Decompositions in doubling weighted Besov-Triebel-Lizorkin spaces and applications

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Contents

Zusammenfassung in deutscher Sprache

Die vorliegende Arbeit beschäftigt sich mit Funktionenräumen vom Besov- und Triebel-Lizorkin-Typ, bezeichnet mit $B^s_{p,q}(\mathbb{R}^n)$ und $F^s_{p,q}(\mathbb{R}^n)$, welche auch mit Gewichten ausgestattet sein können. Diese Räume erlauben eine einheitliche Behandlung vieler schon bekannter Funktionenräume, wie Hölder-Zygmund-Räume, Sobolev-Räume, Slobodeckij-Räume, Bessel-Potential-Räume and Hardy-Räume. In den letzten Jahrzehnten haben diese Skalen ihre Nützlichkeit bewiesen. Es gibt Hunderte von wissenschaftlichen Veröffentlichungen und etliche Bücher, welche diese Skalen in verschiedenster Weise nutzen. So spielen sie zum Beispiel eine wichtige Rolle in der Lösungstheorie von partiellen Differentialgleichungen, der Interpolationstheorie, der Approximationstheorie, der Harmonischen Analysis oder der Spektraltheorie, um nur einige zu nennen. Für eine umfassende Darstellung zu diesen Räumen verknüpft mit historischen Bemerkungen verweisen wir auf die Reihe von Büchern von H. Triebel, [Tri83], [Tri92], [Tri06].

In der Lösungstheorie von elliptischen partiellen Differentialgleichungen für irreguläre Gebiete benötigt man Gewichte, welche lokale Singularitäten aufweisen können. Für Existenz- und Eindeutigkeitsaussagen von Lösungen führt dies zu gewichteten Sobolev-Räumen bzw. allgemeiner (gewichteten) Besov-Räumen $B^s_{p,q}(\mathbb{R}^n, w)$ und Triebel-Lizorkin-Räumen $F^s_{p,q}(\mathbb{R}^n, w)$, wobei in der Definition der Räume der klassische Lebesgue-Raum $L_p(\mathbb{R}^n)$ durch einen gewichteten Lebesgue-Raum $L_p(w) = L_p(\mathbb{R}^n, w)$ ersetzt wird. Hierbei ist w eine fast überall positive und lokal integrierbare Funktion auf $\mathbb{R}^n$. Üblicherweise sind diese Gewichte vom Muckenhoupt-Typ. Wir untersuchen in dieser Arbeit eine Verallgemeinerung, die sogenannten Verdopplungsgewichte. Insbesondere konzentrieren wir uns auf atomare Darstellungen, Wavelet-Charakterisierungen, (kompakte) Einbettungen und Envelopes für diese Räume.

Kapitel 1 beschäftigt sich mit den grundlegenden Begriffen, Definitionen und Eigenschaften der von uns betrachteten Gewichte und Funktionenräume. Wir starten zunächst mit den Muckenhoupt-Gewichten. Eine lokal integrierbare und fast überall positive Funktion w gehört zur Muckenhoupt Klasse $\mathcal{A}_p$, $1 < p < \infty$, falls eine Konstante $0 < A < \infty$ existiert, so dass für alle Kugeln B folgendes gilt

$$\left(\frac{1}{|B|} \int_B w(x) \, \mathrm{d}x \right) \cdot \left(\frac{1}{|B|} \int_B w(x)^{-p'/p} \, \mathrm{d}x \right)^{p/p'} \leq A,$$

wobei $\frac{1}{p} + \frac{1}{p'} = 1$ und $|B|$ bezeichne das Lebesguemaß von B. Diese Gewichte wurden eingeführt von B. Muckenhoupt in [Muc72a]. Für einen umfassenden Überblick verweisen wir beispielsweise auf die Bücher E. M. Stein [Ste93] oder J. Duoandikoetxea [Duo01]. Eine natürliche Erweiterung dieser Gewichte bilden die sogenannten Verdopplungsgewichte

$$w(B(x,2r)) \leq 2^\beta w(B(x,r)), \qquad \text{wobei} \quad w(\Omega) = \int_\Omega w(y)\,\mathrm{d}y, \quad \Omega \subset \mathbb{R}^n.$$

Der Fokus der Arbeit liegt auf verdopplungsgewichteten Funktionenräumen des Besov-Triebel-Lizorkin Typs. Sei $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $\{\varphi_j\}_{j=0}^\infty$ eine glatte dyadische Zerlegung der Eins und w ein Verdopplungsgewicht. Dann ist der gewichtete Besov-Raum $B^s_{p,q}(w) = B^s_{p,q}(\mathbb{R}^n, w)$ gegeben durch

$$B^s_{p,q}(w) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \ : \left(\sum_{j=0}^\infty 2^{jsq} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p(w) \right\|^q \right)^{1/q} < \infty \right\},$$

wobei $\mathcal{F}$ und $\mathcal{F}^{-1}$ die Fouriertransformation bzw. inverse Fouriertransformation bezeichnen. Analog kann man die gewichteten Triebel-Lizorkin-Räume $F^s_{p,q}(w) = F^s_{p,q}(\mathbb{R}^n, w)$ definieren, indem man die ℓ_q-Norm und die gewichtete $L_p(w)$-Norm vertauscht. Diese verdopplungsgewichteten Funktionenräume wurden erstmals von M. Bownik in seinem Paper [Bow05] eingeführt, wobei er dort vorwiegend mit homogenen, anisotropen Besov-Räumen mit erweiterter Streckungsmatrix und allgemeineren Verdopplungsmaßen arbeitete. Wir konnten zeigen, dass diese Räume ebenfalls einige Eigenschaften der ungewichteten $B^s_{p,q}$ und $F^s_{p,q}$ Räume besitzen. So gilt zum Beispiel auch hier die übliche Einbettung zwischen dem Schwartz-Raum $\mathcal{S}(\mathbb{R}^n)$ und dessen Dualraum $\mathcal{S}'(\mathbb{R}^n)$,

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n, w), F^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

siehe Proposition 1.44. Dies ist eine wichtige Eigenschaft, da sie uns die Wohldefiniertheit der dualen Paarung in der Wavelet-Charakterisierung von $B^s_{p,q}(w)$ und $F^s_{p,q}(w)$ sichert, welche ebenfalls ein Resultat dieser Arbeit ist.

In der Theorie der Funktionenräume haben sich atomare, subatomare und Wavelet-Zerlegungen als ein nützliches Werkzeug herausgestellt. In dieser Arbeit beschäftigen wir uns speziell mit atomaren und Wavelet-Darstellungen. Die Grundidee hierbei ist die „Übersetzung“ des Funktionenraumes in einen passenden, äquivalenten Folgenraum. Im Falle der atomaren Darstellung zum Beispiel zerlegt man die Funktion f wie folgt

$$f = \sum_{j,m} \lambda_{j,m} a_{j,m},$$

wobei die Funktionen $a_{j,m}$ vorteilhafte Eigenschaften, wie Glattheit und kompakte Träger, besitzen. Sämtliche Information von f steckt dann in den Koeffizienten $\lambda_{j,m}$, so dass man die Frage $f \in B^s_{p,q}(w)$ auf die meist wesentlich einfachere Frage $\lambda = (\lambda_{j,m})_{j,m} \in \bar{b}^s_{p,q}(w)$ zurückführen kann, wobei die Folgenräume $\bar{b}^s_{p,q}(w)$ vom ℓ_p-Typ sind. In die Forderungen an die Qualität der Atome gehen hierbei nicht nur die Parameter s, p, q der Räume, sondern auch das Gewicht ein.

Die Idee der atomaren Zerlegung für die ungewichteten Besov-Triebel-Lizorkin-Räume geht im wesentlichen auf die Arbeiten [FJ85], [FJ90], [FJW91] von M. Frazier und B. Jawerth zurück. Für einen detailierten Überblick über die komplexe Historie der Atome in verschiedenen Funktionenräumen verweisen wir auf [Tri92, Section 1.9].

Für die atomare Darstellung in verdopplungsgewichteten $B^s_{p,q}(w)$ und $F^s_{p,q}(w)$ Räumen verweisen wir auf die Paper [Bow05] und [Bow07] von M. Bownik, siehe auch Abschnitt 2.2.

Neben den atomaren Zerlegungen interessieren wir uns in dieser Arbeit auch für Wavelet-Charakterisierungen. Genauer gesagt betrachten wir sogenannte Daubechies Wavelets mit kompaktem Träger. Für die Definition und entsprechende Notationen verweisen wir auf Abschnitt 2.1. Die Standardreferenzen sind hier Y. Meyer [Mey92], I. Daubechies [Dau92] und P. Wojtaszczyk [Woj97].

Eine der wesentlichen Eigenschaften von Wavelets ist, dass sie auch als Atome betrachtet werden können. Mit anderen Worten, falls wir eine Wavelet-Charakterisierung haben, können wir diese auch immer als atomare Darstellung auffassen. In dieser Arbeit beschäftigen wir uns mit der umgekehrten Frage, d.h. unter welchen (evtl. zusätzlichen) Bedingungen erhalten wir einen Wavelet-Isomorphismus, wenn wir eine atomare Darstellung haben. Hierzu führen wir ein komplett neues Konzept ein, die sogenannten $\varkappa$-Folgenräume, siehe Definition 2.17. Die Definition ist sehr technisch und ergibt sich aus dem Beweis. Das Hauptresultat sieht hier wie folgt aus:

Sei $A(\mathbb{R}^n)$ ein (isotroper, inhomogener) Funktionraum, welcher eine L-atomare Darstellung besitzt: $f \in \mathcal{S}'(\mathbb{R}^n)$ gehört zu $A(\mathbb{R}^n)$ genau dann, wenn f dargestellt werden kann als

$$f = \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \mu_{j,m}\, a_{j,m}, \qquad \mu \in a(\mathbb{R}^n), \tag{1}$$

mit unbedingter Konvergenz in $\mathcal{S}'(\mathbb{R}^n)$ und

$$\|f\,|A(\mathbb{R}^n)\| \sim \inf \|\mu\,|a(\mathbb{R}^n)\| \tag{2}$$

wobei $a(\mathbb{R}^n)$ der zugehörige Folgenraum ist und die $\{a_{j,m}\}$ L-Atome sind. Das Infimum in (2) wird über alle zulässigen Darstellungen (1) gebildet.
Zusätzlich sei $a(\mathbb{R}^n)$ ein $\varkappa$-Folgenraum gemäß Definition 2.17 mit $0 < \varkappa < L \in \mathbb{N}$.
Dann gilt, dass $f \in \mathcal{S}'(\mathbb{R}^n)$ zu $A(\mathbb{R}^n)$ gehört genau dann, wenn f mit L-Wavelets dargestellt werden kann

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m\, \Psi_m + \sum_{G\in G^*} \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda^{j,G}_m\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in a^w(\mathbb{R}^n), \tag{3}$$

mit unbedingter Konvergenz in $\mathcal{S}'(\mathbb{R}^n)$. Die Darstellung in (3) ist eindeutig,

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2}\big(f, \Psi^j_{G,m}\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m),$$

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, $G \in G^*$, und

$$I: \quad f \mapsto \left\{\lambda_m(f),\, \lambda_m^{j,G}(f)\right\}$$

ist eine isomorphe Abbildung von $A(\mathbb{R}^n)$ auf $a^w(\mathbb{R}^n)$, wobei $a^w(\mathbb{R}^n)$ die Wavelet-Version von $a(\mathbb{R}^n)$ ist.

In Abschnitt 2.3.4 zeigen wir, dass die bekannten atomaren Folgenräume $\bar{b}^s_{p,q}$ und $\bar{f}^s_{p,q}$ der ungewichteten Funktionenräumen $B^s_{p,q}(\mathbb{R}^n)$ und $F^s_{p,q}(\mathbb{R}^n)$ solche $\varkappa$-Folgenräume sind. Die zugehörige Wavelet-Charakterisierung, welche aus dem obigen Theorem abgeleitet werden kann, fällt dann mit den bereits bekannten Resulten zusammen, siehe Korollar 2.38. Im Falle der verdopplungsgewichteten atomaren Folgenräume haben wir folgendes Resultat:

Seien $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ und w ein Verdopplungsgewicht mit Verdopplungskonstante γ. Dann ist $\bar{b}^s_{p,q}(w)$ ein $\varkappa$-Folgenraum für jedes $\varkappa$ mit

$$\varkappa > \max\left(s + \frac{n}{p}, \frac{n\gamma}{p} - s\right),$$

und $\bar{f}^s_{p,q}(w)$ ist ein $\varkappa$-Folgenraum für jedes $\varkappa$ mit

$$\varkappa > \max(s, \gamma\, \sigma_{p,q} + (\gamma - 1)n - s, \frac{n\gamma}{p} - s).$$

Die zugehörige Wavelet-Charakterisierung kann man in Abschnitt 2.4, Korollar 2.41, nachlesen. Die Ergebnisse aus Kapitel 2 sind in einer gemeinsamen Arbeit [HST16] mit D. D. Haroske und H. Triebel zusammengefasst.

In Kapitel 3 diskutieren wir notwendige und hinreichende Bedingungen für stetige und kompakte Einbettungen in verdopplungsgewichteten Besov-Räumen $B^s_{p,q}(w)$. Grundlage hierfür ist eine Reihe von Papern [HS08, HS11a, HS11b] von D. D. Haroske und L. Skrzypczak, in denen dieses Problem für Muckenhoupt-gewichtete Funktionenräume betrachtet wurde. Wir konnten sowohl für stetige als auch kompakte Einbettungen scharfe und somit optimale Bedingungen zeigen, siehe Theorem 3.5. So gilt zum Beispiel im Falle von stetigen Einbettungen:

Seien $-\infty < s_2 \le s_1 < \infty$, $0 < p_1, p_2 \le \infty$, $0 < q_1, q_2 \le \infty$ und w_1, w_2 Verdopplungsgewichte. Dann ist die Einbettung $B^{s_1}_{p_1,q_1}(w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(w_2)$ stetig genau dann, wenn

$$\left\{2^{-j(s_1-s_2)} \left\| \left\{ w_1(Q_{j,m})^{-1/p_1} w_2(Q_{j,m})^{1/p_2} \right\}_m | \ell_{p^*} \right\| \right\}_j \in \ell_{q^*},$$

wobei p^* und q^* gegeben sind durch

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+, \qquad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+.$$

Neben diesem allgemeinen Einbettungsresultat haben wir noch einige Spezialfälle betrachtet. Diese können in Abschnitt 3.3 und 3.4 nachgelesen werden.

Kapitel 4 widmet sich dem Studium der Growth Envelope Funktionen. Das Ziel ist es, das singuläre Verhalten von Funktionen eines Funktionenraumes X zu charakterisieren. Um verschiedene Funktionen mit unterschiedlichen Singularitäten vergleichen zu können, benutzt man die monoton fallende Umordnung f^*, welche die Singularitäten in den Nullpunkt befördert. Das Verhalten der Singularitäten kann man nun mit Hilfe der Growth Envelope Funktion $\mathcal{E}_{\mathsf{G}}^X$ messen

$$\mathcal{E}_{\mathsf{G}}^X(t) := \sup_{f \in X, \|f|X\| \leq 1} f^*(t), \qquad t > 0.$$

Dieses Konzept wurde von Triebel eingeführt und in [Tri01] bzw. [Har02] erstmals studiert. Für detailierte Informationen verweisen wir auf das Buch [Har07] von D. D. Haroske. Dort findet man unter anderem auch die Ergebnisse für die klasssischen (ungewichteten) Besov-Triebel-Lizorkin-Räume $B^s_{p,q}(\mathbb{R}^n)$ und $F^s_{p,q}(\mathbb{R}^n)$. Bereits in [Ska10] haben wir uns mit Growth Envelope Funktionen in Muckenhoupt-gewichteten $B^s_{p,q}(w)$ und $F^s_{p,q}(w)$ Räumen beschäftigt. Für einen umfassenden Überblick über die Ergebnisse im Muckenhoupt-gewichteten Besov-Triebel-Lizorkin-Räumen verweisen wir auf das Paper [Har10] von D. D. Haroske. Unser Hauptresultat sieht hier wie folgt aus:

Seien $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ und w ein Verdopplungsgewicht. Es gelte

$$\max\left(\frac{n}{p} - n, 0\right) + \frac{n}{p}(\gamma - 1) < s < \frac{n}{p}\gamma \qquad \text{und} \qquad \inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c_w > 0.$$

Dann erhalten wir für die Abschätzung von oben

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \leq c\, t^{-\frac{\gamma}{p} + \frac{s}{n}}, \quad t \to 0,$$

und für die Abschätzung von unten

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \geq c\, t^{-\frac{1}{p} + \frac{s}{n}} \sup_{x^0 \in \mathbb{R}^n, t \sim 2^{-jn}} \left(\frac{w(B(x^0, 2^{-j}))}{|B(x^0, 2^{-j})|}\right)^{-1/p}, \quad t \to 0,$$

siehe Proposition 4.13 und Proposition 4.14.

Introduction

This work is mainly concerned with function spaces of Besov and Triebel-Lizorkin type, denoted by $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, possibly connected with some weight functions. These spaces have been investigated for several decades and they play an important role, for instance, in the study of partial differential equations, interpolation theory, approximation theory, harmonic analysis and spectral operator theory. They constitute an indispensable part in many research papers and books.

In particular, the two scales of Besov $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ cover many well-known function spaces such as Hölder-Zygmund spaces, (fractional) Sobolev spaces, Slobodeckij spaces, Bessel-potential spaces and Hardy spaces. For a detailed study together with historical remarks we refer to the monographs of H. Triebel, [Tri83], [Tri92], [Tri06].

In the theory of elliptic partial differential equations one uses weight functions in several models, which may have local singularities.

For example one considers the following differential equation with „disturbed“ ellipticity

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(w_i(x)\frac{\partial u}{\partial x_i}\right) + w_0(x)\frac{\partial u}{\partial x_i} = f$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, with homogeneous Dirichlet-boundary conditions $u_{|_{\partial\Omega}} = 0$. Here the functions $w_i(x)$ of type

$$w_i(x) = (\operatorname{dist}(x, \partial\Omega))^{\varepsilon_i}, \quad \varepsilon_i \in \mathbb{R},$$

are of particular interest. For existence and uniqueness of solutions and regularity questions this leads to weighted Sobolev spaces $W^2_1(\Omega, w)$, $w = (w_0, \ldots, w_n)$, or more general weighted Besov $B^s_{p,q}(\mathbb{R}^n, w)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n, w)$, respectively. The weight function is in general usually a Muckenhoupt weight. We consider in this work the more general doubling weights.

In the definition of these weighted spaces the classical Lebesgue space $L_p(\mathbb{R}^n)$ is replaced by the weighted Lebesgue space $L_p(w) = L_p(\mathbb{R}^n, w)$, where w is here a locally integrable and positive a.e. function on $\mathbb{R}^n$.

In Chapter 1 the basic concepts and definitions related to weights and function spaces are provided. We start with the very famous Muckenhoupt weights. A locally integrable and almost everywhere positive function w belongs to the class $\mathcal{A}_p$, $1 < p < \infty$, if there exists a constant $0 < A < \infty$ such that for all balls B the following inequality holds

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)\cdot\left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \le A,$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $|B|$ stands for the Lebesgue measure of B. It is well-known that this weight class is closely connected with the boundedness of the *Hardy-Littlewood maximal operator* $\mathcal{M}$

$$(\mathcal{M}f)(x) = \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|\,\mathrm{d}y, \quad x\in\mathbb{R}^n,$$

from $L_p(w)$ to $L_p(w)$. These weights were introduced by B. Muckenhoupt in [Muc72a]. For a comprehensive treatment about Muckenhoupt weights we refer to the monographs by E. M. Stein [Ste93] and J. Duoandikoetxea [Duo01]. In this context we collect some properties of Muckenhoupt weights, which partially will be used later on, including that every Muckenhoupt weight satisfies the doubling property

$$w(B(x,2r)) \le 2^{\beta} w(B(x,r)), \qquad \text{where} \quad w(\Omega) = \int_\Omega w(y)\,\mathrm{d}y, \quad \Omega\subset\mathbb{R}^n.$$

In addition, based on a paper from I. Wik, [Wik89], we show that there exists a weight w, which has the doubling property, but does not belong to any Muckenhoupt class $\mathcal{A}_p$. This leads to doubling weights, which naturally extend the Muckenhoupt weights. Here we collect also some basic properties, like

$$0 < w(\Omega) < \infty, \qquad \text{for any } \Omega\subset\mathbb{R}^n \text{ with } 0<|\Omega|<\infty$$

and

$$\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y = \infty,$$

see Proposition 1.23 and 1.25.

The more general doubling measures have a rich history, too; see for example [VK87, LS98, BG00, Sta92]. In these papers one also deals with a more general setting. We consider here doubling measures, which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, and these doubling weights in connection with the already mentioned Besov-Triebel-Lizorkin spaces.

In the theory of function spaces several other classes of weight functions are considered. In the end of Section 1.2 we mention two further weight classes, on the one hand the so-called admissible weights, we refer to the book of D. E. Edmunds and H. Triebel, [ET96] and on the other hand the so-called local Muckenhoupt weights $\mathcal{A}_p^{loc}$, we refer to V. S. Rychkov [Ryc01], T. Schott [Sch98] and A. Wojciechowska [Woj12a].

The focus in this work lies on doubling weighted function spaces of Besov-Triebel-Lizorkin type. Therefore we introduce in Section 1.3 at first the unweighted function spaces of Besov and Triebel-Lizorkin type and give a short overview about these spaces.

Afterwards we extend this by their doubling weighted counterparts. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $\{\varphi_j\}_{j=0}^{\infty}$ a smooth dyadic resolution of unity and let w be a doubling weight. Then the weighted Besov space $B^s_{p,q}(w) = B^s_{p,q}(\mathbb{R}^n, w)$ is given by

$$B^s_{p,q}(w) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \ : \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p(w) \right\|^q \right)^{1/q} < \infty \right\},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. The definition for the weighted Triebel-Lizorkin spaces $F^s_{p,q}(w) = F^s_{p,q}(\mathbb{R}^n, w)$ is similar, one changes the order of the ℓ_q-norm and the weighted Lebesgue space $L_p(w)$-norm. These doubling weighted function spaces were first introduced by M. Bownik in the paper [Bow05]. There he mainly dealt with homogeneous, anisotropic Besov spaces with expansive dilation matrices and more general doubling measures. We show that several properties from the unweighted $B^s_{p,q}$ and $F^s_{p,q}$ spaces remain true. For example, the spaces are embedded between the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and the dual space $\mathcal{S}'(\mathbb{R}^n)$,

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n, w), F^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

see Proposition 1.44. This is an important property and ensures us the well-definedness of the dual pairing in the wavelet characterization for $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$, which is one of the main goals in this thesis.

In the theory of function spaces it is useful to have various representations of a function f from the underlying function space. In the last years it turned out that so-called atomic, sub-atomic or wavelet decompositions are very promising. In our work we amplify the atomic and wavelet representation. The basic idea here is the „translation“ from the function space to appropriate sequence spaces. For example in the case of an atomic representation one decomposes the function f into special building blocks

$$f = \sum_{j,m} \lambda_{j,m} a_{j,m},$$

where these building blocks $a_{j,m}$ are „nice“ functions with convenient properties such as smoothness or compact supports. Then all the information about f are in the coefficients $\lambda_{j,m}$, such that the question about $f \in B^s_{p,q}(w)$ can be reduced to the question $\lambda = (\lambda_{j,m})_{j,m} \in \bar{b}^s_{p,q}(w)$, where the sequence spaces $\bar{b}^s_{p,q}(w)$ are of ℓ_p-type. All the parameters s, p, q of the function space as well as the weight have influence on the quality of the atoms.

The idea of the atomic decomposition from the $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ function spaces goes essentially back to M. Frazier and B. Jawerth in their series of papers [FJ85], [FJ90], [FJW91]. For a detailed overview about the complex history of atoms in various function spaces we refer to [Tri92, Section 1.9].

The atomic representation of our doubling weighted $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$ spaces can be found in the papers [Bow05] and [Bow07] by M. Bownik, see Section 2.2.

Atoms have nice properties, for example, sufficiently high smoothness, compact support and moment conditions. The disadvantage of the atoms is that the representation is not unique, i.e., for a fixed function f one can find different decompositions

$$f = \sum_{j,m} \lambda_{j,m} a_{j,m}.$$

On the other side one has more freedom at the choice of the functions $a_{j,m}$ since the structure is not completely fixed. Sometimes this is advantageous, for example, if one works with traces, because there one does not need the isomorphism between the function space and the corresponding sequence space. But if one is interested in embeddings, as we do, then it is better to work with wavelet isomorphisms. For our purpose we consider compactly supported Daubechies wavelets. For the definition and the notation we refer to Section 2.1 and the standard references Y. Meyer [Mey92], I. Daubechies [Dau92] and P. Wojtaszczyk [Woj97].
One property of wavelets is that they can always be considered as atoms. In other words, if we have a wavelet characterization, then we have also an atomic representation. In this thesis we discuss in Section 2.3 the converse question, that is, under which (additional) conditions we obtain a wavelet isomorphism, when we have an atomic representation. For this we introduce a completely new concept, the so-called $\varkappa-$sequence spaces, cf. Definition 2.17. The definition is very technical and turns out from the proof. The main theorem here is the following.

Let $A(\mathbb{R}^n)$ be an (isotropic, inhomogeneous) function space which can be represented by an L-atomic representation: $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $A(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \mu_{j,m}\, a_{j,m}, \qquad \mu \in a(\mathbb{R}^n), \tag{4}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$ with

$$\|f\,|A(\mathbb{R}^n)\| \sim \inf \|\mu\,|a(\mathbb{R}^n)\| \tag{5}$$

where $a(\mathbb{R}^n)$ is a corresponding sequence space and $\{a_{j,m}\}$ are L-atoms. The infimum in (5) is taken over all admissible representations (4).
Additionally $a(\mathbb{R}^n)$ is a $\varkappa$-sequence space according to Definition 2.17 with $0 < \varkappa < L \in \mathbb{N}$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $A(\mathbb{R}^n)$ if, and only if, it can be represented in terms of L-wavelets as

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m\, \Psi_m + \sum_{G\in G^*} \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in a^w(\mathbb{R}^n), \tag{6}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (6) is unique,

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\big(f, \Psi^j_{G,m}\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m),$$

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, $G \in G^*$, and

$$I: \quad f \mapsto \left\{ \lambda_m(f),\, \lambda_m^{j,G}(f) \right\}$$

is an isomorphic map of $A(\mathbb{R}^n)$ onto $a^w(\mathbb{R}^n)$, where $a^w(\mathbb{R}^n)$ is the wavelet version of $a(\mathbb{R}^n)$.

In Section 2.3.4 we show that the well-known atomic sequence spaces $\bar{b}^s_{p,q}$ and $\bar{f}^s_{p,q}$ from the corresponding unweighted function spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ are such $\varkappa$-sequence spaces. The related wavelet characterization from the theorem above coincides with well-known results, see Corollary 2.38.

In the case of doubling weights we have the following outcome using our new approach:

Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and w be a doubling weight with doubling constant γ. Then $\bar{b}^s_{p,q}(w)$ is a $\varkappa$-sequence space for any $\varkappa$

$$\varkappa > \max\left(s + \frac{n}{p}, \frac{n\gamma}{p} - s \right),$$

and $\bar{f}^s_{p,q}(w)$ is a $\varkappa$-sequence space for any $\varkappa$

$$\varkappa > \max(s, \gamma\, \sigma_{p,q} + (\gamma - 1)n - s, \frac{n\gamma}{p} - s).$$

The related wavelet characterization can be found in Corollary 2.41:

Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and w be a doubling weight with doubling constant γ. We assume

$$L > \max\left(s + \frac{n}{p}, \frac{n}{p}\, c_n\, \gamma - s \right).$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(w)$ if, and only if, it can be represented in terms of L-wavelets as

$$f = \sum_{m \in \mathbb{Z}^n} \lambda_m \Psi_m + \sum_{G \in G^*} \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in b^s_{p,q}(w), \tag{7}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (7) is unique,

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\big(f, \Psi^j_{G,m}\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m),$$

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, $G \in G^*$, and

$$I: \quad f \mapsto \left\{ \lambda_m(f),\, \lambda_m^{j,G}(f) \right\}$$

is an isomorphic map of $B^s_{p,q}(w)$ onto $b^s_{p,q}(w)$.

The results from Chapter 2 are contained in a joint paper with D. D. Haroske and H. Triebel, [HST16].

In Chapter 3 we discuss necessary and sufficient conditions for continuous and compact embeddings for doubling weighted Besov spaces $B^s_{p,q}(w)$. Here we follow the approach from

the series of papers [HS08, HS11a, HS11b] by D. D. Haroske and L. Skrzypczak, where function spaces with Muckenhoupt weights were considered. We use the technique of wavelet characterization, which we proved in Chapter 2. This allows us to transform the problem from the function spaces to the simpler context of the sequence spaces. Moreover we apply an assertion for general weighted sequence spaces which can be found in the paper [KLSS06b] by T. Kühn, H.-G. Leopold, W. Sickel and L. Skrzypczak. Therefore we obtain sharp and optimal conditions for continuous as well as compact embeddings, cf. Theorem 3.5. For example in the case of continuous embeddings we have:
Let $-\infty < s_2 \leq s_1 < \infty$, $0 < p_1, p_2 \leq \infty$, $0 < q_1, q_2 \leq \infty$ and let w_1, w_2 be doubling weights. The embedding $B^{s_1}_{p_1,q_1}(w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(w_2)$ is continuous if, and only if,

$$\left\{2^{-j(s_1-s_2)} \left\| \left\{ w_1(Q_{j,m})^{-1/p_1} w_2(Q_{j,m})^{1/p_2} \right\}_m |\ell_{p^*} \right\| \right\}_j \in \ell_{q^*},$$

where p^* and q^* are given by

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+, \qquad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+.$$

Although the conditions for the embeddings in Theorem 3.5 are sharp and optimal, we discuss later in this chapter two special cases of this general embedding result, since the conditions are very technical and difficult to prove. Here we obtain, for instance, results of the following type:
The embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ is continuous, if

(a) $\inf_{l\in\mathbb{Z}^n} w(Q_{0,l}) \geq c > 0,$

(b) $\begin{cases} \delta > \frac{n}{p_1}(\gamma - 1), & \text{if} \quad q^* < \infty, \\ \delta \geq \frac{n}{p_1}(\gamma - 1), & \text{if} \quad q^* = \infty, \end{cases}$

(c) $p_1 \leq p_2,$

where the difference of the differential dimensions δ is given by

$$\delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2}.$$

We refer to Section 3.3 and Section 3.4 for further results.

Chapter 4 is devoted to the study of growth envelope functions. The aim is to characterize the singularity behaviour of functions belonging to a function space X, in particular, when this space contains essentially unbounded functions. In order to compare various functions regardless of the location of their singularities one uses the non-increasing rearrangement f^* of a function f, which puts the singularities of f into 0. This leads to the growth envelope functions $\mathcal{E}^X_{\mathsf{G}}$ defined by

$$\mathcal{E}^X_{\mathsf{G}}(t) := \sup_{f\in X, \|f|X\|\leq 1} f^*(t), \qquad t > 0.$$

This concept was introduced and first studied in [Tri01] and [Har02]. For detailed information about growth envelopes and more general approaches we refer to the book [Har07] by D. D. Haroske. There one finds among others the results for the classsical (unweighted) Besov-Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$. In [Ska10] we already dealt with growth envelope functions in Muckenhoupt weighted $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$ spaces. For an extensive overview about the results in the context of Muckenhoupt weighted Besov-Triebel-Lizorkin spaces we refer to the paper [Har10] by D. D. Haroske. Our main results, for instance for the Besov spaces, here are the following:

Let $0 < p < \infty$*,* $0 < q \leq \infty$*,* $s \in \mathbb{R}$ *and* w *be doubling. We assume*

$$\max(\frac{n}{p} - n, 0) + \frac{n}{p}(\gamma - 1) < s < \frac{n}{p}\gamma \qquad \text{and} \qquad \inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c_w > 0.$$

Then we have for the estimate from above

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \leq c\, t^{-\frac{\gamma}{p} + \frac{s}{n}}, \quad t \to 0,$$

and for the estimate from below

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \geq c\, t^{-\frac{1}{p} + \frac{s}{n}} \sup_{x^0 \in \mathbb{R}^n, t \sim 2^{-jn}} \Big(\frac{w(B(x^0, 2^{-j}))}{|B(x^0, 2^{-j})|} \Big)^{-1/p}, \quad t \to 0,$$

cf. Proposition 4.13 and Proposition 4.14.

In [Har10] one finds similar results for both estimates from above and from below for $\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t)$, $t \to 0$, if w is a general Muckenhoupt weight, see [Har10, Prop. 4.3., Prop. 4.12. and Rem. 4.14.]. This is not surprising, since we do not use weight-specific properties except for the embedding result and the atomic decomposition, but there we have also similar results as in the Muckenhoupt case. However, nothing was known so far in case of (general) doubling weights. One can think of further applications of those results, but this is postponed to future research.

1 Weighted Function Spaces

1.1 Preliminaries

In this section we collect some notation, which remain fixed throughout this work. By $\mathbb{N}$ we mean the set of natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. $\mathbb{R}^n$ denotes the Euclidean $n-$space, where $n \in \mathbb{N}$, and $\mathbb{C}$ denotes the complex plane. As usual $\mathbb{Z}^n$, where $n \in \mathbb{N}$, is the collection of all lattice points in $\mathbb{R}^n$ having integer components. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$ and $|\alpha| = \sum_{j=1}^n \alpha_j$. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ then we put $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ (monomials). The positive part of a real function f is denoted by $f_+(x) = \max(f(x), 0)$ and the integer part of $a \in \mathbb{R}$ by $\lfloor a \rfloor = \max\{k \in \mathbb{Z} : k \leq a\}$. If $0 < u \leq \infty$, the number u' is given by $\frac{1}{u'} = (1 - \frac{1}{u})_+$. For two non-negative functions ϕ, ψ we mean by $\phi(t) \sim \psi(t)$ that there exist constants $c_1, c_2 > 0$ such that $c_1\phi(t) \leq \psi(t) \leq c_2\phi(t)$ for all admitted values of t. Moreover $\phi(t) \lesssim \psi(t)$ stands for that there exists a constant $c > 0$ such that $\phi(t) \leq c\,\psi(t)$ for all admitted values of t. Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

Let for $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$, $Q_{j,m}$ denote the n-dimensional (open) cube with sides parallel to the axes of coordinates, centred at $2^{-j}m$ and with side length 2^{-j}. Occasionally we shall also deal with n-dimensional (open) cubes $Q = Q(x, l)$ with sides parallel to the axes of coordinates, centred at x and with side length l. Then $2\,Q$ stands for the cube centred at x and with doubled side-length $2\,l$, i.e., $2\,Q = Q(x, 2\,l)$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball $B(x, r) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$.

All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. The characteristic function of a measurable set Ω is denoted by χ_Ω. For any measurable subset $\Omega \subset \mathbb{R}^n$ the Lebesgue space $L_p(\Omega), 0 < p \leq \infty$, consists of all measurable functions for which

$$\|f\ |L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{1/p} \tag{1.1}$$

is finite, where we use in the limiting case $p = \infty$ the usual modification with the essential supremum

$$\|f\ |L_\infty(\Omega)\| = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

Taking $\Omega = \mathbb{N}, \mathbb{Z}$ or $\Omega = \{1, \ldots, n\}$ and replacing the Lebesgue measure by the counting measure produces Lebesgue sequence spaces denoted as usual by ℓ_p and ℓ_p^n, respectively. In the sequel we shall always deal with function spaces on $\mathbb{R}^n$, we may often omit the '$\mathbb{R}^n$' from their notation for convenience.

Let $C(\mathbb{R}^n)$ be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the supremum norm

$$\|f|C(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)|.$$

For $m \in \mathbb{N}$, $C^m(\mathbb{R}^n)$ denotes the collection of all complex-valued functions f which have bounded continuous derivatives $\mathrm{D}^\alpha f$ on $\mathbb{R}^n$ for all $|\alpha| \le m$, i.e.

$$C^m(\mathbb{R}^n) = \{f : \mathbb{R}^n \mapsto \mathbb{C} \mid \mathrm{D}^\alpha f \in C(\mathbb{R}^n) \quad \text{for all} \quad |\alpha| \le m\}.$$

$C^m(\mathbb{R}^n)$ is equipped with the norm

$$\|f|C^m(\mathbb{R}^n)\| = \sum_{|\alpha|\le m} \|\mathrm{D}^\alpha f|C(\mathbb{R}^n)\|.$$

In addition we denote by $C^\infty(\mathbb{R}^n)$ the class of all infinitely differentiable functions f mapping from $\mathbb{R}^n$ to $\mathbb{C}$.

$\mathcal{D}(\mathbb{R}^n)$ or $C_0^\infty(\mathbb{R}^n)$, respectively, denotes the space of all C^∞ functions with compact support. The space of continuous linear functionals on $\mathcal{D}(\mathbb{R}^n)$ will be denoted by $\mathcal{D}'(\mathbb{R}^n)$, the space of distributions and the topological dual of $\mathcal{D}(\mathbb{R}^n)$. The Schwartz space of all complex-valued, rapidly decreasing C^∞ functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$ and is endowed with the semi-norms

$$||\varphi||_{k,l} := \sup_{x\in\mathbb{R}^n} (1+|x|)^k \sum_{|\alpha|\le l} |\mathrm{D}^\alpha\varphi(x)|, \quad \forall\, k,l \in \mathbb{N}_0, \tag{1.2}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The topological dual of $\mathcal{S}(\mathbb{R}^n)$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$, the space of all complex-valued tempered distributions on $\mathbb{R}^n$.

We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ix\xi}\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n,$$

and the inverse Fourier transform by

$$\mathcal{F}^{-1}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{ix\xi}\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n.$$

The Fourier transform is a one to one mapping from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. Moreover, $\mathcal{F}^{-1}(\mathcal{F}f) = f$, $f \in \mathcal{S}(\mathbb{R}^n)$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way.

