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An indefinite inverse spectral problem of Stieltjes type

Andreas Fleige and Henrik Winkler

Dedicated to Heinz Langer on the occasion of his 80th birthday

Abstract. We consider a regular indefinite Krein-Feller differential expression of Stieltjes type $-D_m D_x$. This can be regarded as an indefinite generalization of a vibrating Stieltjes string wearing only concentrated (now positive or negative) "masses" which accumulate at a finite right endpoint. From the general theory of indefinite Krein-Feller operators we conclude a number of spectral properties. In particular, we obtain a spectral function σ which is non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, \infty)$ and which allows the existence of all moments $\int \lambda^n \, d\sigma$ for $n \in \mathbb{N}$. The main result of the present paper is an inverse statement: Starting from a function τ with properties like σ, the (unique) "masses" of an indefinite Krein-Feller operator of Stieltjes type are reconstructed such that τ belongs to the same so-called spectral class like the associated spectral function. All elements of this class are identified as the spectral functions of similar operators with a generally "heavy" right endpoint (and one additional function).

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1. Introduction

A basic inverse spectral result of M.G. Krein states that each Stieltjes function is the principal Titchmarsh-Weyl coefficient of a vibrating string which is determined by a non-decreasing mass distribution function m on some interval $[0, b)$ and certain self-adjoint boundary conditions. The behavior of the string is then determined by the so-called Krein-Feller differential expression $-D_m D_x f$ where the derivative D_m has to be understood in the Radon-Nikodym sense. By the result mentioned above there is a one-to-one correspondence between all such strings and all Stieltjes functions, see e.g. [7, Theorem 11.2]. According to [7, Theorem 11.1], this correspondence can also be expressed in terms of the string and its (nondecreasing) spectral function.

A particular case is a string with only concentrated masses accumulating at the right endpoint. It is convenient to call it *Stieltjes string* (cf. [7]) since its study goes back to the famous work [13] of T.J. Stieltjes, involving moment problems and continued fractions. A condensed presentation of M.G. Krein's inverse result can be found in the book [3] by H. Dym and H.P. McKean, starting with a string with only finitely many point masses and then going over to the general case via a Stieltjes string. In this approach, up to the Stieltjes case the reconstruction of the string from its spectral function is a constructive procedure with explicit formulas for the point masses.

Since the 1970s a number of attempts were made in order to allow also "negative masses", or to be more precise, to allow an indefinite setting. At first *finitely* many "negative masses" or, in other words, negative squares of the associated inner product were considered, see, e.g. [8, 12]. The main result of [12] states that a generalized Stieltjes function Q from the class $\mathbb{N}_\kappa^+$ is the principal Titchmarsh-Weyl coefficient of a generalized string with κ negative "masses", singularities or dipoles.

Recently, J. Eckhardt and A. Kostenko presented in [4] the solution of the inverse spectral problem for a generalized indefinite string without any restrictions on the numbers of negative squares. There a non-linear eigenvalue problem is considered, including an additional term with the square of the eigenvalue forming a generalization of the dipoles from [8, 12]. In particular, it is shown in [4] that the generalized indefinite string is uniquely associated to a two-dimensional canonical system as considered by L. de Branges [2], and then the inverse spectral result of L. de Branges [2, 14] is used in order to show that each Herglotz-Nevanlinna function is the Titchmarsh-Weyl coefficient of such a generalized indefinite string. What is still missing is the indefinite Stieltjes case (with infinitely many "negative masses") allowing explicit reconstruction formulas for the point masses. This setting is studied in the present paper. To this end, an *indefinite* Krein-Feller differential expression $-D_m D_x f$ is considered where m is a *signed* measure. Such expressions generalize the concept of strings indicated above (where m is a nonnegative measure) to an indefinite situation and were studied in detail e.g. in [9, 10, 5]. As mentioned before, we restrict ourselves to Krein-Feller expressions *of Stieltjes type* which means that m has only concentrated (positive or negative) "masses" m_n at "mass points" x_n, accumulating at a finite right endpoint $b \in (0, \infty)$.

Results from [5] are used in order to establish a number of spectral properties for the J-self-adjoint operator A which is associated with the expression $-D_m D_x f$ in the Krein space L^2_m, imposing certain boundary conditions. In particular, a spectral function σ is constructed which is (in contrast to the definite setting) non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, \infty)$. Additionally, we obtain a nondecreasing so-called A_--spectral function σ_- which is connected with σ by $\sigma(t) = \int_0^t s \; d\sigma_-(s)$. A particular feature of the Stieltjes case (in the definite as well as in the indefinite setting) is the existence of all moments $s_k := \int_{-\infty}^{\infty} t^k \, d\sigma_-$ $(k \in \mathbb{N} \cup \{0\})$. Then, in analogy to [3, Chapter 5] certain orthogonal polynomials are constructed in the Hilbert space $L^2_{\sigma_-}$ and also in the Krein space L^2_σ allowing reconstruction formulas for m_n and x_n in terms of the inner products of these polynomials.

In order to treat the inverse statement we start with functions τ_- and $\tau(t) := \int_0^t s \, d\tau_-(s)$ and impose some conditions which were previously observed as properties of σ_- and σ including the existence of all moments $t_k := \int_{-\infty}^{\infty} t^k \, d\tau_-$ $(k \in \mathbb{N} \cup \{0\})$ and the existence of certain orthogonal polynomials. These conditions again allow the formulas mentioned before and lead to "masses" m_n and "mass points" x_n inducing an indefinite Krein-Feller operator of Stieltjes type as above. Furthermore, by the conditions on τ_- and τ the space of all polynomials is dense in $L^2_{\tau_-}$. This means that τ_- is an N-extremal solution of the Hamburger moment problem generated by the sequence (t_k). Each other N-extremal solution $\widehat{\tau}_-$ induces a function $\widehat{\tau}(t) := \int_0^t s \, d\widehat{\tau}_-(s)$ with similar properties like τ. In particular, $\widehat{\tau}$ generates the same "point masses" and hence also the same Krein-Feller operator of Stieltjes type. All the functions $\widehat{\tau}$ form an equivalence class $[\tau]$. As the main result of this paper it turns out that the spectral function σ associated with the constructed Krein-Feller operator indeed also belongs to the class $[\tau]$. This leads to a one-to-one correspondence between all classes $[\tau]$ constructed as above and all indefinite Krein-Feller operators A of Stieltjes type. Furthermore, this result allows us to call this class the *spectral class* of A.

By a more detailed study of the moment problem all N-extremal solutions can be characterized by means of all self-adjoint extensions of a certain symmetric operator in a Hilbert space. As a consequence, the spectral class appears as the collection of the spectral functions associated with all indefinite Krein-Feller operators induced by m_n, x_n and a generally "heavy mass" at b and in addition the spectral function of an exceptional operator.

Finally, note some restrictions of the present approach. First, we use different boundary conditions as in the classical setting. This was done in order to apply the results from [5] which originally were not designed for inverse questions. For the same reason dipoles are avoided here by allowing only non-neutral polynomials. Furthermore, an explicit condition on τ is still missing which guarantees that τ does not belong to a "heavy" right endpoint, i.e. $\tau = \sigma$. This seems to involve the critical point ∞ of the Krein-Feller operators and will be studied in a forthcoming paper.

Here, in analogy to [3, Section 5.8] instead of determinants or continued fractions we use recurrence equations involving two simple kinds of functions as the key tool for the reconstruction formulas.

2. Some known results from the spectral theory of indefinite Krein-Feller operators

In this section some results from [5] are recalled where indefinite Krein-Feller operators are studied in detail. We restrict ourselves to the case of a *regular* differential expression and Neumann boundary condition at the right endpoint.

2.1. Regular Krein-Feller operators

Let $0 < b < \infty$ and let m be a real left-continuous function of bounded variation on $\mathbb{R}$ which is constant on $(-\infty, 0)$ and also on (b, ∞). Then, m induces a signed Lebesgue-Stieltjes measure which is again denoted by m. Denote its total variation by $||m||$ and let the space L^2_m of all (equivalence classes of) $||m||$-measurable complex functions f on $\mathbb{R}$ with $\int |f|^2\, d||m|| < \infty$ be equipped with the inner products

$$[f, g]_m := \int f\overline{g}\, dm, \quad (f, g)_m := \int f\overline{g}\, d||m|| \quad (f, g \in L^2_m). \tag{2.1}$$

Then, according to [5, Appendix C] $(L^2_m, [\cdot,\cdot]_m)$ is a Krein space and its topology is induced by the Hilbert space inner product $(\cdot,\cdot)_m$. The associated fundamental symmetry is given by $Jf = (\chi_{M^+} - \chi_{M^-})f$ $(f \in L^2_m)$ where $\mathbb{R} = M^+ \dot\cup M^-$ is the Hahn decomposition for m and $\chi_{M^\pm}$ is the characteristic function of $M^\pm$. According to [5, Appendix A], a complex function f on $\mathbb{R}$ is called *absolutely continuous with respect to* m if there is a number $f_0 \in \mathbb{C}$ and an $||m||$-integrable function g such that $f(x) = f_0 + \int_0^x g\, dm$ $(x \in \mathbb{R})$ (with the notation $\int_\alpha^\beta := \int_{[\alpha,\beta)}$ if $\alpha \le \beta$ and $\int_\alpha^\beta := -\int_\beta^\alpha$ if $\alpha > \beta$). In this case $D_m f := g$ is called the *Radon-Nikodym derivative of* f *with respect to* m. Then, the so-called *Krein-Feller differential expression* $D_m D_x f$ is well defined ($||m||$-a.e.) for all locally absolutely continuous complex functions f on $\mathbb{R}$ with a derivative $D_x f$ which is absolutely continuous with respect to m. For $f, h \in L^2_m$ the function f belongs to this class and we have $h = D_m D_x f$ if and only if there are numbers $f_0, f_0' \in \mathbb{C}$ such that

$$f(x) = f_0 + f_0' x + \int_0^x \left(\int_0^t h\, dm \right) dt \quad (x \in \mathbb{R});$$

cf. [5, Proposition 2.1]. In this case we have $f_0 = f(0)$ and $f' := D_x f$ satisfies

$$f'(t) = f'(t-0) = f_0' + \int_0^t h\, dm \quad (t \in \mathbb{R}), \qquad f'(0) = f'(0-0) = f_0'.$$

In particular, f is linear on $(-\infty, 0)$ and on (b, ∞). Now, with $\alpha \in (0, \pi/2)$ consider the space $D(A)$ of all such functions f satisfying $D_m D_x f \in L^2_m$ and

$$\cos\alpha f(0) - \sin\alpha f'(0-0) = 0, \tag{2.2}$$

$$f'(b+0) = 0. \tag{2.3}$$

Then, by [5, Theorem 2.13] the operator $Af := -D_m D_x f$, defined for all $f \in D(A)$ is J-self-adjoint, J-non-negative and boundedly invertible (and hence definitizable) in the Krein space $(L^2_m, [\cdot,\cdot]_m)$ with

$$[Af,g]_m = \cot\alpha f(0)\overline{g(0)} + \int_0^b f'\overline{g}'\,dx =: \{f,g\}_+ \qquad (f,g\in D(A)). \quad (2.4)$$

This means that JA is self-adjoint, non-negative and boundedly invertible in the Hilbert space $(L^2_m, (\cdot,\cdot)_m)$. Furthermore, the spectrum $\sigma(A)$ of A is discrete.

