) and interchanging the remaining two. These configurations are distinct in their shapes, but they do have the same inner product matrix $N_P$, i.e., the map $\nu : \mathcal{K}_2^6 \to \mathbf{Sym}(6)$ mapping $[P]$ to $N_P$ is not injective, and thus no embedding. $\diamond$

For $d = 1$, the original shape $[p]$ can be reconstructed from $\nu([p]) = N_P$ resp. $N_P^1$ which has been pointed out by Kent and Mardia (2012). However, the following explicit reconstruction has not been considered before: for a Tyler standardized configuration $P \in \mathcal{K}_1^k$, recall that $N_P^1$ is the inner product matrix to the centered configuration $\left(P_{1\cdot}^t P_{1\cdot}, \dots, P_{k\cdot}^t P_{k\cdot}, -\mathbf{I}_2\right)^t$ of symmetric matrices, i.e., in the Euclidean space $\left(\mathbf{Sym}(2), \langle\cdot,\cdot\rangle_F\right)$. Note that $\mathbf{Sym}(2)$ is naturally isomorphic to the Euclidean space $\mathbf{R}^3$ by mapping the upper triangle of a symmetric matrix to $\mathbf{R}^3$ by multiplying the off-diagonal entry with $\sqrt{2}$, i.e.,

$$\begin{array}{ccc} \mathbf{Sym}(2) & \longrightarrow & \mathbf{R}^3 \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} & \longmapsto & \left(a_{11}, a_{22}, \sqrt{2}a_{12}\right), \end{array} \tag{4.17}$$

so augmented configurations in $\mathbf{Sym}(2)$ can be represented as $(k+1) \times 3$-dimensional matrices with the rows corresponding to symmetric matrices.

The inner product matrix $N_P^1 = (n_{ij})$ is a positive semi-definite similarity matrix, i.e., $N_P^1 \geqslant 0$ and

$$n_{ij} \leqslant n_{ii} \quad \text{for all } i, j. \tag{4.18}$$

Due to results from multidimensional scaling, see (Mardia et al.; 1995, Ch. 14), a centered configuration in $\mathbf{Sym}(2) \cong \mathbf{R}^3$ of $k+1$ landmarks can be constructed which has $N_P^1$ as its inner product matrix. Such a configuration is the $(k+1) \times 3$-dimensional matrix $S$ comprising of eigenvectors to the three largest eigenvalues of $N_P^1$ with their norms being $\sqrt{\frac{d+1}{k}}$. Any other centered configuration with inner product matrix $N_P^1$ is given by a rotation resp. reflexion of $S$ in $\mathbf{Sym}(2)$, i.e., $S$ is only unique up to the action of the orthogonal group $\mathbf{O}(3)$. However, we know that the $(k+1)$-st landmark of a feasible configuration is the negative identity matrix, whence the ambiguity is reduced to an action of $\mathbf{O}(2)$. This is, of course, also the ambiguity of the Tyler standardized configuration $\iota(P)$; in particular, $S$ is a rotation or reflexion of the original configuration $\left(P_{1\cdot}^t P_{1\cdot}, \dots, P_{k\cdot}^t P_{k\cdot}, -\mathbf{I}_2\right)^t$.

**Remark 4.14.** Multidimensional scaling can also be used to define a sample mean of projective shapes $[P_1], \dots, [P_n] \in \mathcal{K}_1^k$: the arithmetic mean of the inner product matrices $N_1^1, \dots, N_n^1 \in \mathbf{Sym}(k)$ is again a positive semi-definite similarity matrix, so it makes sense to look for the configuration in $\mathbf{Sym}(2)$ whose inner product matrix is the closest to this arithmetic mean. Again, multidimensional scaling is concerned with this topic, and, as above, the solution to this problem is given by a matrix comprising of eigenvectors to the three largest eigenvalues with their norms being the square root of the respective eigenvalue. The last landmark can be restandardized to the negative identity matrix. Then, the first $k$ landmarks may not be rank 1 projection matrices (and thus embedded elements of $\mathbf{RP}^1$), but they can be projected to a rank 1 projection matrix resp. to $\mathbf{RP}^1$ by mapping them to the eigenvector of the largest eigenvalue. The shape of the obtained configuration can then be considered as the mean of the sample $[P_1], \ldots, [P_n] \in \mathscr{k}_1^k$. Of course, the largest eigenvalues might not be unique in this computation, so the mean might not be unique in some cases.

This sample mean will not be discussed any further in this thesis, partly because it is unclear if this definition of a mean fits into the framework of the so-called Fréchet mean (see Chapter 5).$\diamond$

## 4.2 Metrization

The space of Tyler standardizable shapes $\mathscr{k}_d^k$ is a differentiable Hausdorff manifold, whence it is metrizable both by a differentiable embedding into Euclidean space (Lee; 2013, Thm. 6.15) and by definition of a Riemannian metric (Lee; 2013, Prop. 13.3).

### Embedding into metric space

In Section 4.1 we have already seen that $\mathscr{k}_1^k$ can be topologically embedded into the Euclidean space $\mathbf{Sym}(k)$ by mapping to inner product matrices $N$, see Equation (4.16). Kent and Mardia (2012) have shown that the distance on $\mathscr{k}_1^4$ induced by the Frobenius norm on $\mathbf{Sym}(k)$ matches then the Euclidean geometry of a planar triangle with its vertices corresponding to the Tyler semi-regular shapes (double pair coincidences). In particular, the topological embedding $\nu : \mathscr{k}_1^4 \hookrightarrow \mathbf{Sym}(4)$ is not a differentiable embedding.

Of course, $\mathscr{k}_d^k$ may be smoothly embedded into some Euclidean space as a differentiable Hausdorff manifold. However, we will discuss only a topological embedding into a metric space.

As mentioned above, the space $\mathbf{TSP}\big(\mathscr{k}_d^k\big)$ of Tyler standardized projection matrices is a topological subspace of $\mathbf{Sym}(k)$ of symmetric matrices which is a Euclidean space when equipped with the Frobenius inner product $\langle A, B\rangle_F = \operatorname{tr}(AB)$. Since $\mathbf{TSP}\big(\mathscr{k}_d^k\big)$ is the space of orthogonal projection matrices fulfilling Equation 4.10, it consists of matrices with trace $d + 1$ and norm $\sqrt{d+1}$. Hence, $\mathbf{TSP}\big(\mathscr{k}_d^k\big)$ is closed, bounded, and compact in this metric space. This fact is quite helpful to construct a topological embedding of the space $\mathscr{k}_d^k$ of Tyler standardizable shapes into a metric space. The remaining ambiguity of a Tyler standardized projection matrix to a projective shape is the action of the finite group $\mathbf{C}_2^k$ by conjugation, i.e., $\mathscr{k}_d^k$ is naturally homeomorphic to the space of equivalence classes

$$\mathbf{tsp}\big(\mathscr{k}_d^k\big) = \mathbf{TSP}\big(\mathscr{k}_d^k\big) \,/\, \mathbf{C}_2^k \tag{4.19}$$

This action only changes the sign pattern of a projection matrix and can be naturally carried forward to the space $\mathbf{Sym}(k)$ of symmetric matrices. In $\mathbf{Sym}(k)$ orthants are then identified by this group action, whence the quotient space

$$\mathbf{sym}(k) = \mathbf{Sym}(k) \,/\, \mathbf{C}_2^k. \tag{4.20}$$

is a space of orthants which are conglutinated along hyperplanes. Thus, $\mathscr{k}_d^k \cong \mathbf{tsp}\big(\mathscr{k}_d^k\big)$ is naturally topologically embedded in $\mathbf{sym}(k)$. Note that $\mathbf{sym}(k)$ consists of $2^{(k^2-k-2)/2}$ orthants since $\mathbf{Sym}(k)$ comprises of $2^{k(k+1)/2}$ orthants with $2^{k-1}$ of those each being identified by the action of $\mathbf{C}_2^k$.

Let $[\![M]\!] = \big\{sMs : s \in \mathbf{C}_2^k\big\} \in \mathbf{sym}(k)$ be the equivalence class of $M \in \mathbf{Sym}(k)$, and define a map $\mathbf{d} : \mathbf{sym}(k) \times \mathbf{sym}(k) \to \mathbf{R}_{\geqslant 0}$ by

$$\begin{aligned}\mathbf{d}^2\big([\![M_1]\!],[\![M_2]\!]\big) &= \min\big\{\|A-B\|_F^2 : A \in [\![M_1]\!], B \in [\![M_2]\!]\big\}\\ &= \min\big\{\|M_1 - sM_2s\|_F^2 : s \in \mathbf{C}_2^k\big\}\\ &= \min\big\{\|M_1\|_F^2 + \|sM_2s\|_F^2 - 2\operatorname{tr}(M_1sM_2s) : s \in \mathbf{C}_2^k\big\}\\ &= 2\big(d+1-\min\big\{\operatorname{tr}(M_1sM_2s) : s \in \mathbf{C}_2^k\big\}\big).\end{aligned}$$

**Proposition 4.15.** $\big(\mathbf{sym}(k),\mathbf{d}\big)$ *is a metric space.*

Proof. The map $\mathbf{d}$ is obviously a non-negative function, i.e., $\mathbf{d}\big([\![M_1]\!],[\![M_2]\!]\big) \geqslant 0$ for all $[\![M_1]\!],[\![M_2]\!] \in \mathbf{sym}(k)$, and symmetric in its arguments, i.e., $\mathbf{d}\big([\![M_1]\!],[\![M_2]\!]\big) = \mathbf{d}\big([\![M_2]\!],[\![M_1]\!]\big)$ for all $[\![M_1]\!],[\![M_2]\!] \in \mathbf{sym}(k)$.

