Entropy and approximation numbers of weighted Sobolev embeddings - a bracketing technique -

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Zusammenfassung in deutscher Sprache

Die vorliegende Arbeit befasst sich mit dem asymptotischen Verhalten von Entropie- und Approximationszahlen der kompakten Einbettung

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B), \qquad 1 \leq p < \infty, m \in \mathbb{N},$$

wobei B die offene Einheitskugel in $\mathbb{R}^n$ ist. Der gewichtete Sobolev-Raum $E^m_{p,\psi}(B)$ ist definiert als der Abschluss von $C^m_0(B)$ in $L_p(B)$ bezüglich der Norm

$$\|f|E^m_{p,\psi}(B)\| = \Big(\int_B |x|^{mp}\psi(|x|)^p \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}}.$$

Dabei ist $\psi : (0,1] \to (0,\infty)$ eine langsam variierende Funktion, die nach unten durch eine positive Konstante beschränkt und in einer Umgebung von Null monoton fallend ist. Einfache Beispiele für die Funktion $\psi(t)$ sind der Logarithmus $(1 + |\log t|)^\sigma, \sigma > 0$, sowie die Funktion $\exp(|\log t|^c), 0 < c < 1$. Notwendiges und hinreichendes Kriterium für die Kompaktheit von id ist die Unbeschränkheit der Funktion $\psi(t)$ bei Null

$$\lim_{t \to 0^+} \psi(t) = \infty.$$

Dieses unbeschränkte Wachstum von ψ gleicht in gewisser Weise das singuläre Verhalten von $|x|^m \to 0$ für $x \to 0$ aus. Erste Untersuchungen dieser Art gehen auf Triebel [Tr12b] zurück. Um Abschätzungen für Entropie- und Approximationszahlen $a_k(\mathrm{id}), e_k(\mathrm{id})$ zu erhalten, kann man oft grundlegende Eigenschaften wie Multiplikativität und Additivität in Kombination mit bekannten Resultaten klassischer Einbettungen verwenden. Allerdings liefert dieses Vorgehen nur exakte Ergebnisse, wenn die Funktion ψ eine hinreichend starke Wachstumsrate aufweist. Im Fall von Hilberträumen kann man die Ergebnisse verbessern, indem man das Problem auf die Eigenwertverteilung entarteter elliptischer Operatoren A^m_ψ der Form

$$\begin{aligned} \mathrm{dom}((A^m_\psi)^{1/2}) &= E^m_{2,\psi}(B) \\ A^m_\psi f(x) &= (-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha \big[|x|^{2m}\psi(|x|)^2 \mathrm{D}^\alpha f(x) \big] \end{aligned}$$

zurückführt und das sogenannte *Dirichlet-Neumann-Bracketing* verwendet. In dieser Arbeit wird eine Zerlegungsmethode vorgestellt, die es ermöglicht diese Herangehensweise auf Banachräume auszudehnen. Angeregt wurde diese Verallgemeinerung des *Dirichlet-Neumann-Bracketing* auf L_p-Räume durch die Arbeiten von Edmunds, Evans und Harris [EH93, EE04]. Die Methode beruht im Wesentlichen darauf Größen einzuführen, die eine *Bracketing*-Eigenschaft in L_p besitzen. Der Name *'Bracketing'* bezieht sich hier auf eine endliche Zerlegung des Gebiets B, die es ermöglicht die Singularität des Gewichts im Ursprung „herauszuschneiden" ohne dabei das asymptotische Verhalten zu beeinflussen. Genauer gesagt betrachtet man die Anzahl der Approximationszahlen, die größer oder gleich einer bestimmten Grenze $\varepsilon > 0$ sind

$$\nu_0(\varepsilon, B) := \max\{k \in \mathbb{N} : a_k(\mathrm{id}) \geq \varepsilon\}$$

und untersucht deren Verhalten für $\varepsilon \searrow 0$. Als zentrales Ergebnis dieser Arbeit wird das exakte asymptotische Verhalten der Entropie- und Approximationszahlen $e_k(\mathrm{id}), a_k(\mathrm{id})$ der kompakten Einbettung $\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$ durch