2.2. A space triplet

According to [5, Theorem 2.30] $D((JA)^{1/2})$ coincides with the space H^1_m of all locally absolutely continuous complex functions f on $\mathbb{R}$ which are linear on each interval I where m vanishes (i.e. $f(x_2)-f(x_1)=c(x_2-x_1)$ for all $x_1,x_2\in I$ with some $c\in\mathbb{C}$) and which satisfy $f'\in L^2[0,b]$ and the "essential" boundary condition (2.2). Then, H^1_m is a Hilbert space with the inner product $\{\cdot,\cdot\}_+$ (defined as in (2.4) for $f,g\in H^1_m$) and it is continuously (and densely) embedded in the Krein space $(L^2_m, [\cdot,\cdot]_m)$; cf. [5, Corollary 2.33, Section 4.1]. Furthermore, its dual space K_- of all continuous linear functionals φ on $(H^1_m, \{\cdot,\cdot\}_+)$ is a Hilbert space with the inner product $\{\cdot,\cdot\}_-$ given by the polar formula

$$\{\varphi,\psi\}_- := \frac{1}{4}(\{\varphi+\psi\}_-^2 - \{\varphi-\psi\}_-^2 + i\{\varphi+i\psi\}_-^2 - i\{\varphi-i\psi\}_-^2)$$

with the norm

$$\{\varphi\}_- := \sup_{\{f,f\}_+\le 1} |\varphi(f)|,$$

for $\varphi,\psi\in K_-$. In this space we define the scalar multiplication by $(c\cdot\varphi)(f) := \bar{c}\cdot\varphi(f)$ $(f\in H^1_m)$. Now, as in [5, Section 4.1], put $K_+ := H^1_m$ and $K := L^2_m$. For $f\in K$ the functional $[\cdot,f]_m$, restricted to K_+, belongs to K_- and in this sense the space K can be regarded as a subspace of K_- and the inner product $[\cdot,\cdot]_m$ on K can be extended to a sesquilinear form on $K_+\times K_-$ by $[g,\varphi]_m := \varphi(g)$. Then we obtain the space triplet

$$K_+ \subset K \subset K_-$$

where each inclusion is continuous and dense. On the other hand, by the Riesz representation theorem

$$A_-g := \{\cdot,g\}_+ \in K_- \quad (g\in K_+)$$

defines a Hilbert space isomorphism from $(K_+, \{\cdot,\cdot\}_+)$ to $(K_-, \{\cdot,\cdot\}_-)$. This is an extension of A since by (2.4) we have

$$[f,Ag]_m = \{f,g\}_+ = [f,A_-g]_m \quad (f\in K_+,\ g\in D(A)).$$

Furthermore, by [5, Section 4.1] the operator A_- considered as an operator in $(K_-, \{\cdot,\cdot\}_-)$ with $D(A_-) := K_+\,(=H^1_m)$ is self-adjoint and has the same spectrum like A (i.e. $\sigma(A_-)=\sigma(A)$). Similarly, the range restriction $A_+ := A|_{D(A_+)}$ with $D(A_+) := A^{-1}(K_+)$ is self-adjoint in $(K_+, \{\cdot,\cdot\}_+)$ with $\sigma(A_+)=\sigma(A)$; cf. [5, Section 4.1]. A particular element of K_- is given by the functional

$$\delta_0(f) := f(0) \quad (f\in K_+) \quad (2.5)$$

which belongs to K if and only if $m(\{0\}) \neq 0$; cf. [5, Proposition 4.2].

2.3. The spectral function

Now, let $E_-(t)$ denote the left-continuous resolution of the identity for the self-adjoint operator A_- in the Hilbert space $(K_-, \{\cdot,\cdot\}_-)$ and, according to [5, Section 4.2], put

$$\begin{aligned} \sigma_-(t) &:= \{E_-(t)\delta_0, \delta_0\}_-, \\ \sigma(t) &:= \int_0^t s\, d\sigma_-(s), \\ \sigma_+(t) &:= \int_0^t s^2\, d\sigma_-(s). \end{aligned} \tag{2.6}$$

Then σ_+ and σ_- are non-decreasing "infinite step functions" with jumps at the eigenvalues of A. Moreover, σ is non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, \infty)$. Put

$$(F, G)_{\sigma_\pm} := \int_{-\infty}^{\infty} F\overline{G} d\sigma_\pm\,,\ [F, G]_\sigma := \int_{-\infty}^{\infty} F\overline{G} d\sigma,\ (F, G \in L^2_{\sigma_\pm}, L^2_\sigma). \tag{2.7}$$

Then, again we have a space triplet with continuous and dense inclusions

$$L^2_{\sigma_+} \subset L^2_\sigma \subset L^2_{\sigma_-} \tag{2.8}$$

where $(L^2_{\sigma_+},\ (\cdot,\cdot)_{\sigma_+})$ and $(L^2_{\sigma_-},\ (\cdot,\cdot)_{\sigma_-})$ are Hilbert spaces and $(L^2_\sigma,\ [\cdot,\cdot]_\sigma)$ is a Krein space; cf. [5, Section 4.4]. According to [5, Section 4.2] (or [9]) we call σ the *spectral function* of the Krein-Feller operator A and $\sigma_\pm$ the $A_\pm$*-spectral function*.

2.4. The Fourier transformation

By [5, Section 2.2] for a fixed $\lambda \in \mathbb{C}$ the equation

$$-D_m D_x f = \lambda f \tag{2.9}$$

has unique solutions $\varphi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ with

$$\varphi(0, \lambda) = \sin\alpha, \quad \varphi'(0-0, \lambda) = \cos\alpha, \tag{2.10}$$

$$\psi(0, \lambda) = \cos\alpha, \quad \psi'(0-0, \lambda) = -\sin\alpha. \tag{2.11}$$

If $\lambda = 0$ $(\in \rho(A))$ these functions are linear (cf. [5, (2.42), (2.43)])

$$\varphi(x, 0) = \sin\alpha + x\cos\alpha, \quad \psi(x, 0) = \cos\alpha - x\sin\alpha. \tag{2.12}$$

In [5, Section 4.3] it is shown that for $f \in L^2_m$ $(= K)$ the function

$$\mathcal{F}_-(f)(\lambda) := \frac{1}{\sin\alpha} \int_0^{b+0} \varphi(x, \lambda)\, f(x)\, dm(x) \tag{2.13}$$

belongs to $L^2_{\sigma_-}$ and

$$(\mathcal{F}_-(f), \mathcal{F}_-(g))_{\sigma_-} = \{f, g\}_- \quad (f, g \in K).$$

Therefore, $\mathcal{F}_-$ can be extended by continuity to K_-. Then, $\mathcal{F}_-$ and the restriction $\mathcal{F}_+ := \mathcal{F}_-|_{K_+}$ are isometric Hilbert space isomorphisms from

$(K_\pm,\ \{\cdot,\cdot\}_\pm)$ to $(L^2_{\sigma_\pm},\ (\cdot,\cdot)_{\sigma_\pm})$ and $\mathcal{F} := \mathcal{F}_-|_K$ is a weak Krein space isomorphism from $(K,\ [\cdot,\cdot]_m)$ to $(L^2_\sigma,\ [\cdot,\cdot]_\sigma)$, i.e. a isometric mapping between dense subspaces. In fact, the weak isomorphism $\mathcal{F}$ is at least defined on K_+ with $\mathcal{F}(K_+) = L^2_{\sigma_+}$ $(\subset L^2_\sigma)$ and it is a (usual) Krein space isomorphism if and only if infinity is not a singular critical point of A. This property is further equivalent to $\mathcal{F}(K) = L^2_\sigma$ and for certain Krein-Feller operators it may indeed happen that $\mathcal{F}(K) \neq L^2_\sigma$. Furthermore, $\mathcal{F}_\pm$ transforms $A_\pm$ into the operator of multipication with the independent variable in $L^2_{\sigma_\pm}$ and a similar transformation result also holds true for A and L^2_σ if $\mathcal{F}(K) = L^2_\sigma$. The above results were obtained in [5, Theorems 3.12, 4.12, 4.16, 4.17, 4.20]. For the terminology of "critical points" we refer to H. Langer's theory of definitizable operators in Krein spaces; cf. [11]. Additionally, consider the Weyl coefficient

$$\Omega(\lambda) := \frac{\psi'(b+0,\lambda)}{\varphi'(b+0,\lambda)}$$

for all $\lambda \in \mathbb{C}$ such that $\varphi'(b+0,\lambda) \neq 0$. Then, the solution

$$\chi(x,\lambda) := \psi(x,\lambda) - \Omega(\lambda)\varphi(x,\lambda)$$

of (2.9) always satisfies $\chi'(b+0,\lambda) = 0$; cf. [5, Section 2.3].

2.5. Krein-Feller operators of Stieltjes type

Now, a particular class of Krein-Feller operators is introduced which we call *of Stieltjes type* (a terminology which was not used in [5]). To this end, fix $\alpha \in (0,\pi/2)$ and consider a sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ with

$$0 = x_0 < x_1 < x_2 < \dots < b, \qquad x_n \longrightarrow b\ (n \to \infty). \tag{2.14}$$

Moreover, let $(m_n)_{n\in\mathbb{N}\cup\{0\}}$ be a real sequence with

$$\sum_{n=0}^{\infty} |m_n| < \infty, \quad m_n \neq 0 \quad (n \in \mathbb{N}\cup\{0\}) \tag{2.15}$$

and put

$$m(x) := \sum_{x_n<x,\, n\in\mathbb{N}\cup\{0\}} m_n, \quad ||m||(x) := \sum_{x_n<x,\, n\in\mathbb{N}\cup\{0\}} |m_n|.$$

