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Zusammenfassung

Wir untersuchen relative Entropie im Rahmen von Modellen, die auf (geometrischen) Lévy-Prozessen basieren. Sei $S = (S_t)_{s\in[0,T]}$, $T > 0$, ein auf einem Wahrscheinlichkeitsraum $(\Omega, \mathscr{F}, \mathbb{P})$ definierter geometrischer Lévy-Prozess; das bedeutet, S sei ein stochastischer Prozess der Form

$$S_t = S_0 \exp(X_t),\ t \in [0,T],$$

wobei $S_0 > 0$ konstant ist und $X = (X_t)_{t\in[0,T]}$ ein Lévy-Prozess mit charakteristischem Tripel (b, σ^2, ν) verknüpft mit der abgeschnittenen Funktion h, welche durch $h(x) = x1_{\{|x|\leq 1\}}$, $x \in \mathbb{R}$, definiert ist.

Dies ist das Basismodell unserer Untersuchung. Wir interessieren uns für diejenigen Martingalmaße $\mathbb{Q}$, für die der Prozess S ein $\mathbb{Q}$-Martingal ist und $\mathbb{Q}$ äquivalent zum ursprünglichen Maß $\mathbb{P}$.

Die Menge der absolut stetigen Martingalmaße $\mathscr{M}_a$ kann eine der folgenden drei verschiedenen Formen annehmen: 1) leer; 2) bestehend aus nur einem Maß (im Fall ohne Sprünge, klassisches Black–Scholes–Modell); 3) bestehend aus einer unendlichen Anzahl an Martingalmaßen. Wir interessieren uns besonders für den letzten Fall.

Seien $\mathbb{Q}$ und $\mathbb{P}$ zwei Wahrscheinlichkeitsmaße auf $(\Omega, \mathscr{F})$ und sei $\mathscr{G}$ eine Unter-σ-Algebra von $\mathscr{F}$. Die relative Entropie $I(\mathbb{Q}, \mathbb{P})_{\mathscr{G}}$ von $\mathbb{Q}$ bezüglich $\mathbb{P}$ über $\mathscr{G}$ ist definiert durch

$$I_{\mathscr{G}}(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathscr{G}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathscr{G}}\right], & \text{if } \mathbb{Q}|_{\mathscr{G}} \ll \mathbb{P}|_{\mathscr{G}}, \\ +\infty, & \text{sonst.} \end{cases}$$

Ein Wahrscheinlichkeitsmaß $\mathbb{Q}_0 \in \mathscr{M}$ heißt Minimales Entropie Martingalmaß (MEMM), falls es

$$I(\mathbb{Q}_0, \mathbb{P}) = \min_{\mathbb{Q}\in\mathscr{M}_a} I(\mathbb{Q}, \mathbb{P}) = \min_{\mathbb{Q}\in\mathscr{M}_a} \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

erfüllt.

Der Abstand zwischen $\mathbb{P}$ und $\mathscr{M}_a$ wird definiert durch

$$I(\mathscr{M}_a, \mathbb{P}) = \inf_{\mathbb{Q}\in\mathscr{M}_a} I(\mathbb{Q}, \mathbb{P}).$$

In dieser Arbeit suchen wir nach dem MEMM $\mathbb{Q}^*$ in der Klasse $\mathscr{M}_a$ für unser Basismodell und untersuchen die Verbindung zwischen diesem MEMM $\mathbb{Q}^*$ und dem sogenannten Esscher Martingalmaß (EMM) $\mathbb{Q}^E$, welches folgendermaßen definiert ist:
Sei $\kappa \in \mathbb{R}$ so, dass $\mathbb{E}[\exp(\kappa X_t)] < \infty$ für $t \in [0, T]$, und setze

$$Z_t^\kappa := \frac{\mathrm{d}\mathbb{Q}^\kappa}{\mathrm{d}\mathbb{P}}|_{\mathscr{F}_t} = \frac{\exp(\kappa X_t)}{\mathbb{E}[\exp(\kappa X_t)]} \text{ für } t \in [0, T].$$

Das Wahrscheinlichkeitsmaß $\mathbb{Q}^\kappa$ ist das EMM, falls X ein $\mathbb{Q}^\kappa$-Martingal auf $[0, T]$ ist. Wir bezeichnen das EMM auf $[0, T]$, so es denn existiert, mit $\mathbb{Q}^E$.

Eines der Hauptresultate der Arbeit ist die Gleichheit der Begriffe MEMM und EMM für unser Basismodell.

Die Arbeit ist in vier Kapitel gegliedert, gefolgt von zwei Appendix. In Kapitel 1 sammeln wir Hauptresultate der Maßtheorie und stochastischen Analysis. In Kapitel 2 liefern wir eine Einführung in das Problem des EMM. In Kapitel 3 untersuchen wir das Modell von Preisprozessen, welches auf einem exponentiellen (geometrischen) zusammengesetzten Poisson-Prozess basiert, und zeigen seine Verbindung zum Ein-Schritt-Modell. In Kapitel 4 sind die Hauptuntersuchungsobjekte Modelle, die von linearen und exponentiellen (geometrischen) Lévy-Prozessen getrieben werden. Hier werden die Hauptresultate der Arbeit aufgeführt, einschließlich einer hinreichenden Bedingung für die Existenz des EMM, des Zusammenfallens des EMM und des MEMM für von linearen und exponentiellen Lévy-Prozessen getriebenen Modelle und einer Reihe wichtiger expliziter Gleichungen für den Wert der relativen Entropie des EMM bezüglich des ursprünglichen Wahrscheinlichkeitsmaßes. In Appendix A sind die wichtigsten Eigenschaften der technischen Funktionen ψ und φ, welche im Haupttext der Arbeit umfangreich benutzt werden, gesammelt. In Appendix B wenden wir die in Kapitel 2 entwickelte allgemeine Theorie auf ein spezielles Modell, das Ein-Schritt-Modell, an.

Abstract

We investigate relative entropy in the frame of models based on (geometric) Lévy processes.

Let $S = (S_t)_{s \in [0,T]}$, $T > 0$, be a geometric Lévy process defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, that is, S is a stochastic process of the form:

$$S_t = S_0 \exp(X_t),\ t \in [0,T],$$

where $S_0 > 0$ is a constant and $X = (X_t)_{t \in [0,T]}$ is a Lévy process with characteristic triplet (b, σ^2, ν) associated with the standard truncation function h defined by $h(x) = x 1_{\{|x| \leq 1\}}$, $x \in \mathbb{R}$.

This is our basic model for investigation. We are interested in such martingale measures $\mathbb{Q}$ that process S is $\mathbb{Q}$-martingale and $\mathbb{Q}$ is equivalent to the original measure $\mathbb{P}$. The set of the absolutely continuous martingale measures $\mathscr{M}_a$ may have one of three different forms: 1) empty; 2) consists of just one measure (case without jumps, the classic Black–Scholes model); 3) consists of the infinite number of martingale measures. We have special interest in the last case.

Let $\mathbb{Q}$ and $\mathbb{P}$ be two probability measures on $(\Omega, \mathscr{F})$ and $\mathscr{G}$ some sub-σ-field of $\mathscr{F}$. The relative entropy $I(\mathbb{Q}, \mathbb{P})_{\mathscr{G}}$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathscr{G}$ is defined by

$$I_{\mathscr{G}}(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{G}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{G}}\right], & \text{if } \mathbb{Q}|_{\mathscr{G}} \ll \mathbb{P}|_{\mathscr{G}}, \\ +\infty, & \text{otherwise.} \end{cases}$$

A probability measure $\mathbb{Q}_0 \in \mathscr{M}_a$ will be called the minimal entropy martingale measure (MEMM) if it satisfies

$$I(\mathbb{Q}_0, \mathbb{P}) = \min_{\mathbb{Q} \in \mathscr{M}_a} I(\mathbb{Q}, \mathbb{P}) = \min_{\mathbb{Q} \in \mathscr{M}_a} \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right].$$

The distance between $\mathbb{P}$ and $\mathscr{M}_a$ is defined by

$$I(\mathscr{M}_a, \mathbb{P}) = \inf_{\mathbb{Q} \in \mathscr{M}_a} I(\mathbb{Q}, \mathbb{P}).$$

In this thesis we are looking for the MEMM $\mathbb{Q}^*$ in the class $\mathscr{M}_a$ for our basic model and are investigating the connection between such MEMM $\mathbb{Q}^*$ and the so-called Esscher Martingale Measure (EMM) $\mathbb{Q}^E$ defined as follows: Let $\kappa \in \mathbb{R}$ with $\mathbb{E}[\exp(\kappa X_t)] < \infty$ for $t \in [0, T]$ and define

$$Z_t^\kappa := \frac{\mathrm{d}\mathbb{Q}^\kappa}{\mathrm{d}\mathbb{P}}|_{\mathscr{F}_t} = \frac{\exp(\kappa X_t)}{\mathbb{E}[\exp(\kappa X_t)]} \text{ for } t \in [0, T].$$

The probability measure $\mathbb{Q}^\kappa$ is the EMM if X is a $\mathbb{Q}^\kappa$-martingale on $[0, T]$. The EMM on $[0, T]$, if it exists, is denoted by $\mathbb{Q}^E$.

In fact, one of the main result of the thesis states the identity of the notions MEMM and EMM for our basic model.

The thesis is divided in four chapters which are followed by two appendices. In Chapter 1 we collect main results of measure theory and stochastic analysis. In Chapter 2 we give an introduction to the problem of the MEMM. In Chapter 3 we investigate the model of the price processes based on the exponential (geometric) compound Poisson process and show its connection with the one-step model. In Chapter 4 the main objects of the investigations are models driven by the linear and exponential (geometric) Lévy processes. There are stated the main results of the thesis, including a sufficient condition of the existence of the EMM, coincidence of the EMM and the MEMM for the models driven by the linear and exponential Lévy processes and a series of important explicit equalities for the value of the relative entropy of the MEMM with respect to the original probability measure. In Appendix A are collected the most important properties of the technical functions ψ and φ which are widely used in the main body of the thesis. In Appendix B we apply the general theory developed in Chapter 2, to a particular model, the one-step model.

Introduction

The main aim of this PhD thesis is to investigate relative entropy in the frame of models based on (geometric) Lévy processes.

Let $S = (S_t)_{s\in[0,T]}$, where $T > 0$ is a finite horizon, be a geometric Lévy process defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, that is, S is a stochastic process of the form:

$$S_t = S_0 \exp(X_t),\ t \in [0,T],$$

where $S_0 > 0$ is a constant and $X = (X_t)_{t\in[0,T]}$ is a Lévy process with characteristic triplet (b, σ^2, ν) associated with the standard truncation function h defined by $h(x) = x1_{\{|x|\leq 1\}}$, $x \in \mathbb{R}$.

This is our basic model for investigation. We are interested in such probability measures $\mathbb{Q}$ that the process S is $\mathbb{Q}$-martingale and $\mathbb{Q}$ is equivalent (or, at least, absolutely continuous) to the original measure $\mathbb{P}$. In financial mathematics the notion of a martingale measure is very important because of the no-arbitrage property. If $\nu \equiv 0$ we get the famous Black–Scholes model with volatility σ and drift $\mu = b - \frac{1}{2}\sigma^2$ which was already studied for decades. The well-known fact is that in this case there exists a unique absolutely continuous martingale measure, but it is not the case for the general Lévy process X. The set of the absolutely continuous martingale measures $\mathscr{M}_a$ may have one of three different forms: 1) the set $\mathscr{M}_a$ could be empty (in case when X or $-X$ is a subordinator, Proposition 4.4); 2) the set $\mathscr{M}_a$ consists of just one measure (case without jumps, the classic Black–Scholes model); 3) the set $\mathscr{M}_a$ consists of an infinite number of martingale measures.

The last case means that one has to make a decision which martingale measure should be used as a pricing measure. And here comes into play the relative entropy.

Let $\mathbb{Q}$ and $\mathbb{P}$ be two probability measures on $(\Omega, \mathscr{F})$ and $\mathscr{G}$ some sub-σ-field of $\mathscr{F}$. The relative entropy $I_{\mathscr{G}}(\mathbb{Q}, \mathbb{P})$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathscr{G}$ is defined by

$$I_{\mathscr{G}}(\mathbb{Q},\mathbb{P}) := \begin{cases} \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\big|_{\mathscr{G}} \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\big|_{\mathscr{G}}\right], & \text{if } \mathbb{Q}|_{\mathscr{G}} \ll \mathbb{P}|_{\mathscr{G}}, \\ +\infty, & \text{otherwise.} \end{cases}$$

In other words, the relative entropy, also known as a Hellinger distance, plays the role of the "distance" between measures. For the sake of simplicity of notation, we will often omit the subindex $\mathscr{G}$ if $\mathscr{G} = \mathscr{F}$ or if it is clear from the context which sub-σ-field $\mathscr{G}$ is used.

We are interested in such an absolutely continuous martingale measure $\mathbb{Q}_0$ that is "the closest" to the original measure $\mathbb{P}$: A probability measure $\mathbb{Q}_0 \in \mathscr{M}_a$ will be called the minimal entropy martingale measure (MEMM) if it satisfies

$$I(\mathbb{Q}_0,\mathbb{P}) = \min_{\mathbb{Q}\in\mathscr{M}_a} I(\mathbb{Q},\mathbb{P}) = \min_{\mathbb{Q}\in\mathscr{M}_a} \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right].$$

The distance between $\mathbb{P}$ and $\mathscr{M}_a$ is defined by

$$I(\mathscr{M}_a,\mathbb{P}) = \inf_{\mathbb{Q}\in\mathscr{M}_a} I(\mathbb{Q},\mathbb{P}).$$

For the definition of the relative entropy there was used the function $x\log x$, but it is also possible to use another function, for instance, x^2 (in this case the measure $\mathbb{Q}_0$ would be named square-optimal). Nevertheless, the relative entropy has very important advantages. First of all, the MEMM, if it exists, is always equivalent to the original measure $\mathbb{P}$ which is not the case for the square-optimal. The second important advantage comes from the motivation side: there is a strong connection with the portfolio-optimization problem, in particular, there could be built the duality problem between the problem of finding the MEMM and the problem of portfolio optimization in case of the exponential utility function. This fact was widely investigated, in particular, in Delbaen et al. (2002) and Kabanov & Stricker (2002).

The third important advantage is the form of the MEMM. The main result of the thesis is the remarkable fact of the coincidence of the MEMM and the so-called Esscher martingale measure (EMM), that is described below: Let $\kappa \in \mathbb{R}$ with $\mathbb{E}[\exp(\kappa X_t)] < \infty$ for $t \in [0,T]$ and define

$$Z_t^\kappa := \frac{\mathrm{d}\mathbb{Q}^\kappa}{\mathrm{d}\mathbb{P}}|_{\mathscr{F}_t} = \frac{\exp(\kappa X_t)}{\mathbb{E}[\exp(\kappa X_t)]} \text{ for } t \in [0,T].$$

The process Z^κ is called Esscher density process and the measure $\mathbb{Q}^\kappa$ defined on $\mathscr{F}_T$ is called the Esscher measure. One says that the probability measure

$\mathbb{Q}^\kappa$ is the Esscher martingale measure (EMM) if S is a $\mathbb{Q}^\kappa$-martingale on $[0,T]$. The EMM on $[0,T]$, if it exists, will be denoted by $\mathbb{Q}^E$.

The notions of the Esscher transformation and the Esscher measure were introduced in actuarial mathematics in Esscher (1932) in 1932. They have many useful properties, in particular, the Esscher transformation preserves the "Lévy property": if X is $\mathbb{P}$-Lévy process then it is also a $\mathbb{Q}^\kappa$-Lévy process. In Gerber & Shiu (1994) it was suggested to use the EMM for the option pricing. This idea was significantly developed, in particularly, because of the duality between the problem of portfolio optimization and the minimization of relative entropy. Using the results of Frittelli (2000), there were investigated for models based on (geometric) Lévy processes the cases when the MEMM is the EMM and vice verse. We mention just a few of the numerous papers in this field: Fujiwara & Miyahara (2003) , Grandits & Rheinländer (2002) and Hubalek & Sgarra (2006). A very important result was obtained in Esche & Schweizer (2005): it was shown that the MEMM, if it exists, always preserves the "Lévy property". The disadvantage of the last paper is its very sophisticated approach based on so-called Girsanov quantities and that the proof of their result that the MEMM is the EMM is not rigorous enough (as it is mentioned by the authors, their proof is "a little bit more rigorous as by Gerber and Shiu").

We have used a completely different approach, widely exploiting an approximation procedure and the connection between the models based on the (geometric) Lévy processes and the simple one-step model. As a first step, we have focused on the simple model, that was already widely investigated in the literature, in particular, in Cherny & Maslov (2003) and Cherny & Shiryaev (2002), but is still significantly undervalued because of its simplicity. All main facts on this model are collected in Appendix B in a form that is useful for application in the chapters below. The second step was to show the equivalence of the models based on the geometric Lévy processes and on the linear Lévy processes. It simplified the variety of the considered models. In the following, it is therefore sufficient to focus on the linear Lévy process X as asset price.

The third step was to find the appropriate collection of the initial assumptions on the basic model and to investigate what happens when these assumptions do not hold. We follow the literature (cf., e.g. Cherny & Shiryaev (2002)) that it is sufficient to assume just that neither X nor $-X$ is a subordinator. We give a new proof that this condition is necessary and sufficient for the no-arbitrage property and we obtain further conclusions. It was already mentioned above that if this condition fails there exist no absolutely continuous martingale measures at all.

Then we have shown that if the EMM exists, then it is the MEMM (see

Theorem 4.7). Further, we have constructed special sequences of probability measures $\mathbb{P}_n$ that have the following properties: $\mathbb{P}_n$ is equivalent to $\mathbb{P}$, the process X has finite $\mathbb{P}_n$-exponential moments of all order and, therefore, there exists a sequence of EMMs $\mathbb{Q}_n^E$, which, according to Theorem 4.7, are the MEMMs for the respective $\mathbb{P}_n$ (but not the original $\mathbb{P}$). Using these sequences, we have estimated the distance between the set of equivalent martingale measures and the original measure $\mathbb{P}$.

The next step was the crucial in our investigation: we have used the connection between the one-step model and the model based on the Lévy processes to show that if the MEMM exists, then there exists the EMM and both measures coincide. If the MEMM does not exist, the EMM also does not exist.

The last step was to extend the results to the wider class of measures, with respect to which the process X is just a local martingale.

Many important facts were collected as conclusions of the main result, in particular, there was defined the notion of a sufficient subclass of measures for the solution of the minimization problem. Amongst others we have determined the largest class for which the solution of the minimization problem is the MEMM and we have given a series of identities for quantities of relative entropy.

The thesis is divided into four chapters which are followed by two appendices.

In Chapter 1 we collect basic results of measure theory and stochastic analysis which will be used in the thesis. There is also a section dedicated to the well-known properties of relative entropy.

The aim of Chapter 2 is to give an introduction to the problem of the minimal entropy martingale measures (MEMM). The chapter is based on results of Frittelli (2000) which are natural generalization of ideas of Csiszár (1975). We consider a general incomplete security market model, with possibly an infinite number of price processes defined on a general filtered probability space. When the processes are bounded, it is proven that the MEMM exists and is unique (Proposition 2.3). It is also shown that the MEMM (if it exists and the relative entropy is finite) is always equivalent to the original measure. The main result of this chapter is formulated in Theorem 2.6 and gives the characterization of the density of the MEMM. The results mentioned above are quite general and abstract, but they are useful in applications to particular models. In our case the results of this chapter will be applied to the one-step model (see Appendix A). Despite its simplicity, the one-step model plays an important role and is used in Chapter 4 for solving the more general problem when the price process is driven by a linear Lévy process.

In Chapter 3 we shall investigate the model of the price processes based on the exponential (geometric) compound Poisson process and show its connection with the one-step model. The surprising result is obtained for the case of compound Poisson process without drift: the problem splits onto two independent problems: find the MEMM for the jump size distribution and find the "best" intensity. There is also investigated the problem of approximation of the optimal measure in case when MEMM does not exist.

At the beginning of Chapter 4, in Section 4.1 the main objects of the investigations are models driven by the linear and geometric (exponential) Lévy processes. Our first aim is to show the equivalence of the problem of finding the martingale measures for such models. In particular, it is used the well-known approach based on the properties of stochastic exponentials and logarithms for reformulation of the linear case problem in terms of exponential case problem and vice verse (see Proposition 4.1). This is the reason why we choose just the linear model for the further consideration.

In the following section (Section 4.2), we are going to introduce the key notion of this thesis – the Esscher martingale measure (EMM) – and provide a sufficient condition of its existence (existence of exponential moments, Proposition 4.4). The problem of existence of at least one equivalent martingale measure is discussed in Section 4.3 (cf. Proposition 4.5). Afterwards, we are going to show the coincidence of the EMM and the minimal entropy martingale measure (MEMM) for the model driven by the linear Lévy process. In Section 4.4, we show that the EMM, if it exists, is the MEMM.

Our next aim is to complete the proof of the main result of the thesis: the coincidence of the EMM and the MEMM for the models driven by a linear Lévy process. More precisely, we are going to prove that the EMM exists if and only if the MEMM exists and if one (hence both) of these conditions is satisfied then we have the coincidence of these probability measures. This will be the subject of Sections 4.5 – 4.7.

The idea of the proof of this basic fact is the following: We construct approximation sequences of probability measures $\mathbb{P}_n$ via their densities $Z_T^{(n)}$ with respect to the original probability measure $\mathbb{P}$ in such a way that there always exists the EMM $\mathbb{Q}_n^E$ with respect to $\mathbb{P}_n$. Furthermore, we show that the infimum of relative entropy in the class of martingale measures $\mathscr{M}_a(T)$ coincides with the upper limit of relative entropy of the sequence $I(\mathbb{Q}_n^E, \mathbb{P})$. Using the connection with the one-step model we get the equivalence of the existence of the EMM and the MEMM and their equality for the models driven by the linear Lévy process.

Finally, in Section 4.8, we state some important corollaries.

The aim of Appendix A is to collect the most important properties of the functions ψ and φ which are widely used in the main body of the thesis:

$$\begin{aligned}\varphi(\kappa) &= \mathbb{E}[\exp(\kappa(\xi - \xi_0))], \quad \kappa \in \mathbb{R}, \\ \psi(\kappa) &= \mathbb{E}[(\xi - \xi_0)\exp(\kappa(\xi - \xi_0))], \quad \kappa \in \mathbb{R}\end{aligned}$$

Note that we work here just with random variables but not with general stochastic processes.

In Appendix B we apply the general theory developed in Chapter 2, to a particular model, the one-step model.

The appendices are included for easier reference. The results are slight extensions of basically known facts.

1 Preliminaries

In this chapter we collect some basic notions and results of measure theory and stochastic analysis which will be used in the thesis. There is also a section dedicated to the well-known properties of relative entropy.

1.1 Measure Theory

We consider an arbitrary nonempty set Ω. If $A \subseteq \Omega$ we denote by A^c the complement of A in Ω. A system $\mathscr{A}$ of subsets of the set Ω is called a *σ-algebra* (in Ω) if it has the following properties: (i) $\Omega \in \mathscr{A}$; (ii) from $A \in \mathscr{A}$ it follows that $A^c \in \mathscr{A}$; (iii) from $(A_n)_{n\in\mathbb{N}} \subset \mathscr{A}$ it follows that $\cup_{n\in\mathbb{N}} A_n \in \mathscr{A}$.

A system $\mathscr{R}$ of subsets of Ω is called a *ring* (in Ω) if it has the following properties: (i) $\emptyset \in \mathscr{R}$; (ii) from $A, B \in \mathscr{R}$ it follows that $A \setminus B \in \mathscr{R}$; (iii) from $A, B \in \mathscr{R}$ it follows that $A \cup B \in \mathscr{R}$. Before we recall the definition of a measure, we start from the notion of a pre-measure: let $\mathscr{R}$ be a ring in Ω and μ a function on $\mathscr{R}$ with values in $[0, +\infty]$. The function μ is called a *pre-measure* on $\mathscr{R}$ if (i) $\mu(\emptyset) = 0$; (ii) for every sequence (A_n) of pairwise disjoint sets from $\mathscr{R}$ whose union lies in $\mathscr{R}$ we have

$$\mu\Big(\bigcup_{i=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

1.1 Definition. A pre-measure defined on a σ-algebra $\mathscr{A}$ of subsets of a set Ω is called a *measure*. If $\mu(\Omega) < +\infty$, the measure μ is called *finite*. If for some sequence $(A_n) \subset \mathscr{A}$ holds $\cup_{n=1}^{\infty} A_n = \Omega$ and for any n we have $\mu(A_n) < \infty$, the measure μ is called *σ-finite*. If $\mu(\Omega) = 1$, the measure μ is called a *probability measure*.

If Ω is a set and $\mathscr{A}$ a σ-algebra in Ω, the pair $(\Omega, \mathscr{A})$ will be called a *measurable space* and the sets from $\mathscr{A}$ *measurable sets*. If in addition a measure μ is given on the σ-algebra $\mathscr{A}$, then the triple $(\Omega, \mathscr{A}, \mu)$ arising from the measurable space $(\Omega, \mathscr{A})$ is called a *measure space*. If μ is a probability measure, the measure space $(\Omega, \mathscr{A}, \mu)$ is called a *probability space*.

1.2 Definition. Let $(\Omega, \mathscr{A})$ and $(\Omega', \mathscr{A}')$ be measurable spaces, and $T : \Omega \to \Omega'$ a mapping of Ω into Ω'. Then T is called *$\mathscr{A}$-$\mathscr{A}'$-measurable* or simply measurable if $T^{-1}(A') \in \mathscr{A}$ for every $A' \in \mathscr{A}'$.

We do not go into details of the definition of the integral of a measurable function with values in $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and we refer to Bauer (2001), Chapter II. We introduce the notation

$$\mu(f) := \int_\Omega f \, \mathrm{d}\mu := \int_\Omega f(x)\mu(\mathrm{d}x)$$

if the integral on the right-hand side exists. In particular, $\mu(f)$ is well defined if f is *nonnegative*. We say that a measurable function f of arbitrary sign is *μ-integrable* or simply *integrable* if $\mu(|f|) < +\infty$.

By *functions*, if not otherwise specified, we mean functions with values in $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, that is, *numerical functions*. Let f be a measurable function. By $\|f\|_q$ we denote the following norm

$$\|f\|_q := \begin{cases} \mu(|f|^q)^{\frac{1}{q}}\,, & q \in [1, +\infty), \\ \\ \operatorname{ess\,sup}_{x\in\Omega} |f(x)|\,, & q = +\infty\,, \end{cases}$$

and we put

$$L^q(\mu) := \{f \text{ measurable} : \|f\|_q < +\infty\}\,, \quad q \in [1, +\infty].$$

We recall that $f \in L^q(\mu)$ is uniquely determined up to equivalence μ-a.e.

A function f belonging to $L^1(\mu)$ is called *integrable*, while it is called *square integrable* if it belongs to $L^2(\mu)$. In general, we say that f is q-integrable if it belongs to $L^q(\mu)$, $q \in [1, +\infty)$. Let $(f_n)_{n\geq 1}$ be a sequence of measurable functions on the measure space $(\Omega, \mathscr{F}, \mu)$. We say that $(f_n)_{n\geq 1}$ converges (μ-a.e.) *pointwise* to the measurable function f if $\lim_{n\to+\infty} |f_n(x) - f(x)| = 0$ for (μ-almost all) $x \in \Omega$. We write $f_n \longrightarrow f$ *pointwise* to mean that the sequence $(f_n)_{n\geq 1}$ converges pointwise to f. If the sequence $(f_n)_{n\geq 1}$ is monotonically increasing (resp., decreasing), i.e., $f_n \leq f_{n+1}$ (resp., $f_n \geq f_{n+1}$), we write $f_n \uparrow f$ (resp., $f_n \downarrow f$) to mean that it converges pointwise to

f. If $(f_n)_{n\geq 1} \subseteq L^q(\mu)$ and $f \in L^q(\mu)$, we say that $(f_n)_{n\geq 1}$ converges to f in $L^q(\mu)$ if $\lim_{n\to+\infty} \|f_n - f\|_q = 0$.

Now we can formulate the theorem of Lebesgue on dominated convergence and the theorem of B. Levi on monotone convergence. We refer to Bauer (2001), §11 and §15.

1.3 Theorem. *We fix $q \in [1, +\infty)$ and consider a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^q(\mu)$ such that $f_n \longrightarrow f$ μ-a.e. pointwise as $n \to +\infty$. If there exists a function $g \geq 0$ in $L^q(\mu)$ such that $|f_n| \leq g$, for every $n \in \mathbb{N}$, then $f \in L^q(\mu)$ and the convergence takes place also in $L^q(\mu)$.*

1.4 Theorem. *Let $(f_n)_{n\in\mathbb{N}}$ be a monotone sequence of nonnegative measurable functions such that $f_n \uparrow f$ pointwise as $n \to +\infty$. Then f is measurable and $\mu(f_n) \uparrow \mu(f)$ as $n \to +\infty$.*

1.5 Definition. If f is a non-negative $\mathscr{A}$-measurable, numerical function on Ω, then the measure ν defined on $\mathscr{A}$ by

$$\nu(A) := \int_A f \mathrm{d}\mu, \quad A \in \mathscr{A},$$

is called the *measure having density f with respect to μ*. It will be denoted by

$$\nu = f\mu.$$

1.6 Theorem. *Let f and φ be non-negative measurable functions on Ω, $\nu := f\mu$. Then*

$$\int \varphi \mathrm{d}\nu = \int \varphi f \mathrm{d}\mu \tag{1.1}$$

which can be also written

$$\int \varphi \mathrm{d}(f\mu) = \int \varphi f \mathrm{d}\mu.$$

Moreover, an $\mathscr{A}$-measurable function $\varphi : \Omega \to \bar{\mathbb{R}}$ is ν-integrable if and only if φf is μ-integrable. In this case (1.1) *is again valid.*

For the proof we refer to Bauer (2001), Theorem 17.3.

1.7 Definition. Given two measures μ and ν defined on the same σ-algebra $\mathscr{A}$, we say that μ is *absolutely continuous* with respect to ν, written $\mu \ll \nu$, if $\mu(A) = 0$, whenever $\nu(A) = 0$ for $A \in \mathscr{A}$. Measures μ and ν are called *equivalent*, written $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.

The following theorem, known as the Radon-Nikodym theorem, explains the connection between the absolute continuity of measures and the existence of a non-negative function f satisfying (1.1). For the proof we refer to Bauer (2001), Theorem 17.10.

1.8 Theorem. *Let μ and ν be measures on a σ-algebra $\mathscr{A}$ in a set Ω. If μ is σ-finite, the following two assertions are equivalent:*
(i) *ν has a density f with respect to μ;*
(ii) *ν is absolutely continuous with respect to μ.*

Obviously, for probability measures the assumption of σ-finiteness is always satisfied.

1.2 Stochastic Processes

Let $(\Omega, \tilde{\mathscr{F}}, \mathbb{P})$ be a probability space. By $\mathscr{N}(\mathbb{P})$ we denote the null sets of $\mathbb{P}$, i.e., $\mathscr{N}(\mathbb{P}) := \{A \subseteq \Omega : \exists B \in \tilde{\mathscr{F}},\ A \subseteq B,\ \mathbb{P}(B) = 0\}$. If $\mathscr{N}(\mathbb{P})$ is not contained in $\tilde{\mathscr{F}}$ we enlarge the σ-algebra by setting $\mathscr{F} := \tilde{\mathscr{F}} \vee \mathscr{N}(\mathbb{P})$. We call $\mathscr{F}$ the completion of $\tilde{\mathscr{F}}$ (*in itself*) with respect to $\mathbb{P}$ or simply $\mathbb{P}$-completion of $\tilde{\mathscr{F}}$ and we say that $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space. If not otherwise specified, we assume a probability space to be complete. In the remaining part of this chapter we assume that a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is fixed.

A measurable mapping ξ on $(\Omega, \mathscr{F})$ into $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is called a *random variable*. The expectation of the random variable with respect to the probability measure $\mathbb{P}$ is defined by

$$\mathbb{E}[\xi] = \int_\Omega \xi \mathrm{d}\mathbb{P},$$

provided the integral exists. If ξ is an integrable random variable, i.e, $\mathbb{E}[|\xi|] < +\infty$, and $\mathscr{G}$ is a sub-σ-algebra of $\mathscr{F}$, we denote by $\mathbb{E}[\xi|\mathscr{G}]$ the conditional expectation with respect to $\mathscr{G}$. Sometimes we write $\mathbb{E}_\mathbb{P}$ or $\mathbb{E}_\mathbb{P}[\cdot|\mathscr{G}]$ to emphasize the dependence on the probability measure $\mathbb{P}$.

Now we recall the notion of the *uniform integrability*. We say that a family $\mathscr{K} \subseteq L^1(\Omega, \mathscr{F}, \mathbb{P})$ is uniformly integrable if

$$\sup_{X \in \mathscr{K}} \mathbb{E}[|X| 1_{\{|X| \geq N\}}] \longrightarrow 0 \quad \text{as} \quad N \to +\infty.$$

If $\mathscr{K}$ is dominated in $L^1(\mathbb{P})$, i.e., there exists $Y \in L^1(\mathbb{P})$ such that $|X| \leq Y$, $X \in \mathscr{K}$, then $\mathscr{K}$ is uniformly integrable. Clearly, any finite family of integrable random variables is uniformly integrable.

By a *filtration* $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ we mean a family of σ-algebras $(\mathscr{F}_t)_{t\geq 0}$ that is increasing, i.e.,$\mathscr{F}_s \subseteq \mathscr{F}_t$ if $s \leq t$. By convention, we set: $\mathscr{F}_\infty = \mathscr{F}$ and $\mathscr{F}_{\infty-} = \bigvee_{s\in\mathbb{R}_+} \mathscr{F}_s$. With filtration $\mathbb{F}$ we associate the filtration

$$\mathbb{F}_+ = (\mathscr{F}_{t+})_{t\geq 0} \text{ by } \mathscr{F}_{t+} := \cap_{\epsilon>0}\mathscr{F}_{t+\epsilon}.$$

The filtration $\mathbb{F}$ is called *right-continuous* if $\mathbb{F} = \mathbb{F}_+$, that is, if $\mathscr{F}_{t+} = \mathscr{F}_t$, $t \geq 0$. Note that $\mathscr{F}_{0+} = \mathscr{F}_0$ if $\mathbb{F}$ is right-continuous.

1.9 Definition. A probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F}$ is called a *filtered probability space*. The filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, equipped with a right-continuous filtration $\mathbb{F}$, is called *complete*, or equivalently is said to *satisfy the usual conditions* if the σ-algebra $\mathscr{F}$ is $\mathbb{P}$-complete and if every $\mathscr{F}_t$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$.

A *stochastic process* X is a collection of $\mathbb{R}$-valued random variables $(X_t)_{t\geq 0}$. A stochastic process X can be interpreted as an application $X : (t, \omega) \longmapsto X(t, \omega)$ of $\mathbb{R}_+ \times \Omega$ into $\mathbb{R}$. We use the notation $X_t(\omega) := X(t, \omega)$ and in most cases we omit ω. If we endow the space $\mathbb{R}_+ \times \Omega$ with the σ-algebra $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}$, we say that the process $X = (X_t)_{t\geq 0}$ is *measurable* if X is a $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}$-measurable application of $\mathbb{R}_+\times\Omega$ into $\mathbb{R}$. A process X is *adapted* to a filtration $\mathbb{F}$ if X_t is $\mathscr{F}_t$-measurable, for every $t \geq 0$. For two stochastic processes X and Y, there exist different concepts of equality:

(i) X is *equal* to Y if $X_t(\omega) = Y_t(\omega)$ for every $t \geq 0$ and for every $\omega \in \Omega$.

(ii) X and Y are *modifications* if $\mathbb{P}(X_t = Y_t) = 1$, for every $t \geq 0$.

(iii) X and Y are *indistinguishable* if $\mathbb{P}(X_t = Y_t, \text{ for every } t \geq 0) = 1$.

If X and Y are modifications there exists a null set, N_t, such that if $\omega \notin N_t$, then $X_t(\omega) = Y_t(\omega)$. The null set N_t depends on t. Since the interval $[0, \infty)$ is uncountable the set $N = \bigcup_{0\leq t<\infty} N_t$ could have any probability between 0 and 1, and it could even be non-measurable. If X and Y are indistinguishable, however, there exists a null set N such that if $\omega \notin N$, then $X_t(\omega) = Y_t(\omega)$, for all t. In other words, the function $t \longmapsto X_t(\omega)$ and $t \longmapsto Y_t(\omega)$ are the same for all $\omega \notin N$, where $\mathbb{P}(N) = 0$.

We call the application $t \mapsto X_t(\omega)$ *path* or *trajectory* of the process X. We say that the stochastic process X is right-continuous with left-hand limits or càdlàg, if every trajectory is right-continuous with left-hand limits. Analogously, we say that X is *continuous* or *left-continuous* if all its trajectories have this property. An adapted stochastic process X which is càdlàg is $\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}$-measurable. For a càdlàg process X, we define the random variable X_{t-} for every $t > 0$ as $X_{t-} := \lim_{s\uparrow t} X_s$ which is finite. Adopting the convention $X_{0-} := X_0$, we can introduce the process $X_- = (X_{t-})_{t\geq 0}$ on the whole positive real line. With a càdlàg process X we associate the process

$\Delta X := (\Delta X_t)_{t\geq 0}$ of jumps of X by $\Delta X_t := X_t - X_{t-}, \quad t \geq 0$. A consequence of the convention $X_{0-} = X_0$ is that $\Delta X_0 = 0$. If X is a continuous stochastic process, then we have $X_- = X$ and $\Delta X = 0$.

The σ-algebra $\mathscr{F}_t$ can be thought of as representing all (theoretically) observable events up to and including time t. We would like to have an analogous notion of events that are observable before a random time.

1.10 Definition. A random variable T on Ω into $[0, +\infty]$ is called a *stopping time* if the event $\{T \leq t\} := \{\omega \in \Omega : T(\omega) \leq t\}$ is $\mathscr{F}_t$-measurable, for every $t \geq 0$. The *σ-algebras* $\mathscr{F}_T$ and $\mathscr{F}_{T-}$ *associated with a stopping time* T are defined to be

$$\mathscr{F}_T := \{A \in \mathscr{F} : A \cap \{T \leq t\} \in \mathscr{F}_t, t \geq 0\}$$

and

$$\mathscr{F}_{T-} := \mathscr{F}_0 \vee \sigma\{A \cap \{t < T\},\, A \in \mathscr{F}_t,\, t \geq 0\}.$$

Let S, T be two stopping times. We define the *stochastic interval* $[S, T)$ by $[S, T) := \{(t, \omega) \in \mathbb{R}_+ \times \Omega : S(\omega) \leq t < T(\omega)\}$. The notions $(S, T]$, (S, T) and $[S, T]$ can be defined in a similar way.

1.11 Definition. The σ-field on $\Omega \times \mathbb{R}_+$ generated by all càdlàg adapted processes is called *optional σ-field* and is denoted by $\mathscr{O}$. The σ-field on $\Omega \times \mathbb{R}_+$ generated by all left-continuous adapted processes is called *predictable σ-field* and is denoted by $\mathscr{P}$. A stochastic process or a set is said to be *optional* (resp., *predictable*), if it is $\mathscr{O}$-measurable (resp., $\mathscr{P}$-measurable).

It is easy to observe that both $\mathscr{O}$ and $\mathscr{P}$ are σ-algebras of the product space $\mathbb{R}_+ \times \Omega$ and the inclusions $\mathscr{P} \subseteq \mathscr{O} \subseteq \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}$ hold. We would like to state some important facts concerning the properties of optional and predictable processes and sets without giving the proofs. First of all, optional processes are also adapted (Jacod & Shiryaev (2000), Proposition I.1.21). Another important fact concerns the predictable sets and processes: for any stopping time T, the stochastic interval $[0, T]$ is a predictable set (Dellacherie (1972), Theorem IV.3) and consequently the process $1_{[0,T]}$ is predictable.

1.12 Definition. Let X be a stochastic process and T a stopping time with values in $[0, +\infty]$. On $\{T < +\infty\}$, we define the random variable X_T by

$$X_T(\omega) := X_{T(\omega)}(\omega) = X(T(\omega), \omega)$$

and, consequently, the stochastic process $X^T = (X^T_t)_{t\geq 0}$ by

$$X^T_t := X_{t \wedge T},\, t \geq 0,$$

where the symbol " $\wedge$ " denotes the minimum function. We say that X^T is the *stopped process at time T*.

Very often the stopped processes have similar properties to the original processes, for instance, if X is an optional (resp., predictable) process, then the stopped process X^T is optional (resp., predictable) (Jacod & Shiryaev (2000), Proposition I.1.21 and I.2.4). A class $\mathscr{C}$ of processes is called *stable under stopping* if for every $X \in \mathscr{C}$ the stopped process X^T belongs again to $\mathscr{C}$, for every stopping time T.