1.2 Weights

By a weight w we shall always mean a locally integrable and positive a.e. function w on $\mathbb{R}^n$, in the sequel. To prevent trivialities we assume always that w is nonzero. As usual, we use the abbreviation

$$w(\Omega) = \int_\Omega w(x)\,\mathrm{d}x, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is some bounded, measurable set.
For such a weight w we extend the usual Lebesgue space $L_p(\mathbb{R}^n)$, $0 < p \le \infty$, with the weighted L_p-norm,

$$\|f\ |L_p(\mathbb{R}^n, w)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}, \tag{1.4}$$

with the usual modification for $p = \infty$, and obtain the weighted Lebesgue space $L_p(w) = L_p(\mathbb{R}^n, w)$, $0 < p \le \infty$.

We are mainly interested in doubling weights, but for later use we briefly recall, in addition, the notion of Muckenhoupt weights and some of their characteristic features.

1.2.1 Muckenhoupt weights

The purpose of this section is to review the definition of the Muckenhoupt weights and the collection of some known properties. For more information about Muckenhoupt weights we refer for example to [Duo01, GCRdF85, Ste93, ST89, Tor86].

For a locally integrable function f the *Hardy-Littlewood maximal operator* $\mathcal{M}$ is given by

$$(\mathcal{M}f)(x) = \sup_{B(x,r)\in\mathcal{B}} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.5}$$

where here $\mathcal{B}$ is the collection of all open balls $B(x,r)$ centred at $x \in \mathbb{R}^n$, $r > 0$. Sometimes we will use the maximal operator with cubes instead of balls. So we define

$$(\mathcal{M}'f)(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y, \tag{1.6}$$

where the supremum is taken over all cubes containing x. There exist constants c_n and C_n, depending only on n, such that

$$c_n\ (\mathcal{M}'f)(x) \le (\mathcal{M}f)(x) \le C_n\ (\mathcal{M}'f)(x). \tag{1.7}$$

Because of inequality (1.7), the two operators $\mathcal{M}$ and $\mathcal{M}'$ are interchangeable, and we will use whichever is more appropriate, depending on the circumstances. Alternatively, one could define centered versions of the maximal functions with centered balls and cubes, respectively. However we do not want to distinguish between $\mathcal{M}$ and $\mathcal{M}'$ in the following.

It is well-known that the Muckenhoupt weight class is closely connected with the boundedness of this operator $\mathcal{M}$ acting in weighted Lebesgue spaces, $L_p(w)$,

$$\int_{\mathbb{R}^n} |(\mathcal{M}f)(x)|^p w(x)\,\mathrm{d}x \le A \int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x, \tag{1.8}$$

for some p, $1 < p < \infty$. Moreover there exist a lot of characterizations of Muckenhoupt weights. We use here the standard definition, which has been proved as very useful in the last years.

Definition 1.1. Let w be a weight on $\mathbb{R}^n$.

(i) Then w belongs to the Muckenhoupt class $\mathcal{A}_p$, $1 < p < \infty$, if there exists a constant $0 < A < \infty$ such that for all balls B the following inequality holds

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right) \cdot \left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \le A. \tag{1.9}$$

The smallest such A is called the Muckenhoupt constant $A_p = A_p(w)$.

(ii) Then w belongs to the Muckenhoupt class $\mathcal{A}_1$ if there exists a constant $0 < A < \infty$ such that the inequality

$$(\mathcal{M}w)(x) \le Aw(x) \tag{1.10}$$

holds for almost all $x \in \mathbb{R}^n$. The smallest such A is called the Muckenhoupt constant $A_1 = A_1(w)$.

(iii) The Muckenhoupt class $\mathcal{A}_\infty$ is given by

$$\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p. \tag{1.11}$$

Since the pioneering work of Muckenhoupt [Muc72a, Muc72b, Muc74], these classes of weight functions have been studied in great detail, we refer, in particular, to the monographs [Duo01, GCRdF85, Ste93, ST89, Tor86] for a complete account on the theory of Muckenhoupt weights.
Note, that the $\mathcal{A}_p$ condition (1.9) can also be defined by cubes instead of balls

$$\left(\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x\right) \cdot \left(\frac{1}{|Q|}\int_Q w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} < \infty, \tag{1.12}$$

for all cubes Q. Both conditions (1.9) and (1.12) are equivalent, see Remark 1.16 below. Moreover, there exists an equivalent characterization for the $\mathcal{A}_1$ weights,

$$\frac{w(Q)}{|Q|} \le Cw(x), \quad \text{a.e. } x \in Q, \tag{1.13}$$

for any cube Q, cf. [Duo01, p. 134] formula (7.4). Certainly (1.13) also holds for balls instead of cubes, because the respective maximal operators are equal.
We give a short overview of some fundamental properties. We start with a series of easy observations, cf. [Ste93, Chapt. V] or [Duo01, Chapt. 7].

Proposition 1.2.

(i) *The class $\mathcal{A}_p$ is invariant concerning translation, dilation and multiplication by a positive scalar, where the Muckenhoupt constant is the same as that of w.*

(ii) *If $w \in \mathcal{A}_p$, then the function $\sigma := w^{-p'/p} = w^{1-p'}$ belongs to $\mathcal{A}_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

(iii) *If $w \in \mathcal{A}_{p_1}$, then $w \in \mathcal{A}_{p_2}$, for $p_1 < p_2$; moreover $A_{p_1}(w) \le A_{p_2}(w)$.*

(iv) *If $w_0, w_1 \in \mathcal{A}_1$, then $w_0 w_1^{1-p} \in \mathcal{A}_p$.*

Proof. The proof of **(i)** is straightforward and **(ii)** is an easy observation by changing the order of the two factors on the left-hand side in (1.9).
Furthermore **(iii)** is a direct consequence of the definition (1.9), Hölder's inequality, and the fact that if $p_1 < p_2$ then $p_2'/p_2 < p_1'/p_1$.
Finally for **(iv)** we need to prove that

$$\left(\frac{1}{|B|}\int_B w_0(x)\, w_1(x)^{1-p}\,\mathrm{d}x\right)\cdot\left(\frac{1}{|B|}\int_B w_0(x)^{1-p'}\, w_1(x)\,\mathrm{d}x\right)^{p/p'} \le A. \tag{1.14}$$

Note, that $1-p' = -\frac{p'}{p}$ and $1-p = -\frac{p}{p'}$. By the $\mathcal{A}_1$ condition (1.13) it holds for $x \in B$ and $i = 0, 1$,

$$w_i(x)^{-1} \le \sup_{x\in B} w_i(x)^{-1} = \left(\inf_{x\in B} w_i(x)\right)^{-1} \le C\left(\frac{w_i(B)}{|B|}\right)^{-1}.$$

If we substitute this into the left-hand side of (1.14) for the negative exponents we get the desired inequality. ■

Example 1.3. One of the most prominent examples of a Muckenhoupt weight $w \in \mathcal{A}_\infty$ is given by $w(x) = |x|^\varrho$, $\varrho > -n$. We modified this example by

$$w_{a,b}(x) = \begin{cases} |x|^a, & |x| < 1,\\ |x|^b, & |x| \ge 1,\end{cases} \tag{1.15}$$

where $a, b > -n$. Straightforward calculation shows that

$$w_{a,b} \in \mathcal{A}_p \quad \text{if, and only if,} \quad \begin{cases} -n < a, b < n(p-1), & \text{if} \quad 1 < p < \infty,\\ -n < a, b \le 0, & \text{if} \quad p = 1.\end{cases}$$

A proof of this one can find for example in [Baa07]. For further examples we refer to [HP08, HS08, HS11a].

There is an alternative way to define $\mathcal{A}_p$ that is closer related to the boundedness of the maximal operator, we described in (1.8). For any locally integrable function f and any ball B the weight w belongs to $\mathcal{A}_p$ exactly when the p-th power of the mean value of f on B is bounded by the mean value of f^p taken with respect to the measure $w(x)\,\mathrm{d}x$.

Proposition 1.4. *Let $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ positive a.e. and $1 < p < \infty$. Then $w \in \mathcal{A}_p$, if and only if, there exists a constant $c > 0$ such that for all non-negative functions $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and for all balls B the following inequality holds*

$$\left(\frac{1}{|B|}\int_B f(x)\,\mathrm{d}x\right)^p \le \frac{c}{w(B)}\int_B f^p(x)w(x)\,\mathrm{d}x. \tag{1.16}$$

Proof.

(i) Let $w \in \mathcal{A}_p$, then

$$\begin{aligned}
\left(\frac{1}{|B|}\int_B f(x)\,\mathrm{d}x\right)^p &= \left(\frac{1}{|B|}\int_B f(x)w^{\frac{1}{p}}(x)w^{-\frac{1}{p}}(x)\,\mathrm{d}x\right)^p \\
&\overset{\text{Hölder}}{\le} \left(|B|^{-1}\left(\int_B f^p(x)w(x)\,\mathrm{d}x\right)^{1/p}\cdot\left(\int_B w(x)^{-\frac{p'}{p}}\,\mathrm{d}x\right)^{1/p'}\right)^p \\
&= |B|^{-p}\cdot\int_B f^p(x)w(x)\,\mathrm{d}x\cdot\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}\,\mathrm{d}x\right)^{p/p'}\cdot|B|^{\frac{p}{p'}} \\
&\overset{w\in\mathcal{A}_p}{\le} \int_B f^p(x)w(x)\,\mathrm{d}x\cdot A_p|B|w(B)^{-1}\cdot|B|^{\frac{p}{p'}}\cdot|B|^{-p} \\
&= |B|^{1-p+\frac{p}{p'}}\cdot\frac{A_p}{w(B)}\int_B f^p(x)w(x)\,\mathrm{d}x = \frac{A_p}{w(B)}\int_B f^p(x)w(x)\,\mathrm{d}x,
\end{aligned}$$

since $1-p = -\frac{p}{p'}$. Besides we have $c \le A_p$.

(ii) On the other hand, assume (1.16). Let $\epsilon > 0$. We choose $f := (w+\epsilon)^{-p'/p}$, then

$$\int_B f(x)\,\mathrm{d}x = \int_B (w+\epsilon)^{-\frac{p'}{p}}(x)\,\mathrm{d}x \overset{w+\epsilon\ge\epsilon}{<} \epsilon^{-\frac{p'}{p}}|B| < \infty \tag{1.17}$$

and so we have $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $f \ge 0$. The assumption (1.16) yields us for this f

$$\left(\frac{1}{|B|}\int_B (w+\epsilon)^{-\frac{p'}{p}}\,\mathrm{d}x\right)^p \le \frac{c}{w(B)}\int_B (w+\epsilon)^{-p'}w(x)\,\mathrm{d}x. \tag{1.18}$$

Additionally we have

$$\int_B (w+\epsilon)^{-p'}(x)w(x)\,\mathrm{d}x \overset{w<w+\epsilon}{<} \int_B (w+\epsilon)^{-p'+1}(x)\,\mathrm{d}x = \int_B (w+\epsilon)^{-\frac{p'}{p}}(x)\,\mathrm{d}x \tag{1.19}$$

is finite, because of (1.17), where we here use that $1-p' = -\frac{p'}{p}$.
Altogether delivers us

$$\begin{aligned}
&|B|^{-p}\,w(B)\left(\int_B (w+\epsilon)^{-p'}(x)\,w(x)\ \mathrm{d}x\right)^p \\
\overset{(1.19)}{<}\ &|B|^{-p}\,w(B)\left(\int_B (w+\epsilon)^{-\frac{p'}{p}}(x)\ \mathrm{d}x\right)^p \\
\overset{(1.18)}{\le}\ &c\int_B (w+\epsilon)^{-p'}\,w(x)\ \mathrm{d}x.
\end{aligned}$$

Since the latter integral is finite, we bring it on the other side and obtain for all $\epsilon > 0$

$$|B|^{-1}\, w(B)\, |B|^{1-p} \left(\int_B (w+\epsilon)^{-p'}(x)\, w(x)\ \mathrm{d}x \right)^{\frac{p}{p'}} \leq c,$$

where the c is independent of ϵ. We use again $1-p = -\frac{p}{p'}$ and $\epsilon \to 0$ finishes the proof. In particular we have $A_p \leq c$. ■

Remark 1.5. The proof of Proposition 1.4 shows us, that the constant c in (1.16) coincides with the Muckenhoupt constant $A_p = A_p(w)$.

A direct consequence of this Proposition is the following useful property, that will be used later more frequently.

Proposition 1.6. *Let $1 \leq p < \infty$ and $w \in \mathcal{A}_p$. Then there exists a constant $c' > 0$ such that for all balls B*

$$\left(\frac{|E|}{|B|} \right)^p \leq c' \frac{w(E)}{w(B)}, \qquad \forall\, E \subset B. \tag{1.20}$$

Proof. Let $B \subset \mathcal{B}$, $E \subset B$. We use Proposition 1.4 with $f := \chi_E$, then

$$\left(\frac{1}{|B|} \int_B \chi_E(x)\, \mathrm{d}x \right)^p \leq \frac{c}{w(B)} \int_B \chi_E^p(x)\, w(x)\, \mathrm{d}x,$$

this implies

$$\left(\frac{|E|}{|B|} \right)^p \leq c' \, \frac{w(E)}{w(B)},$$

where $c' := c$. ■

Proposition 1.7. *Let $w \in \mathcal{A}_\infty$. Then there exists an $r > 1$ and a $c > 0$ such that*

$$\left(\frac{1}{|B|} \int_B w^r(x)\, \mathrm{d}x \right)^{1/r} \leq \frac{c}{|B|} \int_B w(x)\, \mathrm{d}x. \tag{1.21}$$

Proof. A proof of this one can find for example in [Ste93, Ch. §5.3, Prop. 3.4, page 203]. ■

Remark 1.8. This is the so-called reverse Hölder's inequality, because (except for the constant c) this is the reverse of the Hölder's inequality, which holds automatically for all nonnegative functions. Proposition 1.7 is one of the fundamental properties of $\mathcal{A}_p$ weights, which leads us to the next surprising consequence.

Corollary 1.9. *Let $1 < p < \infty$ and $w \in \mathcal{A}_p$. Then there exists a $p_1 < p$ such that*

$$w \in \mathcal{A}_{p_1}. \tag{1.22}$$

Proof. Let $\frac{1}{p}+\frac{1}{p'}=1$ then $\sigma := w^{-\frac{p'}{p}} \in \mathcal{A}_{p'} \subset \mathcal{A}_\infty$. We use Proposition 1.7 for σ, then there exists an $r>1$ and a $c>0$ such that

$$\left(\frac{1}{|B|}\int_B \sigma^r(x)\,\mathrm{d}x\right)^{1/r} \le \frac{c}{|B|}\int_B \sigma(x)\,\mathrm{d}x$$

holds. We take the $\frac{p}{p'}$-th power an both sides and use the $\mathcal{A}_p$ condition (1.9) for w, then

$$\left(\frac{1}{|B|}\int_B w^{-\frac{rp'}{p}}(x)\,\mathrm{d}x\right)^{\frac{p}{rp'}} \le c'\left(\frac{1}{|B|}\int_B w^{-\frac{p'}{p}}(x)\,\mathrm{d}x\right)^{\frac{p}{p'}} \overset{w\in\mathcal{A}_p}{\le} c''\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)^{-1}$$

Thus we get

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w^{-\frac{rp'}{p}}(x)\,\mathrm{d}x\right)^{\frac{p}{rp'}} \le c'' \tag{1.23}$$

Furthermore it holds

$$\frac{1}{p}+\frac{1}{p'}=1 \quad\Leftrightarrow\quad p'+p=pp' \quad\Leftrightarrow\quad p=(p-1)p' \quad\Leftrightarrow\quad \frac{1}{p-1}=\frac{p'}{p}.$$

Then we have

$$\frac{rp'}{p}=\frac{r}{p-1}=\frac{1}{\frac{p-1}{r}+1-1}=\frac{1}{p_1-1}=\frac{p_1'}{p_1}, \tag{1.24}$$

where $p_1 := \frac{p-1}{r}+1$. Moreover

$$1<r \quad\Leftrightarrow\quad p-1<r(p-1) \quad\Leftrightarrow\quad p-1+r<pr \quad\Leftrightarrow\quad \frac{p-1+r}{r}<p \quad\Leftrightarrow\quad p_1<p.$$

So we can write (1.23) together with (1.24) in this way

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w^{-\frac{p_1'}{p_1}}(x)\,\mathrm{d}x\right)^{\frac{p_1}{p_1'}} \le A_{p_1}.$$

■

Corollary 1.10. *Let $1<p<\infty$ and $w\in\mathcal{A}_p$. Then there exists $p_1<p$ and a $c>0$ such that for all balls B*

$$\left(\frac{|E|}{|B|}\right)^{p_1} \le c\,\frac{w(E)}{w(B)}, \qquad E\subset B. \tag{1.25}$$

Proof. This is an immediate conclusion of Proposition 1.6 and Corollary 1.9. ■

Remark 1.11. It even holds the reverse of Corollary 1.10, see [Wik89, Cor. 1, page 250].

Remark 1.12. In view of Corollary 1.9 it is natural to ask for the smallest r, which satisfies (1.22). So we introduce the number

$$r_w := \inf\{r\ge 1\ : w\in\mathcal{A}_r\}, \quad w\in\mathcal{A}_\infty, \tag{1.26}$$

that plays some role later on. Obviously, $1\le r_w<\infty$, and $w\in\mathcal{A}_{r_w}$ implies $r_w=1$.

Example 1.13. For our weight $w_{a,b}$, given by (1.15), here we have $r_{w_{a,b}} = 1 + \frac{\max(a,b,0)}{n}$.

A special case of Proposition 1.4 or Proposition 1.6, respectively, shows us the next property of Muckenhoupt weights, which leads us as well to the weight class we are mainly interested in.

Proposition 1.14. *Let $1 < p < \infty$ and $w \in \mathcal{A}_p$. Then there exists a constant $c > 0$ such that for all $x \in \mathbb{R}^n$ and for all $r > 0$ holds*

$$w(B(x, 2r)) \leq c\, w(B(x, r)). \tag{1.27}$$

Proof. We use Proposition 1.6 with $B = B(x, 2r)$ and $E = B(x, r) \subset B(x, 2r)$, then

$$\begin{aligned}
\left(\frac{|B(x,r)|}{|B(x,2r)|}\right)^p &\leq \frac{c'}{w(B(x,2r))} \cdot w(B(x,r)) \\
\left(\frac{|\omega_n|\, n^{-1}\, r^n}{|\omega_n|\, n^{-1}\, (2r)^n}\right)^p &\leq \frac{c'}{w(B(x,2r))} \cdot w(B(x,r)) \\
w(B(x,2r)) &\leq c'\, 2^{np}\, w(B(x,r)).
\end{aligned}$$

In particular $c = c'\, 2^{np} = A_p(w)\, 2^{np}$. ■

Remark 1.15. Condition (1.27) is called **doubling property** or **doubling condition**, respectively. We see that all Muckenhoupt weights have the doubling property. In the next section we give a finer characterization of what is a doubling weight.

Remark 1.16. In view of this doubling property and the fact that by Proposition 1.2 $w^{-p'/p}$ is also a doubling weight or measure, respectively, together with the characterization (1.16) we could replace the family of balls by the family of cubes or other such equivalent families.

Before we come to the most important weight class of this work, the doubling weights, we discuss a weight w which does not belong to $\mathcal{A}_\infty$ but still has the doubling property.

1.2.2 An Example

In this section we will give a function which does not belongs to $\mathcal{A}_\infty$ but still has the doubling property. This Example is based on a paper of Wik from 1989, see [Wik89].

I. Construction For convenience we consider only the 1-dimensional case in $\mathbb{R}$. We start with the function w_0 given by

$$w_0(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 2 - x, & 1 \leq x \leq 2, \\ 0, & \textit{elsewhere}. \end{cases} \tag{1.28}$$

We construct the graph of w_1 by dividing step by step the graph of w_0 by 2 in both the x and y direction and translate it in that way, that it adjoins to the existing graph. We do this till such time as we reach the point $x = 4$, then we reflect (symmetrically) the existing graph at the line $x = 4$.
In general we construct w_n from w_{n-1} in the same way. In formulas this is given by

$$w_{n+1}(x) = \begin{cases} \sum_{k=0}^{\infty} 2^{-k} w_n\left(2^k(x - 4^{n+1}) + 4^{n+1}\right), & 0 \le x \le 4^{n+1}, \\ w_{n+1}(2 \cdot 4^{n+1} - x), & 4^{n+1} \le x \le 2 \cdot 4^{n+1}, \\ 0, & elsewhere. \end{cases} \tag{1.29}$$

Then we define

$$w(x) := \lim_{n\to\infty} w_n(x), \qquad \text{for} \quad x > 0,$$

and

$$w(x) := w(-x), \qquad \text{for} \quad x < 0.$$

II. Properties At first we prove this easy observation, for $n \in \mathbb{N}_0$ we have

$$\int_0^{2\cdot 4^n} w_n(x)\,\mathrm{d}x = \left(\frac{8}{3}\right)^n. \tag{1.30}$$

Proof. We do this easily by induction. For $n = 0$ we have

$$\int_0^2 w_0(x)\,\mathrm{d}x = \int_0^1 x \ \mathrm{d}x + \int_1^2 (2 - x) \ \mathrm{d}x = 1.$$

Let

$$\int_0^{2\cdot 4^n} w_n(x)\,\mathrm{d}x = \left(\frac{8}{3}\right)^n \tag{1.31}$$

our induction hypothesis. Then

$$\begin{aligned} &\int_0^{2\cdot 4^{n+1}} w_{n+1}(x)\,\mathrm{d}x \\ =&2 \cdot \int_0^{4^{n+1}} w_{n+1}(x)\,\mathrm{d}x \\ =&2 \cdot \sum_{k=0}^{\infty} 2^{-k} \int_0^{4^{n+1}} w_n\left(2^k(x - 4^{n+1}) + 4^{n+1}\right) \mathrm{d}x, \end{aligned}$$

the substitution $y = 2^k(x - 4^{n+1}) + 4^{n+1},\ \mathrm{d}y = 2^k\,\mathrm{d}x$ yields us

$$=2 \cdot \sum_{k=0}^{\infty} 2^{-2k} \int_{4^{n+1}(1-2^k)}^{4^{n+1}} w_n(y)\,\mathrm{d}y.$$

Since $w_n(y) = 0,\ y \notin [0, 2 \cdot 4^n]$ and $4^{n+1} > 2 \cdot 4^n$, $4^{n+1}(1 - 2^k) \le 0$, $k \in \mathbb{N}_0$ we have

$$=2 \cdot \sum_{k=0}^{\infty} 2^{-2k} \int_0^{2\cdot 4^n} w_n(y)\,\mathrm{d}y$$

and the induction hypothesis (1.31) delivers us

$$=2 \cdot \left(\frac{8}{3}\right)^n \underbrace{\sum_{k=0}^{\infty} 2^{-2k}}_{\frac{1}{1-\frac{1}{4}}} = \left(\frac{8}{3}\right)^{n+1}.$$

■

Now let us consider these sets

$$E_n := \left\{x \in [0, 2 \cdot 4^n] : w(x) < 2^{-n}\right\}. \tag{1.32}$$

We make an estimation of the Lebesgue measure of these sets.
For the set E_0 its clear, that we have

$$|E_0| = |\{x \in [0,2] : w_0(x) < 1\}| = 2.$$

In the case of E_1 we have

$$E_1 = \left\{x \in [0,8] : w_1(x) < 2^{-1}\right\}.$$

Let us consider the graph of E_1. We see it contains infinitely many pyramids. We would like to call all pyramids with a rise of 2^{-k} as $P(k+1)$. The graph of w_1 contains exactly 2 pyramids $P1$, $P2$,... In our estimation we would like to consider only the pyramids whose tops lie over the line 2^{-1}. But for these pyramids we have only to consider the part which lies under the line 2^{-1}. So we have in the case of E_1 that only the both pyramids $P1$ rise over the line 2^{-1}. They have an area of 2 and a rise of 1. Thus

$$|E_1| > 2 \cdot 1 \cdot 2^{-1} \cdot 2 = 2.$$

In general we only consider all the pyramids whose tops lie over the line 2^{-n} and for these the part which lies under the line 2^{-n}. The part of the pyramids Pk which lies under the line 2^{-n} is:

$$2^{-k+2} \cdot \frac{1}{2^{-k+1}} \cdot 2^{-n} = 2^{-n+1}. \tag{1.33}$$

We see this part is for all admissible pyramids Pk equal (only depends on n). So we have only to count the admissible pyramids Pk. We see that the number of the pyramids $P1$ is doubled in every step, i.e. the graph of w_n contains 2^n pyramids $P1$. Further the number of the pyramids Pk depends on the number of pyramids $P1$. Because of the construction of w_n from w_{n-1} it emerge iterated sums at counting the sets of pyramids. So we have

$$\begin{aligned}
|E_2| &> 2^{-1}\left[2^2 + 2 \cdot 2^2\right] = 2^{-1} \cdot 2^2 [1+2] = 6, \\
|E_3| &> 2^{-2} \cdot 2^3 [1 + (1+2) + (1+2+3)] = 20, \\
|E_n| &> 2^{n+1} \cdot 2^n \sum_{k_1=1}^{n} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} k_{n-1} \\
&> 2 \sum_{k_1=1}^{n} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \sum_{k_n=1}^{k_{n-1}} 1.
\end{aligned}$$

The iterated sums correspond to this combinatoric question:

$$\#\left\{(k_1,\dots,k_n) : n \geq k_1 \geq k_2 \geq \dots \geq k_{n-1} \geq k_n \geq 1\right\}.$$

This is the selection of n elements from an $n-$element set with repeating.

$$\binom{n+k-1}{k} \overset{n=k}{=} \binom{2n-1}{n} = \frac{(2n-1)!}{(n-1)!n!}.$$

Hence we have

$$|E_n| > 2 \cdot \frac{(2n-1)!}{(n-1)!n!}.$$

Using Stirling's formula, we obtain

$$|E_n| \gtrsim \frac{4^n}{\sqrt{n}}. \tag{1.34}$$

III. $w \notin \mathcal{A}_\infty$ Our purpose is to violate (1.25) if p_1 is large enough. For this we take $p_1 = \ln(n)$, $B = Q_n = [0, 2\cdot 4^n]$, $E = E_n$ as above. Then it holds:

$$\begin{aligned} |Q_n| &= 2\cdot 4^n \quad , & w(Q_n) &= \left(\frac{8}{3}\right)^n, \\ |E_n| &\gtrsim \frac{4^n}{\sqrt{n}} \quad , & w(E_n) &\leq 2^{-n}|E_n|. \end{aligned}$$

This leads to

$$\begin{aligned} \frac{w(E_n)}{w(Q_n)}\left(\frac{|Q_n|}{|E_n|}\right)^{p_1} &\lesssim 2^{-n}\cdot |E_n|^{-(p_1-1)}\cdot \left(\frac{3}{8}\right)^n \cdot 2^{p_1}\cdot 4^{n\,p_1} \\ &\lesssim \left(\frac{3}{16}\right)^n \cdot 2^{p_1}\cdot 4^{n\,p_1}\cdot 4^{-n\,p_1}\cdot 4^n\cdot \left(\sqrt{n}\right)^{p_1-1} \\ &\lesssim \left(\frac{3}{4}\right)^n \cdot 2^{p_1}\left(\sqrt{n}\right)^{p_1-1} \\ &= \left(\frac{3}{4}\right)^n \cdot 2^{\ln(n)}\left(\sqrt{n}\right)^{\ln(n)-1} \overset{n\to\infty}{\to} 0. \end{aligned}$$

Thus w does not belong to any $\mathcal{A}_p$, $p \geq 1$.

IV. w **is doubling** In the first step we consider intervals $I = [a,b]$ of the form

$$a = m\cdot 2^k, \quad b = (m+1)\cdot 2^k, \quad m\in\mathbb{N}_0,\ k\in\mathbb{Z}.$$

Then $|I| = 2^k$. For $I = [a,b]$ we denote by $3I = [2a-b, 2b-a]$.
It holds

$$w(3I) < 6\,w(I). \tag{1.35}$$

Proof. At first we will prove this important fact about the area of the pyramids Pk.

$$\sum_{l=k+1}^{\infty} |Pl| = \frac{1}{3}|Pk|, \tag{1.36}$$

where $|Pk|$ denotes the area of Pk.

$$\begin{aligned}
\sum_{l=k+1}^{\infty} |Pl| &= |P(k+1)| + |P(k+2)| + |P(k+3)| + \dots \\
&= \frac{1}{4} \cdot |Pk| + \frac{1}{4} \cdot |P(k+1)| + \frac{1}{4} \cdot |P(k+2)| + \dots \\
&= \frac{1}{4} \cdot |Pk| + \left(\frac{1}{4}\right)^2 \cdot |Pk| + \left(\frac{1}{4}\right)^3 \cdot |Pk| + \dots \\
&= |Pk| \cdot \sum_{l=1}^{\infty} \left(\frac{1}{4}\right)^l = |Pk| \cdot \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}|Pk|.
\end{aligned}$$

Now we have some cases:

(i) Let $m = 0$ and $k \in \mathbb{N}$ then $I = \left[0, 2^k\right]$. If k is even we have $2^{2l} = 4^l,\ l \in \mathbb{N}$, then $w|_I$ consists of exactly one half of a copy of w_l. Therefore $w|_{3I}$ consists of exactly 3 of these half copies of w_l. It holds

$$\begin{aligned}
&w(\left[-4^l, 0\right]) = w(\left[0, 4^l\right]), \\
&w(\left[4^l, 2 \cdot 4^l\right]) = w(\left[0, 4^l\right]).
\end{aligned}$$

Consequently

$$w(3I) = 3\, w(I) < 6\, w(I).$$

If k is odd we have $2^{2l+1} = 2 \cdot 4^l,\ l \in \mathbb{N}_0$, then $w|_I$ consists of exactly one copy of w_l. Thus $w|_{3I}$ consists of exactly 2 of these copies of w_l and an infinite succession of smaller copies of w_l. So we have

$$\begin{aligned}
&w(\left[-2 \cdot 4^l, 0\right]) = w(\left[0, 2 \cdot 4^l\right]), \\
&w(\left[2 \cdot 4^l, 4^{l+1}\right]) \overset{(1.36)}{=} \frac{1}{3}\, w(\left[0, 2 \cdot 4^l\right]) < w(\left[0, 2 \cdot 4^l\right]),
\end{aligned}$$

and thus

$$w(3I) < 3\, w(I) < 6\, w(I).$$

Here we have used (1.36). Let $m \neq 0$ in the following.

(ii) Let $a = 4^n$ or $b = 4^n$ (w.l.o.g. $a = 4^n$), then $w|_I$ consists of an infinite succession of copies of w_{n-1}. Hence $w|_{3I}$ consists of 2 of these infinite successions of copies of w_{n-1} and in the worst case of one bigger copy of w_{n-1}. It holds

$$\begin{aligned} w([2a-b,a]) &= w([a,b]), \\ w([b,2b-a]) &\overset{(1.36)}{\leq} 3\,w([a,b]). \end{aligned}$$

Consequently

$$w(3I) \leq 5\,w(I) < 6\,w(I).$$

(iii) $w|_I$ consists of exactly one copy of w_n, then we have 2 cases. On the one side $w|_{3I}$ consists of this copy of w_n and 2 infinite successions of smaller copies of w_n. Because of (1.36) it is clear that we have

$$w(3I) \overset{(1.36)}{=} w(I) + \frac{1}{3}\,w(I) + \frac{1}{3}\,w(I) < 6\,w(I).$$

On the other side $w|_{3I}$ consists of this copy of w_n, an infinite succession of smaller copies of w_n and one half bigger copy of w_n. It is also evident, that $w(3I) < 6\,w(I)$,

$$w(3I) \overset{(1.36)}{=} w(I) + \frac{1}{3}\,w(I) + 2\,w(I) < 6\,w(I).$$

(iv) $w|_I$ consists of exactly one half copy of w_n. If $w|_I$ consists of exactly one half copy of w_0, it is clear that $w(3I) \leq 3\,w(I) < 6\,w(I)$. Otherwise we have still 2 cases. On the one side $w|_{3I}$ consists of 2 of these half copies of w_n and one smaller copy of w_n. Thus we obtain

$$w(3I) \overset{(1.36)}{=} w(I) + w(I) + \frac{1}{2}\,w(I) < 6\,w(I).$$

On the other side $w|_{3I}$ consists of 2 these half copies of w_n and one bigger copy of w_{n-1}. (1.36) yields us

$$w(3I) \overset{(1.36)}{=} w(I) + w(I) + 3\,w(I) < 6\,w(I).$$

(v) The remaining cases are: $w|_I$ consists of $\frac{1}{4}$ or $\frac{1}{8}$ or $\ldots$ of a copy of w_0. But for this it is easy to prove that $w(3I) < 6\,w(I)$.

Since w is even, (1.35) also holds for $m < 0$. So we have (1.35) shown for all $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

In the second step let I be an arbitrary interval. Then we find a $k \in \mathbb{Z}$ such that $2^k \leq |I| < 2^{k+1}$ and then we choose an $m \in \mathbb{Z}$ such that

$$I' := \left[m \cdot 2^{k-1}, (m+1) \cdot 2^{k-1}\right] \subset I. \tag{1.37}$$

It is easily seen that $2I \subset 16I' = 2I''$, where $I'' := 8I'$. Since I'' satisfies step 1, we have

$$w(2I) < w(2I'') < w(3I'') \overset{(1.35)}{<} 6\,w(I'').$$

Using (1.35) 3 more times, we obtain

$$w(2I) < 6\,w(I'') \overset{(1.35)}{<} 6^4\,w(I') \overset{(1.37)}{<} 6^4 w(I).$$

This finished the proof. ■

1.2.3 Doubling weights

We come to the most important weight class in this work which naturally extends Muckenhoupt weights.

Definition 1.17. We say that a nonnegative Borel measure μ on $\mathbb{R}^n$ is doubling (concerning balls) if there exists a constant $\beta > 0$ such that

$$\mu(B(x,2r)) \leq 2^{n\beta}\mu(B(x,r)), \qquad \text{for all} \quad x \in \mathbb{R}^n,\ r > 0, \tag{1.38}$$

The smallest such β is called doubling constant of μ.

Remark 1.18. Note that the doubling measure μ does not need to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, cf. [BM00]. On the other hand, any weight $w \in \mathcal{A}_\infty$ defines a doubling measure μ by $\mathrm{d}\mu = w(x)\,\mathrm{d}x$ in view of Proposition 1.14, see also Example 1.21 below.

In the following we are only interested in doubling measures, which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. So we introduce the so-called doubling weights. We remind, that we shall mean by a weight a locally integrable and positive a.e. function on $\mathbb{R}^n$.

Definition 1.19. Let w be a weight on $\mathbb{R}^n$. w is called doubling (concerning balls) if there exists a constant $\beta > 0$ such that

$$w(B(x,2r)) \leq 2^{n\beta}w(B(x,r)), \qquad \text{for all} \quad x \in \mathbb{R}^n,\ r > 0. \tag{1.39}$$

The smallest such β is called doubling constant of w (concerning balls).

Example 1.20. It is clear that $w \equiv 1$ is doubling, because $|B(x,2r)| = 2^n|B(x,r)|$ for arbitrary balls $B(x,r)$, i.e. $\beta = 1$.

Example 1.21. Our weight $w_{a,b}$ from Example 1.3 is doubling, if $a, b > -n$, i.e. $w_{a,b}$ is doubling, if and only if, $w_{a,b} \in \mathcal{A}_\infty$. In view of Proposition 1.14 all Muckenhoupt weights $w \in \mathcal{A}_\infty$ are doubling. Short calculation using (1.20) yields us $\beta = p\,\log A_p(w)$, see also the proof of Proposition 1.14, or, in view of later use, $\beta = c\, r_w$, respectively, whereas the $c = c_w > 1$ depends on w. On the contrary, there exist doubling weights which do not belong to $\mathcal{A}_\infty$ as we have seen in Section 1.2.2. Hence $\mathcal{A}_\infty$ is a proper subset of all doubling weights which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.

Now we introduce another definition of doubling weights with respect to cubes. This is an equivalent definition. The constants depend on the dimension, see Proposition 1.23 below.

Definition 1.22. Let w be a weight on $\mathbb{R}^n$. w is called doubling (concerning cubes) if there exists a constant $\gamma > 0$ such that for all cubes Q

$$w(2\,Q) \le 2^{n\gamma} w(Q). \tag{1.40}$$

The smallest such γ is called doubling constant of w (concerning cubes).

Proposition 1.23. *Let w be a weight on $\mathbb{R}^n$.*

(i) *The conditions* (1.39) *and* (1.40) *are equivalent.*

(ii) *For the doubling constants holds*

$$\frac{1}{c}\,\beta \le \gamma \le c\,\beta, \tag{1.41}$$

where $c = \lfloor \log_2(\sqrt{n}) \rfloor + 2$.

(iii) *The doubling constants satisfy* $\beta \ge 1$ *and* $\gamma \ge 1$.

Proof. *Step 1.* First assume that w satisfies (1.39). Let $Q = Q(x,l) = l \cdot \left[-\frac{1}{2}, \frac{1}{2}\right]^n + x$, $x \in \mathbb{R}^n$, $l > 0$, be an arbitrary cube. Then there exist balls $B_1 = B(x, \frac{l}{2})$, $B_2 = B(x, \sqrt{n}\,l)$, such that the outer ball B_2 touches the corners of the cube $Q(x, 2l)$ and the inner ball B_1 touches the inner sidewalls of the cube $Q(x,l)$. Thus we have

$$\begin{aligned} w(2\,Q) &= \int_{Q(x,2l)} w(y)\,\mathrm{d}y \\ &\le \int_{B(x,\sqrt{n}\,l)} w(y)\,\mathrm{d}y = w(B(x, \sqrt{n}\,l)) \\ &\overset{(1.39)}{\le} 2^{n\beta} w(B(x, \frac{\sqrt{n}}{2}\,l)) \\ &\overset{(1.39)}{\le} 2^{n\beta(k+1)} w(B(x, \frac{\sqrt{n}}{2^k}\,\frac{l}{2})), \end{aligned}$$

where we applied (1.39) and $k \in \mathbb{N}$ is chosen such that $\frac{\sqrt{n}}{2^k} \le 1 \Leftrightarrow \log_2 \sqrt{n} \le k$, say, $k = \lfloor \log_2(\sqrt{n}) \rfloor + 1$. Thus we can continue our estimate by

$$\begin{aligned} w(2\,Q) &\le 2^{n\beta(k+1)} w(B(x, \frac{l}{2})) \\ &\le 2^{n\beta(k+1)} w(Q) \end{aligned}$$

and obtain for the doubling constants $\gamma \le (\lfloor \log_2(\sqrt{n}) \rfloor + 2) \cdot \beta$.