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{falls } \left([\psi(2^{-j})]^{-1}\right)_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}}, \\ k^{-\frac{m}{n}}\, H(k)^{-\frac{1}{2}}, & \text{sonst}, \end{cases}$$

beschrieben. Die Funktion H hängt dabei nur von n, m und ψ (nicht von p) ab. Jedoch benötigt man zur Bestimmung der Funktion H bestimmte Stammfunktionen sowie inverse Funktionen. Um daher Anwendungen in einigen Fällen zu erleichtern, kann man die Wachstumsrate der Funktion ψ mit Hilfe der Zahl

$$c := \lim_{t\to 0^+} \frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})}$$

ausdrücken und erhält folgendes Resultat

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{falls } 1 < c \leq \infty, \\ \psi(2^{-k})^{-1}, & \text{falls } 0 \leq c < 1. \end{cases}$$

Weite Teile dieser Dissertation sind in [Mi15a, Mi15b, Mi15c] veröffentlicht.

1 Introduction

In this thesis we deal with compact embeddings of type

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B), \qquad 1 \le p < \infty, m \in \mathbb{N}, \tag{1.1}$$

defined on the open unit ball B of $\mathbb{R}^n$. Here the weighted Sobolev space $E^m_{p,\psi}(B)$ is the closure of $C^m_0(B)$ in the Lebesgue space $L_p(B)$ with respect to the norm

$$\|f|E^m_{p,\psi}(B)\| = \Big(\int_B |x|^{mp}\psi(|x|)^p \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}} \tag{1.2}$$

and $\psi : (0,1] \to (0,\infty)$ is a slowly varying function which is bounded from below by a positive constant. The observations made in this thesis join a long history of compact embeddings between (weighted) function spaces on a domain $\Omega \subset \mathbb{R}^n$. Especially the asymptotic behaviour of entropy and approximation numbers has been of considerable interest within the last decades due to their applications to spectral theory. We refer to the monograph by Edmunds and Triebel [ET96]. Many authors dealt with these problems in different situations of (weighted) Besov and Triebel-Lizorkin spaces $A^s_{p,q}(\mathbb{R}^n, w)$, including Sobolev spaces. Thereby it is well known that the weights may influence the compactness and often emerge limiting situations. We refer to [HT94] and [Ha95] by Haroske and Triebel. More recently the technique of wavelet representation, see [HT05], has been successfully applied to deal with entropy numbers of embeddings of this type. It allows to transform problems in function spaces to the simpler context of sequence spaces. Based on this approach we want to mention the series of papers [HS08, HS11a, HS11b] by Haroske and Skrzypczak where function spaces with Muckenhoupt weights were considered and the series of papers [KLSS06a, KLSS06b, KLSS07] by Kühn, Leopold, Sickel and Skrypczak where the weights are bounded below away from zero and have no singularities. In all these papers the spaces were defined on $\mathbb{R}^n$. Our situation (1.1) is different. The weight is a positive power weight perturbed by a slowly varying function ψ, that means a function of almost logarithmic growth. In particular, it has a singularity at the origin $x = 0$. The first corresponding Hardy inequalities, that ensure the continuity of the embedding (1.1), were established by Triebel in [Tr12b] where the function $\psi(t)$ is a positive power of the logarithm. Furthermore, the compactness of the embedding, measured in terms of entropy and approximation numbers, is the main content of [Tr12b]. This last named paper can be seen as a starting point for the investigation made in this thesis. On the one hand we deal with more general weights. On the other hand we extend sharp results from [Tr12b] obtained for Hilbert spaces, if $p = 2$, to the general case of Banach spaces $1 \leq p < \infty$.

This thesis is organised as follows. In Section 2 we deal with basic properties of slowly varying functions since those functions serve as a compensation of the singular behaviour of the power weight in (1.2). Note that slowly varying functions extend the class of so-called admissible functions, introduced and treated in Section 2.1. The Hardy inequalities, that show the continuity of the embedding (1.1), are deduced in Corollary 2.15. There we prove the existence of a constant $0 < \delta < 1$ such that for all $f \in C_0^m(B_\delta)$

$$\int_{\mathbb{R}^n} \psi(|x|)^p |f(x)|^p \mathrm{d}x \ \leq \ c \int_{\mathbb{R}^n} |x|^{mp} \psi(|x|)^p \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \qquad (1.3)$$

where $B_\delta = \{x \in \mathbb{R}^n : |x| < \delta\}$. In Section 3.1 we introduce the embedding (1.1) and discuss the compactness of the embeddings. Theorem 3.4 states that $\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$ is compact if, and only if,