1.13 Definition. If $\mathscr{C}$ is a class of processes, we denote by $\mathscr{C}_{loc}$ the *localized class*, defined as such: a process X belongs to $\mathscr{C}_{loc}$ if and only if there exists an increasing sequence (T_n) of stopping times (depending on X) such that $\lim_{n\to\infty} T_n = \infty$ a.s. and that each stopped process X^{T_n} belongs to $\mathscr{C}$. The sequence (T_n) is called a *localizing* or *reducing sequence* for X (relative to $\mathscr{C}$).

1.3 Martingales

For this section we fix the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $\mathbb{F}$ satisfying the usual conditions.

1.14 Definition. A *martingale* (resp., *submartingale*, *supermartingale*) relative to the filtration $\mathbb{F}$ is an adapted process X such that X_t is integrable for every $t \geq 0$ and for all $0 \leq s \leq t$, we have

$$\mathbb{E}[X_t|\mathscr{F}_s] = X_s \quad (\text{resp., } \mathbb{E}[X_t|\mathscr{F}_s] \geq X_s,\ \mathbb{E}[X_t|\mathscr{F}_s] \leq X_s) \quad \mathbb{P}\text{-a.s.}$$

For shortness, we simply say that a martingale (resp., a submartingale or a supermartingale) relative to a filtration $\mathbb{F}$ is an $\mathbb{F}$-martingale (resp., $\mathbb{F}$-submartingale or $\mathbb{F}$-supermartingale) to emphasize the filtration or, if the filtration is fixed, a $\mathbb{P}$-martingale (resp., $\mathbb{P}$-submartingale or $\mathbb{P}$-supermartingale) to emphasize the probability measure. Sometimes, in the definition of a martingale it is also required that $\mathbb{P}$-almost all paths are càdlàg (e.g., in Jacod & Shiryaev (2000)), but because the filtration $\mathbb{F}$ satisfies the usual conditions, every $\mathbb{F}$-martingale admits a càdlàg modification which is again an $\mathbb{F}$-martingale (cf., e.g., Meyer (1966), Chapter VI, or He, Wang & Yan (1992), Chapter II). If not otherwise specified, we always consider càdlàg martingales.

We say that a process X has a *terminal variable* X_∞ if X_t converges a.s. to a limit X_∞ as $t \uparrow +\infty$. If T is a stopping time, the random variable X_T given by $X_T(\omega) := X_{T(\omega)}(\omega)$ is, in general, only defined on $\{T < +\infty\}$. If the process X admits the terminal variable X_∞, the random variable X_T is defined also on $\{T = +\infty\}$ setting $X_T = X_\infty$ on this set.

We denote by $\mathscr{M}$ the class of *uniformly integrable martingales*, i.e., the class of all martingales M such that the family of random variables $(M_t)_{t\geq 0}$ is uniformly integrable. By $\mathscr{M}_0$, we denote the subset of uniformly integrable martingales starting at 0. The terminal value M_∞ of a uniformly integrable martingale exists and M_t converges in $L^1(\mathbb{P})$ to M_∞. Furthermore, for every stopping times T, $M_T = \mathbb{E}[M_\infty|\mathscr{F}_T]$. For a proof of these facts, cf. Jacod & Shiryaev (2000), Theorem I.1.42. Doob's Stopping Theorem (cf. Jacod & Shiryaev (2000), Theorem I.1.39) implies that for every stopping time T the random variable M_T, $M \in \mathscr{M}$, is integrable and, if S is also a stopping time, we have $\mathbb{E}[M_T|\mathscr{F}_S] = M_S$ on $\{S \leq T\}$. A first consequence of these facts is that $\mathscr{M}$ is stable under stopping. Indeed, if $M \in \mathscr{M}$ and T is a stopping time, we have that $T \wedge t$ is a stopping time and that M_∞ and $M_{T\wedge t}$ are integrable, $t \geq 0$. Moreover,

$$|M_t^T| = |M_{T\wedge t}| = |\mathbb{E}[M_\infty|\mathscr{F}_{T\wedge t}]| \leq \mathbb{E}[|M_\infty||\mathscr{F}_{T\wedge t}],$$

where we used Jensen's inequality in the last passage. Because of He, Wang & Yan (1992), Theorem 1.8, the right-hand side in the previous estimation is uniformly integrable and so $M^T = (M_t^T)_{t\geq 0}$ is uniformly integrable. By Doob's Stopping Theorem, we get $M_t = \mathbb{E}[M_T|\mathscr{F}_t]$ on $\{t \leq T\}$ and, by the properties of the conditional expectation, $\mathbb{E}[M_T|\mathscr{F}_t] = M_T$ on $\{t > T\}$. Hence $M_t^T = \mathbb{E}[M_T|\mathscr{F}_t]$, $t \geq 0$, and so the process M^T is an $\mathbb{F}$-martingale. By localization from $\mathscr{M}$ we can introduce the space $\mathscr{M}_{loc}$. A local martingale will be an element of $\mathscr{M}_{loc}$. We observe that the class $\mathscr{M}_{loc}$ is stable under stopping.

We denote by $\mathscr{V}^+$ (resp., by $\mathscr{V}$) the set of all real-valued processes A that are càdlàg, adapted, with $A_0 = 0$ and whose paths are *non-decreasing* (resp., have finite variation on each finite interval $[0,t]$). We say that a process in $\mathscr{V}^+$ (resp., in $\mathscr{V}$) is an *increasing process* (resp., a process of finite variation).

For a process $A \in \mathscr{V}$, by $\mathrm{Var}(A) = (\mathrm{Var}(A)_t)_{t\geq 0}$ we denote the associated variation process, that is, the process such that $\mathrm{Var}(A)_t(\omega)$ is the total variation of the function $s \mapsto A_s(\omega)$ on the interval $[0,t]$. Of course, $\mathrm{Var}(A) = A$ if $A \in \mathscr{V}^+$. In Jacod & Shiryaev (2000), Proposition I.3.3, the following important relation between $\mathscr{V}$ and $\mathscr{V}^+$ is established: If $A \in \mathscr{V}$, then there exists a unique pair of processes $B, C \in \mathscr{V}^+$ such that $A = B - C$ and $\mathrm{Var}(A) = B + C$ (hence, $\mathrm{Var}(A) \in \mathscr{V}^+$ and $\mathscr{V} = \mathscr{V}^+ \ominus \mathscr{V}^+$). Moreover, if A is predictable, then B, C and $\mathrm{Var}(A)$ are also predictable.

For a process $A \in \mathscr{V}^+$ the function $t \mapsto A_t(\omega)$ is a measure-generating function, that is, it is the distribution of a measure, say μ_ω^A, defined on $\mathbb{R}_+$ by

$$\mu_\omega^A([0,t]) := A_t(\omega), \quad t \in \mathbb{R}_+, \quad \omega \in \Omega,$$

which is a locally finite measure.

The stochastic integral with respect to $A \in \mathscr{V}^+$ of a measurable process H is defined as a *Stiltjes–Lebesgue integral*, i.e., by fixing $\omega \in \Omega$ and defining the integral pathwise with respect to the trajectory $t \mapsto A_t(\omega)$. We say that a measurable process H is integrable with respect to A if $\int_0^t |H_s(\omega)|\mu_\omega^A(\mathrm{d}s) < +\infty$ for every $t \geq 0$ and for every $\omega \in \Omega$. If H is integrable with respect to A by

$$\int_0^t H_s(\omega)\,\mathrm{d}A_s(\omega) := \int_0^t H_s(\omega)\,\mu_\omega^A(\mathrm{d}s), \quad t \geq 0,$$

we denote the integral of H with respect to A up to time t. We introduce the *integral process* $H \cdot A = (H \cdot A_t)_{t \geq 0}$ by

$$H \cdot A_t(\omega) := \begin{cases} \int_0^t H_s(\omega)\,\mathrm{d}A_s(\omega), & \text{if } \int_0^t |H_s(\omega)|\,\mathrm{d}A_s(\omega) < +\infty, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

If $A \in \mathscr{V}$, we can introduce the integral for measurable processes in a similar way. Indeed, there exist two unique processes $B, C \in \mathscr{V}^+$ such that $A = B - C$ and $\mathrm{Var}(A) = B + C$. In this case, we say that a measurable process H is integrable with respect to A if it is integrable with respect to $\mathrm{Var}(A)$ and we introduce the integral process $H \cdot A = (H \cdot A_t)_{t \geq 0}$ by

$$H \cdot A_t(\omega) := \begin{cases} \int_0^t H_s(\omega)\,\mathrm{d}A_s(\omega), & \text{if } \int_0^t |H_s(\omega)|\,\mathrm{d}\,\mathrm{Var}(A)_s(\omega) < +\infty, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

If H is a (resp., nonnegative) measurable process which is integrable with respect to $A \in \mathscr{V}$ (resp., $A \in \mathscr{V}^+$), then the process $H \cdot A$ belongs to $\mathscr{V}$ (resp., $\mathscr{V}^+$). If moreover A and H are predictable, then $H \cdot A$ is predictable (cf. Jacod & Shiryaev (2000), Proposition I.3.5). We notice that locally bounded measurable processes are always integrable with respect to $A \in \mathscr{V}$. Indeed, if H is a locally bounded measurable process and $(T_n)_{n \in \mathbb{N}}$ is a reducing sequence, say, such that $|H^{T_n}| \leq c_n$, we have, for every $n \geq 1$ and every fixed $t \geq 0$, $|H^{T_n}| \cdot \mathrm{Var}(A)_t \leq c_n \mathrm{Var}(A)_t < +\infty$. On the other side, $T_n \uparrow +\infty$. Hence, for every fixed $t \geq 0$ and ω, there exists $n(t) \in \mathbb{N}$ such that $T_n(\omega) \geq t$ for every $n \geq n(t)$ and so $|H| \cdot \mathrm{Var}(A)_{t(\omega)} \leq c_n \mathrm{Var}(A)_{t(\omega)} < +\infty$, $n \geq n(t)$.

Now we introduce two other classes of processes.

(i) $\mathscr{A}^+ := \{A \in \mathscr{V}^+ : \mathbb{E}[A_\infty] < +\infty\}$: set of *integrable* processes from $\mathscr{V}^+$.

(ii) $\mathscr{A} := \{A \in \mathscr{V} : \mathbb{E}[\mathrm{Var}(A)_\infty] < +\infty\}$: set of processes of *integrable variation.*

We denote by $\mathscr{H}^2$ the class of square integrable martingales, that is, we put

$$\mathscr{H}^2 := \left\{M \in \mathscr{M}_{loc} : \mathbb{E}[\sup_{t\geq 0}(M_t)^2] < +\infty\right\}.$$

The space $\mathscr{H}^2_{loc}$ of locally square integrable local martingales is introduced from $\mathscr{H}^2$ by localization. The space $\mathscr{H}^2$ is a Hilbert space with the scalar product

$$(M,N)_{\mathscr{H}^2} := \mathbb{E}[M_\infty N_\infty], \tag{1.2}$$

where the terminal value M_∞ is defined because $\mathscr{H}^2 \subset \mathscr{M}$.

Now we can mention the following well-known result (cf. Jacod & Shiryaev (2000), Theorem I.4.2): For any M and N belonging to $\mathscr{H}^2_{loc}$, there exists a predictable process $\langle M,N\rangle \in \mathscr{V}$, called point brackets (or predictable process of finite variation) associated with M and N, which is unique up to an evanescent set, such that $MN - \langle M,N\rangle \in \mathscr{M}_{loc}$. The so-called *polarization identity* holds:

$$\langle M,N\rangle = \frac{1}{4}\left(\langle M+N,M+N\rangle - \langle M-N,M-N\rangle\right). \tag{1.3}$$

If $M,N \in \mathscr{H}^2$, then $\langle M,N\rangle \in \mathscr{A}$ and $MN - \langle M,N\rangle \in \mathscr{M}$. Furthermore, the identity $\langle M,N\rangle = \langle M-M_0, N-N_0\rangle$ holds and $\langle M,M\rangle$ belongs to $\mathscr{V}^+$. The next relation explains the behavior of the point brackets with respect to the stopping procedure (cf. He, Wang & Yan (1992), Theorem 6.31). For every stopping time T it follows that

$$\langle M,N^T\rangle = \langle M^T,N\rangle = \langle M^T,N^T\rangle = \langle M,N\rangle^T. \tag{1.4}$$

Furthermore, because of the definition of the point brackets, we have

$$(M,N)_{\mathscr{H}^2} := \mathbb{E}[\langle M,N\rangle_\infty] + \mathbb{E}[M_0N_0], \quad M,N \in \mathscr{H}^2. \tag{1.5}$$

The following theorem is known as *Kunita–Watanabe inequality* and plays an important role in the construction of the stochastic integral.

1.15 Theorem. *Let M and N be square integrable martingales and H, K be two measurable processes. Then*

$$\mathbb{E}\Big[\int_{(0,\infty)} |H_u|\,|K_u|\mathrm{d}|\langle M,N\rangle|_u\Big] \leq \mathbb{E}\Big[\int_{(0,\infty)} H_u^2\mathrm{d}\langle M,M\rangle_u\Big]^{1/2}\mathbb{E}\Big[\int_{(0,\infty)} K_u^2\mathrm{d}\langle N,N\rangle_u\Big]^{1/2}.$$

For the proof we refer to He, Wang & Yan (1992), Theorem 6.33.

Our aim is to define a stochastic integration of predictable process with respect to locally square integrable martingales. Recall that $\mathscr{H}^2$ is a Hilbert space with scalar product $(M, N)_{\mathscr{H}^2} := \mathbb{E}[M_\infty N_\infty]$ for any N, M in $\mathscr{H}^2$. For $M \in \mathscr{H}^2$, the space of integrands is defined as follows:

$$L^2(M) := \{H \text{ predictable } : H^2 \cdot \langle M, N \rangle \in \mathscr{A}^+\}.$$

Let us introduce a norm $||H\|_{L^2(M)}\|$:

$$|H\|_{L^2(M)}\| := \mathbb{E}\Big[\int_{(0,\infty)} |H_u|\,|H_u| \mathrm{d}|\langle M, N\rangle|_u\Big].$$

For $M \in \mathscr{H}^2$ and $H \in L^2(M)$ we introduce the functional $C_{M,H}$ by

$$C_{M,H}(N) := \mathbb{E}[H \cdot \langle M, N \rangle_\infty], \quad N \in \mathscr{H}^2.$$

Obviously, $C_{M,H}$ is a linear functional on $\mathscr{H}^2$. Moreover, it is continuous. Indeed, by the Kunita–Watanabe inequality,

$$|C_{M,H}(N)| \le \|H\|_{L^2(M)} \|N\|_{\mathscr{H}^2} < +\infty \quad \text{for any } N \in \mathscr{H}^2,$$

where for the last estimation we used that $H \in L^2(M)$ and that $M \in \mathscr{H}^2$. Therefore, $C_{M,H}$ is bounded and linear, hence continuous. By the stheorem of Riez, there exist a unique element $X \in \mathscr{H}^2$, such that

$$C_{M,H}(N) = (X, N)_{\mathscr{H}^2} \quad \text{for any } N \in \mathscr{H}^2.$$

We call this process $X \in \mathscr{H}^2$ the *stochastic integral process* of H with respect to M and we write

$$X =: H \cdot M.$$

By definition, $X \in \mathscr{H}^2$. If $M \in \mathscr{H}^2 \cap \mathscr{V}$, then $H \cdot M$ coincides with the Stieltjes–Lebesgue integral (cf. Jacod (1979), Remark 2.47). Let us define the space $L^2_{loc}(M)$ of all predictable H for which there is an increasing sequence (T_n) of stopping times converging to infinity and such that $H1_{[0,T_n]}$ belongs to $L^2(M)$. It is easy to see that the space of integrands $L^2_{loc}(M)$ is obtained from $L^2(M)$ by localization. In a natural way we can extend the notion of stochastic integration with respect to the martingale M and integrands H from $L^2_l oc(M)$.

An adapted process S is called a *semimartingale* if:

$$S = S_0 + M + A, \tag{1.6}$$

where S_0 is an $\mathscr{F}_0$-measurable random variable, $M \in \mathscr{M}_{loc}$, $M_0 = 0$ and $A \in \mathscr{V}$. We call M the martingale part and A the finite-variation part of the semimartingale S, respectively. We call a representation as (1.6) a *semimartingale decomposition* of S. Note that every local martingale can be decomposed (but not in a unique way) into a locally square integrable martingale and a martingale with finite variation (cf. Jacod & Shiryaev (2000), Proposition 4.17). Therefore, for locally bounded and predictable H we can introduce the stochastic integral process of H with respect to a semimartingale S:

$$H \cdot S := H \cdot M + H \cdot A = H \cdot M' + H \cdot M'' + H \cdot A, \tag{1.7}$$

where $M' \in \mathscr{H}^2_{loc}$ and $M'' \in \mathscr{V}$ with $M = M' + M''$. The property of local boundedness of H ensures the existence of $H \cdot M''$ and $H \cdot A$, and in combination with the property of predictability guaranties that $H \in L^2_{loc}(M')$. The integral process does not depend on the martingale decomposition. For the proof we refer to Jacod & Shiryaev (2000), Proposition 4.40.

1.4 Lévy Process

We fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $\mathbb{F}$. For the moment, $\mathbb{F}$ is a general filtration and we do not assume that the probability space is complete. We recall that an $\mathbb{F}$-adapted stochastic process X is said to have *homogeneous one-dimensional increments* if $(X_t - X_s)$ is distributed as $X_{t-s} - X_0$, for every $0 \le s \le t$, while it is said to have *independent increments* if the random vector $(X_{t_0}, X_{t_1} - X_{t_0}, \ldots X_{t_n} - X_{t_{n-1}})$ is independent, for every $0 \le t_0 < t_1 < \ldots < t_n$, $n \in \mathbb{N}$. We say that an adapted process X has $\mathbb{F}$-*independent increments* if $(X_t - X_s)$ is independent of $\mathscr{F}_s$, for every $0 \le s \le t$. Notice that a process with independent and homogeneous one-dimensional increments has also homogeneous n-dimensional increments, for every $n \ge 1$. In this case we simply say that X has homogeneous increments. It is easy to see that if X is a process with independent increments with respect to the filtration $\mathbb{F}$, so is X with respect to the completion of $\mathbb{F}$. To verify that an adapted process with $\mathbb{F}$-independent increments is an $\mathbb{F}$-martingale is a simple task:

1.16 Lemma. *Let X be an adapted process with $\mathbb{F}$-independent increments such that $X_0 = 0$. Then X is a martingale (not necessarily càdlàg) if and only if the random variable X_t is integrable and $\mathbb{E}[X_t] = 0$, for every $t \ge 0$.*

Proof. If X is a process with $\mathbb{F}$-independent increments such that $X_0 = 0$ and a (not necessarily càdlàg) martingale then $\mathbb{E}[X_t] = \mathbb{E}[X_0] = 0$, $t \ge 0$.

Conversely, if X is a process with $\mathbb{F}$-independent increments such that $X_0 = 0$ and that $\mathbb{E}[X_t] = 0$, $t \geq 0$, we get

$$\mathbb{E}[X_t|\mathscr{F}_s] = \mathbb{E}[X_t - X_s|\mathscr{F}_s] + X_s = \mathbb{E}[X_t - X_s] + X_s = X_s, \quad 0 \leq s \leq t,$$

proving that X is an $\mathbb{F}$-martingale. □

The $\mathbb{F}$-independence of the increments and the homogeneity of the one-dimensional increments are stable under convergence in probability, as the following lemma shows.

1.17 Lemma. *Let X be an $\mathbb{F}$-adapted process. If $(X^n)_{n\geq 1}$ is a sequence of processes with $\mathbb{F}$-independent increments (resp., homogeneous one-dimensional increments) such that X^n_t converges to X_t in probability, for every $t \geq 0$, as $n \to +\infty$, then X has $\mathbb{F}$-independent increments (resp., homogeneous one-dimensional increments).*

Proof. We assume that the sequence $(X^n)_{n\geq 1}$ has $\mathbb{F}$-independent increments (resp., homogeneous one-dimensional increments). For every $0 \leq s \leq t$, we have

$$\mathbb{E}[\mathrm{e}^{iu(X^n_t - X^n_s)}|\mathscr{F}_s] = \mathbb{E}[\mathrm{e}^{iu(X^n_t - X^n_s)}] \quad (\text{resp.,} \quad \mathbb{E}[\mathrm{e}^{iu(X^n_t - X^n_s)}] = \mathbb{E}[\mathrm{e}^{iu(X^n_{t-s} - X^n_0)}]), \quad u \in \mathbb{R}.$$

Letting n converge to $+\infty$ in the previous formula and applying the theorem of Lebesgue on dominated convergence we get

$$\mathbb{E}[\mathrm{e}^{iu(X_t - X_s)}|\mathscr{F}_s] = \mathbb{E}[\mathrm{e}^{iu(X_t - X_s)}], \quad (\text{ resp.,} \quad \mathbb{E}[\mathrm{e}^{iu(X_t - X_s)}] = \mathbb{E}[\mathrm{e}^{iu(X_{t-s} - X_0)}]), \quad u \in \mathbb{R},$$

which concludes the proof. □

If $\mathbb{F}$ is a filtration satisfying *the usual conditions*, a stochastically continuous adapted process with $\mathbb{F}$-independent increments has a unique càdlàg modification which is again a stochastically continuous adapted process with $\mathbb{F}$-independent increments (cf., e.g., He, Wang & Yan (1992), Theorem 2.68). We observe that any process X with one-dimensional homogeneous increments such that $X_0 = 0$ a.s. and that $X_t \longrightarrow 0$ in probability as $t \downarrow 0$ is stochastically continuous. Indeed,

$$\lim_{s\to t} |X_t - X_s| = \lim s \to t|X_{|t-s|}| = \lim_{u\to 0} |X_u| = 0,$$

where the limits are considered in probability and the equalities in distribution. In particular, any càdlàg process with one-dimensional homogeneous increments which starts at zero is stochastically continuous.

1.18 Definition. Let X be an *adapted* and *stochastically continuous* process such that $X_0 = 0$.

(i) We say that X is an *additive process in law* if it has independent increments.

(ii) If X is a *càdlàg* additive process in law, we simply call it an *additive process.*

(iii) We say that X is an *additive process in law relative to the filtration* $\mathbb{F}$ if it has $\mathbb{F}$-independent increments. If X is also *càdlàg*, we simply call it an *additive process relative to* $\mathbb{F}$.

A relevant subclass of additive processes, which we are going to introduce, are *Lévy processes.*

1.19 Definition. (i) We say that an additive process (resp., an additive process in law) is a *Lévy process (resp., a Lévy process in law)* if it has also homogeneous increments.

(ii) We say that an additive process (resp., an additive process in law) relative to $\mathbb{F}$ is a *Lévy process (resp., a Lévy process in law) relative to* $\mathbb{F}$ if it has also homogeneous increments.

Let L be a Lévy process (resp., a Lévy process in law) relative to $\mathbb{F}$. The notation $(L, \mathbb{F})$ emphasizes the filtration with respect to which L is a Lévy process (resp., a Lévy process in law) and sometimes we simply say that $(L, \mathbb{F})$ is a Lévy process (resp., a Lévy process in law) to mean that L is a Lévy process (resp., a Lévy process in law) relative to $\mathbb{F}$.

For Lévy processes the notions of a martingale and a local martingale are equivalent:

1.20 Proposition. Let $(L, \mathbb{F})$ be a Lévy process. Then $(L, \mathbb{F})$ is a local martingale if and only if $(L, \mathbb{F})$ is a martingale.

Proof. Let us prove this statement under the additional assumption that $\mathbb{E}|L_t| < \infty$, $t \geq 0$ (the general case can be found in He, Wang and Yan (1992), Theorem 11.46). We assume that $(L, \mathbb{F})$ is a local martingale and would like to prove that $(L, \mathbb{F})$ is a martingale.

1) There exists a sequence of stopping times (τ_n) such that $\tau_n \uparrow \infty$ and $(L^{\tau_n}, \mathbb{F})$ are martingales.

2) Set $M_t := L_t - \mathbb{E}[L_t]$ for $t \geq 0$. Then $(M, \mathbb{F})$ and $(M^{\tau_n}, \mathbb{F})$ are martingales.

3) From 1) and 2) we have that $(L^{\tau_n} - M^{\tau_n}, \mathbb{F})$ is a martingale. The process L is a Lévy process and hence $\mathbb{E}[L_t] = t\mathbb{E}[L_1]$.

4) Using 2) and 3) we can represent the stopped process M^{τ_n} as

$$M_{t\wedge\tau_n} = L_{t\wedge\tau_n} - (t \wedge \tau_n)\mathbb{E}[L_1],\ t \geq 0.$$

Finally, we calculate the expectation of $L^{\tau_n} - M^{\tau_n}$:

$$0 = \mathbb{E}[L_t^{\tau_n} - M_t^{\tau_n}] = \mathbb{E}[L_t^{\tau_n} - L_t^{\tau_n} + (t \wedge \tau_n)\mathbb{E}L_1] = \mathbb{E}[t \wedge \tau_n]\mathbb{E}[L_1],$$

but, if n is sufficiently large, $\mathbb{E}[t \wedge \tau_n] > 0$ and hence $\mathbb{E}[L_1] = 0 = t\mathbb{E}[L_1] = \mathbb{E}[L_t]$. Therefore the process L is a true martingale, as a Lévy process with expectation zero. □

We recall that we call a real-valued function c *truncation function* if it satisfies the following conditions:
(i) c is bounded;
(ii) $c(x) = 1 + O(x)$ when $x \to 0$;
(iii) $c(x) = O(1/x)$ when $|x| \to +\infty$. The following theorem is known as *the Lévy–Chintchine formula* or *the Lévy–Chintchine decomposition*:

1.21 Theorem. *Let L be a Lévy process and c a truncation function. Then there exist parameters $b \in \mathbb{R}$, $\sigma^2 \geq 0$ and a measure ν on $\mathbb{R}$ satisfying*
(a) $\nu(\{0\}) = 0$,
(b) $\int_{\mathbb{R}} (x^2 \wedge 1)\nu(\mathrm{d}x) < +\infty$,
such that, for every $u \in \mathbb{R}$, one has

$$\mathbb{E}\exp(iL_t u) = \exp\Big[t\Big(iub - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iuxc(x))\nu(\mathrm{d}x)\Big)\Big]. \quad (1.8)$$

For the fixed function c the triplet (b, σ^2, ν) is uniquely determined.

The proof can be found in Sato (1999), Theorem 8.1. The measure ν from the Lévy–Chintchine decomposition is called the *Lévy measure* and the triplet $b, \sigma^2, \nu)$ is called the *characteristic triplet* of L. The next result we want to mention concerns the characteristic triplet under an equivalent change of measure:

1.22 Theorem. *Let $(L, \mathbb{P})$ and $(L, \mathbb{P}')$ be two Lévy processes on $\mathbb{R}$ with characteristic triplets (b, σ^2, ν) and (b', σ'^2, ν'), respectively. Then $\mathbb{P}|_{\mathscr{F}_t}$ and $\mathbb{P}'|_{\mathscr{F}_t}$ are equivalent for all t (or equivalently for one $t > 0$) if and only if the following three conditions are satisfied:*
(i) *$\sigma^2 = \sigma'^2$;*
(ii) *the Lévy measures ν and ν' are equivalent with*

$$\int_{\mathbb{R}} (e^{\beta(x)/2} - 1)^2 \nu(\mathrm{d}x) < \infty,$$

where $\beta(x) = \log \frac{\mathrm{d}\nu'}{\mathrm{d}\nu}$;

(iii) *If* $\sigma = 0$ *then*

$$b' - b = \int_{-1}^{1} x(\nu' - \nu)(\mathrm{d}x).$$

For the proof we refer to Sato (1999), Theorem 33.1.

We define the g-moments of a random variable and discuss finiteness of the g-moment of X_t for a Lévy process X.

1.23 Definition. Let $g(x)$ be a nonnegative measurable function on $\mathbb{R}$. We call $\int g(x)\mu(\mathrm{d}x)$ the *g-moment* of a measure μ on $\mathbb{R}$. We call $\mathbb{E}[g(X)]$ the *g-moment* of a random variable X on $\mathbb{R}$.

1.24 Definition. A function $g(x)$ on $\mathbb{R}$ is called *submultiplicative* if it is nonnegative and there is a constant $a > 0$ such that

$$g(x + y) \leq a g(x) g(y) \text{ for } x, y \in \mathbb{R}. \tag{1.9}$$

A function bounded on every compact set is called *locally bounded.*

1.25 Proposition. Let g be a submultiplicative, locally bounded, measurable function on $\mathbb{R}$. Then, finiteness of the g-moments is not a time dependent property in the class of Lévy processes. Let $(L, \mathbb{F})$ be a Lévy process on $\mathbb{R}$ with Lévy measure ν. Then, L_t has finite g-moments for every $t > 0$ if and only if $\nu 1_{\{|x|>1\}}$ has finite g-moments.

The proof of Proposition 1.25 can be found in Sato (1999), Proposition 25.3. As an important example of a submultiplicative function it is necessary to mention the function $\exp(\alpha x)$ for any $\alpha \in \mathbb{R}$. Therefore, it follows that the Lévy process L has finite (α-)exponential moments if and only if

$$\int_{\{|x>1|\}} \exp(\alpha x)\nu(\mathrm{d}x) < \infty \text{ for } \alpha \in \mathbb{R}.$$

1.5 Poisson Random Measures

We devote this section to Poisson random measures relative to a filtration. We do not consider general Poisson random measures. Rather we restrict our attention to random measures associated with the jumps of adapted càdlàg processes and consider only the *homogeneous* case. Before we need to introduce the notion of a random measure and of an integer-valued random measure. Of particular interest will be the part concerning the definition of the stochastic integral of deterministic functions with respect to a Poisson random measure and with respect to a compensated Poisson random measure. We recall that we fixed a *complete* probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $\mathbb{F}$ satisfying the *usual conditions*. For the sake of simplicity, we introduce the following notation:

$$(E, \mathscr{B}(E)) := (\mathbb{R}_+ \times \mathbb{R}, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathbb{R})). \tag{1.10}$$

1.26 Definition. A random measure μ on $(E, \mathscr{B}(E))$ is a mapping on $\Omega \times \mathscr{B}(E)$ in $[0, +\infty]$ such that:

(i) $\mu(\cdot, A)$ is a random variable for every $A \in \mathscr{B}(E)$.

(ii) $\mu(\omega, \cdot)$ is a measure on $(E, \mathscr{B}(E))$ such that $\mu(\omega; \{0\} \times \mathbb{R}) = 0$, $\omega \in \Omega$.

If μ is a random measure on $(E, \mathscr{B}(E))$, we write $\mu(A) := \mu(\omega, A)$, $A \in \mathscr{B}(E)$.

For any measurable set A, $\mu(A)$ is a nonnegative random variable on $(\Omega, \mathscr{F}, \mathbb{P})$. We can therefore introduce the expectation of $\mu(A)$ (note that, by definition, $\mu(A) \geq 0$). We call *intensity measure* of μ the mapping m on $\mathscr{B}(E)$ in $[0, +\infty]$ defined by

$$\mathrm{m}(A) := \mathbb{E}[\mu(A)]. \tag{1.11}$$

The intensity measure m is a (deterministic) measure on $(E, \mathscr{B}(E))$. Indeed, we have $\mathrm{m}(\emptyset) = 0$ because $\mu(\omega, \emptyset) = 0$, for every ω, $\mu(\omega, \cdot)$ being a measure. The σ-additivity of m follows from the theorem of B. Levi on monotone convergence (cf. Theorem 1.4).

We say that a random measure μ on $(E, \mathscr{B}(E))$ is an *integer-valued* random measure if $\mu(A)$ takes values in $\mathbb{N} \cup \{+\infty\}$, for every $A \in \mathscr{B}(E)$. Integer-valued random measures are of special importance because of the relation that they have with càdlàg adapted processes. Let X be a càdlàg adapted process. For every $A \in \mathscr{B}(E)$ we define on $(E, \mathscr{B}(E))$ the random measure μ by

$$\mu(\omega; A) = \sum_{s \geq 0} 1_{\{\Delta X_s(\omega) \neq 0\}} 1_A(s, \Delta X_s(\omega)), \quad \omega \in \Omega, \quad A \in \mathscr{B}(E). \tag{1.12}$$

1.27 Proposition. Let X be an adapted càdlàg process with values in $\mathbb{R}$. Then the random measure μ defined on $(E, \mathscr{B}(E))$ by (1.12) is an integer-valued random measure.

Proof. Cf. Jacod & Shiryaev (2000), Proposition II.1.16. □

We call the integer-valued random measure μ defined in (1.12) the *jump measure* of X. Let X be an $\mathbb{F}$-adapted càdlàg process and let μ be its jump measure. It is easy to see that $\mu(\{t\}\times\mathbb{R}) \in \{0,1\}$. Indeed, from the definition of μ, we get

$$\begin{aligned}\mu(\{t\}\times\mathbb{R}) &= \textstyle\sum_{s\geq 0} 1_{\{\Delta X_s\neq 0\}} 1_{\{t\}\times\mathbb{R}}(s, \Delta X_s)\\ &= 1_{\{\Delta X_t\neq 0\}} 1_{\{t\}\times\mathbb{R}}(t, \Delta X_t)\\ &= 1_{\{\Delta X_t\neq 0\}} \in \{0,1\}.\end{aligned}$$

If $A \in \mathscr{B}(E)$, we define the process $N^A = (N^A_t)_{t\geq 0}$ by

$$N^A_t := \mu(A \cap [0,t]\times\mathbb{R}). \tag{1.13}$$

1.28 Definition. We say that an integer-valued random measure μ is a *(homogeneous) Poisson random measure relative to the filtration* $\mathbb{F}$ if:

1. The intensity measure m is of the form $\mathrm{m} = \lambda_+ \otimes \nu$, where λ_+ is the Lebesgue measure on $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$ and ν is a σ-finite measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$,
2. For every fixed $s \in \mathbb{R}_+$ and every $A \in \mathscr{B}(E)$ such that $A \subseteq (s,+\infty)\times\mathbb{R}$, $\mathrm{m}(A) < +\infty$, the random variable $\mu(A)$ is independent of $\mathscr{F}_s$.

Now we would like to introduce the stochastic integral for Poisson random measures. Let X be an $\mathbb{F}$-adapted càdlàg process. We assume that the jump measure of X defined by (1.12) is a Poisson random measure relative to the filtration $\mathbb{F}$ with intensity measure $\mathrm{m} = \lambda_+ \otimes \nu$. We observe that this part remains valid also if μ is a general Poisson random measure relative to the filtration $\mathbb{F}$ (cf. Jacod & Shiryaev (2000), Chapter II) and not only a homogeneous Poisson random measure which moreover is the jump measure of an adapted càdlàg process. We recall that the definition of $(E, \mathscr{B}(E))$ was given in (1.10). For a deterministic *numerical* function f which is $\mathscr{B}(E)$-measurable we have introduced the notation

$$\mathrm{m}(f) := \int_E f(t,x)\,\mathrm{m}(\mathrm{d}t, \mathrm{d}x)$$

if the integral on the right-hand side exists. In particular $\mathrm{m}(f)$ is well defined if f is nonnegative. We define the integral of f with respect to μ *ω-wise* in an analogous way, because $\mu(\omega,\cdot)$ is a (nonnegative) measure on $(E, \mathscr{B}(E))$ for every $\omega \in \Omega$. If f is a nonnegative measurable function, then the integral $\int_E f(t,x)\,\mu(\omega, \mathrm{d}t, \mathrm{d}x)$ always exists. We shall use the notation $\mu(f)$ for this random variable with values in $[0,+\infty]$. This definition extends to functions f of arbitrary sign. More precisely, for *any* measurable function f on $(E, \mathscr{B}(E))$, by Ω_f we denote the set of all $\omega \in \Omega$ such that $\int_E f(t,x)\,\mu(\omega, \mathrm{d}t, \mathrm{d}x)$ exists and is finite a.s. Obviously $\Omega_f \in \mathscr{F}$. We say that the integral of f with respect to μ exists and is finite a.s. if $\mathbb{P}[\Omega_f] = 1$. In this case the random variable $\mu(f)$ defined by

$$\mu(\omega, f) := \mu(f)(\omega) := \begin{cases} \int_E f(t,x)\,\mu(\omega, \mathrm{d}t, \mathrm{d}x), & \text{if } \omega \in \Omega_f; \\ \\ 0, & \text{otherwise;} \end{cases} \tag{1.14}$$

is called the *stochastic integral* of f with respect to the Poisson random measure μ. Note that the stochastic integral $\mu(f)$ exists and is finite a.s. if and only if $\mu(|f|) < +\infty$ a.s. We now state the so-called *exponential formula* (cf. Kallenberg (1997), Lemma 10.2).

1.29 Lemma (Exponential Formula)**.** *Let f be a function on $(E, \mathscr{B}(E))$. If $f \geq 0$, then*

$$\mathbb{E}[\mathrm{e}^{-\mu(f)}] = \exp(\mathrm{m}(\mathrm{e}^{-f} - 1)). \tag{1.15}$$

We now show how to compute the expectation of the random variable $\mu(f)$, where f is a function which belongs to $L^1(\mathrm{m})$.

1.30 Lemma. *Let $f \in L^1(\mathrm{m})$. Then*

$$\mathbb{E}[\mu(f)] = \mathrm{m}(f). \tag{1.16}$$

Moreover, the stochastic integral with respect to μ is a continuous operator on $L^1(\mathrm{m})$ into $L^1(\mathbb{P})$.

Proof. For every nonnegative function $f \in L^1(\mathrm{m})$ formula (1.16) holds. Indeed, this is true for indicator functions of the form 1_A, $A \in \mathscr{B}(E)$, $\mathrm{m}(A) < +\infty$, and hence for nonnegative simple functions f. For an arbitrary nonnegative function f we can find a sequence $(f_n)_{n\geq 1}$ of nonnegative simple functions such that $f_n \uparrow f$ pointwise as $n \to +\infty$. The result follows applying the theorem of B. Levi on monotone convergence (cf. Theorem 1.4). Clearly, formula (1.16) extends to functions f such that $\mathrm{m}(|f|) < +\infty$. The statement on the continuity follows from

$$\mathbb{E}[|\mu(f)|] \leq \mathbb{E}[\mu(|f|)] = \mathrm{m}(|f|) < +\infty.$$

□

Now we characterize, in terms of the intensity measure m, under which conditions the integral of a deterministic function f with respect to μ exists and is a.s. finite (cf. Kallenberg (1997), Lemma 10.2).

1.31 Proposition. Let μ be a Poisson random measure on $(E, \mathscr{B}(E))$ with intensity measure m. Then $\mu(f)$ exists and is finite a.s. if and only if $\mathrm{m}(|f| \wedge 1) < +\infty$.

In the following proposition we show that, for every $A \in \mathscr{B}(E)$ such that $\mathrm{m}(A) < +\infty$, the process $(N^A, \mathbb{F})$ defined by (1.13) is a Poisson process.

1.32 Proposition. Let μ be the jump measure of a Lévy process $(L, \mathbb{F})$ with intensity measure $\mathrm{m} = \lambda_+ \otimes \nu$ and let $A \in \mathscr{B}(E)$ be such that $(\lambda_+ \otimes \nu)(A) < +\infty$. Then the process $N^A := (N^A_t)_{t \geq 0}$ defined by $N^A_t := \mu(A \cap [0, t] \times \mathbb{R})$, $t \geq 0$, is a Poisson process relative to $\mathbb{F}$ and $a^A(\cdot) := (\lambda_+ \otimes \nu)(A \cap [0, \cdot] \times \mathbb{R})$ is its intensity function.

Proof. For every set $A \in \mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{R})$ consider $\mu(A \cap [0, t])$. It is a simple point process (cf. Jacod & Shiryaev (2000), Example 3, page 34):

(i) the process $\mu([0, t] \cap A)$ is increasing;

(ii) $\Delta\mu([0, t] \cap A) = \mu(\{t\} \times A) \in \{0, 1\}$ because μ is an integer-valued random measure (cf. Jacod & Shiryaev (2000), Proposition II.1.16);

(iii) the process $\mu([0, t] \cap A)$ is adapted:

$$\begin{aligned} \mu([0, t] \cap A) = \mu(A \cap [0, t] \times \mathbb{R}) &= \textstyle\sum_{0 \leq s \leq t} 1_{\{\Delta X_s \neq 0\}} 1_A(s, \Delta X_s) \\ &= \textstyle\sum_{n: T_n \leq t}^{\infty} 1_A(T_n, \Delta X_{T_n}), \end{aligned}$$

where $(T_n)_{n \geq 1}$ is an exhausting sequence for the jumps of X such that $[T_n] \cap [T_m] = \emptyset$, $m \neq n$. The process ΔX is an optional process and therefore the random variable ΔX_{T_n} is $\mathscr{F}_{T_n}$-measurable, for every $n \geq 1$ (cf. Jacod & Shiryaev (2000), Proposition I.1.21). Hence $1_A(T_n, \Delta X_{T_n})$ is $\mathscr{F}_{T_n}$-measurable and therefore $1_A(T_n, \Delta X_{T_n}) 1_{\{T_n \leq t\}}$ is $\mathscr{F}_t$-measurable, because T_n is a stopping time for every $n \geq 1$. By Jacod & Shiryaev (2000), Theorem II.4.5, the process $\mu([0, t] \cap A)$ is a Poisson process if its intensity function is $a^A(\cdot) := (\lambda_+ \otimes \nu)(A \cap [0, \cdot] \times \mathbb{R})$ is the compensator or, equivalently, that $(\mu([0, \cdot] \cap A) - a^A(\cdot), \mathbb{F})$ is a martingale. For this it is sufficient to show the $\mathbb{F}$-independence of the increments of N^A, which follows from the definition of the Poisson random measure, and to apply Lemma 1.30. □

1.6 Stochastic Exponential and Logarithm

Let us consider the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ satisfying *the usual conditions*. In the following section we would like to focus the attention on the stochastic alternative of the exponential transformation, which plays an important role in this thesis.