Then, m is left-continuous and of bounded variation and $||m||$ induces the total variation of m. On L^2_m the inner products $[\cdot,\cdot]_m$ and $(\cdot,\cdot)_m$ are given by

$$[f,g]_m = \int f\overline{g}\, dm = \sum_{n=0}^{\infty} f(x_n)\overline{g(x_n)} m_n,$$

$$(f,g)_m = \int f\overline{g}\, d||m|| = \sum_{n=0}^{\infty} f(x_n)\overline{g(x_n)} |m_n|,$$

note that in this case the associated fundamental symmetry J is given by $(Jf)(x_n) = \operatorname{sgn}(m_n) f(x_n)$ $(n \in \mathbb{N}\cup\{0\})$. A function f in the domain of definition of the Krein-Feller operator A from Subsection 2.1 (i.e. $f \in D(A)$) is continuous and piecewise linear on $\mathbb{R}$ and differentiable everywhere except for the points x_n for $n \in \mathbb{N}\cup\{0\}$. In particular, f is differentiable at b since $m(\{b\}) = 0$ and by (2.3) we have $f'(b) = 0$. Here, condition (2.2) is not really a restriction on $D(A)$ but a rule how to extend the function on the left of 0: linear such that $f(-\tan\alpha) = 0$. Now, the operator A is given by $Af = -D_m D_x f$ where

$$\begin{aligned}(D_m D_x f)(0) &= \frac{1}{m_0}\left(\frac{f(x_1)-f(0)}{x_1} - \cot\alpha f(0)\right),\\ (D_m D_x f)(x_n) &= \frac{f'(x_n+0)-f'(x_n-0)}{m_n}\end{aligned}$$

$$= \frac{(x_n - x_{n-1})f(x_{n+1}) - (x_{n+1}-x_{n-1})f(x_n) + (x_{n+1}-x_n)f(x_{n-1})}{m_n(x_{n+1}-x_n)(x_n - x_{n-1})}$$

for $n \in \mathbb{N}$. Consequently, the eigenvalue problem $Af = \lambda f$ is a difference equation system. Note that in this situation all elements f from $D(A)$, H^1_m and L^2_m can be regarded as piecewise linear functions on $(-\infty, b)$ with $f(-\tan\alpha) = 0$ but different behaviour at the right endpoint:

$$\begin{aligned}&\text{for } f \in L^2_m: && \sum_{n=0}^{\infty} |f(x_n)|^2 |m_n| < \infty,\\ &\text{for } f \in H^1_m: && \sum_{n=0}^{\infty} \frac{|f(x_{n+1}) - f(x_n)|^2}{x_{n+1}-x_n} < \infty \text{ and } f(b) = \lim_{n\to\infty} f(x_n),\\ &\text{for } f \in D(A): && \lim_{n\to\infty}\frac{f(x_{n+1})-f(x_n)}{x_{n+1}-x_n} = 0 \text{ and } f(b) = \lim_{n\to\infty} f(x_n) \text{ and}\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{|(x_n - x_{n-1})f(x_{n+1}) - (x_{n+1}-x_{n-1})f(x_n) + (x_{n+1}-x_n)f(x_{n-1})|^2}{|m_n|(x_{n+1}-x_n)^2(x_n-x_{n-1})^2} < \infty.$$

The details of the above statements are similar to [5, Example 2.16, Example 2.40].

3. A collection of characteristic spectral properties

Let A be a Krein-Feller operator of Stieltjes type according to Subsection 2.5 and let $A_\pm$ and σ and $\sigma_\pm$ be given as above. Now, some spectral properties of A are collected which will later allow the reconstruction of the operator.

3.1. Characteristic properties induced by δ_n

Recall some known properties (see [5]) of the functions $\delta_0\,(= 1/m_0\,\chi_{\{x_0\}})$ from (2.5) and

$$\delta_n := \frac{1}{m_n}\chi_{\{x_n\}} \in L^2_m \quad (n \in \mathbb{N}) \tag{3.1}$$

(where $\chi_{\{x_n\}}$ denotes the characteristic function of the point set $\{x_n\}$). Then, δ_n has the piecewise linear representation in L^2_m

$$\delta_n(x) = \begin{cases} 0 & (x \le x_{n-1}), \\ \frac{x - x_{n-1}}{m_n(x_n - x_{n-1})} & (x_{n-1} \le x \le x_n), \\ \frac{x_{n+1} - x}{m_n(x_{n+1} - x_n)} & (x_n \le x \le x_{n+1}), \\ 0 & (x \ge x_{n+1}) \end{cases} \qquad ||m||\text{-a.e.}$$

and hence, $\delta_n \in D(A)$ $(\subset D(A_-))$. Here, we formally put $x_{-1} := -\tan\alpha$. Note that $A\delta_n$ is again an element of L^2_m which is different from 0 at only finitely many mass points. Therefore, $\delta_n \in D(A^2)$ and by the same argument $\delta_n \in D(A^k)$ for all $k \in \mathbb{N}$. By [5, Proposition 4.9, Proposition 4.10] we have

$$\mathcal{F}(A^k\delta_0)(t) = \mathcal{F}_-(A^k\delta_0)(t) = t^k\mathcal{F}_-(\delta_0) = t^k \in L^2_{\sigma_-} \tag{3.2}$$

for all $k \in \mathbb{N} \cup \{0\}$ where the equalities hold true σ_--a.e.; see also Subsection 2.4. Furthermore, from [5, Section 4.5] it follows for all δ_n $(\in K_+)$

$$\mathcal{F}(\delta_n) = \mathcal{F}_+(\delta_n) = \frac{\varphi(x_n,\cdot)}{\sin\alpha} \in L^2_{\sigma_+}, \tag{3.3}$$

$$\frac{1}{m_n} = [\delta_n, \delta_n]_m = [\mathcal{F}(\delta_n), \mathcal{F}(\delta_n)]_\sigma = \int_0^b \frac{|\varphi(x_n,\lambda)|^2}{\sin^2\alpha}\, d\sigma(\lambda), \tag{3.4}$$

$$\tan\alpha + x_n = (\mathcal{F}(\delta_n), \mathcal{F}(\delta_n))_{\sigma_-} = \int_0^b \frac{|\varphi(x_n,\lambda)|^2}{\sin^2\alpha}\, d\sigma_-(\lambda) \tag{3.5}$$

$(n \in \mathbb{N} \cup \{0\})$ where by the definition in (2.13) the equalities in (3.3) hold true on the whole complex plane. Using (2.12) this implies additionally

$$\mathcal{F}(\delta_n)(0) = \frac{\varphi(x_n, 0)}{\sin\alpha} = 1 + x_n\cot\alpha = \cot\alpha(\mathcal{F}(\delta_n), \mathcal{F}(\delta_n))_{\sigma_-}. \tag{3.6}$$

In particular, the function $\mathcal{F}(\delta_0) = 1$ is constant (which was already used in (3.2)). The functions δ_n $(n \ge 0)$ form an orthogonal system in the Krein space L^2_m with respect to $[\cdot,\cdot]_m$ since $[\delta_n, \delta_k]_m = \delta_n(x_k) = 0$ for $k \ne n$. Furthermore, they form a basis of L^2_m since each $f \in L^2_m$ can immediately be written as $f = \sum_{n=0}^\infty f(x_n)m_n\delta_n$ converging with respect to $(\cdot,\cdot)_m$. Using the weak Krein space isomorphism $\mathcal{F}$ we can conclude that also $\mathcal{F}(\delta_n)$ $(n \ge 0)$ form an orthogonal system in the Krein space L^2_σ (i.e. $[\mathcal{F}(\delta_n), \mathcal{F}(\delta_k)]_\sigma = 0$ for $k \ne n$). However, we cannot conclude a basis property since in general $\mathcal{F}(K) \ne L^2_\sigma$; cf. [5, Theorem 4.20].

3.2. Characteristic properties induced by γ_n

Now, consider the functions

$$\gamma_n(x) := \begin{cases} 0 & (x \le x_n), \\ \frac{x - x_n}{x_{n+1} - x_n} & (x_n \le x \le x_{n+1}), \\ 1 & (x \ge x_{n+1}) \end{cases} \tag{3.7}$$

(which were not studied in [5]). Here, we allow $n \in \mathbb{N} \cup \{0, -1\}$, again with $x_{-1} = -\tan\alpha$. Then, $\gamma_n \in D(A)$ since by $\gamma_n(-\tan\alpha) = 0$ and $\gamma_n'(b+0) = 0$

the boundary conditions are satisfied. Note that $||m||$-a.e. γ_n can be written as $\gamma_n(x) = \int_{-\infty}^{x} \delta_n \, dm$ for $n \geq 0$.

Lemma 3.1. *For $n \geq -1$ and $\lambda \in \mathbb{C}$ we have*

$$\mathcal{F}(A\gamma_n)(\lambda) = \frac{\varphi'(x_n + 0, \lambda)}{\sin\alpha}, \quad \mathcal{F}(\gamma_n)(t) = \frac{\varphi'(x_n + 0, t)}{t \sin\alpha} \quad \sigma - a.e. \ on \ \mathbb{R}$$

and in particular, $\mathcal{F}(\gamma_{-1})(t) = 1/t \ \cot\alpha$ σ-a.e. on $\mathbb{R}$.

Proof. For $n \geq 0$ the first statement follows from

$$\begin{aligned} \mathcal{F}(A\gamma_n)(\lambda)\sin\alpha &= [\varphi(\cdot,\lambda), A\gamma_n]_m = \{\varphi(\cdot,\lambda), \gamma_n\}_+ \\ &= \int_{x_n}^{x_{n+1}} \varphi'(x,\lambda)\,\gamma_n'(x)\,dx = \varphi'(x_n + 0, \lambda). \end{aligned}$$

However, the same holds true for $n = -1$ since then

$$\{\varphi(\cdot,\lambda), \gamma_{-1}\}_+ = \cot\alpha\varphi(0,\lambda)\gamma_{-1}(0) = \cos\alpha = \varphi'(x_{-1} + 0, \lambda)$$

by the boundary conditions for $\varphi(x,\lambda)$. Now, the first statement implies the second since by [5, Proposition 4.9] $\mathcal{F}$ transforms A into the multiplication by the independent variable in $L^2_{\sigma_-}$ (and the sets of measure zero coincide for $\sigma_\pm$ and σ). For γ_{-1} note again $\varphi'(x_{-1}+0,t) = \varphi'(0-0,t)$ and (2.10). □

Introducing a new notation for $n \geq -1$ and $\lambda \in \mathbb{C}$, we obtain

$$\widetilde{\mathcal{F}(\gamma_n)}(\lambda) := \frac{\varphi'(x_n + 0, \lambda)}{\sin\alpha} = \lambda\mathcal{F}(\gamma_n)(\lambda). \tag{3.8}$$

where the second equality only holds for σ-a.a. $\lambda \in \mathbb{R}$ by Lemma 3.1. Then, in particular we have

$$\widetilde{\mathcal{F}(\gamma_{-1})}(\lambda) = \cot\alpha, \qquad \widetilde{\mathcal{F}(\gamma_n)}(0) = \cot\alpha \tag{3.9}$$

for $\lambda \in \mathbb{C}$ by (2.10), (2.12). As mentioned in Subsection 2.4, $\mathcal{F} : K_+ \longrightarrow L^2_{\sigma_+}$ is isometric with respect to $\{\cdot,\cdot\}_+$ and $(\cdot,\cdot)_{\sigma_+}$ and also with respect to $[\cdot,\cdot]_m$ and $[\cdot,\cdot]_\sigma$. This result implies the following properties.