For the triangle inequality note that $\|M_1\|_F = \|sM_1s\|_F$ for all $s \in \mathbf{C}_2^k$, whence

$$\begin{aligned}\mathbf{d}\big([\![M_1]\!],[\![M_2]\!]\big) &= \min\big\{\|M_1 - sM_2s\|_F : s \in \mathbf{C}_2^k\big\}\\ &= \min\big\{\|sM_1s - tM_2t\|_F : s,t \in \mathbf{C}_2^k\big\}.\end{aligned}$$

Let $[\![M_1]\!],[\![M_2]\!],[\![M_3]\!] \in \mathbf{sym}(k)$. Then, there are $M_i \in [\![M_i]\!]$ for $i \in \{1,2,3\}$ such that both $\mathbf{d}\big([\![M_1]\!],[\![M_2]\!]\big) = \|M_1 - M_2\|_F$ and $\mathbf{d}\big([\![M_2]\!],[\![M_3]\!]\big) = \|M_2 - M_3\|_F$. Consequently,

$$\begin{aligned}\mathbf{d}\big([\![M_1]\!],[\![M_2]\!]\big) + \mathbf{d}\big([\![M_2]\!],[\![M_3]\!]\big) &= \|M_1 - M_2\|_F + \|M_2 - M_3\|_F\\ &\geqslant \|M_1 - M_3\|_F\\ &\geqslant \mathbf{d}\big([\![M_1]\!],[\![M_3]\!]\big)\end{aligned}$$

finishing the proof. □

The metric $\mathbf{d}$ has been considered before by Kent and Mardia (2012) and is a so-called *Procrustes metric*, i.e., the distance of two equivalence classes is given by the shortest distance between representatives.

Of course, one could also consider the affine subspace of $\mathbf{Sym}(k)$ comprising of those symmetric matrices with constant diagonal equal to $\frac{d+1}{k}$ as a surrounding space. However, the embedding above seems to be more convenient for our purposes, see Chapter 5.

**Example 4.16.** In the case $d = 1$ and $k = 4$, the action of $\mathbf{C}_2^4$ identifies the spherical triangles of the space $\mathbf{TSP}\big(\mathcal{K}_1^4\big)$, see Example 4.10. Hence, the space $\mathbf{tsp}\big(\mathcal{K}_1^4\big)$ with the metric $\mathbf{d}$ introduced above is geometrically a spherical triangle with Euclidean distance, as was noted before by Kent and Mardia (2012). ◊

As we have discussed on page 54, the space $\mathbf{TSP}\big(\mathcal{K}_d^k\big) \subset \mathbf{Sym}(k)$ of Tyler projection matrices is closed and bounded in the Frobenius norm (and thus compact) for any $d$ and $k$. Consequently, $\mathbf{tsp}\big(\mathcal{K}_d^k\big)$ is compact as a topological subspace of $\mathbf{sym}(k)$, whence $\big(\mathbf{tsp}\big(\mathcal{K}_d^k\big),\mathbf{d}\big)$ is a complete metric space (Arkhangel'skiĭ and Fedorchuk; 1990, Sect. 5.3, Thm. 7).

## Riemannian metric

Tyler standardization offers two distinct ways to define Riemannian metrics on $\boldsymbol{t}_d^k$.

First, consider the space of Tyler regular standardized configuration matrices $\mathcal{TSC}\big(\boldsymbol{t}_d^k\big)$. As mentioned in Section 4.1, this space is an embedded submanifold of the (orthogonal) Stiefel manifold, and both are embedded submanifolds of the Euclidean space $\mathbf{R}^{k\times(d+1)}$ which is a Riemannian manifold in the natural way. Hence, $\mathcal{TSC}\big(\boldsymbol{t}_d^k\big)$ inherits the subspace metric (also called pullback or induced metric). Since the elements of $\mathbf{O}(d+1)$ and of $\mathbf{C}_2^k$ act as isometries on $\mathbf{R}^{k\times(d+1)}$, this Riemannian metric is well-defined on the quotient space.

Let $P$ be a Tyler standardized projection matrix. The tangent space $T_P\,\mathcal{TSC}\big(t_d^k\big)$ at $P$ is then given by

$$T_P\,\mathcal{TSC}\big(t_d^k\big) = \Big\{A \in \mathbf{R}^{k\times(d+1)} : P^tA + A^tP = 0,\ A_{i\cdot}P_{i\cdot}^t + P_{i\cdot}A_{i\cdot}^t = 0,\ i \in \{1,\dots,k\}\Big\}. \tag{4.21}$$

This can easily be seen by differentiating Equations (4.1) and (4.2). The induced Riemanninan metric is given by the Frobenius inner product $\langle A, B\rangle_F = \operatorname{tr}(A^tB)$ for $A, B \in T_P\,\mathcal{TSC}\big(t_d^k\big)$.

Alternatively, one may consider the space $\mathbf{TSP}\big(t_d^k\big)$ of Tyler standardized projection matrices. This space is a submanifold of the embedded Grassmannian $\iota(\mathbf{Gr}(k, d+1))$, and both are embedded submanifolds of the Euclidean space $\mathbf{Sym}(k)$ of symmetric matrices with scalar product $\langle A, B\rangle_F = \operatorname{tr}(AB)$ which is a Riemannian manifold in the natural way. Hence, the space of Tyler standardized projection matrices inherits the subspace metric. Since the elements of $\mathbf{C}_2^k$ act as isometries on $\mathbf{Sym}(k)$, this Riemannian metric is well-defined on the quotient space $\mathbf{tsp}\big(t_d^k\big) \cong t_d^k$. The tangent spaces $T_M\,\mathbf{TSP}\big(t_d^k\big)$ and $T_{[\![M]\!]}\,\mathbf{tsp}\big(t_d^k\big)$ are identical since $\mathbf{C}_2^k$ is a finite group.

Let $M$ be a Tyler standardized projection matrix. The tangent space $T_M\,\mathbf{TSP}\big(t_d^k\big)$ at $M$ in the space $\mathbf{TSP}\big(t_d^k\big)$ of Tyler standardized projection matrices is then a linear subspace of the tangent space

$$T_M\,\mathbf{Gr}(k, d+1) = \Big\{[M, A] \in \mathbf{Sym}(k) : A \in \mathfrak{o}(k)\Big\}$$

of $M$ considered in the Grassmannian with $[A, B] = AB - BA$ denoting the Lie bracket. The tangent vectors of $M$ in the space $\mathbf{TSP}\big(t_d^k\big)$ of Tyler standardized projection matrices additionally preserve the constant diagonal of $M$, i.e.,

$$\begin{aligned} T_M\,\mathbf{TSP}\big(t_d^k\big) &= \Big\{[M, A] \in \mathbf{Sym}(k) : A \in \mathfrak{o}(k),\ \operatorname{diag}[M, A] = 0\Big\} \\ &= \Big\{[M, A] \in \mathbf{Sym}(k) : A \in \mathfrak{o}(k),\ \operatorname{tr}\big(e_ie_i^t[M, A]\big) = 0\ \forall i \in 1,\dots,k\Big\}. \end{aligned} \tag{4.23}$$

Again, the Riemannian metric is given by the Frobenius inner product $\langle A, B\rangle_F = \operatorname{tr}(AB)$ for $A, B \in T_M\,\mathbf{TSP}\big(t_d^k\big)$.

Of course, a classical result of Cartan (1952) states that Riemannian metrics on the Grassmannian invariant under $\mathbf{O}(k)$ are unique up to positive scale. As it turns out, the Riemannian metrics presented here are identical up to a scale of 2, cf. e.g. (Harandi et al.; 2013).

If $d+1$ and $k$ are relatively prime, then $\mathbf{TSP}\big(t_d^k\big) = \mathbf{TSP}\big(\mathcal{K}_d^k\big)$ is a compact space, whence the Riemannian metric is complete by the Hopf-Rinow theorem (Jost; 2011, Thm. 1.7.1). Unfortunately, the Riemannian metric on $\mathbf{TSP}\big(t_d^k\big)$ is not complete if $d+1$ and $k$ have a common divisor greater than 1 (i.e., if $\mathbf{TSP}\big(t_d^k\big) \neq \mathbf{TSP}\big(\mathcal{K}_d^k\big)$).

The Riemannian metric above cannot be continued on $\mathbf{TSP}\big(\mathcal{K}_d^k\big)$ resp. $\mathbf{tsp}\big(\mathcal{K}_d^k\big)$ in the case that $d+1$ and $k$ are not relatively prime; recall that in this case $\mathbf{TSP}\big(\mathcal{K}_d^k\big)$ is not a manifold.

**Example 4.17.** As we have seen in Example 4.10, $\mathbf{TSP}\big(\mathcal{K}_1^4\big)$ can be seen as three great circles on the sphere in $\mathbf{R}^3$ with radius $\frac{1}{2}$ which intersect orthogonally. Recall that the intersection points correspond to the Tyler semi-regular shapes. The Riemannian metric on $\mathbf{TSP}\big(t_1^4\big)$ discussed above fits into this representation, i.e., $\mathbf{tsp}\big(t_1^4\big)$ with the Riemannian metric above is a spherical triangle with orthogonal edges and removed vertices. In particular, $\mathbf{tsp}\big(t_1^4\big)$ is not complete in this metric. Further, the metric cannot be extended to $\mathbf{tsp}\big(\mathcal{K}_1^4\big)$. $\diamond$

Note that the space $\mathcal{K}_d^k$ for not relatively prime $d+1$ and $k$ might be smoothly embeddable into a Euclidean space and metrizable by a Riemannian metric, but neither can be achieved through Tyler standardization. The reason for that is that Tyler standardization handles Tyler semi-regular shapes not “in a differentiable way.”