1.33 Proposition. Let Y be a real-valued semimartingale and consider the stochastic differential equation

$$\mathrm{d}Z_t = Z_{t-}\mathrm{d}Y_t,\ t \geq 0,$$

with $Z_0 = 1$. This equation has a unique (up to indistinguishability) càdlàg adapted solution, called *the stochastic exponential* of Y, which is a semimartingale and is denoted by $\mathscr{E}(Y)$. Explicitly,

$$\mathscr{E}(Y)_t = \exp(Y_t - Y_0 - \frac{1}{2}\langle Y_t^c\rangle)\prod_{s\leq t}(1+\Delta Y_s)\exp(-\Delta Y_s),\ t \geq 0.$$

If we define $\tau := \inf\{t \geq 0 : \Delta Y_t = -1\}$, then $\mathscr{E}(Y) \neq 0$ on $[0,\tau)$, $\mathscr{E}(Y)_- \neq 0$ on $[0,\tau]$ and $\mathscr{E}(Y) = 0$ on $[\tau,\infty)$, $t \geq 0$.

For the proof we refer to Jacod & Shiryaev (2000), Theorem I.4.61.

The mapping $Y \mapsto \mathscr{E}(Y)$ can be inverted. The following proposition shows this fact.

1.34 Proposition. Let Z be a semimartingale such that Z, Z_- do not vanish. Then there exists an up to indistinguishability unique semimartingale Y with $Y_0 = 0$ and $Z = Z_0\mathscr{E}(Y)$. It is given by

$$Y_t = \int_0^t \frac{1}{Z_{u-}}\mathrm{d}Z_u,\ t \geq 0.$$

The proof of this proposition can be found in Jacod & Shiryaev (2000), Theorem II.8.3, or Kallsen & Shiryaev (2002), Lemma 2.2. The process Y from the previous proposition is called *the stochastic logarithm of Z* and is written $\mathscr{L}(Z) := Y$. But such transformations preserve not just the semimartingale property of the underlying process, but even the local martingale property:

1.35 Proposition. Let $\mathscr{E}(Y) > 0$ $\mathbb{P}$-a.s. The process Y is a local martingale if and only if $\mathscr{E}(Y)$ is a local martingale.

Proof. Let us assume that $\mathscr{E}(Y)$ is a local martingale. Then there exists a sequence of stopping times $\{\tau_n\}$ which localizes the local martingale $\mathscr{E}(Y)$ (i.e., $\mathscr{E}(Y)^{\tau_n}$ is a martingale for any n). Let $\sigma_n = \tau_n \wedge \inf\{t : \mathscr{E}(Y)_t < \frac{1}{n}\}$. So $\{\sigma_n\}$ is still a sequence of stopping times such that $\sigma_n \uparrow \infty$ $\mathbb{P}$-a.s. and

$\mathscr{E}(Y)^{\sigma_n}$ are martingales and the (stopped) process $\dfrac{1_{[0,\sigma_n]}}{\mathscr{E}(Y)_-^{\sigma_n}}$ is predictable and bounded. By the definition of $\mathscr{E}(Y)$,

$$Y_t = \int_0^t \frac{1}{\mathscr{E}(Y)_{u-}} \mathrm{d}\mathscr{E}(Y)_u,\ t \geq 0.$$

Using localization we have

$$Y_t^{\sigma_n} = \int_0^t \frac{1_{[0,\sigma_n]}(u)}{\mathscr{E}(Y)_{u-}^{\sigma_n}} \mathrm{d}\mathscr{E}(Y)_u^{\sigma_n},\ t \geq 0,$$

so $Y_t^{\sigma_n}$ is a martingale and hence Y_t is a local martingale. Here we have used that the stochastic integral of a bounded predictable process with respect to a martingale is again a martingale.

Similarly in the converse direction. □

If the underlying process in the stochastic exponential is a Lévy process, then the stronger result is true:

1.36 Proposition. If L is a Lévy process and a martingale, then its stochastic exponential $Z = \mathscr{E}(L)$ is also a martingale.

The proof of this fact can be found in Cont & Tankow (2003), Proposition 8.23.

1.7 Relative Entropy

In this section we introduce the notion of relative entropy and discuss some well-known properties of it.

1.37 Definition. Let $\mathbb{Q}$ and $\mathbb{P}$ be two probability measures on $(\Omega, \mathscr{F})$ and $\mathscr{G}$ some sub-σ-field of $\mathscr{F}$. The *relative entropy* $I(\mathbb{Q},\mathbb{P})_{\mathscr{G}}$ *of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$ *on* $\mathscr{G}$ is defined by

$$I_{\mathscr{G}}(\mathbb{Q},\mathbb{P}) := \begin{cases} \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathscr{G}} \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathscr{G}}\right], & \text{if } \mathbb{Q}|_{\mathscr{G}} \ll \mathbb{P}|_{\mathscr{G}}, \\ +\infty, & \text{otherwise.} \end{cases}$$

For $\mathscr{G} = \mathscr{F}$ we simply write $I_{\mathscr{G}}(\mathbb{Q},\mathbb{P}) = I(\mathbb{Q},\mathbb{P})$ and call $I(\mathbb{Q},\mathbb{P})$ *the relative entropy of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$.

If we are given a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ for $T > 0$ and $\mathbb{Q}|_{\mathscr{F}_T} \ll \mathbb{P}|_{\mathscr{F}_T}$, we define the *entropy process of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$ by $I_t(\mathbb{Q}, \mathbb{P}) = I_{\mathscr{F}_t}(\mathbb{Q}, \mathbb{P})$, $t \geq 0$.

Now we state some properties of the relative entropy.

1.38 Lemma. *Let* $\mathbb{Q} \ll \mathbb{P}$ *on some* σ*-field* $\mathscr{G} \subseteq \mathscr{F}$ *and let* $\mathscr{H} \subseteq \mathscr{G}$ *be another* σ*-field. Then*

$$I_{\mathscr{H}}(\mathbb{Q}, \mathbb{P}) \leq I_{\mathscr{G}}(\mathbb{Q}, \mathbb{P}).$$

Proof. The density $Z_{\mathscr{G}} = \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathscr{G}}$ is $\mathbb{P}$-integrable, and without loss of generality we may assume that $I_{\mathscr{G}}(\mathbb{P}, \mathbb{Q}) < \infty$. Thus $Z_{\mathscr{G}} \log Z_{\mathscr{G}}$ is $\mathbb{P}$-integrable as well, and by Jensen's inequality for conditional expectations we have

$$\begin{aligned} I_{\mathscr{H}}(\mathbb{Q}, \mathbb{P}) &= \mathbb{E}_{\mathbb{P}}[Z_{\mathscr{H}} \log Z_{\mathscr{H}}] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[Z_{\mathscr{G}}|\mathscr{H}] \log \mathbb{E}_{\mathbb{P}}[Z_{\mathscr{G}}|\mathscr{H}]] \\ &\leq \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[Z_{\mathscr{G}} \log Z_{\mathscr{G}}|\mathscr{H}]] = \mathbb{E}_{\mathbb{P}}[Z_{\mathscr{G}} \log Z_{\mathscr{G}}] \\ &= I_{\mathscr{G}}(\mathbb{Q}, \mathbb{P}), \end{aligned}$$

and hence the claim. □

1.39 Lemma. *Let* $\mathbb{Q}_1, \mathbb{Q}_2 \ll \mathbb{P}$ *on some* σ*-field* $\mathscr{G} \subseteq \mathscr{F}$*. Then for all* $0 \leq \alpha \leq 1$

$$I_{\mathscr{G}}((1-\alpha)\mathbb{Q}_1 + \alpha\mathbb{Q}_2, \mathbb{P}) \leq (1-\alpha) I_{\mathscr{G}}(\mathbb{Q}_1, \mathbb{P}) + \alpha I_{\mathscr{G}}(\mathbb{Q}_2, \mathbb{P}),$$

i.e., relative entropy is a convex functional in the first argument.

Proof. Let $\mathbb{Q} := (1-\alpha)\mathbb{Q}_1 + \alpha\mathbb{Q}_2$ and Z_1 and Z_2 be the densities of $\mathbb{Q}_1$ and $\mathbb{Q}_2$ on $\mathscr{G}$, respectively. Then the density of $\mathbb{Q}$ on $\mathscr{G}$ is $Z = (1-\alpha)Z_1 + \alpha Z_2$. The function $\varphi(z) := z \log z$ is convex on $[0, \infty)$, hence

$$\begin{aligned} I_{\mathscr{G}}(\mathbb{Q}, \mathbb{P}) &= \mathbb{E}_{\mathbb{P}}\varphi(Z) = \mathbb{E}_{\mathbb{P}}[\varphi((1-\alpha)Z_1 + \alpha Z_2)] \\ &\leq \mathbb{E}_{\mathbb{P}}[(1-\alpha)\varphi(Z_1) + \alpha\varphi(Z_2)] \\ &= (1-\alpha)\mathbb{E}_{\mathbb{P}}\varphi(Z_1) + \alpha\mathbb{E}_{\mathbb{P}}\varphi(Z_2), \end{aligned}$$

therefore the statement is proven. □

1.40 Lemma. *Let* $\mathbb{Q}|_{\mathscr{F}_T} \ll \mathbb{P}|_{\mathscr{F}_T}$ *with density process* Z *and finite-valued entropy process. Then* $Z \log Z$ *is a* $\mathbb{P}$*-submartingale on* $[0, T]$ *relative to* $\mathbb{F}$*.*

Proof. Let $0 \leq s \leq t \leq T$. Note that $Z_t \log Z_t$ is $\mathbb{P}$-integrable due to finiteness of $I_t(\mathbb{Q}, \mathbb{P})$. So by Jensen's inequality and the fact that Z is a $\mathbb{P}$-martingale we have

$$\mathbb{E}_{\mathbb{P}}[Z_t \log Z_t|\mathscr{F}] \geq \mathbb{E}_{\mathbb{P}}[Z_t|\mathscr{F}_s] \log \mathbb{E}_{\mathbb{P}}[Z_t|\mathscr{F}] = Z_s \log Z_s$$

$\mathbb{P}$-a.s., hence $\mathbb{Q}$-a.s. The statement is proven. □

In Section 1.3 we have defined the notions of a martingale and a local martingale for time parameters $t \geq 0$. The notion of a martingale can be easily adopted for finite intervals, but with local martingales we have to take some care. If we consider open from the right interval $[0, T)$, for the definition of a local martingale it is sufficient to change the limit point for the reducing sequence from ∞ to T. But for the closed interval $[0, T]$ this condition is not appropriate. Let us consider the following example: Let W be a Wiener process on $[0, T]$. We define a process X in the following way

$$X_t := \begin{cases} W_t, & \text{if } t \in [0, T), \\ \\ 1, & \text{if } t = T. \end{cases}$$

It is easy to see that the process X is a martingale on $[0, T)$, but it is not the case for the closed interval $[0, T]$. Therefore, we say that the process M is a *local martingale on the closed interval* $[0, T]$, if there exists a sequence of stopping times (τ_n) such that $\tau_n \uparrow T$ when $n \to \infty$, the processes M^{τ_n} are martingales for every n and $\{\tau_n = T\} \uparrow \Omega$.

Consider a real-valued adapted stochastic process X defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t \in [0,T]}, \mathbb{P})$ for some fixed $T > 0$.

Let us introduce the following notations for classes of probability measures:

(i) The class of absolutely continuous measures $\mathbb{Q}$ with respect to $\mathbb{P}$ defined on $\mathscr{F}_T$ such that the stochastic process $(X, \mathbb{F})$ is a $\mathbb{Q}$-martingale on $[0, T]$:

$$\mathscr{M}_a(T) := \{\mathbb{Q} : (X, \mathbb{F}) \text{ is a } \mathbb{Q}\text{-martingale on } [0, T],\ \mathbb{Q} \ll \mathbb{P} \text{ on } \mathscr{F}_T\}.$$

(ii) The class of absolutely continuous measures $\mathbb{Q}$ with respect to $\mathbb{P}$ defined on $\mathscr{F}_T$ such that the stochastic process $(X, \mathbb{F})$ satisfies the moment condition $\mathbb{E}_Q X_T = 0$ at time T:

$$\widetilde{\mathscr{M}_a}(T) := \{\mathbb{Q} : \mathbb{E}_Q X_T = 0 \text{ and } \mathbb{Q} \ll \mathbb{P} \text{ on } \mathscr{F}_T\}.$$

In the case $X_0 = 0$ (as it holds for Lévy processes), one can easily observe that $\mathscr{M}_a(T) \subseteq \widetilde{\mathscr{M}_a}(T)$.

(iii) The class of absolutely continuous measures $\mathbb{Q}$ with respect to $\mathbb{P}$ defined on $\mathscr{F}_T$ such that the stochastic process $(X, \mathbb{F})$ is a $\mathbb{Q}$-local martingale on $[0, T]$:

$$\mathscr{M}_a^{loc}(T) := \{\mathbb{Q} : (X, \mathbb{F}) \text{ is a } \mathbb{Q}\text{-local martingale on } [0, T],\ \mathbb{Q} \ll \mathbb{P} \text{ on } \mathscr{F}_T\}.$$

Note that here we use the definition of the local martingale M on the closed interval $[0, T]$: there exists a sequence of monotonically increasing stopping times (τ_n) such that $\tau_n \uparrow T$: $(M^{\tau_n}, \mathbb{F})$ is a martingale on $[0,T]$ and $\{\tau_n = T\} \uparrow \Omega$.

(iv) The class of equivalent measures $\mathbb{Q}$ with respect to $\mathbb{P}$ defined on $\mathscr{F}_T$ such that the stochastic process $(X, \mathbb{F})$ is a $\mathbb{Q}$-local martingale on $[0,T]$:

$$\mathscr{M}_e(T) := \{\mathbb{Q} \in \mathscr{M}_a^{loc}(T) : \mathbb{Q} \sim \mathbb{P} \text{ on } \mathscr{F}_T\}.$$

(v) The class of measures $\mathbb{Q}$ from $\mathscr{M}_a^{loc}(T)$ such that the relative entropy is finite:

$$\mathscr{M}_f(T) := \{\mathbb{Q} \in \mathscr{M}_a^{loc}(T) : I(\mathbb{Q}, \mathbb{P}) < +\infty \text{ on } \mathscr{F}_T\}.$$

(vi) The class of measures $\mathbb{Q}$ from $\mathscr{M}_a^{loc}(T)$ such that the process X is still a $\mathbb{Q}$-Lévy process on $[0,T]$:

$$\mathscr{M}_l(T) := \{\mathbb{Q} \in \mathscr{M}_a^{loc}(T) : (X, \mathbb{F}) \text{ is a } \mathbb{Q}\text{-Lévy process on } [0,T]\}.$$

Taking into account Proposition 1.20 we can easily observe that in the last definition we can substitute the class $\mathscr{M}_a^{loc}(T)$ by the class $\mathscr{M}_a(T)$: if the Lévy process is a local martingale on $[0,T]$, then it is a martingale on $[0,T]$.

Let $\mathbb{Q}_n$ be defined similarly to the definition of $\mathbb{P}_n$:

$$\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{Q}}(\mathscr{F}_T) = Z_T^{(n)},\ n \geq 1,$$

where $\mathbb{Q}$ is absolutely continuous measure with respect to $\mathbb{P}$. We assume that $I(\mathbb{Q}, \mathbb{P}) < +\infty$.

1.41 Definition. A probability measure $\mathbb{Q}_0 \in \mathscr{M}_a$ will be called the minimal entropy martingale measure (MEMM) if it satisfies

$$I(\mathbb{Q}_0, \mathbb{P}) = \min_{\mathbb{Q} \in \mathscr{M}_a} I(\mathbb{Q}, \mathbb{P}) = \min_{\mathbb{Q} \in \mathscr{M}_a} \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right].$$

The distance between $\mathbb{P}$ and $\mathscr{M}_a$ is defined by

$$I(\mathscr{M}_a, \mathbb{P}) = \inf_{\mathbb{Q} \in \mathscr{M}_a} I(\mathbb{Q}, \mathbb{P}).$$

2 General Approach

The aim of this chapter is to give an introduction to the problem of the minimal entropy martingale measures (MEMM). The chapter is based on results of Frittelli (2000) which are natural generalizations of ideas of Csiszár (1975). We consider a general incomplete security market model, with possibly an infinite number of price processes defined on a general filtered probability space. When the processes are bounded, it is proven that the MEMM exists and is unique (Proposition 2.3). It is also shown that the MEMM (if it exists and the relative entropy is finite) is always equivalent to the original measure. The main result of this part is formulated in Theorem 2.6 and gives the characterization of the density of the MEMM. The results mentioned above are quite general and abstract, but they are useful in applications to particular models. In our case the results of this chapter will be applied to the one-step model (see Appendix A). Despite its simplicity, the one-step model plays an important role and is later used for solving the more general problem when the price process is driven by a geometrical Lévy process.

2.1 Some Definitions and Notations

Let $(\Omega, \mathscr{F})$ be a measurable space and $\mathbb{Q}$ any probability measure on it. We denote the set of time parameters by $\mathscr{T} \subseteq [0, +\infty)$. Let us assume that $0 \in \mathscr{T}$. We introduce the filtration $\mathbb{F} := (\mathscr{F}_t)_{t\in\mathscr{T}}$ and consider a family $\mathscr{X}$ of real-valued $\mathbb{F}$-adapted stochastic processes $X = (X_t)_{t\in\mathscr{T}}$ such that X_0 is bounded $\mathbb{Q}$-a.s. The expectation with respect to $\mathbb{Q}$ is denoted by $\mathbb{E}_{\mathbb{Q}}$.

The set of all real-valued random variables on $(\Omega, \mathscr{F}, \mathbb{Q})$ is denoted by $L^0(\mathbb{Q})$. For the set of all essentially bounded random variables on $(\Omega, \mathscr{F}, \mathbb{Q})$ we shall use the notation $L^\infty(\mathbb{Q})$. The set of integrable random variables on

$(\Omega, \mathscr{F}, \mathbb{Q})$ is denoted by $L^1(\mathbb{Q}) = \{\eta : \mathbb{E}_{\mathbb{Q}}[|\eta|] < \infty\}$.

2.1 Definition. A probability measure $\mathbb{Q}$ is called a martingale measure, if all processes $X \in \mathscr{X}$ are martingales under $\mathbb{Q}$.

From now on we fix a probability measure $\mathbb{P}$ on the space $(\Omega, \mathscr{F})$: This is the reference probability measure and by $\mathbb{E}$ we denote the expectation with respect to it.

2.2 Definition. The set of all martingale measures $\mathbb{Q}$ defined on $(\Omega, \mathscr{F})$ which are absolutely continuous with respect to $\mathbb{P}$ is denoted by $\mathscr{M}_a$, that is,

$$\mathscr{M}_a := \{\mathbb{Q} : \mathbb{Q} \ll \mathbb{P},\ (X_t, \mathscr{F}_t)_{t \in \mathscr{T}} \text{ is a } \mathbb{Q}\text{-martingale for any } X \in \mathscr{X}\},$$

while $\mathscr{M}_e$ denotes the set of all martingale measures $\mathbb{Q}$ defined on $(\Omega, \mathscr{F})$ which are equivalent to $\mathbb{P}$, i.e.,

$$\mathscr{M}_e := \{\mathbb{Q} : \mathbb{Q} \in \mathscr{M}_a,\ \mathbb{Q} \sim \mathbb{P}\}.$$

Let us introduce the linear space K of random variables defined by

$$K = \mathrm{Span}(\{\zeta(X_t - X_s) : \zeta \in L^\infty(\Omega, \mathscr{F}_s, \mathbb{P}), \quad s, t \in \mathscr{T}, \quad s \le t, \quad X \in \mathscr{X}\}).$$

It is easy to prove the following identity:

$$\mathscr{M}_a = \{\mathbb{Q} : \mathbb{Q} \ll \mathbb{P},\ \mathbb{E}_{\mathbb{Q}}[Z] = 0, \quad Z \in K\}. \tag{2.1}$$

Indeed, if we set $\mathscr{N} := \{\mathbb{Q} : \mathbb{Q} \ll \mathbb{P},\ \mathbb{E}_{\mathbb{Q}}[Z] = 0,\ Z \in K\}$ and we consider $\mathbb{Q} \in \mathscr{M}_a$, then from $\mathbb{E}_{\mathbb{Q}}[\zeta(X_t - X_s)] = \mathbb{E}_{\mathbb{Q}}[\zeta\, \mathbb{E}_{\mathbb{Q}}[X_t - X_s | \mathscr{F}_s]] = 0$ for every $\mathscr{F}_s$-measurable bounded ζ, it follows that $\mathbb{E}_{\mathbb{Q}}[Z] = 0$ for every $Z \in K$, meaning that $\mathscr{M}_a \subset \mathscr{N}$. Conversely, if $\mathbb{Q} \in \mathscr{N}$ and $X \in \mathscr{X}$, then $\mathbb{E}_{\mathbb{Q}}[1_A(X_t - X_s)] = 0$ for any $A \in \mathscr{F}_s$, thus X is a $\mathbb{Q}$-martingale, i.e., $\mathscr{N} \subset \mathscr{M}_a$. Hence, $\mathscr{M}_a = \mathscr{N}$.

2.2 Construction of the MEMM

Before we describe the particular case of existence and uniqueness of the MEMM, it is necessary to introduce some additional notations. The function φ is defined as

$$\varphi(x) := x \log x$$

for any x from the interval $(0, +\infty)$, $\varphi(0) := 0$. We recall the Definition 1.37: the relative entropy $I_{\mathscr{G}}(\mathbb{Q}, \mathbb{P})$ of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathscr{G} \subset \mathscr{F}$ is defined by

$$I_{\mathscr{G}}(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{G}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{G}}\right], & \text{if } \mathbb{Q}|_{\mathscr{G}} \ll \mathbb{P}|_{\mathscr{G}}, \\ +\infty, & \text{otherwise.} \end{cases}$$

In this chapter we always work with entropy on the σ-algebra $\mathscr{F}$, therefore for the sake of simplicity of notations the σ-algebra will be omitted. We also want to recall the notation of the class $\mathscr{M}_f$ of all absolutely continuous with respect to $\mathbb{P}$ martingale measures with finite relative entropy:

$$\mathscr{M}_f = \{\mathbb{Q} \in \mathscr{M}_a : I(\mathbb{Q}, \mathbb{P}) < +\infty\}.$$

In the following proposition we give a simple condition for the existence of the MEMM.

2.3 Proposition. If $I(\mathscr{M}_a, \mathbb{P}) < +\infty$ and if all $X \in \mathscr{X}$ are bounded then there exists the MEMM and it is unique.

Proof. Let us choose a sequence $(\mathbb{Q}_k)_{k\geq 1} \subset \mathscr{M}_f$ such that $I(\mathbb{Q}_k, \mathbb{P}) \downarrow I(\mathscr{M}_a, \mathbb{P})$. For the probability densities $\frac{\mathrm{d}\mathbb{Q}_k}{\mathrm{d}\mathbb{P}}$ we have the properties

$$\frac{\mathrm{d}\mathbb{Q}_k}{\mathrm{d}\mathbb{P}} \geq 0 \text{ and } \mathbb{E}_{\mathbb{P}} \frac{\mathrm{d}\mathbb{Q}_k}{\mathrm{d}\mathbb{P}} = 1.$$

By the Komlós theorem (see Kabanov & Pergamenshchikov (2013), Theorem A7.1) there is a subsequence k_j such that $Z^n := \frac{1}{n}\sum_{j=1}^{n} \mathrm{d}\mathbb{Q}_{k_j}/\mathrm{d}\mathbb{P}$ converges almost surely to a certain $\eta \in L^1(\mathbb{P})$ as $n \to +\infty$. Using Fatou's lemma and the convexity of φ we obtain

$$\begin{aligned}\mathbb{E}\varphi(\eta) &= \mathbb{E}\Big[\lim_{n\to\infty} \varphi(Z^n)\Big] \leq \liminf_{n\to\infty} \mathbb{E}\Big[\varphi(Z^n)\Big] \leq \limsup_{n\to\infty} \mathbb{E}\Big[\varphi(Z^n)\Big] \\ &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\Big[\varphi\Big(\frac{\mathrm{d}\mathbb{Q}_{k_j}}{\mathrm{d}\mathbb{P}}\Big)\Big] = I(\mathscr{M}_a, \mathbb{P}) < +\infty.\end{aligned} \tag{2.2}$$

Hence, by the Valleé-Poussin criterion, $(Z^n)_{n\geq 1}$ is uniformly integrable and therefore converges in $L^1(\mathbb{P})$ to η. In particular, this implies that $\mathbb{E}[\eta] = 1$. From this and the assumption that $\zeta \in K$ is bounded, it follows that we can define a probability measure $\mathbb{Q}$ with density η with respect to the measure $\mathbb{P}$, i.e., $\mathrm{d}\mathbb{Q} := \eta \mathrm{d}\mathbb{P}$, and

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\zeta] = \mathbb{E}[\zeta\eta] &= \lim_{n\to\infty} \mathbb{E}\left(\zeta \frac{1}{n} \sum_{j=1}^{n} \frac{\mathrm{d}\mathbb{Q}_{k_j}}{\mathrm{d}\mathbb{P}}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\left(\zeta \frac{\mathrm{d}\mathbb{Q}_{k_j}}{\mathrm{d}\mathbb{P}}\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}_{\mathbb{Q}_{k_j}}(\zeta) = 0.\end{aligned}$$

Moreover, the infimum of the relative entropy between the measure $\mathbb{P}$ and the set of all absolutely continuous martingale measures with respect to $\mathbb{P}$ is achieved at $\mathbb{Q}$. This follows from the fact that $\mathbb{Q}$ is a martingale measure, the equality $I(\mathbb{Q},\mathbb{P}) = \mathbb{E}\varphi(\eta)$ and (2.2). Consequently, $I(\mathbb{Q},\mathbb{P}) = I(\mathscr{M}_a,\mathbb{P})$. The uniqueness follows from strict convexity of φ. $\square$

It is worth to mention that in the last proof we follow the approach of Kabanov and Stricker (see Kabanov & Stricker (2002), Proposition 3.2) which is direct and simpler in comparison to the approach of Frittelli (see Frittelli (2000), Theorem 2.1), where are used arguments of the Dunford–Pettis compactness theorem and convergence in the $\sigma(L^1(\mathbb{P}), L^\infty(\mathbb{P}))$ topology.

In general, the condition of the boundedness of the price-processes is very rarely satisfied and usually we don't have so strong limitations. Our next aim is to show an important property of the MEMM: if it exists, then it is equivalent to the original measure $\mathbb{P}$. For other types of Hellinger's distance, for example, so-called variation distance when the function φ is quadratic, this property does not hold. The following lemma will be needed later.

2.4 Lemma. *Let* $\mathbb{Q}_0, \mathbb{Q}_1 \in \mathscr{M}_f$ *and*

$$f(t) := \varphi\left(t\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} + (1-t)\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right),\ t \in [0,1].$$

Then $\mathbb{E}_{\mathbb{Q}_1}\left[\log\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right]$ *is well-defined and*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}f(t)\right)\Big|_{t=0} = \mathbb{E}f'(0) = \mathbb{E}_{\mathbb{Q}_1}\left[\log\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right] - I(\mathbb{Q}_0,\mathbb{P}) < +\infty. \tag{2.3}$$

Proof. Since f is convex, the ratio $\frac{f(t)-f(0)}{t}$ is not decreasing, moreover it is converging to $f'(0)$ as $t \downarrow 0$ and

$$f'(0) \le \frac{f(t)-f(0)}{t} \le f(1) - f(0) = \varphi\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right) - \varphi\left(\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right),$$

therefore $\frac{f(t)-f(0)}{t}$ is dominated by an integrable random variable and, hence, integrable. Indeed, by assumption, the measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$ have finite relative entropy with respect to the measure $\mathbb{P}$, consequently $\mathbb{E}[\varphi\left(\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right)] = I(\mathbb{Q}_0,\mathbb{P}) < +\infty$ and $\mathbb{E}[\varphi\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right)] = I(\mathbb{Q}_1,\mathbb{P}) < +\infty$. Using the theorem on monotone convergence, from these two facts it follows that

$$\lim_{t\downarrow 0}\mathbb{E}\left[\frac{f(t)-f(0)}{t}\right] = \mathbb{E}f'(0)$$

but

$$\begin{aligned} f'(t) &= \left(\varphi\left(t\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} + (1-t)\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right)\right)' \\ &= \varphi'\left(t\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} + (1-t)\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right)\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right) \\ &= \left(\log\left(t\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} + (1-t)\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right) + 1\right)\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right). \end{aligned} \tag{2.4}$$

Putting $t = 0$ into (2.4) we get

$$f'(0) = \left(\log\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right) + 1\right)\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right)$$

and hence

$$\begin{aligned} \mathbb{E}f'(0) &= \mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right)\left(\log\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}} + 1\right) \\ &= \mathbb{E}_{\mathbb{Q}_1}\left[\log\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right] - I(\mathbb{Q}_0, \mathbb{P}). \end{aligned}$$

The statement is proven. □

Now we can prove a statement about the property of the MEMM announced earlier.

2.5 Theorem. *Suppose that $\mathbb{Q}_0$ is the MEMM. If there exists a probability $\mathbb{Q}_1 \in \mathscr{M}_e$ such that $I(\mathbb{Q}_1, \mathbb{P}) < +\infty$, then $\mathbb{Q}_0$ is equivalent to $\mathbb{P}$.*

Proof. To prove this fact it is necessary to show, that the probability of the set $\left\{\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}} = 0\right\}$ is zero with respect to $\mathbb{P}$. For this it is enough to use the fact that $\varphi'(0) = -\infty$ and the previous lemma. Indeed, because $\mathbb{Q}_0$ is the MEMM we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}f(t)\right)\Big|_{t=0} \geq 0, \tag{2.5}$$

otherwise there exists such t_0 and corresponding measure $\mathbb{Q}_{t_0}$, defined by

$$\mathbb{Q}_{t_0} := t_0\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} + (1-t_0)\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}$$

with the property $I(\mathbb{Q}_{t_0}, \mathbb{P}) < I(\mathbb{Q}_0, \mathbb{P})$. However, from the equation (2.3), we see that (2.5) cannot be satisfied if $\mathbb{Q}_0$ is not equivalent to $\mathbb{P}$, since then $\mathbb{P}\left(\left\{\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}} = 0\right\}\right) > 0$ and hence $\mathbb{Q}_1\left(\left\{\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}} = 0\right\}\right) > 0$. □

The next step is the characterization of the MEMM. For this reason let us define the class $\mathscr{L}$ of all integrable random variables with respect to all absolutely continuous (with respect to $\mathbb{P}$) martingale measures with finite entropy,

$$\mathscr{L} := \bigcap_{\mathbb{Q}\in\mathscr{M}_f} L^1(\mathbb{Q}),$$

and the set C_0 of random variables with non-positive expectation under every absolutely continuous (with respect to $\mathbb{P}$) martingale measure with finite entropy,

$$C_0 = \{f \in \mathscr{L} : \mathbb{E}_{\mathbb{Q}}[f] \leq 0, \forall \mathbb{Q} \in \mathscr{M}_f\}. \tag{2.6}$$

2.6 Theorem. *Assume that* $I(\mathscr{M}_e, \mathbb{P}) < +\infty$. *A probability* $\mathbb{Q}_0$ *is the MEMM if and only if:*

(i) $\mathbb{Q}_0 \in \mathscr{M}_a$;

(ii) $\mathrm{d}\mathbb{Q}_0/\mathrm{d}\mathbb{P} = c\exp(-f_0)\mathbb{P}$-*a.s., where* $f_0 \in L^1(\mathbb{Q}_0), \mathbb{E}_{\mathbb{Q}_0} f_0 = 0$ *and* $c > 0$;

(iii) $f_0 \in C_0$.

Proof. Let us first prove the sufficiency of the conditions (i)–(iii) for the probability measure $\mathbb{Q}_0$ to be the MEMM. Let the conditions (i)–(iii) be satisfied. Then the relative entropy of $\mathbb{Q}_0$ with respect to $\mathbb{P}$ is

$$I(\mathbb{Q}_0, \mathbb{P}) = \mathbb{E}[c\exp(-f_0)\log(c\exp[-f_0])] = \mathbb{E}_{\mathbb{Q}_0}[\log c + (-f_0)] = \log c. \tag{2.7}$$

Note that, because of (ii), $\mathbb{Q}_0$ is equivalent to $\mathbb{P}$ and hence every $\mathbb{Q} \in \mathscr{M}_f$ is abolutely continuous with respect to $\mathbb{Q}_0$. Now for any $\mathbb{Q} \in \mathscr{M}_f$ the relative entropy with respect to $\mathbb{P}$ is

$$\begin{aligned}
I(\mathbb{Q}, \mathbb{P}) &= \mathbb{E}_{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}_0}\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}_0}\right] + \mathbb{E}_{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}_0}\right] + \mathbb{E}_{\mathbb{Q}}[\log c - f_0] \\
&= I(\mathbb{Q}, \mathbb{Q}_0) + \log c - \mathbb{E}_{\mathbb{Q}} f_0 \geq \log c,
\end{aligned}$$

where the condition (iii) was used. In the last line it was also used the non-negativity of the relative entropy: $I(\mathbb{Q}, \mathbb{Q}_0) \geq 0$. The sufficiency is proven.

Now let us prove the necessity of the conditions (i)–(iii). Suppose that $\mathbb{Q}_0$ is the MEMM. It is necessary to show that $I(\mathbb{Q}_0, \mathbb{P}) < +\infty$, but from the definition of the MEMM $I(\mathbb{Q}_0, \mathbb{P}) \leq I(\mathscr{M}_e, \mathbb{P}) < +\infty$. The condition (i) holds by the definition of the MEMM. Let $\mathbb{Q}_1 \in \mathscr{M}_f$ be arbitrarily given and define

$$f_0 := I(\mathbb{Q}_0, \mathbb{P}) - \log \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}} \tag{2.8}$$

as well as

$$\mathbb{Q}_t := t\mathbb{Q}_1 + (1-t)\mathbb{Q}_0.$$

By Lemma 2.4 it follows that $\mathbb{E}_{\mathbb{Q}_1}\left[\log \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right]$ is well-defined. Let the function f be defined as in Lemma 2.4, then $I(\mathbb{Q}_t, \mathbb{P}) = \mathbb{E}f(t)$ and since $\mathbb{Q}_0$ is the MEMM,

$$\frac{\mathrm{d}}{\mathrm{d}t} I(\mathbb{Q}_t, \mathbb{P})\Big|_{t=0} \geq 0.$$

Using Lemma 2.4 once again, we obtain

$$0 \leq \left(\frac{\mathrm{d}}{\mathrm{d}t} I(\mathbb{Q}_t, \mathbb{P})\right)\Big|_{t=0} \quad = \quad \mathbb{E}_{\mathbb{Q}_1}\left[\log \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}} - I(\mathbb{Q}_0, \mathbb{P})\right]$$

Taking into account (2.8) we get

$$\begin{aligned} 0 \leq \mathbb{E}_{\mathbb{Q}_1}\left[\log \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right] - I(\mathbb{Q}_0, \mathbb{P}) &= -\mathbb{E}_{\mathbb{Q}_1}[f_0] \\ &= \mathbb{E}_{\mathbb{Q}_1}\left[\log \frac{\mathrm{d}\mathbb{Q}_0}{\mathrm{d}\mathbb{P}}\right] - I(\mathbb{Q}_0, \mathbb{P}) < +\infty. \end{aligned}$$

So (ii) with $c = \exp(I(\mathbb{Q}_0, \mathbb{P}))$ and (iii) hold and the proof of the theorem is finished. □

The set C_0 can be characterized in another way. Let us define the set C as follows:

$$C = K - L^\infty_+(\mathbb{P}) = \left\{\eta - h : \eta \in K,\, h \in L^\infty_+(\mathbb{P})\right\}, \tag{2.9}$$

where

$$L^\infty_+(\mathbb{P}) = \{h : h \in L^\infty(\mathbb{P}),\, h \geq 0\}.$$

By $\bar{C}^{\mathbb{Q}}$ the closure of C in the $L^1(\mathbb{Q})$-norm topology is denoted.

The next proposition gives an important characterization of C_0.

2.7 Proposition. Let C_0 be defined as in (2.6) and the closure of C in $L^1(\mathbb{Q})$-norm topology as above. Then the following identity holds:

$$C_0 = \bigcap_{\mathbb{Q}\in\mathscr{M}_f} \bar{C}^{\mathbb{Q}}.$$

Proof. Indeed, if $\mathbb{Q} \in \mathscr{M}_f$ then for all $f \in C$ it follows $\mathbb{E}_{\mathbb{Q}} f \leq 0$, so $\bigcap_{\mathbb{Q}\in\mathscr{M}_f} \bar{C}^{\mathbb{Q}} \subseteq C_0$.

Let us fix $\mathbb{Q} \in \mathscr{M}_f$. To complete the proof it is sufficient to show that if $f_0 \in \mathscr{L}$ and $f_0 \notin \bar{C}^{\mathbb{Q}}$ then $f_0 \notin C_0$. Note that this statement is equivalent to $\mathscr{L} \cap (\bar{C}^{\mathbb{Q}})^c) \subseteq C_0^c$, hence to $C_0 \subseteq \mathscr{L}^c \cup \bar{C}^{\mathbb{Q}}$, but $C_0 \subseteq \mathscr{L}$, so also to $C_0 \subseteq \bar{C}^{\mathbb{Q}}$ which is just to verify.

The set $\bar{C}^{\mathbb{Q}}$ is a closed convex cone in the Banach space $L^1(\mathbb{Q})$ and by the Hahn-Banach separation theorem there exists a continuous linear functional on $L^1(\mathbb{Q})$ that strictly separates f_0 from $\bar{C}^{\mathbb{Q}}$. This means that there exists $\zeta \in L^\infty(\mathbb{Q})$, $\zeta \neq 0$, such that $0 = \sup_{f\in C} \mathbb{E}_{\mathbb{Q}}[\zeta f] < \mathbb{E}_{\mathbb{Q}}[\zeta f_0]$. Now we want to construct a probability measure which is an absolutely continuous martingale measure with respect to the measure $\mathbb{P}$, but at the same time the expectation of f_0 under this measure is strictly positive.
Since $-1_{\{\zeta<0\}} \in C$ (take in the definition of C the random variable $\eta = 0$), $\zeta \geq 0$ $\mathbb{Q}$-a.s. The random variable ζ is in $L^\infty(\mathbb{Q})$, so it is possible to normalize ζ, i.e., to put $c = \mathbb{E}_{\mathbb{Q}}\zeta > 0$ and then to introduce $\tilde{\zeta} = \frac{\zeta}{c} \in L^\infty(\mathbb{Q})$. Now we define the probability $\mathbb{Q}_1 \ll \mathbb{Q}$ by setting $\frac{d\mathbb{Q}_1}{d\mathbb{Q}} = \tilde{\zeta}$. Since $K \subseteq C$ is a linear space, $\sup_{f\in C} \mathbb{E}_{\mathbb{Q}}[\tilde{\zeta} f] = 0$ implies $\mathbb{E}_{\mathbb{Q}_1} f = 0$ for $f \in K$, so that $\mathbb{Q}_1$ is an absolutely continuous martingale measure. Since $\tilde{\zeta} \in L^\infty(\mathbb{Q})$ and $I(\mathbb{Q}, \mathbb{P}) < +\infty$,

$$\begin{aligned}
I(\mathbb{Q}_1,\mathbb{P}) &= \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\log\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\log\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{Q}}\left(\log\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{Q}}+\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[\tilde{\zeta}\log\tilde{\zeta}\right]+\mathbb{E}_{\mathbb{Q}}\left[\tilde{\zeta}\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right]\\
&\leq \mathbb{E}_{\mathbb{Q}}\left[\tilde{\zeta}\log\tilde{\zeta}\right]+\mathbb{E}_{\mathbb{Q}}\left[\tilde{\zeta}\left(\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^+\right]\\
&\leq \mathbb{E}_{\mathbb{Q}}\left[\tilde{\zeta}\log\tilde{\zeta}\right]+d\,\mathbb{E}_{\mathbb{Q}}\left[\left(\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^+\right]<+\infty,
\end{aligned}$$

where we used that $\tilde{\zeta}$ is nonnegative and bounded $\mathbb{Q}$-a.s. from above by a constant d, that the function $x\log x$ is bounded on compact sets and that $\mathbb{Q}$ is in $\mathscr{M}_f$ which ensures that

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^+\right]<+\infty.$$

Hence $\mathbb{Q}_1\in\mathscr{M}_f$ and

$$\mathbb{E}_{\mathbb{Q}_1}[f_0]=\mathbb{E}_{\mathbb{Q}}[\tilde{\zeta}f_0]=\frac{1}{c}\mathbb{E}_{\mathbb{Q}}[\zeta f_0]>0,$$

consequently $f_0\notin C_0$. $\square$

3

Compound Poisson Case

In this chapter we would like to investigate the model of the price processes based on the exponential (geometric) compound Poisson process and to show its connection with the one-step model.