Proposition 3.2. *For $k \geq -1$, $n \geq 0$ we have*

$$(\widetilde{\mathcal{F}(\gamma_k)}, \widetilde{\mathcal{F}(\gamma_k)})_{\sigma_-} = (\mathcal{F}(\gamma_k), \mathcal{F}(\gamma_k))_{\sigma_+} = \{\gamma_k, \gamma_k\}_+ = \frac{1}{x_{k+1} - x_k}, \tag{3.10}$$

$$(\mathcal{F}(\delta_n), \widetilde{\mathcal{F}(\gamma_k)})_{\sigma_-} = [\mathcal{F}(\delta_n), \mathcal{F}(\gamma_k)]_\sigma = [\delta_n, \gamma_k]_m = \gamma_k(x_n) \tag{3.11}$$

(where $\gamma_k(x_n) = 0$ for all $n \leq k$ and $\gamma_k(x_n) = 1$ for all $n \geq k+1$ by definition of γ).

Proof. The calculations are clear. For $k = -1$ observe that $1/(x_0 - x_{-1}) = \cot\alpha$. □

Note that some of the relations (3.3) - (3.11) play a similar role as [3, Section 5.7, (1) - (6)] in the definite situation (where all $m_n > 0$).

Proposition 3.3. (i) *γ_n $(n \geq -1)$ form an orthogonal basis of $(K_+, \{\cdot,\cdot\}_+)$,*
(ii) *$A\gamma_n$ $(n \geq -1)$ form an orthogonal basis of $(K_-, \{\cdot,\cdot\}_-)$,*

(iii) $\widetilde{\mathcal{F}(\gamma_n)}$ $(n \geq -1)$ *form an orthogonal basis of* $(L^2_{\sigma_-},\ (\cdot,\cdot)_{\sigma_-})$.

Proof. (i) For $k \neq n$ the functions γ_k and γ_n have no interval in common where the derivatives do not vanish. Therefore, they are orthogonal in $(K_+,\ \{\cdot,\cdot\}_+)$. Now, let $f \in K_+ \,(= H^1_m)$ and put $\alpha_k := f'(x_k+0)(x_{k+1}-x_k)$ for $k \geq 0$ and $\alpha_{-1} := f(x_0)$. Then, for $n \in \mathbb{N}$ and $0 \leq j \leq n$ we have

$$\sum_{k=-1}^{n-1} \alpha_k \gamma_k(x_j) = \sum_{k=-1}^{j-1} \alpha_k = f(x_0) + \sum_{k=0}^{j-1} f'(x_k+0)(x_{k+1}-x_k) = f(x_j).$$

Consequently, the piecewise linear functions f and $f_n := \sum_{k=-1}^{n-1} \alpha_k \gamma_k$ coincide on $[x_0, x_n]$. Since additionally, f_n is constant on $[x_n, b]$ we obtain $\{f - f_n, f - f_n\}_+ = \int_{x_n}^{b} |f'|^2\, dx \longrightarrow 0$ for $n \to \infty$. This is the basis property in $(K_+,\ \{\cdot,\cdot\}_+)$.

(ii), (iii) By definition, the operator A_- is an isometric isomorphism from $(K_+,\ \{\cdot,\cdot\}_+)$ to $(K_-,\ \{\cdot,\cdot\}_-)$ and by [5, Theorem 4.12] also $\mathcal{F}_-$ is an isometric isomorphism from $(K_-,\ \{\cdot,\cdot\}_-)$ to $(L^2_{\sigma_-},\ (\cdot,\cdot)_{\sigma_-})$. Therefore, by (i) the functions $A\gamma_n$ $(n \geq -1)$ form an orthogonal basis in $(K_-,\ \{\cdot,\cdot\}_-)$ and hence, $\widetilde{\mathcal{F}(\gamma_n)}\ (= \mathcal{F}(A\gamma_n))$ form an orthogonal basis in $(L^2_{\sigma_-},\ (\cdot,\cdot)_{\sigma_-})$. □

3.3. Recurrence relations

Now, it is observed that δ_n and γ_n as well as $\mathcal{F}(\delta_n)$ and $\widetilde{\mathcal{F}(\gamma_n)}$ are connected by recurrence equations.

Lemma 3.4. *For $n \geq 0$ we have*

$$\delta_n = \frac{1}{m_n}(\gamma_{n-1} - \gamma_n), \tag{3.12}$$

$$A\gamma_n = \frac{1}{x_{n+1} - x_n}(\delta_{n+1} - \delta_n). \tag{3.13}$$

Proof. For the first equation check the functions at x_n and for the second at x_n and x_{n+1}. Obviously, the functions vanish at all other mass points. □

Lemma 3.5. (i) *For $n \geq 0$ and $\lambda \in \mathbb{C}$ we have*

$$\lambda \mathcal{F}(\delta_n)(\lambda) = \frac{1}{m_n}(\widetilde{\mathcal{F}(\gamma_{n-1})}(\lambda) - \widetilde{\mathcal{F}(\gamma_n)}(\lambda)), \tag{3.14}$$

$$\widetilde{\mathcal{F}(\gamma_n)} = \frac{1}{x_{n+1} - x_n}(\mathcal{F}(\delta_{n+1}) - \mathcal{F}(\delta_n)). \tag{3.15}$$

(ii) *The function $\mathcal{F}(\delta_n)$ is a polynomial of exact degree n $(n \geq 0)$.*

(iii) *The function $\widetilde{\mathcal{F}(\gamma_n)}$ is a polynomial of exact degree $n+1$ $(n \geq -1)$.*

Proof. Equation (3.15) follows from (3.13) and Lemma 3.1 by the linearity of $\mathcal{F}$. Equation (3.14) follows from (3.3) by a direct calculation since $\varphi(\cdot, \lambda)$

is a solution of (2.9):

$$\begin{aligned} \lambda\mathcal{F}(\delta_n) &= \frac{\lambda\varphi(x_n,\lambda)}{\sin\alpha} = \frac{\varphi'(x_n-0,\lambda)-\varphi'(x_n+0,\lambda)}{m_n\sin\alpha} \\ &= \frac{1}{m_n}(\widetilde{\mathcal{F}(\gamma_{n-1})}(\lambda)-\widetilde{\mathcal{F}(\gamma_n)}(\lambda)). \end{aligned}$$

This is (i) and it implies (ii) and (iii) by induction. Indeed, we have $\mathcal{F}(\delta_0)=1$ and $\widetilde{\mathcal{F}(\gamma_{-1})}=\cot\alpha$ by (3.3), (3.9) and therefore, $\widetilde{\mathcal{F}(\gamma_0)}(\lambda)=\cot\alpha-m_0\lambda$ by (3.14) and $\mathcal{F}(\delta_1)(\lambda)=(x_1-x_0)(\cot\alpha-m_0\lambda)+1$ by (3.15). Clearly, if $\mathcal{F}(\delta_n)$ and $\widetilde{\mathcal{F}(\gamma_{n-1})}$ are polynomials of degree n then $\widetilde{\mathcal{F}(\gamma_n)}$ has degree $n+1$ by (3.14), and $\mathcal{F}(\delta_{n+1})$ has degree $n+1$ by (3.15). □

3.4. A moment problem associated with the A_--spectral function

By (3.2) we have

$$e_k := A^k\delta_0 \in D(A), \quad \mathcal{F}(e_k)(t)=t^k\in L^2_{\sigma_-} \quad (k\in\mathbb{N}\cup\{0\}). \tag{3.16}$$

Consequently, for σ_- the power moments of all orders exist satisfying

$$\int_{-\infty}^{\infty} t^k\, d\sigma_- = (\mathcal{F}(e_k),\mathcal{F}(e_0))_{\sigma_-} = \{e_k,e_0\}_- =: s_k \quad (k\in\mathbb{N}\cup\{0\}). \tag{3.17}$$

With these moments which can also be expressed as

$$s_k = [A^{-1}e_k,e_0]_m = \begin{cases} [e_{k-1},\delta_0]_m = e_{k-1}(0), \\ [A^{-1}\delta_0,\delta_0]_m = \sin\alpha\chi(0,0)=\tan\alpha \end{cases} \tag{3.18}$$

(cf. [5, Proposition 4.3, Proposition 2.11]) $\mu:=\sigma_-$ is a solution of the classical Hamburger moment problem

$$\int_{-\infty}^{\infty} t^k\, d\mu = s_k \quad (k\in\mathbb{N}\cup\{0\}). \tag{3.19}$$

Furthermore, by Proposition 3.3 and Lemma 3.5 the space $\mathcal{P}[\mathbb{C}]$ of the algebraic polynomials with complex coefficients is dense in $L^2_{\sigma_-}$. According to M. Riesz' theorem (cf. [1, Theorem 2.3.3]), all solutions μ of the moment problem (3.19) such that $\mathcal{P}[\mathbb{C}]$ is dense in L^2_μ are precisely the so-called N-extremal solutions of (3.19). Here, we use this density property as the definition of an *N-extremal* solution. Then, in particular, σ_- is N-extremal.

Now, let $\widehat{\sigma}_-$ be another non-decreasing left-continuous N-extremal solution of the moment problem (3.19). Put $\widehat{\sigma}(t) := \int_0^t s\, d\widehat{\sigma}_-(s)$ and $\widehat{\sigma}_+(t) := \int_0^t s^2\, d\widehat{\sigma}_-(s)$ and let the inner products $(\cdot,\cdot)_{\widehat{\sigma}_\pm}$ and $[\cdot,\cdot]_{\widehat{\sigma}}$ in $L^2_{\widehat{\sigma}_\pm}$ and $L^2_{\widehat{\sigma}}$ be given as in (2.7). Then, for all polynomials P, Q we have $(P,Q)_{\widehat{\sigma}_\pm}=(P,Q)_{\sigma_\pm}$ and $[P,Q]_{\widehat{\sigma}}=[P,Q]_\sigma$ since these values are linear combinations of the moments s_k $(k\in\mathbb{N}\cup\{0\})$. Therefore the following properties, obtained in Subsections 3.1 and 3.2 for $\sigma_\pm$ and σ, remain true for $\widehat{\sigma}_\pm$ and $\widehat{\sigma}$: Recall that δ_n is defined for $n\geq 0$ and γ_n is defined for $n\geq -1$.

(i) The polynomials $\widetilde{\mathcal{F}(\gamma_n)}$ form an orthogonal basis of $(L^2_{\widehat{\sigma}_-},\ (\cdot,\cdot)_{\widehat{\sigma}_-})$.

(ii) The polynomials $\mathcal{F}(\delta_n)$ form an orthogonal system of $(L^2_{\widehat{\sigma}},\ [\cdot,\cdot]_{\widehat{\sigma}})$.

(iii) $\sum_{k=0}^{\infty} 1/(\widetilde{\mathcal{F}(\gamma_k)},\widetilde{\mathcal{F}(\gamma_k)})_{\widehat{\sigma}_-} = \sum_{k=0}^{\infty} x_{k+1}-x_k = b < \infty$,

(iv) $\widetilde{\mathcal{F}(\gamma_n)}(0) = \cot\alpha$,

(v) $[\mathcal{F}(\delta_n),\mathcal{F}(\delta_n)]_{\widehat{\sigma}} \neq 0, \quad \sum_{k=0}^{\infty} 1/|[\mathcal{F}(\delta_k),\mathcal{F}(\delta_k)]_{\widehat{\sigma}}| = \sum_{k=0}^{\infty} |m_k| < \infty$,

(vi)

$$\begin{aligned}\mathcal{F}(\delta_n)(0) &= 1+\cot\alpha\sum_{k=0}^{n-1}(x_{k+1}-x_k)\\ &= 1+\cot\alpha\sum_{k=0}^{n-1} 1/(\widetilde{\mathcal{F}(\gamma_k)},\widetilde{\mathcal{F}(\gamma_k)})_{\widehat{\sigma}_-},\end{aligned}$$

(vii) $(\mathcal{F}(\delta_n),\widetilde{\mathcal{F}(\gamma_k)})_{\widehat{\sigma}_-} = 1 \quad (k=-1,0,...,n-1,\ n\geq 0)$.