Step 2. On the other side if (1.40) holds. Let $B = B(x,r)$, $x \in \mathbb{R}^n$, $r > 0$ be an arbitrary ball. Then we have in the same way 2 cubes $Q_1 = Q(x, \frac{2r}{\sqrt{n}})$, $Q_2 = Q(x, 4r)$ with

$$\begin{aligned} w(B(x,2r)) &\le w(Q(x,4r)) \\ &\overset{(1.40)}{\le} 2^{n\gamma} w(Q(x,2r)) = 2^{n\gamma} w(Q(x, \frac{2r}{\sqrt{n}} \cdot \sqrt{n})) \\ &\overset{(1.40)}{\le} 2^{n\gamma(k+1)} w(Q(x, \frac{2r}{\sqrt{n}} \cdot \frac{\sqrt{n}}{2^k})), \end{aligned}$$

where we applied (1.40) and $k \in \mathbb{N}$ have been chosen again such that $\frac{\sqrt{n}}{2^k} \leq 1 \Leftrightarrow \log_2 \sqrt{n} \leq k$, say, $k = \lfloor \log_2(\sqrt{n}) \rfloor + 1$. Hence,

$$\begin{aligned} w(B(x,2r)) &\leq 2^{n\gamma(k+1)} w(Q(x, \frac{2r}{\sqrt{n}})) \\ &\leq 2^{n\gamma(k+1)} w(B(x,r)) \end{aligned}$$

and we get $\beta \leq (\lfloor \log_2(\sqrt{n}) \rfloor + 2) \cdot \gamma$. This concludes the proof of (i) and (ii). It remains to verify (iii).

Step 3. Let w be doubling (concering cubes). Let Q be a cube with sidelength 1. Moreover let l be an arbitrary natural number. Q contains 2^{nl} disjoint cubes Q_i with sidelength 2^{-l}. It holds

$$\bigcup_{i=1}^{2^{nl}} Q_i \subset Q, \quad |Q_i| = 2^{-nl}, \quad Q_i \cap Q_j = \emptyset.$$

Let Q_i be an arbitrary small cube in Q, then the big cube Q is covered by $x \cdot Q_i$, whereas $x \cdot 2^{-l} \geq 2$, i.e. $x \geq 2^{l+1}$. Then it applies for a fixed $l \in \mathbb{N}$ and all $i \in \{1, \dots, 2^{nl}\}$ that

$$\begin{aligned} 2^{nl} \min_{j=1,\dots,2^{nl}} w(Q_j) &\leq \sum_{j=1}^{2^{nl}} w(Q_j) \leq w(\bigcup_{j=1}^{2^{nl}} Q_j) \leq w(Q) \\ &\leq w(2^{l+1} Q_i) \leq 2^{n\gamma(l+1)} w(Q_i), \qquad i = 1, \dots, 2^{nl}, \ \forall\, l \in \mathbb{N}. \end{aligned}$$

Choose i such that $w(Q_i)$ is minimal, then it holds

$$2^{nl} \leq 2^{n\gamma(l+1)} \quad \Rightarrow \quad \frac{l}{l+1} \leq \gamma \quad \forall l \in \mathbb{N},$$

i.e. $\gamma \geq 1$.

The proof for β is similar. Let w be now a doubling weight concerning balls. Let B a ball with radius 1, then B contains a cube Q with sidelength $\sqrt{n}$ and this cube Q contains as mentioned above 2^{nl} disjoint cubes Q_i with sidelength $2^{-l}\sqrt{n}$, $\bigcup_{i=1}^{2^{nl}} Q_i \subset Q$, $|Q_i| = 2^{-nl}$, $Q_i \cap Q_j = \emptyset$. Moreover every of the small cubes Q_i contains a ball B_i with radius $r_i = 2^{-l}\frac{\sqrt{n}}{2}$. Consequently the ball B contains 2^{nl} disjoint balls B_i with radius $2^{-l}\frac{\sqrt{n}}{2}$ and $\bigcup_{i=1}^{2^{nl}} B_i \subset B$. We take an arbitrary small ball B_i, blow it $x-$times up to cover the big ball B, whereby $x \cdot r_i = x \cdot 2^{-l}\frac{\sqrt{n}}{2} > 2$, i.e. $x > 2^{l+2} > 2^{l+1}\frac{\sqrt{n}}{2}$. Then holds for a fixed $l \in \mathbb{N}$ and all $i \in \{1, \dots, 2^{nl}\}$ that

$$\begin{aligned} 2^{nl} \min_{j=1,\dots,2^{nl}} w(B_j) &\leq \sum_{j=1}^{2^{nl}} w(B_j) \leq w(\bigcup_{j=1}^{2^{nl}} B_j) \leq w(B) \\ &\leq w(2^{l+1} B_i) \leq 2^{n\beta(l+2)} w(B_i), \qquad i = 1, \dots, 2^{nl}, \ \forall\, l \in \mathbb{N}. \end{aligned}$$

Choose i again such that $w(B_i)$ is minimal, then

$$2^{nl} \leq 2^{n\beta(l+2)} \quad \Rightarrow \quad \frac{l}{l+2} \leq \beta \quad \forall l \in \mathbb{N},$$

i.e. $\beta \geq 1$. ■

Remark 1.24. In the following we will not distinguish between doubling weights concerning balls or concerning cubes as long as their doubling constants do not play any role. Otherwise we stick to our convention to label the doubling constant concerning balls with β, and the one concerning cubes with γ.

We prove another feature of doubling weights which will be used below.

Proposition 1.25. *Let w be a doubling weight. Then*

(i) $\forall\, E \subset \mathbb{R}^n$ *with* $0 < |E| < \infty:$ $\qquad 0 < w(E) < \infty,$

(ii) $\displaystyle\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y = \infty.$

Proof. **(i)** This is a immediate consequence of the doubling property.
(ii) Let w be a doubling weight with $w(B(x,2r)) \le c\, w(B(x,r))$ for arbitrary $x \in \mathbb{R}^n$, $r > 0$, here we denote the doubling constant $2^{n\beta}$ by c. Let $R_0 > 0$ be an arbitrary positive number and $x_0 = \left(\frac{R_0}{2}, 0, \cdots, 0\right)^T$ and $x_1 = (2\,R_0, 0, \cdots, 0)^T$. Then

$$B(x_0, \frac{R_0}{2}) \subset \{y \in \mathbb{R}^n : R_0 \le |y - x_1| \le 2\,R_0\}\,. \tag{1.42}$$

Since w is doubling, remember that $0 < w(B) < \infty$ for all balls B; see **(i)**. So in particular $w(B(x_0, \frac{R_0}{2})) \ge a > 0$. Now it holds

$$\begin{aligned} w(B(x_1, 2\,R_0)) &= \int_{B(x_1, 2\,R_0)} w(y)\,\mathrm{d}y \\ &= \int_{B(x_1, R_0)} w(y)\,\mathrm{d}y + \int_{R_0 \le |y-x_1| \le 2\,R_0} w(y)\,\mathrm{d}y \\ &\overset{(1.42)}{\ge} \frac{1}{c} w(B(x_1, 2\,R_0)) + \underbrace{\int_{B(x_0, \frac{R_0}{2})} w(y)\,\mathrm{d}y}_{\ge a > 0} \\ &\ge \frac{1}{c} w(B(x_1, 2\,R_0)) + a. \end{aligned}$$

We bring the first summand on the other side and obtain

$$w(B(x_1, 2\,R_0)) \ge a\,\frac{c}{c-1}.$$

Next we set $R_1 := 4R_0$, then $x_1 = \left(\frac{R_1}{2}, 0, \cdots, 0\right)^T$. In general we set $R_{k+1} = 4R_k$, $x_k = \left(\frac{R_k}{2}, 0, \cdots, 0\right)^T$ for $k \in \mathbb{N}_0$. Repeat the upper calculation for x_1, R_1 instead of x_0, R_0 and receive

$$w(B(x_2, \frac{R_2}{2})) \ge a\,\left(\frac{c}{c-1}\right)^2.$$

Iteratively we obtain in the k-th step

$$w(B(x_k, \frac{R_k}{2})) \ge a\,\left(\frac{c}{c-1}\right)^k. \tag{1.43}$$

Finally we get

$$\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y \geq \lim_{k\to\infty} \int_{B(x_k,\frac{R_k}{2})} w(y)\,\mathrm{d}y \overset{(1.43)}{\geq} a\left(\frac{c}{c-1}\right)^k = \infty.$$

■

1.2.4 Further weight classes

In the history of Sobolev, Besov and Triebel-Lizorkin type spaces several classes of weights play an important role. In this section we have a look at some of these weight classes and briefly discuss their relationship to Muckenhoupt weights and doubling weights, respectively, to get a better overview.

Admissible weights

We start with the so-called admissible weights, which have a long history in the theory of function spaces.

We use the abbreviation $\langle x\rangle = (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^n$.

Definition 1.26. The class of admissible weight functions is the collection of all positive C^∞ functions w on $\mathbb{R}^n$ with the following properties:

(i) for all $\eta \in \mathbb{N}_0^n$ there exists a positive constant c_η with

$$|\mathrm{D}^\eta w(x)| \leq c_\eta\, w(x) \quad \text{for all} \quad x \in \mathbb{R}^n;$$

(ii) there exist two constants $c > 0$ and $\alpha \geq 0$ such that

$$0 < w(x) \leq c\, w(y)\, \langle x-y\rangle^\alpha \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

Remark 1.27. Note that for admissible weights w and v, also $1/w$ and vw are admissible weights. For further details about admissible weights we refer for example to [HT94, HT05] or also to [ET96, KLSS06a, KLSS06b, KLSS07].

Example 1.28. Obviously, $v_\alpha(x) = \langle x\rangle^\alpha$, $\alpha \in \mathbb{R}$, is an admissible weight. Note that $v_\alpha \in \mathcal{A}_\infty$ for $\alpha > -n$ unlike in case of $\alpha \leq -n$. Conversely, $w_{a,b}$ given by (1.15) with $-n < a < 0$, $b > -n$, is not admissible in the above sense, but belongs to $\mathcal{A}_\infty$ or is doubling, respectively.

There exists a generalization of definition 1.26.

Definition 1.29. The class of general locally regular weight functions is the collection of all positive C^∞ functions w on $\mathbb{R}^n$ with the following properties:

(i) for all $\eta \in \mathbb{N}_0^n$ there exists a positive constant c_η with

$$|\mathrm{D}^\eta w(x)| \le c_\eta \, w(x) \quad \text{for all} \quad x \in \mathbb{R}^n;$$

(ii) there exist two constants $C > 0$ and $0 < \beta \le 1$ such that

$$0 < w(x) \le C \, w(y) \, \exp\left(C \, |x-y|^\beta\right) \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

Remark 1.30. Of course any admissible weight is locally regular. For further details we refer to [Sch98].

Example 1.31. For example the weight

$$w(x) = \exp(|x|^\beta), \quad 0 < \beta \le 1, \tag{1.44}$$

is locally regular but not admissible, see [Sch98].

Local Muckenhoupt weights

We extend these weight classes by the so-called local Muckenhoupt weights, which also contain the already mentioned Muckenhoupt weights.

Definition 1.32. Let w be a weight on $\mathbb{R}^n$.

(i) Then w belongs to the local Muckenhoupt class $\mathcal{A}_p^{\mathrm{loc}}$, $1 < p < \infty$, if there exists a constant $0 < A < \infty$ such that for all balls B with $|B| \le 1$ the following inequality holds

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right) \cdot \left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{p/p'} \le A. \tag{1.45}$$

The smallest such A is called the Muckenhoupt constant $A_p^{\mathrm{loc}} = A_p^{\mathrm{loc}}(w)$.

(ii) Then w belongs to the Muckenhoupt class $\mathcal{A}_1^{\mathrm{loc}}$ if there exists a constant $0 < A < \infty$ such that the inequality

$$\mathcal{M}^{\mathrm{loc}} w(x) \le A w(x) \tag{1.46}$$

holds for almost all $x \in \mathbb{R}^n$, where here $\mathcal{M}^{\mathrm{loc}}$ stands for the local Hardy-Littlewood maximal operator given by

$$\mathcal{M}^{\mathrm{loc}} f(x) = \sup_{B(x,r)\in\mathcal{B}, 0<r\le 1} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

The smallest such A is called the Muckenhoupt constant $A_1^{\mathrm{loc}} = A_1^{\mathrm{loc}}(w)$.

(iii) The Muckenhoupt class $\mathcal{A}_\infty^{\mathrm{loc}}$ is given by

$$\mathcal{A}_\infty^{\mathrm{loc}} = \bigcup_{p>1} \mathcal{A}_p^{\mathrm{loc}}. \tag{1.47}$$

Remark 1.33. These weights were introduced 2001 by Rychkov in [Ryc01]. Obviously it holds that $\mathcal{A}_p \subset \mathcal{A}_p^{\mathrm{loc}}$ and $A_p^{\mathrm{loc}}(w) \leq A_p(w)$ for any $w \in \mathcal{A}_p$, $1 \leq p < \infty$. They do not only extend the Muckenhoupt weights, but also contain the above introduced admissible and locally regular weights, cf. [Sch98, Ryc01, HT05]. Moreover $\mathcal{A}_1^{\mathrm{loc}}$ contains the regular weights, see [Woj12a, Prop. 2.4].

Example 1.34. A typical example which is contained in $\mathcal{A}_\infty^{\mathrm{loc}}$, but not in $\mathcal{A}_\infty$ and is also not locally regular, is given by

$$w_{a,\exp}(x) = \begin{cases} |x|^a, & \text{if} \quad |x| \leq 1, \\ \exp(|x|-1), & \text{if} \quad |x| > 1, \end{cases}$$

where $a > -n$, see [Woj12a]. If $-n < a < n(p-1)$ and $1 < p < \infty$ then $w \in \mathcal{A}_p^{\mathrm{loc}}$. If $-n < a \leq 0$ then $w \in \mathcal{A}_1^{\mathrm{loc}}$.

1.3 Function spaces

1.3.1 Spaces of Besov and Triebel-Lizorkin type

This section gives an introduction to the classical function spaces of Besov and Triebel-Lizorkin type. There are various ways to define these spaces, e.g. by derivatives, differences of functions, the Fourier analytical representation, local means, atomic decomposition, etc. We present the most common Fourier analytical approach. For this we need the concept of a smooth dyadic resolution of unity in $\mathbb{R}^n$. This is a system of functions $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset C^\infty(\mathbb{R}^n)$ with the following properties.

$$\operatorname{supp}\varphi_0 \subset \{x\in\mathbb{R}^n : |x|\le 2\}\,, \tag{1.48}$$

$$\operatorname{supp}\varphi_j \subset \left\{x\in\mathbb{R}^n : 2^{j-1}\le |x|\le 2^{j+1}\right\}, \qquad j\in\mathbb{N}, \tag{1.49}$$

$$|(\mathrm{D}^\alpha\varphi_j)(x)| \le c_\alpha\, 2^{-j|\alpha|}, \qquad \forall\, x\in\mathbb{R}^n,\ \forall\,\alpha\in\mathbb{N}_0^n,\ j\in\mathbb{N}_0, \tag{1.50}$$

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \qquad \forall\, x\in\mathbb{R}^n. \tag{1.51}$$

It is rather easy to construct such a resolution of unity: Let $\varphi_0=\varphi\in\mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\varphi\subset\{y\in\mathbb{R}^n : |y|<2\} \quad \text{and} \quad \varphi(x)=1,\ \text{ if }\ |x|\le 1,$$

and for each $j\in\mathbb{N}$ let $\varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a system with the required properties. Let $f_j := \mathcal{F}^{-1}(\varphi_j\mathcal{F}f)$, $f\in\mathcal{S}'(\mathbb{R}^n)$. Then $\operatorname{supp}\mathcal{F}f_j\subset\operatorname{supp}\varphi_j$. Since φ_j has a compact support f_j is well-defined for any $f\in\mathcal{S}'(\mathbb{R}^n)$ and f_j is by the Paley-Wiener-Schwartz theorem an entire analytic function with respect to $x\in\mathbb{R}^n$. Furthermore it holds for all $f\in\mathcal{S}'(\mathbb{R}^n)$

$$f(x)=\sum_{j=0}^\infty \mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(x)=\sum_{j=0}^\infty f_j(x) \qquad \text{with convergence in } \mathcal{S}'(\mathbb{R}^n).$$

The (unweighted) Besov and Triebel-Lizorkin spaces are defined in the following way.

Definition 1.35. Let $\{\varphi_j\}_{j=0}^\infty$ be a smooth dyadic resolution of unity.

(i) Let $0<p\le\infty$, $0<q\le\infty$, $s\in\mathbb{R}$. The Besov space $B^s_{p,q}=B^s_{p,q}(\mathbb{R}^n)$ is the set of all distributions $f\in\mathcal{S}'$ such that

$$\left\|f|B^s_{p,q}\right\| = \left(\sum_{j=0}^\infty 2^{jsq}\left\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)|L_p\right\|^q\right)^{1/q}$$

is finite (with the usual modification in the limiting case $q=\infty$).

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. The Triebel-Lizorkin space $F^s_{p,q} = F^s_{p,q}(\mathbb{R}^n)$ is the set of all distributions $f \in \mathcal{S}'$ such that

$$\left\| f | F^s_{p,q} \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot) \right|^q \right)^{1/q} |L_p \right\|$$

is finite (with the usual modification in the limiting case $q = \infty$).

Remark 1.36. The spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ appearing in their definitions (in the sense of equivalent norms). A proof may be found in [Tri92, Section 2.3.2, pp. 93-96]. They are quasi-Banach spaces (Banach spaces for $p, q \ge 1$) and it holds $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n), F^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, where the first embedding is dense if $p < \infty$ and $q < \infty$, cf. [Tri83, Section 2.3.3].
Moreover we have some elementary embeddings for these spaces. For this purpose we adopt the usual convention to write $A^s_{p,q}(\mathbb{R}^n)$ instead of $B^s_{p,q}(\mathbb{R}^n)$ or $F^s_{p,q}(\mathbb{R}^n)$, respectively, when both scales of spaces are meant simultaneously in some context. Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, then

$$\begin{aligned} A^{s_0}_{p,q_0}(\mathbb{R}^n) &\hookrightarrow A^{s_1}_{p,q_1}(\mathbb{R}^n), \quad && \text{if} \quad -\infty < s_1 < s_0 < \infty, \ \ 0 < q_0, q_1 \le \infty, \\ A^{s}_{p,q_0}(\mathbb{R}^n) &\hookrightarrow A^{s}_{p,q_1}(\mathbb{R}^n), \quad && \text{if} \quad 0 < q_0 \le q_1 \le \infty, \end{aligned}$$

and

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n),$$

cf. [Tri83, Section 2.3.2, Prop. 2].
If one compares the two parts of the above definitions then $p = \infty$ is missing in connection with the space $F^s_{p,q}$. It comes out that a direct extension of the above definition of $F^s_{p,q}$ to $p = \infty$ does not make sense if $0 < q < \infty$ (in particular, a corresponding space is not independent of the choice of $\{\varphi_j\}_j$). However, using a modification it is possible to define spaces $F^s_{\infty,q}(\mathbb{R}^n)$, cf. [Tri92, 1.5.2]. Note that the spaces $A^s_{p,q}(\mathbb{R}^n)$ contain tempered distributions which can only be interpreted as regular distributions (functions) for sufficiently high smoothness. More precisely, for B-spaces we have

$$B^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} s > \sigma_p, & \text{for } 0 < p \le \infty, \ 0 < q \le \infty, \\ s = \sigma_p, & \text{for } 0 < p \le 1, \ 0 < q \le 1, \\ s = \sigma_p, & \text{for } 1 < p \le \infty, \ 0 < q \le \min(p, 2), \end{cases} \tag{1.52}$$

and in case of F-spaces

$$F^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} s \ge \sigma_p, & \text{for } 0 < p < 1, \ 0 < q \le \infty, \\ s > \sigma_p, & \text{for } 1 \le p < \infty, \ 0 < q \le \infty, \\ s = \sigma_p, & \text{for } 1 \le p < \infty, \ 0 < q \le 2, \end{cases} \tag{1.53}$$

cf. [ST95, Thm. 3.3.2], where σ_p is given as usual by

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ .$$

The scale $F^s_{p,q}(\mathbb{R}^n)$ contains many well-known function spaces. We list a few special cases. Let $1 < p < \infty$, then

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \qquad s \in \mathbb{R},$$

where the latter are the (*fractional*) *Sobolev spaces* containing all $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\mathcal{F}^{-1}\big((1+|\xi|^2)^{s/2}\mathcal{F}f\big) \in L_p(\mathbb{R}^n).$$

In particular, for $k \in \mathbb{N}_0$, we obtain the *classical Sobolev spaces*

$$F^k_{p,2}(\mathbb{R}^n) = W^k_p(\mathbb{R}^n) \qquad \text{and} \qquad F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

usually normed by

$$\|f|W^k_p(\mathbb{R}^n)\| = \left(\sum_{|\alpha|\le k} \|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\|^p\right)^{1/p},$$

where here $\mathrm{D}^\alpha f$ are generalized derivatives in the sense of distributions. Otherwise for $0 < p < \infty$ we obtain the *Hardy spaces*

$$F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n).$$

For comprehensive treatment of the Besov-Triebel-Lizorkin spaces we refer, in particular, to the series of monographs by Triebel, [Tri78, Tri83, Tri92, Tri97, Tri01, Tri06, Tri08].

Remark 1.37. As already mentioned there are different ways to define Besov and Triebel-Lizorkin spaces, respectively. The *classical Besov* spaces, in particular, when $1 \le p,q \le \infty$ and $s > 0$, are characterized by iterated differences and derivatives.
For an arbitrary function f on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $r \in \mathbb{N}$ let

$$(\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta^{r+1}_h f)(x) = \Delta^1_h(\Delta^r_h f)(x)$$

be the iterated differences.
Let $1 \le p,q \le \infty$ and $s > 0$. We put

$$s = [s]^- + \{s\}^+, \tag{1.54}$$

where $[s]^-$ is an integer and $0 < \{s\}^+ \le 1$. Then the *classical Besov* space $B^s_{p,q}(\mathbb{R}^n)$ contains all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| = \|f|W^{[s]^-}_p(\mathbb{R}^n)\| + \sum_{|\alpha|=[s]^-}\left(\int_{\mathbb{R}^n} |h|^{-\{s\}^+q}\|\Delta^2_h \mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}h}{|h|^n}\right)^{1/q} \tag{1.55}$$

is finite, where we use for $q=\infty$ the following norm modification

$$\|f|B^s_{p,\infty}(\mathbb{R}^n)\| = \|f|W_p^{[s]^-}(\mathbb{R}^n)\| + \sum_{|\alpha|=[s]^-} \sup_{0\neq h\in\mathbb{R}^n} |h|^{-\{s\}^+}\|\Delta_h^2 \mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\|. \tag{1.56}$$

These Banach spaces have been introduced by Besov in 1959/60, see [Bes59, Bes61], and have a comprehensive history, see for example in [Tri92, Section 1.2.5].
However (1.55) can be modified in the following way. Given a function $f\in L_p(\mathbb{R}^n)$ the *r-th order of modulus of smoothness* is defined by

$$\omega_r(f,t)_p = \sup_{|h|\le t}\|\Delta_h^r f|L_p(\mathbb{R}^n)\|, \quad t>0, \quad 0<p\le\infty. \tag{1.57}$$

Then (1.55) and (1.56) can be replaced by

$$\|f|B^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq}\omega_r(f,t)_p^q\,\frac{\mathrm{d}t}{t}\right)^{1/q} \tag{1.58}$$

(with the usual modification if $q=\infty$). The study for all admitted s, p and q goes back to [SO78], we also refer to [BS88, Ch. 5, Def. 4.3] and [DL93, Ch. 2, §10]. In view of the characterization (1.56) there is obviously an analogy from $B^s_{\infty,\infty}(\mathbb{R}^n)$ with the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$, which are given by all $f\in C(\mathbb{R}^n)$ such that

$$\|f|\mathcal{C}^s(\mathbb{R}^n)\| = \|f|C^{[s]^-}(\mathbb{R}^n)\| + \sum_{|\alpha|=[s]^-} \sup_{0\neq h\in\mathbb{R}^n} |h|^{-\{s\}^+}\|\Delta_h^2 \mathrm{D}^\alpha f|C(\mathbb{R}^n)\|$$

is finite. This means, that

$$B^s_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), \quad s>0. \tag{1.59}$$

This can be extended to all $s\in\mathbb{R}$. For more details about these *classical Besov* spaces we refer to [Tri83, Section 2.2.2, 2.5.12].
The approach by differences for the spaces $F^s_{p,q}(\mathbb{R}^n)$ has been described in detail in [Tri83, Section 2.5.10]. Otherwise one finds in [Tri06], Section 9.2.2, pp. 386-390, the necessary explanations and references to the relevant literature.

1.3.2 Weighted function spaces

In this section we define doubling weighted Besov and Triebel-Lizorkin spaces and collect some basic properties, which have a later use. As already introduced in the beginning of Section 1.2 the weighted Lebesgue space $L_p(w)=L_p(\mathbb{R}^n,w)$ is defined with the weighted $L_p(w)$-norm, $0<p\le\infty$,

$$\|f\ |L_p(w)\| = \|f\ |L_p(\mathbb{R}^n,w)\| = \left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p},$$

with the usual modification for $p = \infty$, whereas w is now a doubling weight. Unless otherwise mentioned, w is usually a doubling weight in the following.

Then $L_p(w)$ equipped with this norm are quasi-Banach spaces (Banach spaces for $p, q \geq 1$). It is clear that $L_p(\mathbb{R}^n, w) = L_p(\mathbb{R}^n)$ for $w = 1$. Moreover for $p = \infty$ one obtains the classical (unweighted) Lebesgue space, $L_\infty(\mathbb{R}^n, w) = L_\infty(\mathbb{R}^n)$ with equality of norms, i.e. $||f|L_\infty(\mathbb{R}^n, w)|| \sim ||f|L_\infty(\mathbb{R}^n)||$, more precisely,

$$\inf_{\substack{N\subset\mathbb{R}^n,\\ w(N)=0}} \sup_{x\in\mathbb{R}^n\setminus N} |f(x)| \sim \inf_{\substack{N\subset\mathbb{R}^n,\\ |N|=0}} \sup_{x\in\mathbb{R}^n\setminus N} |f(x)|.$$

Thus it is sufficient to show $w(N) = 0 \quad \Leftrightarrow \quad |N| = 0$. This is an immediate consequence of Proposition 1.25 **(i)** and the fact, that we here only consider measures, which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.
We thus mainly restrict ourselves to $p < \infty$ in what follows.

Example 1.38. For $0 < p < \infty$ and w doubling the function $(1+|x|)^{-L}$ belongs to $L_p(w)$ for sufficiently large $L \in \mathbb{N}$.

Proof. We use the notation $B_j := \{x \in \mathbb{R}^n : |x| \leq 2^j\}$ for $j \in \mathbb{N}_0$. Then $B_j \setminus B_{j-1} = \{x \in \mathbb{R}^n : 2^{j-1} < |x| \leq 2^j\}$, $j \in \mathbb{N}$, denotes the annuli. For $x \in B_j \setminus B_{j-1}$, we have $(1 + 2^j)^{-Lp} \leq (1 + |x|)^{-Lp} < (1 + 2^{j-1})^{-Lp}$, i.e. $(1 + |x|)^{-Lp} \sim 2^{-Lpj}$. Furthermore we can use the doubling property $w(B_j \setminus B_{j-1}) \leq w(B_j) = w(2^j B_0) \leq 2^{jn\beta} w(B_0)$.
Both together lead us to

$$\begin{aligned} ||(1+|x|)^{-L}|L_p(w)||^p &= \int_{\mathbb{R}^n} (1+|x|)^{-Lp} w(x)\,\mathrm{d}x \\ &= \sum_{j=1}^{\infty} \int_{B_j\setminus B_{j-1}} \underbrace{(1+|x|)^{-Lp}}_{\leq c2^{-Lpj}} w(x)\,\mathrm{d}x + \int_{B_0} \underbrace{(1+|x|)^{-Lp}}_{\leq 1} w(x)\,\mathrm{d}x \\ &\leq c_1 \sum_{j=1}^{\infty} 2^{-jLp} \underbrace{w(B_j\setminus B_{j-1})}_{\leq 2^{jn\beta}w(B_0)} + w(B_0) \\ &\leq c_2 \sum_{j=0}^{\infty} 2^{-jLp} 2^{jn\beta} \underbrace{w(B_0)}_{\sim 1} < \infty, \end{aligned}$$

if $L > \frac{n\beta}{p}$. ■

We use the Fourier analytical approach for the definition of the doubling weighted Besov and Triebel-Lizorkin spaces. We refer to the beginning of Section 1.3.1 for explaining the concept of a smooth dyadic resolution of unity and their properties.

Definition 1.39. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $\{\varphi_j\}_{j=0}^{\infty}$ a smooth dyadic resolution of unity and let w be a doubling weight.

(i) The weighted Besov space $B^s_{p,q}(w) = B^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'$ such that

$$\left\| f|B^s_{p,q}(w)\right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)|L_p(w)\right\|^q\right)^{1/q}$$

is finite (with the usual modification in the limiting case $q = \infty$).

(ii) The weighted Triebel-Lizorkin space $F^s_{p,q}(w) = F^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'$ such that

$$\left\| f|F^s_{p,q}(w)\right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)\right|^q\right)^{1/q} |L_p(w)\right\|$$

is finite (with the usual modification in the limiting case $q = \infty$).

Remark 1.40. The spaces $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$ are independent of the choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ appearing in their definitions, cf. [Bow05] and [BH06], respectively. They are quasi-Banach spaces (Banach spaces for $p, q \geq 1$). Moreover, for $w \equiv 1$ we re-obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces, for this we refer to Section 1.3.1.

Weighted function spaces have also a preceding history. A general approach for Besov-Triebel-Lizorkin spaces with weights is given in [ST87, Chapter 5]. In [ET96, Chapter 4] one finds function spaces especially with admissible weights, which we introduced in Section 1.2.4.

Doubling weighted Besov-Triebel-Lizorkin spaces were first introduced by Bownik in the papers [Bow05, BH06, Bow07, Bow08]. There he mainly dealt with homogeneous, anisotropic Besov spaces with expansive dilation matrices and more general doubling measures, but he showed that some of these result also hold for inhomogeneous spaces. For more details about the differences between his and our approach we refer to Remark 2.10 in Chapter 2 below.

Remark 1.41. As already mentioned there exist further types of Besov-Triebel-Lizorkin spaces with other weight classes, for example Muckenhoupt weights or admissible weights. They can be introduced in the same way like our doubling weighted spaces. One considers the weighted Lebesgue spaces $L_p(w)$, where the Lebesgue measure is replaced by the measure $w(x)\,\mathrm{d}x$, as we introduced it in Section 1.2. For Muckenhoupt weights and admissible weights, respectively, $L_p(w)$ are again (quasi-)Banach spaces and it also holds $w(N) = 0 \quad\Leftrightarrow\quad |N| = 0$, $N \subset \mathbb{R}^n$, thus $L_\infty(w) = L_\infty$. Then for these weights the Besov and Triebel-Lizorkin spaces can be defined as in Definition 1.39 by replacing the doubling weighted $L_p(w)$-norm by the respective one.

The spaces with weights of Muckenhoupt type have been studied systematically by Bui in [Bui81, Bui82, Bui83, Bui84, Bui94]. There exist many counterparts of the results from the unweighted situation, compare with Remark 1.36 in Section 1.3.1. For example we have

$$F^0_{p,2}(w) = h_p(w), \qquad \text{for } \ 0 < p < \infty, \ w \in \mathcal{A}_p,$$

see [Bui82, Thm. 1.4], in particular,

$$F^0_{p,2}(w) = L_p(w) = h_p(w), \qquad \text{for } \ 1 < p < \infty, \ w \in \mathcal{A}_p,$$

see [ST89, Chapter 6, Thm. 1].
Concerning Sobolev spaces $W^k_p(w)$ it holds

$$F^k_{p,2}(w) = W^k_p(w), \qquad \text{for } \ k \in \mathbb{N}_0, \ 1 < p < \infty, \ w \in \mathcal{A}_p,$$

see [Bui82, Thm. 2.8]. Further results, concerning, for instance, embeddings, real interpolation, extrapolation, lift operators and duality assertions may be found in [Bui82, Bui84, GCRdF85, Rou04].
Concerning admissible weights exist also many respective counterparts of the results from the unweighted Besov-Triebel-Lizorkin spaces, which we mentioned in Section 1.3.1. We refer to [ET96, Chapter 4] and [Tri06, Chapter 6] for the necessary explanations. Because of a later use, we have in particular,

$$B^s_{\infty,\infty}(\mathbb{R}^n, w_\alpha) = \mathcal{C}^s(\mathbb{R}^n, w_\alpha), \tag{1.60}$$

cf. [Tri06, Remark 6.14]. Whereas $w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$, is the admissible weight from Example 1.28.

Now we consider again function spaces with doubling weights. We have the usual elementary embeddings for these weighted spaces. For this purpose we adopt the usual convention to write $A^s_{p,q}(w)$ instead of $B^s_{p,q}(w)$ or $F^s_{p,q}(w)$, respectively, when both scales of spaces are meant simultaneously in some context.

Proposition 1.42. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and w be a doubling weight.*

(i) *Let $0 < q_0 \le q_1 \le \infty$. Then*

$$A^s_{p,q_0}(w) \hookrightarrow A^s_{p,q_1}(w). \tag{1.61}$$

(ii) *Let $0 < q_0 \le \infty$, $0 < q_1 \le \infty$ and $\varepsilon > 0$. Then*

$$A^{s+\varepsilon}_{p,q_0}(w) \hookrightarrow A^s_{p,q_1}(w). \tag{1.62}$$

(iii) *We have*

$$B^s_{p,\min(p,q)}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^s_{p,\max(p,q)}(w). \tag{1.63}$$

Proof. **(i)** is a simple consequence of the monotonicity of the ℓ_q–spaces, since $\ell_{q_0} \hookrightarrow \ell_{q_1}$, for $0 < q_0 \leq q_1 \leq \infty$. The **(ii)** assertion works in a similar way and follows from

$$\begin{aligned}\left(\sum_{j=0}^{\infty} 2^{sjq_1}|b_j|^{q_1}\right)^{1/q_1} &\leq \sup_{j\in\mathbb{N}_0} 2^{(s+\varepsilon)j}|b_j| \left(\sum_{j=0}^{\infty} 2^{-\varepsilon jq_1}\right)^{1/q_1} \\ &\leq c \sup_{j\in\mathbb{N}_0} 2^{(s+\varepsilon)j}|b_j| = c\,\|2^{(s+\varepsilon)j}|b_j| \mid \ell_\infty\| \leq c\,\|2^{(s+\varepsilon)j}|b_j| \mid \ell_{q_0}\|\end{aligned}$$

(modification if $q_1 = \infty$).
We prove **(iii)**. Let $a_j(x) := 2^{sj}\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(x)$ with $j \in \mathbb{N}_0$, $x \in \mathbb{R}^n$. We shall use the generalized triangle inequality for Banach spaces.
Thus we assume first $0 < q \leq p < \infty$, i.e. $\frac{p}{q} \geq 1$. Then we have

$$\begin{aligned}||f|B^s_{p,\max(p,q)}(w)|| = ||f|B^s_{p,p}(w)|| &= ||a_j|\ell_p(L_p(w))|| = ||a_j|L_p(w,\ell_p)|| \\ &\leq ||a_j|L_p(w,\ell_q)|| = ||f|F^s_{p,q}(w)||.\end{aligned}$$

Furthermore

$$\begin{aligned}||f|F^s_{p,q}(w)|| &= \left\|\left(\sum_{j=0}^{\infty}|a_j(\cdot)|^q\right)^{1/q}|L_p(w)\right\| = \left\|\sum_{j=0}^{\infty}|a_j(\cdot)|^q|L_{p/q}(w)\right\|^{1/q} \\ &\leq \left(\sum_{j=0}^{\infty}\left\||a_j(\cdot)|^q|L_{p/q}(w)\right\|\right)^{1/q} = \left(\sum_{j=0}^{\infty}\|a_j(\cdot)|L_p(w)\|^q\right)^{1/q} = ||f|B^s_{p,q}(w)||.\end{aligned}$$

For $0 < p < q \leq \infty$ it works analogously with the Minkowski's inequality. ■

Lemma 1.43. *Let $\mathcal{K} \subset \mathbb{R}^n$ compact, $0 < p < \infty$ and w be a doubling weight. Then there exist $c, N > 0$ such that for all $j \in \mathbb{N}_0$,*

$$\sup_{x\in\mathbb{R}^n}\frac{|f(x)|}{(1+|x|)^N} \leq c^{j+1}\,||f|L_p(w)|| \qquad \textit{for all } f \in \mathcal{S}'(\mathbb{R}^n) \textit{ with } \operatorname{supp}\mathcal{F}f \subset 2^j\mathcal{K}. \tag{1.64}$$

Proof. The proof works similarly to the proof of Corollary 3.1 in [Bow05]. ■

Proposition 1.44. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and w be a doubling weight.*

(i)

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L_p(w) \tag{1.65}$$

(ii)

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow A^s_{p,q}(w) \tag{1.66}$$

(iii)

$$A^s_{p,q}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{1.67}$$

Proof. **(i)** Let $f \in \mathcal{S}(\mathbb{R}^n)$. The semi-norms in $\mathcal{S}(\mathbb{R}^n)$ are given by

$$||f||_{k,l} := \sup_{x\in\mathbb{R}^n} (1+|x|)^k \sum_{|\alpha|\le l} |\mathrm{D}^\alpha f(x)|, \quad \forall\, k,l \in \mathbb{N}_0.$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} ||f|L_p(w)||^p &= \int_{\mathbb{R}^n} \left[(1+|x|)^L |f(x)|\right]^p (1+|x|)^{-Lp} w(x)\,\mathrm{d}x \\ &\le ||f||_{L,0}^p \, ||(1+|x|)^{-L}|L_p(w)||^p < \infty, \end{aligned}$$

for $L > \frac{n\beta}{p}$, see Example 1.38. This proves **(i)**.