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, where T is a finite horizon. Let us introduce the compound Poisson process $(X, \mathbb{F})$ with the drift b by its characteristic function

$$\mathbb{E}\exp(iuX_t) = \exp\left(ibt + \int_{\mathbb{R}} (e^{iux} - 1)\nu(\mathrm{d}x)\right),\ u \in \mathbb{R},\ t \in [0, T].$$

The characteristic triplet is $(b, 0, \nu)$ where ν is a finite measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, $\nu(\mathbb{R}) > 0$. We have used here the truncation function $c = 0$, because every compound Poisson process has a finite number of jumps on finite intervals and we do not need to manipulate with the infinite number of small jumps in the neighborhood of zero. The process X can be also represented in the following form:

$$X_t = \sum_{i=1}^{N_t} \xi_i + bt,\ t \in [0, T],$$

where N is a Poisson process with intensity $\lambda = \nu(\mathbb{R})$, the sequence (ξ_i) is independent and identically distributed, independent of N, with distribution law $\mathbb{P}_{\xi_1} = \frac{1}{\lambda}\nu$.

Let us define the price process S by

$$S_t := S_0 \exp(X_t),\ t \in [0, T],$$

for some finite time horizon T and $S_{0-} = S_0 > 0$.

We recall that a probability measure $\mathbb{Q}$, defined on $(\Omega, \mathscr{F}, \mathbb{Q})$, is called a martingale measure, if $(S, \mathbb{F})$ is a $\mathbb{Q}$-(local) martingale. The following sets of martingale measures where already defined and discussed in Section 1.7, so we just recall the definition:

$$\begin{aligned}
\mathscr{M}_a(T) &:= \{\mathbb{Q} : (S, \mathbb{F}) \text{ is a } \mathbb{Q}\text{-local martingale}, \mathbb{Q} \ll \mathbb{P} \text{ on } \mathscr{F}_T\}, \\
\mathscr{M}_e(T) &:= \{\mathbb{Q} \in \mathscr{M}_a(T) : \mathbb{Q} \sim \mathbb{P} \text{ on } \mathscr{F}_T\}, \\
\mathscr{M}_l(T) &:= \{\mathbb{Q} \in \mathscr{M}_a(T) : (X, \mathbb{F}) \text{ is a } \mathbb{Q}\text{-Lévy process}\}, \\
\mathscr{M}_f(T) &:= \{\mathbb{Q} \in \mathscr{M}_a(T) : I_T(\mathbb{Q}, \mathbb{P}) < \infty\}.
\end{aligned}$$

We are interested in the MEMM $\mathbb{Q}^*$ in the class of martingale measures which preserve the Lévy property of the price-generating process X:

$$\mathbb{Q}^* = \operatorname{argmin}_{\mathbb{Q} \in \mathscr{M}_l(T)} I_T(\mathbb{Q}, \mathbb{P}).$$

The interest is naturally caused by the properties of the process with independent and homogeneous increments that make the model intuitively close to the real behavior of the price processes. Nevertheless, in Chapter 4 we prove that

$$\operatorname{argmin}_{\mathbb{Q} \in \mathscr{M}_a(T)} I_T(\mathbb{Q}, \mathbb{P}) = \operatorname{argmin}_{\mathbb{Q} \in \mathscr{M}_l(T)} I_T(\mathbb{Q}, \mathbb{P})$$

and therefore the desired martingale measure $\mathbb{Q}^*$, if it exists, is also the MEMM in the class $\mathscr{M}_a(T)$.

In our particular model, the price process is always a positive semimartingale and $S_0 > 0$. These are important properties, which give us the possibility to investigate the price process as a stochastic exponential of some other semimartingale and use the already known relation between stochastic exponentials and stochastic logarithms stated in Section 1.6.

Now we would like to show the explicit relation between the process X and the stochastic logarithm of the process S.

3.1 Proposition. The stochastic logarithm $\mathscr{L}(S)$ of S is equal to

$$\mathscr{L}(S)_t = \int_0^t [\exp(\Delta X_u) - 1] \mathrm{d}N_u + bt = \sum_{i=1}^{N_t} [\exp(\xi_i) - 1] + bt =: L_t, \; t \in [0, T].$$

The process $(L, \mathbb{F})$ is a compound Poisson process with drift b and Lévy measure

$$\nu^L = \nu \circ F^{-1}, \; F(x) = e^x - 1, \; x \in \mathbb{R}.$$

The measure ν^L can also be written as

$$\nu^L = \lambda \mathbb{P}_{\eta_1}, \quad \eta_1 := \exp(\xi_1) - 1.$$

Proof. Because of the correspondence between stochastic exponentials and stochastic logarithms (cf., Theorem 1.34), to prove the statement it is sufficient to show that the stochastic exponential of the process L is equal to the process S $\mathbb{P}$-a.s. According to Theorem 1.33, the stochastic exponential of some semimartingale Y has the form

$$\mathscr{E}(Y)_t = \exp(Y_t - Y_0 - \frac{1}{2}\langle Y^c \rangle) \prod_{s \le t} (1 + \Delta Y_s) e^{-\Delta Y_s},\ t \in [0, T],$$

where Y^c stands for the continuous martingale part of the process Y. Let us introduce the process $(L, \mathbb{F})$ by

$$L_t := \sum_{i=1}^{N_t} [\exp(\xi_i) - 1] + bt = \int_0^t [\exp(\Delta X_u) - 1] \mathrm{d}N_u + bt,\ t \in [0, T].$$

The stochastic exponential of the process $(L, \mathbb{F})$ has the form

$$\begin{aligned}
\mathscr{E}(L)_t &= \exp\left(\sum_{i=1}^{N_t} [\exp(\xi_i) - 1] + bt \right) \\
&\quad \times \prod_{0 \le s \le t} [1 + \Delta N_s [\exp(\xi_{N_s}) - 1]] \exp(-\Delta N_s [\exp(\xi_{N_s}) - 1]) \\
&= \exp\left(\sum_{i=1}^{N_t} [\exp(\xi_i) - 1] + bt \right) \\
&\quad \times \prod_{i=1}^{N_t} [1 + [\exp(\xi_i) - 1]] \exp(-[\exp(\xi_i) - 1]) \\
&= \exp\left(\sum_{i=1}^{N_t} [\exp(\xi_i) - 1] + bt + \sum_{i=1}^{N_t} \xi_i - \sum_{i=1}^{N_t} [\exp(\xi_i) - 1] \right) \\
&= \exp\left(bt + \sum_{i=1}^{N_t} \xi_i \right) = S_t.
\end{aligned}$$

So L is the stochastic logarithm of S. □

In the following we will give some preparations before the consideration of our concrete model, in particular there will be stated some general results about Lévy processes and (local) martingales.

Proposition 1.35 gives us an alternative formulation of the original problem: Instead of the price process S being a local martingale we may look for measures $\mathbb{Q}$ under which the stochastic logarithm of the price process

$\mathscr{L}(S) = L$ is a local martingale. Because of Proposition 3.1, the process L is a Lévy process under the original measure $\mathbb{P}$, moreover L is a compound Poisson process with drift. At the same time, the MEMM $\mathbb{Q}^*$ is in class $\mathscr{M}_l(T)$ and by definition preserves the Lévy property of the process X, hence the process L is also a $\mathbb{Q}^*$-Lévy process.

In the general case, the price process L must be just a local martingale with respect to the MEMM $\mathbb{Q}^*$, but in our particular model we can prove that it is even a true martingale under $\mathbb{Q}^*$. For this the key property is the preservation of the Lévy property of L with respect to $\mathbb{Q}^*$ because for Lévy processes the conditions of being a martingale and a local martingale are equivalent (cf. Theorem 1.20). The price process L is a compound Poisson process with drift under the original probability measure $\mathbb{P}$. Using this fact, we show that under the MEMM $\mathbb{Q}^*$, if it exists, L is also a compound Poisson with drift and there cannot appear a Gaussian component. For this we need to know the relation between the parameters of Lévy processes under equivalent change of measure that preserves the Lévy property.

3.2 Theorem. *Let $(L_t, \mathscr{F}_t, t \geq 0)$ be a $\mathbb{P}$- and $\mathbb{Q}$-Lévy process on $\mathbb{R}$ with generating triplets $(\tilde{b}, 0, \nu)$ and $(\tilde{b}_{\mathbb{Q}}, \sigma^2, \nu_{\mathbb{Q}})$, respectively, and standard truncation function $c(x)$. The filtration $\mathbb{F}^0 = (\mathscr{F}_t^0)_{t\geq 0}$ is a natural filtration: $\mathscr{F}_t^0 = \sigma(L_s : 0 \leq s \leq t)$, $t \geq 0$. Then the following two statements* (i) *and* (ii) *are equivalent:*

(i) $\mathbb{P}|_{\mathscr{F}_t} \sim \mathbb{Q}|_{\mathscr{F}_t}$ *for every* $t \in (0, \infty)$*;*

(ii) *The generating triplets satisfy*

(a) $\sigma^2 = 0$*;*

(b) $\nu \sim \nu_{\mathbb{Q}}$*,*

(c) $\int\limits_{\mathbb{R}} \left(\exp\left[\frac{\beta(x)}{2}\right] - 1\right)^2 \nu(\mathrm{d}x) < \infty$

with the function β defined by $\dfrac{\mathrm{d}\nu_{\mathbb{Q}}}{\mathrm{d}\nu} = \exp[\beta]$*,*

(d) $\tilde{b}_{\mathbb{Q}} - \tilde{b} - \int\limits_{\{|x|\leq 1\}} x(\nu_{\mathbb{Q}} - \nu)(\mathrm{d}x) = 0.$

Proof. This is the special case of Theorem 1.22 for a Lévy process without Gaussian part. □

Note that this theorem is only valid for the natural filtration $\mathbb{F} = \mathbb{F}^0$ where $\mathbb{F}^0 = (\mathscr{F}_t^0)_{t\geq 0}$ and $\mathscr{F}_t^0 = \sigma(X_s : 0 \leq s \leq t)$, $t \geq 0$. Otherwise the statement is true just in the direction (i)⇒ (ii) and in general does not hold in the direction (ii)$\Rightarrow$ (i). It is worth to mention that in case of a compound Poisson process with a drift we can omit the truncation function (in other words, we put $c \equiv 0$). Therefore, the new drift parameters $b_{\mathbb{Q}}$ and b can be easily calculated:

$$\begin{aligned} b_{\mathbb{Q}} &= \tilde{b}_{\mathbb{Q}} - \int\limits_{\{|x|\leq 1\}} x\nu_{\mathbb{Q}}(\mathrm{d}x), \\ b &= \tilde{b} - \int\limits_{\{|x|\leq 1\}} x\nu(\mathrm{d}x). \end{aligned}$$

Taking into account condition (d) of the previous theorem we get $b_{\mathbb{Q}} = b$.

In the following $(L, \mathbb{F})$ denotes an arbitrary compound Poisson process with drift.

The following proposition shows that under the MEMM, if it exists, the linear price process $(L, \mathbb{F})$ is still a compound Poisson process with drift.

3.3 Proposition. Assume that $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$ and $(L, \mathbb{F})$ is a $\mathbb{Q}$-Lévy process. Then X (resp., L) is compound Poisson with drift b and Lévy measure $\nu_{\mathbb{Q}} \sim \nu$ (resp., $\nu^L_{\mathbb{Q}} = \nu_{\mathbb{Q}} \circ F^{-1}$).

Proof. Because of Proposition 3.1 it is sufficient to prove the statement either for the process X or for the process L. Let us prove it for the process X. According to Theorem 3.2 the characteristic triplets must satisfy the relations (a), (b), (c) and (d). The process $(X, \mathbb{F})$ is $\mathbb{P}$-compound Poisson, hence $\sigma^2 = 0$ (i.e., $(X, \mathbb{F})$ has no Gaussian part under the measure $\mathbb{Q}$). From the statement (b) we have the equivalence of the Lévy measures. The parameters b and $b_{\mathbb{Q}}$ are finite, moreover $\nu(\mathbb{R}) < +\infty$. We now show that $\nu_{\mathbb{Q}}(\mathbb{R}) < +\infty$ which would imply that the process X is a $\mathbb{Q}$-compound Poisson process with the drift $b_{\mathbb{Q}}$. Indeed, the density $\exp[\beta(x)]$ can be estimated:

$$\begin{aligned} \exp[\beta(x)] &= (\exp[\beta(x)/2])^2 \\ &= (\exp[\beta(x)/2] - 1 + 1)^2 \\ &\leq 2[(\exp[\beta(x)/2] - 1)^2 + 1]. \end{aligned}$$

Taking into account the condition (c) and the fact that $\nu(\mathbb{R}) < +\infty$ we have:

$$\begin{aligned} \nu_{\mathbb{Q}}(\mathbb{R}) &= \int\limits_{\mathbb{R}} \exp\left(\beta(x)\right)\nu(\mathrm{d}x) \\ &\leq 2\left[\int\limits_{\mathbb{R}} \left(\exp\left[\frac{\beta(x)}{2}\right] - 1\right)^2 \nu(\mathrm{d}x) + \nu(\mathbb{R})\right] < +\infty. \end{aligned}$$

The statement is proven. $\square$

3.1 Construction of the density for the measure transformation

Let the process $(L, \mathbb{F})$ be a general compound Poisson process with drift, the jump sizes are denoted by (η_n), the Lévy measure (intensity measure) by ν^L.

Our next goal is to find conditions, under which the Lévy preserving measure is also a martingale measure. We assume that the following condition holds:

$$\mathscr{M}_l(T) \cap \mathscr{M}_f(T) \cap \mathscr{M}_e(T) \neq \emptyset.$$

If $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$ and $(L, \mathbb{F})$ is a $\mathbb{Q}$-Lévy process, we always denote by $\nu_{\mathbb{Q}}$ (respectively, $\nu^L_{\mathbb{Q}}$) the Lévy measure of the compound Poisson process $(X, \mathbb{F})$ (respectively, $(L, \mathbb{F})$) with respect to $\mathbb{Q}$ (cf., Proposition 3.3). We also introduce the intensity $\lambda^{\mathbb{Q}}$ of the compound Poisson process $(X, \mathbb{F})$ and $(L, \mathbb{F})$ with respect to $\mathbb{Q}$. Note that $\lambda^{\mathbb{Q}} = \nu_{\mathbb{Q}}(\mathbb{R}) = \nu^L_{\mathbb{Q}}(\mathbb{R})$. The next proposition gives a characterization of martingale measures in the class of Lévy-preserving probability measures $\mathbb{Q}$ on $\mathscr{F}_T$ equivalent to $\mathbb{P}$.

3.4 Proposition. Let $(L, \mathbb{F})$ be a $\mathbb{Q}$-Lévy process, $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$. Then the probability measure $\mathbb{Q} \in \mathscr{M}_e(T) \cap \mathscr{M}_l(T)$ iff $\mathbb{Q}$ is a martingale measure. Moreover, $\mathbb{Q}$ is a martingale measure iff

$$\mathbb{E}_{\mathbb{Q}}[\exp(\xi_1)] = -\frac{b}{\lambda^{\mathbb{Q}}} + 1. \tag{3.1}$$

Proof. We will prove this statement in several steps.

<u>*Step 1.*</u> Let us show the equivalence of the following conditions (1) and (2) for any compound Poisson process $(Y, \mathbb{F})$ with Lévy measure ν:

(1) $\mathbb{E}[|Y_t|] < +\infty$ for any $t \geq 0$;

(2) $\int_{\mathbb{R}} |x| \nu(\mathrm{d}x) < +\infty$.

Let us assume that $\mathbb{E}[|Y_t|] < +\infty$ for some $t \geq 0$. The process Y at fixed time point is a compound Poisson random variable, so

$$Y_t = \sum_{i=1}^{N_t} \zeta_i,$$

where N_t is a Poisson random variable with intensity $\lambda t := \nu(\mathbb{R})t$, the sequence $\{\zeta_i\}$ is independent and identically distributed, independent of N_t, with the distribution law $\zeta_i \sim \dfrac{\nu}{\nu(\mathbb{R})}$. By assumption, for any $t \geq 0$ we have

$$\begin{aligned} +\infty &> \mathbb{E}[|Y_t|] \\ &= \mathbb{E}[|\sum_{i=1}^{N_t} \zeta_i|] \\ &= \sum_{n=0}^{\infty} \mathbb{E}[|\sum_{i=1}^{n} \zeta_i \frac{(\lambda t)^n}{n!} e^{-\lambda t}|] \\ &\geq \mathbb{E}[|\zeta_1|]\lambda t e^{-\lambda t}. \end{aligned}$$

From this follows that $\mathbb{E}[|\zeta_1|] < +\infty$. But $\zeta_i \sim \dfrac{\nu}{\nu(\mathbb{R})}$ and hence

$$\mathbb{E}[|\zeta_1|] = \int_{\mathbb{R}} |x| \frac{\nu(\mathrm{d}x)}{\nu(\mathbb{R})} < +\infty$$

from which we have

$$\int_{\mathbb{R}} |x|\nu(\mathrm{d}x) < +\infty.$$

Let us prove the converse statement. Assume that $\int_{\mathbb{R}} |x|\nu(\mathrm{d}x) < +\infty$. We have to show that $\mathbb{E}[|Y_t|] < +\infty$ for any $t \geq 0$. But for any $t \geq 0$ we have

$$\begin{aligned} \mathbb{E}[|Y_t|] &= \mathbb{E}[|\sum_{i=1}^{N_t} \zeta_i|] \\ &\leq \mathbb{E}[\sum_{i=1}^{N_t} |\zeta_i|] \\ &= \mathbb{E}[|\zeta_1|]\,\mathbb{E}[N_t] = \mathbb{E}[|\zeta_1|]\lambda t < +\infty, \end{aligned}$$

where we have used Wald's identity.

Step 2. The application of Proposition 3.3 ensures that under the assumptions of this proposition $(L, \mathbb{F})$ is a compound Poisson process with drift b with respect to $\mathbb{Q}$. Now we can apply the result of Step 1 to the compound Poisson process L with drift b with the characteristic triplet $(b, 0, \nu^L_{\mathbb{Q}})$. Indeed, it is possible to estimate the expectation $\mathbb{E}_{\mathbb{Q}}[|L_t|]$ for any $t \geq 0$ from above and below:

$$\mathbb{E}_{\mathbb{Q}}[|L_t - bt| - |bt|] \leq \mathbb{E}_{\mathbb{Q}}[|L_t|] \leq \mathbb{E}_{\mathbb{Q}}[|L_t - bt|] + |bt|,\ t \geq 0,$$

where $(L_t - bt)_{t\geq 0}$ is a compound Poisson process. Therefore the condition

$$\mathbb{E}_{\mathbb{Q}}[|L_t|] < +\infty,\ t \geq 0,$$

is equivalent to

$$\int_{\mathbb{R}} |x| \nu_{\mathbb{Q}}^L(\mathrm{d}x) < \infty.$$

<u>*Step 3.*</u> The distribution of η_i is just $\mathbb{Q}_{\eta_i} = \frac{\nu_{\mathbb{Q}}^L}{\nu_{\mathbb{Q}}^L(\mathbb{R})}$. This yields that $\mathbb{E}_{\mathbb{Q}}[|L_t|] < +\infty$, $t \geq 0$, if and only if $\mathbb{E}_{\mathbb{Q}}[|\eta_i|] < +\infty$.

<u>*Step 4.*</u> Assume $\mathbb{Q} \in \mathscr{M}_e(T) \cap \mathscr{M}_l(T)$. It follows that $\mathbb{E}_{\mathbb{Q}}[L_t] = 0$ for any $t > 0$ and, in particular, $\mathbb{E}_{\mathbb{Q}}[|L_t|] < +\infty$, $t \geq 0$. Therefore

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{Q}}[L_t] \\ &= \mathbb{E}_{\mathbb{Q}}\Big[\sum_{i=1}^{N_t}(\eta_i)\Big] + bt \\ &= \mathbb{E}_{\mathbb{Q}}[N_t]\,\mathbb{E}_{\mathbb{Q}}[\eta_i] + bt \\ &= \lambda^{\mathbb{Q}} t\,\mathbb{E}_{\mathbb{Q}}[\eta_1] + bt, \end{aligned}$$

where $\lambda^{\mathbb{Q}} = \nu_{\mathbb{Q}}^L(\mathbb{R})$. Here Wald's identity is used which can be applied because $\mathbb{E}_{\mathbb{Q}} N_t < +\infty$ and $\mathbb{E}_{\mathbb{Q}}|\eta_i| < +\infty$. Finally, we obtain

$$\mathbb{E}_{\mathbb{Q}}[\eta_1] = -\frac{b}{\lambda^{\mathbb{Q}}}.$$

<u>*Step 5.*</u> Now we want to prove the statement of Step 4 in the converse direction. By assumption, $\mathbb{E}_{\mathbb{Q}}[\eta_1] = -\frac{b}{\lambda^{\mathbb{Q}}}$. Then $\mathbb{E}_{\mathbb{Q}}[\eta_1] < +\infty$ and therefore $\mathbb{E}_{\mathbb{Q}}[|L_t|] < +\infty$ which follows from Step 3. Similarly to Step 4, we can compute $\mathbb{E}_{\mathbb{Q}}[L_t]$ with the result

$$\mathbb{E}_{\mathbb{Q}}[L_t] = \lambda^{\mathbb{Q}} t\,\mathbb{E}_{\mathbb{Q}}\,[\eta_i] + bt$$

which is equal to zero because of the assumption. As a Lévy process with expectation zero, $(L, \mathbb{F})$ is a $\mathbb{Q}$-martingale. Since $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$ and $(L, \mathbb{F})$ is a $\mathbb{Q}$-Lévy process by assumption we obtain $\mathbb{Q} \in \mathscr{M}_e(T) \cap \mathscr{M}_l(T)$. □

Note that the last proposition gives us a necessary and sufficient condition for a Lévy preserving measure to be a martingale measure. At the beginning of this chapter it was assumed that the following condition is satisfied:

$$\mathscr{M}_l(T) \cap \mathscr{M}_f(T) \cap \mathscr{M}_e(T) \neq \emptyset,$$

and the condition of finite entropy we will discuss later. The next step is to find the explicit form of the density of the measure transformation. We have already characterized it in Theorem 3.2 and Proposition 3.3, but without giving the explicit form of the density.

3.5 Theorem. *Let $(L, \mathbb{F})$ be a compound Poisson process with drift b with respect to the probability measure $\mathbb{P}$.*

(1) *If $(L, \mathbb{F})$ is a compound Poisson process with drift b with respect to $\mathbb{Q}$ and $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$, then $\nu^L_{\mathbb{Q}} \sim \nu^L$ and*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}^L_T) = \exp\left[T(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) + \sum_{0<r\leq T} \log \rho(\Delta L_r)\right], \tag{3.2}$$

where $\rho := \dfrac{\mathrm{d}\nu^L_{\mathbb{Q}}}{\mathrm{d}\nu^L}$.

(2) *If the measure $\mathbb{Q}$ is defined on $\mathscr{F}_T$ by*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) = \exp\left[T(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) + \sum_{0<r\leq T} \log \rho(\Delta L_r)\right], \tag{3.3}$$

where $\nu^L_{\mathbb{Q}}$ is a finite measure on $\mathbb{R}$ such that $\nu^L_{\mathbb{Q}} \sim \nu^L$ and $\rho := \dfrac{\mathrm{d}\nu^L_{\mathbb{Q}}}{\mathrm{d}\nu^L}$, then the process $(L, \mathbb{F})$ is a compound Poisson process with Lévy measure $\nu^L_{\mathbb{Q}}$ and drift b with respect to $\mathbb{Q}$.

Proof. First we prove part (2). Let us define the process

$$U_t := \exp\left[t(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) + \sum_{0\leq r\leq t} \log \rho(\Delta L_r)\right], \quad t \in [0, T].$$

Inputting η_i and $\beta = \log \rho$ we can write

$$\begin{aligned} U_t &:= \exp\left[t(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) + \sum_{i=1}^{N_t} \beta(\eta_i)\right] \\ &= \exp\left[t(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R}))\right] \prod_{i=1}^{N_t} \rho(\eta_i), \quad t \in [0, T]. \end{aligned}$$

This shows that $(U, \mathbb{F})$ is an exponential compound Poisson process with drift, $U_0 = 1$, and for proving that it is a density process (a martingale) it is sufficient to verify that $\mathbb{E}[U_t] = 1$, $t \in [0,T]$. For this let us compute

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{N_t}\rho(\eta_i)\right] &= \sum_{k=0}^{\infty}\mathbb{E}\left[\prod_{i=1}^{k}\rho(\eta_i)1_{\{N_t=k\}}\right] \\
&= \sum_{k=0}^{\infty}\mathbb{E}\left[\prod_{i=1}^{k}\rho(\eta_i)\right]\mathbb{P}(\{N_t=k\}) \\
&= \sum_{k=0}^{\infty}\left(\mathbb{E}\left[\rho(\eta_1)\right]\right)^k\frac{(\lambda t)^k}{k!}\exp(-\lambda t) \\
&= \exp(\lambda t(\mathbb{E}[\rho(\eta_1)-1])) \\
&= \exp(t[\nu_{\mathbb{Q}}^L(\mathbb{R})-\nu^L(\mathbb{R})])
\end{aligned}$$

since $\lambda = \nu^L(\mathbb{R})$ and $\lambda\mathbb{E}[\rho(\eta_1)] = \nu_{\mathbb{Q}}^L(\mathbb{R})$. This proves the martingale property of $(U, \mathbb{F})$.

Now let us define the probability measure $\mathbb{Q}$ on $\mathscr{F}_T$ by

$$\mathrm{d}\mathbb{Q} = U_T \mathrm{d}\mathbb{P}.$$

We want to prove that the process L is a Lévy process with respect to $\mathbb{Q}$ on $[0,T]$ with characteristic triplet $(b, 0, \nu_{\mathbb{Q}}^L)$. In order to prove that L is a $\mathbb{Q}$-Lévy process with prescribed characteristics, we consider the following lemma:

3.6 Lemma. *For any $0 \le s \le t$ and $u \in \mathbb{R}$ the following identity holds*

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}[\exp(iu(L_t - L_s))|\mathscr{F}_s] \\
&\quad = \exp[i(t-s)ub + (t-s)\int_{\mathbb{R}}(e^{iux}-1)\nu_{\mathbb{Q}}^L(\mathrm{d}x)] \quad \mathbb{P}\text{-}a.s. \text{ and } \mathbb{Q}\text{-}a.s.
\end{aligned}$$

Proof. Because of the homogeneity of increments of the process L in time and their independence the process L^s defined by $L^s := L_{s+t} - L_s$, $t \ge 0$, is again a Lévy process with respect to $\mathbb{F}^s := (\mathscr{F}_{s+t})_{t\ge 0}$ and has the same distribution as L for all $s \ge 0$. In particular, L^s is independent of $\mathscr{F}_s$. Now we calculate

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}[\exp(iu(L_t - L_s))|\mathscr{F}_s] \\
&= \mathbb{E}_{\mathbb{P}}[\exp(iu(L_t - L_s))\frac{U_t}{U_s}|\mathscr{F}_s] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(iu\Big(\sum_{s<r\leq t}\Delta L_r + (t-s)b\Big) + (t-s)(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) \\
&\quad + \sum_{s<r\leq t}\log\rho(\Delta L_r)\Big)|\mathscr{F}_s\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(iu\Big(\sum_{0<r\leq t-s}\Delta L^s_r + (t-s)b\Big) + (t-s)(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) \\
&\quad + \sum_{0<r\leq t-s}\log\rho(\Delta L^s_r)\Big)|\mathscr{F}_s\Big].
\end{aligned}
$$

Because of the independence of the process L^s of $\mathscr{F}_s$, the conditional expectation in the last expression is equal to the usual expectation:

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}[\exp(iu(L_t - L_s))|\mathscr{F}_s] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(iu\Big(\sum_{0<r\leq t-s}\Delta L^s_r + (t-s)b\Big) + (t-s)(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) \\
&\quad + \sum_{0<r\leq t-s}\log\rho(\Delta L^s_r)\Big)\Big].
\end{aligned}
$$

Taking into account the fact that processes L and L^s are equal in distribution, we can replace L^s by L:

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\exp\Big(iu\Big(\sum_{0<r\leq t-s}\Delta L^s_r + (t-s)b\Big) + (t-s)(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) \\
&\quad + \sum_{0<r\leq t-s}\log\rho(\Delta L^s_r)\Big)\Big] \\
&= \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(iu\Big(\sum_{0<r\leq t-s}\Delta L_r + (t-s)b\Big) + (t-s)(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) \\
&\quad + \sum_{0<r\leq t-s}\log\rho(\Delta L_r)\Big)\Big] \\
&= \mathbb{E}_{\mathbb{P}}[\exp(iuL_{t-s})U_{t-s}].
\end{aligned}
$$

On the other side,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[\exp(iuL_t)U_t] \\
&= \exp(t(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})))\,\mathbb{E}_{\mathbb{P}}\Big[\exp\Big(iu\Big(\sum_{j=1}^{N_t}\eta_j + bt\Big) + \sum_{j=1}^{N_t}\beta(\eta_j)\Big)\Big] \\
&= \exp(t(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) + tiub)\,\mathbb{E}_{\mathbb{P}}\Big[\exp\Big(\sum_{j=1}^{N_t}(iu\eta_j + \beta(\eta_j))\Big)\Big].
\end{aligned}$$

Expanding the expression under the expectation for the formula of total probability, we continue

$$\begin{aligned}
&= \exp(t(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) + tiub)\sum_{k=0}^{\infty}\Big[\mathbb{E}_{\mathbb{P}}\Big[\exp\Big(iu\eta_1 + \beta(\eta_1)\Big)\Big]\Big]^k\frac{(\lambda t)^k}{k!}e^{-\lambda t} \\
&= \exp(t(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) + tiub + (\lambda t)\mathbb{E}_{\mathbb{P}}[\exp(iu\eta_1 + \beta(\eta_1))] - \lambda t).
\end{aligned}$$

Now we obtain

$$\begin{aligned}
\lambda\mathbb{E}_{\mathbb{P}}[\exp(iu\eta_1 + \beta(\eta_1))] &= \lambda\int_{\mathbb{R}}\exp(iux + \beta(x))\frac{\nu^L(\mathrm{d}x)}{\nu^L(\mathbb{R})} \\
&= \int_{\mathbb{R}}\exp(iux)\rho(x)\nu^L(\mathrm{d}x) \\
&= \int_{\mathbb{R}}\exp(iux)\nu^L_{\mathbb{Q}}(\mathrm{d}x).
\end{aligned}$$

At the last step we have used the facts that $\lambda = \nu^L(\mathbb{R})$ and $\rho = \exp(\beta)$. Hence

$$\mathbb{E}_{\mathbb{P}}[\exp(iuL_t)U_t] = \exp(itub + t\int_{\mathbb{R}}(\exp(iux) - 1)\nu^L_{\mathbb{Q}}(\mathrm{d}x)).$$

Inserting $t - s$ instead of t yields the statement of the lemma. □

Hence the process $(L, \mathbb{F})$ is a Lévy process with respect to $\mathbb{Q}$ on $[0, T]$ with characteristic triplet $(b, 0, \nu^L_{\mathbb{Q}})$ relative to the truncation function $c = 0$. This completes the proof of part (2).

We now prove part (1) of the theorem. Let $\mathbb{Q}$ be a probability measure such that $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$ and $(L, \mathbb{F})$ is a compound Poisson process with Lévy measure $\nu^L_{\mathbb{Q}}$ and drift b. First we will show that then $\nu^L_{\mathbb{Q}} \sim \nu^L$. To this end let M^L be the jump measure of L which is a Poisson random measure with intensity measure $\lambda \otimes \nu^L$ with respect to $\mathbb{P}$ and $\lambda \otimes \nu^L_{\mathbb{Q}}$ with respect to $\mathbb{Q}$. Consequently, for every $B \in \mathscr{B}(\mathbb{R})$, $M^L([0, T] \times B)$ is a Poisson-distributed random variable with parameter $T\nu^L(B)$ with respect to $\mathbb{P}$ and $T\nu^L_{\mathbb{Q}}(B)$ with respect to $\mathbb{Q}$. This yields

$$\mathbb{P}(\{M^L([0,T] \times B) = 0\}) = \exp(-T\nu^L(B))$$

and

$$\mathbb{Q}(\{M^L([0,T] \times B) = 0\}) = \exp(-T\nu^L_{\mathbb{Q}}(B)).$$

If $\nu^L(B) = 0$ from the first equality it follows that

$$\mathbb{P}(\{M^L([0,T] \times B) = 0\}) = 1$$

and because of $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$, $\mathbb{Q}(\{M^L([0,T] \times B) = 0\}) = 1$. Now the second equality implies $\nu^L_{\mathbb{Q}}(B) = 0$. Hence $\nu^L_{\mathbb{Q}} \ll \nu^L$. Analogously, it can be shown that $\nu^L \ll \nu^L_{\mathbb{Q}}$. Thus $\nu^L \sim \nu^L_{\mathbb{Q}}$ and the claim is proven.

Now we define the probability measure $\tilde{\mathbb{Q}}$ on $\mathscr{F}_T$ by

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) = \exp\left[T(\nu^L(\mathbb{R}) - \nu^L_{\mathbb{Q}}(\mathbb{R})) + \sum_{0 \le r \le T} \log \rho(\Delta L_r)\right],$$

where $\rho = \dfrac{\mathrm{d}\nu^L_{\mathbb{Q}}}{\mathrm{d}\nu^L}$. Because of part (2) of the theorem and its proof, $\tilde{\mathbb{Q}}$ is indeed a probability measure on $\mathscr{F}_T$. Moreover, the process $(L, \mathbb{F})$ is a compound Poisson process on $[0,T]$ with Lévy measure $\nu^L_{\mathbb{Q}}$ and drift b with respect to $\tilde{\mathbb{Q}}$. But by assumption $(L, \mathbb{F})$ is also a compound Poisson process with Lévy measure $\nu^L_{\mathbb{Q}}$ and drift b with respect to $\mathbb{Q}$. from this follows that the finite dimensional distributions are equal:

$$\tilde{\mathbb{Q}}(\{L_{t_1} \in B_1, \ldots, L_{t_n} \in B_n\}) = \mathbb{Q}(\{L_{t_1} \in B_1, \ldots, L_{t_n} \in B_n\})$$

for $B_1, \ldots, B_n \in \mathscr{B}$, $t_1, \ldots, t_n \le T$. Hence $\tilde{\mathbb{Q}} = \mathbb{Q}$ on $\mathscr{F}_T$. Consequently,

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}^L_T) = \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}^L_T) = \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T), \tag{3.4}$$

the second equality being true because by definition of $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)$ we see that $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)$ is $\mathscr{F}^L_T$-measurable. Again using the definition of $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)$ we obtain (3.2) from (3.4). Part (1) of the theorem is proven. □

From the last theorem we observe that on $\mathscr{F}^L_T = \mathscr{F}^X_T$ the Lévy preserving equivalent martingale measure is uniquely determined by the characteristic triplet of the process X (or the process L, because by Proposition 3.1 from the characteristic triplet of the process X we can easily derive the characteristic triplet of the process L and vice versa). Moreover, there is no Gaussian part and the Lévy measure is finite, therefore on $\mathscr{F}_T^L$ the Lévy preserving equivalent martingale measure $\mathbb{Q}$ can be defined just by two parameters: the intensity $\lambda^{\mathbb{Q}}$ and the distribution of the size of the jump η_1 with respect to the measure $\mathbb{Q}$, denoted as $\mu^{\mathbb{Q}}$.

3.2 Well-determined Systems of Measures

However, since we introduced the entropy $I_T(\mathbb{Q},\mathbb{P})$ with respect to the σ-field $\mathscr{F}_T$ instead of $\mathscr{F}_T^L$ we need an additional property of a measure $\mathbb{Q} \in \mathscr{M}_e(T)$ that $\mathbb{Q}$ is well-determined on $\mathscr{F}_T$ by its values on $\mathscr{F}_T^L$. This gives rise to introduce the next definition.

3.7 Definition. (i) Let $\mathbb{Q}$ be a probability measure on $\mathscr{F}_T$ such that $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$. Then $\mathbb{Q}$ is said to be *well-defined by L on* $[0,T]$ if

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T^L) \quad \mathbb{P}\text{-a.s.}$$

or, equivalently, if $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)$ is $\mathscr{F}_T^L$-measurable.

(ii) We shall denote by $\mathscr{M}_e^L(T)$ the system of all $\mathbb{Q} \in \mathscr{M}_e(T)$ such that $\mathbb{Q}$ is well-determined by L on $[0,T]$.

For all $\mathbb{Q} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_e(T)$ it is now possible to calculate its entropy $I_T(\mathbb{Q},\mathbb{P})$.

It is worth recalling the definition of the relative entropy with respect to the one-dimensional distribution(cf. Section 1.7): Let μ and μ' be probability distributions on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$. The relative entropy $H(\mu',\mu)$ of μ' with respect to μ is defined by

$$H(\mu',\mu) = \begin{cases} \mathbb{E}_\mu\left[\frac{\mathrm{d}\mu'}{\mathrm{d}\mu}\log\frac{\mathrm{d}\mu'}{\mathrm{d}\mu}\right] & \text{if } \mu' \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

In more details the one-step model was discussed in Appendix B.

3.8 Proposition. Let us assume that there exists an equivalent martingale measure $\mathbb{Q}$ such that it preserves the Lévy property: $\mathbb{Q} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$. Then the relative entropy $I_T(\mathbb{Q},\mathbb{P})$ has the form

$$I_T(\mathbb{Q},\mathbb{P}) = T[\lambda - \lambda^{\mathbb{Q}} + \lambda^{\mathbb{Q}}\log\frac{\lambda^{\mathbb{Q}}}{\lambda} + \lambda^{\mathbb{Q}} H(\mu^{\mathbb{Q}},\mu)],$$

where $\mu^{\mathbb{Q}}$ and μ are the distributions of η_1 under the probability measures $\mathbb{Q}$ and $\mathbb{P}$, respectively.

Proof. We can write $I_T(\mathbb{Q},\mathbb{P})$ as

$$\begin{aligned}
I_T(\mathbb{Q},\mathbb{P}) &= \mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\Big] \\
&= \mathbb{E}_{\mathbb{Q}}\Big[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\Big] \\
&= \mathbb{E}_{\mathbb{Q}}\Big[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T^L)\Big] \\
&= \mathbb{E}_{\mathbb{Q}}\Big[\log\exp\Big[T(\nu^L(\mathbb{R})-\nu^L_{\mathbb{Q}}(\mathbb{R}))+\sum_{0<r\le T}\log\rho(\Delta L_r)\Big]\Big] \\
&= T(\lambda-\lambda^{\mathbb{Q}})+\mathbb{E}_{\mathbb{Q}}\Big[\sum_{0<r\le T}\log\rho(\Delta L_r)\Big] \\
&= T(\lambda-\lambda^{\mathbb{Q}})+\mathbb{E}_{\mathbb{Q}}\Big[\sum_{i=1}^{N_t}\beta(\eta_i)\Big] \\
&= T(\lambda-\lambda^{\mathbb{Q}})+\lambda^{\mathbb{Q}}T\,\mathbb{E}_{\mathbb{Q}}[\beta(\eta_1)],
\end{aligned}$$

where we have used that $\mathbb{Q}$ is well-determined by L on $[0,T]$ and the explicit form of the density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T^L)$ from Theorem 3.5, (1). Let us calculate the last part of the expression:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\beta(\eta_1)] &= \int_{\mathbb{R}}\log\frac{\mathrm{d}\nu^Q}{\mathrm{d}\nu}(x)\mu^{\mathbb{Q}}(\mathrm{d}x) \\
&= \int_{\mathbb{R}}\log\frac{\lambda^{\mathbb{Q}}\mathrm{d}\mu^{\mathbb{Q}}}{\lambda\mathrm{d}\mu}(x)\mu^{\mathbb{Q}}(\mathrm{d}x) \\
&= \log\frac{\lambda^{\mathbb{Q}}}{\lambda}+H(\mu^{\mathbb{Q}},\mu).
\end{aligned}$$

Finally, the entropy can be represented as

$$I_T(\mathbb{Q},\mathbb{P})=T[\lambda-\lambda^{\mathbb{Q}}+\lambda^{\mathbb{Q}}\log\frac{\lambda^{\mathbb{Q}}}{\lambda}+\lambda^{\mathbb{Q}}I(\mu^{\mathbb{Q}},\mu)]. \tag{3.5}$$

The theorem is proven. □

Remark. As we can easily observe from the equation (3.5), the relative entropy $I_T(\mathbb{Q},\mathbb{P})$ is finite if and only if $H(\mu^{\mathbb{Q}},\mu)$ is finite. Indeed, $\lambda^{\mathbb{Q}}$ and λ are finite and strictly positive and, hence the finiteness of the relative entropy $I_T(\mathbb{Q},\mathbb{P})$ depends just on the last term. Moreover, the equation (3.5) is true not just for martingale measures, but for all Lévy preserving measures.