(Here, in fact, (iv) is a property which does not involve $\widehat{\sigma}_-$.) Obviously, the above properties are independent of the choice of $\widehat{\sigma}_-$ (but depend on the moments in (3.19)). Below we shall see that these properties characterize the Krein-Feller operator A completely. Therefore, the following definition is justified.

Definition 3.6. The set $[\sigma]$ of all functions $\widehat{\sigma}$ which arise from a nondecreasing left-continuous N-extremal solution $\widehat{\sigma}_-$ of (3.19) by $\widehat{\sigma}(t) = \int_0^t s\ d\widehat{\sigma}_-(s)$ is called the *spectral class* of A.

Obviously, we have $\sigma \in [\sigma]$. Furthermore, the spectral function σ is completely characterized in $[\sigma]$ by the following integral representation.

Proposition 3.7. *Assume that $\widehat{\sigma} \in [\sigma]$. Then, we have $\widehat{\sigma} = \sigma$ if and only if*

$$\int_{-\infty}^{\infty} \frac{d\widehat{\sigma}(t)}{t-\lambda} = \sin^2\alpha\left(\cot\alpha - \frac{\psi'(b+0,\lambda)}{\varphi'(b+0,\lambda)}\right) \tag{3.20}$$

for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proof. First note that by [5, Theorem 4.6] the representation (3.20) holds true for $\widehat{\sigma} := \sigma$. Now, let $\widehat{\sigma}$ be induced by an arbitrary N-extremal solution $\widehat{\sigma}_-$ in the form $\widehat{\sigma}(t) = \int_0^t s\ d\widehat{\sigma}_-(s)$ satisfying (3.20). Since the first moments coincide for $\widehat{\sigma}_-$ and σ_- we have

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\lambda}{t-\lambda}d\widehat{\sigma}_-(t) &= \int_{-\infty}^{\infty}\left(\frac{t}{t-\lambda}-1\right)d\widehat{\sigma}_-(t) = \int_{-\infty}^{\infty}\frac{d\widehat{\sigma}(t)}{t-\lambda} - \int_{-\infty}^{\infty}d\widehat{\sigma}_-(t)\\ &= \int_{-\infty}^{\infty}\frac{d\sigma(t)}{t-\lambda} - \int d\sigma_-(t) = \int_{-\infty}^{\infty}\frac{\lambda}{t-\lambda}d\sigma_-(t).\end{aligned}$$

Then, by the generalized inversion formula of Stieltjes-Livsic (see e.g. [11, Section II.1, Corollary 2], [5, Proposition A.10]) we have $\widehat{\sigma}(t) = \int_0^t s\, d\widehat{\sigma}_-(s) = \int_0^t s\, d\sigma_-(s) = \sigma(t)$. □

4. The inverse spectral problem

4.1. The backwards problem

The properties (i) - (vii) of subsection 3.4 are properties of certain polynomials given by the Fourier transform of functions associated with a Krein-Feller operator of Stieltjes type. Now we start with a left-continuous nondecreasing real function τ_- and an associated non-monotone function τ. Then, assuming the existence of certain polynomials with properties like (i) - (vii), we reconstruct a Krein-Feller operator of Stieltjes type by adapting the techniques from the definite case; cf. [3, Section 5.8]. First we fix $\alpha \in (0, \pi/2)$ and introduce a class of non-monotone functions which we will call "candidate of a spectral function".

To this end, consider a left-continuous nondecreasing real function τ_- on $\mathbb{R}$ such that the power moments of all orders

$$t_k := \int_{-\infty}^{\infty} t^k\, d\tau_-(t) \quad (k \in \mathbb{N} \cup \{0\}) \tag{4.1}$$

exist. Put

$$\tau(t) := \int_0^t s\, d\tau_-(s), \quad \tau_+(t) := \int_0^t s^2\, d\tau_-(s) \quad (t \in \mathbb{R}). \tag{4.2}$$

Then, τ_+ is non-decreasing and τ is non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, \infty)$. With

$$(F, G)_{\tau_\pm} := \int_{-\infty}^{\infty} F\overline{G}\, d\tau_\pm\,, \quad [F, G]_\tau := \int_{-\infty}^{\infty} F\overline{G}\, d\tau, \quad (F, G \in L^2_{\tau_\pm}, L^2_\tau).$$

$(L^2_{\tau_+},\ (\cdot,\cdot)_{\tau_+})$ and $(L^2_{\tau_-},\ (\cdot,\cdot)_{\tau_-})$ are Hilbert spaces and $(L^2_\tau,\ [\cdot,\cdot]_\tau)$ is a Krein space. Although an analogue of the space triplet (2.8) cannot be guaranteed so far we have the inclusion $\mathcal{P}[\mathbb{C}] \subset (L^2_{\tau_-} \cap L^2_\tau \cap L^2_{\tau_+})$ and for polynomials $P, Q \in \mathcal{P}[\mathbb{C}]$ and $\widetilde{P}(t) := tP(t)$ the relations

$$(\widetilde{P}, Q)_{\tau_-} = [P, Q]_\tau, \quad [\widetilde{P}, Q]_\tau = (P, Q)_{\tau_+}.$$

Now assume that there are polynomials $\widetilde{\Gamma}_n$ of exact degree $n+1$ ($n \in \mathbb{N} \cup \{-1, 0\}$) forming an orthogonal basis of $(L^2_{\tau_-},\ (\cdot,\cdot)_{\tau_-})$ such that

$$\sum_{k=-1}^{\infty} \frac{1}{(\widetilde{\Gamma}_k, \widetilde{\Gamma}_k)_{\tau_-}} \quad < \quad \infty, \tag{4.3}$$

$$\widetilde{\Gamma}_n(0) \quad = \quad \cot\alpha \qquad (n \geq -1). \tag{4.4}$$

Additionally, assume that there are polynomials Δ_n of exact degree n ($n \in \mathbb{N} \cup \{0\}$) forming an orthogonal system in the Krein space $(L^2_\tau,\ [\cdot,\cdot]_\tau)$ (i.e.

$[\Delta_n, \Delta_k]_\tau = 0$ for $n \neq k$) such that all Δ_n are not neutral (i.e. $[\Delta_n, \Delta_n]_\tau \neq 0$) and

$$\sum_{k=0}^{\infty} \frac{1}{|[\Delta_k, \Delta_k]_\tau|} \quad < \quad \infty, \tag{4.5}$$

$$\Delta_n(0) \quad = \quad 1 + \cot\alpha \sum_{k=0}^{n-1} \frac{1}{(\widetilde{\Gamma}_k, \widetilde{\Gamma}_k)_{\tau_-}}, \tag{4.6}$$

$$(\Delta_n, \widetilde{\Gamma}_k)_{\tau_-} \quad = \quad 1 \qquad (k = -1, 0, ..., n-1). \tag{4.7}$$

In (4.6) and below, an "empty" sum is considered as 0. As announced before, we call each function τ obtained from a function τ_- as above by (4.2) and which allows the existence of polynomials as described above satisfying (4.3)-(4.7) a *candidate for a spectral function.*

Now, for the given candidate τ and $n \in \mathbb{N} \cup \{0\}$ put

$$m_n := \frac{1}{[\Delta_n, \Delta_n]_\tau}, \qquad x_n := \sum_{k=0}^{n-1} \frac{1}{(\widetilde{\Gamma}_k, \widetilde{\Gamma}_k)_{\tau_-}}, \qquad b := \sum_{k=0}^{\infty} \frac{1}{(\widetilde{\Gamma}_k, \widetilde{\Gamma}_k)_{\tau_-}}. \tag{4.8}$$

Then the conditions (2.14), (2.15) from Subsection 2.5 are satisfied and (x_n) and (m_n) together with $\alpha \in (0, \pi/2)$ induce an indefinite Krein-Feller operator A of Stieltjes type according to Subsection 2.5. Let again σ denote its spectral function according to Subsection 2.3.

Before we state the "backwards statement" note that $\mu := \tau_-$ is an N-extremal solution of the Hamburger moment problem

$$\int_{-\infty}^{\infty} t^k \, d\mu = t_k \quad (k \in \mathbb{N} \cup \{0\}). \tag{4.9}$$

Let $\widehat{\tau}_-$ be another left-continuous nondecreasing N-extremal solution of (4.9) and put $\widehat{\tau}(t) = \int_0^t s \, d\widehat{\tau}_-(s)$. Then, as in Subsection 3.4, the conditions above are satisfied again for $\widehat{\tau}_-$ and $\widehat{\tau}$ with the same polynomials and with the same values in (4.3) - (4.8) as observed for τ_- and τ. In particular, $\widehat{\tau}$ is another candidate for a spectral function. Furthermore, the moments (4.1) induce an equivalence relation on the set of all candidates for a spectral function if we relate two candidates belonging to N-extremal solutions of the same moment problem. Then, as in Definition 3.6, the set $[\tau]$ of all functions $\widehat{\tau}$ which arise from a left-continuous nondecreasing N-extremal solution $\widehat{\tau}_-$ of (4.9) by $\widehat{\tau}(t) = \int_0^t s \, d\widehat{\tau}_-(s)$ is an associated equivalence class. Obviously, we have $\tau \in [\tau]$ again. Note that each spectral class of a Krein Feller operator of Stieltjes type is one of these equivalence classes.

Theorem 4.1. *Let $\alpha \in (0, \pi/2)$, let τ be a candidate for a spectral function and let A be the Krein-Feller operator of Stieltjes type induced by* (4.8)*. Then the class $[\tau]$ coincides with the spectral class $[\sigma]$ of A, i.e. $[\tau] = [\sigma]$.*

A proof of this theorem will be given in the following subsection. Additionally, note that a characterization of the spectral function σ in the class $[\tau]$ is given by Proposition 3.7.

4.2. Proof of the backwards statement

First, in analogy to [3, Lemma 5.8.1] we show that Δ_n and $\widetilde{\Gamma}_n$ satisfy the same recurrence equations as already observed in Lemma 3.5.