# Chapter 5

# Averaging projective shapes

The mean of a random value $X \in \mathcal{M}$ or of a sample $X_1, \ldots, X_n \in \mathcal{M}$ in a metric, but non-Euclidean space $(\mathcal{M}, d)$ cannot be defined as the usual population resp. sample mean since $\mathcal{M}$ does generally not carry a vector space structure. As a remedy, Fréchet (1948) introduced the *Fréchet population mean* $\hat{\mu}$ as the set of minimizers

$$\mu = \operatorname*{arg\,min}_{p \in \mathcal{M}} \mathbf{E}\, d^2(X, p) \tag{5.1}$$

of the expected quadratic distance to the random value resp. the *Fréchet sample mean*

$$\bar{\mu}_n = \operatorname*{arg\,min}_{p \in \mathcal{M}} \sum_{i=1}^{n} d^2(X_i, p) \tag{5.2}$$

in the case of an empirical distribution. This generalizes the usual population and sample means for random values on Euclidean spaces.

On the space $\mathcal{K}_d^k$ of Tyler standardizable shapes two different metrics have been discussed in Section 4.2: first, there is the Riemannian metric on $\boldsymbol{t}_d^k$ which would lead to a so-called *intrinsic mean shape*. This mean will not be discussed in this thesis. Secondly, there is the metric given by embedding Tyler standardizable shapes into the metric space $\big(\mathbf{sym}(k), \mathbf{d}\big)$. The corresponding Fréchet mean is then called *extrinsic mean shape* since the it uses the metric $\mathbf{d}$ of the surrounding metric space. The computation of this mean will be discussed in this chapter. As we will see in Section 5.1, the Fréchet function

$$F\big([\![R]\!]\big) = \tfrac{1}{n} \sum_{i=1}^{n} \mathbf{d}^2\big([\![M_i]\!], [\![R]\!]\big) \tag{5.3}$$

for a sample $[\![M_1]\!], \ldots, [\![M_n]\!] \in \mathbf{tsp}\big(\mathcal{K}_d^k\big)$ decomposes into a Euclidean and a projection term. For concentrated data the computation of the Euclidean term is rather straight-forward. Meanwhile, the projection term can only be estimated numerically as of now. As a remedy, *Tyler population and sample mean shapes* are introduced in Section 5.2 for which consistency can be proven. Finally, both means are discussed in a few examples in Section 5.3.

## 5.1 Extrinsic mean shape

Kent's shape space $\mathcal{K}_d^k$ can be topologically embedded into the metric space $\big(\mathbf{sym}(k), \mathbf{d}\big)$ introduced in Section 4.2, so in this setup the *extrinsic population mean shape* $[\![\mu]\!]$ of a random variable $[\![M]\!]$ with values in $\mathbf{tsp}\big(\mathcal{K}_d^k\big) \cong \mathcal{K}_d^k$ is

$$[\![\mu]\!] = \operatorname*{arg\,min}_{[\![R]\!] \in \mathbf{tsp}(\mathcal{K}_d^k)} \mathbf{E}\, \mathbf{d}^2\big([\![M]\!], [\![R]\!]\big). \tag{5.4}$$

Analogously, the *extrinsic sample mean shape* $[\![\bar{\mu}_n]\!]$ of a sample $[\![M_1]\!], \ldots, [\![M_n]\!] \in \mathbf{tsp}\big(\mathcal{K}_d^k\big)$ of projective shapes is then the minimizer in $\mathbf{tsp}\big(\mathcal{K}_d^k\big) \cong \mathcal{K}_d^k$ of the *Fréchet function*

$$\begin{array}{rccc} F: & \mathbf{sym}(k) & \longrightarrow & \mathbf{R}_{\geqslant 0} \\ & [\![R]\!] & \longmapsto & \frac{1}{n}\sum_{i=1}^{n} \mathbf{d}^2\big([\![M_i]\!], [\![R]\!]\big), \end{array} \tag{5.5}$$

i.e.,

$$[\![\bar{\mu}_n]\!] = \underset{[\![R]\!] \in \mathbf{tsp}(\mathcal{K}_d^k)}{\arg\min} \frac{1}{n} \sum_{i=1}^{n} \mathbf{d}^2\big([\![M_i]\!], [\![R]\!]\big). \tag{5.6}$$

Note that both $[\![\mu]\!]$ and $[\![\bar{\mu}_n]\!]$ might be sets, but there always is an extrinsic population resp. sample mean since $\mathbf{tsp}\big(\mathcal{K}_d^k\big)$ is compact, i.e., $[\![\mu]\!], [\![\bar{\mu}_n]\!] \neq \varnothing$.

**Proposition 5.1.** *Let $[\![M_1]\!], \ldots, [\![M_n]\!] \in \mathbf{tsp}\big(\mathcal{K}_d^k\big)$ be independent, identically distributed random variables with unique extrinsic population mean shape $[\![\mu]\!]$. Every measurable choice from the extrinsic sample mean shape $[\![\bar{\mu}_n]\!]$ is then a strongly consistent estimator of the extrinsic population mean shape $[\![\mu]\!]$, i.e.,*

$$[\![\bar{\mu}_n]\!] \overset{n \to \infty}{\longrightarrow} [\![\mu]\!] \quad a.\ s. \tag{5.7}$$

Proof. Recall that $\mathbf{tsp}\big(\mathcal{K}_d^k\big)$ is compact. Then, the statement immediately follows from more general results by Ziezold (1977) and Bhattacharya and Patrangenaru (2003) (Thm. 2.3). □

Unfortunately, the computation of an extrinsic sample mean is not straightforward. The function $F$ can be decomposed into two parts: let $R \in [\![R]\!]$ and choose $M_i \in [\![M_i]\!]$, $i \in \{1, \ldots, n\}$, such that $\mathbf{d}\big([\![R]\!], [\![M_i]\!]\big) = \|R - M_i\|_F$ for all $i \in \{1, \ldots, n\}$. Of course, the choice of $M_i \in [\![M_i]\!]$ might not be unique. Further, let

$$M = \underset{A \in \mathbf{Sym}(k)}{\arg\min} \frac{1}{n} \sum_{i=1}^{n} \big\|A - M_i\big\|_F^2 = \frac{1}{n} \sum_{i=1}^{n} M_i \tag{5.8}$$

be the Euclidean sample mean of the representing matrices $M_i$. Then,

$$\begin{aligned} F\big([\![R]\!]\big) &= \frac{1}{n} \sum_{i=1}^{n} \big\|R - M_i\big\|_F^2 \\ &= \big\|R - M\big\|_F^2 + \frac{1}{n} \sum_{i=1}^{n} \big\|M - M_i\big\|_F^2, \end{aligned} \tag{5.9}$$

so $F$ decomposes into a term measuring the distance from $R \in \mathbf{Sym}(k)$ to some "Euclidean" sample mean $M$ and a term measuring the distance from $M$ to the data.

To find the minimum of $F$ in $\mathbf{tsp}(\mathcal{K}_d^k)$ for a given sample, there are two problems remaining: first, the representing Tyler standardizations $M_i$ in Equation (5.9) depend on $[\![R]\!]$. In particular, it does not suffice to minimize the first term of Equation (5.9), but the sum of both terms has to minimized. However, there are at most $2^{k-1}$ choices for $M_i$, and thus at most $\big(2^{k-1}\big)^n = 2^{n(k-1)}$ possibilities for $M$. Note that there might not be an $[\![R]\!] \in \mathbf{tsp}(\mathcal{K}_d^k)$ for all possible "Euclidean" means $M$. Hence, a solution to this problem is to compare the minimizers of the term $\|R - M\|_F^2$ for all these valid choices for $M$. Of course, one may compute all $2^{n(k-1)}$ possible Euclidean means and their corresponding minimizers of the term $\|R - M\|_F^2$, but then one has to check the minimizer afterward if they have indeed the corresponding $M_i$, $i \in \{1, \ldots, n\}$, as closest representatives of the data (see Algorithm 1). Unfortunately, the number of possibilities $2^{n(k-1)}$ increases exponentially with the sample size $n$ and might be rather large. As we will see in Proposition 5.2, this problem simplifies if the data are sufficiently concentrated.

The second problem is—of course—to identify the minimizers of $\|R - M\|_F^2$, i.e., to compute the projection in the sense of best approximation of $M \in \mathbf{Sym}(k)$ to the topological subspace of Tyler standardized projection matrices $\mathbf{TSP}\big(\mathcal{K}_d^k\big) \subset \mathbf{Sym}(k)$. Note that this projection exists and is unique since $\mathbf{TSP}\big(\mathcal{K}_d^k\big)$ is compact.

The first problem vanishes if the sample is sufficiently concentrated.