To show **(ii)** and **(iii)** we remind a useful trick. As a consequence of Proposition 1.42 it holds for all $\varepsilon > 0$

$$B^{s+\varepsilon}_{p,\infty}(w) \hookrightarrow B^s_{p,q}(w) \hookrightarrow B^s_{p,\infty}(w)$$

and

$$B^{s+\varepsilon}_{p,\infty}(w) \hookrightarrow B^s_{p,\min(p,q)}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^s_{p,\max(p,q)}(w) \hookrightarrow B^s_{p,\infty}(w).$$

So it is sufficient to show, that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

(ii) Let $f \in \mathcal{S}(\mathbb{R}^n)$.

$$\begin{aligned} ||f\,|B^s_{p,\infty}(w)|| &= \sup_{j\in\mathbb{N}_0} 2^{js} ||\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\,|L_p(w)|| \\ &= \sup_{j\in\mathbb{N}_0} 2^{js} ||(1+|x|)^L \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(1+|x|)^{-L}\,|L_p(w)|| \\ &\le \sup_{j\in\mathbb{N}_0} 2^{js} ||\underbrace{(1+|x|)^L}_{=\sum_\alpha c_\alpha x^\alpha} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\,|L_\infty||\; ||(1+|x|)^{-L}\,|L_p(w)||, \end{aligned}$$

where the latter part is bounded, see Example 1.38. Then we take the polynomial inside the Fourier transform and get derivatives

$$\le c_1 \sup_{j\in\mathbb{N}_0} 2^{js} ||\mathcal{F}^{-1}\Big[\sum_{|\alpha|\le L} \underbrace{c_\alpha}_{\le \max c_\alpha = c} |\mathrm{D}^\alpha(\varphi_j \mathcal{F} f)|\Big]\,|L_\infty||,$$

use Riemann-Lebesgue and Leibniz formula for derivatives and obtain

$$\begin{aligned} &\le c_2 \sup_{j\in\mathbb{N}_0} 2^{js} ||\sum_{|\alpha|\le L} |\mathrm{D}^\alpha(\varphi_j \mathcal{F} f)|\,|L_1|| \\ &\le c_3 \sup_{j\in\mathbb{N}_0} 2^{js} ||\sum_{|\alpha|\le L} |\mathrm{D}^\alpha \varphi_j| \sum_{|\eta|\le L} |\mathrm{D}^\eta(\mathcal{F} f)|\,|L_1||. \end{aligned}$$

Since $\operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\}$, it holds $2^{js} \sim (1+|x|)^s$, thus

$$||f\,|B^s_{p,\infty}(w)|| \le c_4 \sup_{j\in\mathbb{N}_0} ||\underbrace{(1+|x|)^s \sum_{|\alpha|\le L} |\mathrm{D}^\alpha \varphi_j|}_{<\infty} \sum_{|\eta|\le L} |\mathrm{D}^\eta(\mathcal{F} f)|\,|L_1||.$$

Choose M large enough, such that $||(1+|x|)^{-M}|L_1||$ is bounded, then

$$
\begin{aligned}
&\leq c_5 \sup_{j\in\mathbb{N}_0} ||\underbrace{(1+|x|)^s(1+|x|)^M}_{\sim(1+|x|)^{s+M}} \sum_{|\eta|\leq L} |\mathrm{D}^\eta(\mathcal{F}f)|(1+|x|)^{-M}\,|L_1|| \\
&\leq c_6\, ||\mathcal{F}f||_{s+M,L}\, ||(1+|x|)^{-M}\,|L_1|| \leq c\, ||f||_{L+n+1,s+M},
\end{aligned}
$$

since $\mathcal{F}$ is bijective on $\mathcal{S}(\mathbb{R}^n)$.

(iii) Similar to step **(ii)** it is sufficient to show, that $B^s_{p,\infty}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, i.e. $\exists\ k,l \in \mathbb{N}_0\ \forall\ \Phi \in \mathcal{S}(\mathbb{R}^n)$:

$$|f(\Phi)| \leq c\, ||f|B^s_{p,\infty}(w)||\ ||\Phi||_{k,l}.$$

We mention that $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity. We define

$$
\begin{aligned}
\psi_0 &:= \varphi_0+\varphi_1, \\
\psi_j &:= \varphi_{j-1}+\varphi_j+\varphi_{j+1}, \quad j\in\mathbb{N}.
\end{aligned}
\tag{1.68}
$$

Thus

$$
\begin{aligned}
&\psi_0 \equiv 1 \quad \text{on}\ \ \operatorname{supp}\varphi_0 \quad \text{and} \quad \operatorname{supp}\psi_0 \subset \left\{x\in\mathbb{R}^n : |x|\leq 2^2\right\}, \\
&\psi_j \equiv 1 \quad \text{on}\ \ \operatorname{supp}\varphi_j \quad \text{and} \quad \operatorname{supp}\psi_j \subset \left\{x\in\mathbb{R}^n : 2^{j-2}\leq |x|\leq 2^{j+2}\right\}.
\end{aligned}
\tag{1.69}
$$

Let $f\in\mathcal{S}'(\mathbb{R}^n)$ and $\Phi\in\mathcal{S}(\mathbb{R}^n)$, then

$$
\begin{aligned}
|f(\Phi)| &= \Big|\sum_{j=0}^\infty \mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(\Phi)\Big| \\
&= \Big|\sum_{j=0}^\infty (\mathcal{F}f)(\varphi_j\mathcal{F}^{-1}\Phi)\Big| \\
&\overset{(1.69)}{=} \Big|\sum_{j=0}^\infty (\mathcal{F}f)(\varphi_j\psi_j\mathcal{F}^{-1}\Phi)\Big| \\
&= \Big|\sum_{j=0}^\infty (\mathcal{F}f)(\varphi_j\mathcal{F}^{-1}\mathcal{F}\psi_j\mathcal{F}^{-1}\Phi)\Big| \\
&= \Big|\sum_{j=0}^\infty \underbrace{(\mathcal{F}^{-1}(\varphi_j\mathcal{F}f))}_{=:f_j}\underbrace{(\mathcal{F}(\psi_j\mathcal{F}^{-1}\Phi))}_{=:\Phi_j}\Big|.
\end{aligned}
\tag{1.70}
$$

Now we use Lemma 1.43 with $\mathcal{K} = \operatorname{supp}\varphi_0$ and $f = f_j$, since $\operatorname{supp}\mathcal{F}f_j \subset \operatorname{supp}\varphi_j \subset 2^j\operatorname{supp}\varphi_0 = 2^j\mathcal{K}$. Then there exist $c, N > 0$ such that for all $j\in\mathbb{N}_0$

$$\sup_{x\in\mathbb{R}^n}\frac{|f_j(x)|}{(1+|x|)^N} \leq c^{j+1}\, ||f_j|L_p(w)||. \tag{1.71}$$

We obtain

$$\begin{aligned}
|f_j(\Phi_j)| &\le \int_{\mathbb{R}^n} |f_j(x)|(1+|x|)^{-N}|\Phi_j(x)|(1+|x|)^N \, \mathrm{d}x \\
&\le c^{j+1} \, ||f_j|L_p(w)|| \, ||(1+|x|)^N \Phi_j|L_1|| \\
&\le c^{j+1} \, 2^{-js} \underbrace{2^{js}||f_j|L_p(w)||}_{\le ||f|B^s_{p,\infty}(w)||} \, ||(1+|x|)^N \Phi_j|L_1|| \\
&\le c_1 2^{j(\max(\log_2(c),0)-s)} \, ||f|B^s_{p,\infty}(w)|| \, ||(1+|x|)^N \Phi_j|L_1||. \qquad (1.72)
\end{aligned}$$

Now we consider the last part

$$\begin{aligned}
||(1+|x|)^N \Phi_j|L_1|| &\le \int_{\mathbb{R}^n} |\mathcal{F}(\psi_j \mathcal{F}^{-1}\Phi)(x)|(1+|x|)^N (1+|x|)^{n+1}(1+|x|)^{-n-1} \, \mathrm{d}x \\
&\le ||\mathcal{F}(\psi_j \mathcal{F}^{-1}\Phi)||_{N+n+1,0} \underbrace{\int_{\mathbb{R}^n} (1+|x|)^{-n-1} \, \mathrm{d}x}_{<\infty} .
\end{aligned}$$

Since $\mathcal{F}$ is bijective on $\mathcal{S}(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\le c_2 \, ||\psi_j \mathcal{F}^{-1}\Phi||_{n+1,N+n+1} \\
&= c_2 \sup_{x \in \mathbb{R}^n} (1+|x|)^{n+1} \sum_{|\alpha| \le N+n+1} |\mathrm{D}^\alpha (\psi_j(\mathcal{F}^{-1}\Phi))(x)|.
\end{aligned}$$

Use first the support of the ψ_j, (1.69),

$$= c_2 \sup_{2^{j-2} \le |x| \le 2^{j+2}} (1+|x|)^{n+1} \sum_{|\alpha| \le N+n+1} |\mathrm{D}^\alpha (\psi_j)(x)| \sum_{|\eta| \le N+n+1} |\mathrm{D}^\eta (\mathcal{F}^{-1}\Phi)(x)|$$

and then the boundedness of ψ_j, (1.68), (1.50), hence we obtain for any $s_1 > 0$

$$\begin{aligned}
&\le c_3 \sup_{|x| \sim 2^j} (1+|x|)^{n+1+s_1} \sum_{|\eta| \le N+n+1} |\mathrm{D}^\eta (\mathcal{F}^{-1}\Phi)(x)| \underbrace{(1+|x|)^{-s_1}}_{\sim 2^{-js_1}} \\
&\le c_4 \, 2^{-js_1} \, ||\mathcal{F}^{-1}\Phi||_{n+1+s_1,N+n+1} \\
&\le c_5 \, 2^{-js_1} \, ||\Phi||_{N+2n+2,n+1+s_1}.
\end{aligned}$$

Note, that $s_1 > 0$ is at the moment arbitrary and we can choose it later, if necessary. So we have for arbitrary $s_1 > 0$

$$||(1+|x|)^N \mathcal{F}(\psi_j \mathcal{F}^{-1}\Phi)|L_1|| \le c_5 \, 2^{-js_1} \, ||\Phi||_{N+2n+2,n+1+s_1}. \qquad (1.73)$$

Insert this in (1.70), (1.72) and we obtain

$$\begin{aligned}
|f(\Phi)| &\le \sum_{j=0}^{\infty} |f_j(\Phi_j)| \\
&\le c_1 \sum_{j=0}^{\infty} 2^{j(\max(\log_2(c),0)-s)} \, ||f|B^s_{p,\infty}(w)|| \, ||(1+|x|)^N \Phi_j|L_1|| \\
&\le c_6 \sum_{j=0}^{\infty} 2^{-j(s_1-\max(\log_2(c),0)+s)} \, ||\Phi||_{N+2n+2,n+1+s_1} \, ||f|B^s_{p,\infty}(w)||.
\end{aligned}$$

We choose $s_1 > 0$ such that $s_1 - \max(\log_2(c), 0) + s =: s_2 > 0$, then

$$\leq c_6 \underbrace{\sum_{j=0}^{\infty} 2^{-js_2}}_{<\infty} \; ||\Phi||_{N+2n+2,n+1+s_1} \; ||f|B^s_{p,\infty}(w)||.$$

So there exist $K := N + 2n + 2 \geq 0$, $L := n + 1 + s_1 \geq 0$ such that for all $\Phi \in \mathcal{S}(\mathbb{R}^n)$

$$|f(\Phi)| \leq c_7 \; ||\Phi||_{K,L} \; ||f|B^s_{p,\infty}(w)||.$$

■

2 Decompositions

The main goal of this section is to prove a wavelet characterization for spaces of type $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$, where w is a doubling weight. For this we determine in Section 2.3 a new and equally useful tool, the so-called $\varkappa$-sequence spaces, which yields us, under some additional conditions, a wavelet isomorphism, when we have an atomic decomposition. This part is the heart of the thesis and we deal here with a more general setting. So the main theorem in Section 2.3.3 can be applied for many different function spaces, when they satisfy the $\varkappa$-condition, see Definition 2.17.
As a preparation we introduce in Section 2.1 the concept of atoms and wavelets and what we understand by this, since there exist many different kinds of these.
In Section 2.2 we collect the atomic representations for the function spaces, in which we are particularly interested. Finally in Section 2.4 we show the wavelet characterization for our doubling weighted Besov-Triebel-Lizorkin spaces by using the main Theorem 2.23. Additionally we prove the wavelet characterization for some further function spaces and compare this with the well-known results from the literature.
The results of this chapter are contained in the joined paper [HST16] which is submitted for publication.

2.1 Atoms and wavelets

In the theory of function spaces it is useful to have various representations of a function f from the underlying function space,

$$f = \sum_{j,m} \lambda_{j,m} a_{j,m}.$$

In most of them one decomposes the function f into special building blocks, for example, atoms, wavelets, quarks, molecules. For more information about this we refer for example to [Tri06, Tri01, Woj97]. These building blocks are „nice“ functions with convenient properties such as smoothness or compact supports. Here we amplify the atomic and wavelet representation. We start with the $L_\infty-$normalized (K, L, d)-atoms, where $K \in \mathbb{N}_0, L \in \mathbb{N}_0, d > 1$, cf. [Tri06, Tri08].

Definition 2.1. Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ and $d > 1$.
The complex-valued functions $a_{j,m} \in C^K(\mathbb{R}^n)$ are called (K, L, d)-*atoms* if

$$\operatorname{supp} a_{j,m} \subset d\, Q_{j,m}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{2.1}$$

$$|\mathrm{D}^\eta a_{j,m}(x)| \le 2^{j|\eta|}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad |\eta| \le K, \tag{2.2}$$

$$\int_{\mathbb{R}^n} x^\theta a_{j,m}(x)\,\mathrm{d}x = 0, \qquad j \in \mathbb{N}, \quad m \in \mathbb{Z}^n, \quad |\theta| < L. \tag{2.3}$$

Choosing $L = 0$ in (2.3) means that no moment conditions are required. For convenience the fixed number $d > 1$ will not be indicated in the sequel. Furthermore, if $K = L \in \mathbb{N}$ we denote the respective functions $a_{j,m}$ as L-*atoms*, [Tri08, Tri10].

The idea of atomic decompositions in $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ goes essentially back to Frazier and Jawerth in their series of papers [FJ85], [FJ90], [FJW91], see also [Tri97, Section 13] for an alternative way based on so-called local means. For a detailed overview about the complex history of atoms in various function spaces we refer to [Tri92, Section 1.9], see also [Tri06, Remark 1.48].

Atoms have nice properties, for example, sufficiently high smoothness, compact support and moment conditions. The disadvantage of the atoms is that the representation is not unique, i.e., for a fixed function f one can find different decompositions

$$f = \sum_{j,m} \lambda_{j,m} a_{j,m}.$$

On the other side one has more freedom at the choice of the functions $a_{j,m}$ since the structure is not completely fixed. Sometimes this is advantageous, for example, if one works with traces, because there one does not need the isomorphism between the function space and the corresponding sequence space. But if one is interested in embeddings, as we do, then it is better to work with wavelet isomorphisms. Thus next we introduce the concept of (smooth) wavelet systems. For this we give a brief description of some well-known assertions about wavelet bases in $L_2(\mathbb{R}^n)$ and multiresolution analysis, see [Tri06, Section 1.7]. The standard references here are [Mal89], [Mal98], [Mey87], [Mey92], [Dau88], [Dau92], [Woj97].
We look first at the one-dimensional case.

Definition 2.2. An (inhomogeneous) multiresolution analysis is a sequence $\{V_j : j \in \mathbb{N}_0\}$ of subspaces of $L_2(\mathbb{R}^n)$ such that

(i) $V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots$; span $\bigcup_{j=0}^\infty V_j = L_2(\mathbb{R})$,

(ii) $f \in V_0$ if, and only if, $f(x - m) \in V_0$ for any $m \in \mathbb{Z}$,

(iii) $f \in V_j$ if, and only if, $f(2^{-j}x) \in V_0$ for all $j \in \mathbb{N}$,

(iv) there is a function $\psi_F \in V_0$ such that $\{\psi_F(x - m) : m \in \mathbb{Z}\}$ is an orthonormal basis in V_0.

Remark 2.3. The function ψ_F is called scaling function (or father wavelet, where the F comes from.) By (iii) and (iv) it follows that

$$\left\{2^{j/2}\psi_F(2^j x - m) : m \in \mathbb{Z}\right\}, \quad j \in \mathbb{N}_0,$$

is an orthonormal basis in V_j. Let $W_j \subset L_2(\mathbb{R})$ the orthogonal complement such that

$$V_{j+1} = W_j \oplus V_j; \quad j \in \mathbb{N}_0.$$

Then (i) can be reformulated as

$$L_2(\mathbb{R}) = V_0 \oplus \bigoplus_{j=0}^{\infty} W_j,$$

the orthogonal decomposition.
One of the main assertions of multiresolution analysis is to prove that there are functions $\psi_M \in L_2(\mathbb{R})$, called an associated function (or mother wavelet), such that

$$\{\psi_M(x-m) : m \in \mathbb{Z}\} \quad \text{is an orthonormal basis in } W_0,$$

and to construct them starting from ψ_F. Then it holds

$$\psi_m^j(x) = \begin{cases} \psi_F(x-m), & \text{if } j=0,\ m \in \mathbb{Z}, \\ 2^{j-1/2}\psi_M(2^{j-1}x - m), & \text{if } j \in \mathbb{N},\ m \in \mathbb{Z}, \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R})$.
The extension from one dimension to n dimensions follows by the standard procedures of tensor products. Let $G = (G_1, \dots, G_n) \in G^* := \{F, M\}^{n*}$, where G_r is either F or M and where * indicates that at least one of the components of G must be an M. Then we set

$$\Psi_{G,m}^j(x) := 2^{jn/2} \prod_{r=1}^{n} \psi_{G_r}\big(2^j x_r - m_r\big), \qquad G \in G^* = \{F, M\}^{n*},\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \quad (2.4)$$

and the starting terms are given by

$$\Psi_m(x) := \prod_{r=1}^{n} \psi_F(x_r - m_r), \qquad m \in \mathbb{Z}^n. \quad (2.5)$$

Then $\left\{\Psi_m, \Psi_{G,m}^j :\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0,\ G \in G^*\right\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$.

For our purpose we consider here smooth wavelets, more precisely compactly supported Daubechies wavelets.

Definition 2.4. Let $L \in \mathbb{N}$. Let ψ_F, $\psi_M \in C^L(\mathbb{R})$ are real-valued compactly supported (L_2-normalized) functions with

$$\int_{\mathbb{R}} \psi_F(t)\mathrm{d}t = 1, \qquad \int_{\mathbb{R}} \psi_M(t)\, t^l\, \mathrm{d}t = 0, \qquad l < L. \quad (2.6)$$

Then $\left\{\Psi_m, \Psi_{G,m}^j :\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0,\ G \in G^*\right\}$, constructed in the above sense, is called a (Daubechies) wavelet system.

The existence of the functions ψ_F and ψ_M in Definition 2.4 is given by [Mey92] and [Dau92], see also [Woj97] or [Tri06, Thm. 1.61]. Hence the above definition makes sense.

Remark 2.5. The structure of the wavelet system $\{\Psi_m, \Psi^j_{G,m} :\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0,\ G \in G^*\}$ is rather fixed. We start with two (not explicitly known) functions ψ_M, ψ_F and build the rest in a fixed pattern. This structure saves us the isomorphism. We know that this system of functions $\{\Psi_m, \Psi^j_{G,m} :\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0,\ G \in G^*\}$ builds an orthonormal basis in $L_2(\mathbb{R}^n)$, if the starting functions ψ_F, ψ_M are L_2-normalized. Furthermore, for some $c > 0$,

$$\{c\Psi_m,\ c2^{-jn/2}\Psi^j_{G,m}\} \quad \text{are } L_\infty\text{-normalized } L\text{-atoms,}$$

cf. [Tri06, Chapter 3]. This means, that wavelets can always be considered as atoms. In other words, if we have a wavelet characterization, then we have also an atomic representation. Later in Section 2.3 we shall discuss the converse question, that is, under which (additional) conditions we obtain a wavelet isomorphism, when we have an atomic representation.

2.2 Atomic decomposition

In connection with atoms and function spaces we always have sequence spaces for the sequences of the coefficients, which will appear in atomic decompositions.

Recall our definition of $Q_{j,m}$, where $m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, in the beginning. For $0 < p < \infty$, $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ we denote by $\chi^{(p)}_{j,m}$ the p-normalised characteristic function of the cube $Q_{j,m}$ defined by

$$\chi^{(p)}_{j,m}(x) = 2^{\frac{jn}{p}} \chi_{j,m}(x) = \begin{cases} 2^{\frac{jn}{p}}, & \text{if} \quad x \in Q_{j,m}, \\ 0, & \text{if} \quad x \notin Q_{j,m}. \end{cases} \tag{2.7}$$

It is easy to see that $||\chi^{(p)}_{j,m}|L_p(\mathbb{R}^n)|| = 1$.

Definition 2.6.

(i) Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and w be a doubling weight. Then

$$\bar{b}^s_{p,q}(w) = \left\{ \lambda = \{\lambda_{j,m}\}_{j,m} : \ \lambda_{j,m} \in \mathbb{C}, \quad \big\|\lambda|\bar{b}^s_{p,q}(w)\big\| < \infty \right\}$$

and

$$\big\|\lambda|\bar{b}^s_{p,q}(w)\big\| = \left\| \left\{ 2^{j(s-\frac{n}{p})} \Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}| \chi^{(p)}_{j,m} \Big| L_p(w) \Big\| \right\}_{j\in\mathbb{N}_0} \Big| \ell_q \right\| \tag{2.8}$$

(with obvious modification for $p = \infty$ or $q = \infty$).

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and w be a doubling weight. Then

$$\bar{f}^s_{p,q}(w) = \left\{ \lambda = \{\lambda_{j,m}\}_{j,m} : \ \lambda_{j,m} \in \mathbb{C}, \quad \big\|\lambda|\bar{f}^s_{p,q}(w)\big\| < \infty \right\}$$

and

$$\big\|\lambda|\bar{f}^s_{p,q}(w)\big\| = \left\| \left(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \Big| \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}| \chi^{(p)}_{j,m}(\cdot) \Big|^q \right)^{1/q} \Big| L_p(w) \right\| \tag{2.9}$$

(with obvious modification for $q = \infty$).

Remark 2.7. We can rewrite the $\bar{b}^s_{p,q}(w)$-norm as follows

$$\big\|\lambda|\bar{b}^s_{p,q}(w)\big\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p w(Q_{j,m}) \Big)^{\frac{q}{p}} \right)^{1/q} \tag{2.10}$$

(with obvious modification for $p = \infty$ or $q = \infty$).

Remark 2.8. Note that the cubes $Q_{j,m}$ have no overlap on the same level j. Modify $\|\cdot|\bar{f}^s_{p,q}(w)\|$ by

$$\|\lambda|\bar{f}^s_{p,q}(w)\| = \Big\| \Big(\sum_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q \chi_{j,m}(\cdot)\Big)^{1/q} \Big| L_p(w)\Big\|. \tag{2.11}$$

Remark 2.9. In the unweighted case, i.e., when $w\equiv 1$, we get

$$\|\lambda|\bar{b}^s_{p,q}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p\Big)^{\frac{q}{p}}\Big)^{1/q} \tag{2.12}$$

and

$$\|\lambda|\bar{f}^s_{p,q}\| = \Big\|\Big(\sum_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q \chi_{j,m}(\cdot)\Big)^{1/q} \Big| L_p\Big\| \tag{2.13}$$

(with obvious modification for $p=\infty$ or $q=\infty$).

Remark 2.10. The next result that we want to apply is from Bownik, see [Bow05] and [Bow07], respectively. Note, that Bownik dealt with anisotropic Besov-Triebel-Lizorkin spaces with expansive dilation matrices and more general doubling measures. The difference is that there are used quasi-norms ϱ_A associated with an expansive matrix A. In the standard dyadic case $A = 2I$ a quasi-norm ϱ_A satisfies $\varrho_A(2x) = 2^n\varrho_A(x)$ instead of the usual scalar homogeneity. In particular, $\varrho_A(x) = |x|^n$ is an example for a quasi-norm for $A = 2I$.

Instead of this quasi-norm $|\cdot|^n$ we will use the usual Euclidean norm $|\cdot|$ in $\mathbb{R}^n$. For more details we refer to [Bow03, LR94]. We recall that all quasi-norms associated to a fixed dilation matrix A are equivalent. Moreover, there always exists a quasi-norm ϱ_A, which is C^∞ on $\mathbb{R}^n$ except the origin.

Note also that Bownik dealt with a different decomposition of unity, but we get equivalent quasi-norms. In the main part of [Bow05] and [Bow07], respectively, Bownik works with homogeneous spaces, later he showed that these results also hold for inhomogeneous spaces. Furthermore the atoms and the sequence spaces are L_2-normalised. In our case we have an L_∞-normalisation.

For convenience we adopt the usual notations

$$\sigma_p = n\left(\frac{1}{p}-1\right)_+, \quad \sigma_{p,q} = n\left(\frac{1}{\min(p,q)}-1\right)_+, \tag{2.14}$$

for $0<p,q\le\infty$.

Then the atomic decomposition result used below reads as follows, see [Bow05, Thm. 5.10] and [Bow07, Theorem 5.7, Remark 5.8] with the above-described modifications.

Proposition 2.11. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and w be a doubling weight with doubling constant β.*

(i) *Let $K \in \mathbb{N}_0, L \in \mathbb{N}_0, d \in \mathbb{R}$ with*

$$K > s \qquad \text{and} \qquad L > \frac{n(\beta - 1)}{p} + \sigma_p - s \tag{2.15}$$

and $d > 1$ be fixed. Then a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(w)$ if, and only if, it can be written as a series

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}, \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{2.16}$$

where $a_{j,m}$ are (K, L)-atoms according to Definition 2.1 and $\lambda = \{\lambda_{j,m}\}_{j,m} \in \bar{b}^s_{p,q}(w)$. Furthermore

$$\inf ||\lambda|\bar{b}^s_{p,q}(w)|| \tag{2.17}$$

is an equivalent quasi-norm in $B^s_{p,q}(w)$, where the infimum ranges over all admissible representations (2.16).

(ii) *Let $K \in \mathbb{N}_0, L \in \mathbb{N}_0, d \in \mathbb{R}$ with*

$$K > s \quad \text{and} \quad L > \frac{n(\beta - 1)}{p} + \sigma_{p,q} - s \tag{2.18}$$

and $d > 1$ be fixed. Then a tempered distribution $f \in \mathcal{S}(\mathbb{R}^n)$ belongs to $F^s_{p,q}(w)$ if, and only if, it can be written as a series

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} a_{j,m}, \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{2.19}$$

where $a_{j,m}$ are (K, L)-atoms according to Definition 2.1 and $\lambda = \{\lambda_{j,m}\}_{j,m} \in \bar{f}^s_{p,q}(w)$. Furthermore

$$\inf ||\lambda|\bar{f}^s_{p,q}(w)|| \tag{2.20}$$

is an equivalent quasi-norm in $F^s_{p,q}(w)$, where the infimum ranges over all admissible representations (2.19).

We exemplify the above result in two cases and compare it with known results.

Example 2.12. Let $w \equiv 1$. Then we have by Example 1.20 that $\beta = 1$, such that (2.15) reads as $K > s$ and $L > \sigma_p - s$. Then the result coincides with [Tri97, Theorem 13.8] or [Tri06, Theorem 1.19], respectively.

Example 2.13. Let $w \in \mathcal{A}_\infty$. Then by Example 1.21 we have $\beta = c\,r_w$ such that (2.15) reads as

$$K > s \qquad \text{and} \qquad L > \frac{n(c\,r_w - 1)}{p} + \sigma_p - s. \tag{2.21}$$

This result is contained in [HP08, Theorem 3.10], because assumption (2.21) is in this case slightly stronger than the assumption on L in [HP08, Theorem 3.10]. Consequently, the result of [HP08] has better quantitative characteristics than the ones obtained here as long as we stay in the realm of $\mathcal{A}_\infty$ weights. This is a prize to pay by studying Besov-Triebel-Lizorkin spaces with doubling weights instead of $\mathcal{A}_\infty$ weights.

Remark 2.14. Weighted Besov spaces and their atomic (and wavelet) decompositions in case of admissible weights have been studied in some detail in [HT94, HT05, KLSS06a, KLSS06b, KLSS07]. As far as local Muckenhoupt weights $\mathcal{A}_p^{\mathrm{loc}}$ are concerned, we refer to [Ryc01, Woj12a, Woj11, Woj12b].

2.3 From atoms to wavelets: the $\varkappa$-connection

2.3.1 The Setting

We begin our exposition in a general setting. Let $a(\mathbb{R}^n)$ be a quasi-Banach sequence space, consisting of all sequences

$$\mu = \{\mu_{j,m} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\}, \quad \|\mu \,|a(\mathbb{R}^n)\| < \infty, \tag{2.22}$$

with the standard properties of quasi-Banach lattices:
If $|\mu| = \{|\mu_{j,m}|\}$ and $\mu' = \{\mu'_{j,m}\}$ with $|\mu'_{j,m}| \le |\mu_{j,m}|$, then

$$\big\| \, |\mu| \, |a(\mathbb{R}^n)\big\| = \|\mu \,|a(\mathbb{R}^n)\|, \qquad \|\mu' \,|a(\mathbb{R}^n)\| \le \|\mu \,|a(\mathbb{R}^n)\|. \tag{2.23}$$

Let $a_0(\mathbb{R}^n)$ be the subspace of $a(\mathbb{R}^n)$ consisting of all sequences

$$\big\{\mu_{j,m} : \ \mu_{0,m} = \mu_m, \ \ \mu_{j,m} = 0 \text{ where } j \in \mathbb{N}, \, m \in \mathbb{Z}^n\big\}. \tag{2.24}$$

Then the wavelet version $a^w(\mathbb{R}^n)$ of $a(\mathbb{R}^n)$ collects all sequences

$$\mu = \big\{\mu_m \in \mathbb{C}, \ \mu_m^{j,G} \in \mathbb{C} : \ m \in \mathbb{Z}^n, \ j \in \mathbb{N}_0, \ G \in G^*\big\}, \tag{2.25}$$

quasi-normed by

$$\|\mu \,|a^w(\mathbb{R}^n)\| = \|\{\mu_m\}_m \,|a_0(\mathbb{R}^n)\| + \sum_{G \in G^*} \big\|\{\mu_m^{j,G}\}_{j,m} \,|a(\mathbb{R}^n)\big\| < \infty. \tag{2.26}$$

For example the classical ℓ_p-spaces fit into this scheme. We are interested in sequence spaces of $\bar{b}^s_{p,q}$, $\bar{f}^s_{p,q}$-type, also with some weight functions, especially with doubling weights. These spaces also fit into this scheme of quasi-Banach sequence spaces $a(\mathbb{R}^n)$.

On the other side we deal with (isotropic, inhomogeneous) quasi-Banach function spaces $A(\mathbb{R}^n)$ in $\mathbb{R}^n$, which satisfy

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow A(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \tag{2.27}$$

where $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ have their usual meaning here, cf. Section 1.1. Additionally $A(\mathbb{R}^n)$ can be characterized in terms of L-atomic representations:
$f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $A(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \mu_{j,m} \, a_{j,m}, \qquad \mu \in a(\mathbb{R}^n), \tag{2.28}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$ with

$$\|f \,|A(\mathbb{R}^n)\| \sim \inf \|\mu \,|a(\mathbb{R}^n)\|, \tag{2.29}$$

where $a(\mathbb{R}^n)$ *is a sequence space as introduced above and* $\{a_{j,m}\}$ *are* L*-atoms. The infimum in* (2.29) *is taken over all admissible representations* (2.28).

The question arises under which issue L-atomic representations of $A(\mathbb{R}^n)$, based on $a(\mathbb{R}^n)$, admit corresponding L-wavelet characterizations now based on the wavelet version of $a^w(\mathbb{R}^n)$ of $a(\mathbb{R}^n)$ according to (2.25), (2.26). The desired result in this context reads as follows:

$f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $A(\mathbb{R}^n)$ *if, and only if, it can be represented in terms of* L*-wavelets as*

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m \Psi_m + \sum_{G\in G^*} \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G} 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in a^w(\mathbb{R}^n), \tag{2.30}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. *The representation* (2.30) *is unique,*

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2} \big(f, \Psi^j_{G,m}\big), \qquad \lambda_m = (f, \Psi_m), \tag{2.31}$$

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, $G \in G^*$ *and*

$$I: \quad f \mapsto \big\{\lambda_m,\, \lambda_m^{j,G}\big\} \tag{2.32}$$

is an isomorphic map of $A(\mathbb{R}^n)$ *onto* $a^w(\mathbb{R}^n)$.

As already mentioned, if one has such an L-wavelet representation, then this automatically provides an L-atomic counterpart. However, the step from L-atomic representations to L-wavelet characterizations causes several problems. Formally one has to show that

$$\|\lambda\,|a^w(\mathbb{R}^n)\| \le c\,\|\mu\,|a(\mathbb{R}^n)\|, \tag{2.33}$$

with λ as in (2.31) and a constant $c > 0$ which is independent of all admitted sequences μ in (2.28), (2.29). For this purpose one has not only to clarify what is meant by the dual pairings $(f, \Psi^j_{G,m})$, (f, Ψ_m) of $f \in A(\mathbb{R}^n)$ and L-wavelets, but also to ensure $f = g \in A(\mathbb{R}^n)$ if

$$\big(f, \Psi^j_{G,m}\big) = \big(g, \Psi^j_{G,m}\big), \qquad (f, \Psi_m) = (g, \Psi_m), \tag{2.34}$$

for all $\Psi^j_{G,m}$, Ψ_m. This is a matter of duality which requires some care.

Later in this work we want to apply the above mentioned concept of quasi-Banach sequence spaces $a(\mathbb{R}^n)$ and quasi-Banach function spaces $A(\mathbb{R}^n)$ to such prominent examples like $B^s_{p,q}(w)$, $F^s_{p,q}(w)$ and $\bar{b}^s_{p,q}(w)$, $\bar{f}^s_{p,q}(w)$, where $B^s_{p,q}(w)$, $F^s_{p,q}(w)$ and $\bar{b}^s_{p,q}(w)$, $\bar{f}^s_{p,q}(w)$ are the above introduced sequence and function spaces.

We have the essential embeddings

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(w), F^s_{p,q}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \tag{2.35}$$

for all $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and w be doubling, cf. Proposition 1.44. Thus the (doubling weighted) spaces $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$ should fit into our scheme of (isotropic, inhomogeneous) quasi-Banach function spaces $A(\mathbb{R}^n)$ (if we have in addition an atomic representation of them).

Remark 2.15. Let $0 < p, q \le \infty$ be fixed. Then

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{\alpha\in\mathbb{R}, s\in\mathbb{R}} B^s_{p,q}(\mathbb{R}^n, w_\alpha) \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^n) = \bigcup_{\alpha\in\mathbb{R}, s\in\mathbb{R}} B^s_{p,q}(\mathbb{R}^n, w_\alpha), \tag{2.36}$$

where $w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$, is the admissible weight from Example 1.28. A detailed proof of this more or less known assertion may be found in [Kab08]. We remind, that

$$B^s_{\infty,\infty}(\mathbb{R}^n, w_\alpha) = \mathcal{C}^s(\mathbb{R}^n, w_\alpha),$$

cf. (1.60) in Remark 1.41. Of interest for us is a special case from (2.36)

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{\alpha\in\mathbb{R}, s\in\mathbb{R}} \mathcal{C}^s(\mathbb{R}^n, w_\alpha) = \bigcap_{\alpha, s\in\mathbb{R}} \mathring{\mathcal{C}}^s(\mathbb{R}^n, w_\alpha), \tag{2.37}$$

where $\mathring{\mathcal{C}}^s(\mathbb{R}^n, w_\alpha)$ is the completion of $\mathcal{D}(\mathbb{R}^n) = C^\infty_0(\mathbb{R}^n)$ in $\mathcal{C}^s(\mathbb{R}^n, w_\alpha)$. The second equality follows from

$$\mathcal{C}^{s+\varepsilon}(\mathbb{R}^n, w_{\alpha+\varepsilon}) \hookrightarrow \mathring{\mathcal{C}}^s(\mathbb{R}^n, w_\alpha), \qquad \varepsilon > 0, \tag{2.38}$$

which can be justified by the wavelet characterization for spaces of type $B^s_{p,q}(\mathbb{R}^n, w_\alpha)$, [Tri06, Theorem 6.15].

$$\|g\,|\mathcal{C}^s(\mathbb{R}^n, w_\alpha)\| \sim \sup_{j,G,m} 2^{js}\big(1 + |2^{-j}m|\big)^\alpha |\lambda^{j,G}_m(g)|. \tag{2.39}$$

Remark 2.16. If $A(\mathbb{R}^n)$ is a Banach space, then it follows from (2.27), (2.37) and well known properties of embeddings of locally convex spaces according to [Yos80, Theorem 1, Section I,6, p.42] that

$$\mathring{\mathcal{C}}^s(\mathbb{R}^n, w_\alpha) \hookrightarrow A(\mathbb{R}^n), \tag{2.40}$$

for some $s \in \mathbb{R}$ and some $\alpha \in \mathbb{R}$. If in addition $\mathcal{S}(\mathbb{R}^n)$ is dense in the Banach spaces $A(\mathbb{R}^n)$, then (2.40) can be complemented by

$$A'(\mathbb{R}^n) \hookrightarrow \big(\mathring{\mathcal{C}}^s(\mathbb{R}^n, w_\alpha)\big)' = B^{-s}_{1,1}(\mathbb{R}^n, w_{-\alpha}). \tag{2.41}$$

If $A(\mathbb{R}^n)$ is a quasi-Banach space (in particular not necessarily locally convex), then it is not clear whether (2.27) ensures (2.40) for some s, α and (2.41). But, we shall see that a weak local duality of (2.40) will be sufficient to justify $\big(f, \Psi^j_{G,m}\big)$ and $f = g$ if one has (2.34).