Lemma 4.2. *For $n \geq 0$ and $\lambda \in \mathbb{C}$ we have*

$$\lambda\Delta_n(\lambda) = \frac{1}{m_n}(\widetilde{\Gamma}_{n-1}(\lambda) - \widetilde{\Gamma}_n(\lambda)), \tag{4.10}$$

$$\widetilde{\Gamma}_n = \frac{1}{x_{n+1} - x_n}(\Delta_{n+1} - \Delta_n). \tag{4.11}$$

Proof. In addition to $\widetilde{\Gamma}_n$ we introduce the functions

$$\Gamma_n(\lambda) := \frac{\widetilde{\Gamma}_n(\lambda)}{\lambda}$$

for $n \geq -1$ and $\lambda \in \mathbb{C} \setminus \{0\}$. First check (4.10): By (4.4) we have $\widetilde{\Gamma}_{n-1}(0) - \widetilde{\Gamma}_n(0) = 0$. Therefore, $\Gamma_{n-1} - \Gamma_n$ is a polynomial of degree n and hence, $\Gamma_{n-1} - \Gamma_n = c_0\Delta_0 + ... + c_n\Delta_n$ with some $c_0, ..., c_n \in \mathbb{C}$. Furthermore, for all $k \leq n-1$ we have

$$\begin{aligned} c_k[\Delta_k, \Delta_k]_\tau &= [\Gamma_{n-1} - \Gamma_n, \Delta_k]_\tau = (\widetilde{\Gamma}_{n-1} - \widetilde{\Gamma}_n, \Delta_k)_{\tau_-} \\ &= (\widetilde{\Gamma}_{n-1}, \Delta_k)_{\tau_-} - (\widetilde{\Gamma}_n, \Delta_k)_{\tau_-} = 0 \end{aligned}$$

since the polynomials Δ_n form an orthogonal system in $(L^2_\tau,\ [\cdot,\cdot]_\tau)$ and the polynomials $\widetilde{\Gamma}_n$ form an orthogonal basis in $(L^2_{\tau_-},\ (\cdot,\cdot)_{\tau_-})$. Consequently, $c_0 = ... = c_{n-1} = 0$ since none of the Δ_k is neutral. Hence, $\Gamma_{n-1} - \Gamma_n = c_n\Delta_n$ with $c_n \neq 0$. Calculating c_n, we observe as before that

$$\frac{c_n}{m_n} = c_n[\Delta_n, \Delta_n]_\tau = [\Gamma_{n-1} - \Gamma_n, \Delta_n]_\tau = (\widetilde{\Gamma}_{n-1}, \Delta_n)_{\tau_-} = 1$$

by (4.7). This implies $\Gamma_{n-1} - \Gamma_n = m_n\Delta_n$ and hence, (4.10).

Now, check (4.11): For $n \in \mathbb{N} \cup \{0\}$ consider the polynomial $p_n(\lambda) := (\Delta_{n+1}(\lambda) - \Delta_n(\lambda))(\widetilde{\Gamma}_n, \widetilde{\Gamma}_n)_{\tau_-}$ of degree $n+1$. Let $-1 \leq k \leq n-1$. By (4.4), the function $q_k(\lambda) := \Gamma_k(\lambda) - \cot\alpha/\lambda \,(= (\widetilde{\Gamma}_k(\lambda) - \cot\alpha)/\lambda)$ is a polynomial of degree k. This implies

$$(\widetilde{\Gamma}_k, p_n)_{\tau_-} = (\widetilde{\Gamma}_k - \cot\alpha, p_n)_{\tau_-} + (\cot\alpha, p_n)_{\tau_-} = [q_k, p_n]_\tau + (\widetilde{\Gamma}_{-1}, p_n)_{\tau_-}$$

using (4.4) again. Here, $[q_k, p_n]_\tau = 0$ since the polynomials Δ_n form an orthogonal system in $(L^2_\tau,\ [\cdot,\cdot]_\tau)$. Additionally, we have $(\widetilde{\Gamma}_{-1}, p_n)_{\tau_-} = 0$ by (4.7). Hence, the polynomial p_n is orthogonal to $\widetilde{\Gamma}_{-1}, ..., \widetilde{\Gamma}_{n-1}$ in $(L^2_{\tau_-},\ (\cdot,\cdot)_{\tau_-})$. Consequently, $p_n = c\widetilde{\Gamma}_n$ with some $c \in \mathbb{C}$. Calculating c, we observe that by (4.6) and (4.4)

$$p_n(0) = (\Delta_{n+1}(0) - \Delta_n(0))(\widetilde{\Gamma}_n, \widetilde{\Gamma}_n)_{\tau_-} = \cot\alpha = \widetilde{\Gamma}_n(0).$$

This implies $c = 1$ and hence, $p_n = \widetilde{\Gamma}_n$ which is (4.11). □

Now, starting with the Krein-Feller operator of Stieltjes type induced by (4.8) and $\alpha \in (0, \pi/2)$, let σ_-, σ, σ_+ be given by (2.6). Furthermore, let the solutions φ, ψ and the Fourier transformations $\mathcal{F}_+$, $\mathcal{F}$, $\mathcal{F}_-$ be given according to Subsection 2.4 and δ_n, γ_n, $\widetilde{\mathcal{F}(\gamma_n)}$ according to (3.1), (3.7), (3.8).

Lemma 4.3. *We have*

$$\mathcal{F}(\delta_n) = \Delta_n \quad (n \in \mathbb{N} \cup \{0\}), \qquad \widetilde{\mathcal{F}(\gamma_n)} = \widetilde{\Gamma}_n \quad (n \in \mathbb{N} \cup \{-1, 0\}). \quad (4.12)$$

Proof. By Lemma 3.5 and Lemma 4.2, $\mathcal{F}(\delta_n)$, $\widetilde{\mathcal{F}(\gamma_n)}$ and Δ_n, $\widetilde{\Gamma}_n$ satisfy the same recurrence equations. Since additionally, $\mathcal{F}(\delta_0) = 1 = \Delta_0$ and $\widetilde{\mathcal{F}(\gamma_{-1})} = \cot\alpha = \widetilde{\Gamma}_{-1}$ the functions $\mathcal{F}(\delta_n)$ and Δ_n as well as $\widetilde{\mathcal{F}(\gamma_n)}$ and $\widetilde{\Gamma}_n$ coincide. □

We know that the set $\mathcal{P}[\mathbb{C}]$ of all polynomials is included in $L^2_{\sigma_-}$ as well as in $L^2_{\tau_-}$ and $\mathcal{P}[\mathbb{C}]$ is dense in both spaces.

Lemma 4.4. *On the space* $\mathcal{P}[\mathbb{C}]$ $(\subset L^2_{\sigma_-} \cap L^2_{\tau_-})$ *the inner products* $(\cdot,\cdot)_{\sigma_-}$ *and* $(\cdot,\cdot)_{\tau_-}$ *coincide.*

Proof. By Proposition 3.3 and by the assumptions on τ_- the polynomials $\widetilde{\Gamma}_n$ $(= \widetilde{\mathcal{F}(\gamma_n)})$ form an orthogonal system with respect to $(\cdot,\cdot)_{\tau_-}$ as well as to $(\cdot,\cdot)_{\sigma_-}$. Then the statement follows from $(\widetilde{\Gamma}_n, \widetilde{\Gamma}_n)_{\tau_-} = 1/(x_{n+1} - x_n) = (\widetilde{\Gamma}_n, \widetilde{\Gamma}_n)_{\sigma_-}$; cf. Proposition 3.2. □

From Lemma 4.4 we conclude that the power moments of all orders for σ_- and for τ_- coincide, or, in other words, σ_- and τ_- are N-extremal solutions of the same moment problem, i.e (3.19) is equivalent to (4.9). Then, by the definition of the classes $[\sigma]$ and $[\tau]$ these classes coincide, i.e. $[\sigma] = [\tau]$. This completes the proof of Theorem 4.1.

4.3. The uniqueness statement

Now, for some $\alpha \in (0, \pi/2)$ and a given candidate of a spectral function τ according to Subsection 4.1 it is shown that the operator A induced by the sequences (m_n) and (x_n) from (4.8) is the only Krein-Feller operator of Stieltjes type such that $[\tau]$ coincides with its spectral class. Adapting again the techniques from the definite case (cf. [3, Section 5.9]), we obtain this result in the following formulation.

Theorem 4.5. *Let* $\alpha \in (0, \pi/2)$ *and let* τ *be a candidate of a spectral function associated with* τ_- *and* τ_+ *according to Subsection* 4.1. *Furthermore, let* (x_n) *and* (m_n) *be two arbitrary sequences satisfying* (2.14), (2.15) *such that for the associated Krein-Feller operator* A *of Stieltjes type the spectral class* $[\sigma]$ *coincides with* $[\tau]$. *Then,* (x_n) *and* (m_n) *are determined by* (4.8).

Proof. Let σ_-, σ, σ_+ be induced by A according to (2.6). Then, $[\sigma] = [\tau]$ implies the coincidence of the moments associated with σ_- and τ_-. Indeed, for $k \geq 1$ we obtain $\int_\infty^\infty t^k \, d\sigma_- = \int_\infty^\infty t^{k-1} \, d\sigma = \int_\infty^\infty t^k \, d\tau_-$ and $\int_\infty^\infty \, d\sigma_- = \tan\alpha = \int_\infty^\infty \, d\tau_-$ for $k = 0$ by (3.18), (4.4), (4.6) and (4.7). Consequently, the inner products $(\cdot,\cdot)_{\sigma\pm}$ and $(\cdot,\cdot)_{\tau\pm}$ as well as $[\cdot,\cdot]_\sigma$ and $[\cdot,\cdot]_\tau$ coincide on $\mathcal{P}[\mathbb{C}]$. Moreover, let the solutions φ, ψ and the Fourier transformations $\mathcal{F}_+$, $\mathcal{F}$, $\mathcal{F}_-$ be induced by A according to Subsection 2.4 and δ_n, γ_n, $\widetilde{\mathcal{F}(\gamma_n)}$ according to (3.1), (3.7), (3.8). In order to show (4.8), we first consider $n = 0$: By Subsection 3.1 and (4.6) we have $\mathcal{F}(\delta_0) = 1 = \Delta_0$ and hence,

$$m_0 = \frac{1}{[\mathcal{F}(\delta_0),\mathcal{F}(\delta_0)]_\sigma} = \frac{1}{[\Delta_0,\Delta_0]_\tau}$$

by (3.4). According to (3.9) and (4.4) we find $\widetilde{\mathcal{F}(\gamma_{-1})} = \cot\alpha = \widetilde{\Gamma}_{-1}$ and hence, by Lemma 3.5 and Lemma 4.2

$$\begin{aligned}\widetilde{\mathcal{F}(\gamma_0)}(\lambda) &= \widetilde{\mathcal{F}(\gamma_{-1})}(\lambda) - \lambda m_0 \mathcal{F}(\delta_0)(\lambda) = \cot\alpha - \lambda m_0\\ &= \widetilde{\Gamma}_{-1}(\lambda) - \lambda m_0\Delta_0(\lambda) = \widetilde{\Gamma}_0(\lambda).\end{aligned}$$

This implies with $x_0 = 0$

$$x_1\,(= x_1 - x_0) = \frac{1}{(\widetilde{\mathcal{F}(\gamma_0)},\widetilde{\mathcal{F}(\gamma_0)})_{\sigma_-}} = \frac{1}{(\widetilde{\Gamma}_0,\widetilde{\Gamma}_0)_{\tau_-}}$$

by (3.10). Now, assume that for some $n \in \mathbb{N}\cup\{0\}$ we already know that

$$\mathcal{F}(\delta_n) = \Delta_n, \quad \widetilde{\mathcal{F}(\gamma_n)} = \widetilde{\Gamma}_n, \quad m_n = \frac{1}{[\Delta_n,\Delta_n]_\tau}, \quad x_{n+1} - x_n = \frac{1}{(\widetilde{\Gamma}_n,\widetilde{\Gamma}_n)_{\tau_-}}.$$

Then, we can apply the second equation in Lemma 3.5 and Lemma 4.2 with the same value of $x_{n+1} - x_n$. It follows that $\mathcal{F}(\delta_{n+1}) = \Delta_{n+1}$ and hence, $m_{n+1} = 1/[\Delta_{n+1},\Delta_{n+1}]_\tau$ by (3.4). Then, we can also apply the first equation in Lemma 3.5 and Lemma 4.2 for $n+1$ with the same value of m_{n+1} and we obtain, as above, $\widetilde{\mathcal{F}(\gamma_{n+1})} = \widetilde{\Gamma}_{n+1}$. This implis $x_{n+2} - x_{n+1} = 1/(\widetilde{\Gamma}_{n+1},\widetilde{\Gamma}_{n+1})_{\tau_-}$ by (3.10). This completes the proof. □

A combination of Theorem 4.1 and Theorem 4.5 gives our main result.