3.3 Case without Drift

Let us come back to the original problem stated at the beginning: we are interested in the MEMM $\mathbb{Q}^*$ in the class $\mathscr{M}_a(T)$ such that

$$I_T(\mathbb{Q}^*, \mathbb{P}) = \inf_{\mathbb{Q} \in \mathscr{M}_a(T)} I_T(\mathbb{Q}, \mathbb{P}). \tag{3.6}$$

As we have already seen in Proposition 3.1, there is a unique correspondence between the original process X and the stochastic logarithm of its exponential $L := \mathscr{L}(\exp(X))$, therefore Theorem 1.35 gives us an alternative formulation of the original problem: we are allowed to replace the price process $S = S_0 \exp(X)$ by the process L. As we have seen, $(L, \mathbb{F})$ is a compound Poisson process with drift. For the convenience of the calculations we consider the original problem with respect to the compound Poisson process $(L, \mathbb{F})$. With this in mind, we can dispense with the origin of the process $(L, \mathbb{F})$ as the stochastic logarithm of $(S, \mathbb{F})$ and allow that $(L, \mathbb{F})$ is an arbitrary compound Poisson process with drift. In this way, we shall investigate a more general problem than the original one for the exponential price process $(S, \mathbb{F})$. Because of Theorem 2.5 we find that under the condition that the relative entropy between the original measure $\mathbb{P}$ and the class of equivalent measures $\mathscr{M}_e(T)$ is finite, i.e., $I_T(\mathscr{M}_e(T), \mathbb{P}) < \infty$, then $\mathbb{Q}^* \in \mathscr{M}_e(T)$. In this chapter we are looking for the MEMM in the class of Lévy preserving martingale measures $\mathscr{M}_l(T)$ not just because of heuristic reasons, but this is a sufficient subclass of measures. The last fact will be proven in Theorem 4.21. It means that the problem (3.6) can be reduced to the problem (I):

$$I_T(\mathbb{Q}^*, \mathbb{P}) = \inf_{\mathbb{Q} \in \mathscr{M}_e(T) \cap \mathscr{M}_l(T)} I_T(\mathbb{Q}, \mathbb{P}). \tag{I}$$

Note that the conditions for $\mathscr{M}_e(T) \cap \mathscr{M}_l(T) \neq \emptyset$ are equivalent to the conditions for $\mathscr{M}_e(T) \neq \emptyset$. Indeed, from Proposition **??** it follows that there exists at least one equivalent martingale measure if and only if the Lévy process L is not monotone, or in other words, L and $-L$ are not subordinators. But in case when L is a compound Poisson process, the statement can be simplified, because there is no need of a truncation function. Therefore we can deduce the following conditions in dependence on the sign of the drift in terms of the Lévy measure:

3.9 Proposition. Let $(L, \mathbb{F})$ be a compound Poisson process with drift b with respect to the measure $\mathbb{P}$. Then the class $\mathscr{M}_e(T) \cap \mathscr{M}_l(T)$ is not empty if and only if one of the following conditions is satisfied:
(i) $b > 0$ and $\operatorname{supp} \nu^L \cap (-\infty, 0) \neq \emptyset$;
(ii) $b < 0$ and $\operatorname{supp} \nu^L \cap (0, +\infty) \neq \emptyset$;
(iii) $b = 0$, $\operatorname{supp} \nu^L \cap (-\infty, 0)- \neq \emptyset$ and $\operatorname{supp} \nu^L \cap (0, +\infty) \neq \emptyset$.

Now we will show that $\mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$ is a sufficient subclass of $\mathscr{M}_e(T) \cap \mathscr{M}_l(T)$ for solving the problem (I).

3.10 Proposition. For every $\mathbb{Q} \in \mathscr{M}_e(T) \cap \mathscr{M}_l(T)$ there exists $\tilde{\mathbb{Q}} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$ such that

$$I_T(\tilde{\mathbb{Q}}, \mathbb{P}) \leq I_T(\mathbb{Q}, \mathbb{P}).$$

Proof. Suppose that $\mathbb{Q} \in \mathscr{M}_e(T) \cap \mathscr{M}_l(T)$. We define the probability measure $\tilde{\mathbb{Q}}$ on $\mathscr{F}_T$ by its density

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T^L).$$

Using Theorem 3.5, (1), we get

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) = \exp\left[T(\nu^L(\mathbb{R}) - \nu_{\mathbb{Q}}^L(\mathbb{R})) + \sum_{0<r\leq T} \log \rho(\Delta L_r)\right]$$

with $\rho := \frac{\mathrm{d}\nu_L^{\mathbb{Q}}}{\mathrm{d}\nu^L}$. An application of Theorem 3.5, (2), now shows that $(L, \mathbb{F})$ is a compound Poisson process with Lévy measure $\nu_{\mathbb{Q}}^L$ and drift b with respect to $\tilde{\mathbb{Q}}$.

For proving $\tilde{\mathbb{Q}} \in \mathscr{M}_e(T)$ it suffices to verify that

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[L_t] = 0, \quad t \in [0, T].$$

But

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathbb{Q}}}[L_t] &= \mathbb{E}\Big[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) L_t\Big] \\
&= \mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T^L) L_t\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) | \mathscr{F}_T^L\Big] L_t\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) L_t | \mathscr{F}_T^L\Big]\Big] \\
&= \mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) L_t\Big] \\
&= \mathbb{E}_{\mathbb{Q}}[L_t] = 0
\end{aligned}$$

because $\mathbb{Q} \in \mathscr{M}_e(T) \cap \mathscr{M}_l(T)$. Furthermore, $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)$ is $\mathscr{F}_T^L$-measurable and hence $\tilde{\mathbb{Q}}$ is well-determined by L on $[0, T]$. Summarizing the properties of $\tilde{\mathbb{Q}}$

we observe that $\tilde{\mathbb{Q}} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$. Finally, we estimate

$$\begin{aligned} I_T(\tilde{\mathbb{Q}}, \mathbb{P}) &= \mathbb{E}\Big[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) \log \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)|\mathscr{F}_T^L\Big] \log \mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)|\mathscr{F}_T^L\Big]\Big] \\ &\leq \mathbb{E}\Big[\mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)|\mathscr{F}_T^L\Big]\Big] \\ &= \mathbb{E}\Big[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\Big] \\ &= I_T(\mathbb{Q}, \mathbb{P}) \end{aligned}$$

where in the third line we have applied Jensen's inequality for conditional expectations and the convex function $x \log x$, $x > 0$. This completes the proof of the proposition. □

As a result of the last proposition, the problem (I) is equivalent to the following problem (II):

$$I_T(\mathbb{Q}^*, \mathbb{P}) = \inf_{\mathbb{Q} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)} I_T(\mathbb{Q}, \mathbb{P}). \tag{II}$$

In the following we shall investigate the problem (II) under the assumption of the absence of drift: $b = 0$. Note that the condition (iii) of Proposition 3.9 coincides with the statement of Proposition B.2 that refers us to the one-step model. Recall that the distribution of η_1 (the size of the first jump of L) with respect to $\mathbb{P}$ is denoted by μ. By analogy to the one-step model, let us introduce the class $\mathfrak{M}_e$ of all probability measures μ' on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that $\mu' \sim \mu$ and the moment condition is satisfied:

$$\int_{\mathbb{R}} x \mu'(\mathrm{d}x) = 0.$$

Let us consider another minimization problem, (III), which is motivated by identity (3.5). First of all, we rewrite the right-hand side of the identity (3.5) in the following way:

$$J(\lambda', \mu') := T[\lambda - \lambda' + \lambda' \log \frac{\lambda'}{\lambda} + \lambda' H(\mu', \mu)], \tag{3.7}$$

where λ and μ are given and fixed, $\lambda' > 0$ and $\mu' \in \mathfrak{M}_e$. In particular, μ is the distribution of η_1 with respect to $\mathbb{P}$, the parameter λ is defined as $\lambda = \nu^L(\mathbb{R}) > 0$. Assume that $\mathfrak{M}_e \neq \emptyset$, or in other words, the distribution μ

satisfies the condition of Proposition B.2. We are interested in such a couple (λ^*, μ^*) that minimizes the expression (3.7):

$$J(\lambda^*, \mu^*) = \inf_{\lambda'>0,\, \mu' \in \mathfrak{M}_e} J(\lambda', \mu'). \tag{III}$$

We are now going to investigate the relation between the minimization problems (II) and (III). For this we introduce the mapping

$$\Lambda : \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T) \longmapsto (0, +\infty) \times \mathfrak{M}_e$$

defined by

$$\Lambda(\mathbb{Q}) = (\lambda^{\mathbb{Q}}, \mu^{\mathbb{Q}}),\ \mathbb{Q} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T).$$

We recall that $(L, \mathbb{F})$ is a compound Poisson process on $[0, T]$ with respect to $\mathbb{Q}$ with Lévy measure $\nu_{\mathbb{Q}}^L$ for every $\mathbb{Q} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$ and that $\lambda^{\mathbb{Q}} = \nu_{\mathbb{Q}}^L(\mathbb{R})$ and $\mu^{\mathbb{Q}} = \frac{\nu_{\mathbb{Q}}^L}{\nu_{\mathbb{Q}}^L(\mathbb{R})}$ denote the intensity and the jump size distribution of $(L, \mathbb{F})$ with respect to $\mathbb{Q}$.

3.11 Proposition. The mapping Λ is a bijection between the sets $\mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$ and $(0, +\infty) \times \mathfrak{M}_e$.

Proof. First we show that $\Lambda(\mathbb{Q}) = (\lambda^{\mathbb{Q}}, \mu^{\mathbb{Q}})$ belongs to $(0, +\infty) \times \mathfrak{M}_e$ for every $\mathbb{Q} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$. Obviously, $\lambda^{\mathbb{Q}} > 0$. Furthermore, in view of Proposition 3.4 we get

$$\int_{\mathbb{R}} x \mu^{\mathbb{Q}}(\mathrm{d}x) = \mathbb{E}_{\mathbb{Q}}[\eta_1] = 0$$

because $\mathbb{Q} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$. Using Theorem 3.5, (1), we obtain that $\nu_{\mathbb{Q}}^L \sim \nu^L$ which implies $\mu^{\mathbb{Q}} \sim \mu$. Consequently, $\mu^{\mathbb{Q}} \in \mathfrak{M}_e$. Next we verify that the mapping Λ is one-to-one. For this let $\mathbb{Q}, \tilde{\mathbb{Q}} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)$ such that $(\lambda^{\mathbb{Q}}, \mu^{\mathbb{Q}}) = (\lambda^{\tilde{\mathbb{Q}}}, \mu^{\tilde{\mathbb{Q}}})$. Thus we have $\nu_{\mathbb{Q}}^L = \nu_{\tilde{\mathbb{Q}}}^L$. Applying Theorem 3.5, (1), we observe

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T^L) = \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T^L) \quad \mathbb{P}\text{-a.s.}$$

and since $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ are well-determined by L on $[0, T]$ we get

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) = \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) \quad \mathbb{P}\text{-a.s.}$$

Hence $\mathbb{Q} = \tilde{\mathbb{Q}}$ on $\mathscr{F}_T$.

Finally, we prove that Λ maps onto $(0, +\infty) \times \mathfrak{M}_e$. Let $(\lambda', \mu') \in (0, +\infty) \times \mathfrak{M}_e$ and define $\nu_{\mathbb{Q}}^L := \lambda' \mu'$. Then $\nu_{\mathbb{Q}}^L \sim \nu^L$. Using Theorem 3.5, (2), we obtain that $(L, \mathbb{F})$ is a compound Poisson process with Lévy measure $\nu_{\mathbb{Q}}^L$ with respect to the probability measure $\mathbb{Q}$ on $\mathscr{F}_T$ defined by its density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) = \exp\left[T(\nu^L(\mathbb{R}) - \nu_{\mathbb{Q}}^L(\mathbb{R})) + \sum_{0<r\leq T} \log \rho(\Delta L_r)\right],$$

where $\rho := \frac{\mathrm{d}\nu_{\mathbb{Q}}^L}{\mathrm{d}\nu^L}$. Obviously, $\lambda^{\mathbb{Q}} = \lambda'$ and $\mu^{\mathbb{Q}} = \mu'$. It remains to show that $\mathbb{Q} \in \mathscr{M}_e^L(T) \in \mathscr{M}_l(T)$. By the construction of $\mathbb{Q}$, we have $\mathbb{Q} \sim \mathbb{P}$ on $\mathscr{F}_T$ and $\mathbb{Q}$ is well-determined by L on $[0, T]$ because $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)$ is $\mathscr{F}_T^L$-measurable. Moreover, $\mathbb{Q}$ is a martingale measure because

$$\mathbb{E}_{\mathbb{Q}}[\eta_1] = \int_{\mathbb{R}} x\mu^{\mathbb{Q}}(\mathrm{d}x) = \int_{\mathbb{R}} x\mu'(\mathrm{d}x) = 0$$

since $\mu' \in \mathfrak{M}_e$ (cf. Proposition 3.4). This proves the proposition. □

3.12 Theorem. *The minimization problems* (II) *and* (III) *are equivalent: If* $\mathbb{Q}^*$ *is a solution of the problem* (II) *then* $(\lambda^*, \mu^*) := (\lambda^{\mathbb{Q}^*}, \mu^{\mathbb{Q}^*})$ *is a solution of the problem* (III). *Conversely, if* (λ^*, μ^*) *is a solution of the problem* (III) *then* $\mathbb{Q}^* := \Lambda^{-1}(\lambda^*, \mu^*)$ *is a solution of the problem* (II).

Proof. Let $\mathbb{Q}^*$ be a solution of the problem (II) and put $(\lambda^*, \mu^*) := (\lambda^{\mathbb{Q}^*}, \mu^{\mathbb{Q}^*})$. Then

$$\begin{aligned} J(\lambda^*, \mu^*) &= J(\lambda^{\mathbb{Q}^*}, \mu^{\mathbb{Q}^*}) \\ &= I_T(\mathbb{Q}^*, \mathbb{P}) \\ &\leq I_T(\mathbb{Q}, \mathbb{P}) \\ &= J(\lambda^{\mathbb{Q}}, \mu^{\mathbb{Q}}). \end{aligned}$$

In the second and fourth lines we have used Proposition 3.8. Hence,

$$\begin{aligned} J(\lambda^*, \mu^*) &= \inf_{\mathbb{Q} \in \mathscr{M}_e^L(T) \cap \mathscr{M}_l(T)} J(\lambda^{\mathbb{Q}}, \mu^{\mathbb{Q}}) \\ &= \inf_{(\lambda', \mu') \in (0, +\infty) \times \mathfrak{M}_e} J(\lambda', \mu'), \end{aligned}$$

where in the last equality Proposition 3.11 is used. This implies that (λ^*, μ^*) is a solution of the problem (III). The proof of the converse is similar. □

Since the latter theorem holds, we can consider the alternative, reparametrized problem (III) instead of the problem (II).

If $\mathscr{M}_e(T) \cap \mathscr{M}_l(T) \cap \mathscr{M}_f(T) = \emptyset$ (or, equivalently, $H(\mu', \mu) = +\infty$ for all $\mu' \in \mathfrak{M}_e$) then every $\mathbb{Q} \in \mathscr{M}_e(T) \cap \mathscr{M}_l(T)$ is a solution of the problem (II) (or, equivalently, every $(\lambda', \mu') \in (0, +\infty) \times \mathfrak{M}_e$ is a solution of the problem (III)). Therefore, in the following it can be assumed that $\mathscr{M}_e(T) \cap \mathscr{M}_l(T) \cap \mathscr{M}_f(T) \neq \emptyset$ which is equivalent to $H(\mu', \mu) < \infty$ for some $\mu' \in \mathfrak{M}_e$. Hence the minimization can be restricted to these sets.

3.13 Proposition. The solution (λ^*, μ^*) of the problem (III) exists if and only if there exists a solution $\tilde{\mu}$ of the one-step model problem (D). Moreover, $\mu^* = \tilde{\mu}$ and

$$\lambda^* = \lambda^*(\mu^*) = \lambda \exp(-H(\mu^*, \mu)). \tag{3.8}$$

Proof. Considering the expression (3.7) it is easy to observe that the one-dimensional relative entropy $H(\mu', \mu)$ appears with the strictly positive coefficient $T\lambda'$ and hence the smaller is $H(\mu', \mu)$, the smaller is all the right-hand side in (3.7). This observation is important since the set $\mathfrak{M}_e$ of parameters μ' does not depend on λ' and therefore the problem of minimization of the one-dimensional relative entropy $H(\mu', \mu)$ can be considered independently of the more complicated problem (III). For the one-dimensional problem we can refer to Appendix B where the one-step model is considered, in particular, Theorem B.5 gives us the concrete form of the entropy minimal martingale distribution μ^* satisfying the problem (D) and explains the conditions when such optimal distribution exists. It allows us to make the next step: let us fix the value of $H(\mu', \mu)$. Taking derivative of (3.7) in λ' and equating it to zero we can find the extremal points of the expression:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda'} I(\lambda', \mu') = T[-1 + \log\frac{\lambda'}{\lambda} + \lambda'\frac{1}{\lambda'} + H(\mu', \mu)] = 0.$$

The constants inside the brackets mutually compensate and after the elementary transformations we obtain the extremal point $\lambda^*(\mu')$:

$$\lambda^*(\mu') = \lambda \exp(-H(\mu', \mu)),\ \mu' \in \mathfrak{M}_e.$$

The second derivative is strictly positive and hence this point corresponds to the local minimum. Putting into (3.7) the boundary values of λ' (points 0 and $+\infty$) we find out that at point $\lambda^*(\mu')$ is a global minimum. If the solution $\tilde{\mu}$ of one-step model problem (D) does not exist (cf., Theorem B.5) the minimum of the expression (3.7) cannot be attained and hence there does not exist such a couple (λ^*, μ^*) that solves the problem (III).

We have just proven that the solution (λ^*, μ^*) of (III) exists if there exists a solution $\tilde{\mu}$ of the one-step model problem (D). Now our aim is to show that the reverse statement is also true: there exists a solution $\tilde{\mu}$ of the one-step model problem (D) if the solution (λ^*, μ^*) of (III) exists. But it follows from the construction of (λ^*, μ^*): the distribution $\mu^* \in \mathfrak{M}_e$ is such that it minimizes the relative entropy in the class $\mathfrak{M}_e$, that is

$$(H(\mu^*, \mu) = \min_{\mu' \in \mathfrak{M}_e} H(\mu', \mu))$$

and hence is a solution of the problem (D). □

Assume that there exists the MEMM $\mathbb{Q}^*$ in the class of $\mathscr{M}_e(T)$. By Proposition 3.11 we can reparametrize this measure with the corresponding couple $(\lambda^*, \mu^*) = (\lambda^{\mathbb{Q}^*}, \mu^{\mathbb{Q}^*})$. Proposition 3.13 gives the connection of the couple (λ^*, μ^*) and the solution $\tilde{\mu}$ of the one-step model problem (D) and therefore the measure $\mathbb{Q}^*$ can be characterized just by the one-dimensional distribution $\tilde{\mu}$. Because of Theorem B.5 we know that $\tilde{\mu}$ is the EMM and hence can be characterized just by one simple parameter: the Esscher parameter κ^*.

Let us calculate the relative entropy $I_T(\mathbb{Q}, \mathbb{P})$:

$$\begin{aligned} I_T(\mathbb{Q}, \mathbb{P}) &= J(\lambda^{\mathbb{Q}}, \mu^{\mathbb{Q}}) \\ &= T[\lambda - \lambda^{\mathbb{Q}} + \lambda_{\mathbb{Q}} \log \frac{\lambda^{\mathbb{Q}}}{\lambda} + \lambda^{\mathbb{Q}} H(\mu^{\mathbb{Q}}, \mu)]. \end{aligned}$$

The last formula is true also for the MEMM $\mathbb{Q}^*$. For finding $I_T(\mathbb{Q}^*, \mathbb{P})$ we substitute instead of $\lambda^{\mathbb{Q}}$ the value (3.8) and continue

$$\begin{aligned} I_T(\mathbb{Q}^*, \mathbb{P}) &= T[\lambda - \lambda^* + \lambda^* \log \frac{\lambda^*}{\lambda} + \lambda^* H(\mu^*, \mu)] \\ &= T[\lambda - \lambda \exp(-H(\mu^*, \mu)) + \lambda \exp(-H(\mu^*, \mu)) \log \frac{\lambda \exp(-H(\mu^*, \mu))}{\lambda} \\ &\quad + \lambda \exp(-H(\mu^*, \mu)) H(\mu^*, \mu)] \\ &= T[\lambda - \lambda \exp(-H(\mu^*, \mu))]. \end{aligned}$$

Note that the optimal λ^* guarantees the mutual compensation of the last two terms on the right-hand side of the second equality. At the next step we would like to use the connection with the one-step model (Appendix B). Let us recall the form of the Esscher density $\frac{\mathrm{d}\mu^*}{\mathrm{d}\mu}$:

$$\begin{aligned} \frac{\mathrm{d}\mu^*}{\mathrm{d}\mu} &= \frac{\exp(\kappa^* \eta_1)}{\mathbb{E}_\mu[\exp(\kappa^* \eta_1)]} \\ &= c \exp(\kappa^* \eta_1). \end{aligned}$$

where

$$c = \Big(\int_{\mathbb{R}} \exp(\kappa^* x) \mathrm{d}\mu \Big)^{-1}.$$

Inserting the Esscher density inside the entropy $H(\mu^*, \mu)$ we get

$$\begin{aligned} T\Big[\lambda - \lambda \exp(-H(\mu^*,\mu))\Big] &= T\lambda\Big[1-\exp\Big(-\mathbb{E}_\mu\Big[\frac{\mathrm{d}\mu^*}{\mathrm{d}\mu}\log\frac{\mathrm{d}\mu^*}{\mathrm{d}\mu}\Big]\Big)\Big] \\ &= T\lambda\Big[1-\exp\Big(-\mathbb{E}_{\mu^*}\Big[\log\frac{\mathrm{d}\mu^*}{\mathrm{d}\mu}\Big]\Big)\Big] \\ &= T\lambda[1-\exp(-\log c)] \\ &= T\lambda\Big[1-\exp\Big(-\log\Big(\int_{\mathbb{R}}\exp(\kappa^* x)\mathrm{d}\mu\Big)^{-1}\Big)\Big] \\ &= T\lambda\Big[1-\exp\Big(\log\Big(\int_{\mathbb{R}}\exp(\kappa^* x)\mathrm{d}\mu\Big)\Big)\Big] \\ &= T\lambda\Big[1-\int_{\mathbb{R}}\exp(\kappa^* x)\mathrm{d}\mu\Big]. \end{aligned}$$

Finally, we may state the following theorem:

3.14 Theorem. *If there exists the MEMM $\mathbb{Q}^*$ in the class $\mathscr{M}_e(T)$ then the following identity for the relative entropy $I_T(\mathbb{Q}^*, \mathbb{P})$ holds:*

$$I_T(\mathbb{Q}^*,\mathbb{P}) = T\lambda\Big[1-\int_{\mathbb{R}}\exp(\kappa^* x)\mathrm{d}\mu\Big], \tag{3.9}$$

where $\kappa^* = \kappa^*(\mathbb{Q}^*)$ is the corresponding Esscher parameter.

3.4 Approximation of the Entropy-Optimal Measure

We shall now briefly discuss what happens when the MEMM $\mathbb{Q}^*$ does not exist. In other words, what can one say about the relative entropy when there does not exist the corresponding Esscher parameter for the one-step model problem D? Assume that $\mathfrak{M}_f \neq \emptyset$. Then for every $\varepsilon > 0$ it is always possible to find μ_ε such that

$$H(\mu_\varepsilon, \mu) \le \inf_{\mu' \in \mathfrak{M}_e} H(\mu', \mu) + \varepsilon. \tag{3.10}$$

In other words, the relative entropy $H(\mu_\varepsilon, \mu)$ of the one-dimensional distribution μ_ε with respect to the original distribution μ does not exceed the entropy $H(\mathfrak{M}_e, \mu)$ more than the value ε. Because of Proposition 3.13 (form of the parameter λ^*) and Proposition 3.11 we can construct a probability measure $\mathbb{Q}_\varepsilon$ on $\mathscr{F}_T$ that corresponds to the distribution μ_ε. More precisely, the probability measure $\mathbb{Q}_\varepsilon$ is given by

$$\mathbb{Q}_\varepsilon = \Lambda^{-1}(\lambda^*(\mu_\varepsilon), \mu_\varepsilon),$$

where $\lambda^*(\mu_\varepsilon) = \lambda \exp(-H(\mu_\varepsilon, \mu))$. Therefore it has sense to consider the relative entropy $I_T(\mathbb{Q}_\varepsilon, \mathbb{P})$ and to compare it with the relative entropy $I_T(\mathscr{M}_e(T), \mathbb{P})$:

$$\begin{aligned} & I_T(\mathbb{Q}_\varepsilon, \mathbb{P}) - I_T(\mathscr{M}_e(T), \mathbb{P}) \leq \\ & \leq \; T\lambda[1 - \exp(-\inf_{\mu' \in \mathfrak{M}_e} H(\mu', \mu) - \varepsilon)] - T\lambda[1 - \exp(-\inf_{\mu' \in \mathfrak{M}_e} H(\mu', \mu))] \\ & = \; T\lambda \exp(-\inf_{\mu' \in \mathfrak{M}_e} H(\mu', \mu))[1 - \exp(-\varepsilon)] \\ & \leq \; T\lambda[1 - \exp(-\varepsilon)] \leq T\lambda\varepsilon. \end{aligned} \tag{3.11}$$

In the last inequality, the fact was used that the relative entropy $\inf_{\mu' \in \mathfrak{M}_e} H(\mu', \mu)$ is always nonnegative and hence $\exp(-\inf_{\mu' \in \mathfrak{M}_e} H(\mu', \mu)) \leq 1$. We observe that the error generated by the ε-measure $\mathbb{Q}_\varepsilon$ can be estimated by the deterministic function, has the order of convergence $O(\varepsilon)$ and does not depend on the nature of the class $\mathscr{M}_e(T)$.

3.5 The Cramér–Lundberg Model

As a particular example of the model without drift, we would like to consider the Cramér–Lundberg model with stochastic premiums that comes from the risk theory. The classical Cramér–Lundberg model is usually used in general (non-life) insurance to explain the evolution of the capital of the company and it has the following form

$$U_t = u_0 + c\,t - S_t = u_0 + c\,t - \sum_{j=1}^{N_t} \xi_i, \quad t \geq 0,$$

where U_t is the value of the company assets at the moment t, u_0 stands for the initial value of the company assets, the flow of insurance premiums $c\,t$ is assumed to be homogeneous and constant in time (which corresponds to the assumption of the stable market), the process S is a compound Poisson process with parameters λ and μ, and it stands for the random sum of the positive independent identically distributed (i.i.d.) contingent claims (ξ_i) (with distribution μ), the Poisson process N with parameter λ corresponds to the number of claims.

Despite its simplicity, the model is still used in insurance and there are various generalizations of it adopted to needs of the particular market: with additional income from investment, 'switching' parameter of claims intensity, including inflation and so on. For the case of the new or unstable market corresponds the so-called Cramér–Lundberg model with stochastic premiums. The main aim is to replace the constant flow of the premiums $c\,t$ by the new compound Poisson process P that stands for stochastic premiums:

$$U_t = u_0 + P_t - S_t = u_0 + \sum_{i=1}^{N_t^1} \zeta_i - \sum_{j=1}^{N_t^2} \xi_i, t \geq 0.$$

The Poisson processes N^1 and N^2 have the parameters λ_1 and λ_2, respectively, the distributions μ_1 and μ_2 correspond to the positive i.i.d. random variables (ζ_i) and (ξ_j). Note, that N^1, N^2, (ζ_i) and (ξ_j) are assumed to be independent.

It is easy to see that the sum of two compound Poisson processes $P - S$ is again a compound Poisson process. Indeed, let us consider a characteristic function $\psi(s)$ of this process at the time t:

$$\begin{aligned}
\psi(s) &= \mathbb{E}[\exp(i\,s(P_t - S_t))] \\
&= \mathbb{E}\left[\exp\left(i\,s\left(\sum_{i=1}^{N_t^1} \zeta_i - \sum_{j=1}^{N_t^2} \xi_i\right)\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\,s\sum_{i=1}^{N_t^1} \zeta_i\right) \exp\left(-i\,s\sum_{j=1}^{N_t^2} \xi_i\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\exp\left(i\,s\sum_{i=1}^{N_t^1} \zeta_i\right) \exp\left(-i\,s\sum_{j=1}^{N_t^2} \xi_i\right) \Big| \sigma(S_l,\, 0 \leq l \leq t)\right]\right] \\
&= \mathbb{E}\left[\exp\left(i\,s\sum_{i=1}^{N_t^1} \zeta_i\right) \mathbb{E}\left[\exp\left(-i\,s\sum_{j=1}^{N_t^2} \xi_i\right) \Big| \sigma(S_l,\, 0 \leq l \leq t)\right]\right] \\
&= \mathbb{E}\left[\exp\left(i\,s\sum_{i=1}^{N_t^1} \zeta_i\right)\right] \mathbb{E}\left[\exp\left(-i\,s\sum_{j=1}^{N_t^2} \xi_i\right)\right].
\end{aligned}$$

With $\sigma(S_l,\, 0 \leq l \leq t)$ we have denoted the σ-algebra generated by the process S at time interval $[0, t]$. For the last step we have used the independence of S and P. Let us denote the characteristic functions of distributions of ζ and

$-\xi$ by χ_1 and χ_2, respectively. Continuing our calculation, we get

$$\begin{aligned}
\psi(s) &= \mathbb{E}\left[\exp\left(i\,s\sum_{i=1}^{N_t^1}\zeta_i\right)\right]\mathbb{E}\left[\exp\left(-i\,s\sum_{j=1}^{N_t^2}\xi_i\right)\right] \\
&= \exp(\lambda_1 t(\chi_1(s)-1))\exp(\lambda_2 t(\chi_2(s)-1)) \\
&= \exp\left(it\,(\lambda_1+\lambda_2)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\chi_1(s)+\frac{\lambda_2}{\lambda_1+\lambda_2}\chi_2(s)-1\right)\right).
\end{aligned}$$

The last line corresponds to the characteristic function of the compound Poisson process N with intensity parameter $\lambda = \lambda_1 + \lambda_2$ and the jump distribution $\mu = \frac{\lambda_1}{\lambda_1+\lambda_2}\mu_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\mu_2^*$, where by μ_2^* is denoted the distribution of $(-\xi)$.

Now our model has the form of the compound Poisson process without drift and as it was already Section 3.3, the problem of finding the MEMM reduces to the problem for one-step model for distribution μ.

4

Models driven by the Lévy Processes. General Case

At the beginning of the chapter the main objects of the investigations are models driven by the linear and geometric (exponential) Lévy processes. Our first aim is to show the equivalence of the problem of finding the martingale measures for such models. In particular, it is used the well-known approach based on the properties of stochastic exponentials and logarithms for reformulation of the linear case problem in terms of exponential case problem and vice versa (see Proposition 4.1). This is the reason why we choose just the linear model for the further consideration.

In the following section (Section 4.2)we are going to introduce the key notion of this thesis – the Esscher martingale measure (EMM) – and provide a sufficient condition of its existence (existence of exponential moments, Proposition 4.4). The problem of existence of at least one equivalent martingale measure is discussed in the next section (cf. Proposition 4.5). Afterwords, we are going to show the coincidence of the EMM and the minimal entropy martingale measure (MEMM) for the model driven by the linear Lévy process. We start with the definition of the EMM and then show that the EMM, if it exists, is the MEMM.

Our next aim is prove the main result of the thesis: the coincidence of the EMM and the MEMM for the models driven by a linear Lévy process. More precisely, the EMM exists if and only if the MEMM exists and if one (hence both) of these conditions is satisfied then we have the coincidence of these probability measures.

The idea of the basic fact is the following: We construct approximation sequences of probability measures $(\mathbb{P}_n)$ via their densities $(Z_T^{(n)})$ with respect to the original probability measure $\mathbb{P}$ in such a way that there always exists the EMM $\mathbb{Q}_n^E$ with respect to $\mathbb{P}_n$. Furthermore, we show that the infimum of relative entropy in the class of martingale measures $\mathscr{M}_a(T)$ coincides with the upper limit of relative entropy of the approximating sequence $I(\mathbb{Q}_n^E, \mathbb{P})$. Using the connection with the one-step model we get the equivalence of the existence of the EMM and the MEMM and their equality for the models driven by the linear Lévy process. Finally we state some important corollaries and give several examples.

4.1 Connection between Exponential and Linear Models

In the following section we are going to consider two models driven by the Lévy processes: linear and exponential. The main objective of the section is to show the equivalence of these models in the sense that we can pass on from one to the other without changing the class of local martingale measures. To achieve this aim, we use the well-known standard approach which was discussed, for example, in Fujiwara & Miyahara (2003).

Let $S = (S_t)_{s\in[0,T]}$, $T > 0$, be a geometric Lévy process defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, that is, S is a stochastic process of the form:

$$S_t = S_0 \exp(X_t),\ t \in [0, T],$$

where $S_0 > 0$ is a constant and $X = (X_t)_{t\in[0,T]}$ is a Lévy process and characteristic triplet (b, σ^2, ν) associated with the standard truncation function h defined by $h(x) = x1_{\{|x|\leq 1\}}$, $x \in \mathbb{R}$.

If $\nu \equiv 0$ we get the famous Black–Scholes model with volatility σ and drift $\mu = b - \frac{1}{2}\sigma^2$ which was already studied for decades and there is a huge variety of literature dedicated to this case. To exclude this special case from discussion, we assume that $\nu \not\equiv 0$.

If $\sigma = 0$ we get a pure jump Lévy process with drift b. In particular, this case includes the model driven by a compound Poisson process with drift which was already discussed in the previous chapter and we would like to get a result similar to Proposition 3.1.

Let us denote by F the point process on $\mathbb{R} \setminus \{0\}$ defined by $F_t := \Delta X_t$, where

$$\Delta X_t := X_t - X_{t-}, \quad X_{t-} := \lim_{u\uparrow t} X_u.$$

We denote by $N_F(\mathrm{d}u\mathrm{d}x)$ the counting measure of the point process F:

$$N_F((0, t], A) := \sharp\{u \in \mathbb{D}_F \cap (0, t];\ F_u \in A\} \text{ for } A \in \mathscr{B}(\mathbb{R} \setminus \{0\}),$$

where $\mathbb{D}_F$ denotes the domain of F_t, that is, $\mathbb{D}_F := \{t > 0 : \Delta X_t \neq 0\}$ and $\mathscr{B}(\mathbb{R} \setminus \{0\})$ the Borel σ-field on $\mathbb{R} \setminus \{0\}$. We also denote by $\hat{N}_F(\mathrm{d}u\mathrm{d}x)$ the compensator of $N_F(\mathrm{d}u\mathrm{d}x)$. In fact, $N_F(\mathrm{d}u\mathrm{d}x)$ is a Poisson random measure and

$$\hat{N}_F(\mathrm{d}u\mathrm{d}x) = \mathrm{d}u\nu(\mathrm{d}x).$$

The Lévy–Itô decomposition of the process X yields the representation:

$$X_t = \sigma W_t + bt + \int_{(0,t]} \int_{\{|x| \leq 1\}} x \tilde{N}_F(\mathrm{d}u\mathrm{d}x) + \int_{(0,t]} \int_{\{|x|>1\}} x N_F(\mathrm{d}u\mathrm{d}x), \quad (4.1)$$

where (W_t) is a one-dimensional standard Brownian motion and $\tilde{N}_F(\mathrm{d}u\mathrm{d}x)$ is the compensated measure of $N_F(\mathrm{d}u\mathrm{d}x)$ defined by

$$\tilde{N}_F(\mathrm{d}u\mathrm{d}x) := N_F(\mathrm{d}u\mathrm{d}x) - \hat{N}_F(\mathrm{d}u\mathrm{d}x). \quad (4.2)$$

Now we would like to represent the process S as a stochastic exponential of some other process L: $S_t = S_0 \mathscr{E}(L)_t$. We apply Proposition 1.33 and find the explicit form of the process L:

$$\begin{aligned} L_t &= X_t + \frac{1}{2}\sigma^2 t + \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} (e^x - 1 - x) N_F(\mathrm{d}u\mathrm{d}x) \\ &= \sigma W_t + b_1 t + \int_{(0,t]} \int_{\{|x| \leq 1\}} (e^x - 1) \tilde{N}_F(\mathrm{d}u\mathrm{d}x) \\ &\quad + \int_{(0,t]} \int_{\{|x|>1\}} (e^x - 1) N_F(\mathrm{d}u\mathrm{d}x), \end{aligned}$$

where

$$b_1 := \frac{1}{2}\sigma^2 + b + \int_{\{|x| \leq 1\}} (e^x - 1 - x)\nu(\mathrm{d}x).$$

Therefore, L is still a Lévy process under the measure $\mathbb{P}$.

In order to know the characteristics of L, we transform the point process F into another one G by

$$\mathbb{D}_G := \mathbb{D}_F \text{ and } G_t := \vartheta(F_t),$$

where $\vartheta(x) := e^x - 1$ for $x \in \mathbb{R}$. Then we see that

$$\hat{N}_G(\mathrm{d}u\mathrm{d}y) = \mathrm{d}u\mu(\mathrm{d}y),$$

where $\nu^L(\mathrm{d}y) := \nu \circ \vartheta^{-1}(\mathrm{d}y)$ and that

$$\begin{aligned} L_t &= \sigma W_t + b_1 t + \int_{(0,t]} \int_{\{e^{-1}-1 \le y \le e-1\}} y \tilde{N}_G(\mathrm{d}u\mathrm{d}y) \\ &\quad + \int_{(0,t]} \int_{\{y < e^{-1}-1\} \cup \{y > e-1\}} y N_G(\mathrm{d}u\mathrm{d}y) \\ &= \sigma W_t + b_2 t + \int_{(0,t]} \int_{\{|y| \le 1\}} y \tilde{N}_G(\mathrm{d}u\mathrm{d}y) \\ &\quad + \int_{(0,t]} \int_{\{|y| > 1\}} y N_G(\mathrm{d}u\mathrm{d}y), \end{aligned}$$

where

$$b_2 := b_1 + \int_{\{x<-1\}} (e^x - 1)\nu(\mathrm{d}x) - \int_{\{\log 2 < x \le 1\}} (e^x - 1)\nu(\mathrm{d}x).$$

This gives the Lévy–Itô decomposition of the process L associated with the standard truncation function $h(x) := x 1_{\{|x|\le 1\}}$, and hence its characteristic triplet has the form (b_2, σ^2, ν^L).

4.1 Proposition. Let $S = (S_t)_{s\in[0,T]}$, $T > 0$, be a Geometric Lévy process defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$:

$$S_t = S_0 \exp(X_t),\ t \in [0,T],$$

where $S_0 > 0$ is a constant and $X = (L_t)_{t\in[0,T]}$ is a Lévy process with $X_0 = 0$ and characteristic triplet (b, σ^2, ν) associated with the standard truncation function $h(x) = x 1_{\{|x|\le 1\}}$.
Then $S_t = S_0 \mathscr{E}(L)_t$, where L is a $\mathbb{P}$-Lévy process with characteristic triplet (b_2, σ^2, ν^L),

$$\begin{aligned} b_2 &:= \frac{1}{2}\sigma^2 + b + \int_{\{|x|\le 1\}} (e^x - 1 - x)\nu(\mathrm{d}x) \\ &\quad + \int_{\{x<-1\}} (e^x - 1)\nu(\mathrm{d}x) - \int_{\{\log 2 < x \le 1\}} (e^x - 1)\nu(\mathrm{d}x). \end{aligned}$$

Taking into account Proposition 1.35 we find that for every probability measure $\mathbb{Q}$ the process S is $\mathbb{Q}$-local martingale if and only if the process $\hat{L}$ is $\mathbb{Q}$-local martingale. Moreover, because of Proposition 1.36, it follows that if the measure $\mathbb{Q}$ is a local martingale measure for one of these processes and preserves the Lévy property of it, then it is even a martingale measure for both processes.

As we have seen, the exponential problem seeming at first glance much more complicated than the linear problem, is in fact even simpler because of the limited support of the sizes of the negative jumps and is already included in the linear problem. Therefore, for the further investigation we would like to focus our attention on the model driven by the linear Lévy processes.

4.2 The Esscher Martingale Measure

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let us introduce a Lévy process $(L, \mathbb{F})$ with characteristic triplet (b, σ, ν).

Let us introduce the Esscher martingale measure $\mathbb{Q}^E$:

4.2 Definition. Let $\kappa \in \mathbb{R}$ with $\mathbb{E}[\exp(\kappa L_t)] < \infty$ for $t \in [0, T]$ and define

$$Z_t^\kappa := \frac{\mathrm{d}\mathbb{Q}^\kappa}{\mathrm{d}\mathbb{P}}|_{\mathscr{F}_t} = \frac{\exp(\kappa L_t)}{\mathbb{E}[\exp(\kappa L_t)]} \quad \text{for } t \in [0, T]. \tag{4.3}$$

The process Z^κ is called *Esscher density process* and the measure $\mathbb{Q}^\kappa$ defined on $\mathscr{F}_T$ is called *Esscher measure*. One says that the probability measure $\mathbb{Q}^\kappa$ is the *Esscher martingale measure* (EMM) if $(L, \mathbb{F})$ is a $\mathbb{Q}^\kappa$-martingale on $[0, T]$. The EMM on $[0, T]$, if it exists, will be denoted by $\mathbb{Q}^E$.