2.3.2 Well-definedness of the dual pairing

We deal with sequence spaces $a(\mathbb{R}^n)$ with (2.22), (2.23) adapted to atomic decompositions. Let $m \in \mathbb{Z}^n$, $j, J \in \mathbb{N}_0$, $d > 1$, $C_1 > 0$. For convenience let us denote by

$$I^j_J(m) = \{M \in \mathbb{Z}^n : d\,Q_{J,M} \cap C_1\,Q_{j,m} \neq \emptyset\} \subset \mathbb{Z}^n. \tag{2.42}$$

Note that the cardinality $\# I_J^j(m)$ satisfies

$$\# I_J^j(m) \ \sim \ \begin{cases} 1, & J \le j, \\ 2^{n(J-j)}, & J > j, \end{cases} \tag{2.43}$$

with constants independent of $j, J \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Definition 2.17. Let $\varkappa > 0$. Then $a(\mathbb{R}^n)$ is called a $\varkappa$-sequence space if

(i) for any $d > 1$, $C_1 > 0$, and all $\mu \in a(\mathbb{R}^n)$ any sequence

$$\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$$

with

$$|\lambda_{j,m}| \le C_1 \sum_{J \in \mathbb{N}_0} 2^{-\varkappa |J-j|} \sum_{M \in I_J^j(m)} 2^{-n(J-j)_+} |\mu_{J,M}|, \quad j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{2.44}$$

belongs to $a(\mathbb{R}^n)$ and satisfies

$$\|\lambda | a(\mathbb{R}^n)\| \le C_2 \, \|\mu | a(\mathbb{R}^n)\| \tag{2.45}$$

for some $C_2 > 0$ which may depend on d, C_1, $\varkappa$ and $n \in \mathbb{N}$;

(ii) for any cube Q there is a constant $c_Q > 0$ such that for all $\mu \in a(\mathbb{R}^n)$,

$$|\mu_{J,M}| \le c_Q \ 2^{J\varkappa} \ \|\mu | a(\mathbb{R}^n)\| \quad \text{for all} \quad J \in \mathbb{N}_0 \text{ and } M \in \mathbb{Z}^n \quad \text{with} \quad Q_{J,M} \subset Q. \tag{2.46}$$

Remark 2.18. The definition of the $\varkappa$-sequence space is very technical. It comes out from the proof of our main theorem below, where **(i)** is used in the proof of Theorem 2.23 and **(ii)** in the proof of Proposition 2.20. We add a respective comment in Remark 2.39 below.

Remark 2.19. If $J \le j$, then the sum over $Q_{J,M}$ in (2.44) has only finitely many terms, independent of $Q_{j,m}$. If $J > j$, then this sum has $\sim 2^{n(J-j)}$ terms and

$$\sum_{M \in I_J^j(m)} 2^{-n(J-j)} |\mu_{J,M}| \le c \max_{M \in I_J^j(m)} |\mu_{J,M}| \tag{2.47}$$

for some $c > 0$ which again is independent of $Q_{j,m}$.

If we remind to the last section, then it is our first aim to explain the well-definedness of the dual pairings $(f, \Psi_{G,m}^j)$, (f, Ψ_m) in (2.31) of $f \in A(\mathbb{R}^n)$ and the L-wavelets Ψ_m, $\Psi_{G,m}^j$.

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and $\varrho \in \mathbb{R}$. Then $\overset{\circ}{\mathcal{C}}{}^\varrho(\Omega)$ is the completion of $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ in $\mathcal{C}^\varrho(\mathbb{R}^n)$. Let again (f, φ) with $f \in A(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\Omega)$ be the usual dual pairing in the context of $\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)\big)$. If one has in addition

$$|(f,\varphi)| \le c_\Omega \, \|f\,|A(\mathbb{R}^n)\| \cdot \|\varphi\,|\mathcal{C}^\varrho(\mathbb{R}^n)\|, \qquad f \in A(\mathbb{R}^n), \quad \varphi \in \mathcal{D}(\Omega), \tag{2.48}$$

then one can extend the above dual pairing to (f, g) with $f \in A(\mathbb{R}^n)$ and $g \in \overset{\circ}{\mathcal{C}}{}^\varrho(\Omega)$ by standard arguments in the duality theory of function spaces. We refer to questions of this type in [Tri83, Section 2.11] for a detailed discussion. Then we will say that the dual pairing (f, g) with $f \in A(\mathbb{R}^n)$ and $g \in \overset{\circ}{\mathcal{C}}{}^\varrho(\Omega)$ is well defined.

Proposition 2.20. *Let $A(\mathbb{R}^n)$ be a function space which can be represented by the L-atomic expansions* (2.28), (2.29) *where $a(\mathbb{R}^n)$ is a $\varkappa$-sequence space according to Definition 2.17 with $0 < \varkappa < L \in \mathbb{N}$. Let $\varrho > \varkappa$. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. Then the dual pairing*

$$(f,g) \qquad \text{with} \quad f \in A(\mathbb{R}^n), \quad g \in \overset{\circ}{\mathcal{C}}{}^\varrho(\Omega) \tag{2.49}$$

is well defined and there is a constant $c_\Omega > 0$ such that

$$|(f,g)| \le c_\Omega \, \|f\,|A(\mathbb{R}^n)\| \cdot \|g\,|\mathcal{C}^\varrho(\mathbb{R}^n)\|, \qquad f \in A(\mathbb{R}^n), \quad g \in \overset{\circ}{\mathcal{C}}{}^\varrho(\Omega). \tag{2.50}$$

Proof. We may assume $\varkappa < \varrho < L$. Let $g \in \overset{\circ}{\mathcal{C}}{}^\varrho(\Omega)$. Since $\overset{\circ}{\mathcal{C}}{}^\varrho(\Omega) \hookrightarrow \mathcal{C}^\varrho(\Omega)$ and $\overset{\circ}{\mathcal{C}}{}^\varrho(\Omega)$ is the completion of $\mathcal{D}(\Omega)$, g has compact support in Ω and we can extend g on $\mathbb{R}^n$ (by zero). Thus $g \in \widetilde{\mathcal{C}}{}^\varrho(\overline{\Omega}) := \big\{h \in \mathcal{C}^\varrho(\mathbb{R}^n) : \operatorname{supp}(h) \subset \overline{\Omega}\big\} \subset \mathcal{C}^\varrho(\mathbb{R}^n)$. On $\mathcal{C}^\varrho(\mathbb{R}^n) = B^\varrho_{\infty,\infty}(\mathbb{R}^n)$ we have a wavelet characterization, see [Tri06, Theorem 3.5] or [Tri08, Theorem 1.20]. Let

$$g = \sum_{j,G,m} \lambda^{j,G}_m(g)\, 2^{-jn/2}\, \Psi^j_{G,m} \tag{2.51}$$

be the L-wavelet expansion of $g \in \widetilde{\mathcal{C}}{}^\varrho(\overline{\Omega})$, incorporating now the starting terms Ψ_m, see (2.30). Then

$$\|g\,|\mathcal{C}^\varrho(\mathbb{R}^n)\| \sim \sup_{j,G,m} 2^{j\varrho} \big|\lambda^{j,G}_m(g)\big|. \tag{2.52}$$

We may assume $\|g\,|\mathcal{C}^\varrho(\mathbb{R}^n)\| = 1$. Let $f \in A(\mathbb{R}^n)$ be expanded by L-atoms according to (2.28). Then

$$(f,g) = \sum_{J,M} \mu_{J,M} \big(a_{J,M},\, g_J + g^J\big), \tag{2.53}$$

where

$$g_J = \sum_{\substack{j<J,\\ G,m}} \lambda^{j,G}_m(g)\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad g^J = \sum_{\substack{j\ge J,\\ G,m}} \lambda^{j,G}_m(g)\, 2^{-jn/2}\, \Psi^j_{G,m}. \tag{2.54}$$

Formally (2.53) has to be justified. But one can first deal with finite sums and defines afterwards (2.53) by standard limiting arguments based on what follows.

$$
\begin{aligned}
|(f,g)| \le & \sum_{J,M} |\mu_{J,M}||(a_{J,M}, g_J)| + \sum_{J,M} |\mu_{J,M}||(a_{J,M}, g^J)| \\
\le & \sum_{J,M} |\mu_{J,M}| \sum_{\substack{j<J,\\ G,m}} |\lambda_m^{j,G}(g)|\,|(a_{J,M}, 2^{-jn/2}\,\Psi^j_{G,m})| \\
& + \sum_{J,M} |\mu_{J,M}| \sum_{\substack{j\ge J,\\ G,m}} |\lambda_m^{j,G}(g)|\,|(a_{J,M}, 2^{-jn/2}\,\Psi^j_{G,m})|.
\end{aligned}
$$

The sums over $m \in \mathbb{Z}^n$ have only finitely many terms since both the atoms and wavelets have compact support conditions. We use the index set

$$
I^j_J(M) = \{m \in \mathbb{Z}^n : d\,Q_{J,M} \cap C_1\,Q_{j,m} \ne \emptyset\}\,,
$$

where $d\,Q_{J,M}$ denotes the support of $a_{J,M}$ and $C_1\,Q_{j,m}$ denotes the support of $\widetilde{\Psi}^G_{j,m} := 2^{-jn/2}\,\Psi^j_{G,m}$. Then

$$
\# I^j_J(M) \;\sim\; \begin{cases} 1, & j < J, \\ 2^{n(j-J)}, & j \ge J. \end{cases}
$$

Furthermore both the L-atoms and L-wavelets have classical derivatives up to order L and cancellations of type (2.3), (2.6). For fixed J, M and $j < J$, $m \in I^j_J(M)$ we use a Taylor expansion of $\widetilde{\Psi}^G_{j,m}$ in $x_0 = 2^{-J}M$ up to the order $L-1$,

$$
\widetilde{\Psi}^G_{j,m}(y) \;=\; \sum_{|\alpha|<L} \frac{\mathrm{D}^\alpha \widetilde{\Psi}^G_{j,m}(x_0)}{\alpha!}(y-x_0)^\alpha + \sum_{|\alpha|=L} \frac{\mathrm{D}^\alpha \widetilde{\Psi}^G_{j,m}(\xi)}{\alpha!}(y-x_0)^\alpha
$$

where ξ lies between x_0 and y.

Insert this Taylor expansion and use the moment conditions of $a_{J,M}$ up to the order $L-1$. Then one obtains for fixed J, M and $j < J$, $m \in I^j_J(M)$

$$
\begin{aligned}
\big|\big(a_{J,M},\, \widetilde{\Psi}^G_{j,m}\big)\big| =\; & \Big|\int_{\mathbb{R}^n} a_{J,M}(y)\,\widetilde{\Psi}^G_{j,m}(y)\,\mathrm{d}y\Big| \\
\le\; & c_1 \,\Big|\sum_{|\alpha|<L} c_\alpha \underbrace{\int_{\mathbb{R}^n} y^\alpha a_{J,M}(y)\,\mathrm{d}y}_{=0,\ \forall\,|\alpha|<L}\Big| && (2.55) \\
& + c_1 \sum_{|\alpha|=L} \underbrace{\sup_{x\in\mathbb{R}^n} \big|\mathrm{D}^\alpha \widetilde{\Psi}^G_{j,m}(x)\big|}_{\le c'\,2^{j|\alpha|} = c'\,2^{jL}} \int_{\mathbb{R}^n} \big|a_{J,M}(y)\big|\big|y - 2^{-J}M\big|^L \,\mathrm{d}y \\
\le\; & c_2\, 2^{jL} \int_{d\,Q_{J,M}} \underbrace{\big|a_{J,M}(y)\big|}_{\le 1}\, \underbrace{\big|y-2^{-J}M\big|^L}_{\le c\,2^{-JL}} \,\mathrm{d}y \\
\le\; & c_3\, 2^{jL}\, 2^{-JL}\, 2^{-Jn} = c_3\, 2^{(j-J)L}\, 2^{-Jn}. && (2.56)
\end{aligned}
$$

On the other hand for fixed $j \geq J$ we change the roles, i.e., we use a Taylor expansion of $a_{J,M}$ in $x_0 = 2^{-j}m$ up to the order $L-1$,

$$a_{J,M}(y) \;=\sum_{|\alpha|<L} \frac{\mathrm{D}^\alpha a_{J,M}(x_0)}{\alpha!}(y-x_0)^\alpha + \sum_{|\alpha|=L} \frac{\mathrm{D}^\alpha a_{J,M}(\xi)}{\alpha!}(y-x_0)^\alpha$$

and use then the moment conditions of $\widetilde{\Psi}^G_{j,m}$. Then the counterpart of (2.56) is

$$\begin{aligned}
\big|\big(a_{J,M},\, \widetilde{\Psi}^G_{j,m}\big)\big| &= \Big| \int_{\mathbb{R}^n} a_{J,M}(y)\, \widetilde{\Psi}^G_{j,m}(y)\,\mathrm{d}y\Big| \\
&\leq c_1 \,\Big| \sum_{|\alpha|<L} c_\alpha \underbrace{\int_{\mathbb{R}^n} y^\alpha \widetilde{\Psi}^G_{j,m}(y)\,\mathrm{d}y}_{=0,\ \forall\, |\alpha|<L} \Big| \qquad (2.57)\\
&\quad + c_1 \sum_{|\alpha|=L} \underbrace{\sup_{x\in\mathbb{R}^n} \big|\mathrm{D}^\alpha a_{J,M}(x)\big|}_{\leq c'\, 2^{J|\alpha|} = c'\, 2^{JL}} \int_{\mathbb{R}^n} \big|\widetilde{\Psi}^G_{j,m}(y)\big| \big|y-2^{-j}m\big|^L\,\mathrm{d}y \\
&\leq c_2\, 2^{JL} \int_{C_1\, Q_{j,m}} \underbrace{\big|\widetilde{\Psi}^G_{j,m}(y)\big|}_{\leq c} \underbrace{\big|y-2^{-j}m\big|^L}_{\leq c\, 2^{-jL}}\,\mathrm{d}y \\
&\leq c_3\, 2^{JL}\, 2^{-jL}\, 2^{-jn} = c_3\, 2^{(J-j)L}\, 2^{-jn}. \qquad (2.58)
\end{aligned}$$

We use $|\lambda^{j,G}_m(g)| \leq c\, 2^{-j\varrho}$ and $\varkappa < \varrho < L$. Since Ω is bounded one has $\sim 2^{Jn}$ relevant terms for fixed J in the sum over M. Let $\sum_{J,M}^{\Omega}$ be the corresponding sum. Note that we have ~ 1 relevant j-terms if $j < J$. Then one has by (2.46),

$$\begin{aligned}
\sum_{J,M} |\mu_{J,M}|\, |(a_{J,M},\, g_J)| &\leq c \sum_{J,M}^{\Omega} |\mu_{J,M}| \Big(\sum_{\substack{j<J,\\ G,m}} 2^{-j\varrho}\, 2^{-L(J-j)}\, 2^{-Jn}\Big) \\
&\leq c' \sum_{J,M}^{\Omega} 2^{-Jn}\, |\mu_{J,M}|\, 2^{-LJ} \Big(\underbrace{\sum_{j<J} 2^{-j(\varrho-L)}}_{\sim 2^{-J(\varrho-L)}} \Big) \qquad (2.59)\\
&\leq c_\Omega \sup_{J,M}{}^{\Omega} |\mu_{J,M}|\, 2^{-J\varrho} \\
&\leq c'_\Omega\, \|\mu\,|a(\mathbb{R}^n)\|.
\end{aligned}$$

Furthermore for $j \geq J$ we have $\sim 2^{n(j-J)}$ relevant j-terms, such that

$$\begin{aligned}
\sum_{J,M} |\mu_{J,M}|\, |(a_{J,M},\, g^J)| &\leq c \sum_{J,M}^{\Omega} |\mu_{J,M}| \Big(\sum_{j\geq J} 2^{n(j-J)}\, 2^{-j\varrho}\, 2^{-L(j-J)}\, 2^{-jn}\Big) \\
&\leq c' \sum_{J,M}^{\Omega} 2^{-Jn}\, |\mu_{J,M}| \Big(\sum_{j\geq J} 2^{-j\varrho}\, 2^{-L(j-J)}\Big) \\
&\leq c_\Omega \sup_{J,M}{}^{\Omega} |\mu_{J,M}|\, 2^{-J\varrho} \Big(\underbrace{\sum_{j\geq 0} 2^{-j(L+\varrho)}}_{<\infty} \Big) \qquad (2.60)\\
&\leq c'_\Omega\, \|\mu\,|a(\mathbb{R}^n)\|.
\end{aligned}$$

By (2.53) and (2.29) one obtains

$$|(f,g)| \le c_\Omega \, \|f \,|A(\mathbb{R}^n)\| \tag{2.61}$$

for $g \in \overset{\circ}{\mathcal{C}}{}^{\varrho}(\Omega)$ with $\|g\,|\mathcal{C}^{\varrho}(\mathbb{R}^n)\| = 1$. This proves also (2.50) for all $g \in \overset{\circ}{\mathcal{C}}{}^{\varrho}(\Omega)$. ■

Remark 2.21. The above proposition is a weak and, in particular, local duality assertion. But this will be sufficient to show that (2.34) implies $f = g$.

Remark 2.22. In (2.59) and (2.60) we use (2.46), thus there comes the **(ii)** condition at $\varkappa$ out.

2.3.3 Main theorem

Recall that $a^w(\mathbb{R}^n)$ is the wavelet version of $a(\mathbb{R}^n)$ as introduced in (2.25), (2.26).

Theorem 2.23. *Let $A(\mathbb{R}^n)$ be an (isotropic, inhomogeneous) function space which can be represented by the L-atomic expansions (2.27)-(2.29) where $a(\mathbb{R}^n)$ is a $\varkappa$-sequence space according to Definition 2.17 with $0 < \varkappa < L \in \mathbb{N}$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $A(\mathbb{R}^n)$ if, and only if, it can be represented in terms of L-wavelets as*

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m \Psi_m + \sum_{G\in G^*}\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in a^w(\mathbb{R}^n), \tag{2.62}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (2.62) is unique,

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\big(f, \Psi^j_{G,m}\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m), \tag{2.63}$$

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, $G \in G^*$, and

$$I: \quad f \mapsto \big\{\lambda_m(f),\, \lambda_m^{j,G}(f)\big\} \tag{2.64}$$

is an isomorphic map of $A(\mathbb{R}^n)$ onto $a^w(\mathbb{R}^n)$.

Proof. *Step 1.* By (2.25), (2.26) the right-hand side of (2.62) can be interpreted as an L-atomic representation of f. Hence $f \in A(\mathbb{R}^n)$ and

$$\|f\,|A(\mathbb{R}^n)\| \le c\, \|\lambda\,|a^w(\mathbb{R}^n)\|. \tag{2.65}$$

Step 2. We prove the converse. At first we show the well-definedness of (2.63). The counterpart of (2.38) yields us that for every wavelet $\Psi^j_{G,m}$ (including the starting terms Ψ_m) there exists a bounded C^∞ domain Ω in $\mathbb{R}^n$ and an $\varepsilon > 0$ such that $\Psi^j_{G,m} \in \mathcal{C}^{\varrho+\varepsilon}(\Omega) \hookrightarrow \overset{\circ}{\mathcal{C}}{}^{\varrho}(\Omega)$. Thus we can apply Proposition 2.20 with $\varkappa < \varrho < L$ to $\Psi^j_{G,m}$ with a corresponding bounded C^∞ domain Ω and $f \in A(\mathbb{R}^n)$. Therefore (2.63) is well defined.
Step 3. Next we prove that $\lambda(f) = \big\{\lambda_m(f),\, \lambda_m^{j,G}(f)\big\} \in a^w(\mathbb{R}^n)$ and $\|\lambda(f)\,|a^w(\mathbb{R}^n)\| \le$

$c \, \| f \, | A(\mathbb{R}^n) \|$. Since $f \in A(\mathbb{R}^n)$ we have an admissible atomic representation with (2.28), (2.29)

$$f = \sum_{J \in \mathbb{N}_0} \sum_{M \in \mathbb{Z}^n} \mu_{J,M} \, a_{J,M}, \qquad \mu \in a(\mathbb{R}^n).$$

For convenience we will ignore the starting terms $\lambda_m(f)$ in (2.63) and concentrate on $\lambda_m^{j,G}(f)$. The modifications otherwise are obvious. For fixed j, G, m we insert the atomic representation in $\lambda_m^{j,G}(f)$ according to (2.63), hence

$$\lambda_m^{j,G}(f) = 2^{jn} \sum_{J \in \mathbb{N}_0} \sum_{M \in \mathbb{Z}^n} \mu_{J,M} \big(a_{J,M}, \, 2^{-jn/2} \, \Psi^j_{G,m} \big). \tag{2.66}$$

Formally one may insert first only finite partial sums of (2.28) complemented afterwards by standard limiting arguments based on Proposition 2.20 and what follows. We will not stress this point. The situation now is very similar to the proof of Proposition 2.20. We have two cases for $\big(a_{J,M}, \, 2^{-jn/2} \, \Psi^j_{G,m} \big)$, if $J \le j$ and if $J > j$. We do a Taylor expansion both on $a_{J,M}$ and on $\widetilde{\Psi}^G_{j,m} := 2^{-jn/2} \, \Psi^j_{G,m}$ and use alternately the support, boundary and moment conditions of $a_{J,M}$ and $\widetilde{\Psi}^G_{j,m}$, (2.1),(2.2),(2.3),(2.6). We use the index set

$$I^j_J(m) = \{ M \in \mathbb{Z}^n : d \, Q_{J,M} \cap C_1 \, Q_{j,m} \neq \emptyset \},$$

see (2.42).
Then we obtain for fixed j, m and $J \le j$, $M \in I^j_J(m)$, that

$$\big| \big(a_{J,M}, \, \widetilde{\Psi}^G_{j,m} \big) \big| \le c \, 2^{(J-j)L} \, 2^{-jn} \tag{2.67}$$

and for fixed j, m and $J > j$, $M \in I^j_J(m)$,

$$\big| \big(a_{J,M}, \, \widetilde{\Psi}^G_{j,m} \big) \big| \le c \, 2^{(j-J)L} \, 2^{-Jn}. \tag{2.68}$$

Recall that the set $I^j_J(m)$ has finitely many terms, see (2.43). If $J \le j$ then we apply (2.67) to ~ 1 relevant terms and if $J > j$, then we apply (2.68) to $\sim 2^{n(J-j)}$ relevant terms. Hence

$$\begin{aligned}
|\lambda_m^{j,G}(f)| \le \; & 2^{jn} \sum_{J \le j} \sum_{M \in I^j_J(m)} |\mu_{J,M}| \, \big| \big(a_{J,M}, \, 2^{-jn/2} \, \Psi^j_{G,m} \big) \big| \\
& + 2^{jn} \sum_{J > j} \sum_{M \in I^j_J(m)} |\mu_{J,M}| \, \big| \big(a_{J,M}, \, 2^{-jn/2} \, \Psi^j_{G,m} \big) \big| \\
\le \; & c \, 2^{jn} \sum_{J \le j} \sum_{M \in I^j_J(m)} |\mu_{j,m}| \, 2^{-L(j-J)} \, 2^{-jn} \\
& + c \, 2^{jn} \sum_{J > j} \sum_{M \in I^j_J(m)} |\mu_{j,m}| \, 2^{-L(J-j)} \, 2^{-Jn}.
\end{aligned} \tag{2.69}$$

Recall that $\varkappa < L$. Then one obtains by (2.44), (2.45), in view of (2.69) that $\lambda(f) \in a^w(\mathbb{R}^n)$ and

$$\| \lambda(f) \, | a^w(\mathbb{R}^n) \| \le c \, \| \mu \, | a(\mathbb{R}^n) \|, \tag{2.70}$$

where $c > 0$ is independent of μ. By (2.29) one obtains

$$\|\lambda(f)\,|a^w(\mathbb{R}^n)\| \le c\,\|f\,|A(\mathbb{R}^n)\|. \tag{2.71}$$

Step 4. In particular, if $f \in A(\mathbb{R}^n)$, then one has

$$g := \sum_{m\in\mathbb{Z}^n} \lambda_m(f)\,\Psi_m + \sum_{G\in G^*}\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}(f)\,2^{-jn/2}\,\Psi^j_{G,m} \in A(\mathbb{R}^n), \tag{2.72}$$

since $\{\Psi_m, 2^{-jn/2}\,\Psi^j_{G,m}\}$ can be also considered as atoms and by (2.70) $\lambda(f)$ also belongs to $a(\mathbb{R}^n)$.

But now we are in the same position as in [Tri06, p.155] relying on Proposition 2.20 instead of the duality relations for $A^s_{p,q}(\mathbb{R}^n)$ used there. Recall that $\{\Psi_m, \Psi^j_{G,m}\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$. In particular,

$$\big(g, \Psi^j_{G,m}\big) = 2^{-jn/2}\lambda^{j,G}_m(f) = \big(f, \Psi^j_{G,m}\big) \quad\text{and}\quad (g, \Psi_m) = (f, \Psi_m) \tag{2.73}$$

for all admitted j, G, m. We apply Proposition 2.20 and the L-wavelet expansion (2.51) to $\varphi \in \mathcal{D}(\mathbb{R}^n)$, hence

$$\varphi = \lim_{j\to\infty} \varphi_j \quad\text{with}\quad \varphi_j = \sum_{J\le j, G, M} \lambda^{J,G}_M(\varphi)\,2^{-Jn/2}\Psi^J_{G,M}. \tag{2.74}$$

Since φ_j are finite linear combinations of $\Psi^J_{G,M}$ one can extend (2.73) to

$$(g, \varphi_j) = (f, \varphi_j), \qquad j \in \mathbb{N}. \tag{2.75}$$

Furthermore,

$$\|\varphi - \varphi_j\,|\mathcal{C}^\varrho(\mathbb{R}^n)\| \sim \sup_{J\ge j, M, G} 2^{J\varrho}\,|\lambda^{J,G}_M(\varphi)| \le c\,2^{-j(L-\varrho)}. \tag{2.76}$$

Hence $\varphi_j \to \varphi$ in $\mathcal{C}^\varrho(\mathbb{R}^n)$ with $\varkappa < \varrho < L$. Then one obtains from Proposition 2.20 and (2.75)

$$(f, \varphi) = (g, \varphi), \qquad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{2.77}$$

This can be extended to $\varphi \in \mathcal{S}(\mathbb{R}^n)$ by standard arguments. Hence $f = g$. This proves (2.62). In the same way one obtains the uniqueness of the representation (2.62). From (2.65) and (2.71) it follows that I in (2.64) is an isomorphic map. ■

Remark 2.24. So far we relied on L-atoms (better (L, d)-atoms with some fixed $d > 1$) according to (2.1)-(2.3) with $K = L$. But K and L play in the theory of atomic representations different roles, [Tri08, pp.4/5, Theorem 1.7]. For example $L = 0$ is useful for pointwise multipliers. The situation for wavelets is different where we relied on $K = L = u$ as in [Tri08, p.13].

Remark 2.25. To use the same $L \in \mathbb{N}$ in Theorem 2.23 for atoms, wavelets and the $\varkappa$-sequence spaces with $\varkappa < L$ is convenient. One may ask for better optimal choices (maybe a suitable decoupling). But in the case of $a(\mathbb{R}^n) = \overline{b}^s_{p,q}$ and (even more) $a(\mathbb{R}^n) = \overline{f}^s_{p,q}$ in the proof of [Tri08, Theorem 1.15, pp.7-12] one needs L large enough to compensate $2^{j(s-\frac{n}{p})}$ compared with $2^{J(s-\frac{n}{p})}$ and for $\overline{f}^s_{p,q}$ in addition to compensate estimates for related maximal functions. We recall the typical argument. Let $J > j$,

$$Q_{J,M} \subset Q_{j,m}, \qquad x \in Q_{j,m}, \qquad w = \min(1,p,q). \tag{2.78}$$

Let $\mathcal{M}$ be the usual *Hardy-Littlewood maximal function*, which we introduced in Section 1.2.1. Then the estimate

$$\chi_{j,m}(x) \le c\, 2^{(J-j)\frac{n}{w}} \big(\mathcal{M}\chi_{J,M}\big)(x)^{1/w} \tag{2.79}$$

for the related characteristic functions follows from

$$(\mathcal{M}\chi_{J,M})(x)^{1/w} \ge c\Big(\frac{1}{|Q_{j,m}|}\int_{Q_{J,M}} \mathrm{d}y\Big)^{1/w} = c\, 2^{-(J-j)\frac{n}{w}}, \qquad x \in Q_{j,m}. \tag{2.80}$$

Hence $\varkappa$ in Definition 2.17 must be large enough to ensure (2.45) in these cases.

2.3.4 Applications and Examples

Recall that

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+, \qquad 0 < p \le \infty. \tag{2.81}$$

We consider first the unweighted sequence space $\bar{b}^s_{p,q}$ defined by (2.12).

Proposition 2.26. *Let $0 < p,q \le \infty$, $s \in \mathbb{R}$. Then $\bar{b}^s_{p,q}$ is a $\varkappa$-sequence space for any $\varkappa$*

$$\varkappa > \max\left(s, \frac{n}{p} - s\right). \tag{2.82}$$

Proof. *Step 1.* Note that (2.82) also implies

$$\varkappa > \max\left(s, \sigma_p - s\right) \ge 0. \tag{2.83}$$

This can be seen as follows. We explicate (2.44) and obtain

$$|\lambda_{j,m}| \;\le\; C_1 \sum_{J=0}^{j} 2^{-\varkappa(j-J)} \sum_{M\in I^j_J(m)} |\mu_{J,M}| + C_1 \sum_{J>j} 2^{-(J-j)(\varkappa+n)} \sum_{M\in I^j_J(m)} |\mu_{J,M}|. \tag{2.84}$$

If $0 < p \leq 1$, then this can be continued immediately by

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p \\
&\leq\ C_1 \sum_{J=0}^{j} 2^{-(j-J)\varkappa p} \sum_{m\in\mathbb{Z}^n} \sum_{M\in I_J^j(m)} |\mu_{J,M}|^p + C_1 \sum_{J>j} 2^{-(J-j)(\varkappa+n)p} \sum_{m\in\mathbb{Z}^n} \sum_{M\in I_J^j(m)} |\mu_{J,M}|^p \\
&\leq C_1 \sum_{J=0}^{j} 2^{-(j-J)\varkappa p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \underbrace{\sum_{m\in I_j^J(M)} 1}_{\sim 2^{n(j-J)}} + C_1 \sum_{J>j} 2^{-(J-j)(\varkappa+n)p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \underbrace{\sum_{m\in I_j^J(M)} 1}_{\sim 1} \\
&\leq C_2 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\frac{n}{p})p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p + C_2 \sum_{J>j} 2^{-(J-j)(\varkappa+n)p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p. \qquad (2.85)
\end{aligned}$$

If $1 < p < \infty$, then applying Hölder's inequality twice yields for some $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{J=0}^{j} 2^{-\varkappa(j-J)} \sum_{M\in I_J^j(m)} |\mu_{J,M}| \\
&\leq C_1 \left(\sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon)p} \Big(\sum_{M\in I_J^j(m)} |\mu_{J,M}| \Big)^p \right)^{1/p} \underbrace{\left(\sum_{J=0}^{j} 2^{-(j-J)\varepsilon p'} \right)^{1/p'}}_{<\infty} \\
&\leq c_1 \left(\sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon)p} \sum_{M\in I_J^j(m)} |\mu_{J,M}|^p \underbrace{\Big(\sum_{M\in I_J^j(m)} 1^{p'} \Big)^{p/p'}}_{\sim 1} \right)^{1/p} \\
&\leq c_2 \left(\sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon)p} \sum_{M\in I_J^j(m)} |\mu_{J,M}|^p \right)^{1/p}, \qquad (2.86)
\end{aligned}$$

where we also applied (2.43). On the other hand, when $J > j$, similar arguments lead to

$$\begin{aligned}
&\sum_{J>j} 2^{-(J-j)(\varkappa+n)} \sum_{M\in I_J^j(m)} |\mu_{J,M}| \\
&\leq c_3 \left(\sum_{J>j} 2^{-(J-j)(\varkappa+n-\varepsilon)p} \sum_{M\in I_J^j(m)} |\mu_{J,M}|^p \underbrace{\Big(\sum_{M\in I_J^j(m)} 1^{p'} \Big)^{p/p'}}_{\sim 2^{n(J-j)(p-1)}} \right)^{1/p} \\
&= c_4 \left(\sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p})p} \sum_{M\in I_J^j(m)} |\mu_{J,M}|^p \right)^{1/p}. \qquad (2.87)
\end{aligned}$$

Combining (2.86) and (2.87), the counterpart of (2.85) for $1 < p < \infty$ thus reads as

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p \le C_3 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon-\frac{n}{p})p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \\ + C_3 \sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p})p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p. \end{aligned} \tag{2.88}$$

Using the notation (2.81) we can unify (2.85) and (2.88) by

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p \le C_3 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon-\frac{n}{p})p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \\ + C_3 \sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p}-\sigma_p)p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p. \end{aligned} \tag{2.89}$$

Now assume first $0 < q \le p$, then as before,

$$\begin{aligned} & 2^{j(s-\frac{n}{p})q} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p \right)^{q/p} \\ \le\ & C_4 \sum_{J=0}^{j} 2^{(j-J)(s-\frac{n}{p})q}\, 2^{-(j-J)(\varkappa-\varepsilon-\frac{n}{p})q}\, 2^{J(s-\frac{n}{p})q} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \right)^{q/p} \\ & + C_4 \sum_{J>j} 2^{(j-J)(s-\frac{n}{p})q}\, 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p}-\sigma_p)q}\, 2^{J(s-\frac{n}{p})q} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \right)^{q/p} \\ =\ & C_4 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon-s)q}\, 2^{J(s-\frac{n}{p})q} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \right)^{q/p} \\ & + C_4 \sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+s-\sigma_p)q}\, 2^{J(s-\frac{n}{p})q} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \right)^{q/p}, \end{aligned}$$

such that finally

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p \right)^{q/p} \\
\leq\ & C_4 \sum_{j=0}^{\infty} \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon-s)q}\, 2^{J(s-\frac{n}{p})q} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \right)^{q/p} \\
&+ C_4 \sum_{j=0}^{\infty} \sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+s-\sigma_p)q}\, 2^{J(s-\frac{n}{p})q} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \right)^{q/p} \\
\leq\ & C_5 \sum_{J=0}^{\infty} 2^{J(s-\frac{n}{p})q} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \right)^{q/p} \underbrace{\sum_{j\geq J} 2^{-(j-J)(\varkappa-\varepsilon-s)q}}_{<\infty} \\
&+ C_5 \sum_{J=0}^{\infty} 2^{J(s-\frac{n}{p})q} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p \right)^{q/p} \underbrace{\sum_{j<J} 2^{-(J-j)(\varkappa-\varepsilon+s-\sigma_p)q}}_{<\infty} \\
\leq\ & C_6 \left\| \mu | \bar{b}^s_{p,q} \right\|^q ,
\end{aligned}$$

where we may always choose ε such that

$$0 < \varepsilon < \min\left(\varkappa - s, \varkappa + s - \sigma_p\right) = \varkappa - \max\left(s, \sigma_p - s\right)$$

in view of (2.83). This gives (2.45) for $a = \bar{b}^s_{p,q}$. In case of $p < q < \infty$ and also for $p < \infty, q = \infty$ the argument is similar, we make use of Hölder's inequality again and may choose ε sufficiently small such that, say, $2\varepsilon < \varkappa - \max\left(s, \sigma_p - s\right)$. The case $p = \infty$, $q = \infty$ can be handled analogously with the use of Remark 2.19, see in the following,

$$\begin{aligned}
\sup_{m\in\mathbb{Z}^n} |\lambda_{j,m}| &\overset{(2.47)}{\leq} C_1' \sum_{J=0}^{j} 2^{-(j-J)\varkappa} \sup_{m\in\mathbb{Z}^n} \max_{M\in I^j_J(m)} |\mu_{J,M}| \\
&\quad + C_1' \sum_{J>j} 2^{-(J-j)(\varkappa+n)} \sup_{m\in\mathbb{Z}^n} 2^{n(J-j)} \max_{M\in I^j_J(m)} |\mu_{J,M}| \\
&\leq C_2' \sum_{J=0}^{j} 2^{-(j-J)\varkappa} \sup_{M\in\mathbb{Z}^n} |\mu_{J,M}| + C_2' \sum_{J>j} 2^{-(J-j)\varkappa} \sup_{M\in\mathbb{Z}^n} |\mu_{J,M}|.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sup_{j\in\mathbb{N}_0} 2^{js} \sup_{m\in\mathbb{Z}^n} |\lambda_{j,m}| \le\ & C_1' \sup_{j\in\mathbb{N}_0} \sum_{J=0}^{j} 2^{(j-J)s}\, 2^{-(j-J)\varkappa}\, 2^{Js} \sup_{M\in\mathbb{Z}^n} |\mu_{J,M}| \\
&+ C_1' \sup_{j\in\mathbb{N}_0} \sum_{J>j} 2^{(j-J)s}\, 2^{-(J-j)\varkappa}\, 2^{Js} \sup_{M\in\mathbb{Z}^n} |\mu_{J,M}| \\
\le\ & C_1' \sup_{J\in\mathbb{N}_0} 2^{Js} \sup_{M\in\mathbb{Z}^n} |\mu_{J,M}| \sum_{j\ge J} 2^{-(j-J)(\varkappa-s)} \\
&+ C_1' \sup_{J\in\mathbb{N}_0} 2^{Js} \sup_{M\in\mathbb{Z}^n} |\mu_{J,M}| \sum_{j<J} 2^{-(J-j)(\varkappa+s)} \\
\le\ & C_2' \, ||\mu|\bar{b}^s_{\infty,\infty}||,
\end{aligned}$$

where we have to choose

$$\varkappa > \max(-s, s) = |s|.$$

Step 2. It remains to verify (ii), that is, (2.46). By the monotonicity of the spaces $\bar{b}^s_{p,q}$ in $0 < q \le \infty$, it is sufficient to show

$$2^{-j\varkappa}|\mu_{j,m}| \le c_Q \left\|\mu|\bar{b}^s_{p,\infty}(\mathbb{R}^n)\right\| \sim c_Q \sup_{\nu\in\mathbb{N}_0} 2^{\nu(s-\frac{n}{p})} \left(\sum_{k\in\mathbb{Z}^n} |\mu_{\nu,k}|^p\right)^{1/p}$$

for all $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ with $Q_{j,m} \subset Q$. Let ℓ_Q be the side-length of the cube Q, then the assumption $Q_{j,m} \subset Q$ gives the rough estimate $2^j \ge \ell_Q^{-1}$. Hence

$$\begin{aligned}
\sup_{\nu\in\mathbb{N}_0} 2^{\nu(s-\frac{n}{p})} \left(\sum_{k\in\mathbb{Z}^n} |\mu_{\nu,k}|^p\right)^{1/p} &\ge 2^{j(s-\frac{n}{p})}|\mu_{j,m}| = 2^{j(s-\frac{n}{p}+\varkappa)}2^{-j\varkappa}|\mu_{j,m}| \\
&\ge \ell_Q^{-(s-\frac{n}{p}+\varkappa)} 2^{-j\varkappa}|\mu_{j,m}| =: \frac{1}{c_Q} 2^{-j\varkappa}|\mu_{j,m}|
\end{aligned}$$

if $\varkappa \ge \frac{n}{p} - s$. ∎

We now consider the weighted spaces $\bar{b}^s_{p,q}(w)$ where w is some Muckenhoupt weight.