**Data:** observations $[p_1], \ldots, [p_n] \in \mathfrak{t}_d^k$
**Result:** (set of) extrinsic sample mean shape(s)

compute orbits $[\![M_1]\!], \ldots, [\![M_n]\!]$ of Tyler standardized projection matrices to $[p_1], \ldots, [p_n]$;
compute all arithmetic means $\sum_{i=1}^n M_i$ with $M_i \in [\![M_i]\!]$, $i \in \{1, \ldots, n\}$;
compute all projections in the sense of best approximations of the arithmetic means to $\mathbf{TSP}\big(\mathfrak{t}_d^k\big)$;
check if the projections have indeed the corresponding $M_i$, $i \in \{1, \ldots, n\}$, as closest representatives of the data; if not, remove the projection;
return the valid projection(s) which minimize(s) $F$

**Algorithm 1:** algorithm for computation of extrinsic mean (sets) of projective shape data

**Proposition 5.2.** *Let $[\![N]\!] \in \mathbf{sym}(k)$ be the equivalence class of a symmetric matrix under conjugation with sign matrices, and define*

$$\varepsilon = \varepsilon\big([\![N]\!]\big) = \min\big\{\|N - sNs\|_F : s \in \mathbf{C}_2^k, s \neq \pm\mathbf{I}_k\big\} \tag{5.10}$$

*as the minimal distance between symmetric matrices in $[\![N]\!]$. Further, let $[P_1], \ldots, [P_n] \in \mathfrak{K}_d^k$ be a sample of projective shapes and $[\![M_1]\!], \ldots, [\![M_n]\!] \in \mathbf{tsp}\big(\mathfrak{K}_d^k\big)$ their corresponding equivalence classes of Tyler standardized projection matrices.*

*(i) If $\mathbf{d}\big([\![M_i]\!], [\![N]\!]\big) < \frac{\varepsilon}{4}$ for all $i \in \{1, \ldots, n\}$, i.e., if the data are concentrated in an open ball $B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$ with radius $\frac{\varepsilon}{4}$ and center $[\![N]\!]$, then there are unique Tyler standardized projection matrices $M_i \in [\![M_i]\!]$ corresponding to the data such that*

$$\mathbf{d}\big([\![M_i]\!], [\![M_j]\!]\big) = \big\|M_i - M_j\big\|_F \quad \textit{and} \quad \mathbf{d}\big([\![A]\!], [\![M_i]\!]\big) = \big\|A - M_i\big\|_F \tag{5.11}$$

*for all $i, j \in \{1, \ldots, n\}$ and for all $[\![A]\!] \in B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$ with $\mathbf{d}\big([\![A]\!], [\![N]\!]\big) = \|A - N\|_F$ for $A \in \mathbf{Sym}(k)$. In particular, the minimizer of $F$ in $\mathbf{sym}(k)$ is in $B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$, and it is uniquely given by the equivalence class of*

$$M = \tfrac{1}{n} \sum_{i=1}^n M_i. \tag{5.12}$$

*(ii) If $[\![N]\!] \in \mathbf{tsp}\big(\mathfrak{K}_d^k\big)$ and the data are concentrated in an open ball $B_{\frac{\varepsilon}{8}}\big([\![N]\!]\big)$ with radius $\frac{\varepsilon}{8}$, then the minimizer of $F$ in $\mathbf{tsp}\big(\mathfrak{K}_d^k\big)$ is an element of $B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$ and is the minimizer of $\|R - M\|_F^2$ with $M$ as in Equation* (5.12)*.*

**Remark 5.3.** Let $[p] \in \mathfrak{tsr}_d^k$ be a Tyler semi-regular shape with decomposition

$$\big(\{1, \ldots, i\}, j\big), \big(\{i+1, \ldots, k\}, d+1-j\big) \in C(p).$$

Any Tyler standardized projection matrix $M_P$ to $[p]$ is then block diagonal (see page 54 in Section 4.1). Consequently,

$$sM_Ps = M_P$$

for any sign matrix $s \in \mathbf{C}_2^k$ which is constant on the blocks, i.e., $s_{i_1 i_1} = s_{i_2 i_2}$ for all $i_1, i_2 \in \{1, \ldots, i\}$ resp. for all $i_1, i_2 \in \{i+1, \ldots, k\}$. In particular, $\varepsilon\big([\![M_P]\!]\big) = 0$ for Tyler semi-regular shapes, so Proposition 5.2 is not helpful if the data are concentrated around a Tyler semi-regular shape. ◊

Proof (Proposition 5.2). (i) Choose a matrix $N \in \mathbf{Sym}(k)$ from the orbit of $[\![N]\!] \in \mathbf{sym}(k)$, and let $A, B \in \mathbf{Sym}(k)$ be the matrices in $[\![A]\!] \in B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big) \subset \mathbf{sym}(k)$ resp. $[\![B]\!] \in B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$ which are closest to $N$, i.e.,

$$\mathbf{d}\big([\![A]\!], [\![N]\!]\big) = \big\|A - N\big\|_F < \tfrac{\varepsilon}{4} \qquad \text{and} \qquad \mathbf{d}\big([\![B]\!], [\![N]\!]\big) = \big\|B - N\big\|_F < \tfrac{\varepsilon}{4}.$$

Then,

$$\mathbf{d}\big([\![A]\!],[\![B]\!]\big) \leqslant \mathbf{d}\big([\![A]\!],[\![N]\!]\big) + \mathbf{d}\big([\![N]\!],[\![B]\!]\big) < \tfrac{\varepsilon}{2},$$

and

$$\begin{aligned}\big\|A - sBs\big\|_F &\geqslant \big\|N - sNs\big\|_F - \big\|N - A\big\|_F - \big\|sBs - sNs\big\|_F \\ &> \varepsilon - \tfrac{\varepsilon}{4} - \tfrac{\varepsilon}{4} \\ &= \tfrac{\varepsilon}{2}.\end{aligned}$$

for all sign matrices $s \in \mathbf{C}_2^k$, $s \neq \pm\mathbf{I}_k$, whence necessarily

$$\mathbf{d}\big([\![A]\!],[\![B]\!]\big) = \big\|A - B\big\|_F.$$

In particular, this is true if $[\![A]\!] = [\![M_i]\!]$ and $[\![B]\!] = [\![M_j]\!]$ for $i, j \in \{1, \ldots, n\}$, proving Equations (5.11).

As for the statement that the minimizer of $F$ is in $B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$, let $[\![Z]\!] \notin B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$ with $Z \in \mathbf{Sym}(k)$ such that $\mathbf{d}\big([\![Z]\!],[\![N]\!]\big) = \|Z - N\|_F$, and define

$$\begin{aligned}I &= \big\{i \in \{1, \ldots, n\} : \mathbf{d}\big([\![Z]\!],[\![M_i]\!]\big) = \|Z - M_i\|_F\big\}, \\ J &= \big\{j \in \{1, \ldots, n\} : \mathbf{d}\big([\![Z]\!],[\![M_j]\!]\big) = \|Z - s_j M_j s_j\|_F \neq \|Z - M_j\|_F \text{ for some } s_j \neq \pm\mathbf{I}_k\big\}.\end{aligned}$$

Then,

$$F\big([\![Z]\!]\big) = \tfrac{1}{n}\Big(\sum_{i \in I}\|Z - M_i\|_F^2 + \sum_{j \in J}\|Z - s_j M_j s_j\|_F^2\Big)$$

with $I \,\dot\cup\, J = \{1, \ldots, n\}$ and $s_j \neq \pm\mathbf{I}_k$ for $j \in J$. The summands $\|Z - s_j M_j s_j\|_F^2$ indexed by $j \in J$ are greater than $\big(\tfrac{3\varepsilon}{4} - \|Z - N\|_F\big)^2$ since

$$\begin{aligned}\varepsilon &\leqslant \big\|N - s_j N s_j\big\|_F \\ &\leqslant \big\|N - Z\big\|_F + \big\|Z - s_j M_j s_j\big\|_F + \big\|s_j M_j s_j - s_j N s_j\big\|_F \\ &< \big\|N - Z\big\|_F + \big\|Z - s_j M_j s_j\big\|_F + \tfrac{\varepsilon}{4}\end{aligned}$$

for $s_j \neq \pm\mathbf{I}_k$. If all summands in $F\big([\![Z]\!]\big)$ are greater than $\frac{\varepsilon^2}{16}$, then $F\big([\![N]\!]\big) < \frac{\varepsilon^2}{16} \leqslant F\big([\![Z]\!]\big)$, and if $J = \varnothing$, then $F\big([\![M]\!]\big) < F\big([\![Z]\!]\big)$ for $M = \frac{1}{n}\sum_{i=1}^n M_i$. So, assume that there is an $i \in I$ such that $\|Z - M_i\|_F < \frac{\varepsilon}{4}$ (w.l.o.g.), and thus $\mathbf{d}\big([\![N]\!],[\![Z]\!]\big) = \|N - Z\|_F < \frac{\varepsilon}{2}$. If $\mathbf{d}\big([\![N]\!],[\![Z]\!]\big) = \|Z - N\|_F = \frac{\varepsilon}{4}$, then $\big\|Z - s_j M_j s_j\big\|_F > \frac{3\varepsilon}{4} - \|Z - N\|_F = \frac{\varepsilon}{2} > \big\|Z - M_j\big\|_F$ for all $s_j \neq \pm\mathbf{I}_k$ and for all $j \in \{1, \ldots, k\}$, i.e., $J = \varnothing$, thus $F\big([\![M]\!]\big) < F\big([\![Z]\!]\big)$ for $M = \frac{1}{n}\sum_{i=1}^n M_i$ as we have seen shortly before. Consequently, we can assume $\mathbf{d}\big([\![N]\!],[\![Z]\!]\big) = \|Z - N\|_F > \frac{\varepsilon}{4}$. Then, the orbit $[\![Y]\!]$ of

$$Y = Z - 2\tfrac{\|Z-N\|_F - \frac{\varepsilon}{4}}{\|Z-N\|_F} \cdot (Z - N) = Z - \big(2\|Z - N\|_F - \tfrac{\varepsilon}{2}\big) \cdot \tfrac{Z-N}{\|Z-N\|_F}$$

is in the ball $B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$, i.e., $[\![Y]\!]$ has smaller distance to $[\![N]\!]$ than $[\![Z]\!]$, see Figure 5.1:

$$\begin{aligned}\mathbf{d}\big([\![Y]\!],[\![N]\!]\big) &= \big\|Y - N\big\|_F \\ &= \Big\|Z - 2\tfrac{\|Z-N\|_F - \frac{\varepsilon}{4}}{\|Z-N\|_F} \cdot (Z - N) - N\Big\|_F \\ &= \Big(1 - 2\tfrac{\|Z-N\|_F - \frac{\varepsilon}{4}}{\|Z-N\|_F}\Big) \cdot \big\|Z - N\big\|_F \\ &= \tfrac{\varepsilon}{2} - \big\|Z - N\big\|_F \\ &< \tfrac{\varepsilon}{4}.\end{aligned}$$