The process Z^κ is indeed a strictly positive $\mathbb{P}$-martingale on $[0, T]$ as an exponential of a process with independent increments divided by its expectation. An important fact is that the process $(L, \mathbb{F})$ is still a $\mathbb{Q}^\kappa$-Lévy process. In fact, for our model driven by the linear Lévy process $(L, \mathbb{F})$ the martingale condition can be replaced just by the moment condition

$$\mathbb{E}_{\mathbb{Q}^\kappa}[L_T] = \mathbb{E}_P[cL_T \exp(\kappa L_T)] = 0.$$

Indeed, it is easy to see that for $0 \leq s \leq t \leq T$ we have:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^\kappa}[L_t|\mathscr{F}_s] &= \mathbb{E}_{\mathbb{Q}^\kappa}[L_t - L_s + L_s|\mathscr{F}_s] \\ &= \mathbb{E}_{\mathbb{Q}^\kappa}[L_t - L_s|\mathscr{F}_s] + L_s \\ &= \mathbb{E}_{\mathbb{Q}^\kappa}[L_t - L_s] + L_s, \end{aligned}$$

where the last step follows from the $\mathbb{F}$-independence of increments. Hence, the process L is a $\mathbb{Q}^\kappa$-martingale means that

$$\mathbb{E}_{\mathbb{Q}^\kappa}[L_t - L_s] = 0.$$

Taking into account the homogeneity of L in time and the initial value $L_0 = 0$ we come to the fact that for

$$\mathbb{E}_{\mathbb{Q}^\kappa}[L_t - L_s] = \mathbb{E}_{\mathbb{Q}^\kappa}[L_{t-s} - L_0] = \mathbb{E}_{\mathbb{Q}^\kappa}[L_{t-s}] = 0.$$

Moreover, instead of the fixed time we can take any $t \in (0, T]$.

Let us consider the function φ of t and κ defined by

$$\varphi_t(\kappa) = \mathbb{E}[\exp(\kappa L_t)], \quad \kappa \in \mathbb{R},\, t \in [0, T],$$

in more details. The function φ is called the cummulant function. We can observe that this is the moment generating function of L and hence,

$$\varphi_t(\kappa) = (\varphi_1(\kappa))^t,$$

or in other words, the logarithm of this function is linear in t:

$$\log \varphi_t(\kappa) = t \log \varphi_1(\kappa).$$

We can also explicitly calculate the relative entropy of the EMM $\mathbb{Q}^E$ (if it exists) with respect to $\mathbb{P}$ on $\mathscr{F}_T$:

$$\begin{aligned} I(\mathbb{Q}^E, \mathbb{P}) &= \mathbb{E}_{Q^E}\left[\log \frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] \\ &= \mathbb{E}_{Q^E}[\log \exp(\kappa L_T - T \log(\varphi_1(\kappa)))] \\ &= \mathbb{E}_{Q^E}[\kappa L_T - T \log(\varphi_1(\kappa))] && (4.4) \\ &= \mathbb{E}_{Q^E}[\kappa L_T] - T \log(\varphi_1(\kappa)) && (4.5) \\ &= -T \log(\varphi_1(\kappa)). && (4.6) \end{aligned}$$

At step from (4.4) to (4.5) we used the linearity of the integral and from (4.5) to (4.6) that $\mathbb{Q}^E$ is a martingale measure. We are allowed to use linearity because

$$\mathbb{E}_{Q^E}\left[\left|\log \frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right|\right] < \infty.$$

Indeed, for proving the last claim we estimate

$$\begin{aligned} \left|\log \frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right| &= |\log \exp(\kappa X_T - T \log(\varphi_1(\kappa)))| \\ &\leq |T \log \varphi_1(\kappa)| + |\kappa X_T| \quad \text{for } \kappa \in \mathbb{R}. \end{aligned}$$

But in view of $\mathbb{E}_{Q^E} L_T = 0$ by the definition of the EMM $\mathbb{Q}^E$ it follows that $\mathbb{E}_{Q^E}|L_T| < \infty$.

Note that $0 \leq -T \log(\varphi_1(\kappa)) < +\infty$. Nonnegativity follows from the nonnegativity property of the relative entropy, finiteness follows from the definition of the EMM.

Summarizing the properties of EMM discussed above, we can state the following lemma:

4.3 Lemma. *Let $\kappa_0 \in \mathbb{R}$ with $\mathbb{E}[\exp(\kappa_0 L_t)] < \infty$ for some (and hence for all) $t \in [0,T]$ and $\mathbb{Q}^{\kappa_0}$ is defined by (4.3). Then*

1. *The Esscher measure $\mathbb{Q}^{\kappa_0}$ is a martingale measure $\mathbb{Q}^E$ if and only if*

$$\mathbb{E}_{\mathbb{Q}^{\kappa_0}}[L_t] = 0 \text{ for some (and hence for all) } t \in (0,T]. \tag{4.7}$$

2. *The relative entropy of $\mathbb{Q}^{\kappa_0}$ with respect to $\mathbb{P}$ has the form*

$$I(\mathbb{Q}^E, \mathbb{P}) = -T \log(\varphi_1(\kappa_0)). \tag{4.8}$$

For later use, we would like to give simple sufficient conditions for the existence of the EMM in terms of exponential moments:

4.4 Proposition. Assume that the Lévy process L is not monotone (in other words, neither L nor $-L$ is a *subordinator*) and for any $\kappa \in \mathbb{R}$ we have $\mathbb{E}[\exp(\kappa L_T)] < \infty$. Then there exists the EMM $\mathbb{Q}^E$.

Proof. It is necessary to prove that there exists $\kappa_0 \in \mathbb{R}$ such that $\mathbb{Q}^{\kappa_0}$ is well-defined and $\mathbb{E}_{\mathbb{Q}^{\kappa_0}}[L_T] = 0$. Because of independence of increments and homogeneity of the Lévy process L, for any $0 < t \leq T$ we have

$$\mathbb{E}[\exp(\kappa L_t)] = \big(\mathbb{E}[\exp(\kappa L_T)]\big)^{\frac{t}{T}} < \infty,$$

therefore, for any $\kappa \in \mathbb{R}$ the Esscher measure $\mathbb{Q}^\kappa$ is well-defined.

The process L is not monotone, in particular, means that $\mathbb{P}(\{L_T > 0\}) > 0$ and $\mathbb{P}(\{L_T < 0\}) > 0$. Hence, we can consider the asymptotic behavior of the function $\psi(\kappa) := \mathbb{E}[L_T \exp(\kappa L_T)]$. Applying Proposition A.1, part5, we know that

$$\begin{aligned} \lim_{\kappa \uparrow +\infty} \psi(\kappa) &= +\infty; \\ \lim_{\kappa \downarrow -\infty} \psi(\kappa) &= -\infty. \end{aligned}$$

In other words, $\psi(\mathbb{R}) = (-\infty, +\infty)$. By Proposition A.1, part6, the function ψ is continuous and therefore there exists κ_0 such that $\psi(\kappa_0) = 0$. The statement is proven. □

4.3 Existence of Equivalent Martingale Measures

In this section we would like to consider conditions under which there exists at least one equivalent martingale measure. In fact, there can be stated a necessary and sufficient condition which was already mentioned in the previous proposition.

4.5 Proposition. Let $(L_t, \mathscr{F}_t)_{t\in[0,T]}$ be a nontrivial Lévy process under measure $\mathbb{P}$ with characteristic triplet (b, σ^2, ν). There exists an equivalent martingale measure if and only if L is not monotone.

Proof. First we verify that the condition is necessary. Assume L is monotone and increasing Lévy process, in other words subordinator. It means that

$$\mathbb{P}(\{L_t - L_s \geq 0\}) = 1 \text{ for any } 0 \leq s \leq t \leq T.$$

In particular, it holds for $s = 0$ and $t = T$:

$$\mathbb{P}(\{L_T - L_0 \geq 0\}) = \mathbb{P}(\{L_T \geq 0\}) = 1.$$

Let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent probability measure. By definition, $\mathbb{Q}(\{L_T \geq 0\}) = 1$ and hence $\mathbb{E}_{\mathbb{Q}}[L_T] \geq 0$. Taking into account the fact that L is not trivial, we get a strict inequality $\mathbb{E}_{\mathbb{Q}}[L_T] > 0$. Therefore, in the class of probability measures equivalent to $\mathbb{P}$ there exists no martingale measure. Now let us prove the statement in the reverse direction: if L is not monotone, then there exists at least one equivalent martingale measure. Let $\mathbb{Q}$ be the probability measure that satisfies Theorem 1.22 with $b' = b$

$$r(x) := \begin{cases} 1, & \text{if } |x| \leq 1, \\ \exp(-x^2), & \text{if } |x| > 1. \end{cases}$$

It means that $\mathbb{Q} \sim \mathbb{P}$ and L is $\mathbb{Q}$-Lévy process with characteristic triplet (b, σ^2, ν'), where $r(x) = \log \frac{\mathrm{d}\nu'}{\mathrm{d}\nu}$. In other words, probability measure $\mathbb{Q}$ changes just the probability of the jumps of the process L of sizes larger than 1, keeping the same drift, diffusion part and small jumps (of sizes smaller than 1). According to Theorem 1.25, the process L has finite exponential moments (i.e., $\mathbb{E}_{\mathbb{Q}}[\exp(\kappa X_t)] < +\infty$ for all $t \in [0, T], \kappa \in \mathbb{R}$) with respect to the probability measure $\mathbb{Q}$ if and only if

$$\int_{\{|x|>1\}} \exp(\kappa x)\nu'(\mathrm{d}x) < \infty \quad \text{ for } \kappa \in \mathbb{R}.$$

But

$$\int_{\{|x|>1\}} \exp(\kappa x)\nu'(\mathrm{d}x) = \int_{\{|x|>1\}} \exp(\kappa x)\exp(-x^2)\nu(\mathrm{d}x)$$
$$< \infty) \quad \text{for } \kappa \in \mathbb{R}.$$

The integrand is a continuous function and because of L'Hospital's rule converges to zero if $|x|$ converges to ∞. Hence the integrand is bounded and therefore integrable on $|x| > 1$ because ν is a finite measure on this set. Applying Proposition 4.4 to the probability measure $\mathbb{Q}$ we find that there exists the Esscher martingale measure $\mathbb{Q}^E$ with respect to $\mathbb{Q}$. Thus, $\mathbb{Q}^E \sim \mathbb{Q}$ and hence $\mathbb{Q}^E \sim \mathbb{P}$. The statement is proven. □

Note, that $\mathbb{Q}^E$ is the Esscher martingale measure in correspondence to $\mathbb{Q}$, but in general is not such for the measure $\mathbb{P}$.

If the Lévy process is trivial, then there also exists an equivalent martingale measure: $\mathbb{P}$ itself.

4.6 Proposition. Assume that the no-arbitrage condition holds: $(L, \mathbb{F})$ is a non-monotonic Lévy process. Then we have

$$\mathscr{M}_e(T) \cap \mathscr{M}_f(T) \cap \mathscr{M}_l(T) \neq \emptyset. \tag{4.9}$$

Proof. Indeed, the Esscher martingale measure $\mathbb{Q}^E$ with respect to $\mathbb{Q}$ is a martingale measure equivalent to $\mathbb{P}$, has finite entropy with respect to $\mathbb{P}$ and is preserving the Lévy property. □

Note, that usually (4.9) is used as an assumption, in particular, in Esche & Schweizer (2005) and Frittelli (2000), but now we see that it is sufficient just to assume the no-arbitrage condition.

4.4 The EMM is the MEMM

Our next aim is to show that the EMM, if it exists, is the MEMM in the class $\mathscr{M}_a(T)$.

4.7 Theorem. *The EMM $\mathbb{Q}^E$, if it exists, is the MEMM in the class $\mathscr{M}_a(T)$.*

Proof. Assume that there exists the EMM $\mathbb{Q}^E$. From the above we know that $I(\mathbb{Q}^E, \mathbb{P}) < +\infty$. It means that the class $\mathscr{M}_f(T)$ is not empty. Let $\mathbb{Q}$

be from the class $\mathscr{M}_f(T)$. We can estimate the relative entropy $I(\mathbb{Q},\mathbb{P})$:

$$\begin{aligned} I(\mathbb{Q},\mathbb{P}) &= \mathbb{E}_{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^E}(\mathscr{F}_T)+\log\frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] \qquad (4.10) \\ &= I(\mathbb{Q},\mathbb{Q}^E)+\mathbb{E}_{\mathbb{Q}}[\kappa L_T - T\log(\varphi_1(\kappa))]. \qquad (4.11) \end{aligned}$$

Note that $\mathbb{Q}^E \sim \mathbb{P}$ on $\mathscr{F}_T$ and hence $\mathbb{Q} \ll \mathbb{Q}^E$ on $\mathscr{F}_T$. At step from (4.10) to (4.11) we used the linearity of the integral. Note that the two summands of the integrand in (4.10) are $\mathbb{Q}$-integrable. For the second one, it follows by direct calculation. For the first one, we have

$$\log \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{Q}^E(F_T) = \log \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}(F_T) - \log \mathrm{d}\mathbb{Q}^E/\mathrm{d}\mathbb{P}(F_T),$$

both members of the right hand side are $\mathbb{Q}$-integrable. Hence linearity can be applied for concluding (4.11) from (4.10).

Taking into account that by assumption the process L is a $\mathbb{Q}$-martingale, we conclude

$$\begin{aligned} I(\mathbb{Q},\mathbb{P}) &= I(\mathbb{Q},\mathbb{Q}^E)+\mathbb{E}_Q[\kappa L_T]-T\log(\varphi_1(\kappa)) \\ &= I(\mathbb{Q},\mathbb{Q}^E)-T\log(\varphi_1(\kappa)) \\ &\geq -T\log(\varphi_1(\kappa)) = I(\mathbb{Q}^E,\mathbb{P}), \end{aligned}$$

where the last equality comes from (4.6). The statement is proven. □

Note that in the proof of the last statement, instead of the martingale property of the measure $\mathbb{Q}$, it is used just the moment condition $\mathbb{E}_{\mathbb{Q}}[X_T]=0$. Therefore, the following corollary holds:

4.8 Corollary. The EMM $\mathbb{Q}^E$, if it exists, is the MEMM in the class $\widetilde{\mathscr{M}_a}(T)$.

It is possible to generalize the result of the last theorem for the class $\mathscr{M}_a^{loc}(T)$.

4.9 Theorem. *The EMM $\mathbb{Q}^E$ is the MEMM in the class $\mathscr{M}_a^{loc}(T)$.*

Proof. Assume that there exists the EMM $\mathbb{Q}^E$. Then $\mathbb{Q}^E \in \mathscr{M}_a(T) \subseteq \mathscr{M}_a^{loc}(T)$ and hence the class $\mathscr{M}_a^{loc}(T)$ is not empty. Let $\mathbb{Q}$ be from the class $\mathscr{M}_a^{loc}(T)$. It means that there exists a localizing sequence $(\rho_n)_{n\geq 1}$ for the process L such that $\rho_n \uparrow +\infty$ and L^{ρ_n} is a $\mathbb{Q}$-martingale on $[0,T]$ for any

$n \geq 1$. Now we estimate the ρ_n-entropy of the measure $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathscr{F}_T$:

$$\begin{aligned} I_{\rho_n \wedge T}(\mathbb{Q}, \mathbb{P}) &:= I_{\mathscr{F}_{\rho_n \wedge T}}(\mathbb{Q}, \mathbb{P}) \\ &= \mathbb{E}_Q \left[\log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_{\rho_n \wedge T}) \right]. \end{aligned}$$

It is worth to mention that by definition

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_{\rho_n \wedge T}) = \mathbb{E}_Q \left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) | \mathscr{F}_{\rho_n \wedge T} \right].$$

As in the previous theorem, we would like to use at this step the linearity of the integral and split the expression of the relative entropy into two parts

$$\begin{aligned} \mathbb{E}_Q \left[\log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_{\rho_n \wedge T}) \right] &= \mathbb{E}_Q \left[\log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^E}(\mathscr{F}_{\rho_n \wedge T}) \right] \\ &\quad + \mathbb{E}_Q \left[\log \frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_{\rho_n \wedge T}) \right]. \end{aligned} \tag{4.12}$$

The first term from the right is the ρ_n-entropy of the measure $\mathbb{Q}$ with respect to the EMM $\mathbb{Q}^E$ on $\mathscr{F}_T$ and therefore it is nonnegative. Now we have to calculate the second term. Note that

$$Z_t^\kappa = \frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_t) = \exp(\kappa X_t - t \log \varphi_1(\kappa)) \quad \mathbb{P}\text{-a.s., } t \in [0, T].$$

This is because the right-hand side is a martingale as well as the left-hand side and the equality holds for $t = T$. Using Doob's optional sampling theorem we obtain

$$\begin{aligned} \frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_{\rho_n \wedge T}) &= \mathbb{E} \left[\frac{\mathrm{d}\mathbb{Q}^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) | \mathscr{F}_{\rho_n \wedge T} \right] \\ &= \mathbb{E}[Z_T^\kappa | \mathscr{F}_{\rho_n \wedge T}] \\ &= Z_{\rho_n \wedge T}^\kappa \\ &= \exp(\kappa L_{\rho_n \wedge T} - (\rho_n \wedge T) \log \varphi_1(\kappa)), \end{aligned} \tag{4.13}$$

where $\varphi_1(\kappa)$ is the cumulant function taken at time $t = 1$ and the Esscher parameter κ. From (4.12) and (4.13) we can now conclude

$$\begin{aligned} I_{\rho_n \wedge T}(\mathbb{Q}, \mathbb{P}) &\geq \mathbb{E}_{\mathbb{Q}}[\kappa L_{\rho_n \wedge T} - (\rho_n \wedge T) \log \varphi_1(\kappa)] \\ &= \mathbb{E}_{\mathbb{Q}}[\kappa L_{\rho_n \wedge T}] - \mathbb{E}_{\mathbb{Q}}[(\rho_n \wedge T)] \log \varphi_1(\kappa). \end{aligned}$$

Recall that $\mathbb{Q} \in \mathscr{M}_a^{loc}(T)$ therefore $\mathbb{E}_{\mathbb{Q}}[\kappa L_{\rho_n \wedge T}] = 0$ and we get

$$I_{\rho_n \wedge T}(\mathbb{Q}, \mathbb{P}) \geq -\mathbb{E}_{\mathbb{Q}}[(\rho_n \wedge T)] \log \varphi_1(\kappa). \tag{4.14}$$

In view of $\mathbb{E}[Z_t^\kappa \log Z_t^\kappa] < +\infty$, $t \in [0, T]$, and that $(Z^\kappa, \mathbb{F})$ is a $\mathbb{P}$-martingale on $[0, T]$ we observe that $(Z^\kappa \log Z^\kappa, \mathbb{F})$ is a $\mathbb{P}$-submartingale on $[0, T]$. Using again Doob's optional sampling theorem we get

$$\begin{aligned} I_T(\mathbb{Q}, \mathbb{P}) &= \mathbb{E}[Z_T^\kappa \log Z_T^\kappa] \\ &\geq \mathbb{E}[Z_{\rho_n \wedge T}^\kappa \log Z_{\rho_n \wedge T}^\kappa] \\ &= I_{\rho_n \wedge T}(\mathbb{Q}, \mathbb{P}) \\ &\geq -\mathbb{E}_{\mathbb{Q}}[\rho_n \wedge T] \log \varphi_1(\kappa), \end{aligned}$$

where the last inequality follows from (4.14). Passing to the limit $n \to \infty$ yields

$$I(\mathbb{Q}, \mathbb{P}) \geq -T \log \varphi_1(\kappa).$$

On the other side, in Lemma 4.3 we have calculated

$$I_T(\mathbb{Q}^E, \mathbb{P}) = \log c = -T \log \varphi_1(\kappa)$$

and consequently

$$I_T(\mathbb{Q}, \mathbb{P}) \geq I_T(\mathbb{Q}^E, \mathbb{P})$$

for all $\mathbb{Q} \in \mathscr{M}_a^{loc}(T)$. But $\mathbb{Q}^E \in \mathscr{M}_a(T) \subseteq \mathscr{M}_a^{loc}(T)$ and hence $\mathbb{Q}^E$ is the MEMM in $\mathscr{M}_a^{loc}(T)$ (and also in $\mathscr{M}_a(T)$). The theorem is proven. □

Our next aim (and the main goal of the thesis) is to show the coincidence of the EMM and the MEMM for the models driven by a linear Lévy process. More precisely, we would like to prove the EMM exists if and only if the MEMM exists and if one (hence both) of these conditions is satisfied then we have coincidence of these probability measures.

At first it is necessary to make some preparations. The basic idea is the following: We construct approximation sequences of probability measures $(\mathbb{P}_n)$ via their densities $(Z_T^{(n)})$ with respect to the original probability measure $\mathbb{P}$ in such a way that there always exists the EMM $\mathbb{Q}_n^E$ with respect to $\mathbb{P}_n$. Furthermore, we show that the infimum of relative entropy $I(Q, P)$ where $\mathbb{Q}$ is running through the class of martingale measures $\mathscr{M}_a(T)$ coincides with the upper limit of relative entropy of the approximating sequence $I(\mathbb{Q}_n^E, \mathbb{P})$. Using the connection with the one-step model we get the equivalence of the existence of the EMM and the MEMM and there equality for the models driven by the linear Lévy process. Finally we state some important corollaries.

4.5 Approximation

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let us introduce a Lévy process $(L, \mathbb{F})$ with characteristic triplet (b, σ, ν). The jump measure of the process L is denoted by $\mu = \mu^L$.

Let μ^p be the compensator of μ:

$$\mu^p((0,t] \times B) = t\nu(B), \quad t \geq 0, \quad B \in \mathscr{B}(\mathbb{R}),$$

where ν is the Lévy measure of the process L. In other words, μ^p is the product measure $\lambda \otimes \nu$ where λ denotes the Lebesgue measure on $\mathbb{R}_+$.

The aim is to construct such a sequence of probability densities $(Z_T^{(n)})$ that preserves the Lévy property (i.e., the process $LZ^{(n)}$ is a Lévy process with respect to the measure $\mathbb{P}$) and at the same time allows us to build the EMM with respect to every probability measure $\mathbb{P}^n$ of the sequence. Another important property, which we would like to have, is the equivalence of $\mathbb{P}_n$ to the original measure $\mathbb{P}$.

Let us define the function Y_n as

$$Y_n(x) = \exp(-f_n(x)),\ x \in \mathbb{R},\ n \geq 1,$$

where the functions f_n satisfy the following properties:

(1) $0 \leq f_{n+1},\ f_{n+1} \leq f_n,\ \lim_{n\to\infty} f_n(x) = 0;$

(2) $\dfrac{|x|}{f_n(x)} \to 0$ when $|x| \to \infty$ for $\forall n \geq 1$
(in other words, $|x| = o(f_n(x))$ when $|x| \to \infty$);

(3) $1 - \exp(-f_n) \in L^1(\nu)$.

The family of functions (Y_n) will be used for the construction of the probability densities with the desirable properties. First of all, we observe that $Y_n - 1 \in L^2(\nu) \cap L^1(\nu)$: From (3) follows $Y_n - 1 = \exp(-f_n) - 1 \in L^1(\nu)$. Moreover,

$$(Y_n(x) - 1)^2 = (\exp(-f_n(x)) - 1)^2 \leq |(\exp(-f_n(x)) - 1)|$$

because $0 \leq |(\exp(-f_n(x)) - 1)| \leq 1$, therefore $Y_n - 1 \in L^2(\nu)$. Our observation implies $(Y_n - 1)1_{[0,t]} \in L^2(\mu^p) \cap L^1(\mu^p)$ for all $t \geq 0$. From $(Y_n - 1)1_{[0,t]} \in L^2(\mu^p)$ and the definition of the stochastic integral with respect to the compensated Poisson random measure we get that the stochastic integral

$$N_t^{(n)} = (Y_n - 1) * (\mu - \mu^p)_t, \quad t \geq 0,$$

is well-defined and is a square integrable martingale, moreover, for $t \in [0,T]$ and $n \geq 1$

$$\begin{aligned} N_t^{(n)} &= (Y_n - 1) * (\mu - \mu^p)_t \\ &= (Y_n - 1) * \mu_t - \int\limits_{\mathbb{R}\times[0,t]} (\exp(-f_n(x)) - 1)\mu^p(\mathrm{d}x, \mathrm{d}s) \qquad (4.15) \\ &= (Y_n - 1) * \mu_t - t\int\limits_{\mathbb{R}} (\exp(-f_n(x)) - 1)\nu(\mathrm{d}x). \qquad (4.16) \end{aligned}$$

In (4.15) we use the fact that $(Y_n - 1)1_{[0,t]} \in L^2(\mu^p) \cap L^1(\mu^p)$ for all $t \geq 0$ and hence the integral can be splitted according to Proposition II.1.28 from Jacod & Shiryaev (2000). The last part of the expression in (4.16) is a continuous function of t, the compensator of the first part which is a pure jump Lévy process. The family of density processes $(Z^{(n)})$ we introduce as stochastic exponentials (see Section 1.6, in particular Proposition 1.33) of the martingales (and Lévy processes) $N^{(n)}$:

$$\begin{aligned} Z_t^{(n)} &:= \mathscr{E}(N^{(n)})_t \qquad (4.17) \\ &= \exp(N_t^{(n)}) \prod_{0<u\leq t} (1 + \Delta N_u^{(n)})e^{-\Delta N_u^{(n)}}, \ t \geq 0. \qquad (4.18) \end{aligned}$$

Let us calculate $\prod_{0<u\leq t} (1 + \Delta N_u^{(n)})e^{-\Delta N_u^{(n)}}$:

$$\begin{aligned} &\prod_{0<u\leq t} (1 + \Delta N_u^{(n)})e^{-\Delta N_u^{(n)}} \\ &= \prod_{0<u\leq t} (1 + \Delta((Y_n - 1) * \mu_u))e^{-\Delta((Y_n-1)*\mu_u)} \\ &= \left(\prod_{0<u\leq t} (1 + \Delta((Y_n - 1) * \mu_u))\right) \exp\left(\sum_{0<u\leq t} -\Delta((Y_n - 1) * \mu_u)\right) \\ &= \prod_{0<u\leq t} (1 + (Y_n(\Delta L_u) - 1)) \exp(-(Y_n - 1) * \mu_t) \\ &= \exp\left(- \sum_{0<u\leq t} (f_n(\Delta L_u)) - (Y_n - 1) * \mu_t\right). \end{aligned}$$

We still need to prove the convergence of $\sum_{0<u\leq t} (f_n(\Delta L_u))$, but it follows from

the fact that $1-\exp(-f_n)\in L^1(\nu)$. Indeed, let us define the function h as

$$h(x)=\frac{1-e^{-x}}{x\wedge 1},\quad x\in(0,+\infty]\,,$$

which is strictly positive and continuous on the set $(0,+\infty]$. Moreover,

$$\lim_{x\downarrow 0}h(x)=-(e^{-x})'_{x=0}=1.$$

Hence the function

$$\bar{h}(x)=\begin{cases} h(x), & x\in(0,+\infty]\,,\\ 1, & x=0,\end{cases}$$

is strictly positive and continuous on the compact set $[0,+\infty]$. Consequently,

$$\inf_{x\in[0,+\infty]}\bar{h}(x)=c>0.$$

From this we observe

$$c(x\wedge 1)\le 1-e^{-x},\ x\ge 0,$$

and therefore

$$c(f_n(x)\wedge 1)\le 1-e^{-f_n(x)},\ x\in\mathbb{R}.$$

Hence, if $1-\exp(-f_n)\in L^1(\nu)$ then $f_n\wedge 1\in L^1(\nu)$. Since we assume (3) the expression $\sum_{0<u\le t} f_n(\Delta L_u)=f_n*\mu_t$ converges for $t\ge 0$. Coming back to the stochastic exponential $Z^{(n)}$, we can write

$$\begin{aligned}
Z_t^{(n)} &:= \mathscr{E}(N^{(n)})_t\\
&= \exp(N_t^{(n)})\prod_{0<u\le t}(1+\Delta N_u^{(n)})e^{-\Delta N_u^{(n)}}\\
&= \exp\left((Y_n-1)*(\mu-\mu^p)_t\right)\exp\left(-(Y_n-1)*\mu_t-\sum_{o<u\le t}f_n(\Delta L_u)\right)\\
&= \exp\left(-\sum_{0<u\le t}f_n(\Delta L_u)-(Y_n-1)*\mu_t^p\right)\\
&= \exp\left(-\sum_{0<u\le t}f_n(\Delta L_u)+t\int_{\mathbb{R}}(1-\exp(-f_n(x)))\nu(\mathrm{d}x)\right).
\end{aligned}$$

As a result, we can conclude the following representation of the density process $Z^{(n)}$:

$$Z_t^{(n)} = \exp\left(- \sum_{0<u\leq t} f_n(\Delta L_u) + t \int_{\mathbb{R}} (1 - \exp(-f_n(x)))\nu(\mathrm{d}x) \right). \tag{4.19}$$

Since $(N^{(n)}, \mathbb{F})$ are martingales, because of Theorem 1.35 the processes $(Z^{(n)}, \mathbb{F})$ are also local martingales. To confirm that $(Z^{(n)}, \mathbb{F})$ are indeed density processes it is sufficient to show that they are (nonnegative) martingales and the expectation of $Z_t^{(n)}$ is equal to 1. Using the monotonicity of f_n in n we have

$$\begin{aligned} Z_t^{(n)} &\leq \exp\left(t \int_{\mathbb{R}} (1 - \exp(-f_n(x)))\nu(\mathrm{d}x) \right) \\ &\leq \exp\left(t \int_{\mathbb{R}} (1 - \exp(-f_1(x)))\nu(\mathrm{d}x) \right) < +\infty \end{aligned}$$

hence

$$\sup_{\substack{0\leq t\leq T \\ n\geq 1}} Z_t^{(n)} \leq \exp\left(T \int_{\mathbb{R}} (\exp(-f_1(x)) - 1)\nu(\mathrm{d}x) \right) < +\infty. \tag{4.20}$$

Therefore $(Z^{(n)}, \mathbb{F})$ are local martingales bounded on every finite interval $[0, T]$ and hence are martingales. Therefore $\mathbb{E}[Z_t^{(n)}] = \mathbb{E}[Z_0^{(n)}]$ and by definition of $Z^{(n)}$ we have $Z_0^{(n)} = 1$. Hence $\mathbb{E}[Z_t^{(n)}] = 1$ for all $t \geq 0$.

We define the sequence of probability measures $\mathbb{P}_n$ using their densities with respect to the measure $\mathbb{P}$:

$$\mathrm{d}\mathbb{P}_n = Z_T^{(n)} \mathrm{d}\mathbb{P} \text{ on } \mathscr{F}_T. \tag{4.21}$$

Now we are going to show that such measures preserve the Lévy property of the process L:

4.10 Proposition. $(L, \mathbb{F})$ is a $\mathbb{P}_n$-Lévy process on $[0, T]$.

Proof. Let $A \in \mathscr{F}_t$. We calculate

$$\begin{aligned} \mathbb{P}_n\left(A \cap \{L_{t+s} - L_t \in B\}\right) &= \mathbb{E}\left[Z_T^{(n)} 1_A 1_{\{L_{t+s}-L_t\in B\}} \right] \\ &= \mathbb{E}\left[Z_{t+s}^{(n)} 1_A 1_{\{L_{t+s}-L_t\in B\}} \right], \end{aligned}$$

because $(Z^{(n)}, \mathbb{F})$ is a martingale and we can condition with respect to $\mathscr{F}_{t+s}$ inside the expectation. Now we can write

$$\mathbb{E}\left[Z_{t+s}^{(n)} 1_A 1_{\{L_{t+s}-L_t \in B\}}\right] = \mathbb{E}\left[Z_t^{(n)} 1_A \frac{Z_{t+s}^{(n)}}{Z_t^{(n)}} 1_{\{L_{t+s}-L_t \in B\}}\right]$$

and we have

$$\begin{aligned} \frac{Z_{t+s}^{(n)}}{Z_t^{(n)}} &= \exp\left(-\sum_{t<u\leq t+s} f_n(\Delta L_u) + s\int_{\mathbb{R}} (1-\exp(-f_n(x)))\nu(\mathrm{d}x)\right) \\ &= \exp\left(-\sum_{0<u\leq s} f_n(\Delta {}^t L_u) + s\int_{\mathbb{R}} (1-\exp(-f_n(x)))\nu(\mathrm{d}x)\right), \end{aligned}$$

where ${}^tX = ({}^tX_s)_{s\geq 0}$ is defined by ${}^tL_s := L_{t+s} - L_t$, $s \geq 0$, which is again a Lévy process with characteristics (β, σ^2, ν) independent of $\mathscr{F}_t$. Thus we can write

$$\begin{aligned} &\frac{Z_{t+s}^{(n)}}{Z_t^{(n)}} 1_{\{L_{t+s}-L_t \in B\}} \\ &= \exp\left(-\sum_{0<u\leq s} f_n(\Delta {}^t L_u) + s\int_{\mathbb{R}} (1-\exp(-f_n(x)))\nu(\mathrm{d}x)\right) 1_{\{{}^tL_s \in B\}} \end{aligned}$$

which is independent of $\mathscr{F}_t$ and, consequently, independent of $Z_t^{(n)} 1_A$. From this follows

$$\begin{aligned} \mathbb{P}_n\left(A \cap \{L_{t+s} - L_t \in B\}\right) &= \mathbb{E}[Z_t^{(n)} 1_A]\, \mathbb{E}\Bigg[\exp\Bigg(-\sum_{0<u\leq s} f_n(\Delta {}^t L_u) \\ &\qquad + s\int_{\mathbb{R}} (1-\exp(-f_n(x)))\nu(\mathrm{d}x)\Bigg) 1_{\{{}^tL_s \in B\}}\Bigg] \\ &= \mathbb{E}[Z_t^{(n)} 1_A]\, \mathbb{E}[Z_s^{(n)} 1_{\{L_s \in B\}}], \end{aligned}$$

because ${}^tL$ has the same law as L with respect to $\mathbb{P}$. We also get

$$\begin{aligned} \mathbb{E}[Z_t^{(n)} 1_A]\, \mathbb{E}[Z_s^{(n)} 1_{\{L_s \in B\}}] &= \mathbb{E}[Z_T^{(n)} 1_A]\, \mathbb{E}[Z_T^{(n)} 1_{\{L_s \in B\}}] \\ &= \mathbb{P}_n(A)\, \mathbb{P}_n(\{L_s \in B\}) \end{aligned}$$

and as a result we get

$$\mathbb{P}_n\left(A \cap \{L_{t+s} - L_t \in B\}\right) = \mathbb{P}_n(A)\,\mathbb{P}_n(\{L_s \in B\}). \tag{4.22}$$

Putting $A = \Omega$ yields

$$\mathbb{P}_n\left(\{L_{t+s} - L_t \in B\}\right) = \mathbb{P}_n(\{L_s \in B\}),\ s, t \geq 0,$$

hence L has homogeneous increments with respect to $\mathbb{P}_n$. Inserting this in (4.22) we obtain

$$\mathbb{P}_n\left(A \cap \{L_{t+s} - L_t \in B\}\right) = \mathbb{P}_n(A)\,\mathbb{P}_n(\{L_{t+s} - L_t \in B\})$$

for all $A \in \mathscr{F}_t$ and $B \in \mathscr{B}(\mathbb{R})$. This means that $L_{t+s} - L_t$ is independent of $\mathscr{F}_t$ for all $s, t \geq 0$. Finally, $L_0 = 0$ and L is càdlàg $\mathbb{P}_n$-a.s. since $\mathbb{P}_n \ll \mathbb{P}$. Hence $(L, \mathbb{F})$ is a Lévy process with respect to $\mathbb{P}_n$. □

Our next goal is to investigate the characteristics of the Lévy process $(L, \mathbb{F})$ with respect to the new measures $\mathbb{P}_n$.

4.11 Proposition. With respect to $\mathbb{P}_n$, the Lévy process $(L, \mathbb{F})$ has characteristics (β, σ^2, ν_n) with Lévy measure given by $\nu_n(\mathrm{d}x) = \exp(-f_n(x))\nu(\mathrm{d}x)$.

Proof. Since $(L, \mathbb{F})$ is a $\mathbb{P}_n$-Lévy process its jump measure μ is a Poisson random measure with intensity measure μ_n^p defined by

$$\mu_n^p(A) = \mathbb{E}_{P_n}[\mu(A)], \quad A \in \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathbb{R}),$$

which is given by $\mu_n^p = \lambda_+ \otimes \nu_n$. Here λ_+ denotes the Lebesgue measure on $\mathbb{R}_+$ and ν_n the Lévy measure of $(L, \mathbb{F})$ with respect to $\mathbb{P}_n$. From this immediately follows

$$\nu_n(B) = \mathbb{E}_{P_n}\mu([0,1] \times B),\ B \in \mathscr{B}(\mathbb{R}).$$

Using this formula we calculate ν_n. For this let $B \in \mathscr{B}(\mathbb{R})$ such that $\nu(B) < +\infty$. By the definition of $\mathbb{P}_n$ we get

$$\begin{aligned}
&\mathbb{E}_{P_n}[\exp(-\mu([0,1] \times B))] = \mathbb{E}[Z_1^{(n)} \exp(-\mu([0,1] \times B))] \\
&\quad = \mathbb{E}\left[\exp\left(-\sum_{0<u\leq 1} f_n(\Delta L_u) + \int_{\mathbb{R}} (1 - \exp(-f_n(x)))\nu(\mathrm{d}x) - \mu([0,1] \times B)\right)\right] \\
&\quad = \exp\left(\int_{\mathbb{R}} (1 - \exp(-f_n(x)))\nu(\mathrm{d}x)\right)\mathbb{E}[\exp(-(f_n + 1_B) * \mu_1)]. \qquad (4.23)
\end{aligned}$$

Using the exponential formula for the Poisson random measure μ with respect to $\mathbb{P}$ (cf. Kallenberg (1997), Lemma 10.2) we get

$$\mathbb{E}[\exp(-(f_n + 1_B) * \mu_1)] = \exp(\mu^p(\exp[-(f_n + 1_B)1_{[0,1]}] - 1)), \qquad (4.24)$$

where $\mu^p = \lambda_+ \otimes \nu$ is the intensity measure of μ with respect to $\mathbb{P}$. Now we observe

$$\begin{aligned}
&\mu^p(\exp[-(f_n + 1_B)1_{[0,1]}] - 1) \\
&= \int_{[0,1]\times\mathbb{R}} [\exp(-f_n(x) - 1_B(x)) - 1](\lambda_+ \otimes \nu)(\mathrm{d}s, \mathrm{d}x) \\
&= \int_{\mathbb{R}} [\exp(-f_n(x) - 1_B(x)) - 1]\nu(\mathrm{d}x). \qquad (4.25)
\end{aligned}$$

Pasting together (4.23) – (4.25) yields

$$\begin{aligned}
&\mathbb{E}_{P_n}[\exp(-\mu([0,1]) \times B)] \\
&= \exp\left(\int_{\mathbb{R}} (1 - \exp(-f_n(x)))\nu(\mathrm{d}x) + \int_{\mathbb{R}} [\exp(-f_n(x) - 1_B(x)) - 1]\nu(\mathrm{d}x)\right) \\
&= \exp\left(\int_{\mathbb{R}} [\exp(-1_B(x)) - 1] \exp(-f_n(x))\nu(\mathrm{d}x)\right) \\
&= \exp\left((\exp(-1) - 1) \int_B \exp(-f_n(x))\nu(\mathrm{d}x)\right). \qquad (4.26)
\end{aligned}$$

This equality means that $\mu([0,1] \times B)$ has a Poisson distribution with parameter $\int_B \exp(-f_n(x))\nu(\mathrm{d}x)$ with respect to $\mathbb{P}_n$. On the other side, $(L, \mathbb{F})$ is a Lévy process with respect to $\mathbb{P}_n$ and if we denote its Lévy measure by ν_n as above we conclude

$$\begin{aligned}
\nu_n(B) &= \mathbb{E}_{P_n}[\mu([0,1] \times B)] \qquad (4.27) \\
&= \int_B \exp(-f_n(x))\nu(\mathrm{d}x), \quad \forall B \text{ with } \nu(B) < +\infty, \qquad (4.28)
\end{aligned}$$

as expectation of the Poisson-distributed random variable $\mu([0,1] \times B)$ with parameter $\int_B \exp(-f_n(x))\nu(\mathrm{d}x)$ with respect to $\mathbb{P}_n$. If B is an arbitrary Borel subset of $\mathbb{R}$ we set

$$B_k = B \cap \{|x| \geq frm[o]--/k\}\,,\ k \geq 1.$$

Then $\nu(B_k) < +\infty$ and hence (4.28) holds for B_k:

$$\nu_n(B_k) = \int\limits_{B_k} \exp(-f_n(x))\nu(\mathrm{d}x).$$

Passing to the limit as $k \to \infty$ yields

$$\nu_n(B) = \int\limits_{B} \exp(-f_n(x))\nu(\mathrm{d}x).$$

Note that $\nu_n(\{0\}) = \nu(\{0\}) = 0$. Consequently, we have proven

$$\nu_n(\mathrm{d}x) = \exp(-f_n(x))\nu(\mathrm{d}x).$$

In the next step we calculate

$$\mathbb{E}_{P_n}[\exp(iuL_t)] = \mathbb{E}[Z_t^{(n)} \exp(iuL_t)].$$

For this we use the Lévy–Itô decomposition of L:

$$L_t = \beta t + \sigma W_t + (x1_{\{|x|\leq 1\}}) * \bar{\mu}_t + (x1_{\{|x|>1\}}) * \mu_t,$$

where $\bar{\mu} = \mu - \mu^p$ is the compensated Poisson random measure with respect to $\mathbb{P}$. Here $(W, \mathbb{F})$ is a Brownian motion independent of μ if $\sigma \neq 0$. (If $\sigma = 0$ we use the convention $W \equiv 0$.) Now we observe that $\beta t + \sigma W_t$ is independent of $Z_t^{(n)} \exp(iu[(x1_{\{|x|\leq 1\}}) * \bar{\mu}_t + (x1_{\{|x|>1\}}) * \mu_t])$ and hence

$$\begin{aligned}
&\mathbb{E}_{P_n}[\exp(iuL_t)] \\
&\quad = \mathbb{E}[\exp(iu\beta t + iu\sigma W_t)]\, \mathbb{E}_{P_n}[\exp(iu[(x1_{\{|x|\leq 1\}}) * \bar{\mu}_t + (x1_{\{|x|>1\}}) * \mu_t])] \\
&\quad = \exp(iu\beta t - \frac{1}{2}u^2\sigma^2 t)\, \exp\left(t \int\limits_{\mathbb{R}} (\exp(iux) - 1 - iux1_{\{|x|\leq 1\}}) \right) \nu_n(\mathrm{d}x)
\end{aligned}$$

because $(x1_{\{|x|\leq 1\}}) * \bar{\mu}_t + (x1_{\{|x|>1\}}) * \mu_t)$ is a Lévy process with characteristics $(0, 0, \nu_n)$ with respect to $\mathbb{P}_n$. This follows from the fact proven above that ν_n is the Lévy measure of $(L, \mathbb{F})$ with respect to $\mathbb{P}_n$. Hence

$$\mathbb{E}_{P_n}[\exp(iuL_t)] = \exp\left(iu\beta t - \frac{1}{2}u^2\sigma^2 t + t \int\limits_{\mathbb{R}} (\exp(iux) - 1 - iux1_{\{|x|\leq 1\}})\nu_n(\mathrm{d}x) \right)$$

and by the uniqueness of the Lévy–Chintchine representation we get that $(L, \mathbb{F})$ is a Lévy process with characteristics (β, σ^2, ν_n) with respect to $\mathbb{P}_n$ where we have $\nu_n(\mathrm{d}x) = \exp(-f_n(x))\nu(\mathrm{d}x)$. □

Now it is time to explain the role of condition (2) on the sequence of functions f_n. Let us fix n.