Proposition 2.27. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty$. Then $\bar{b}^s_{p,q}(w)$ is a $\varkappa$-sequence space for any $\varkappa$*

$$\varkappa > \max\left(s + \frac{n}{p}, \frac{n\, r_w}{p} - s\right). \tag{2.90}$$

Proof. *Step 1.* We return to (2.84) and include the weight terms according to (2.10),

$$
\begin{aligned}
|\lambda_{j,m}|\; w(Q_{j,m})^{\frac{1}{p}} \le\; & C_1 \sum_{J=0}^{j} 2^{-\varkappa(j-J)} \sum_{M\in I_J^j(m)} |\mu_{J,M}|\; \underbrace{w(Q_{j,m})^{\frac{1}{p}}}_{\le w(Q_{J,M})^{\frac{1}{p}}} \\
& + C_1 \sum_{J>j} 2^{-(J-j)(\varkappa+n)} \sum_{M\in I_J^j(m)} |\mu_{J,M}|\; w(Q_{J,M})^{\frac{1}{p}} \underbrace{\left(\frac{w(Q_{j,m})}{w(Q_{J,M})}\right)^{\frac{1}{p}}}_{\overset{(1.20)}{\le} 2^{-(j-J)nr\cdot\frac{1}{p}}} \\
\le\; & C_2 \sum_{J=0}^{j} 2^{-\varkappa(j-J)} \sum_{M\in I_J^j(m)} |\mu_{J,M}|\; w(Q_{J,M})^{\frac{1}{p}} \\
& + C_2 \sum_{J>j} 2^{-(J-j)(\varkappa+n-n\frac{r}{p})} \sum_{M\in I_J^j(m)} |\mu_{J,M}|\; w(Q_{J,M})^{\frac{1}{p}},
\end{aligned}
$$

where we used $w \ge 0$ a.e. in $\mathbb{R}^n$ for the first term, and (1.20) for the second term with $r > r_w$. Proceeding now as above we arrive at the counterpart of (2.89),

$$
\begin{aligned}
& 2^{jsp} \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p\; w(Q_{j,m}) \\
\le\; & C_3 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon-s-\frac{n}{p})p}\; 2^{Jsp} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p w(Q_{J,M}) \\
& + C_3 \sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p}-n\frac{r}{p}+s-\sigma_p)p}\; 2^{Jsp} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p w(Q_{J,M}). \qquad (2.91)
\end{aligned}
$$

The rest of the argumentation is now the same as in the proof of Proposition 2.26, that is, application of Hölder's inequality (when $q > p$) or monotonicity (when $q \le p$), such that we arrive at

$$
\left\|\lambda|\bar{b}^s_{p,q}(w)\right\| \le\; C\; \left\|\mu|\bar{b}^s_{p,q}(w)\right\|
$$

assuming that

$$
\begin{aligned}
0 < \varepsilon &< \min\left(\varkappa - s - \frac{n}{p}, \varkappa + s - \sigma_p - n\,\frac{r_w-1}{p}\right) \\
&= \varkappa - \max\left(s + \frac{n}{p}, \sigma_p - s + n\,\frac{r_w-1}{p}\right). \qquad (2.92)
\end{aligned}
$$

Note that (2.90) implies (2.92). Analogously to the unweighted case we obtain the same result for $q = \infty$. For $p = \infty, q = \infty$ the spaces $\bar{b}^s_{p,q}(w)$ coincide with the unweighted $\bar{b}^s_{p,q}$. So we do not consider it here.

Step 2. As for (ii), that is, (2.46), we may again restrict ourselves to the case $q = \infty$ by monotonicity, that is, it is sufficient to show

$$
2^{-j\varkappa}|\mu_{j,m}| \le c_Q \left\|\mu|\bar{b}^s_{p,\infty}(w)\right\| \sim c_Q \sup_{\nu\in\mathbb{N}_0} 2^{\nu s} \left(\sum_{k\in\mathbb{Z}^n} |\mu_{\nu,k}|^p\; w(Q_{\nu,k})\right)^{1/p}
$$

for all $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ with $Q_{j,m} \subset Q$. Let again ℓ_Q be the side-length of the cube Q, then $Q_{j,m} \subset Q$ implies $2^j \geq \ell_Q^{-1}$. Hence

$$\begin{aligned}
&\sup_{\nu\in\mathbb{N}_0} 2^{\nu s} \left(\sum_{k\in\mathbb{Z}^n} |\mu_{\nu,k}|^p \, w(Q_{\nu,k}) \right)^{\frac{1}{p}} \\
&\geq\ 2^{js}|\mu_{j,m}| \, w(Q_{j,m})^{\frac{1}{p}} = 2^{j(s+\varkappa)}2^{-j\varkappa}|\mu_{j,m}| \left(\frac{w(Q_{j,m})}{w(Q)} \right)^{\frac{1}{p}} w(Q)^{\frac{1}{p}} \\
&\geq\ c\, 2^{j(s+\varkappa-\frac{nr}{p})} \, w(Q)^{\frac{1}{p}} |Q|^{-\frac{r}{p}} \, 2^{-j\varkappa}|\mu_{j,m}| \\
&\geq\ \ell_Q^{-(s-\frac{nr}{p}+\varkappa)} \, w(Q)^{\frac{1}{p}} |Q|^{-\frac{r}{p}} \, 2^{-j\varkappa}|\mu_{j,m}| \ =: c_Q^{-1} \, 2^{-j\varkappa}|\mu_{j,m}|
\end{aligned}$$

if $\varkappa \geq \frac{nr}{p} - s$, where we applied (1.20) with $r > r_w$ again. So we arrive at

$$\varkappa > \max\left(s + \frac{n}{p}, \frac{n\, r_w}{p} - s \right).$$

■

In the next example we consider the weighted sequence spaces $\bar{b}^s_{p,q}(w)$ where w is a doubling weight.

Proposition 2.28. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and w be a doubling weight with doubling constant γ. Then $\bar{b}^s_{p,q}(w)$ is a $\varkappa$-sequence space for any $\varkappa$*

$$\varkappa > \max\left(s + \frac{n}{p}, \frac{n\,\gamma}{p} - s \right). \tag{2.93}$$

Proof. *Step 1.* Analogously to the unweighted case we have

$$|\lambda_{j,m}|^p \leq c_1 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon)p} \sum_{M\in I^j_J(m)} |\mu_{J,M}|^p + c_1 \sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p}-\sigma_p)p} \sum_{M\in I^j_J(m)} |\mu_{J,M}|^p.$$

Always we have to consider the two cases $J \leq j$ and $J > j$.
For $J \leq j$ it is clear: $w(Q_{j,m}) \leq w(Q_{J,M})$ (maybe with some constant because of the overlap in $I^j_J(m)$).
For $J > j$ we blow the cube $Q_{J,M}$ l-times up until we cover the cube $Q_{j,m}$. We choose $l = J - j + 1$. Then we have with the use of the doubling property (1.40)

$$w(Q_{j,m}) \leq w(2^l\, Q_{J,M}) \leq 2^{ln\gamma} w(Q_{J,M}). \tag{2.94}$$

Thus we get

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p\ w(Q_{j,m}) \le\ & c_2 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon-\frac{n}{p})p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p w(Q_{J,M}) \\
& + c_2 \sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p}-\sigma_p)p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p\, 2^{(J-j)n\gamma}\, 2^{n\gamma} w(Q_{J,M}) \\
\le\ & c_3 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-\varepsilon-\frac{n}{p})p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p w(Q_{J,M}) \\
& + c_3 \sum_{J>j} 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p}-\sigma_p-\frac{n\gamma}{p})p} \sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p w(Q_{J,M}). \qquad (2.95)
\end{aligned}
$$

Similar to the unweighted case we have to use Hölder's inequality again (when $q > p$) or monotonicity (when $q \le p$). Finally we get

$$
\begin{aligned}
& \sum_{j=0}^{\infty} 2^{jsq} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p\ w(Q_{j,m}) \right)^{q/p} \\
\le\ & c_4 \sum_{J=0}^{\infty} 2^{Jsq} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p w(Q_{J,M}) \right)^{q/p} \sum_{j\ge J} 2^{-(j-J)(\varkappa-\varepsilon-s-\frac{n}{p})q} \\
& + c_4 \sum_{J=0}^{\infty} 2^{Jsq} \left(\sum_{M\in\mathbb{Z}^n} |\mu_{J,M}|^p w(Q_{J,M}) \right)^{q/p} \sum_{j<J} 2^{-(J-j)(\varkappa-\varepsilon+\frac{n}{p}+s-\sigma_p-\frac{n\gamma}{p})q} \\
\le\ & c_5 ||\mu|\bar{b}^s_{p,q}(w)||^q, \qquad (2.96)
\end{aligned}
$$

where we assume ε such that

$$
\begin{aligned}
0 < \varepsilon < \min & \left(\varkappa - s - \frac{n}{p}, \varkappa + s - \sigma_p - \frac{n}{p}(\gamma-1) \right) \\
&= \varkappa - \max\left(s + \frac{n}{p}, \sigma_p - s + \frac{n}{p}(\gamma-1) \right). \qquad (2.97)
\end{aligned}
$$

Note again that (2.93) implies (2.97).

Step 2. Now let us prove (ii).We may again restrict ourselves to the case $q = \infty$ by monotonicity. Let Q be an arbitrary cube with sidelength ℓ_Q. Let $Q_{j,m} \subset Q$ for fixed $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$. This implies $2^j \ge \ell_Q^{-1}$. We blow $Q_{j,m}$ l-times up to cover the cube Q, i.e., $2^l Q_{j,m} = Q_{j-l,2^{-l}m} \supset Q$, where we assume $2^{-j+l} \ge 2\ell_Q$. So we choose $l = \lfloor \log_2(\ell_Q) \rfloor + j + 1$.

Because of $Q_{j,m} \subset Q$, the number l is always natural. Hence

$$\begin{aligned}
||\mu|\bar{b}^s_{p,\infty}(w)|| &= \sup_{\nu\in\mathbb{N}_0} 2^{\nu s}\left(\sum_{k\in\mathbb{Z}^n} |\mu_{\nu,k}|^p\ w(Q_{\nu,k})\right)^{\frac{1}{p}} \\
&\geq 2^{j(s+\varkappa)} 2^{-j\varkappa}|\mu_{j,m}| w(Q_{j,m})^{\frac{1}{p}} \\
&\geq 2^{j(s+\varkappa)}\ 2^{-n\gamma l\frac{1}{p}} w(2^l\ Q_{j,m})^{\frac{1}{p}}\ 2^{-j\varkappa}|\mu_{j,m}| \\
&\geq 2^{j(s+\varkappa-\frac{n\gamma}{p})}\ 2^{-\frac{n\gamma}{p}(\lfloor\log_2(\ell_Q)\rfloor+1)} w(Q)^{\frac{1}{p}}\ 2^{-j\varkappa}|\mu_{j,m}| \\
&\geq \ell_Q^{-(s+\varkappa-\frac{n\gamma}{p})}\ 2^{-\frac{n\gamma}{p}(\lfloor\log_2(\ell_Q)\rfloor+1)} w(Q)^{\frac{1}{p}}\ 2^{-j\varkappa}|\mu_{j,m}| \\
&=: c_Q^{-1}\ 2^{-j\varkappa}|\mu_{j,m}|
\end{aligned}$$

if $\varkappa \geq \frac{n\gamma}{p} - s$. This together with (2.97) yields (2.93). ■

Now let us consider the $\bar{f}^s_{p,q}$-spaces. Therefore we need some preliminary considerations. Recall that $\mathcal{M}$ stands for the *Hardy-Littlewood maximal operator*

$$(\mathcal{M}g)(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |g(y)|\,\mathrm{d}y,$$

where the supremum is taken over all cubes containing x and g is a locally integrable function. In this situation we need cubes. We refer to Section 1.2.1.
Our later arguments rely on the vector-valued maximal inequality of Fefferman-Stein due to [FS71].

Proposition 2.29. *Let* $0 < p < \infty$, $0 < q \leq \infty$, $0 < \varrho < \min(p,q)$. *Then there exists a constant* C *such that*

$$\left\|\left(\sum_{k=0}^{\infty} \mathcal{M}(|g_k|^{\varrho})(\cdot)^{q/\varrho}\right)^{1/q} |L_p(\mathbb{R}^n)\right\| \leq c \left\|\left(\sum_{k=0}^{\infty} |g_k(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\|. \tag{2.98}$$

For a proof we refer to [FS71].
In view of the $\varkappa$-condition for doubling weighted $\bar{f}^s_{p,q}(w)$-spaces we need a little modification of the maximal operator $\mathcal{M}$.

Definition 2.30. Let w be a doubling weight and $g \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The *weighted Hardy-Littlewood maximal operator* $\mathcal{M}_w$ is defined by

$$(\mathcal{M}_w g)(x) = \sup_{Q\ni x} \frac{1}{w(Q)}\int_Q |g(y)|w(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{2.99}$$

where the supremum is taken over all open cubes Q containing x.

For this operator $\mathcal{M}_w$ exists a modified weighted vector-valued maximal inequality of Fefferman-Stein.

Lemma 2.31. *Let $1 < p < \infty$, $1 < q \leq \infty$ and w be a doubling weight. Then there exists a constant C such that*

$$\left\| \left(\sum_{k=0}^{\infty} |\mathcal{M}_w g_k|^q \right)^{1/q} |L_p(w) \right\| \leq C \left\| \left(\sum_{k=0}^{\infty} |g_k|^q \right)^{1/q} |L_p(w) \right\| \tag{2.100}$$

holds for any $(g_k)_k \subset L_p(w)$.

For a proof we refer to [Bow07, Prop. 2.8].
An immediate conclusion of this lemma is the following corollary.

Corollary 2.32. *Let $0 < p < \infty$, $0 < q \leq \infty$, $0 < \varrho < \min(p, q)$ and w be a doubling weight. Then there exists a constant C such that*

$$\left\| \left(\sum_{k=0}^{\infty} \mathcal{M}_w \big(|g_k|^\varrho \big)(\cdot)^{q/\varrho} \right)^{1/q} |L_p(w) \right\| \leq c \left\| \left(\sum_{k=0}^{\infty} |g_k|^q \right)^{1/q} |L_p(w) \right\|. \tag{2.101}$$

Proof. Since $0 < \varrho < \min(p, q)$, it follows that $1 < \frac{p}{\varrho} < \infty$ and $1 < \frac{q}{\varrho} \leq \infty$. Then Corollary 2.32 is a consequence of Lemma 2.31 and $\tilde{g}_k := |g_k|^\varrho$. ∎

Now let us consider the $\varkappa$-condition for the $\bar{f}^s_{p,q}$-space. We start with the unweighted case.

Proposition 2.33. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. Then $\bar{f}^s_{p,q}$ is a $\varkappa$-sequence space for any $\varkappa$*

$$\varkappa > \max \left(s, \frac{n}{p} - s, \sigma_{p,q} - s \right), \tag{2.102}$$

where $\sigma_{p,q}$ is given by

$$\sigma_{p,q} = n \Big(\frac{1}{\min(p, q)} - 1 \Big)_+ = \frac{n}{\min(1, p, q)} - n. \tag{2.103}$$

Proof. *Step 1.* In the first step we have to prove (2.45) under the assumption of (2.44). This part is based on the vector-valued maximal inequality of Fefferman-Stein, cf. (2.98). We return to (2.84)

$$|\lambda_{j,m}| \leq C_1 \sum_{J=0}^{j} 2^{-\varkappa(j-J)} \sum_{M \in I^j_J(m)} |\mu_{J,M}| + C_1 \sum_{J>j} 2^{-(J-j)(\varkappa+n)} \sum_{M \in I^j_J(m)} |\mu_{J,M}|.$$

Let first $J \leq j$. We assume $q < \infty$ and $\varepsilon > 0$, then we obtain (by Hölders inequality or monotonicity),

$$2^{jsq} |\lambda_{j,m}|^q \chi_{j,m}(x) \leq C_2 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-s-\varepsilon)q} \sum_{M \in I^j_J(m)} 2^{Jsq} |\mu_{J,M}|^q \chi_{j,m}(x).$$

Summation over $m \in \mathbb{Z}^n$ delivers

$$\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q \chi_{j,m}(x) \le C_3 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-s-\varepsilon)q} \sum_{M\in\mathbb{Z}^n} 2^{Jsq}|\mu_{J,M}|^q \sum_{m\in\mathbb{Z}^n : M\in I^j_J(m)} \chi_{j,m}(x).$$

Analogously to the proof of Thm. 1.15 in [Tri08], see page 11, we argue that for fixed $x \in \mathbb{R}^n$, $j, J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^n$ the summation $\sum \chi_{j,m}(x)$ over those $m \in \mathbb{Z}^n$ with $M \in I^j_J(m)$ is comparable with $\chi_{J,M}(x)$ and can be estimated from above by its maximal function. Hence we obtain for any $\varrho > 0$,

$$\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q \chi_{j,m}(x) \le C_4 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-s-\varepsilon)q} \sum_{M\in\mathbb{Z}^n} \mathcal{M}\Big(2^{Js\varrho}|\mu_{J,M}|^\varrho \chi_{J,M}(\cdot)\Big)(x)^{q/\varrho}. \tag{2.104}$$

As for the case $J > j$, we have

$$2^{js}|\lambda_{j,m}|\chi_{j,m}(x) \le C_2\, \chi_{j,m}(x) \sum_{J>j} 2^{-(J-j)(\varkappa+s+n)} \sum_{M\in I^j_J(m)} 2^{Js}|\mu_{J,M}|.$$

Assume $0 < \varrho < 1$ and $x \in \mathbb{R}^n$ with $\chi_{j,m}(x) = 1$, then we can estimate the last sum by

$$\begin{aligned}
\Big(\sum_{M\in I^j_J(m)} 2^{Js}|\mu_{J,M}|\Big)^\varrho &\le \sum_{M\in I^j_J(m)} 2^{Js\varrho}|\mu_{J,M}|^\varrho \cdot \frac{1}{|Q_{J,M}|}\int_{\mathbb{R}^n} \chi_{J,M}(y)\,\mathrm{d}y \\
&\le c\, 2^{Jn}\, 2^{-jn}\, 2^{jn} \int_{\mathbb{R}^n} \sum_{M\in I^j_J(m)} 2^{Js\varrho}|\mu_{J,M}|^\varrho \chi_{J,M}(y)\,\mathrm{d}y \\
&\le c'\, 2^{(J-j)n}\, \mathcal{M}\Big(\sum_{M\in I^j_J(m)} 2^{Js\varrho}|\mu_{J,M}|^\varrho \chi_{J,M}(\cdot)\Big)(x).
\end{aligned}$$

Insert this above. Assuming again $q < \infty$ one obtains for any fixed $\varepsilon > 0$ that

$$2^{jsq}|\lambda_{j,m}|^q \chi_{j,m}(x) \le c_3 \sum_{J>j} 2^{-(J-j)(\varkappa+s+n-\frac{n}{\varrho}-\varepsilon)q} \mathcal{M}\Big(\sum_{M\in I^j_J(m)} 2^{Js\varrho}|\mu_{J,M}|^\varrho \chi_{J,M}(\cdot)\Big)(x)^{q/\varrho}.$$

First we consider the summation over $m \in \mathbb{Z}^n$. With $J = j + t$ we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q \chi_{j,m}(x) \\
&\le\ c_3 \sum_{t=1}^{\infty} 2^{-t(\varkappa+s+n-\frac{n}{\varrho}-\varepsilon)q} \sum_{m\in\mathbb{Z}^n} \mathcal{M}\Big(\sum_{M\in I^j_{j+t}(m)} 2^{(j+t)s\varrho}|\mu_{j+t,M}|^\varrho \chi_{j+t,M}(\cdot)\Big)(x)^{q/\varrho}.
\end{aligned} \tag{2.105}$$

Let us consider the sum

$$\sum_{M\in I^j_{j+t}(m)} \big(2^{(j+t)s}|\mu_{j+t,M}|\chi_{j+t,M}(x)\big)^\varrho$$

for fixed $j \in \mathbb{N}_0$, $t \in \mathbb{N}$, $m \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. The cube $Q_{j,m}$ is divided in the cubes $Q_{j+t,M}$. Note, that the „small“ cubes $Q_{j+t,M}$ are all disjoint to each other. So we have

$$\begin{aligned} &\sum_{M \in I^j_{j+t}(m)} \left(2^{(j+t)s}|\mu_{j+t,M}|\chi_{j+t,M}(x)\right)^\varrho \\ =&\Big(\sum_{M \in I^j_{j+t}(m)} 2^{(j+t)s}|\mu_{j+t,M}|\chi_{j+t,M}(x)\Big)^\varrho =: g^t_{j,m}(x)^\varrho. \end{aligned}$$

Thus we get with (2.104) and (2.105) together and summation over j

$$\begin{aligned} &\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q\chi_{j,m}(x) \\ \leq\ & C_5 \sum_{J=0}^{\infty}\sum_{M\in\mathbb{Z}^n} \mathcal{M}\Big(2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}(\cdot)\Big)(x)^{q/\varrho} \underbrace{\sum_{j\geq J} 2^{-(j-J)(\varkappa-s-\varepsilon)q}}_{<\infty} \\ &+ c_4 \sum_{t=1}^{\infty} 2^{-t(\varkappa+s+n-\frac{n}{\varrho}-\varepsilon)q} \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \mathcal{M}\big(g^t_{j,m}(\cdot)^\varrho\big)(x)^{q/\varrho}, \end{aligned}$$

with $0 < \varepsilon < \varkappa - s$. Finally we have

$$\begin{aligned} ||\lambda|\bar{f}^s_{p,q}|| \leq\ & C_6 \Big\|\Big(\sum_{J=0}^{\infty}\sum_{M\in\mathbb{Z}^n} \mathcal{M}\Big(2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}\Big)(\cdot)^{q/\varrho}\Big)^{1/q}\Big|L_p\Big\| \\ &+ c_5 \Big\|\Big(\sum_{t=1}^{\infty} 2^{-t(\varkappa+s+n-\frac{n}{\varrho}-\varepsilon)q} \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \mathcal{M}\big((g^t_{j,m})^\varrho\big)(\cdot)^{q/\varrho}\Big)^{1/q}\Big|L_p\Big\|. \end{aligned}$$

With an additional use of Hölders inequality (for $0 < q < 1$) and an additional ε we can take the sum over t out of the L_p-norm,

$$\begin{aligned} ||\lambda|\bar{f}^s_{p,q}|| \leq\ & C_6 \Big\|\Big(\sum_{J=0}^{\infty}\sum_{M\in\mathbb{Z}^n} \mathcal{M}\Big(2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}\Big)(\cdot)^{q/\varrho}\Big)^{1/q}\Big|L_p\Big\| \\ &+ c_6 \sum_{t=1}^{\infty} 2^{-t(\varkappa+s+n-\frac{n}{\varrho}-2\varepsilon)} \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \mathcal{M}\big((g^t_{j,m})^\varrho\big)(\cdot)^{q/\varrho}\Big)^{1/q}\Big|L_p\Big\|. \end{aligned}$$

Since $0 < \varrho < \min(1, p, q)$ we can apply the vector-valued maximal inequality due to

Fefferman-Stein from (2.98).

$$\begin{aligned}
&||\lambda|\bar{f}^s_{p,q}|| \\
\leq\ & C_7 \Big\| \Big(\sum_{J=0}^{\infty} \sum_{M\in\mathbb{Z}^n} 2^{Jsq} |\mu_{J,M}|^q \chi_{J,M}(\cdot) \Big)^{1/q} \Big| L_p \Big\| \\
&+ c_7 \sum_{t=1}^{\infty} 2^{-t(\varkappa+s+n-\frac{n}{\varrho}-2\varepsilon)} \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \underbrace{\Big(\sum_{M\in I^j_{j+t}(m)} 2^{(j+t)s} |\mu_{j+t,M}| \chi_{j+t,M}(\cdot) \Big)^q}_{= \sum_{M\in I^j_{j+t}(m)} 2^{(j+t)sq} |\mu_{j+t,M}|^q \chi_{j+t,M}(\cdot)} \Big)^{1/q} \Big| L_p \Big\| \\
=\ & C_7 ||\mu|\bar{f}^s_{p,q}|| + c_7 \underbrace{\sum_{t=1}^{\infty} 2^{-t(\varkappa+s+n-\frac{n}{\varrho}-2\varepsilon)}}_{<\infty} \Big\| \Big(\sum_{J=0}^{\infty} \sum_{M\in\mathbb{Z}^n} 2^{Jsq} |\mu_{J,M}|^q \chi_{J,M}(\cdot) \Big)^{1/q} \Big| L_p \Big\| \\
\leq\ & C_8 ||\mu|\bar{f}^s_{p,q}||,
\end{aligned}$$

where $0 < 2\varepsilon < \varkappa + s + n - \frac{n}{\varrho}$ and $0 < \varrho < \min(1, p, q)$. Recall, that we have additionally $0 < \varepsilon < \varkappa - s$. Finally we choose

$$\varkappa > \max(s, \sigma_{p,q} - s). \tag{2.106}$$

Step 2. The second part of Definition 2.17 is easy to show. It holds $\bar{f}^s_{p,q} \hookrightarrow \bar{b}^s_{p,\infty}$. So we have here the same condition for $\varkappa$ as in the $\bar{b}$-case,

$$\varkappa > \frac{n}{p} - s. \tag{2.107}$$

Both (2.106) and (2.107) together lead to (2.102). With some modifications and a similar proof we obtain the same result for $q = \infty$. ■

In our next example we consider the doubling weighted $\bar{f}^s_{p,q}(w)$-space.

Proposition 2.34. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and w be a doubling weight with doubling constant γ. Then $\bar{f}^s_{p,q}(w)$ is a $\varkappa$-sequence space for any $\varkappa$*

$$\varkappa > \max(s, \gamma\,\sigma_{p,q} + (\gamma - 1)n - s, \frac{n\gamma}{p} - s). \tag{2.108}$$

Proof. We recall the norm for the weighted $\bar{f}^s_{p,q}(w)$ from Remark 2.8

$$\big\|\lambda|\bar{f}^s_{p,q}(w)\big\| = \Bigg\| \Bigg(\sum_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n} 2^{jsq} |\lambda_{j,m}|^q \chi_{j,m}(\cdot) \Bigg)^{1/q} \Bigg| L_p(w) \Bigg\|.$$

As usual we split into two cases, $J \leq j$ and $J > j$. Let first $J \leq j$. Analogously to the unweighted case we have for $q < \infty$ and $\varepsilon > 0$

$$\sum_{m\in\mathbb{Z}^n} 2^{jsq} |\lambda_{j,m}|^q \chi_{j,m}(x) \leq C_3 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-s-\varepsilon)q} \sum_{M\in\mathbb{Z}^n} 2^{Jsq} |\mu_{J,M}|^q \sum_{m\in\mathbb{Z}^n : M\in I^j_J(m)} \chi_{j,m}(x).$$

With similar arguments it holds, that for fixed $x \in \mathbb{R}^n$, $j, J \in \mathbb{N}_0$ and $M \in \mathbb{Z}^n$ the summation $\sum \chi_{j,m}(x)$ over $m \in \mathbb{Z}^n$ with $M \in I_J^j(m)$ is comparable with $\chi_{J,M}(x)$ and can be estimated from above by the weighted maximal function $\mathcal{M}_w$. Hence we obtain for any $\varrho > 0$,

$$\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q\chi_{j,m}(x) \le C_4 \sum_{J=0}^{j} 2^{-(j-J)(\varkappa-s-\varepsilon)q} \sum_{M\in\mathbb{Z}^n} \mathcal{M}_w\Big(2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}(\cdot)\Big)(x)^{q/\varrho}. \tag{2.109}$$

In the case $J > j$, then we have

$$2^{js}|\lambda_{j,m}|\chi_{j,m}(x) \le C_2\, \chi_{j,m}(x) \sum_{J>j} 2^{-(J-j)(\varkappa+s+n)} \sum_{M\in I_J^j(m)} 2^{Js}|\mu_{J,M}|.$$

Assume $0 < \varrho < 1$ and $x \in \mathbb{R}^n$ with $\chi_{j,m}(x) = 1$, then

$$\begin{aligned}\Big(\sum_{M\in I_J^j(m)} 2^{Js}|\mu_{J,M}|\Big)^\varrho &\le \sum_{M\in I_J^j(m)} 2^{Js\varrho}|\mu_{J,M}|^\varrho \cdot \frac{1}{w(Q_{J,M})}\int_{\mathbb{R}^n} \chi_{J,M}(y)w(y)\,\mathrm{d}y \\ &\le c\,2^{(J-j)n\gamma}\frac{1}{w(Q_{j,m})}\int_{\mathbb{R}^n} \sum_{M\in I_J^j(m)} 2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}(y)w(y)\,\mathrm{d}y \\ &\le c'\,2^{(J-j)n\gamma}\,\mathcal{M}_w\Big(\sum_{M\in I_J^j(m)} 2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}(\cdot)\Big)(x),\end{aligned}$$

where we here used in the second estimate the doubling property with (2.94). Assume again $q < \infty$ and $\varepsilon > 0$, we receive with this

$$2^{jsq}|\lambda_{j,m}|^q\chi_{j,m}(x) \le c_3 \sum_{J>j} 2^{-(J-j)(\varkappa+s+n-\frac{n\gamma}{\varrho}-\varepsilon)q}\mathcal{M}_w\Big(\sum_{M\in I_J^j(m)} 2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}(\cdot)\Big)(x)^{q/\varrho}.$$

Summation over $m \in \mathbb{Z}^n$ with $J = j + t$ yields

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q\chi_{j,m}(x) \\ \le\ & c_3 \sum_{t=1}^{\infty} 2^{-t(\varkappa+s+n-\frac{n\gamma}{\varrho}-\varepsilon)q} \sum_{m\in\mathbb{Z}^n} \mathcal{M}_w\Big(\sum_{M\in I_{j+t}^j(m)} 2^{(j+t)s\varrho}|\mu_{j+t,M}|^\varrho\chi_{j+t,M}(\cdot)\Big)(x)^{q/\varrho}.\end{aligned} \tag{2.110}$$

As in the proof of Proposition 2.33 we denote by

$$\begin{aligned}g_{j,m}^t(x)^\varrho &:= \Big(\sum_{M\in I_{j+t}^j(m)} 2^{(j+t)s}|\mu_{j+t,M}|\chi_{j+t,M}(x)\Big)^\varrho \\ &= \sum_{M\in I_{j+t}^j(m)} \big(2^{(j+t)s}|\mu_{j+t,M}|\chi_{j+t,M}(x)\big)^\varrho.\end{aligned}$$

Summation over j in (2.109) and (2.110) yields

$$\begin{aligned}
&\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} 2^{jsq}|\lambda_{j,m}|^q\chi_{j,m}(x)\\
\leq\ & C_5\sum_{J=0}^{\infty}\sum_{M\in\mathbb{Z}^n}\mathcal{M}_w\Big(2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}(\cdot)\Big)(x)^{q/\varrho}\underbrace{\sum_{j\geq J}2^{-(j-J)(\varkappa-s-\varepsilon)q}}_{<\infty}\\
&+c_4\sum_{t=1}^{\infty}2^{-t(\varkappa+s+n-\frac{n\gamma}{\varrho}-\varepsilon)q}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mathcal{M}_w\big(g^t_{j,m}(\cdot)^\varrho\big)(x)^{q/\varrho},
\end{aligned}$$

with $0<\varepsilon<\varkappa-s$. Finally we have with an additional use of Hölders inequality (for $0<q<1$) and an additional ε that

$$\begin{aligned}
||\lambda|\bar{f}^s_{p,q}(w)||\leq\ & C_6\Big\|\Big(\sum_{J=0}^{\infty}\sum_{M\in\mathbb{Z}^n}\mathcal{M}_w\Big(2^{Js\varrho}|\mu_{J,M}|^\varrho\chi_{J,M}\Big)(\cdot)^{q/\varrho}\Big)^{1/q}\Big|L_p(w)\Big\|\\
&+c_6\sum_{t=1}^{\infty}2^{-t(\varkappa+s+n-\frac{n\gamma}{\varrho}-2\varepsilon)}\Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\mathcal{M}_w\big((g^t_{j,m})^\varrho\big)(\cdot)^{q/\varrho}\Big)^{1/q}\Big|L_p(w)\Big\|.
\end{aligned}$$

We choose $0<\varrho<\min(1,p,q)$ and use Corollary 2.32, then we obtain

$$\begin{aligned}
&||\lambda|\bar{f}^s_{p,q}(w)||\\
\leq\ & C_7||\mu|\bar{f}^s_{p,q}(w)||\\
&+c_7\sum_{t=1}^{\infty}2^{-t(\varkappa+s+n-\frac{n\gamma}{\varrho}-2\varepsilon)}\Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\Big(\sum_{M\in I^j_{j+t}(m)}2^{(j+t)s}|\mu_{j+t,M}|\chi_{j+t,M}(\cdot)\Big)^q\Big)^{1/q}\Big|L_p(w)\Big\|\\
=\ & C_7||\mu|\bar{f}^s_{p,q}(w)||\\
&+c_7\underbrace{\sum_{t=1}^{\infty}2^{-t(\varkappa+s+n-\frac{n\gamma}{\varrho}-2\varepsilon)}}_{<\infty}\Big\|\Big(\sum_{J=0}^{\infty}\sum_{M\in\mathbb{Z}^n}2^{Jsq}|\mu_{J,M}|^q\chi_{J,M}(\cdot)\Big)^{1/q}\Big|L_p(w)\Big\|\\
\leq\ & C_8||\mu|\bar{f}^s_{p,q}(w)||,
\end{aligned}$$

where $0<2\varepsilon<\varkappa+s+n-\frac{n\gamma}{\varrho}$ and $0<\varrho<\min(1,p,q)$. Furthermore we still need $0<\varepsilon<\varkappa-s$. Finally we choose

$$\varkappa>\max(s,\gamma\,\sigma_{p,q}+(\gamma-1)n-s). \tag{2.111}$$

The second part of Definition 2.17 is easy to show. It holds $\bar{f}^s_{p,q}(w)\hookrightarrow\bar{b}^s_{p,\infty}(w)$. So we have here the same condition for $\varkappa$ as for the $\bar{b}^s_{p,q}(w)$-spaces,

$$\varkappa>\frac{n\gamma}{p}-s. \tag{2.112}$$

Both (2.111) and (2.112) together lead to (2.108). With some modifications and a similar proof we obtain the same result for $q=\infty$. ■

Remark 2.35. Finally we consider again Muckenhoupt weights as special case of doubling weights. One can exactly follow the proof of Proposition 2.33 with the Muckenhoupt weighted vector-valued maximal inequality instead of the unweighted vector-valued maximal inequality of Fefferman-Stein from (2.98).
Let $0 < p < \infty$, $0 < q \leq \infty$ and $w \in \mathcal{A}_\infty$ with $r_w = \inf\{r \geq 1 \ : w \in \mathcal{A}_r\}$. Furthermore let $0 < \varrho < \min(p/r_w, q)$, then holds

$$\left\| \left(\sum_{k=0}^{\infty} \mathcal{M}\big(|g_k|^\varrho\big)(\cdot)^{q/\varrho} \right)^{1/q} | L_p(w) \right\| \leq c \left\| \left(\sum_{k=0}^{\infty} |g_k(\cdot)|^q \right)^{1/q} | L_p(w) \right\|, \tag{2.113}$$

where $\mathcal{M}$ stands here for the usual (unweighted) maximal operator from (1.6). A proof of this interesting result may be found in [AJ81], [Kok78], see also [Bui81, Thm. 3.1], [GCRdF85]. Then one obtains, that $\bar{f}^s_{p,q}(w)$, $w \in \mathcal{A}_\infty$, is a $\varkappa$-sequence space for any $\varkappa$

$$\varkappa > \max(s, \sigma_{p/r_w,q} - s, \frac{n\, r_w}{p} - s). \tag{2.114}$$

Remark 2.36. We have shown that the (classical) sequence spaces $\bar{b}^s_{p,q}$ and $\bar{f}^s_{p,q}$ with $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ fit into the scheme of $\varkappa$-sequence spaces introduced in Section 2.3.2. Moreover we have even proved that doubling weighted sequence spaces of $\bar{b}$-type and $\bar{f}$-type are $\varkappa$-sequence spaces if $\varkappa$ is sufficiently large. One may ask for optimal (or at least sufficient) $\varkappa$ in the context of this spaces? The condition **(ii)** is local, such that one can expect that weighted spaces of type $\bar{b}^s_{p,q}(w)$ and $\bar{f}^s_{p,q}(w)$ can be incorporated for very general weights or measures. The weight properties are only used in the first part of the proofs of Proposition 2.28 or Proposition 2.34.