Further, $[\![Y]\!]$ and has smaller distance to $[\![M_i]\!]$ than $Z$ for all $j \in J$ and $i \in I$:

$$\begin{aligned}\mathbf{d}\big([\![Y]\!],[\![M_j]\!]\big) &\leqslant \mathbf{d}\big([\![Y]\!],[\![N]\!]\big) + \mathbf{d}\big([\![N]\!],[\![M_j]\!]\big) \\ &< \tfrac{\varepsilon}{2} - \big\|Z - N\big\|_F + \tfrac{\varepsilon}{4} \\ &< \big\|Z - s_j M_j s_j\big\|_F \\ &= \mathbf{d}\big([\![Z]\!],[\![M_j]\!]\big)\end{aligned}$$

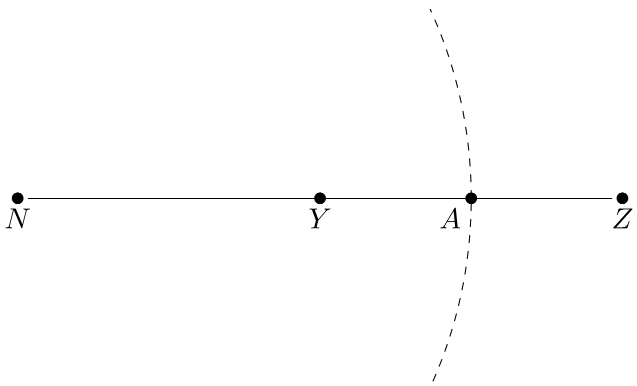



**Figure 5.1:** The construction of $Y$. $A = N + \frac{\varepsilon}{4}\frac{Z-N}{\|Z-N\|}$ is on the boundary of $B_{\frac{\varepsilon}{4}}(N)$. $Y$ is the reflection of $Z$ at $A$.

for all $j \in J$. Additionally,

$$\begin{aligned}
\mathbf{d}^2\big([\![Y]\!],[\![M_i]\!]\big) &= \big\|Y - M_i\big\|_F^2 \\
&= \big\|N - M_i\big\|_F^2 + \big\|Y - N\big\|_F^2 - 2\,\langle Y - N, N - M_i\rangle \\
&= \big\|N - M_i\big\|_F^2 + \big\|Y - N\big\|_F \cdot \Big(\big\|Y - N\big\|_F - 2\underbrace{\big\langle \tfrac{Y-N}{\|Y-N\|_F}, N - M_i\big\rangle}_{=\langle \frac{Z-N}{\|Z-N\|_F}, N-M_i\rangle}\Big) \\
&< \big\|N - M_i\big\|_F^2 + \big\|Z - N\big\|_F \cdot \Big(\big\|Z - N\big\|_F - 2\big\langle \tfrac{Z-N}{\|Z-N\|_F}, N - M_i\big\rangle\Big) \\
&= \mathbf{d}^2\big([\![Z]\!],[\![M_i]\!]\big)
\end{aligned}$$

for all $i \in I$. Hence, $F\big([\![Y]\!]\big) < F\big([\![Z]\!]\big)$, so the minimizer of $F$ in $\mathbf{sym}(k)$ is indeed in $B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$, and it is the orbit of

$$M = \operatorname*{arg\,min}_{A \in \mathbf{Sym}(k)} \tfrac{1}{n}\sum_{i=1}^{n} \big\|A - M_i\big\|_F^2 = \tfrac{1}{n}\sum_{i=1}^{n} M_i$$

which is in $B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$ since the latter ball is convex.

For (ii), let $[\![N]\!] \in \mathbf{tsp}\big(\mathcal{K}_d^k\big)$ be a projective shape, and let the data be concentrated in an open $\frac{\varepsilon}{8}$-ball with center $[\![N]\!]$. The minimizer of $F$ in $\mathbf{tsp}\big(\mathcal{K}_d^k\big)$ is then an element of the $\frac{\varepsilon}{4}$-ball with center $[\![N]\!]$ since, for $[\![R]\!] \notin B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$,

$$F\big([\![R]\!]\big) = \tfrac{1}{n}\sum_{i=1}^{n}\mathbf{d}^2\big([\![R]\!],[\![M_i]\!]\big) \geqslant \tfrac{1}{n}\sum_{i=1}^{n}\Big(\frac{\varepsilon}{8}\Big)^2 > \tfrac{1}{n}\sum_{i=1}^{n}\mathbf{d}^2\big([\![N]\!],[\![M_i]\!]\big) = F\big([\![N]\!]\big).$$

Due to (i),

$$F\big([\![R]\!]\big) = \Big\|R - \tfrac{1}{n}\sum_{i=1}^{n} M_i\Big\|_F^2 + \tfrac{1}{n}\sum_{j=1}^{n}\Big\|\tfrac{1}{n}\sum_{i=1}^{n} M_i - M_j\Big\|_F^2$$

for all $[\![R]\!] \in B_{\frac{\varepsilon}{4}}\big([\![N]\!]\big)$, so the minimizer of $F$ in $\mathbf{tsp}\big(\mathcal{K}_d^k\big)$ minimizes the distance to the "Euclidean" sample mean in this case. □

Note that projective shape data are often highly concentrated, e.g. both in the case of face recognition and in the case of fusion of images, so Proposition 5.2 is a very useful result.

The second problem in the minimization of the Fréchet function $F$ is the computation of the closest Tyler standardized projection matrix to a symmetric matrix, i.e., the computation of the minimizer of

$$\begin{array}{rccc} G: & \mathbf{TSP}\big(\mathcal{K}_d^k\big) & \longrightarrow & \mathbf{R}_{\geqslant 0} \\ & R & \longmapsto & \big\|R - M\big\|_F^2 \end{array} \tag{5.13}$$

for a matrix $M \in \mathbf{Sym}(k)$. The mapping of $M$ to the corresponding minimizer is—of course—the projection in the sense of best approximation. However, this projection might not be unique for some $M$.

To find a local minimizer of $G(R) = \|R - M\|_F^2$, there is the idea of using the method of the steepest descent on a Riemannian manifold which generalizes the usual steepest descent method by conducting a search along a curve in the manifold through the iteration step whose differential at the iteration step equals the gradient of the scalar field which is to minimize: let $g : \mathcal{M} \to \mathbf{R}$ be a differentiable scalar field on a Riemannian manifold $\mathcal{M}$, and let $r$ be a *retraction*, i.e., a smooth mapping from the tangent bundle $T\mathcal{M}$ to $\mathcal{M}$ with restrictions $r_p = r|_{T_p\mathcal{M}}$ such that

(i) $r_p(0_p) = p$ for all $p \in \mathcal{M}$ with $0_p \in T_p\mathcal{M}$ denoting the zero element of $T_p\mathcal{M}$, and

(ii) the differential $\mathrm{D}r_p(0_p)$ is the identity on $T_p\mathcal{M}$, or equivalently, the curve $\gamma_\xi(t) = r_p(t\xi)$ satisfies $\dot{\gamma}_\xi(0) = \xi$ for all $\xi \in T_p\mathcal{M}$.

Then, the update formula is given by

$$p^{[i+1]} = r_{p^{[i]}}\big(-t_i\,\mathbf{grad}\,g\big(p^{[i]}\big)\big), \tag{5.14}$$

so the next iteration step is found by first moving along the negative gradient with a step size $t_i$, and then to "project" this point back to the manifold with the retraction $r$. For suitable step sizes $t_i$, this gradient descent algorithm guarantees convergence to critical points, see (Absil et al.; 2008, Ch. 4). Note that the requirements of the algorithm may be weakened: it suffices if $r$ is defined on a small neighborhood of $0_p \in T_p\mathcal{M}$ for each $p \in \mathcal{M}$, and it suffices if the directions of the iteration step have negative scalar product with the gradient at the iteration step.

For complete Riemannian manifolds there always is a retraction; in this case, the exponential map $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ is defined on all of $T_p\mathcal{M}$ for all $p \in \mathbf{M}$ by the theorem of Hopf-Rinow (Jost; 2011, Thm. 1.7.1). For incomplete Riemannian manifolds, the exponential map is only defined on a neighborhood of $0_p \in T_p\mathcal{M}$ for each $p \in \mathcal{M}$, but—as noted before—this is sufficient for the method of the steepest descent. However, the computation of the exponential map might be numerically challenging since the exponential map is itself defined as a solution of non-linear ordinary differential equations which are, in general, not numerically cheap to compute.