4.12 Proposition. Let $\mathbb{P}_n$ be the probability measure defined in (4.21) and assume that the function f_n satisfies the conditions (1) – (3). Then L has finite exponential moments of all orders with respect to $\mathbb{P}_n$.

Proof. From the last proposition we already know that the process L is a Lévy process with Lévy measure $\nu_n(\mathrm{d}x) = \exp(-f_n(x))\nu(\mathrm{d}x)$ with respect to $\mathbb{P}_n$. Recall the necessary and sufficient conditions of the existence of finite moments $\mathbb{E}_{P_n}[g(L_t)]$ for any $t > 0$, where g is a submultiplicative (hence nonnegative), locally bounded measurable function on $\mathbb{R}$. The condition is

$$\int_{\{|x|>1\}} g(x)\nu(\mathrm{d}x) < +\infty. \tag{4.29}$$

We want to show that the condition (2) guarantees the finiteness of exponential moments of all orders: for any $\alpha \in \mathbb{R}$ and any $t \in [0, T]$ we have

$$\mathbb{E}_{P_n}[\exp(\alpha L_t)] < +\infty.$$

The first step is to prove, that the function $\exp(\alpha x)$ satisfies the conditions necessary and sufficient of existence of finite moments. This function is indeed measurable and bounded on every compact set, submultiplicativity follows from multiplicativity of the exponential function. The last step is to show that (4.29) holds:

$$\int_{\{|x|>1\}} \exp(\alpha x)\nu_n(\mathrm{d}x) \quad = \quad \int_{\{|x|>1\}} \exp(\alpha x)\exp(-f_n(x))\nu(\mathrm{d}x).$$

The Lévy measure ν is, by definition, finite outside any neighborhood of zero and, in particular, $\nu(\{|x| > 1\}) < +\infty$. Hence the condition is satisfied if $\alpha = 0$. Let us now assume that $\alpha \neq 0$. By condition (2), there exists $c > 1$ such that

$$|x| \leq \frac{1}{|\alpha|} f_n(x), \quad |x| > c,$$

and hence

$$|\alpha x| \leq f_n(x), \quad |x| > c.$$

By the linearity and monotonicity of the integral we can split

$$\begin{aligned}
&\int_{\{|x|>1\}} \exp(\alpha x - f_n(x))\nu(\mathrm{d}x) \\
&\le \int_{\{1<|x|\le c\}} \exp(\alpha x - f_n(x))\nu(\mathrm{d}x) + \int_{\{|x|>c\}} \exp(|\alpha x| - f_n(x))\nu(\mathrm{d}x) \\
&\le \int_{\{1<|x|\le c\}} \exp(\alpha x)\nu(\mathrm{d}x) + \int_{\{|x|>c\}} c\nu(\mathrm{d}x) \qquad (4.30) \\
&\le \exp(|\alpha c|)\nu(\{|x|>1\}) + c\nu(\{|x|>1\}) < +\infty.
\end{aligned}$$

In (4.30) we have used the estimate

$$\exp(|\alpha x| - f_n(x)) \le 1 \text{ for } |x| > c.$$

Hence, the process L has finite exponential moments of all orders with respect to the measure $\mathbb{P}_n$. The proposition is proven. □

Let $\mathbb{Q}_n$ be defined similarly to the definition of $\mathbb{P}_n$:

$$\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{Q}}(\mathscr{F}_T) = Z_T^{(n)},\ n \ge 1.$$

4.13 Proposition. $I_T(\mathbb{Q},\mathbb{P}) = \lim_{n\to\infty} I_T(\mathbb{Q}_n,\mathbb{P}_n)$.

Proof. By definition

$$\begin{aligned}
I_T(\mathbb{Q}_n,\mathbb{P}_n) &= \mathbb{E}_{\mathbb{Q}_n}\left[\log\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}_n}\right] \\
&= \mathbb{E}_{\mathbb{Q}_n}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right], \qquad (4.31)
\end{aligned}$$

because

$$\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}_n} = \frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{Q}}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T) = Z_T^{(n)}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(Z_T^{(n)})^{-1}(\mathscr{F}_T) = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T).$$

Now we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_n}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{Q}}(\mathscr{F}_T)\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[Z_T^{(n)}\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right].
\end{aligned}$$

By the assumption that $I_T(\mathbb{Q},\mathbb{P}) < \infty, \dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)$ is $\mathbb{Q}$-integrable and by (4.20) and (4.31), $\lim\limits_{n\to\infty} Z_T^{(n)} = 1$ and $Z_T^{(n)}$ is uniformly bounded. Combining these facts and letting n to infinity, we can use the theorem of Lebesgue:

$$\mathbb{E}_{\mathbb{Q}}\left[Z_T^{(n)} \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] \longrightarrow \mathbb{E}_{\mathbb{Q}}\left[\log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] = I_T(\mathbb{Q},\mathbb{P}).$$

Hence,

$$\lim_{n\to\infty} I_T(\mathbb{Q}_n,\mathbb{P}_n) = I_T(\mathbb{Q},\mathbb{P}).$$

The statement is proven. □

4.6 The Esscher Martingale Measures with Respect to $\mathbb{P}_n$

Let us introduce the EMM $\mathbb{Q}_n^E$ with respect to the probability measure $\mathbb{P}_n$:

$$\frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T) = c_n \exp(\kappa_n X_T),$$

where κ_n is such that the moment condition

$$\mathbb{E}_{\mathbb{P}_n}[L_T \exp(\kappa_n L_T)] = 0$$

is satisfied, c_n is the normalizing constant:

$$c_n = \left(\mathbb{E}_{\mathbb{P}_n}[\exp(\kappa_n L_T)]\right)^{-1}.$$

We know from Proposition 4.12 that the condition (2) on the functions f_n guarantees the finiteness of all exponential moments of L under the probability measure $\mathbb{P}_n$ and therefore, because of Proposition 4.4, the parameters c_n and κ_n always exist. In the next proposition, we compare the asymptotic behavior of measures $\mathbb{Q}_n^E$ and $\mathbb{P}_n$.

4.14 Proposition.

$$\overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E,\mathbb{P}) \le \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E,\mathbb{P}_n).$$

Proof. Let us consider the relative entropy between the measures $\mathbb{Q}_n^E$ and $\mathbb{P}$:

$$\begin{aligned}
I_T(\mathbb{Q}_n^E, \mathbb{P}) &= \mathbb{E}_{\mathbb{Q}_n^E}\left[\log \frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] \\
&= \mathbb{E}_{Q_n^E}\left[\log \frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T) + \log \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] \qquad (4.32) \\
&= I_T(\mathbb{Q}_n^E, \mathbb{P}_n) + \mathbb{E}_{\mathbb{Q}_n^E}\left[\log \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right] \qquad (4.33) \\
&= I_T(\mathbb{Q}_n^E, \mathbb{P}_n) + \mathbb{E}_{\mathbb{Q}_n^E}\left[\log Z_T^{(n)}\right].
\end{aligned}$$

Note that $I_T(\mathbb{Q}_n^E, \mathbb{P}_n)$ is nonnegative (by definition) and finite:

$$\begin{aligned}
0 \le I_T(\mathbb{Q}_n^E, \mathbb{P}_n) &= \mathbb{E}_{\mathbb{Q}_n^E}\left[\log \frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T)\right] \\
&= \mathbb{E}_{\mathbb{Q}_n^E}(\log c_n + \kappa_n L_T) \\
&= \log c_n < +\infty.
\end{aligned}$$

The expression $\log \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) = \log Z_T^{(n)}$ is bounded from above by the constant

$$T \int_{\mathbb{R}} 1 - \exp(-f_n(x))\nu(\mathrm{d}x).$$

Hence $\mathbb{E}_{Q_n^E}\left[\log \frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T)\right]$ exists and is less than $+\infty$. Therefore we are allowed to use the linearity of the integral and to split the expression in (4.32) into on two parts in (4.33). Now, by Lebesgue's theorem

$$\overline{\lim_{n\to\infty}} \mathbb{E}_{Q_n^E}\left[\log Z_T^{(n)}\right] \le \overline{\lim_{n\to\infty}} T \int_{\mathbb{R}} (1 - \exp(-f_n(x)))\nu(\mathrm{d}x) = 0,$$

and finally we get

$$\overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}) \le \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}_n).$$

The proposition is proven. □

Our next goal is to compare the asymptotics of $I_T(\mathbb{Q}_n, \mathbb{P}_n)$ and $I_T(\mathbb{Q}_n^E, \mathbb{P}_n)$. Recall that $\mathbb{Q}_n$ are defined similarly to the definition of $\mathbb{P}_n$:

$$\frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{Q}}(\mathscr{F}_T) = Z_T^{(n)},\ n \ge 1.$$

But to get a better result we need to state additional assumptions on the probability measure $\mathbb{Q}$.

4.15 Proposition. Let $\mathbb{Q}$ be from $\widetilde{\mathscr{M}}_a(T)$: $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}[L_T] = 0$. Then

$$\overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}_n) \leq \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n, \mathbb{P}_n).$$

Proof. Let us start from the right hand side:

$$\begin{aligned}
I_T(\mathbb{Q}_n, \mathbb{P}_n) &= \mathbb{E}_{\mathbb{Q}_n}\left[\log \frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T)\right] &(4.34)\\
&= \mathbb{E}_{\mathbb{Q}_n}\left[\log \frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{Q}_n^E}(\mathscr{F}_T) + \log \frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T)\right] &(4.35)\\
&= \mathbb{E}_{\mathbb{Q}_n}\left[\log \frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{Q}_n^E}(\mathscr{F}_T)\right] + \mathbb{E}_{\mathbb{Q}_n}\left[\log \frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T)\right] &(4.36)\\
&= I_T(\mathbb{Q}_n, \mathbb{Q}_n^E) + \log c_n + \mathbb{E}_{\mathbb{Q}_n}[\kappa_n L_T] &(4.37)\\
&\geq \log c_n + \mathbb{E}_{Q_n}[\kappa_n L_T], &(4.38)
\end{aligned}$$

where $\log c_n = I(\mathbb{Q}_n^E, \mathbb{P}_n)$. For step from (4.35) to (4.36) we used the linearity of the integral. We are allowed to use it because

$$\mathbb{E}_{\mathbb{Q}_n}\left[\log \frac{\mathrm{d}\mathbb{Q}_n}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T)\right] = I_T(\mathbb{Q}_n, \mathbb{Q}_n^E) \geq 0$$

and

$$\mathbb{E}_{\mathbb{Q}_n}\left[\left|\log \frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T)\right|\right] < +\infty. \tag{4.39}$$

Indeed, for proving the second claim, we estimate

$$\begin{aligned}
\left|\log \frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T)\right| &= |\log[c_n \exp(\kappa_n L_T)]|\\
&\leq |\log c_n| + |\kappa_n L_T|
\end{aligned}$$

for some $c_n > 0$ and $\kappa \in \mathbb{R}$. Now we observe

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_n}[|L_T|] &= \mathbb{E}_{\mathbb{Q}}[Z_T^{(n)}|L_T|]\\
&\leq K\,\mathbb{E}_{\mathbb{Q}}[|L_T|]
\end{aligned}$$

where K is the constant given on the right-hand side of (4.20). But in view of $\mathbb{E}_{\mathbb{Q}}[L_T] = 0$ by the assumption of the proposition it follows that $\mathbb{E}_{\mathbb{Q}_n}[|L_T|] < +\infty$. Pasting together the above estimates yields the claim (4.39).

In (4.37) we have used the fact that the entropy $I_T(\mathbb{Q}_n, \mathbb{Q}_n^E)$ is nonnegative. For proving the statement of Proposition 4.15, it now suffices to verify that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\kappa_n L_T] = 0.$$

By definition of the EMM $\mathbb{Q}_n^E$ we have:

$$\mathbb{E}_{\mathbb{P}_n}\left[L_T \exp(\kappa_n L_T)\right] = 0.$$

We now introduce the function $\varphi_{T,n}$ by

$$\varphi_{T,n}(\kappa) = \mathbb{E}_{\mathbb{P}_n}[\exp(\kappa L_T)] < +\infty,\ \kappa \in \mathbb{R}.$$

Then $\varphi_{T,n}$ is twice continuously differentiable (by Proposition A.1, parts (2),(3),(6) and (7))and it is possible to write explicit forms of the derivatives

$$\varphi'_{T,n}(\kappa) = \mathbb{E}_{\mathbb{P}_n}[L_T \exp(\kappa L_T)],\ \varphi''_{T,n}(\kappa) = \mathbb{E}_{\mathbb{P}_n}[L_T^2 \exp(\kappa L_T)],\ \kappa \in \mathbb{R}$$

(see Appendix A). From the definition of κ_n now follows $\varphi'_{T,n}(\kappa_n) = 0$ and, because of $\varphi''_{T,n}(\kappa_n) > 0$, $\varphi_{T,n}$ reaches its minimum at the point κ_n.

We define the sets K_n by

$$K_n := \left\{ \kappa \in \mathbb{R} :\ \varphi_{T,n}(\kappa) \exp\left(-T \int_{\mathbb{R}} (1 - \exp(-f_n))\nu(\mathrm{d}x) \right) \leq 1 \right\}.$$

We notice that

$$\begin{aligned}
&\varphi_{T,n}(\kappa) \exp\left(-T \int_{\mathbb{R}} (1 - \exp(-f_n))\nu(\mathrm{d}x) \right) \\
&= \mathbb{E}_{\mathbb{P}_n}[\exp(\kappa L_T)]\ \exp\left(-T \int_{\mathbb{R}} (1 - \exp(-f_n))\nu(\mathrm{d}x) \right) \\
&= \mathbb{E}_{\mathbb{P}}[Z_T^{(n)} \exp(\kappa L_T)]\ \exp\left(-T \int_{\mathbb{R}} (1 - \exp(-f_n))\nu(\mathrm{d}x) \right) \\
&= \mathbb{E}_{\mathbb{P}}\left[\exp\left(- \sum_{0<u\leq T} f_n(\Delta L_u) \right) \exp(\kappa L_T) \right],
\end{aligned}$$

where we have used the representation (4.19) for the density process $Z^{(n)}$. The last term being monotonically increasing in n shows that the sets K_n are monotonically decreasing and, in particular, that $K_n \subseteq K_1$ for all $n \geq 1$. The function φ_n is strictly positive, $\varphi_n(0) = 1$, hence its minimum is not larger then 1 and $\kappa_n \in K_n \subseteq K_1$. We can easily observe that K_1 is compact:

(a) K_1 is bounded because $\lim_{|\kappa|\to\infty} \varphi_1(\kappa) = +\infty$ (Proposition A.1, part (1));

(b) K_1 is closed because φ_1 is continuous (Proposition A.1, part (2)).

This implies that the sequence (κ_n) is bounded and therefore we can estimate

$$|\mathbb{E}_{\mathbb{Q}_n}[\kappa_n L_T]| \leq \sup_{k\geq 1} |\kappa_k|\, |\mathbb{E}_{\mathbb{Q}_n}[L_T]|.$$

But

$$\mathbb{E}_{\mathbb{Q}_n}[L_T] = \mathbb{E}_{\mathbb{Q}}[Z_T^{(n)} L_T]$$

and using Lebesgue's theorem on dominating convergence we obtain that the right-hand side converges to $\mathbb{E}_{\mathbb{Q}}[L_T] = 0$. Indeed, we have

$$\lim_{n\to\infty} Z_T^{(n)} L_T = L_T$$

and

$$|Z_T^{(n)} L_T| \leq K|L_T|$$

with a constant $K > 0$ given in (4.20). Obviously, the right-hand side $K|L_T|$ is integrable with respect to $\mathbb{Q}$. In summary, we can conclude

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\kappa_n L_T] = 0.$$

Taking the upper limit in (4.38) and noting that $\log c_n = I_T(\mathbb{Q}_n^E, \mathbb{P}_n)$ we obtain the required result. □

In the proof of the last proposition, an important fact was proven, which we would like to use also later and formulate it as a corollary:

4.16 Corollary. The sequence of Esscher parameters $(\kappa_n)_{n\geq 1}$ is bounded.

Summarizing Proposition 4.13, Proposition 4.14 and Proposition 4.15 we get the following statement.

4.17 Proposition. The following identity holds:

$$\overline{\lim_{n\to\infty}}\, I_T(\mathbb{Q}_n^E, \mathbb{P}) = \inf_{\mathbb{Q}\in\mathscr{M}_a(T)} I_T(\mathbb{Q}, \mathbb{P}) = \inf_{\mathbb{Q}\in\widetilde{\mathscr{M}}_a(T)} I_T(\mathbb{Q}, \mathbb{P}).$$

Proof. From Proposition 4.14 we have

$$\overline{\lim_{n\to\infty}}\, I_T(\mathbb{Q}_n^E, \mathbb{P}) \leq \overline{\lim_{n\to\infty}}\, I_T(\mathbb{Q}_n^E, \mathbb{P}_n).$$

Applying Proposition 4.15 we get

$$\overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}_n) \leq \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n, \mathbb{P}_n)$$

for any $\mathbb{Q} \in \widetilde{\mathscr{M}_a}(T)$, but from Proposition 4.13 it follows

$$\overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n, \mathbb{P}_n) = I_T(\mathbb{Q}, \mathbb{P}).$$

Hence

$$\overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}) \leq \inf_{\mathbb{Q}\in\widetilde{\mathscr{M}_a}(T)} I_T(\mathbb{Q}, \mathbb{P}).$$

The sequence of EMM $\mathbb{Q}_n^E$ is contained in the class of all martingale measures $\mathscr{M}_a(T)$, therefore

$$\inf_{Q\in\mathscr{M}_a(T)} I_T(\mathbb{Q}, \mathbb{P}) \leq \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}).$$

On the other side, $\mathscr{M}_a(T) \subseteq \widetilde{\mathscr{M}_a}(T)$ and hence

$$\inf_{\mathbb{Q}\in\widetilde{\mathscr{M}_a}(T)} I_T(\mathbb{Q}, \mathbb{P}) \leq \inf_{\mathbb{Q}\in\mathscr{M}_a(T)} I_T(\mathbb{Q}, \mathbb{P}).$$

Finally, we get

$$\inf_{\mathbb{Q}\in\widetilde{\mathscr{M}_a}(T)} I_T(\mathbb{Q}, \mathbb{P}) \leq \inf_{\mathbb{Q}\in\mathscr{M}_a(T)} I_T(\mathbb{Q}, \mathbb{P}) \leq \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}) \leq \inf_{\mathbb{Q}\in\widetilde{\mathscr{M}_a}(T)} I_T(\mathbb{Q}, \mathbb{P})$$

and, consequently,

$$\inf_{\mathbb{Q}\in\widetilde{\mathscr{M}_a}(T)} I_T(\mathbb{Q}, \mathbb{P}) = \inf_{\mathbb{Q}\in\mathscr{M}_a(T)} I_T(\mathbb{Q}, \mathbb{P}) = \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}).$$

The statement is proven. □

4.7 Coincidence of the EMM and the MEMM

Finally, we can prove the main result of the present thesis: the coincidence of the EMM and the MEMM. First we prove the simpler case, for martingale measures and then we will apply a localization procedure and generalize the result also to local martingale measures.

4.18 Theorem. *(i) If there exists the EMM $\mathbb{Q}^E$ then it is the MEMM in the class $\tilde{\mathscr{M}_a}(T)$.*

(ii) If there exists the MEMM $\mathbb{Q}^*$ in the class $\mathscr{M}_a(T)$ then it is the EMM.

(iii) If there exists the MEMM $\mathbb{Q}^*$ *in the class* $\widetilde{\mathscr{M}_a}(T)$ *then it is the EMM.*

Proof. The identity proven in the last proposition gives us an opportunity to reduce the problem of finding the MEMM in the classes $\mathscr{M}_a(T)$ and $\widetilde{\mathscr{M}_a}(T)$ to the one-step problem (D) (see Appendix B.5). Let us start from (iii). Assume that $\mathbb{Q}^*$ is the MEMM in the class $\widetilde{\mathscr{M}_a}(T)$ (as a solution of the moment problem). Then by Theorem 2.6 we find that

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = c\exp(f),$$

where $f \in \overline{K}^Q$, $\mathbb{Q} \in \mathscr{M}^0$ (using the notations of Definition 2.2). But this is the formulation of the one-step problem (D) and hence from Theorem 2.7 we know the structure of the space $\overline{K}^Q$:

$$\overline{K}^Q = K = \{\kappa L_T : \kappa \in \mathbb{R}\}.$$

Taking into account the last fact we observe the explicit form of the density:

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = c\exp(\kappa L_T).$$

Note that parameters κ and c are deterministic, therefore the obtained density coincides with the definition of the Esscher density. Because $\mathbb{E}_{\mathbb{Q}^*}[L_T] = 0$ by assumption, the Esscher measure $\mathbb{Q}^*$ is a martingale measure ((L, F) is again a Lévy process w.r.t. $\mathbb{Q}^*$) and hence it is the Esscher martingale measure. Let us prove now the converse direction, part (i) of the theorem. Assume that there exists the EMM $\mathbb{Q}^E$, then we can consider our problem in terms of the one-step problem (D), setting $\xi = L_T$ and $\mathscr{F}_1 = \mathscr{F}_T$. In this case the EMM $\mathbb{Q}^E$ coincides the Esscher martingale measure of the one-step problem, which is, by Theorem B.5, the MEMM in the class $\mathfrak{M}_a$ or in other words, the solution of the one-step problem (D). Taking into account that $\xi = L_T$ we observe that classes of measures $\mathfrak{M}_a$ and $\widetilde{\mathscr{M}_a}(T)$ coincide and hence $\mathbb{Q}^E$ is the solution of the moment problem.

The statement (ii) is based on Proposition 4.17 and part (iii): if there exists the MEMM $\mathbb{Q}^*$ in the class $\mathscr{M}_a(T)$ then by Proposition 4.17 it is the MEMM in class $\widetilde{\mathscr{M}_a}(T)$ and we can use part (iii). □

The last step is to localize the previous statement.

4.19 Theorem. *If the MEMM* $\mathbb{Q}^*$ *in the class* $\mathscr{M}_a^{loc}(T)$ *exists then it is the EMM.*

Proof. Let us assume that the MEMM $\mathbb{Q}^*$ in the class $\mathscr{M}_a^{loc}(T)$ exists. Define the sequence of only finite measures $(\mathbb{Q}_n^*)_{n\geq 1}$ by

$$\frac{\mathrm{d}\mathbb{Q}_n^*}{\mathrm{d}\mathbb{Q}^*}(\mathscr{F}_T) = Z_T^{(n)},\ n \geq 1, \tag{4.40}$$

where $Z_T^{(n)}$ is defined in (4.17). In a similar way we define the sequence of probability measures $(\mathbb{P}_n)_{n\geq 1}$:

$$\frac{\mathrm{d}\mathbb{P}_n}{\mathrm{d}\mathbb{P}}(\mathscr{F}_T) = Z_T^{(n)},\ n \geq 1. \tag{4.41}$$

From Proposition 4.12 it follows that the process L has finite exponential moments with respect to every probability measure $\mathbb{P}_n$ and hence, applying Proposition 4.4, we find that for every $\mathbb{P}_n$ there exists the appropriate EMM $\mathbb{Q}_n^E$:

$$\frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_T) \quad = \quad \exp(\kappa_n L_T - T\log\varphi_{1,n}(\kappa_n)). \tag{4.42}$$

Choose a reducing sequence $(\rho_i)_{i\geq 1}$: $(\rho_i)_{i\geq 1}$ is a sequence of stopping times such that $\rho_i \uparrow +\infty$ and $(L^{\rho_i\wedge T}, \mathbb{F})$ is a $\mathbb{Q}^*$-martingale on $[0,T]$ for every $i \geq 1$. We then get

$$\begin{aligned}
I_T(\mathbb{Q}_n^*, \mathbb{P}_n) &\geq I_{\rho_i\wedge T}(\mathbb{Q}_n^*, \mathbb{P}_n) && (4.43)\\
&= \mathbb{E}_{\mathbb{Q}_n^*}\left[\log\frac{\mathrm{d}\mathbb{Q}_n^*}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_{\rho_i\wedge T})\right]\\
&= \mathbb{E}_{\mathbb{Q}_n^*}\left[\log\frac{\mathrm{d}\mathbb{Q}_n^*}{\mathrm{d}\mathbb{Q}_n^E}(\mathscr{F}_{\rho_i\wedge T})\right] + \mathbb{E}_{\mathbb{Q}_n^*}\left[\log\frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_{\rho_i\wedge T})\right] && (4.44)\\
&\geq \mathbb{E}_{\mathbb{Q}_n^*}\left[\log\frac{\mathrm{d}\mathbb{Q}_n^E}{\mathrm{d}\mathbb{P}_n}(\mathscr{F}_{\rho_i\wedge T})\right], && (4.45)
\end{aligned}$$

In (4.43) we used the monotonicity of the relative entropy in time, in (4.44) we used linearity of the integral, which is allowed because of

$$0 \leq \mathbb{E}_{Q_n^*}\left[\log\frac{\mathrm{d}\mathbb{Q}_n^*}{\mathrm{d}\mathbb{Q}_n^E}(\mathscr{F}_{\rho_i\wedge T})\right] < +\infty$$

as the property of the relative entropy, and in (4.45) we used the nonnegativity of the relative entropy. Inserting (4.42) in (4.45) and applying Doob's optional sampling theorem, we obtain

$$I_T(\mathbb{Q}_n^*, \mathbb{P}_n) \geq \mathbb{E}_{\mathbb{Q}_n^*}[\kappa_n L_{\rho_i\wedge T}] - \log\varphi_{1,n}(\kappa_n)\mathbb{E}_{\mathbb{Q}_n^*}[\rho_i \wedge T].$$

Letting n to infinity we get

$$\overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^*, \mathbb{P}_n) \geq \overline{\lim_{n\to\infty}} \mathbb{E}_{\mathbb{Q}_n^*}[\kappa_n L_{\rho_i \wedge T}] - \underline{\lim}_{n\to\infty} \log \varphi_n(\kappa_n) \mathbb{E}_{\mathbb{Q}_n^*}[\rho_i \wedge T].$$

Our next step is to prove that

$$\overline{\lim_{n\to\infty}} \mathbb{E}_{\mathbb{Q}_n^*}[\kappa_n L_{\rho_i \wedge T}] = 0.$$

We use the same approach as in the proof of Proposition 4.15. First of all, it is worth to mention that according to Corollary 4.16 our sequence of Esscher parameters is bounded. Hence the following estimate holds:

$$|\mathbb{E}_{\mathbb{Q}_n^*}[\kappa_n L_{\rho_i \wedge T}]| \leq \sup_{m\geq 1} |\kappa_m| |\mathbb{E}_{\mathbb{Q}_n^*}[L_{\rho_i \wedge T}]|.$$

But

$$\mathbb{E}_{\mathbb{Q}_n^*}[L_{\rho_i \wedge T}] = \mathbb{E}_{\mathbb{Q}^*}[Z_T^{(n)} L_{\rho_i \wedge T}]$$

and using Lebesgue's theorem on dominated convergence we obtain that the right-hand side converges to $\mathbb{E}_{\mathbb{Q}^*}[L_{\rho_i \wedge T}] = 0$. As a result, we get:

$$\overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^*, \mathbb{P}_n) \geq -\underline{\lim}_{n\to\infty} \log \varphi_{1,n}(\kappa_n) \mathbb{E}_{\mathbb{Q}_n^*}[\rho_i \wedge T].$$

Using Propositions 4.13 and 4.14, and letting i to infinity we get

$$\begin{aligned} I_T(\mathbb{Q}^*, \mathbb{P}) &= \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^*, \mathbb{P}_n) \\ &\geq \overline{\lim}_{n\to\infty} - \log \varphi_{1,n}(\kappa_n) T = \overline{\lim_{n\to\infty}} I_T(\mathbb{Q}_n^E, \mathbb{P}_n) \geq \inf_{Q \in \mathscr{M}_a^{loc}(T)} I_T(\mathbb{Q}, \mathbb{P}), \end{aligned}$$

where the last inequality follows from the fact, that the Esscher martingale measures $(\mathbb{Q}_n^E)$ are included in the class $\mathscr{M}_a^{loc}(T)$. From the last expression we find that

$$\inf_{Q \in \mathscr{M}_a^{loc}(T)} I_T(\mathbb{Q}, \mathbb{P}) = \overline{\lim_{n\to\infty}} [-\log \varphi_{1,n}(k_n) T].$$

Proposition A.1, part 4 implies the existence of a minimal point of the function φ_n, moreover κ_n is the minimal point of $\varphi_{1,n}$ (Proposition A.1, part 4) guarantees the convergence of the sequence (κ_n) to some κ_0 and the convergence of the minimal values $\varphi_{1,n}(\kappa_n)$ to $\varphi_1(\kappa_0)$. Hence

$$\inf_{Q \in \mathscr{M}_a^{loc}(T)} I_T(\mathbb{Q}, \mathbb{P}) = \overline{\lim_{n\to\infty}} [-\log \varphi_{1,n}(k_n) T] = -T \log \varphi_1(\kappa_0). \qquad (4.46)$$

Let us define $\mathbb{Q}^{\kappa_0}$ as the Esscher measure with parameter κ_0 (in general, it need not be a martingale measure, but it is the measure defined by the

Esscher density). Then we get

$$
\begin{aligned}
I_T(\mathbb{Q}^*,\mathbb{P}) &\geq I_{\rho_i\wedge T}(\mathbb{Q}^*,\mathbb{P})\\
&= \mathbb{E}_{\mathbb{Q}^*}\left[\log\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}}(\mathscr{F}_{\rho_i\wedge T})\right]\\
&= \mathbb{E}_{\mathbb{Q}^*}\left[\log\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}^{\kappa_0}}(\mathscr{F}_{\rho_i\wedge T})\right]+\mathbb{E}_{\mathbb{Q}^*}\left[\log\frac{\mathrm{d}\mathbb{Q}^{\kappa_0}}{\mathrm{d}\mathbb{P}}(\mathscr{F}_{\rho_i\wedge T})\right]\\
&= I_{\rho_i\wedge T}(\mathbb{Q}^*,\mathbb{Q}^{\kappa_0})+\mathbb{E}_{\mathbb{Q}^*}\left[\kappa_0 L_{\rho_i\wedge T}-\log\varphi_1(\kappa_0)(\rho_i\wedge T)\right]\\
&= I_{\rho_i\wedge T}(\mathbb{Q}^*,\mathbb{Q}^{\kappa_0})-\log\varphi_1(\kappa_0)\mathbb{E}_{\mathbb{Q}^*}[\rho_i\wedge T].
\end{aligned}
$$

Passing to the limit for $i\to\infty$ we find

$$I_T(\mathbb{Q}^*,\mathbb{P})\geq\lim_{i\to\infty}I_{\rho_i\wedge T}(\mathbb{Q}^*,\mathbb{Q}^{\kappa_0})-\log\varphi_1(\kappa_0)T=\lim_{i\to\infty}I_{\rho_i\wedge T}(\mathbb{Q}^*,\mathbb{Q}^{\kappa_0})+I_T(\mathbb{Q}^*,\mathbb{P}).$$

This implies

$$\lim_{i\to\infty}I_{\rho_i\wedge T}(\mathbb{Q}^*,\mathbb{Q}^{\kappa_0})=0$$

and since the entropy process is increasing in time

$$I_{\rho_i\wedge T}(\mathbb{Q}^*,\mathbb{Q}^{\kappa_0})=0\ \text{ for }\forall i\geq 1.$$

The entropy is zero just in case when the measures coincide on the given σ-algebras. Thus

$$\mathbb{Q}^*|_{\mathscr{F}_{\rho_i\wedge T}}=\mathbb{Q}^{\kappa_0}|_{\mathscr{F}_{\rho_i\wedge T}}\ \text{ for }\forall i\geq 1.$$

Let us recall that by definition of a local martingale we have $\{\rho_i\wedge T=T\}\uparrow\Omega$ as $n\to\infty$. The set $\{\rho_i\wedge T=T\}$ belongs to $\mathscr{F}_{\rho_i\wedge T}$, take any set $A\in\mathscr{F}_T$, then $A\cap\{\rho_i\wedge T=T\}\in\mathscr{F}_{\rho_i\wedge T}$, letting i to ∞ the increasing sequence $A\cap\{\rho_i\wedge T\}$ converges to A and we get

$$\mathbb{Q}^*(A)=\mathbb{Q}_0(A)\ \text{ for any }A\in\mathscr{F}_T.$$

This means that the Esscher measure, defined by its parameter κ_0, coincides with the MEMM $\mathbb{Q}^*$. As a result the Esscher measure $\mathbb{Q}_0$ is a martingale measure, namely, the MEMM $\mathbb{Q}^*$. Hence $\mathbb{Q}^*$ is the EMM. The proposition is proven. □

4.8 Conclusions

Combining all the propositions above we can generalize the statement of Proposition 4.17 and Proposition 4.18:

4.20 Theorem. *The following identity holds:*

$$\inf_{n\geq 1} I_T(\mathbb{Q}_n^E,\mathbb{P}) = \inf_{Q\in\mathscr{M}_a(T)} I_T(\mathbb{Q},\mathbb{P}) \tag{4.47}$$

$$= \inf_{\mathbb{Q}\in\widetilde{\mathscr{M}_a}(T)} I_T(\mathbb{Q},\mathbb{P}) \tag{4.48}$$

$$= \inf_{\mathbb{Q}\in\mathscr{M}_a^{loc}(T)} I_T(\mathbb{Q},\mathbb{P}) \tag{4.49}$$

$$= -T\log\varphi(\kappa_0) \tag{4.50}$$

$$= \inf_{\mathbb{Q}\in\mathscr{M}_{efl}(T)} I_T(\mathbb{Q},\mathbb{P}), \tag{4.51}$$

where $\mathscr{M}_{efl}(T)$ denotes $\mathscr{M}_e(T)\cap\mathscr{M}_f(T)\cap\mathscr{M}_l(T)$.

Proof. Identities (4.47) and(4.48) follow from Proposition 4.17. Identity of (4.49) and (4.50) was proven in Theorem 4.19 in (4.46). Identity of (4.47) and (4.50) follows from Proposition (A.1), part (4). The last identity, (4.51), follows from the facts that the Esscher martingale measures $\mathbb{Q}_n^E$ preserve the Lévy property and, hence, belong to the class $\mathscr{M}_l(T)$. At the same time, $\mathscr{M}_l(T)\subset\mathscr{M}_a(T)$. □

Theorem 4.19 also gives us the clue for sufficient classes of measures that should be considered for finding the MEMM: The key role is played by the Esscher measures. We stress the very important fact that Esscher measures preserve the Lévy property of the process $(L,\mathbb{F})$.

4.21 Theorem. *The probability measure $\mathbb{Q}^*$ is the minimal entropy martingale measure in the class $\mathscr{M}_a^{loc}(T)$ if and only if it is the minimal entropy martingale measure in the class $\mathscr{M}_{efl}(T)$.*

Proof. The proof follows from the fact, that the Esscher measures preserve the Lévy property.

If there exists a probability measure $\mathbb{Q}^*$ which is the minimal entropy martingale measure in the class $\mathscr{M}_a^{loc}(T)$, then by Theorem 4.19 we find that $\mathbb{Q}^*$ is the Esscher martingale measure and hence $\mathbb{Q}^*\in\mathscr{M}_l(T)$, and by the fact that $\mathscr{M}_l(T)\subseteq\mathscr{M}_a^{loc}(T)$ it is the MEMM in the class $\in\mathscr{M}_l(T)$. Moreover, applying (4.51) we get that the MEMM is even in class $\mathscr{M}_{efl}(T)$.

Now let us prove the statement in the reverse direction, but it obviously follows from identities (4.49)-(4.51). □

4.22 Theorem. *The EMM $\mathbb{Q}^E$ exists if and only if there exists the MEMM $\mathbb{Q}^*$ in the class $\mathscr{M}_a^{loc}(T)\cup\widetilde{\mathscr{M}_a}(T)$. These measures coincide: $\mathbb{Q}^E=\mathbb{Q}^*$.*

Proof. Assume there exists the MEMM $\mathbb{Q}^*$ in class $\mathscr{M}_a^{loc}(T) \cup \widetilde{\mathscr{M}_a}(T)$. It means that either $\mathbb{Q}^* \in \mathscr{M}_a^{loc}(T)$ or $\mathbb{Q}^* \in \widetilde{\mathscr{M}_a}(T)$. If $\mathbb{Q}^* \in \mathscr{M}_a^{loc}(T)$, then by Theorem 4.18, part (iii), $\mathbb{Q}^*$ is the EMM. If $\mathbb{Q}^* \in \widetilde{\mathscr{M}_a}(T)$, then by Theorem 4.19 $\mathbb{Q}^*$ is the EMM.

Now let us prove in the reverse direction. Assume there exists the EMM $\mathbb{Q}^E$. Then according to Lemma 4.3 we have $I_T(\mathbb{Q}, \mathbb{P}) = -T \log \varphi(\kappa_0)$. Note, that

$$\begin{aligned} \inf_{\mathbb{Q} \in \widetilde{\mathscr{M}_a}(T) \cup \mathscr{M}_a^{loc}(T)} I_T(\mathbb{Q}, \mathbb{P}) &= \min\left(\inf_{\mathbb{Q} \in \widetilde{\mathscr{M}_a}(T)} I_T(\mathbb{Q}, \mathbb{P}), \inf_{\mathbb{Q} \in \mathscr{M}_a^{loc}(T)} I_T(\mathbb{Q}, \mathbb{P})\right) \\ &= -T \log \varphi(\kappa_0), \end{aligned}$$

where the last equality holds because of Theorem 4.20. While the EMM $\mathbb{Q}^E \in \mathscr{M}_a^{loc}(T) \cup \widetilde{\mathscr{M}_a}(T)$, it is the MEMM in this class. The statement is proven. □

Last two theorems characterize the value of the minimal entropy and the connection between the EMM and the MEMM in the class $\mathscr{M}_a^{loc}(T) \cup \widetilde{\mathscr{M}_a}(T)$. Our next statement is based on the collection of the Theorems 4.21 – 4.22 and concerns the preservation of infima of the entropy by the sufficient subclass for the solution of all the minimization problems.