2.4 Wavelet characterization

In this section we apply our main Theorem 2.23 from the last section to obtain the wavelet characterization for specific function spaces, more precisely for doubling weighted Besov and Triebel-Lizorkin spaces.

At first we have to modify the sequence space norm from Definition 2.6 a little bit to get the „wavelet version“ of them.

Definition 2.37.

(i) Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and w be a doubling weight. Then $b^s_{p,q}(w)$ is the collection of all sequences

$$\lambda = \left\{ \lambda_m \in \mathbb{C}, \lambda^{j,G}_m \in \mathbb{C} : m \in \mathbb{Z}^n, j \in \mathbb{N}_0, G \in G^* \right\}$$

such that

$$\left\| \lambda | b^s_{p,q}(w) \right\| = \left(\sum_{m \in \mathbb{Z}^n} |\lambda_m|^p w(Q_{0,m}) \right)^{1/p} + \left(\sum_{j=0}^{\infty} 2^{jsq} \sum_{G \in G^*} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{j,G}_m|^p w(Q_{j,m}) \right)^{\frac{q}{p}} \right)^{1/q}$$

is finite (with obvious modification for $p = \infty$ or $q = \infty$).

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and w be a doubling weight. Then $f^s_{p,q}(w)$ is the collection of all sequences

$$\lambda = \left\{ \lambda_m \in \mathbb{C}, \lambda^{j,G}_m \in \mathbb{C} : m \in \mathbb{Z}^n, j \in \mathbb{N}_0, G \in G^* \right\}$$

such that

$$\left\| \lambda | f^s_{p,q}(w) \right\| = \Big\| \sum_{m \in \mathbb{Z}^n} |\lambda_m| \chi_{0,m}(\cdot) \, \Big| L_p(w) \Big\| + \left\| \left(\sum_{m \in \mathbb{Z}^n, j \in \mathbb{N}_0, G \in G^*} 2^{jsq} |\lambda^{j,G}_m|^q \chi_{j,m}(\cdot) \right)^{1/q} \middle| L_p(w) \right\|$$

is finite (with obvious modification for $q = \infty$).

As a conclusion of Theorem 2.23 we obtain a wavelet characterization for the unweighted Besov and Triebel-Lizorkin spaces.

Corollary 2.38.

(i) *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. *We assume*

$$L > \max\left(s, \frac{n}{p} - s \right). \tag{2.115}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented in terms of* L*-wavelets as*

$$f = \sum_{m \in \mathbb{Z}^n} \lambda_m \Psi_m + \sum_{G \in G^*} \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}, \qquad \lambda \in b^s_{p,q}, \tag{2.116}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$*. The representation* (2.116) *is unique,*

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\big(f, \Psi_{G,m}^j\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m), \tag{2.117}$$

$m \in \mathbb{Z}^n$*,* $j \in \mathbb{N}_0$*,* $G \in G^*$*, and*

$$I: \quad f \mapsto \big\{\lambda_m(f),\, \lambda_m^{j,G}(f)\big\} \tag{2.118}$$

is an isomorphic map of $B_{p,q}^s(\mathbb{R}^n)$ *onto* $b_{p,q}^s$*.*

(ii) *Let* $0 < p < \infty$*,* $0 < q \leq \infty$*,* $s \in \mathbb{R}$*. We assume*

$$L > \max\left(s, \frac{n}{p} - s, \sigma_{p,q} - s\right). \tag{2.119}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $F_{p,q}^s(\mathbb{R}^n)$ *if, and only if, it can be represented in terms of* L*-wavelets as*

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m \Psi_m + \sum_{G\in G^*}\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_{G,m}^j, \qquad \lambda \in f_{p,q}^s, \tag{2.120}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$*. The representation* (2.120) *is unique,*

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\big(f, \Psi_{G,m}^j\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m), \tag{2.121}$$

$m \in \mathbb{Z}^n$*,* $j \in \mathbb{N}_0$*,* $G \in G^*$*, and*

$$I: \quad f \mapsto \big\{\lambda_m(f),\, \lambda_m^{j,G}(f)\big\} \tag{2.122}$$

is an isomorphic map of $F_{p,q}^s(\mathbb{R}^n)$ *onto* $f_{p,q}^s$*.*

Proof. *Step 1.* Since $L > \max(s, \frac{n}{p} - s)$ it exists a $\varkappa > 0$ such that $L > \varkappa > \max(s, \frac{n}{p} - s)$. As a consequence of Proposition 2.26 $\bar{b}_{p,q}^s$ is a $\varkappa$-sequence space for this $\varkappa$ and $b_{p,q}^s$ is the wavelet version of $\bar{b}_{p,q}^s$. Moreover, there exists an L-atomic representation for $B_{p,q}^s(\mathbb{R}^n)$, for $L > \max(s, \frac{n}{p} - s)$, see for example in [Tri08, Theorem 1.7, Section 1.1.2, p.5] or [Tri97, Theorem 13.8], respectively. In addition, $B_{p,q}^s(\mathbb{R}^n)$ satisfies the essential embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{p,q}^s(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, cf. Remark 1.36 or [Tri83, Section 2.3.3]. Thus Theorem 2.23 yields us the desired result.

Step 2. Analogously there exists a $\varkappa$ such that $L > \varkappa > \max(s, \frac{n}{p} - s, \sigma_{p,q} - s)$ and $\bar{f}_{p,q}^s$ is a $\varkappa$-sequence space for this $\varkappa$, because of Proposition 2.33. Additionally $f_{p,q}^s$ is the wavelet version of $\bar{f}_{p,q}^s$ and it holds $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F_{p,q}^s(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, cf. Remark 1.36 or [Tri83, Section 2.3.3]. Furthermore we have also an L-atomic representation for $F_{p,q}^s(\mathbb{R}^n)$, for $L > \max(s, \frac{n}{p} - s, \sigma_{p,q} - s)$, see for example in [Tri08, Theorem 1.7, Section 1.1.2, p.5] or [Tri97, Theorem 13.8], respectively. Then Theorem 2.23 yields us the L-wavelet characterization for $F_{p,q}^s(\mathbb{R}^n)$. ■

Remark 2.39. The condition for L comes out from the condition for $\varkappa$. If one compares this result with the well-known results for $b^s_{p,q}$ and $f^s_{p,q}$ spaces from [Tri08, Theorem 1.20] and takes a look in the proofs of Proposition 2.26 or Proposition 2.33, respectively, then one sees that the **(i)** condition at $\varkappa$ is really sharp and coincides with the condition at $L = u$ in [Tri08, Theorem 1.20]. But the **(ii)** condition at $\varkappa$ is stronger and therefore the result in Corollary 2.38 is slightly weaker than in [Tri08, Theorem 1.20]. In summary it can be said, that condition **(i)** at $\varkappa$ is sharp and (perhaps) one can find optimal values for $\varkappa$, but it is more technical and harder to prove. On the other hand condition **(ii)** at $\varkappa$ is easy to prove, but maybe too weak to get optimal values for $\varkappa$ in connection with condition **(i)**. Maybe it is suitable to decouple both conditions to get optimal values for $\varkappa$.

Next we get a wavelet characterization for Muckenhoupt weighted Besov-Triebel-Lizorkin spaces. This result is not new, for example one can find it in [HS08]. But here we have a new approach.

Corollary 2.40. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty$ be a weight with r_w given by* (1.26)*.*

(i) *We assume*

$$L > \max\left(s + \frac{n}{p}, \frac{n}{p}\, r_w - s\right). \tag{2.123}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(w)$ if, and only if, it can be represented in terms of L-wavelets as

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m\, \Psi_m + \sum_{G\in G^*}\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda^{j,G}_m\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in b^s_{p,q}(w), \tag{2.124}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (2.124) *is unique,*

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2}\big(f, \Psi^j_{G,m}\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m), \tag{2.125}$$

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, $G \in G^*$, and

$$I: \quad f \mapsto \big\{\lambda_m(f),\, \lambda^{j,G}_m(f)\big\} \tag{2.126}$$

is an isomorphic map of $B^s_{p,q}(w)$ onto $b^s_{p,q}(w)$.

(ii) *We assume*

$$L > \max(s, \sigma_{p/r_w,q} - s, \frac{n}{p}\, r_w - s). \tag{2.127}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^s_{p,q}(w)$ if, and only if, it can be represented in terms of L-wavelets as

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m\, \Psi_m + \sum_{G\in G^*}\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}^n} \lambda^{j,G}_m\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in f^s_{p,q}(w), \tag{2.128}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. *The representation* (2.128) *is unique,*

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\big(f, \Psi_{G,m}^j\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m), \tag{2.129}$$

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, $G \in G^*$, *and*

$$I: \quad f \mapsto \big\{\lambda_m(f),\, \lambda_m^{j,G}(f)\big\} \tag{2.130}$$

is an isomorphic map of $F_{p,q}^s(w)$ *onto* $f_{p,q}^s(w)$.

Proof. *Step 1.* Since $L > \max(s+\frac{n}{p}, \frac{n}{p}\, r_w - s)$ it exists a $\varkappa > 0$ such that $L > \varkappa > \max(s+\frac{n}{p}, \frac{n}{p}\, r_w - s)$. Then $\bar{b}_{p,q}^s(w)$ is a $\varkappa$-sequence space for this $\varkappa$ concerning Proposition 2.27. It holds that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{p,q}^s(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, see [Bui82, Thm. 2.4], and $b_{p,q}^s(w)$ is the wavelet version of $\bar{b}_{p,q}^s(w)$. Moreover there exists an L-atomic representation for $B_{p,q}^s(w)$, since (2.123) implies $L > \max(s, \sigma_{p/r_w} - s)$, cf. [HS08, Proposition 1.12] or [HP08, Theorem 3.10]. Thus Theorem 2.23 yields us the desired result.
Step 2. Analogously there exists a $\varkappa$ such that $L > \varkappa > \max(s, \sigma_{p/r_w,q} - s, \frac{n}{p}\, r_w - s)$ and $\bar{f}_{p,q}^s(w)$ is a $\varkappa$-sequence space for this $\varkappa$, see Remark 2.35. It holds also that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F_{p,q}^s(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, see [Bui82, Thm. 2.4], and $f_{p,q}^s(w)$ is the related wavelet version of $\bar{f}_{p,q}^s(w)$. Furthermore we have also an L-atomic representation for $F_{p,q}^s(w)$, since $L > \max(s, \sigma_{p/r_w,q} - s, \frac{n\, r_w}{p} - s)$, cf. [HP08, Theorem 3.10]. Then Theorem 2.23 yields us the L-wavelet characterization for $F_{p,q}^s(w)$. ■

Now we consider doubling weighted Besov and Triebel-Lizorkin spaces and obtain for these spaces a wavelet characterization. This is a new result.

Corollary 2.41. *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and* w *be a doubling weight with doubling constant* γ.

(i) *We assume*

$$L > \max\left(s + \frac{n}{p}, \frac{n}{p}\, c\, \gamma - s\right), \tag{2.131}$$

where $c = \lfloor \log_2(\sqrt{n}) \rfloor + 2$ *is the same as in Proposition 1.23.*
Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B_{p,q}^s(w)$ *if, and only if, it can be represented in terms of L-wavelets as*

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m\, \Psi_m + \sum_{G\in G^*} \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_{G,m}^j, \qquad \lambda \in b_{p,q}^s(w), \tag{2.132}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. *The representation* (2.132) *is unique,*

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\big(f, \Psi_{G,m}^j\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m), \tag{2.133}$$

$m \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$, $G \in G^*$, *and*

$$I: \quad f \mapsto \big\{\lambda_m(f),\, \lambda_m^{j,G}(f)\big\} \tag{2.134}$$

is an isomorphic map of $B_{p,q}^s(w)$ *onto* $b_{p,q}^s(w)$.

(ii) *We assume*

$$L > \max(s, \gamma\,\sigma_{p,q} + (\gamma-1)n - s, \frac{n}{\min(p,q)}\, c\,\gamma - s), \tag{2.135}$$

where $c = \lfloor \log_2(\sqrt{n}) \rfloor + 2$ *is the same as in Proposition 1.23.*
Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $F^s_{p,q}(w)$ *if, and only if, it can be represented in terms of* L*-wavelets as*

$$f = \sum_{m\in\mathbb{Z}^n} \lambda_m\, \Psi_m + \sum_{G\in G^*} \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda^{j,G}_m\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad \lambda \in f^s_{p,q}(w), \tag{2.136}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$*. The representation* (2.136) *is unique,*

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2}\big(f, \Psi^j_{G,m}\big), \qquad \lambda_m = \lambda_m(f) = (f, \Psi_m), \tag{2.137}$$

$m \in \mathbb{Z}^n$*,* $j \in \mathbb{N}_0$*,* $G \in G^*$*, and*

$$I: \quad f \mapsto \big\{\lambda_m(f),\, \lambda^{j,G}_m(f)\big\} \tag{2.138}$$

is an isomorphic map of $F^s_{p,q}(w)$ *onto* $f^s_{p,q}(w)$*.*

Proof. *Step 1.* Since $L > \max(s + \frac{n}{p}, \frac{n}{p}\, c\,\gamma - s) \geq \max(s + \frac{n}{p}, \frac{n\gamma}{p} - s)$ it exists a $\varkappa > 0$ such that $L > \varkappa > \max(s + \frac{n}{p}, \frac{n\gamma}{p} - s)$. As a consequence of Proposition 2.28 $\bar{b}^s_{p,q}(w)$ is a $\varkappa$-sequence space for this $\varkappa$ and $b^s_{p,q}(w)$ is the wavelet version of $\bar{b}^s_{p,q}(w)$. On the other side we have the essential embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, see Proposition 1.44. It holds that $L > \max(s + \frac{n}{p}, \frac{n}{p}\, c\,\gamma - s) \geq \max(s + \frac{n}{p}, \frac{n}{p}\,\beta - s) \geq \max(s, \frac{n(\beta-1)}{p} + \sigma_p - s)$, where β is the doubling constant concerning balls, see Section 1.2.3. Hence, there exists an L-atomic representation for $B^s_{p,q}(w)$, cf. Proposition 2.11 (i) or see in [HS14, Proposition 2.21, p. 10] or [Bow05, Theorem 5.10], respectively. Thus Theorem 2.23 yields us the wavelet isomorphism for doubling weighted Besov spaces $B^s_{p,q}(w)$.
Step 2. Analogously there exists a $\varkappa$ such that $L > \varkappa > \max(s, \gamma\,\sigma_{p,q} + (\gamma-1)n - s, \frac{n\gamma}{p} - s)$, since $L > \max(s, \gamma\,\sigma_{p,q} + (\gamma-1)n - s, \frac{n}{\min(p,q)}\, c\gamma - s) \geq \max(s, \gamma\,\sigma_{p,q} + (\gamma-1)n - s, \frac{n}{p}\,\gamma - s)$. Thus $\bar{f}^s_{p,q}(w)$ is a $\varkappa$-sequence space for this $\varkappa$, see Proposition 2.33, and $f^s_{p,q}(w)$ is the related wavelet version of $\bar{f}^s_{p,q}(w)$. Furthermore we have also the essential embedding $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, see Proposition 1.44, and an L-atomic representation for $F^s_{p,q}(w)$, since $L > \max(s, \gamma\,\sigma_{p,q} + (\gamma-1)n - s, \frac{n}{\min(p,q)}\, c\,\gamma - s) \geq \max(s, \gamma\,\sigma_{p,q} + (\gamma-1)n - s, \frac{n}{\min(p,q)}\,\beta - s) \geq \max(s, \gamma\,\sigma_{p,q} + (\gamma-1)n - s, \frac{n(\beta-1)}{p} + \sigma_{p,q} - s)$, cf. Proposition 2.11 (ii) or see in [BH06, Theorem 5.11]. Then Theorem 2.23 yields us **(ii)**. ■

Remark 2.42. On the one side the condition for L comes out from the condition for $\varkappa$ and on the other side it comes out from the condition for $L = K$ in the atomic representation. One can slightly optimize the condition for L in corollary 2.41 by using both doubling constants γ (concerning cubes) and β (concerning balls), then (2.131) can be replaced by

$$L > \max\left(s + \frac{n}{p}, \frac{n}{p}\,\gamma - s, \frac{n(\beta-1)}{p} + \sigma_p - s\right)$$

and (2.135) can be replaced by

$$L > \max(s, \gamma\, \sigma_{p,q} + (\gamma - 1)n - s, \frac{n}{p}\,\gamma - s, \frac{n(\beta - 1)}{p} + \sigma_{p,q} - s).$$

On the other hand (2.135) can also be replaced by

$$L > \max(s, c\,\gamma\, \sigma_{p,q} + (c\,\gamma - 1)n - s, \frac{n}{p}\,\gamma - s).$$

But if one ask for optimal values for L, then the theory of $\varkappa$-sequence spaces is not the best choice. In this case it is the best way to prove the wavelet characterization directly.

3 Continuous and compact embeddings

The aim of this Chapter is to study necessary and sufficient conditions for continuous and compact embeddings for doubling weighted Besov spaces $B^s_{p,q}(w)$. We follow the approach from the series of papers [HS08, HS11a, HS11b] by HAROSKE and SKRZYPCZAK. Therefore we apply the wavelet characterization, which we proved in the last chapter. This allows us to transform the problem from the function spaces to the simpler context of the sequence spaces. Additionally we use a result for general weighted sequence spaces from the paper [KLSS06b, Thm. 3.1] by KÜHN LEOPOLD SICKEL and SKRZYPCZAK.

3.1 Embeddings of general weighted sequence spaces

Before we come to state our embedding results, we introduce a notation for sequence spaces, which is used in the paper [KLSS06b].

Definition 3.1. Let $\xi = (\xi_j)_j$ and $w = (w_{j,m})_{j,m}$ be sequences of positive numbers. Then $\ell_q(\xi_j \ell_p(w))$ is the collection of all sequences

$$\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$$

such that

$$\big\|\lambda|\ell_q(\xi_j\ell_p(w))\big\| = \Big(\sum_{j=0}^{\infty} \xi_j^q \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m} w_{j,m}|^p\Big)^{\frac{q}{p}}\Big)^{1/q} \tag{3.1}$$

is finite (with the usual modifications for $p = \infty$ or $q = \infty$).

Remark 3.2. At first we adapt our sequence spaces $b^s_{p,q}(w)$ with w doubling to this description. Let $\lambda = \big(\lambda^{j,G}_m\big)_{j,G,m} \subset \mathbb{C}$, $s \in \mathbb{R}$, $0 < p < \infty$, and assume $0 < q < \infty$ for convenience. Then

$$\begin{aligned}\big\|\lambda|b^s_{p,q}(w)\big\| &= \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_m|^p w(Q_{0,m})\Big)^{1/p} + \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{G\in G^*} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda^{j,G}_m|^p w(Q_{j,m})\Big)^{\frac{q}{p}}\Big)^{1/q} \\ &\sim \Big(\sum_{j=0}^{\infty} \xi_j^q \Big(\sum_{m\in\mathbb{Z}^n} |\widetilde{\lambda}_{j,m}|^p |w_{j,m}|^p\Big)^{\frac{q}{p}}\Big)^{1/q} = \Big\|\widetilde{\lambda}|\ell_q(\xi_j\ell_p(\widetilde{w}))\Big\| \end{aligned} \tag{3.2}$$

with $\widetilde{\lambda} = \big(\widetilde{\lambda}_{j,m}\big)_{j,m} \in \mathbb{C}$, $\xi = (\xi_j)_j = (2^{js})_j$ and $\widetilde{w} = (w_{j,m})_{j,m}$, $w_{j,m} = w(Q_{j,m})^{1/p}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$. Note that if w is doubling, then $0 < w(B) < \infty$ for all balls B, cf. Proposition 1.25. Therefore $\xi = (\xi_j)_j$ and $\widetilde{w} = (w_{j,m})_{j,m}$ are sequences of positive numbers.

Remark 3.3. Let $\xi^{(k)} = \left(\xi_j^{(k)}\right)_j$ and $w^{(k)} = \left(w_{j,m}^{(k)}\right)_{j,m}$, $k = 1, 2$, be sequences of positive numbers. One can easily verify that

$$\ell_{q_1}(\xi_j^{(1)}\ell_{p_1}(w^{(1)})) \hookrightarrow \ell_{q_2}(\xi_j^{(2)}\ell_{p_2}(w^{(2)}))$$

holds if, and only if,

$$\ell_{q_1}(\frac{\xi_j^{(1)}}{\xi_j^{(2)}}\ell_{p_1}(\frac{w^{(1)}}{w^{(2)}})) \hookrightarrow \ell_{q_2}(\ell_{p_2}).$$

So it is sufficient to consider unweighted target spaces.

Corollary 3.4. *Let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$ and let w_1, w_2 be doubling weights. We put*

$$\frac{1}{p^*} := \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+, \qquad \frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+. \tag{3.3}$$

(i) *The embedding $b^{s_1}_{p_1,q_1}(w_1) \hookrightarrow b^{s_2}_{p_2,q_2}(w_2)$ is continuous if, and only if,*

$$\left\{2^{-j(s_1-s_2)}\left\|\left\{w_1(Q_{j,m})^{-1/p_1}w_2(Q_{j,m})^{1/p_2}\right\}_m|\ell_{p^*}\right\|\right\}_j \in \ell_{q^*}. \tag{3.4}$$

(ii) *The embedding $b^{s_1}_{p_1,q_1}(w_1) \hookrightarrow b^{s_2}_{p_2,q_2}(w_2)$ is compact if, and only if,* (3.4) *holds and, in addition,*

$$\lim_{j\to\infty} 2^{-j(s_1-s_2)}\left\|\left\{w_1(Q_{j,m})^{-1/p_1}w_2(Q_{j,m})^{1/p_2}\right\}_m|\ell_{p^*}\right\| = 0 \quad \textit{if } q^* = \infty \tag{3.5}$$

and

$$\lim_{|m|\to\infty} w_1(Q_{j,m})^{1/p_1}w_2(Q_{j,m})^{-1/p_2} = \infty \quad \textit{for all } j \in \mathbb{N}_0 \quad \textit{if } p^* = \infty. \tag{3.6}$$

Proof. Let $b^{s_k}_{p_k,q_k}(w_k) = \ell_{q_k}(\xi_j^{(k)}\ell_{p_k}(w^{(k)}))$ with $\xi_j^{(k)} = 2^{js_k}$ and $w^{(k)} = \left(w_{j,m}^{(k)}\right)_{j,m}$, $w_{j,m}^{(k)} = w_k(Q_{j,m})^{1/p_k}$, $k = 1, 2$, be given. We apply [KLSS06b, Thm. 3.1] and obtain that

$$\ell_{q_1}(\xi_j^{(1)}\ell_{p_1}(w^{(1)})) = b^{s_1}_{p_1,q_1}(w_1) \hookrightarrow b^{s_2}_{p_2,q_2}(w_2) = \ell_{q_2}(\xi_j^{(2)}\ell_{p_2}(w^{(2)})) \tag{3.7}$$

is continuous if, and only if,

$$\left\{\frac{\xi_j^{(2)}}{\xi_j^{(1)}}\left\|\left\{\frac{w_{j,m}^{(2)}}{w_{j,m}^{(1)}}\right\}_m|\ell_{p^*}\right\|\right\}_j \in \ell_{q^*}, \tag{3.8}$$

which coincides with (3.4). Moreover (3.7) is compact if, and only if, (3.8) holds and, in addition,

$$\lim_{j\to\infty}\frac{\xi_j^{(2)}}{\xi_j^{(1)}}\left\|\left\{\frac{w_{j,m}^{(2)}}{w_{j,m}^{(1)}}\right\}_m|\ell_{p^*}\right\| = 0 \quad \text{if } q^* = \infty$$

and

$$\lim_{|m|\to\infty}\frac{w_{j,m}^{(1)}}{w_{j,m}^{(2)}} = \infty \quad \text{for all } j \in \mathbb{N}_0 \quad \text{if } p^* = \infty,$$

which coincides with (3.4), (3.5) and (3.6). ■

3.2 The main embedding result

At first we write down the general result for doubling weighted embeddings. Later we discuss two special cases of embeddings. Here we follow the approach from [HS11a].

Theorem 3.5. *Let $-\infty < s_2 \le s_1 < \infty$, $0 < p_1, p_2 \le \infty$, $0 < q_1, q_2 \le \infty$ and let w_1, w_2 be doubling weights.*

(i) *The embedding $B^{s_1}_{p_1,q_1}(w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(w_2)$ is continuous if, and only if,*

$$\left\{2^{-j(s_1-s_2)} \left\| \left\{ w_1(Q_{j,m})^{-1/p_1} w_2(Q_{j,m})^{1/p_2} \right\}_m |\ell_{p^*} \right\| \right\}_j \in \ell_{q^*}, \tag{3.9}$$

where p^* and q^* are given by (3.3).

(ii) *The embedding $B^{s_1}_{p_1,q_1}(w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(w_2)$ is compact if, and only if,* (3.9) *holds and, in addition,*

$$\lim_{j\to\infty} 2^{-j(s_1-s_2)} \left\| \left\{ w_1(Q_{j,m})^{-1/p_1} w_2(Q_{j,m})^{1/p_2} \right\}_m |\ell_{p^*} \right\| = 0 \quad \text{if } q^* = \infty \tag{3.10}$$

and

$$\lim_{|m|\to\infty} w_1(Q_{j,m})^{1/p_1} w_2(Q_{j,m})^{-1/p_2} = \infty \quad \text{for all } j \in \mathbb{N}_0 \quad \text{if } p^* = \infty, \tag{3.11}$$

where p^* and q^* are given by (3.3).

Proof. It follows from Theorem 2.41 that we have isomorphic maps T between $B^{s_1}_{p_1,q_1}(w_1)$ and $b^{s_1}_{p_1,q_1}(w_1)$ and S between $B^{s_2}_{p_2,q_2}(w_1)$ and $b^{s_2}_{p_2,q_2}(w_2)$. Moreover Corollary 3.4 yields, that the embedding $b^{s_1}_{p_1,q_1}(w_1) \hookrightarrow b^{s_2}_{p_2,q_2}(w_2)$ is continuous if, and only if, (3.9) holds, and the embedding $b^{s_1}_{p_1,q_1}(w_1) \hookrightarrow b^{s_2}_{p_2,q_2}(w_2)$ is compact if, and only if, (3.9), (3.10) and (3.11) holds. Consequently we have the following commutative diagrams

$$\begin{tikzcd}
B^{s_1}_{p_1,q_1}(w_1) \arrow[r, "T"] \arrow[d, "\mathrm{Id}"'] & b^{s_1}_{p_1,q_1}(w_1) \arrow[d, "\mathrm{id}"] \\
B^{s_2}_{p_2,q_2}(w_2) & b^{s_2}_{p_2,q_2}(w_2) \arrow[l, "S"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
b^{s_1}_{p_1,q_1}(w_1) \arrow[r, "T^{-1}"] \arrow[d, "\mathrm{id}"'] & B^{s_1}_{p_1,q_1}(w_1) \arrow[d, "\mathrm{Id}"] \\
b^{s_2}_{p_2,q_2}(w_2) & B^{s_2}_{p_2,q_2}(w_2) \arrow[l, "S^{-1}"']
\end{tikzcd}.$$

Because of the two isomorphic maps T and S we get the same conditions for the function space embeddings $B^{s_1}_{p_1,q_1}(w_1) \hookrightarrow B^{s_2}_{p_2,q_2}(w_2)$. ■

Remark 3.6. In view of what we said in the beginning of Section 1.3.2 we obtain the unweighted Besov spaces if $p_1 = p_2 = \infty$. We exclude this case in the sequel, since the unweighted situation is well-known already.

Example 3.7. For $w_1 \equiv w_2 \equiv 1$ we have $w_1(Q_{j,m}) = w_2(Q_{j,m}) = 2^{-jn}$. Thus we get in (3.9)

$$2^{-j(s_1-s_2)} || \left\{ 2^{\frac{jn}{p_1} - \frac{jn}{p_2}} \right\}_m |\ell_{p^*}|| = 2^{-j(s_1-\frac{n}{p_1})+j(s_2-\frac{n}{p_2})} || \{1\}_m |\ell_{p^*}||.$$

Then $\|1|\ell_{p^*}\| < \infty$ immediately implies $p^* = \infty$, that is $p_1 \le p_2$. We set

$$\delta := s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \tag{3.12}$$

as the difference of the differential dimensions, as usual. Then it remains to consider $\left\{2^{-j\delta}\right\}_\nu \in \ell_{q^*}$. For $q^* = \infty$, i.e., $q_1 \le q_2$, we need $\delta \ge 0$. Otherwise, for $q_1 > q_2$, $\delta > 0$ is required. Altogether the embedding $B^{s_1}_{p_1,q_1} \hookrightarrow B^{s_2}_{p_2,q_2}$ is continuous if, and only if,

$$p_1 \le p_2, \quad s_2 \le s_1, \quad \begin{cases} \delta \ge 0, & \text{if} \quad q_1 \le q_2, \\ \delta > 0, & \text{if} \quad q_1 > q_2. \end{cases}$$

Moreover the embedding is never compact, since (3.11) for all $j \in \mathbb{N}_0$ failed. This is a generalization of [Tri83, Theorem 2.7.1, p. 129].

Theorem 3.5 is sharp and optimal in view of the embeddings. But the conditions (3.9), (3.10) and (3.11) are very technical and difficult to prove. Therefore we ask now for simpler sufficient or necessary conditions.

In the literature mainly two special cases of weighted embeddings are of further interest. Firstly, when only the source space is weighted and the target space is unweighted, i.e., we consider embeddings of type

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n), \tag{3.13}$$

where w is doubling and the parameters are given by

$$-\infty < s_2 \le s_1 < \infty,\ 0 < p_1 < \infty,\ 0 < p_2 \le \infty,\ 0 < q_1, q_2 \le \infty.$$

As mentioned above, we assume that $p_1 < \infty$, since otherwise we have $B^{s_1}_{p_1,q_1}(w) = B^{s_1}_{p_1,q_1}$ and we arrive in the unweighted situation which is already well-known. Secondly, we consider the so-called „double-weighted“ situation, where both spaces are weighted in the same way, i.e. $w_1 = w_2 = w$. The corresponding setting is to consider embeddings of type

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w), \tag{3.14}$$

with

$$-\infty < s_2 \le s_1 < \infty,\ 0 < p_1, p_2 < \infty,\ 0 < q_1, q_2 \le \infty$$

and w doubling.

Before we write down the associated results, we insert a short preparation. In view of Theorem 3.5 we have to check the three conditions (3.9), (3.10) and (3.11). In (3.9) and (3.10) we have to consider expressions of type $w_1(Q_{j,m})^{-1/p_1} w_2(Q_{j,m})^{1/p_2}$. In case of (3.13) this reads as $2^{-jn/p_2} w(Q_{j,m})^{-1/p_1}$ and in case of (3.14) that equals $w(Q_{j,m})^{\frac{1}{p_2} - \frac{1}{p_1}}$. For $p^* = \infty$, i.e., $p_1 \le p_2$, we get there for both cases expressions of type $w(Q_{j,m})^\kappa$ with $\kappa < 0$, where in case of (3.14) we exclude $p_1 = p_2$, because this is trivial, see Corollary

3.12 below. Furthermore we get in (3.11) for both cases expressions of type $w(Q_{j,m})^\lambda$ with $\lambda > 0$, if $p^* = \infty$, where we here exclude $p_1 = p_2$ again in case of (3.14).
Let $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$ be fixed. Assume $m \in \mathbb{Z}^n$ such that $Q_{j,m} \cap Q_{0,l} \neq \emptyset$. Note that there exist $\sim 2^{jn}$ such cubes $Q_{j,m}$. We blow the little cube $Q_{j,m}$ α-times up until we cover the cube $Q_{0,l}$, i.e. $Q_{0,l} \subset 2^\alpha \cdot Q_{j,m}$. It is sufficient to choose $\alpha = j+1$. Let w be a doubling weight. Then we obtain via $(j+1)$-times application of the doubling property (1.40) for fixed $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$

$$w(Q_{0,l}) \leq 2^{(j+1)n\gamma}\, w(Q_{j,m}) \qquad \forall\, m \in \mathbb{Z}^n \text{ with } Q_{j,m} \cap Q_{0,l} \neq \emptyset. \tag{3.15}$$

So, for any $\kappa < 0$ we have

$$w(Q_{j,m})^\kappa \leq 2^{-(j+1)n\gamma\kappa}\, w(Q_{0,l})^\kappa \qquad \forall\, m \in \mathbb{Z}^n \text{ with } Q_{j,m} \cap Q_{0,l} \neq \emptyset.$$

Then

$$\begin{aligned}
\|\, \{w(Q_{j,m})^\kappa\}_m \,|\ell_\infty\| &= \sup_{m\in\mathbb{Z}^n} w(Q_{j,m})^\kappa \\
&\leq \sup_{l\in\mathbb{Z}^n} \max_{\substack{m\in\mathbb{Z}^n: \\ Q_{j,m}\cap Q_{0,l}\neq\emptyset}} w(Q_{j,m})^\kappa \\
&\leq 2^{-(j+1)n\gamma\kappa} \sup_{l\in\mathbb{Z}^n} \max_{\substack{m\in\mathbb{Z}^n: \\ Q_{j,m}\cap Q_{0,l}\neq\emptyset}} w(Q_{0,l})^\kappa \\
&\leq c\, 2^{-jn\gamma\kappa} \sup_{l\in\mathbb{Z}^n} w(Q_{0,l})^\kappa \\
&\leq c\, 2^{-jn\gamma\kappa} \left(\inf_{l\in\mathbb{Z}^n} w(Q_{0,l}) \right)^\kappa
\end{aligned} \tag{3.16}$$

for any $\kappa < 0$, $j \in \mathbb{N}_0$. Moreover, for any $\lambda > 0$, we have by (3.15) for $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$

$$w(Q_{j,m})^\lambda \geq 2^{-(j+1)n\gamma\lambda} w(Q_{0,l})^\lambda \qquad \forall\, m \in \mathbb{Z}^n \text{ with } Q_{j,m} \cap Q_{0,l} \neq \emptyset.$$

This leads to

$$\lim_{|m|\to\infty} w(Q_{j,m})^\lambda = \infty \quad \text{for all } j \in \mathbb{N}_0, \quad \text{if, and only if,} \quad \lim_{|l|\to\infty} w(Q_{0,l}) = \infty, \tag{3.17}$$

where $\lambda > 0$ is fixed. The necessity is clear with $j = 0$ and the sufficiency follows from the above estimate. Altogether for embeddings of type (3.13) and (3.14) we have to require conditions

$$\inf_{l\in\mathbb{Z}^n} w(Q_{0,l}) \geq c > 0 \tag{3.18}$$

and

$$\lim_{|l|\to\infty} w(Q_{0,l}) = \infty \tag{3.19}$$

if $p^* = \infty$. For $p^* < \infty$ we consider the embeddings (3.13) and (3.14) separately. We start with the „double-weighted“ situation. When $p^* < \infty$, i.e., $p_1 > p_2$, we have $0 < \frac{1}{p_2} - \frac{1}{p_1} =$

$\frac{1}{p^*}$. For any fixed $j \in \mathbb{N}_0$ we obtain

$$\left\| \left\{ w(Q_{j,m})^{\frac{1}{p_2}-\frac{1}{p_1}} \right\}_m |\ell_{p^*} \right\| = \left(\sum_{m\in\mathbb{Z}^n} w(Q_{j,m})^{\frac{1}{p^*}p^*} \right)^{\frac{1}{p^*}}$$
$$= \left(\sum_{m\in\mathbb{Z}^n} w(Q_{j,m}) \right)^{\frac{1}{p^*}}$$
$$= \left(\sum_{m\in\mathbb{Z}^n} \int_{Q_{j,m}} w(y)\,\mathrm{d}y \right)^{\frac{1}{p^*}} = \left(\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y \right)^{\frac{1}{p^*}}.$$

So we have to demand for our weight that $\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y < \infty$. But this is impossible for a doubling weight, recall Proposition 1.25. So the situation $p^* < \infty$ does not appear in (3.14), unlike in case of (3.13). Here we have the following estimate

$$|| \left\{ w(Q_{j,m})^{-1/p_1} \right\}_m |\ell_{p^*}|| = \Big(\sum_{m\in\mathbb{Z}^n} w(Q_{j,m})^{-\frac{p^*}{p_1}} \Big)^{1/p^*}$$
$$\leq \Big(\sum_{l\in\mathbb{Z}^n} \sum_{\substack{m\in\mathbb{Z}^n: \\ Q_{j,m}\cap Q_{0,l}\neq\emptyset}} w(Q_{j,m})^{-\frac{p^*}{p_1}} \Big)^{1/p^*}$$
$$\leq \Big(\sum_{l\in\mathbb{Z}^n} 2^{-(j+1)n\gamma(-\frac{p^*}{p_1})} w(Q_{0,l})^{-\frac{p^*}{p_1}} \underbrace{\sum_{\substack{m\in\mathbb{Z}^n: \\ Q_{j,m}\cap Q_{0,l}\neq\emptyset}} 1}_{\sim 2^{jn}} \Big)^{1/p^*}$$
$$\leq c\, 2^{j\frac{n\gamma}{p_1}+j\frac{n}{p^*}} \left(\sum_{l\in\mathbb{Z}^n} w(Q_{0,l})^{-\frac{p^*}{p_1}} \right)^{1/p^*}$$
$$= c\, 2^{j\frac{n\gamma}{p_1}+j\frac{n}{p^*}} || \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l |\ell_{p^*}||,$$

where we use (3.15) again. Therefore the condition

$$|| \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l |\ell_{p^*}|| < \infty \tag{3.20}$$

is necessary for embeddings of type (3.13) if $p^* < \infty$. Note that in this case condition (3.11) in Theorem 3.5 disappears.