In the case of minimizing $G(R) = \|R - M\|_F^2$, this steepest descent gradient algorithm is applicable on $\mathbf{TSP}\big(t_d^k\big)$ as an embedded manifold with well-known tangent bundle (see Section 4.2). It is not applicable on $\mathbf{TSP}\big(\mathcal{K}_d^k\big)$ in case that there are Tyler semi-regular shapes since $\mathbf{TSP}\big(\mathcal{K}_d^k\big)$ is then not a manifold.

The gradient of $G$ at $R \in \mathbf{TSP}\big(t_d^k\big)$ is the orthogonal projection of $2(R - M)$ to the tangent space at $R$.

Regarding the retraction, one can of course use the Riemannian exponential. However, we suggest using the following map

$$\mathrm{TylP} : \quad \mathbf{Sym}(k) \quad \longrightarrow \quad \mathbf{tsp}\big(\mathcal{K}_d^k\big) \tag{5.15}$$

which maps a symmetric matrix $A \in \mathbf{Sym}(k)$ to the equivalence class of Tyler standardized projection matrices corresponding to a configuration matrix $P$ whose column space spans the space spanned by the eigenvectors to the $d + 1$ largest eigenvalues of $A$. Then, it seems sensible to map a tangent vector $V \in T_{M_1}\mathbf{TSP}\big(t_d^k\big)$ at $M_1$ to the matrix $M_2 \in [\![M_2]\!] = \mathrm{TylP}(M_1 + V)$ which is closest to $M_1$, i.e., $\mathbf{d}\big([\![M_1]\!], [\![M_2]\!]\big) = \|M_1 - M_2\|_F$. Unfortunately, it is unclear if this procedure defines a retraction.

**Remarks 5.4.** (a) The choice of $P$ in the definition of TylP is irrelevant since $PB$ for $B \in \mathbf{GL}(d+1)$ gives the same equivalence class of Tyler standardized projection matrices;

(b) TylP is not well-defined if the configuration matrix $P$ is Tyler extended- or Tyler irregular, and it might be set-valued if the $(d + 1)$-st and $(d + 2)$-nd eigenvalues of $A$ are equal; we call

**Data:** symmetric matrix $M \in \mathbf{Sym}(k)$; step size $\gamma$; stopping criteria $\delta$ and $t$;
**Result:** a local minimizer $M^{\mathrm{loc}}$ of the function $G(R) = \|R - M\|_F^2$

compute Tyler standardized projection matrix $M^{[0]} \in \mathrm{TylP}(M)$ which is closest to $M$;
$M^{\mathrm{loc}} \leftarrow M^{[0]}$;
$i \leftarrow 0$;
**while** $\left\|\mathrm{proj}_{T_{M^{\mathrm{loc}}}\mathbf{TSP}(t_d^k)}\big(M - M^{\mathrm{loc}}\big)\right\|_F > \delta$ *and* $i < t$ **do**
  $N \leftarrow M^{\mathrm{loc}} + \gamma \cdot \mathrm{proj}_{T_{M^{\mathrm{loc}}}\mathbf{TSP}(t_d^k)}\big(M - M^{\mathrm{loc}}\big)$;
  compute Tyler standardized projection matrix $M^{[i+1]} \in \mathrm{TylP}(N)$ which is closest to $M$;
  $M^{\mathrm{loc}} \leftarrow M^{[i+1]}$;
  $i \leftarrow i + 1$
**end**

**Algorithm 2:** algorithm for computation of a local minimizer of $G$

symmetric matrices with identical $(d+1)$-st and $(d+2)$-nd eigenvalues *Tyler focal points*. The part of the domain of TylP where TylP is well-defined and unique is open and dense in $\mathbf{Sym}(k)$. On the part of the domain, where TylP maps uniquely to Tyler standardized matrices of Tyler regular shapes, TylP is differentiable since Tyler standardization is continuously differentiable (see Remark 4.8). For our purposes it suffices to define TylP on positive semi-definite, symmetric matrices;

(c) TylP is invariant under $\mathbf{C}_2^k$, i.e., $\mathrm{TylP}(A) = \mathrm{TylP}(sAs)$ for all $s \in \mathbf{C}_2^k$ and $A \in \mathbf{Sym}(k)$ (if $A = UDU^t$ is an eigendecomposition of $A$, then $sAs = (sU)D(sU)^t$ is an eigendecomposition of $sAs$); in particular, TylP is well-defined on $\mathbf{sym}(k)$. $\diamond$

As for the initial value recall that the Frobenius norm of $R - M$ is small if $R$ and $M$ have similar eigenvalues to similar eigenvectors. Since we are looking for the closest Tyler standardized projection matrix to $M$, the matrix $M^{[0]} \in \mathrm{TylP}(M)$ which is closest to $M$ should be a good guess for the minimizer of $G(R) = \|R - M\|_F^2$. Thus, we use $M^{[0]}$ as the initial value for the algorithm.

Starting with $M^{[0]}$, the algorithm works by projecting the negative gradient $2\big(M - M^{[i]}\big)$ of $G$ in $\mathbf{Sym}(k)$ to the tangent space of $\mathbf{TSP}\big(t_d^k\big)$ at the point $M^{[i]}$ of the current iterate. The next iterate $M^{[i+1]}$ is found by proceeding with a small step size $\gamma > 0$ on $\mathbf{TSP}\big(t_d^k\big)$ into the direction of the projected negative gradient. This last step is done by computing the Tyler standardized projection matrix to the eigenvectors corresponding to the $d+1$ largest eigenvalues of

$$M^{[i]} + 2\gamma \cdot \mathrm{proj}_{T_{M^{[i]}}\mathbf{TSP}(t_d^k)}\big(M - M^{[i]}\big), \tag{5.16}$$

which is closest to $M$.

While it is unclear if this algorithm (see Algorithm 2) always converges to the global minimizer or even to a critical value, our examples in Section 5.3 hint at that this is indeed a valid construction. Note that one can check the type of critical value again numerically.

## 5.2 Tyler mean shape

As an alternative to Fréchet means, we introduce another mean: for a random variable $[\![M]\!] \in \mathbf{tsp}\big(\mathcal{k}_d^k\big)$ define the *Tyler population mean shape* to be

$$[\![\tau]\!] = \mathrm{TylP}\Big(\operatorname*{arg\,min}_{[\![A]\!] \in \mathbf{sym}(k)} \mathbf{E}\,\mathbf{d}^2\big([\![M]\!], [\![A]\!]\big)\Big), \tag{5.17}$$

and analogously for a sample $[\![M_1]\!], \ldots, [\![M_n]\!] \in \mathbf{tsp}\big(\mathfrak{K}_d^k\big)$ define the *Tyler sample mean shape* to be

$$[\![\bar{\tau}_n]\!] = \operatorname{TylP}\Big( \operatorname*{arg\,min}_{[\![A]\!] \in \mathbf{sym}(k)} \sum_{i=1}^{n} \mathbf{d}^2\big([\![M_i]\!], [\![A]\!]\big)\Big). \tag{5.18}$$

The Tyler mean shape can be understood as the "Tyler standardization" of the Fréchet mean in $\big(\mathbf{sym}(k), \mathbf{d}\big)$, i.e. of the minimizer in $\mathbf{sym}(k)$ of the Fréchet function $F$ (see Equation (5.3)). Consequently, the Tyler sample mean shape is easier to compute than the extrinsic sample mean shape from Section 5.1 since there is no projection anymore. However, there are still $2^{n(k-1)}$ possible values for the extrinsic sample mean in $\mathbf{sym}(k)$. In the case that the data are highly concentrated, the Tyler sample mean shape can be computed using Proposition 5.2, and it equals the initial value $M^{[0]}$ from the previous section.

The Tyler mean shape $[\![\tau]\!]$ resp. $[\![\bar{\tau}_n]\!]$ is a set in two cases: the Fréchet mean in $\big(\mathbf{sym}(k), \mathbf{d}\big)$ might itself be a set or it might be a Tyler focal point such that the mapping TylP is set-valued.

Again, strong consistency can be proven for this definition of a population resp. sample mean shape.

**Proposition 5.5.** *Let $[\![M_1]\!], \ldots, [\![M_n]\!] \in \mathbf{tsp}\big(\mathfrak{K}_d^k\big)$ be independent, identically distributed random variables with unique extrinsic population mean $[\![R]\!]$ in $\mathbf{sym}(k)$ such that $\operatorname{TylP}\big([\![R]\!]\big)$ is well-defined and unique. Every measurable choice from the Tyler sample mean shape $[\![\bar{\tau}_n]\!]$ is then a strongly consistent estimator of the Tyler population mean shape $[\![\tau]\!]$, i.e.,*

$$[\![\bar{\tau}_n]\!] \overset{n\to\infty}{\longrightarrow} [\![\tau]\!] \quad a.\ s. \tag{5.19}$$

Proof. Due to (Ziezold; 1977) resp. (Bhattacharya and Patrangenaru; 2003, Thm. 2.3), the Fréchet sample mean in $\big(\mathbf{sym}(k), \mathbf{d}\big)$ is a strongly consistent estimator of the corresponding Fréchet population mean since the data lie in the closed ball $\overline{B_{\sqrt{d+1}}\big([\![0]\!]\big)} \subset \mathbf{sym}(k)$. Further, recall that TylP is a continuous mapping on its domain, see Remarks 5.4. Then, the result is an immediate consequence of the continuous mapping theorem. □

## 5.3 Examples in $\mathfrak{K}_2^5$

For the discussion of the presented methods, we will compute some extrinsic and Tyler sample means in the case $k = 5$ and $d = 2$. For these $k$ and $d$, there are no Tyler semi-regular shapes since $k = 5$ and $d + 1 = 3$ are relatively prime, so $\mathfrak{K}_2^5 = \boldsymbol{t}_2^5$.