Theorem 4.6. *For a given $\alpha \in (0,\pi/2)$ the mapping $A \to [\sigma]$ which associates the operator with its spectral class defines a one-to-one correspondence between all Krein-Feller operators A of Stieltjes type (according to Subsection* 2.5*) and all classes $[\tau]$ where τ is a candidate of a spectral function according to Subsection* 4.1*. The inverse mapping is induced by* (4.8)*.*

Remark 4.7. Since the classes $[\tau]$ are completely characterized by the associated moments (4.1) this result also implies a one-to-one correspondence by (3.18) between all Krein-Feller operators A of Stieltjes type and all moment problems (4.9) induced by functions τ_- satisfying the conditions from Subsection 4.1. Then, the moment problems (3.19) and (4.9) are equivalent.

5. Characterizations of all N-extremal solutions of the moment problem

Let again, $\alpha \in (0, \pi/2)$ and (x_n), (m_n) satisfy (2.14), (2.15) and consider the associated Krein-Feller operator A of Stieltjes type in the Krein space $(L^2_m, [\cdot,\cdot]_m)$ according to Subsection 2.5. We use the same notations as in the Sections 2 and 3, in particular, for the self-adjoint operator A_- in the Hilbert space $(K_-, \{\cdot,\cdot\}_-)$ according to Subsection 2.2 and for the elements $\delta_n \in D(A)$ $(n \geq 0)$ from (2.5), (3.1) and $\gamma_n \in D(A)$ $(n \geq -1)$ from (3.7). In this section all N-extremal solutions of the moment problem (3.19) (or, equivalently (4.9)) and hence, the elements of the spectral class of A are characterized in detail.

5.1. A characterization by self-adjoint extensions of a symmetric operator

In view of the inverse results from Section 4 it first remains to clarify whether the moment problem (3.19) is determinate or not. Recall that a moment problem is called *determinate* if it has only a single solution in the sense that two functions solving the moment problem are considered to be the same solution if the difference of them is a constant function. A moment problem which is not determinate is called *indeterminate*; cf [1, Chapter 2]. Here, the question of determinism is also the question whether the spectral class $[\sigma]$ includes only a single function. In order to see that the answer is "no" consider the elements

$$g_k = \eta_k A\gamma_k \in D(A) \quad (k \geq -1) \tag{5.1}$$

where the real numbers η_k are given by

$$\eta_{-1} := \frac{1}{\sqrt{\{A\gamma_{-1}, A\gamma_{-1}\}_-}}, \quad \eta_{k+1} := -\frac{\operatorname{sgn}(\eta_k m_{k+1})}{\sqrt{\{A\gamma_{k+1}, A\gamma_{k+1}\}_-}} \quad (k \geq -1).$$

Then, by Proposition 3.3 and Lemma 3.4 the elements g_k $(k \geq -1)$ form an orthonormal basis of $(K_-, \{\cdot,\cdot\}_-)$ satisfying

$$\{Ag_k, g_i\}_- = [g_k, g_i]_m = \eta_k\eta_i[A\gamma_k, A\gamma_i]_m = \frac{\eta_k\eta_i[\delta_{k+1} - \delta_k, \delta_{i+1} - \delta_i]_m}{(x_{k+1} - x_k)(x_{i+1} - x_i)} \tag{5.2}$$

where the last equation only holds for $k, i \geq 0$. In addition to the representation of $A\gamma_k$ for $k \geq 0$ from Lemma 3.4 we have

$$A\gamma_{-1} = \delta_0 \cot\alpha$$

since $\gamma_{-1} = 1$ $||m||$−a.e. and hence $(D_m D_x \gamma_{-1})(x_n) = 0$ for $n \geq 1$ and $(D_m D_x \gamma_{-1})(x_0) = -(\cot\alpha)/m_0$. Therefore, (5.2) implies for $k \geq 0$

$$\begin{aligned}
\{Ag_k, g_i\}_- &= 0, \qquad \text{if } |i-k| > 1,\\
b_k := \{Ag_k, g_{k+1}\}_- &= -\frac{\eta_k\eta_{k+1}}{m_{k+1}(x_{k+1} - x_k)(x_{k+2} - x_{k+1})} > 0,\\
a_k := \{Ag_k, g_k\}_- &= \frac{\eta_k^2(m_{k+1} + m_k)}{m_{k+1}m_k(x_{k+1} - x_k)^2}
\end{aligned}$$

and for $k = -1$

$$\begin{aligned} \{Ag_{-1}, g_i\}_- &= 0, \qquad\qquad \text{if } i > 0, \\ b_{-1} := \{Ag_{-1}, g_0\}_- &= -\frac{\eta_{-1}\eta_0 \cot\alpha}{m_0 x_1} > 0, \\ a_{-1} := \{Ag_{-1}, g_{-1}\}_- &= \frac{\eta_{-1}^2 \cot^2\alpha}{m_0}. \end{aligned}$$

Consequently,

$$\begin{aligned} J_- &:= (\{A_- g_k, g_i\}_-)_{k,i=-1}^{\infty} = (\{Ag_k, g_i\}_-)_{k,i=-1}^{\infty} \\ &= \begin{pmatrix} a_{-1} & b_{-1} & 0 & 0 & \dots \\ b_{-1} & a_0 & b_0 & 0 & \dots \\ 0 & b_0 & a_1 & b_1 & \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \end{aligned}$$

is a $\mathcal{J}$-matrix in the sense of [1], i.e. a symmetric tridiagonal matrix with real entries on the diagonal and positive entries on the subdiagonal. According to [1, Section 1.1 in Chapter 4] J_- together with the orthonormal basis g_k ($k \geq -1$) defines a closed symmetric operator S_- in the Hilbert space $(K_-, \{\cdot,\cdot\}_-)$ as the minimal closed symmetric linear operator satisfying

$$S_- g_k = b_{k-1} g_{k-1} + a_k g_k + b_k g_{k+1} \qquad (k \geq -1)$$

with $b_{-2} := 0$. Here "minimal" means that any other closed symmetric operator with this property is an extension of S_-. By the construction we have

$$A_- g_k = \sum_{i=-1}^{\infty} \{A_- g_k, g_i\}_- g_i = S_- g_k \qquad (k \geq -1)$$

and hence, A_- is a self-adjoint extension of S_-. According to [1] every $\mathcal{J}$-matrix induces a (so-called positive) sequence of moments such that the associated Hamburger moment problem has a solution. In our case, by [1, Section 1.3 in Chapter 4] the moments induced by J_- are given by

$$\{S_-^k g_{-1}, g_{-1}\}_- = \eta_{-1}^2 \cot^2\alpha \{A^k \delta_0, \delta_0\}_- = \frac{\cot^2\alpha \{e_k, e_0\}_-}{[\gamma_{-1}, A\gamma_{-1}]_m} = \cot\alpha\, s_k \quad (5.3)$$

for $k \geq 0$ using the moments s_k from (3.17). Therefore, up to the factor $\cot\alpha$ the moment problem (3.19) coincides with the moment problem induced by J_- and hence, this also holds true for the solutions. This observation allows us to apply the results from [1, Chapter 4], in particular [1, Theorem 4.1.4]. Then, observing that the functional

$$\delta_b(f) := f(b) \qquad (f \in K_+)$$

belongs to K_- by [5, Section 4.1] we obtain the following standard results from [1] for the operator S_- induced by the elements from (5.1).

Proposition 5.1. (i) *The moment problem* (3.19) *(as well as* (4.9)*) is indeterminate.*

(ii) *The symmetric operator* S_- *in* $(K_-, \{\cdot,\cdot\}_-)$ *has defect* $(1,1)$.

(iii) *A nondecreasing left-continuous function* $\widehat{\sigma}_-$ *is an N-extremal solution of* (3.19) *if and only if*

$$\widehat{\sigma}_-(t) = \{\widehat{E}_-(t)\delta_0, \delta_0\}_- \qquad (t \in \mathbb{R})$$

where $\widehat{E}_-$ *is the left-continuous resolution of the identity associated with a self-adjoint extension of* S_- *in* $(K_-,\ \{\cdot,\cdot\}_-)$.

(iv) *We have* $\overline{R(S_-)} = \delta_b^{\{\perp\}_-}$ *(i.e. the orthogonal complement of* δ_b *in* $(K_-,\ \{\cdot,\cdot\}_-)$*).*

Proof. In order to show (iv), for $k \geq -1$ one finds that $\{S_- g_k, \delta_b\}_- = \{A_- g_k, \delta_b\}_- = [g_k, \delta_b]_m = \eta_k(A\gamma_k)(b) = -\eta_k(D_m D_x \gamma_k)(b) = 0$ and hence $R(S_-)\{\perp\}_-\delta_b$. This implies $\overline{R(S_-)} \subset \delta_b^{\{\perp\}_-}$ and $S_- \neq A_-$ since $R(A_-) = K_-$. Then, we obtain (ii) since by [1, Section 1.2 in Chapter 4] only defect $(1,1)$ or $(0,0)$ is possible and the second case would imply $S_- = A_-$. Consequently, we also get (iv) since $\overline{R(S_-)}$ has at most codimension 1. Here we use the fact that $0 \in \rho(A_-)$ and hence, the kernel of the one-dimensional extension S_-^* of A_- has at most dimension 1.