4.23 Theorem. *The MEMM $\mathbb{Q}^*$ in the class $\mathscr{M}_a^{loc}(T) \cup \widetilde{\mathscr{M}_a}(T)$ exists if and only if there exists the MEMM $\mathbb{Q}_l^*$ in the class $\mathscr{M}_l(T)$. Moreover, in case of existence these measures coincide between themselves and with the EMM, which also exists: $\mathbb{Q}_l^* = \mathbb{Q}^* = \mathbb{Q}^E$.*

Proof. The prove is trivial and follows from Theorem 4.22, Theorem 4.20, part 4.51, and the fact that the EMM preserves the Lévy property of the process (and hence, if the EMM exists, it belongs to the class $\mathscr{M}_l(T)$). □

In the last theorem it is explicitly mentioned the smallest, sufficient class in searching of the MEMM – the class of Lévy preserving measures $\mathscr{M}_l(T)$ – and the largest class, for which this measure, if it exists, is also the MEMM. It is worth to mention, that if there does not exist the EMM, then there does not exist a MEMM in all these classes. What happens in this case? Surprisingly, the situation is the same as in one-step model, in particular, it depends on the exponential integrability and there may happen 4 situations (among which just 3 are indeed different), which are discussed in Proposition A.2.

Now we will conduct a small comparison of our results with results stated in Esche & Schweizer (2005).

The authors have proven that the MEMM $\mathbb{Q}^*$ is Lévy preserving (Theorem A). In fact, they prove, but not state it explicitly, that Lévy preserving (local) martingale measures are a sufficient subclass. In our case we obtain this property as a corollary (cf. Theorem 4.21) and show that the subclass $\mathscr{M}_{efl}(T)$ is sufficient for the search of the MEMM. Theorem B states that the EMM is the MEMM. Theorem A is actually used to justify that for finding the MEMM it is enough to minimize the entropy over all (local) martingale measures $\mathbb{Q}$ that preserve the Lévy property. This yields the minimization problem over the deterministic parameters (called β and Γ) for finding the MEMM. This minimization problem was solved by using formal arguments. The formal solution is the EMM. In other words, the paper does not give a strict proof that the MEMM (if it exists) implies the existence of the EMM (and equality between them).

Another paper in this field, Fujiwara & Miyahara (2003), states the following: EMM, if it exists, is the MEMM. Moreover, there are also conditions on existence of the EMM: The EMM exists if there exists $\kappa^* \in \mathbb{R}$ such that

(1) $\displaystyle\int_{\mathbb{R}} |x \exp \kappa^* x - h(x)| \nu(\mathrm{d}x) < +\infty,$

(2) $\displaystyle\beta + \sigma\kappa^* + \int_{\mathbb{R}} (x \exp \kappa^* x - h(x)) \nu(\mathrm{d}x) = 0.$

These are analytical conditions in terms of the characteristics (β, σ, ν). In probabilistic terms it means:

(1') $\mathbb{E}[|L_T| \exp(\kappa^* L_T)] < +\infty,$

(2') $\mathbb{E}[L_T \exp(\kappa^* L_T)] = 0.$

Appendix A. Properties of the Functions φ and ψ

The aim of this appendix is to collect the most important properties of the functions φ and ψ which are widely used in the main body of the thesis. Note that we work here just with random variables but not with general stochastic processes.

Let be given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a non-degenerated random variable $\xi - \xi_0$ defined on it. It looks a little bit confusing to consider here the difference $\xi - \xi_0$ instead of just ξ, but the motivation comes from the one-step model (cf. Appendix B): ξ_0 is an initial value of some price process X, $\xi_0 = X_0$, may be either constant or a random variable measurable with respect to the initial σ-algebra $\mathscr{F}_0$, random variable ξ stands for the value of the price process X at the end of the step, $\xi = X_1$, and is measurable with respect to a σ-algebra $\mathscr{F}_1 = \mathscr{F}$. In this appendix we are interested in the behavior of the functions, that use as an argument an increment $\xi - \xi_0$ of the value of the price process.

Assume that for the support of the distribution of the increment $\xi - \xi_0$ the following condition holds: $\operatorname{supp}\mathscr{L}(\xi - \xi_0) \cap (-\infty, 0) \neq \emptyset$ and $\operatorname{supp}\mathscr{L}(\xi - \xi_0) \cap (0, +\infty) \neq \emptyset$. According to Proposition B.2, this is a necessary and sufficient condition for the existence of at least one martingale measure for the one-step model with initial value ξ_0 and terminal value ξ. Define

$$\varphi(\kappa) = \mathbb{E}[\exp(\kappa(\xi - \xi_0))], \quad \kappa \in \mathbb{R}, \tag{A.1}$$
$$\psi(\kappa) = \mathbb{E}[(\xi - \xi_0)\exp(\kappa(\xi - \xi_0))], \quad \kappa \in \mathbb{R}. \tag{A.2}$$

Let $I = \{\kappa \in \mathbb{R} : \varphi(\kappa) < +\infty\}$. Then I is convex and $0 \in I$. Therefore I is an interval with endpoints a and b, $a \leq 0 \leq b$. The set of interior points of I is denoted by I^0. Clearly, $I^0 = (a, b)$. Note that I^0 can be the empty set which happens in the case if $a = b = 0$. Let $E = \{\kappa \in \mathbb{R} : |\psi(\kappa)| < +\infty\}$.

A.1 Proposition. Let functions φ and ψ be defined by (A.1) and (A.2), respectively. Then the following properties hold:

1. φ is convex,
$$\varphi(-\infty) = \varphi(+\infty) = +\infty;$$
2. φ is continuous on I;
3. φ is differentiable on (a, b). If $a \in I$ then φ differentiable from the right at a with right derivative $+\infty > \varphi'_+(a) \geq -\infty$. If $b \in I$ then φ is differentiable from the left at b with left derivative $+\infty \geq \varphi'_-(b) > -\infty$;
4. φ has a unique minimum point κ_0;
5. ψ is monotonically increasing,
$$\psi(-\infty) = -\infty, \quad \psi(+\infty) = +\infty;$$
6. ψ is continuous on E;
7. ψ is differentiable on (a, b).

Proof. (1) In case $I^0 = \emptyset$ the statement of the proposition is trivial. Therefore we assume that $I^0 = (a, b)$ with $a < b$. Since $\varphi''(\kappa) > 0$, the function φ is strictly convex on (a, b). The function $(\xi - \xi_0) \exp(\kappa(\xi - \xi_0))$ is increasing in κ and therefore $\mathbb{E}[(\xi - \xi_0) \exp(\kappa(\xi - \xi_0))]$ exists for all $\kappa \geq b$ and

$$-\infty < \mathbb{E}[(\xi - \xi_0) \exp(\kappa(\xi - \xi_0))].$$

Analogously, $\mathbb{E}[(\xi - \xi_0) \exp(\kappa(\xi - \xi_0))]$ exists and

$$\mathbb{E}[(\xi - \xi_0) \exp(\kappa(\xi - \xi_0))] < +\infty, \quad \kappa \leq a.$$

Hence the function ψ introduced by

$$\psi(\kappa) = \mathbb{E}[(\xi - \xi_0) \exp(\kappa(\xi - \xi_0))], \quad \kappa \in \mathbb{R},$$

is well-defined and monotonically increasing. Because of the monotonicity it follows that

$$-\infty = \psi(\kappa), \quad \kappa < a; \quad \psi(\kappa) = +\infty, \quad b < \kappa.$$

To prove that for $\kappa < a \leq 0$ holds $\psi(\kappa) = -\infty$. It is sufficient to verify that

$$\mathbb{E}[(\xi - \xi_0)^- \exp(\kappa(\xi - \xi_0))] = +\infty,$$

since $(\xi - \xi_0)^- \exp(\kappa(\xi - \xi_0))$ is negative part of $(\xi - \xi_0)\exp(\kappa(\xi - \xi_0))$.

$$\begin{aligned}\mathbb{E}[(\xi - \xi_0)^- \exp(\kappa(\xi - \xi_0))] &\geq \mathbb{E}[(\xi - \xi_0)^- 1_{\{(\xi-\xi_0)^- \geq 1\}} \exp(\kappa(\xi - \xi_0))] \\ &\geq \mathbb{E}[1_{\{(\xi-\xi_0)^- \geq 1\}} \exp(\kappa(\xi - \xi_0))] = \infty\end{aligned}$$

since $\varphi(\kappa) = \mathbb{E}[\exp(\kappa(\xi - \xi_0))] = +\infty$ and using the assumption $\kappa < 0$ we have

$$\mathbb{E}[1_{\{(\xi-\xi_0)^- < 1\}} \exp(\kappa(\xi - \xi_0))] \leq \mathbb{E}[1_{\{(\xi-\xi_0)^- \geq 1\}} \exp(-\kappa(\xi - \xi_0)^-)] < +\infty.$$

Similar arguments we use for $\kappa > b$. We can observe the asymptotic behavior of the function ψ:

$$\lim_{\kappa \to +\infty} \psi(\kappa) = +\infty, \lim_{\kappa \to -\infty} \psi(\kappa) = -\infty.$$

Indeed,

$$\mathbb{E}[(\xi - \xi_0)^+ \exp(\kappa(\xi - \xi_0))] = \mathbb{E}[(\xi - \xi_0)^+ 1_{\{(\xi-\xi_0)^+ > 0\}} \exp(\kappa(\xi - \xi_0)^+)]$$

which converges to $+\infty$ as $\kappa \to \infty$ by monotone convergence. Here we used one part of the no-arbitrage property of Proposition B.2: $\mathbb{P}\left(\{(\xi - \xi_0)^+ > 0\}\right) > 0$. Similarly for $\kappa \to -\infty$. The property (1) is proven.

(2) We would like to show that the function φ is continuous on I. In view of its convexity, φ is continuous on I^0. Now assume that $b \in I$. We then have to show that φ is continuous at b. Let $\kappa_n \to b \quad (a < \kappa_n \leq b)$. We have

$$0 \leq \exp(\kappa_n(\xi - \xi_0)) \leq \exp(\kappa_n(\xi - \xi_0)^+) \leq \exp(b(\xi - \xi_0)^+)$$

and since $\exp(b(\xi - \xi_0)^+)$ is integrable, by dominated convergence we obtain

$$\lim_{n \to \infty} \mathbb{E}[\exp(\kappa_n(\xi - \xi_0))] = \mathbb{E}[\exp(b(\xi - \xi_0))].$$

(3) Because of the theorem of Lebesgue the function φ is differentiable on (a, b) and

$$\varphi'(\kappa) = \mathbb{E}[(\xi - \xi_0)\exp(\kappa(\xi - \xi_0)], \; \kappa \in (a, b).$$

Similarly, the second derivative is

$$\varphi''(\kappa) = \psi'(\kappa) = \mathbb{E}[(\xi - \xi_0)^2 \exp(\kappa(\xi - \xi_0))], \; \kappa \in (a, b).$$

Moreover, φ is infinitely differentiable on (a, b). Therefore the properties (7) and (3) hold.

(4) We now show that φ always has a minimum. Let us define the set

$$E := \{\kappa \in \mathbb{R} : \ \mathbb{E}[|\xi - \xi_0| \exp(\kappa(\xi - \xi_0))] < +\infty\} .$$

Obviously, $E \subseteq I$, and $E^0 = I^0$.

The next aim is to show that function φ attains its minimum on E.

The value of the function φ converges to infinity while the argument converges to infinity:

$$\lim_{|\kappa|\to\infty} \varphi(\kappa) = +\infty.$$

Indeed, if $\kappa > 0$

$$\varphi(\kappa) \geq \mathbb{E}[1_{\{\xi - xi_0 > 0\}} \exp(\kappa(\xi - \xi_0)^+)]$$

and the right-hand side converges to $+\infty$ if $\kappa \to \infty$ by monotone convergence and $\mathbb{P}(\{(\xi - \xi_0)^+ > 0\}) > 0$. The proof is similar for $\kappa \to -\infty$.

The set $K = \{\kappa \in \mathbb{R} : \varphi(\kappa) \leq 1\}$ is a nonempty compact subset of $\mathbb{R}$. Indeed, $0 \in K$ and K is bounded in view of the above verified convergence to ∞. Let $(\kappa_n)_{n\geq 1}$ be a sequence from K such that $\kappa_n \to \kappa$. By the lemma of Fatou,

$$\varphi(\kappa) = \mathbb{E}[\exp(\kappa(\xi - \xi_0))] \leq \underline{\lim}_{n\to\infty} \mathbb{E}[\exp(\kappa_n(\xi - \xi_0))] \leq 1.$$

Hence K is closed. As a bounded and closed subset of $\mathbb{R}$, the set K must be compact. Therefore, it follows from (2) that the function φ is continuous on the compact set K. Thus φ has a minimum on K which coincides with the minimum of φ on $\mathbb{R}$ which follows from the definition of K. Uniqueness follows from the convexity of the function.

Let κ_0 be the minimal point of φ. Then $\kappa_0 \in I$. If $\kappa_0 \in I^0 = (a, b)$ then as a necessary condition to be the minimal point of φ we have

$$\psi(\kappa_0) = \varphi'(\kappa_0) = \mathbb{E}[(\xi - \xi_0) \exp(\kappa_0(\xi - \xi_0))] = 0.$$

Conversely, if $\kappa_0 \in I^0$ satisfies

$$\psi(\kappa_0) = \varphi'(\kappa_0) = \mathbb{E}[(\xi - \xi_0) \exp(\kappa_0(\xi - \xi_0))] = 0$$

then, because of the strict convexity of φ and $\varphi'(\kappa_0) = 0$, κ_0 is a minimum point of φ. The latter is also true if κ_0 is only a boundary point of I and does not belong to the interior points I^0.

Let us consider two cases: 1) $a = b = 0$; 2) $a < b$. The first case is trivial, we have $0 \notin I^0$ and the function φ attains its minimum at point 0. Otherwise, if $a < b$ and

$$\mathbb{E}[(\xi - \xi_0)\exp(b(\xi - \xi_0))] = 0 \tag{A.3}$$

then we must have $\varphi(b) < +\infty$ and there exists the left derivative $\varphi_l'(b)$ and

$$\varphi_l'(b) = \mathbb{E}[(\xi - \xi_0)\exp(b(\xi - \xi_0))] = 0.$$

Similarly we obtain

$$\varphi_l''(b) = \mathbb{E}[(\xi - \xi_0)^2\exp(b(\xi - \xi_0))] > 0.$$

This implies that φ has a minimum at point b. The last argument can be replaced by the following: φ must have a minimum at b because

$$\varphi_l'(\kappa) = \psi(\kappa) = \mathbb{E}[(\xi - \xi_0)\exp(\kappa(\xi - \xi_0))] < 0, \quad \kappa \in I,\, \kappa < b,$$

where the strict inequality follows from monotonicity of ψ on (a, b) (because of the strict positivity of ψ' on (a, b)) and (A.3), hence $\varphi(b) < \varphi(\kappa)$ for such κ. The left differentiability can be seen as follows: For $h > 0$, we estimate

$$\begin{aligned} |\exp(-h(\xi - \xi_0)) - 1| &= \Big| -\int_0^h (\xi - \xi_0)\exp(-u(\xi - \xi_0))\mathrm{d}u\Big| \\ &\le \int_0^h |\xi - \xi_0|\exp(-u(\xi - \xi_0))\mathrm{d}u \\ &\le |\xi - \xi_0| h \exp(h(\xi - \xi_0)^-) \end{aligned}$$

and hence

$$\begin{aligned} &\left|\frac{\exp((b-h)(\xi - \xi_0)) - \exp(b(\xi - \xi_0))}{h}\right| \\ &= \exp(b(\xi - \xi_0))\left|\frac{\exp(-h(\xi - \xi_0)) - 1}{h}\right| \\ &\le \exp(b(\xi - \xi_0))|\xi - \xi_0|\exp(h(\xi - \xi_0)^-) \\ &\le \exp(b(\xi - \xi_0)^+)|\xi - \xi_0| \end{aligned}$$

for $h \le b$ which is integrable because of $\mathbb{E}[(\xi - \xi_0)\exp(b(\xi - \xi_0))] = 0$. □

Summarizing, if

$$\mathbb{E}[(\xi - \xi_0)\exp(\kappa_0(\xi - \xi_0))] = 0 \tag{A.4}$$

for some $\kappa_0 \in I$ then κ_0 is the unique minimum point of φ and the converse is true if $\kappa_0 \in (a, b)$. However, there can be points κ_0 on the boundary of I (a or b if $\varphi(a) < +\infty$ or $\varphi(b) < +\infty$) such that κ_0 is a minimum point but the equality (A.4) does not hold:

$$\mathbb{E}[(\xi - \xi_0)\exp(\kappa_0(\xi - \xi_0))] \neq 0.$$

More precisely, it can happen that

$$\mathbb{E}[(\xi - \xi_0)\exp(b(\xi - \xi_0))] < 0$$

or

$$\mathbb{E}[(\xi - \xi_0)\exp(a(\xi - \xi_0))] > 0.$$

But the structure of the set E gives more information about the existence of the MEMM. It is easy to see, that if $E = \emptyset$, i.e., there is no Esscher martingale measure, by Theorem 2.6 the MEMM does not exist. Assume E is not empty. Then it may have one of the following form: (i) (a, b), (ii) $[a, b]$, (iii) $(a, b]$, (iv) $[a, b)$, where $-\infty \leq a \leq 0 \leq b \leq +\infty$. If the endpoints are not included in the set E, the function $\psi(\kappa)$ smoothly converges to $\pm\infty$, the sign depending on whether it is the right or left endpoint, while κ converges to the endpoint. Taking into account the continuity of the function ψ in κ, we find that for the case (i) there always exists the MEMM. Indeed,

$$\lim_{\kappa \to a+} \psi(\kappa) = -\infty, \ \lim_{\kappa \to b-} \psi(\kappa) = +\infty$$

and the function ψ is continuous on (a, b), therefore there always exists κ_0 such that $\psi(\kappa_0) = 0$, i.e., κ_0 is the required Esscher parameter and the martingale condition is satisfied. If one of the endpoints is included in E, then there can arise some problems because the minimum of the function φ can be attained exactly at the endpoint, but the martingale condition is not necessarily satisfied. In case (iii) (respectively, (iv)) the existence of the MEMM depends on the sign of the $\psi(\kappa)$ at the endpoint $\kappa = a$ (respectively, $\kappa = b$). We have $\lim_{\kappa \to b-} \psi(\kappa) = +\infty$ (respectively, $\lim_{\kappa \to a+} \psi(\kappa) = -\infty$), therefore for the existence of the MEMM it is necessary and sufficient that $\psi(a) \leq 0$ (respectively, $\psi(b) \geq 0$), i.e., it is necessary and sufficient that the function ψ crosses the level 0. In case of equality $\psi(a) = 0$ (respectively, $\psi(b) = 0$) the minimum of φ is attained at the endpoint a (respectively, b) and the martingale condition is satisfied.

The last case (ii) happens when both endpoints are included in the set E: $E = [a, b]$. Obviously, we obtain the following condition for the existence of the MEMM: $\psi(a)\psi(b) \leq 0$. In other words, we still need that ψ crosses the level 0. If the last condition is not satisfied, we get the situation that φ attains its minimum on the endpoint, but the martingale condition does not hold. In terms of the function ψ it means, that it "jumps over" the level 0: $\psi(\kappa)$ is either always greater then zero or always smaller then zero for $\kappa \in E$. Therefore there does not exist the Esscher parameter which satisfies the martingale condition and consequently the MEMM does not exist.

Summarizing all we have mentioned above, we may state the following proposition:

A.2 Proposition. Let the set E be defined as

$$E := \{\kappa \in \mathbb{R} : \ \mathbb{E}[|\xi - \xi_0| \exp(\kappa(\xi - \xi_0))] < +\infty\} .$$

There are four possible cases:

1. The set E is not empty and has the form of an open interval:

   $$E = (a, b), \text{ where } a, b \in \mathbb{R} \cup \{\pm\infty\}.$$

   Then there always exists $\kappa_0 \in E$ such that $\psi(\kappa_0) = 0$;

2. The set E is not empty and has the form of a semi-open interval:

   $$\begin{aligned} E &= (a, b], \text{ where } a \in \mathbb{R} \cup \{\pm\infty\}, \quad b \in \mathbb{R}, \text{ or} \\ E &= [a, b), \text{ where } b \in \mathbb{R} \cup \{\pm\infty\}, \quad a \in \mathbb{R}. \end{aligned}$$

   Then there exists such $\kappa_0 \in E$ that $\psi(\kappa_0) = 0$ if and only if $\psi(b) \geq 0$ (for the case $E = (a, b]$) or $\psi(a) \leq 0$ (for the case $E = [a, b)$).

3. The set E has the form of a closed interval:

   $$E = [a, b], \text{ where } a, b \in \mathbb{R}.$$

   Then there exists such $\kappa_0 \in E$ that $\psi(\kappa_0) = 0$ if and only if

   $$\psi(a)\psi(b) \leq 0.$$

4. The set E is empty: $E = \emptyset$. Then there does not exist a point κ_0 that satisfies the martingale property.

B

Appendix B. One-Step Model

In this appendix we would like to apply the general theory developed in Chapter 2, to a particular model, the one-step model. The appendix is included for easier reference. The results are slight extensions of basically known facts.

Let us consider a probability space $(\Omega, \mathscr{F}, \mu)$ and random variables ξ and ξ_0 defined on it. There are two time points: the initial time $t = 0$ and the terminal time $t = 1$. The filtration $\mathbb{F}$ ("the flow of information") consists of two σ-algebras: $\mathscr{F}_0$ and $\mathscr{F}_1$, $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}$, $\mathbb{F} = (\mathscr{F}_t)_{t=0,1}$. Then $(\Omega, \mathscr{F}, \mathbb{F}, \mu)$ is a filtered probability space. The price process is defined by $X = (X_t)_{t=0,1}$ with $X_0 = \xi_0$ and $X_1 = \xi$. By assumption, the process X is $\mathbb{F}$-adapted and the increment $X_1 - X_0 = \xi - \xi_0$ is independent of $\mathscr{F}_0$ with respect to the given probability measure μ. The special case: $\mathscr{F}_0$ is trivial (or only μ-trivial). Then the assumption of independence of increment is automatically satisfied and ξ_0 is (μ-a.s.) equal to a constant. Similarly as in Section 2.1, let us define the set K. In our particular case it has a much simpler structure, because there are just two time-points:

$$\begin{aligned} K &= \operatorname{Span}(\{\zeta(X_t - X_s) : \zeta \in L^\infty(\Omega, \mathscr{F}_s, \mathbb{P}), s, t \in \mathscr{T}, s \leq t, \quad X \in \mathscr{X}\}) \\ &= \{\eta : \eta = k(\xi - \xi_0) \text{ for } k \in L^\infty(\Omega, \mathscr{F}_0, \mu)\}. \end{aligned}$$

The definition of the martingale measure is also simplified:

B.1 Definition. A probability measure ν on $(\Omega, \mathscr{F})$ is called a martingale measure, if $\mathbb{E}_\nu[\xi|\mathscr{F}_0] = \xi_0$.

Note, that in the special case when $\mathscr{F}_0$ is trivial, the variable ξ_0 is a fixed constant, therefore the martingale condition can be formulated without the conditional expectation:

$$\mathbb{E}_\nu[\xi|\mathscr{F}_0] = \mathbb{E}_\nu[\xi] = \xi_0.$$

If $\mathscr{F}_0$ is μ-trivial, the statement holds for absolutely continuous measures.

Now we state the problem (D) which we would like to investigate in the current Appendix: We would like to find such a martingale measure ν_0, that is absolutely continuous with respect to μ and minimizes the relative entropy:

$$H(\nu_0, \mu) = \inf_{\nu \in \mathscr{M}_a} H(\nu, \mu). \tag{D}$$

The existence of at least one equivalent martingale measure means that the financial market is arbitrage-free. This is a very important condition for the further investigation and it is possible to state necessary and sufficient conditions for the existence of an equivalent martingale measure in our particular model.

B.2 Proposition. The set of equivalent martingale measures $\mathscr{M}_e$ is not empty if and only if one of the following conditions is satisfied:
(i) $\mu(\{\xi = \xi_0\}) = 1$;
(ii) $\mu(\{\xi - \xi_0 > 0\}) > 0$ and $\mu(\{\xi - \xi_0 < 0\}) > 0$.

Proof. Assume that the set of equivalent martingale measures $\mathscr{M}_e$ is not empty and hence there exists a martingale measure $\nu \in \mathscr{M}_e$. Let us further assume that condition (i) is not satisfied. Then we have to verify that condition (ii) holds. The expectation of $(\xi - \xi_0)^+$ and $(\xi - \xi_0)^-$ under the measure ν must be equal (by definition of a martingale measure) and must be strictly positive, otherwise it contradicts the assumption that (i) is not satisfied. Hence, $\nu(\{\xi - \xi_0 > 0\}) > 0$ and $\nu(\{\xi - \xi_0 < 0\}) > 0$. But $\nu \sim \mu$, therefore $\mu(\{\xi - \xi_0 > 0\}) > 0$ and $\mu(\{\xi - \xi_0 < 0\}) > 0$. This proves the necessity of the condition that (i) or (ii) holds.
Now let us prove the sufficiency. If the condition (i) is satisfied, then $\mu \in \mathscr{M}_e$ and the statement holds. Let us assume that the condition (ii) is satisfied, i.e., $\mu(\{\xi - \xi_0 > 0\}) > 0$ and $\mu(\{\xi - \xi_0 < 0\}) > 0$ holds. Define

$$l^-(c) := \mathbb{E}[|\xi - \xi_0| 1_{\{\xi - \xi_0 < 0\}} \exp(c(\xi - \xi_0))]$$

and similarly

$$l^+(c) := \mathbb{E}[(\xi - \xi_0) 1_{\{\xi - \xi_0 > 0\}} \exp(c(\xi - \xi_0))].$$

For $c < 0$ the expression $l^+(c)$ is always finite and strictly positive, for $c > 0$ the expression $l^-(c)$ is always finite and strictly positive, so let us fix a couple (c_1, c_2) such that $-\infty < c_1 < 0 < c_2 < +\infty$. Now we would like to construct a density, that will make the tails of the distribution lighter and at the same time the new measure defined by this density will be a martingale measure. Let us define a function $a(c_1, c_2)$ as

$$\begin{aligned} a(c_1, c_2) \;:=\; & \exp(c_1(\xi - \xi_0))1_{\{\xi-\xi_0>0\}} l^-(c_2) \\ & + \exp(c_2(\xi - \xi_0))1_{\{\xi-\xi_0<0\}} l^+(c_1) \\ & + 1_{\{\xi=\xi_0\}}. \end{aligned}$$

Obviously, we always have $a(c_1, c_2) > 0$. Then define ν as

$$\eta = \frac{\mathrm{d}\nu}{\mathrm{d}\mu},$$

where

$$\eta = \frac{a(c_1, c_2)}{\mathbb{E}[a(c_1, c_2)]}.$$

In view of $\eta > 0$ we get $\nu \sim \mu$. Furthermore, a simple calculation shows that

$$\begin{aligned} \mathbb{E}_\nu[\xi - \xi_0|\mathscr{F}_0] &= \frac{\mathbb{E}_\mu[\eta(\xi - \xi_0)|\mathscr{F}_0]}{\mathbb{E}_\mu[\eta|\mathscr{F}_0]} \\ &= \mathbb{E}_\mu[\eta(\xi - \xi_0)|\mathscr{F}_0] \qquad \text{(B.1)} \\ &= \mathbb{E}_\mu[\eta(\xi - \xi_0)] = 0, \qquad \text{(B.2)} \end{aligned}$$

where we use twice the independence of the increment $\xi - \xi_0$ of $\mathscr{F}_0$ with respect to the measure μ. Hence ν is an equivalent martingale measure. □

Note, if condition (i) or (ii) is satisfied, it is always possible to build an equivalent martingale measure, using the density η, with the finite entropy, that preserves the "independence property". The independence of the increment is explicitly shown in the proof. Finiteness of the entropy follows from the way in which we choose parameters c_1 and c_2. Our next aim is to apply the general approach introduced in Chapter 2 to our particular case. Let us recall the definition of the set C_0:

$$C_0 = \{f \in \mathscr{L} : \mathbb{E}_\mathbb{Q}[f] \le 0, \forall \mathbb{Q} \in \mathscr{M}_f\}.$$

In the following lemma we explicitly use the structure of the set K for the one-step model and improve the result of Lemma 2.4.

B.3 Lemma. *Let $\nu_0 \in \mathscr{M}_f$, $\nu_0 \in \mathscr{M}_e$, $f_0 \in C_0$, $\mathbb{E}_{\nu_0}[f_0] = 0$. Then $f_0 \in \overline{K}^{\nu_0}$ and*

$$\overline{K}^{\nu_0} = \{k(\xi - \xi_0) : k(\xi - \xi_0) \text{ integrable w.r.t. } \nu_0,\ k\, \mathscr{F}_0\text{-measurable}\}. \qquad \text{(B.3)}$$

Proof. The first part of the statement that $f_0 \in \overline{K}^{\nu_0}$, follows from Lemma 2.4. The principle difference is the additional information about the structure of the set $\overline{K}^{\nu_0}$. Let us denote the right-hand part of the (B.3) by $\widetilde{K}$:

$$\widetilde{K} := \{k(\xi - \xi_0) : k(\xi - \xi_0) \text{ integrable w.r.t. } \nu_0, \; k\,\mathscr{F}_0\text{-measurable}\}.$$

We have to prove that $\overline{K}^{\nu_0} = \widetilde{K}$. We start by proving the inclusion $\widetilde{K} \subseteq \overline{K}^{\nu_0}$. Let $k(\xi - \xi_0) \in \widetilde{K}$. We can define $k_n = (k \wedge n) \vee (-n)$ and by the definition of the set $\overline{K}^{\nu_0}$ we get $k_n(\xi - \xi_0) \in \overline{K}^{\nu_0}$. Obviously, we have

$$|k_n(\xi - \xi_0)| \leq |k||\xi - \xi_0|.$$

The right hand side being integrable with respect to ν_0 we can apply the theorem of Lebesgue on dominated convergence and obtain that $k_n(\xi - \xi_0)$ converges to $k(\xi - \xi_0)$ in $L^1(\nu_0)$. Hence, $k(\xi - \xi_0) \in \overline{K}^{\nu_0}$. Therefore, the inclusion $\widetilde{K} \subseteq \overline{K}^{\nu_0}$ holds. Our next aim is to show that every converging in $L^1(\nu_0)$ sequence from $\overline{K}^{\nu_0}$ has a limit in $\widetilde{K}$:

$$\eta_n := k_n(\xi - \xi_0) \in \overline{K}^{\nu_0}, \, \eta_n \longrightarrow \eta \text{ in } L^1(\nu_0) \text{ when } n \longrightarrow \infty, \text{ then } \eta \in \widetilde{K}.$$

Let (η_n) be such a sequence, then

$$\begin{aligned}
\mathbb{E}_{\nu_0}[|\eta_n - \eta_m|] &= \mathbb{E}_{\nu_0}[|k_n(\xi - \xi_0) - k_m(\xi - \xi_0)|] && \text{(B.4)}\\
&= \mathbb{E}_{\nu_0}[\mathbb{E}_{\nu_0}[|\xi - \xi_0||k_n - k_m||\mathscr{F}_0]] && \text{(B.5)}\\
&= \mathbb{E}_{\nu_0}[|k_n - k_m|\mathbb{E}_{\nu_0}[|\xi - \xi_0||\mathscr{F}_0]] && \text{(B.6)}\\
&= \mathbb{E}_{\nu_1}[|k_n - k_m|] \longrightarrow 0 \quad \text{when } n, m, \to \infty, && \text{(B.7)}
\end{aligned}$$

where $\mathrm{d}\nu_1 := Z\mathrm{d}\nu_0$ and

$$Z := \mathbb{E}_{\nu_0}[|\xi - \xi_0||\mathscr{F}_0] \geq 0.$$

From (B.7) it follows that (k_n) is a Cauchy sequence in $L^1(\nu_1)$. Hence there exists a limit k of (k_n) in $L^1(\nu_1)$. As a limit, k is $\mathscr{F}_0^{\nu_0}$-measurable, where $\mathscr{F}_0^{\nu_0}$ is the completion of $\mathscr{F}_0$ with respect to ν_0. We can assume that $\mathscr{F}_0 = \mathscr{F}_0^{\nu_0}$, otherwise we choose an $\mathscr{F}_0$-measurable version of k, again denoted by k. We put $\eta = k(\xi - \xi_0)$. Inserting in (B.4)-(B.7) η instead of η_m we observe that η_n converges to η in $L^1(\nu_0)$. Hence, η belongs to $\overline{K}^{\nu_0}$ as required. The statement is proven. □

We recall the notion of the Esscher martingale measure for the one-step model and investigate its relation to the MEMM. Let us start from the definitions:

B.4 Definition. Given a probability space $(\Omega, \mathscr{F}, \mu)$, assume ξ, ξ_0 are random variables and $\kappa \in \mathbb{R}$ such that $\mathbb{E}[\exp(\kappa(\xi - \xi_0))] < +\infty$. Then ζ defined by

$$\zeta := \frac{\exp(\kappa(\xi - \xi_0))}{\mathbb{E}[\exp(\kappa(\xi - \xi_0))]}$$

is called an Esscher density and κ is called an Esscher parameter or coefficient.

The random variable ζ is positive μ-a.s., integrable and normalized ($\mathbb{E}[\zeta] = 1$), hence ν defined by $\zeta = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is a probability measure, $\nu \sim \mu$. This probability measure ν is called an Esscher measure. If $\mathbb{E}_\nu[\xi - \xi_0] = 0$, then ν is a martingale measure for the process $(X, \mathbb{F})$ and it is called an Esscher martingale measure (EMM).

The next step is to investigate the relation between the MEMM and the Esscher martingale measure. The following theorem states that for a one-step model the notions of the minimal entropy martingale measure and the Esscher martingale measure coincide.

B.5 Theorem. *Let μ be an arbitrary probability measure on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t=1,2})$, ξ is the random variable defined on this space and ξ is independent of $\mathscr{F}_0$, $\xi_0 \in \mathbb{R}$. A probability measure ν_0 on $(\Omega, \mathscr{F})$ is the MEMM if and only if ν_0 is the Esscher martingale measure.*

Proof. Let us prove that if there exists the MEMM, then it is the Esscher martingale measure. Let ν_0 be the MEMM. By Theorem 2.6 we know the form of the density of the MEMM with respect to the given probability measure μ: $\mathrm{d}\nu_0/\mathrm{d}\mu = c\exp(-f_0)\quad \mu$-a.s., where $f_0 \in C_0$ and $\mathbb{E}_{\nu_0}[f_0] = 0$, $c > 0$.

Lemma B.3 guarantees that $f_0 \in \bar{K}^{\nu_0}$ and provides us with the explicit structure of the space $\bar{K}^{\nu_0}$: there exists an $\mathscr{F}_0$-measurable random variable k such that $f_0 = -k(\xi - \xi_0)\quad \nu_0$-a.s. From the properties of the density of a probability measure we get the expression for the constant c: $c^{-1} = \mathbb{E}[\exp(-f_0)]$. Taking into account that ν_0 is a martingale measure, we obtain the identity:

$$\mathbb{E}_{\nu_0}[\xi - \xi_0|\mathscr{F}_0] = c\mathbb{E}_\mu[(\xi - \xi_0)\exp(k(\xi - \xi_0))|\mathscr{F}_0](\mathbb{E}[\mathrm{d}\nu_0/\mathrm{d}\mu|\mathscr{F}_0])^{-1} = 0.$$

Using that k is $\mathscr{F}_0$-measurable and $\xi - \xi_0$ is independent of $\mathscr{F}_0$, we can rewrite the conditional expectation above as follows

$$\begin{aligned} 0 &= \mathbb{E}_\mu[(\xi - \xi_0)\exp(k(\xi - \xi_0))|\mathscr{F}_0](\omega) \\ &= \mathbb{E}_\mu[(\xi - \xi_0(\cdot))\exp(k(\omega)(\xi - \xi_0)(\cdot))] \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

where the expectation $\mathbb{E}_\mu$ is taken with respect to the free variable $(\cdot)$. The right-hand side being equal to zero μ-a.s. and recalling the definition of function ψ (see (A.2)) we obtain $\psi(k(\omega)) = 0 \quad \mu$-a.s. We know (see Proposition A.1) that because of monotonicity the function ψ has at most one point $\kappa \in \mathbb{R}$ such that $\psi(\kappa) = 0$. As a result, a random variable k is constant μ-a.s. Setting $\kappa_0 = \mathbb{E}[k]$, finally we get that the given MEMM ν_0 has density

$$\frac{\mathrm{d}\nu_0}{\mathrm{d}\mu} = c\exp(\kappa(\xi - \xi_0)) =: \zeta.$$

From this it follows, that

$$c = \frac{1}{\mathbb{E}[\exp(\kappa(\xi - \xi_0))]}$$

and

$$\zeta = \frac{\exp(\kappa(\xi - \xi_0))}{\mathbb{E}[\exp(\kappa(\xi - \xi_0))]}$$

is the Esscher transformation, ν_0 is the Esscher martingale measure and κ is the Esscher parameter.

Now we are going to prove, that the statement also is true in the reverse direction: if there exists an Esscher martingale measure, then it is the MEMM. Assume that there exists the Esscher martingale measure ν_0. This means that there exists κ_0 such that $\mathbb{E}_\mu[(\xi - \xi_0)\exp(\kappa(\xi - \xi_0))]$. Then it is easy to verify that the Esscher martingale measure ν_0 with Esscher parameter κ_0 satisfies the conditions of Theorem 2.6 with $f_0 = -\kappa_0(\xi - \xi_0)$ and $c = (\mathbb{E}[\exp(\kappa_0(\xi - \xi_0))])^{-1}$. Theorem 2.6 implies that ν_0 is the MEMM. □

Now we can summarize different characterizations of the existence of the MEMM in the following corollary:

B.6 Corollary. Assume that set $\mathscr{M}_e$ is not empty. Then the following conditions are equivalent:

(a) There exists the MEMM.
(b) There exists the Esscher martingale measure.
(c) There exists $\kappa \in \mathbb{R}$ such that $\mathbb{E}_\mu[(\xi - \xi_0)\exp(\kappa(\xi - \xi_0))] = 0$.
(d) There exists the solution of the problem (D).

Proof. Conditions (a) and (b) are equivalent by Theorem B.5. Conditions (b) and (c) are equivalent by the definition of the Esscher martingale measure. Conditions (a) and (d) are equivalent by the definition of the MEMM. □

Note that we can rewrite condition (c) in a more useful form:

$$\mathbb{E}_\mu[(\xi - \xi_0)\exp(\kappa(\xi - \xi_0))] = \psi(\kappa) = 0.$$

A similar problem was considered in Cherny & Maslov (2003) but using completely another approach. In particular, there was used the method of Lagrange multipliers for minimization of the expression $-\mathbb{E}_\nu[\xi] + H(\nu, \mu)$ over all absolutely continuous distributions ν with respect to μ, where ξ is a random variable such that $\mathbb{E}[\exp(\xi)] < +\infty$ and $\mathbb{E}[|\xi|\exp(\xi)] < +\infty$. As a result there was obtained the form of the optimal density:

$$\mathrm{d}\nu_0 = c\exp(\xi)\mathrm{d}\mu, \quad c \in \mathbb{R},$$

that corresponds to the Esscher transformation and that ν_0 is the Esscher measure. Applying the martingale condition on the measure ν_0 it was found that the MEMM coincides with the EMM.

Now let us consider two very similar cases of distributions μ but with different results of existence of the MEMM:

Example 1. Let ξ be a Cauchy distributed symmetric random variable. Its distribution is:

$$\mu(\mathrm{d}x) = \frac{1}{\pi(1 + x^2)}\mathrm{d}x.$$

In this case there does not exist the MEMM. For this it is sufficient to show that for all $\kappa : \mathbb{E}[\exp(\kappa\xi)] = +\infty$. Obviously,

$$\mathbb{E}[\exp(\kappa\xi)] = \mathbb{E}[1_{\xi<0}\exp(\kappa\xi)] + \mathbb{E}[1_{\{\xi>0\}}\exp(\kappa\xi)].$$

If $\kappa > 0$, then $\mathbb{E}[1_{\{\xi>0\}}\exp(\kappa\xi)] = +\infty$. If $\kappa < 0$, then $\mathbb{E}[1_{\{\xi<0\}}\exp(\kappa\xi)] = +\infty$. If $\kappa = 0$, we have $\psi(\kappa) = +\infty$ and, consequently, there exists no Esscher measure, hence, no MEMM.

Example 2. Let ξ is "shifted" one-side Cauchy distributed random variable.

$$\mu(\mathrm{d}x) = 1_{\{x\in[-a,+\infty)\}}\frac{2}{\pi(1 + (x + a)^2)}\mathrm{d}x$$

where $a > 0$. Then, similarly to the proof of Lemma B.3, it is possible to construct $l^-(\kappa)$ and $l^+(\kappa)$ just for negative κ and show that there exists a solution. Another way is to analyze the behavior of ψ and to apply Proposition A.2, in particular, the set E has the form of an open interval. According to it, there exists such κ_0 and corresponding EMM.

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