All computations have been performed using our own code based on the software package `R` (version 3.3.1) (R Core Team; 2016). For the computation of a solution of Equation (4.5), the package `ICSNP` (Nordhausen et al.; 2015) has been used. The extrinsic mean shapes have been computed with Algorithms 1 and 2 (step size $\gamma = .01$, stop criteria $\delta = .0001$ and $t = 1000$).

First, we will discuss the computation of weighted means of two shapes. For the computation of the mean of $n$ shapes randomly drawn from $\big\{[p], [q]\big\} \subset \boldsymbol{t}_2^5$, it suffices to compute the projection of $2^{k-1} = 16$ arithmetic means of Tyler standardized projection matrices since we can choose a fixed representative of $[\![M_P]\!]$ and compute the arithmetic mean with all choices $M_Q \in [\![M_Q]\!]$.

**Example 5.6.** We compute the weighted extrinsic and Tyler mean shapes of two shapes in $\boldsymbol{t}_2^5$.

(a) Let

$$P = \begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ -5 & 8 & 1 \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 8 & 0 & 1 \end{pmatrix}.$$

Figure 5.2 shows the extrinsic and Tyler means of 16 shapes drawn from $\big\{[P], [Q]\big\} \subset \boldsymbol{t}_2^5$. The shapes are presented in the chart given by standardizing the first four landmarks to a square in

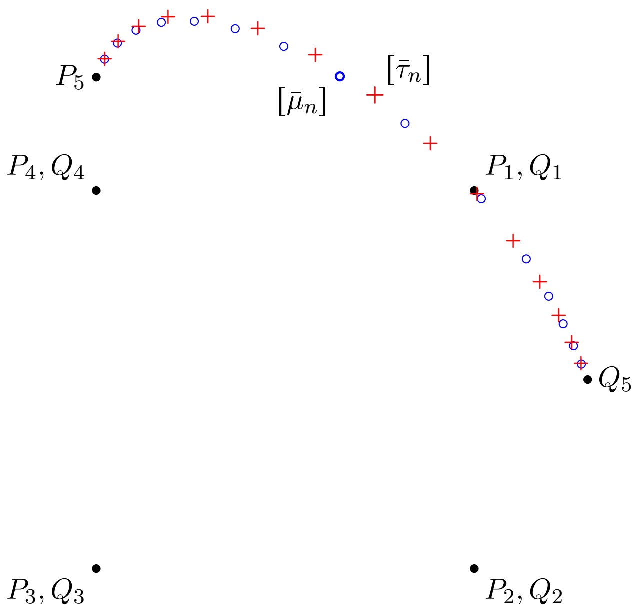



**Figure 5.2:** Two shapes $[P], [Q]$ from Example 5.6 (a) and their weighted extrinsic ($\circ$) resp. Tyler ($+$) means in the chart mapping the frame in their first four landmarks to the square given by $P_{\{1,\ldots,4\},\{1,2\}}$.

$\mathbf{R}^2 \subset \mathbf{RP}^2$, cf. Example 3.16. The means follow almost the same path, but differ in position. While the extrinsic sample mean of $\{[P],[Q]\}$ is approximately

$$[\bar{\mu}_n] = \left[\begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 1.443 & 8.021 & 1 \end{pmatrix}\right],$$

the Tyler sample mean is approximately

$$[\bar{\tau}_n] = \left[\begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 2.373 & 7.527 & 1 \end{pmatrix}\right].$$

(b) As a warning, we consider another pair of shapes: let

$$R = \begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 0 & -4 & 1 \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} 5 & 5 & 1 \\ -5 & -5 & 1 \\ 5 & -5 & 1 \\ -5 & 5 & 1 \\ 3 & 6 & 1 \end{pmatrix}$$

with corresponding Tyler standardized projection matrices

$$M_R = \begin{pmatrix} 0.6 & 0.149 & -0.268 & 0.375 & 0.070 \\ 0.149 & 0.6 & -0.175 & -0.268 & 0.339 \\ -0.268 & -0.175 & 0.6 & 0.149 & 0.339 \\ 0.375 & -0.268 & 0.149 & 0.6 & 0.070 \\ 0.070 & 0.339 & 0.339 & 0.070 & 0.6 \end{pmatrix}$$

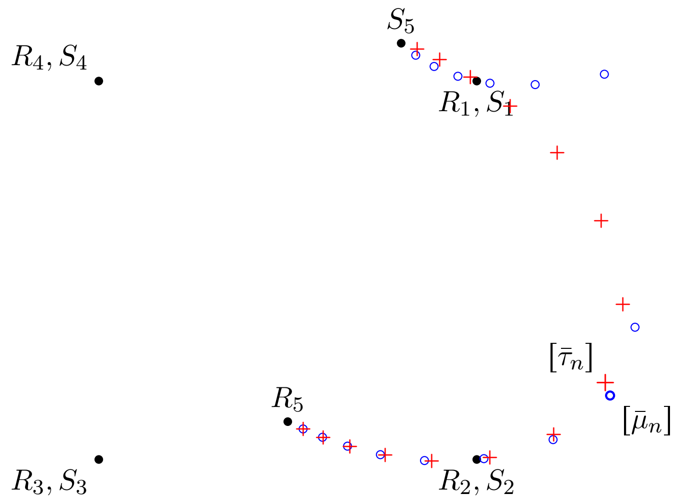



**Figure 5.3:** Two shapes $[R], [S]$ from Example 5.6 (b) and their weighted extrinsic ($\circ$) resp. Tyler ($+$) means in the chart mapping the frame in their first four landmarks to the square given by $R_{\{1,\ldots,4\},\{1,2\}}$.

and

$$M_S = \begin{pmatrix} 0.6 & 0.227 & 0.078 & 0.183 & -0.386 \\ 0.227 & 0.6 & -0.367 & 0.190 & 0.133 \\ 0.078 & -0.367 & 0.6 & 0.313 & -0.027 \\ 0.183 & 0.190 & 0.313 & 0.6 & 0.269 \\ -0.386 & 0.133 & -0.027 & 0.269 & 0.6 \end{pmatrix}.$$

Figure 5.3 shows the extrinsic and Tyler means of 16 shapes drawn from $\big\{[R], [S]\big\} \subset t_2^5$. Again, the shapes are presented in the chart given by standardizing the first four landmarks to a square in $\mathbf{R}^2 \subset \mathbf{RP}^2$. While the extrinsic sample mean of $\big\{[R], [S]\big\}$ is approximately

$$[\bar{\mu}_n] = \left[\begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 8.527 & -3.310 & 1 \end{pmatrix}\right],$$

the Tyler sample mean is approximately

$$[\bar{\tau}_n] = \left[\begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 8.402 & -2.966 & 1 \end{pmatrix}\right].$$

However, the weighted means do not follow the same path since the extrinsic mean in $\mathbf{tsp}(t_2^5)$ is not the projection in the sense of best approximation of the extrinsic mean in $\mathbf{sym}(5)$ for all possible samples. For a sample comprising of six observations of $[R]$ and ten observations of $[S]$, the extrinsic sample mean in $\mathbf{sym}(5)$ is

$$M = \tfrac{1}{16}(6 \cdot M_R + 10 \cdot M_S) = \begin{pmatrix} 0.6 & 0.198 & -0.052 & 0.255 & -0.215 \\ 0.198 & 0.6 & -0.295 & 0.018 & 0.210 \\ -0.052 & -0.295 & 0.6 & 0.252 & 0.110 \\ 0.255 & 0.018 & 0.252 & 0.6 & 0.194 \\ -0.215 & 0.210 & 0.110 & 0.194 & 0.6 \end{pmatrix}$$

with eigenvalues 0.999, 0.949, 0.811, 0.189, and 0.051. The projection in the sense of best approximation of $M$ is farther away from the sample than the projection of the "Euclidean" sample mean

$$N = \tfrac{1}{16}\big(6 \cdot M_R + 10 \cdot sM_S s\big) = \begin{pmatrix} 0.6 & 0.086 & 0.149 & 0.255 & -0.268 \\ 0.086 & 0.6 & -0.295 & 0.219 & 0.210 \\ 0.149 & -0.295 & 0.6 & 0.140 & 0.110 \\ 0.255 & 0.219 & 0.140 & 0.6 & 0.142 \\ -0.268 & 0.210 & 0.110 & 0.142 & 0.6 \end{pmatrix}$$

with

$$s = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}.$$

This simple example shows that the extrinsic mean shape might cause some undesired phenomena when working with unconcentrated data. ♢

Additionally, we discuss an application of Proposition 5.2 for concentrated data.