3.3 The One-weighted situation

At first we recall the setting for the one-weighted situation. We regard embeddings of type

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n),$$

where w is doubling and the parameters are given by

$$-\infty < s_2 \leq s_1 < \infty,\ 0 < p_1 < \infty,\ 0 < p_2 \leq \infty,\ 0 < q_1, q_2 \leq \infty. \tag{3.21}$$

The above considerations yield us three necessary conditions. If $p^* = \infty$ the conditions

$$\inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c > 0 \tag{3.22}$$

and

$$\lim_{|l| \to \infty} w(Q_{0,l}) = \infty \tag{3.23}$$

are essential and if $p^* < \infty$ we need

$$|| \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l |\ell_{p^*}|| < \infty. \tag{3.24}$$

Recall that δ is the difference of the differential dimensions

$$\delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2}.$$

We start with $p^* = \infty$.

Corollary 3.8. *Let the parameters be given by* (3.21) *with* $p_1 \leq p_2$. *Let* w *be a doubling weight with the corresponding doubling constant* γ.

(i) *Then the embedding* $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is continuous, if*

(a) $$\inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c > 0, \tag{3.25}$$

(b) $$\begin{cases} \delta > \frac{n}{p_1}(\gamma - 1), & \textit{if} \quad q^* < \infty, \\ \delta \geq \frac{n}{p_1}(\gamma - 1), & \textit{if} \quad q^* = \infty. \end{cases} \tag{3.26}$$

Conversely, if the embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is continuous, then*

(a) $$\inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c > 0, \tag{3.27}$$

(b) $$\delta \geq 0. \tag{3.28}$$

(ii) *The embedding* $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is compact, if*

(a) $$\inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c > 0, \tag{3.29}$$

(b) $$\lim_{|l| \to \infty} w(Q_{0,l}) = \infty, \tag{3.30}$$

(c) $$\delta > \frac{n}{p_1}(\gamma - 1). \tag{3.31}$$

Conversely, if the embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ is compact, then

$$\textbf{(a)} \qquad \inf_{l\in\mathbb{Z}^n} w(Q_{0,l}) \geq c > 0, \tag{3.32}$$

$$\textbf{(b)} \qquad \lim_{|l|\to\infty} w(Q_{0,l}) = \infty, \tag{3.33}$$

$$\textbf{(c)} \qquad \delta \geq 0. \tag{3.34}$$

(iii) *If $\delta < 0$ or $\delta = 0$ and $q^* < \infty$, then $B^{s_1}_{p_1,q_1}(w)$ is not embedded in $B^{s_2}_{p_2,q_2}$.*

Proof. *Step 1.* We start with (iii). For any $j \in \mathbb{N}_0$ it holds

$$w(Q_{0,l}) \geq \; 2^{jn} \min_{\substack{m\in\mathbb{Z}^n:\\ Q_{j,m}\subset Q_{0,l}}} w(Q_{j,m}), \qquad l \in \mathbb{Z}^n.$$

Then one obtains

$$|| \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l |\ell_\infty|| \leq \; 2^{-jn/p_1} \; || \left\{ w(Q_{j,m})^{-1/p_1} \right\}_m |\ell_\infty||, \quad j \in \mathbb{N}_0.$$

Thus

$$\begin{aligned} & \left\| \left\{ 2^{-j(s_1-s_2)-j\frac{n}{p_2}} || \left\{ w(Q_{j,m})^{-1/p_1} \right\}_m |\ell_\infty|| \right\}_j \middle| \ell_{q^*} \right\| \\ \geq \; & \left\| \left\{ 2^{-j\delta} || \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l |\ell_\infty|| \right\}_j \middle| \ell_{q^*} \right\| \\ = \; & || \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l |\ell_\infty|| \left\| \left\{ 2^{-j\delta} \right\}_j \middle| \ell_{q^*} \right\| = \infty \end{aligned}$$

if $\delta < 0$ or $\delta = 0$ and $q^* < \infty$. This together with Theorem 3.5 yields us (iii).
Step 2. Let (3.25) and (3.26) be satisfied. Thus (3.9) (with $w_1 = w$, $w_2 = 1$) can be reduced by using (3.16), with $\kappa = -1/p_1$, to

$$\begin{aligned} & 2^{-j(s_1-s_2)} \left\| \left\{ w_1(Q_{j,m})^{-1/p_1} w_2(Q_{j,m})^{1/p_2} \right\}_m |\ell_\infty \right\| \\ \leq \; & c \, 2^{-j(s_1-s_2+\frac{n}{p_2}-\frac{n\gamma}{p_1})} \left(\inf_{l\in\mathbb{Z}^n} w(Q_{0,l}) \right)^{-1/p_1} \\ = \; & c \, 2^{-j(\delta-\frac{n}{p_1}(\gamma-1))} \left(\inf_{l\in\mathbb{Z}^n} w(Q_{0,l}) \right)^{-1/p_1} . \end{aligned}$$

So the continuity of the embedding follows from Theorem 3.5 (i) in view of (3.25) and (3.26). Moreover, if additionally (3.30) and (3.31) hold, the compactness of the embedding follows from Theorem 3.5 (ii) and the above estimate, where (3.30) yields us (3.17) and (3.31) together with (3.29) ensure (3.17).
Step 3. If (3.27) does not hold, then $|| \left\{ w(Q_{j,m})^{-1/p_1} \right\}_m |\ell_\infty||$ fails for $j = 0$. Thus there is no embedding (independent of δ) in view of (3.9). Similarly, if (3.33) does not hold, then the embedding cannot be compact in view of (3.11) and (3.17) with $\lambda = 1/p_1$. The rest of (i) and (ii) follows from *step 1.* ■

Remark 3.9. There remains a gap for

$$0 \;<\; \delta \;\leq\; \frac{n}{p_1}(\gamma - 1).$$

This is not surprising, because conditions as (3.25) and (3.30) are general features of w. It makes sense, that we need more information about the weight, than reflected by γ and (3.25) or (3.30) only, to get a full characterization, respectively.

The situation with $p^* < \infty$ is similar.

Corollary 3.10. *Let the parameters be given by* (3.21) *with* $p_1 > p_2$. *Let* w *be a doubling weight with the corresponding doubling constant* γ.

(i) *Then the embedding* $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is continuous, if*

$$\textbf{(a)} \qquad || \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} || < \infty, \tag{3.35}$$

$$\textbf{(b)} \qquad \begin{cases} \delta > \frac{n}{p^*} + \frac{n}{p_1}(\gamma - 1), & \text{if} \quad q^* < \infty, \\ \delta \geq \frac{n}{p^*} + \frac{n}{p_1}(\gamma - 1), & \text{if} \quad q^* = \infty. \end{cases} \tag{3.36}$$

Conversely, if the embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is continuous, then*

$$\textbf{(a)} \qquad || \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} || < \infty, \tag{3.37}$$

$$\textbf{(b)} \qquad \delta \geq \frac{n}{p^*}. \tag{3.38}$$

(ii) *The embedding* $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is compact, if*

$$\textbf{(a)} \qquad || \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} || < \infty, \tag{3.39}$$

$$\textbf{(b)} \qquad \delta > \frac{n}{p^*} + \frac{n}{p_1}(\gamma - 1). \tag{3.40}$$

Conversely, if the embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is compact, then*

$$\textbf{(a)} \qquad || \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} || < \infty, \tag{3.41}$$

$$\textbf{(b)} \qquad \delta \geq \frac{n}{p^*}. \tag{3.42}$$

(iii) *If* $\delta < \frac{n}{p^*}$ *or* $\delta = \frac{n}{p^*}$ *and* $q^* < \infty$, *then* $B^{s_1}_{p_1,q_1}(w)$ *is not embedded in* $B^{s_2}_{p_2,q_2}$.

Proof. The proof works similar to the proof of Corollary 3.8.
Step 1. For any $j \in \mathbb{N}_0$ it holds

$$w(Q_{0,l}) \geq \; 2^{jn} \min_{m \in \mathbb{Z}^n : Q_{j,m} \subset Q_{0,l}} w(Q_{j,m}), \qquad l \in \mathbb{Z}^n.$$

Then one obtains

$$|| \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} || \leq \; c \, 2^{-jn/p_1 - jn/p^*} \; || \left\{ w(Q_{j,m})^{-1/p_1} \right\}_m | \ell_{p^*} ||, \quad j \in \mathbb{N}_0.$$

Thus

$$\begin{aligned}
&\left\| \left\{ 2^{-j(s_1-s_2)-j\frac{n}{p_2}} \left\| \left\{ w(Q_{j,m})^{-1/p_1} \right\}_m | \ell_{p^*} \right\| \right\}_j \Big| \ell_{q^*} \right\| \\
\geq\ & c \left\| \left\{ 2^{-j\delta+j\frac{n}{p^*}} \left\| \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} \right\| \right\}_j \Big| \ell_{q^*} \right\| \\
=\ & c \left\| \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} \right\| \left\| \left\{ 2^{-j(\delta-\frac{n}{p^*})} \right\}_j \Big| \ell_{q^*} \right\| = \infty
\end{aligned}$$

if $\delta < \frac{n}{p^*}$ or $\delta = \frac{n}{p^*}$ and $q^* < \infty$. This together with Theorem 3.5 yields us (iii).
Step 2. Let (3.35) and (3.36) be satisfied. Then

$$\begin{aligned}
& 2^{-j(s_1-s_2)}\, 2^{-jn/p_2} \left\| \left\{ w(Q_{j,m})^{-1/p_1} \right\}_m | \ell_{p^*} \right\| \\
\leq\ & c\, 2^{-j(s_1-s_2+n/p_2-\frac{n\gamma}{p_1}-\frac{n}{p^*})} \left\| \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} \right\| \\
=\ & c\, 2^{-j(\delta-\frac{n}{p^*}-\frac{n}{p_1}(\gamma-1))} \left\| \left\{ w(Q_{0,l})^{-1/p_1} \right\}_l | \ell_{p^*} \right\|
\end{aligned}$$

is finite. Thus the embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ is continuous concerning Theorem 3.5 (i). Moreover the embedding is even compact, if (3.40) holds instead of (3.36). Note, that condition (3.11) has no meaning here.
Step 3. Obviously there is no embedding (independent of δ), if (3.37) or (3.41) do not hold in view of (3.9), respectively. Also if (3.38) or (3.42) does not holds, see *step 1.*, respectively. ■

3.4 The Double-weighted situation

The setting for the double-weighted situation is the following. We regard embeddings of type

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w),$$

where w is doubling and the parameters are given by

$$-\infty < s_2 \leq s_1 < \infty, 0 < p_1, p_2 < \infty, 0 < q_1, q_2 \leq \infty. \tag{3.43}$$

The considerations in Section 3.2 show, that the conditions

$$\inf_{x \in \mathbb{Q}^n} w(Q_{0,x}) \geq c > 0$$

and

$$\lim_{|x| \to \infty} w(Q_{0,x}) = \infty$$

are essential. Furthermore we proved there, that only the case $p^* = \infty$, i.e., $p_1 \leq p_2$, is interesting. Recall that

$$\delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2}$$

has its usual meaning.

Corollary 3.11. *Let the parameters be given by* (3.43) *with* $p_1 < p_2$. *Let* w *be a doubling weight with the corresponding doubling constant* γ.

(i) *Then the embedding* $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}(w)$ *is continuous, if*

(a) $$\inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c > 0, \tag{3.44}$$

(b) $$\begin{cases} \delta > n(\gamma - 1)(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if} \quad q^* < \infty, \\ \delta \geq n(\gamma - 1)(\frac{1}{p_1} - \frac{1}{p_2}), & \text{if} \quad q^* = \infty. \end{cases} \tag{3.45}$$

Conversely, if the embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}(w)$ *is continuous, then*

(a) $$\inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c > 0, \tag{3.46}$$

(b) $$\delta \geq 0. \tag{3.47}$$

(ii) *The embedding* $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}(w)$ *is compact, if*

(a) $$\inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c > 0, \tag{3.48}$$

(b) $$\lim_{|l| \to \infty} w(Q_{0,l}) = \infty, \tag{3.49}$$

(c) $$\delta > n(\gamma - 1)(\frac{1}{p_1} - \frac{1}{p_2}). \tag{3.50}$$

Conversely, if the embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$ *is compact, then*

$$\textbf{(a)} \qquad \inf_{l\in\mathbb{Z}^n} w(Q_{0,l}) \geq c > 0, \tag{3.51}$$

$$\textbf{(b)} \qquad \lim_{|l|\to\infty} w(Q_{0,l}) = \infty, \tag{3.52}$$

$$\textbf{(c)} \qquad \delta \geq 0. \tag{3.53}$$

(iii) *If* $\delta < 0$ *or* $\delta = 0$ *and* $q^* < \infty$, *then* $B^{s_1}_{p_1,q_1}(w)$ *is not embedded in* $B^{s_2}_{p_2,q_2}$.

Proof. The proof is completely parallel to the proof of Corollary 3.8, where we apply (3.16) with $\kappa = \frac{1}{p_2} - \frac{1}{p_1} < 0$ and (3.17) with $\lambda = \frac{1}{p_1} - \frac{1}{p_2}$. ■

Corollary 3.12. *Let the parameters be given by* (3.43) *with* $p_1 = p_2$. *Let* w *be a doubling weight with the corresponding doubling constant* γ. *Then the embedding* $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}(w)$ *is continuous if, and only if,*

$$\begin{cases} s_1 - s_2 > 0, & \textit{if} \quad q^* < \infty, \\ s_1 - s_2 \geq 0, & \textit{if} \quad q^* = \infty. \end{cases}$$

The embedding $B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}(w)$ *is never compact.*

Proof. Corollary 3.12 follows immediately from Theorem 3.5. ■

Remark 3.13. This result is already well-known for Muckenhoupt weights, see [HS08, HS11a]. It is natural to extend this to doubling weights, since there is no direct influence of the weight there.

4 An application: Envelopes

In this section we talk about growth envelope functions in doubling weighted Besov-Triebel-Lizorkin spaces as an application of our atomic decomposition from Proposition 2.11 and the embedding result from Theorem 3.5. The concept of envelopes was introduced and first studied in [Tri01, Sect. 12], [Har02]. For detailed information about envelopes and the proofs for the basic properties we refer to the book from HAROSKE, [Har07].

We start with some preliminaries. Let for some measurable function $f : \mathbb{R}^n \to \mathbb{C}$, finite a.e., its non-increasing rearrangement f^* be defined as usual,

$$f^*(t) := \inf\{s > 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \leq t\}, \quad t \geq 0.$$

For further details about the non-increasing rearrangement f^* we refer to [BS88, Ch. 2, Sect. 1], [DL93, Ch. 2, §2] and [EE04, Ch. 3], for instance.

Definition 4.1. Let X be a quasi-normed function space on $\mathbb{R}^n$.
The **growth envelope function** $\mathcal{E}_{\mathsf{G}}^X : (0,\infty) \to [0,\infty]$ of X is defined by

$$\mathcal{E}_{\mathsf{G}}^X(t) = \sup_{f \in X, ||f|X|| \leq 1} f^*(t), \qquad t > 0. \tag{4.1}$$

Remark 4.2. We put $\mathcal{E}_{\mathsf{G}}^X(\tau) := \infty$ if $\{f^*(\tau) : ||f|X|| \leq 1\}$ is not bounded from above for some $\tau > 0$. Note that it causes some problems when taking into account that we shall always deal with equivalent (quasi-) norms in the underlying function space. Assume we have two different, but equivalent (quasi-) norms $||\cdot|X||_1$ and $||\cdot|X||_2$ in X. Then there exists for every function $f \in X$ with $||f|X||_1 \leq 1$, $f \neq 0$, a function $g_f := cf$, where $c = ||f|X||_1/||f|X||_2$ and it holds $||g_f|X||_2 \leq 1$. From the properties of the non-increasing rearrangement results $g_f^* = c\, f^*$, see [BS88, Ch. 2, Sect. 1, Prop. 1.7].
Now we build on the one hand the growth envelope function with $||\cdot|X||_1$ and on the other hand with $||\cdot|X||_2$. This leads to two different, but equivalent expressions for $\mathcal{E}_{\mathsf{G}}^X$. Therefore it is a matter of equivalence classes of growth envelope functions, where we choose one representative

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \sup_{||f|X|| \leq 1} f^*(t), \qquad t > 0.$$

However we do not want to distinguish between representative and equivalence class in the sequel.
Furthermore, by (4.1) the growth envelope function $\mathcal{E}_{\mathsf{G}}^X(t)$ is defined for all values $t > 0$, but it is of particular interest to consider this function for small $t > 0$, say, $0 < t < 1$, because there accumulate the singularities through the non-increasing rearrangement. This local characterization is reinforced by the so-called index u_{G}^X, which gives a finer measure of the (local) integrability of functions belonging to X. The exact definition of this index u_{G}^X is very technical and not important for this work, since we only look for the growth envelope function $\mathcal{E}_{\mathsf{G}}^X(t)$ here. Both together, the growth envelope function $\mathcal{E}_{\mathsf{G}}^X(t)$ and the index u_{G}^X are called the growth envelope $\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^X(\cdot), u_{\mathsf{G}}^X\right)$ for the function space X. For detailed information about the index u_{G}^X and the growth envelope $\mathfrak{E}_{\mathsf{G}}(X)$ we refer to the book [Har07, Ch. 4] by HAROSKE.
In contrast to the local characterization it turned out, that sometimes also the global behavior of the growth envelope function $\mathcal{E}_{\mathsf{G}}^X(t)$ for $t \to \infty$ is of interest. But in this work we only look for the local characterization.

The classical example for growth envelopes is the Lorentz space $L_{p,q}$.

Definition 4.3. Let $0 < p, q \leq \infty$. The Lorentz space $L_{p,q} = L_{p,q}(\mathbb{R}^n)$ consists of all measurable functions f for which

$$||f|L_{p,q}|| := \begin{cases} \left(\int\limits_0^\infty \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup\limits_{0<t<\infty} t^{\frac{1}{p}} f^*(t), & q = \infty, \end{cases} \tag{4.2}$$

is finite.

Remark 4.4. This definition is well-known and can be found, for instance, in [BS88, Ch. 4, p. 216]. Obviously, $L_{p,p} = L_p$ and $L_{\infty,q} = \{0\}$, $0 < q < \infty$, contains only the zero function. In addition, $L_{\infty,\infty} = L_\infty$ the classical Lebesgue space. Moreover, it holds

$$L_{p,q} \hookrightarrow L_{p,r} \qquad \text{if, and only if,} \qquad q \leq r. \tag{4.3}$$

Note that (4.2) do not give a norm in any case, not even for $p, q \geq 1$. However, replacing the non-increasing rearrangement f^* in (4.2) by its maximal function f^{**}, given by

$$f^{**}(t) = \frac{1}{t} \int\limits_0^t f^*(s)\,\mathrm{d}s, \quad t > 0, \tag{4.4}$$

one obtains for $1 < p < \infty$, $1 \leq q \leq \infty$, a norm, see [BS88, Ch. 4, Thm. 4.6]. The essential advantage of the maximal function f^{**} is that it possesses a certain sub-additivity property,

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \quad t > 0,$$

cf. [BS88, Ch. 2, (3.10)]. Moreover, for $1 < p \le \infty$ and $1 \le q \le \infty$, the corresponding expressions (4.2) with f^* and f^{**}, respectively, are equivalent; cf. [BS88, Ch. 4, Lemma 4.5].

For the Lorentz space $L_{p,q}$ we have the following growth envelope function.

Proposition 4.5. *Let $0 < p < \infty$, $0 < q \le \infty$. Then*

$$\mathcal{E}_{\mathsf{G}}^{L_{p,q}}(t) \sim t^{-\frac{1}{p}}, \quad t \to 0. \tag{4.5}$$

Proof. [Har07, Prop. 3.12]. ∎

Remark 4.6. In particular, it is known that

$$\mathfrak{E}_{\mathsf{G}}(L_{p,q}) = \left(t^{-\frac{1}{p}}, q\right),$$

cf. [Har07, Thm. 4.7]. Hence, this leads to expressions of type

$$\left(\int_0^{\epsilon} \left[t^{\frac{1}{p}} f^*(t)\right]^v \frac{\mathrm{d}t}{t}\right)^{1/v} \le c||f|L_{p,q}||$$

if, and only if, $v \ge q = u_{\mathsf{G}}^{L_{p,q}}$. Here one observes very well that the index u_{G}^X gives a finer local characterization and there is some connection to a Lorentz space embedding.

Let us collect some basic properties of the growth envelope function.

Proposition 4.7. *Let X, X_1, X_2 be some function spaces on $\mathbb{R}^n$.*

(i) *$\mathcal{E}_{\mathsf{G}}^X$ is monotonically decreasing and right-continuous, $(\mathcal{E}_{\mathsf{G}}^X)^* = \mathcal{E}_{\mathsf{G}}^X$.*

(ii) *We have $X \hookrightarrow L_\infty$ if, and only if, $\mathcal{E}_{\mathsf{G}}^X(\cdot)$ is bounded.*

(iii) *If $X_1 \hookrightarrow X_2$ then there exists a constant $c > 0$ such that for all $t > 0$*

$$\mathcal{E}_{\mathsf{G}}^{X_1}(t) \le c\, \mathcal{E}_{\mathsf{G}}^{X_2}(t). \tag{4.6}$$

For a proof we refer to [Har07, Prop. 3.4].

Remark 4.8. For rearrangement-invariant Banach function spaces X with fundamental function φ_X it is proved in [Har07, Sect. 3.3] that

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \frac{1}{\varphi_X(t)} = ||\chi_{A_t}|X||^{-1}, \quad t > 0,$$

where $A_t \subset \mathbb{R}^n$ with $|A_t| = t$. For more information about rearrangement-invariant function spaces and the concept of the fundamental function φ_X we refer to [Har07, Sect. 3.3] or [BS88, Ch. 2], respectively.

In the classical (unweighted) Besov and Triebel-Lizorkin spaces we have this result.

Proposition 4.9. *Let* $0 < p < \infty$, $0 < q \leq \infty$ *and* $\sigma_p = n(\frac{1}{p} - 1)_+ < s < \frac{n}{p}$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}} \tag{4.7}$$

Remark 4.10. The condition to s comes from the fact, that the concept of growth envelopes makes only sense for regular distributions and unbounded growth envelope functions, where the borderline situations are not considered here. For the unweighted Besov and Triebel-Lizorkin spaces are also the growth envelopes known, that is,

$$\mathfrak{E}_{\mathsf{G}}(B^s_{p,q}) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, q\right)$$

and

$$\mathfrak{E}_{\mathsf{G}}(F^s_{p,q}) = \left(t^{-\frac{1}{p}+\frac{s}{n}}, p\right).$$

Proofs of this can be found in [Har07, Thm. 8.1] or [Tri01, Thm. 15.2]. Moreover, there one can also find some results for the borderline situations.

Now we want to characterize the singularity behavior of $A^s_{p,q}(w)$, where w is doubling. As already mentioned the concept of growth envelopes makes only sense for regular distributions, i.e. we need a condition $A^s_{p,q}(w) \subset L_1^{\mathrm{loc}}$ for our function spaces. On the other hand we already know, that we have no singularity behavior in the sense of growth envelope, that is $\mathcal{E}_{\mathsf{G}}^X(t)$ is bounded, if $A^s_{p,q}(w) \hookrightarrow L_\infty$. The borderline situations we do not consider here. So as a preparation we receive Corollary 4.11.

Corollary 4.11. *Let* $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ *and* w *is doubling with*

$$\inf_{l \in \mathbb{Z}^n} w(Q_{0,l}) \geq c_w > 0. \tag{4.8}$$

(i) *Let* $s - \frac{n}{p}(\gamma - 1) > \sigma_p$. *Then*

$$A^s_{p,q}(w) \subset L_1^{\mathrm{loc}}. \tag{4.9}$$

(ii) *Let* $s > \frac{n}{p}\gamma$. *Then*

$$A^s_{p,q}(w) \hookrightarrow L_\infty. \tag{4.10}$$

Proof. The extension to the F-spaces is a direct consequence of (1.63), so it is sufficient to consider B-spaces. We use Corollary 3.8 (i) with $p_1 = p_2 = p$, $q_1 = q_2 = q$, $s_1 = s$ and $s_2 = s - \frac{n}{p}(\gamma - 1)$. Then $p^* = \infty$, $q^* = \infty$ and $\delta = \frac{n}{p}(\gamma - 1)$. Thus (4.8) implies the embedding

$$B^s_{p,q}(w) \hookrightarrow B^{s-\frac{n}{p}(\gamma-1)}_{p,q}. \tag{4.11}$$

Moreover we have

$$B^{s-\frac{n}{p}(\gamma-1)}_{p,q} \subset L^{\mathrm{loc}}_1 \qquad \text{if} \quad s-\frac{n}{p}(\gamma-1) > \sigma_p,$$

cf. (1.52) in Remark 1.36. This completes (i). Otherwise it holds

$$B^{s-\frac{n}{p}(\gamma-1)}_{p,q} \hookrightarrow L_\infty \qquad \text{if} \quad s-\frac{n}{p}(\gamma-1) > \frac{n}{p},$$

see [ET96, 2.3.3. (iii)] or [Har07, Prop. 7.13 (7.42)], respectively. This completes (ii). ■

Remark 4.12. In view of this corollary it makes only sense to consider growth envelopes for parameters

$$0<p<\infty, \quad 0<q\le\infty, \quad \sigma_p+\frac{n}{p}(\gamma-1)<s<\frac{n}{p}\gamma; \tag{4.12}$$

borderline situations are still out of the frame. Furthermore we have the restriction (4.8) for our weight. If one compares this with well known results for Muckenhoupt weights, then we have there similar conditions, see [Ska10, Lemma 4.6, Bem. 4.7] or [Har10, Prop. 4.3].

We start with the estimate from above.

Proposition 4.13. *Let $0<p<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$ and w be doubling.*
We assume

$$\sigma_p+\frac{n}{p}(\gamma-1)<s<\frac{n}{p}\gamma \qquad \textit{and} \qquad \inf_{l\in\mathbb{Z}^n} w(Q_{0,l}) \ge c_w > 0.$$

Then

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \le c\, t^{-\frac{\gamma}{p}+\frac{s}{n}}, \quad t\to 0. \tag{4.13}$$

Proof. Just as in the proof of Corollary 4.11, we use the embedding from Corollary 3.8 (i) with $p_1=p_2=p$, $q_1=q_2=q$, $s_1=s$ and $s_2=s-\frac{n}{p}(\gamma-1)$. Then $p^*=\infty$, $q^*=\infty$ and $\delta=\frac{n}{p}(\gamma-1)$ again and in view of our assumptions is $\sigma_p<s_2=s-\frac{n}{p}(\gamma-1)<\frac{n}{p}$. Thus it follows from the results of the unweighted case, see Proposition 4.9, and property (4.6) of the growth envelope functions

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \overset{(4.6),(4.11)}{\le} c_1\, \mathcal{E}_{\mathsf{G}}^{B^{s-\frac{n}{p}(\gamma-1)}_{p,q}}(t) \overset{(4.7)}{\le} c_2\, t^{-\frac{1}{p}+\frac{1}{n}(s-\frac{n}{p}(\gamma-1))} = c_2\, t^{-\frac{\gamma}{p}+\frac{s}{n}}, \quad t\to 0.$$

The F-space result follows immediately from the embedding (1.63), then

$$\mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w)}(t) \overset{(4.6),(1.63)}{\le} c_1\, \mathcal{E}_{\mathsf{G}}^{B^s_{p,\infty}(w)}(t) \le c_2\, t^{-\frac{\gamma}{p}+\frac{s}{n}}, \quad t\to 0.$$

■

Now we deal with the estimate from below.

Proposition 4.14. *Let $0<p<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$ and w be doubling. We assume*

$$\sigma_p+\frac{n}{p}(\gamma-1)<s<\frac{n}{p}\gamma \qquad \textit{and} \qquad \inf_{l\in\mathbb{Z}^n} w(Q_{0,l})\ge c_w>0.$$

Then

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t)\ge c\,t^{-\frac{1}{p}+\frac{s}{n}}\sup_{x^0\in\mathbb{R}^n,t\sim 2^{-jn}}\left(\frac{w(B(x^0,2^{-j}))}{|B(x^0,2^{-j})|}\right)^{-1/p}, \quad t\to 0. \tag{4.14}$$

Proof. As usual for the proof of the estimate from below we construct special functions $f_{j,x^0}\in B^s_{p,q}(w)$ with $||f_{j,x^0}|B^s_{p,q}(w)||\sim 1$ such that

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(2^{-jn})\ge c\sup_{x^0} f^*_{j,x^0}(2^{-jn}), \quad j\in\mathbb{N}.$$

Let for $x^0\in\mathbb{R}^n$, $j\in\mathbb{N}$,

$$f_{j,x^0}(x):=\lambda_{j,x^0}a_{j,x^0}(x), \qquad x\in\mathbb{R}^n, \tag{4.15}$$

with

$$\lambda_{j,x^0}:=2^{-j(s-\frac{n}{p})}\left(\frac{w(B(x^0,2^{-j}))}{|B(x^0,2^{-j})|}\right)^{-1/p} \quad \text{and} \quad a_{j,x^0}(x):=\psi(2^j(x-x^0)),$$

where $\psi\in C_0^\infty(\mathbb{R}^n)$ is given by

$$\psi(x)=\begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{if} \quad |x|<1,\\ 0, & \text{if} \quad |x|\ge 1.\end{cases}$$

We observe that the a_{j,x^0}, $j\in\mathbb{N}$ (without loss of generality should $j\ge 2$), are special atoms according to Definition 2.1 with $d=4$, $K>s$ and $L=0$, since $\operatorname{supp} a_{j,x^0}\subset B(x^0,2^{-j})\subset 4\,Q_{j,2^{j-2}\lfloor x^0\rfloor}$,

$$|\mathrm{D}^\alpha a_{j,x^0}(x)|\le 2^{j|\alpha|}, \quad |\alpha|\le K$$

(up to a constant depending on ψ) and our assumption on s implies that we do not need moment conditions, see (2.15). Then $f_{j,x^0}(x)=\lambda_{j,x^0}a_{j,x^0}$ is a special atomic decomposition and we obtain for the norm

$$||f_{j,x^0}|B^s_{p,q}(w)||\le c\,||\lambda|\bar{b}^s_{p,q}(w)||\sim\lambda_{j,x^0}2^{js}w(B(x^0,2^{-j}))^{1/p}=1.$$

Thus our functions f_{j,x^0}, $j\in\mathbb{N}$, $x^0\in\mathbb{R}^n$, are admitted to the competition in the supremum of $\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}$.

Furthermore for $j\in\mathbb{N}$ we have

$$\begin{aligned}&\mu_{f_{j,x^0}}(c'\,2^{-js}w(B(x^0,2^{-j}))^{-1/p})\\ &=|\{x\in\mathbb{R}^n:|2^{-js}w(B(x^0,2^{-j}))^{-1/p}\psi(2^j(x-x^0))|>c'\,2^{-js}w(B(x^0,2^{-j}))^{-1/p}\}|\\ &\ge c\,|B(x^0,2^{-j})|=c''\,2^{-jn}.\end{aligned} \tag{4.16}$$

Then

$$f^*_{j,x^0}(c''\,2^{-jn}) = \inf\{s>0 : \mu_{f_{j,x^0}}(s) \le c\,2^{-jn}\} \overset{(4.16)}{\ge} c'\,2^{-js} w(B(x^0,2^{-j}))^{-1/p}, \tag{4.17}$$

since $\mu_{f_{j,x^0}}$ is monotonically decreasing.
Let $0<t<1$ fixed, then there exists a $j_0 \in \mathbb{N}$ such that $t \sim 2^{-j_0 n}$. Thus

$$\begin{aligned}
\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) &\ge \sup_{j\in\mathbb{N},x^0\in\mathbb{R}^n} f^*_{j,x^0}(t)\\
&\ge c \sup_{x^0\in\mathbb{R}^n, t\sim 2^{-jn}} f^*_{j,x^0}(t)\\
&\overset{(4.17)}{\ge} c\,t^{-\frac{1}{p}+\frac{s}{n}} \sup_{x^0\in\mathbb{R}^n, t\sim 2^{-jn}} \left(\frac{w(B(x^0,2^{-j}))}{|B(x^0,2^{-j})|}\right)^{-1/p}
\end{aligned}$$

Just as in the proof of Corollary 4.11 and Proposition 4.13 follows the assertion for the $F-$space with embedding (1.63) and property (4.6)

$$\begin{aligned}
\mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w)}(t) &\overset{(4.6),(1.63)}{\ge} c_1\,\mathcal{E}_{\mathsf{G}}^{B^s_{p,\min(p,q)}(w)}(t)\\
&\ge c_2\,t^{-\frac{1}{p}+\frac{s}{n}} \sup_{x^0\in\mathbb{R}^n, t\sim 2^{-jn}} \left(\frac{w(B(x^0,2^{-j}))}{|B(x^0,2^{-j})|}\right)^{-1/p}, \quad t\to 0.
\end{aligned}$$

■

Remark 4.15. If one uses the approach from the proof of Proposition 4.12. in [Har10], one can refine the result (4.14) a little bit by

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \ge c \sup_{x^0\in\mathbb{R}^n} \left(\sum_{j=1}^{\lfloor \frac{1}{n}|\log t|\rfloor} 2^{-j(s-\frac{n}{p})q'} \left(\frac{w(B(x^0,2^{-j}))}{|B(x^0,2^{-j})|}\right)^{\frac{-q'}{p}} \right)^{1/q'}, \quad t\to 0 \tag{4.18}$$

(with usual modification if $q'=\infty$).
Now we briefly discuss the compatibility of (4.13) and (4.14). Let $x^0\in\mathbb{R}^n$ and $\nu\in\mathbb{N}$. We apply the doubling property respective cubes (1.40) together with (4.8), then

$$w(Q_{\nu,m}) \ge w(Q_{0,l})2^{-\nu n\gamma} \ge c_w\,2^{-\nu n\gamma}$$

and consequently

$$\sup_{x^0\in\mathbb{R}^n}\sup_{Q_{\nu,m}\ni x^0} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|}\right)^{-1/p} \le c\,2^{\nu n\frac{(\gamma-1)}{p}},$$

where c is independent of $\nu\in\mathbb{N}$ and $x^0\in\mathbb{R}^n$. Moreover let $t\sim 2^{-\nu n}$. Thus we have

$$\begin{aligned}
& t^{-\frac{1}{p}+\frac{s}{n}} \sup_{x^0\in\mathbb{R}^n, t\sim 2^{-\nu n}} \left(\frac{w(B(x^0,2^{-\nu}))}{|B(x^0,2^{-\nu})|}\right)^{-1/p}\\
\sim\ & t^{-\frac{1}{p}+\frac{s}{n}} \sup_{x^0\in\mathbb{R}^n}\sup_{Q_{\nu,m}\ni x^0, t\sim 2^{-\nu n}} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|}\right)^{-1/p}\\
\le\ & c\,t^{-\frac{\gamma}{p}+\frac{s}{n}}
\end{aligned}$$

So (4.13) and (4.14) do not contradict each other. The compatibility of (4.13) and (4.18) can be shown in the same way.

Remark 4.16. In [Har10] one finds similar results for both estimates from above and from below for $\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t)$, $t \to 0$, if $w \in \mathcal{A}_\infty$, see [Har10, Prop. 4.3., Prop. 4.12. and Rem. 4.14.]. This is not surprising, since we do not use weight-specific properties except for the embedding (4.11) or (3.9), respectively, and the atomic decomposition from Proposition (2.11) with γ or β instead of r_w, respectively.
Furthermore similar to the Muckenhoupt weights one could introduce the so-called set of singularities $\mathbf{S}_{\text{sing}}(w) = \mathbf{S}_0(w) \cup \mathbf{S}_\infty(w)$, where $\mathbf{S}_0(w)$ and $\mathbf{S}_\infty(w)$ are given by

$$\mathbf{S}_0(w) = \left\{ x^0 \in \mathbb{R}^n : \inf_{Q_{\nu,m} \ni x^0} \frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} = 0 \right\},$$

$$\mathbf{S}_\infty(w) = \left\{ x^0 \in \mathbb{R}^n : \sup_{Q_{\nu,m} \ni x^0} \frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} = \infty \right\}.$$

In case of $w \in \mathcal{A}_\infty$ we know, that $|\mathbf{S}_{\text{sing}}(w)| = 0$, see [HS11a, Prop. 4.5.]. An extension to $|\overline{\mathbf{S}_{\text{sing}}(w)}| = 0$, i.e. $\mathbf{S}_{\text{sing}}(w)$ is not dense in $\mathbb{R}^n$, can be found in [HS16]. In case of doubling weights there exists no statement yet. Then, if $\mathbf{S}_0(w) \neq \emptyset$, (4.18) can be replaced by

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \geq c \sup_{x^0 \in \mathbf{S}_0(w)} \left(\sum_{j=1}^{\lfloor \frac{1}{n} |\log t| \rfloor} 2^{-j(s-\frac{n}{p})q'} \left(\frac{w(B(x^0, 2^{-j}))}{|B(x^0, 2^{-j})|} \right)^{\frac{-q'}{p}} \right)^{1/q'}, \quad t \to 0$$

and (4.14) can be replaced by

$$\mathcal{E}_{\mathsf{G}}^{A^s_{p,q}(w)}(t) \geq c\, t^{-\frac{1}{p}+\frac{s}{n}} \sup_{x^0 \in \mathbf{S}_0(w)} \sup_{Q_{\nu,m} \ni x^0, t \sim 2^{-\nu n}} \left(\frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} \right)^{-1/p} \quad t \to 0,$$

respectively.

Example 4.17. In the unweighted case for $w \equiv 1$ we have $\gamma = 1$ and the supremum in (4.14) vanishes. Thus (4.14) and (4.13) together coincide with the unweighted result (4.7).

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