Furthermore, J_- is of "type C" in the terminology of [1, Section 1.2 in Chapter 4] and hence, (i) follows from [1, Theorem 2.1.2]. Finally, note that (iii) follows from [1, Theorem 4.1.4] since "orthogonal spectral function" in the terminology of [1] means "resolution of the identity associated with a self-adjoint extension remaining in the underlying Hilbert space". Here we use the transformation $\{\widehat{E}_- g_{-1}, g_{-1}\}_- = \cot\alpha\{\widehat{E}_-(t)\delta_0, \delta_0\}_-$ between solutions of (3.19) and solutions of the moment problem induced by J_-; cf. (5.3). □

Note that one of the self-adjoint extensions of S_- is A_-. Then, the N-extremal solution of (3.19) associated with A_- according to (iii) is indeed $\sigma_-(t) = \{E_-(t)\delta_0, \delta_0\}_-$.

Remark 5.2. We know that $\operatorname{span}\{A_-^k\delta_0 \mid k \geq 0\}$ (= $\operatorname{span}\{\delta_k \mid k \geq 0\}$) is dense in $(K_-,\ \{\cdot,\cdot\}_-)$. Therefore, the self-adjoint operator A_- has simple spectrum with generating element δ_0 and hence, by the theorem of Stone [1, Theorem 4.2.3], A_- is generated by a $\mathcal{J}$-matrix of "type D" in the terminology of [1]. This means, that A_- is also associated with some determinate moment problem. However, this moment problem and the corresponding orthonormal basis of $(K_-,\ \{\cdot,\cdot\}_-)$ must be different to (3.19) and the basis g_k $(k \geq -1)$. Here, we do not study this determinate moment problem any further.

5.2. A characterization by operators with a "heavy" right endpoint

In view of Proposition 5.1 the next task is the identification of all self-adjoint extensions of S_-. This will be obtained by means of the left-continuous function

$$m_\beta(x) := \begin{cases} m(x) & (x \leq b), \\ m(b) - \beta & (x > b) \end{cases}$$

where $\beta \in \mathbb{R}$. Then, we have $m = m_0$ (i.e. m_β with $\beta = 0$) and in general, $m_\beta(\{b\}) = -\beta$ if we again denote the induced signed Lebesgue-Stieltjes measure by m_β. (In a setting with $m_\beta(\{b\}) \neq 0$ we speak about a "heavy right endpoint".) Let A_β denote the associated Krein-Feller operator in the Krein space $(L^2_{m_\beta}, [\cdot,\cdot]_{m_\beta})$ according to Subsection 2.1, again determined by the boundary conditions (2.2), (2.3). By definition, the space $H^1_{m_\beta}$ from Subsection 2.2 and its inner product (2.4) do not depend on β. Therefore, for all $\beta \in \mathbb{R}$ the space $(K_+,\ \{\cdot,\cdot\}_+)$ from Subsection 2.2 coincides with $(H^1_m,\ \{\cdot,\cdot\}_+)$ and consequently, also the dual space $(K_-,\ \{\cdot,\cdot\}_-)$ is independent of β. However, the space $L^2_{m_\beta}$ in the "middle" of the space triplet $K_+ \subset L^2_{m_\beta} \subset K_-$ from Subsection 2.2 indeed depends on $\beta \in \mathbb{R}$. In this sense, according to Subsection 2.2, an element $f \in L^2_{m_\beta}$ has to be identified with the functional $[\cdot, f]_{m_\beta} \in K_-$ (restricted to K_+), i.e. with the functional

$$K_+ \ni g \longrightarrow [g,f]_{m_\beta} = \int g\overline{f}\, dm_\beta = \int g\overline{f}\, dm - \beta g(b)\overline{f}(b). \tag{5.4}$$

Now, we fix the point of view to the identification of L^2_m in K_- (i.e. $\beta = 0$) as in the previous sections. Then, identifying $[\cdot, f]_m$ with f, the functional (5.4) coincides with

$$f - \beta f(b)\delta_b. \tag{5.5}$$

Note that here an element $f \in L^2_{m_\beta}$ represents different elements of K_- for different β although as a function (or, more precisely, equivalence class of functions) it is always the same, at least for $\beta \in \mathbb{R}\setminus\{0\}$. The representation (5.5) is valid in particular for all elements of H^1_m $(= K_+)$ which is the domain of definition of the self-adjoint operator $A_{\beta-}$ in $(K_-,\ \{\cdot,\cdot\}_-)$ introduced in Subsection 2.2 by the Riesz representation theorem for m_β. In this sense we can write

$$D(A_{\beta-}) = \{f - \beta f(b)\delta_b \mid f \in H^1_m\}$$

as a subset of K_- and for $f \in H^1_m$ we have

$$A_{\beta-}(f - \beta f(b)\delta_b) = \{\cdot, f\}_+ = A_- f$$

as a functional on $(K_+,\ \{\cdot,\cdot\}_+)$. Furthermore, let $E_{\beta-}$ denote the left-continuous resolution of the identity for $A_{\beta-}$ in $(K_-,\ \{\cdot,\cdot\}_-)$ and let $\sigma_{\beta-}$, σ_β and $\sigma_{\beta+}$ denote the induced ($A_{\beta\pm}$-)spectral functions according to (2.6).

In addition to $A_{\beta-}$ with $\beta \in \mathbb{R}$ we introduce another operator. To this end put

$$H^1_{m,0} := \{f \in H^1_m \mid f(b) = 0\}$$

and let $A_{\infty-}$ be the operator in $(K_-,\ \{\cdot,\cdot\}_-)$ given by

$$D(A_{\infty-}) = H^1_{m,0} + \operatorname{span}\{\delta_b\}, \qquad A_{\infty-}(f + c\delta_b) := \{\cdot, f\}_+ \ (= A_- f)$$

for $f \in H^1_{m,0}$ and $c \in \mathbb{C}$. Now, the self-adjoint extensions of S_- can be characterized by means of [6, Theorem 5.2].

Proposition 5.3. (i) *The symmetric operator S_- in $(K_-,\ \{\cdot,\cdot\}_-)$ satisfies*

$$D(S_-) = H^1_{m,0}, \qquad S_- f = A_- f \quad (f \in D(S_-)).$$

(ii) *The operator $A_{\infty-}$ is self-adjoint in $(K_-,\ \{\cdot,\cdot\}_-)$.*
(iii) *All self-adjoint extensions of S_- are given by $A_{\beta-}$ with $\beta \in \mathbb{R}\cup\{\infty\}$.*

Proof. In order to check the assumptions of [6, Theorem 5.2] we first observe that $\delta_b \notin L^2_m$ since $m(\{b\}) = 0$ and hence, the tuple $(\delta_b, 0)$ does not belong to the graph of A_- in K_-. Then [6, Theorem 5.2] presents a characterization of all self-adjoint extensions of the symmetric restriction $\widetilde{S_-}$ of A_- given by $D(\widetilde{S_-}) := \{f \in D(A_-) \mid \{A_- f, \delta_b\}_- = 0\}$; cf. [6, Lemma 5.1]. Here, we have $D(\widetilde{S_-}) = H^1_{m,0}$ since $\{A_- f, \delta_b\}_- = [f, \delta_b]_m = f(b)$. By Proposition 5.1 (iv) the restriction S_- of A_- also satisfies $\{S_- f, \delta_b\}_- = 0$ $(f \in D(S_-))$ and hence, $S_- \subset \widetilde{S_-}$. Since by Proposition 5.1 (ii) and [6, Lemma 2.2] both symmetric operators are closed and have defect $(1,1)$ they coincide. This implies (i). Furthermore, by [6, Theorem 5.2] all self-adjoint extensions of S_- in $(K_-, \{\cdot,\cdot\}_-)$ are given by the operators $A_-(\beta)$ with $\beta \in \mathbb{R} \cup \{\infty\}$ where $A_-(0) := A_-$ and $c \in \mathbb{C}$

$$\begin{aligned} D(A_-(\infty)) &:= \{h \in K_- \mid h + c\delta_b \in D(A_-),\ \{A_-(h + c\delta_b), \delta_b\}_- = 0\}, \\ A_-(\infty)h &:= A_-(h + c\delta_b) \quad \text{where } h \in D(A_-(\infty)),\ h + c\delta_b \in D(A_-) \end{aligned}$$

and for $\beta \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} D(A_-(\beta)) &:= \{h \in K_- \mid h + c\delta_b \in D(A_-),\ c = \beta\{A_-(h + c\delta_b), \delta_b\}_-\}, \\ A_-(\beta)h &:= A_-(h + c\delta_b) \quad \text{where } h \in D(A_-(\beta)),\ h + c\delta_b \in D(A_-). \end{aligned}$$

Here, for $f := h + c\delta_b \in D(A_-)$ we again have $\{A_- f, \delta_b\}_- = [f, \delta_b]_m = f(b)$. Since $D(A_-) = H^1_m$ this implies

$$\begin{aligned} D(A_-(\infty)) &= \{h \in K_- \mid h + c\delta_b \in H^1_{m,0},\ c \in \mathbb{C}\} = D(A_{\infty-}), \\ D(A_-(\beta)) &= \{h \in K_- \mid h + c\delta_b \in H^1_m,\ c = \beta(h + c\delta_b)(b)\} = D(A_{\beta-}) \end{aligned}$$

for $\beta \in \mathbb{R} \setminus \{0\}$. Therefore, we have $A_-(\beta) = A_{\beta-}$ for all $\beta \in \mathbb{R} \cup \{\infty\}$. This implies (iii) and then, in particular, also (ii). □

The main results of this section are based on a combination of the Propositions 5.1 and 5.3 and the observation that $\{E_{\beta-}(t)\delta_0, \delta_0\}_- = \sigma_{\beta-}(t)$.

Theorem 5.4. *Let $\alpha \in (0, \pi/2)$ and (x_n), (m_n) satisfy* (2.14), (2.15)*. Furthermore, let the functions $\sigma_{\beta-}$ for $\beta \in \mathbb{R}$ be given as above and put*

$$\sigma_{\infty-}(t) := \{E_{\infty-}(t)\delta_0, \delta_0\}_- \qquad (t \in \mathbb{R})$$

where $E_{\infty-}$ denotes the left-continuous resolution of the identity associated with $A_{\infty-}$. Then all nondecreasing left-continuous N-extremal solutions of (3.19) *are given by $\sigma_{\beta-}$ with $\beta \in \mathbb{R} \cup \{\infty\}$.*

Corollary 5.5. *The spectral class of A is given by*

$$[\sigma] = \{\sigma_\beta \mid \beta \in \mathbb{R} \cup \{\infty\}\}$$

where σ_∞ is given by $\sigma_\infty(t) := \int_0^t s\, d\sigma_{\infty-}(s)$ $(t \in \mathbb{R})$.

Now, the inverse statement of Theorem 4.1 can be reformulated in the following way.

Corollary 5.6. *Assume that* $\alpha \in (0, \pi/2)$ *and that* τ *is candidate of a spectral function according to Subsection* 4.1*. Let the sequences* (x_n) *and* (m_n) *be given by* (4.8) *(satisfying* (2.14), (2.15)*). Furthermore, let the functions* σ_β *be given as above. Then there is a number* $\beta_0 \in \mathbb{R} \cup \{\infty\}$ *such that* $\tau = \sigma_{\beta_0}$*.*

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