**Example 5.7.** Let $[P]$ be the shape of

$$P = \begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 10 & 5 & 1 \end{pmatrix},$$

and let $[\![M_P]\!]$ be the equivalence class of Tyler standardized projection matrices of $[P]$. Then, $\varepsilon = \varepsilon\big([\![M_P]\!]\big) \approx 1.13$. We consider a sample of $n = 25$ shapes $[Q_m]$, $1 \leqslant m \leqslant n$, around $[P]$ with

$$Q_m = \begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 10 + x_m & 5 + y_m & 1 \end{pmatrix}.$$

Here, $x_m$ and $y_m$, $1 \leqslant m \leqslant n$, are independently drawn from a uniform distribution on $[-1.5, 1.5] \subset \mathbf{R}$. In the sample we considered, the equivalence classes $[\![M_m]\!]$ to $[Q_m]$, $1 \leqslant m \leqslant n$, were in a ball with center $[\![M_P]\!]$ and radius $\frac{\varepsilon}{4}$, as we have checked numerically. Therefore, Proposition 5.2 can be applied for the computation of both the extrinsic and the Tyler sample mean shapes of $[Q_1], \ldots, [Q_n]$. The results are practically identical:

$$[\bar{\mu}_n] = [\bar{\tau}_n] = \left[\begin{pmatrix} 5 & 5 & 1 \\ 5 & -5 & 1 \\ -5 & -5 & 1 \\ -5 & 5 & 1 \\ 10.118 & 5.243 & 1 \end{pmatrix}\right],$$

see Figure 5.4.

This suggests that the numerically challenging computation of the projection in the sense of best approximation can be avoided with clear conscience. ♢

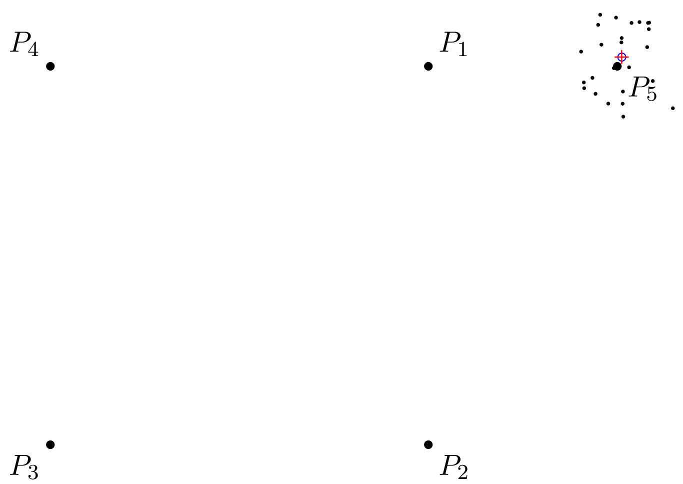



**Figure 5.4:** $[P]$ and the sample from Example 5.7 in the chart mapping the frame in their first 4 landmarks to the square given by $P_{\{1,\ldots,4\},\{1,2\}}$. Its extrinsic ($\circ$) and Tyler ($+$) sample means are practically identical at $(10.118, 5.243)$.

# Chapter 6

# Discussion and outlook

This chapter summarizes and discusses the main results and original contributions of this thesis. Further, it provides a collection of unsolved problems for future research.

## 6.1 Summary

The main objective of this work was to determine reasonable topological subspaces of projective shape space. To accomplish that, a detailed topological discussion of projective shape space was presented. It turned out that the topological entities of projective shape space are more intricate than in similarity or affine shape space where the topological subspace of free shapes is a differentiable Hausdorff manifold. In projective shape space the topological subspace $f_d^k$ of shapes with trivial isotropy group gives rise to a differentiable T1 manifold which is never Hausdorff for any $k > d+2$ (Theorem 3.24). However, the shapes that cannot be separated from another shape by an open neighborhood were characterized (Proposition 3.10), and we consequently determined the topological subspaces which are Hausdorff (Proposition 3.12). Additionally, a reasonable class of topological subspaces was identified, and easy-to-check conditions were determined for which such a topological subspace is a differentiable Hausdorff manifold (Theorem 3.27) and maximal (Corollary 3.28).

The topological subspace of Tyler regular shapes is an element of this class. While this subspace is indeed a differentiable Hausdorff manifold and a sensible choice for a topological subspace of projective shape space, there are cases when the subspace of Tyler regular shapes is not a maximal choice in this class (Proposition 3.29).

The advantage of Tyler regular shapes is the existence of a Tyler standardization, i.e. a projective pre-shape (Theorem 4.3). Using Tyler standardization, the definition of Procrustes metrics on the topological subspace of Tyler regular shapes is possible through embeddings. To one of these metrics, the computation of an extrinsic sample mean for projective shape data was discussed (Section 5.1). Additionally, the Tyler mean was introduced which is easier to compute (Section 5.2). For both means consistency has been proven, and examples have been discussed (Section 5.3).

## 6.2 Contributions

The main contribution of the thesis is the thorough discussion of the topology of projective shape space. Another noteworthy achievement is the definition and computation of a sample mean shape using Tyler standardization.

In detail, the substantial contributions are:

- A sensible list of requirements has been presented which a reasonable topological subspace of projective shape space has to satisfy (page 21).

- Projective subspace constraints are used for the description of irregularity of configurations resp. shapes. Calculation rules for projective subspace constraints have been presented (Lemma 2.5). The notion of "total decomposition" has been introduced. Using the latter, a configuration is called decomposable if its total decomposition is non-trivial. In particular, a decomposable shape possesses a matrix representative which is block diagonal (Proposition 2.7). A key result shows that decomposable shapes are not free, and vice versa (Proposition 3.1). This immediately gives a stratification of projective shape space (Proposition 3.2).

- Using the notion of the blur and a generalization of the method of distinct speeds of convergence introduced by Kent et al. (2011), it was determined which shapes $[q]$ cannot be separated from another shape $[p]$ by an open neighborhood of $[p]$. Indeed, a shape $[p] \in \mathfrak{a}_d^k$ can be separated from all less regular shapes (Proposition 3.10). This result is useful for the determination of T1 subspaces. As it turns out, the largest reasonable T1 subspace is the subspace of free shapes.

- Additionally, the Hausdorff subspaces were characterized, again using the method of distinct speeds of convergence Proposition 3.12.

- The topological subspace of free shapes carries the structure of a topological manifold. Charts were constructed by generalizing the notion of frames to the new notion of pseudo-frames. These charts are compatible rendering the topological subspace of the free a differentiable manifold (Theorem 3.24).

- The idea of Kent and Mardia (2012) of bounding the number of landmarks in a topological subspace was generalized, and the class of topological subspaces bounded by projective subspace numbers was introduced. These subspaces are by definition closed under permutations and respect the hierarchy of projective subspace constraints (Section 3.5). Under simple bounds to the projective subspace numbers, these topological subspaces of projective shape space are Hausdorff and open subsets of the subspace of free shapes, hence differentiable manifolds (Theorem 3.27). Maximality in this class is achieved by exhaustion of these bounds (Corollary 3.28).

- The space of Tyler regular shapes is an example for a topological subspace bounded by projective subspace numbers. Using our more general results, it was shown that this subspace is indeed a differentiable Hausdorff manifold, but only maximal if the greatest common divisor of $k$ and $d + 1$ is at most 2 (Proposition 3.29).

- A complete proof was given for the statement of Kent et al. (2011) that Tyler standardization is only possible for Tyler regular and Tyler semi-regular shapes (Theorem 4.3). The latter are decomposable, but only exist if the greatest common divisor of $k$ and $d + 1$ is larger than 1, i.e., if $k$ and $d + 1$ are not relatively prime. The topological subspace of Tyler standardizable shapes is a differentiable manifold in either case (page 50).

- When representing Tyler standardizable shapes as equivalence classes of Tyler standardized projection matrices, one obtains a Procrustes metric on the space of Tyler standardizable shapes by embedding Tyler standardizable shapes into the space of equivalence classes of symmetric matrices (Section 4.2). In this setup, the corresponding extrinsic mean was discussed (Section 5.1). While the extrinsic sample mean shape is a strongly consistent estimator of the corresponding extrinsic population mean shape (Proposition 5.1), the computation of the extrinsic sample mean is rather difficult. A method to compute this mean was given, but proving its correctness appears difficult.

- As a remedy, a new mean for projective shapes, the Tyler mean, was introduced which is easier to compute. Again, the Tyler sample mean shape is a strongly consistent estimator of the corresponding Tyler population mean shape (Proposition 5.5).

- The computation of both means simplifies if the data are sufficiently concentrated (Proposition 5.2).

- Both means were discussed and compared in elementary examples (Section 5.3).

## 6.3 Outlook

While the objective to find reasonable topological subspaces of projective shape space has been achieved, there remain several interesting questions for future research:

- Are there "natural" standardizations, embeddings or Riemannian metrics for other reasonable topological subspaces? Are there embeddings into Euclidean spaces? Is there a sensible way to embed Tyler standardizable shapes smoothly?

- The construction of confidence regions for the Tyler mean shape should be rather straightforward as the images under TylP of respective confidence regions of the extrinsic mean in $\mathbf{sym}(k)$. The map TylP is differentiable when well-defined and unique, so there should be sufficient estimates for the image of a confidence region around the extrinsic sample mean.

- Is $\mathfrak{a}_d^k = \left(\mathbf{RP}^d\right)^k/\mathbf{PGL}(d)$ the right shape space for uncalibrated cameras? While projective geometry is useful for image analysis, one should always remember that real world cameras are Euclidean devices taking measurements in a Euclidean space. In particular, some effects cannot happen in reality; e.g. landmarks cannot be pushed beyond infinity by a hyperplane-to-hyperplane projective transformation, and often there is information about the camera, e.g. whether object and film are on the same side of the optical center, or not. This extra information should of course be taken into account in applications—and this might lead in turn to new interesting shape spaces...

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