$$\lim_{t \to 0^+} \psi(t) = \infty. \qquad (1.4)$$

Now the special focus lies on the degree of compactness in terms of entropy and approximation numbers. The underlying concept has a long history. The interest of this subject is the arising opportunity of transferring asymptotic results to the spectral theory of diverse types of partial differential operators. The idea is to measure compactness of some linear and bounded operator T acting between two (quasi) Banach spaces with the help of a sequence of numbers, say $(\gamma_k(T))_{k\in\mathbb{N}}$, that is monotonically decreasing and tends to zero. Hence we can quantify compactness in the sense that the rate of decay of $(\gamma_k(T))_{k\in\mathbb{N}}$ can be interpreted as a degree of compactness. We say, the faster $(\gamma_k(T))_{k\in\mathbb{N}}$ tends to zero the better is the compactness of T. The concept of approximation numbers $(a_k(T))_{k\in\mathbb{N}}$ measures how well the operator T can be approximated by finite rank operators whereas entropy numbers $(e_k(T))_{k\in\mathbb{N}}$ are more related to geometric properties of T. In many situations one can use basic properties as sub-additivity and -multiplicativity combined with well-known classical embeddings to get estimates for entropy and approximation numbers. In Section 3.2 we execute these standard methods to our setting of (1.1). The results are Proposition 3.8 and Corollary 3.10. We show that

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim k^{-\frac{m}{n}} \tag{1.5}$$

if the growth rate of ψ is above a critical bound. This is a classical result. However, due to the singular behaviour of the weight, the utilised techniques do not give sharp results for the asymptotic behaviour of entropy and approximation numbers. But in case of Hilbert spaces, TRIEBEL proposed in [Tr12b] an approach via quadratic forms to seal this gap. Mainly COURANT's Max-Min-principle for positive definite self-adjoint operators comes into play which is so effective in determining the asymptotic limit of eigenvalue counting functions in L_2. This strategy is explained and applied in Section 3.3. Proposition 3.14 deals with the distribution of the eigenvalues of some degenerate elliptic operators A_ψ^m defined by

$$\mathrm{dom}((A_\psi^m)^{1/2}) = E_{2,\psi}^m(B)$$
$$A_\psi^m f = (-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha(b_{m,\psi}\mathrm{D}^\alpha f)$$

where $b_{m.\psi}(x) := |x|^{2m}\,\psi(|x|)^2$, $x \in B$. These spectral results can then be applied to entropy and approximation numbers of the embedding id : $E_{2,\psi}^m(B) \hookrightarrow L_2(B)$, see Theorem 3.16. The results are sharp, but as already mentioned the proof techniques depend on specific Hilbert space arguments. Especially the technique of Dirichlet-Neumann bracketing is not available if $p \neq 2$. But in [EH93] the authors EVANS and HARRIS developed a partial analogue for estimating the approximation numbers of Sobolev embeddings on a wide class of domains, see also Chapter 6.3 in [EE04] by EDMUNDS and EVANS. Although an unweighted setting of Sobolev spaces with smoothness parameter 1 was considered in the last named paper, it is presented in Section 4 how one can transfer this idea to control the singularity of the weight arising in (1.1). The essence there is to introduce quantities that obey bracketing properties now valid for $1 \leq p < \infty$ and that are suitable for cutting off the singularity without affecting the asymptotic behaviour. Here 'bracketing' refers to 'finite decomposition of the domain'. One can think of the eigenvalue counting function of the Dirichlet - or Neumann Laplace operator as a corresponding analogue in case of Hilbert spaces. One of the introduced quantities simply counts the number of approximation numbers greater than or equal to a certain bound. It is given by

$$\nu_0(\varepsilon,\Omega) := \max\{k\in\mathbb{N} : a_k(\mathrm{id}_\Omega)\ge\varepsilon\}, \qquad \varepsilon>0. \tag{1.6}$$

The operator id_Ω denotes the restriction of id from (1.1) to a subset $\Omega\subset B$. Since $\lim\limits_{k\to\infty} a_k(\mathrm{id})=0$, clearly the asymptotic behaviour of

$$\lim_{\varepsilon\to 0^+}\nu_0(\varepsilon,B)=\infty \tag{1.7}$$

is of interest now. On the other hand, we consider a quantity that is related to the injectivity of restriction operators to finite-dimensional subspaces, namely

$$\mu_0(\varepsilon,\Omega) := \max\left\{\dim S : \alpha(S)=\sup_{u\in S\setminus\{0\}}\frac{\|u|E^m_{p,\sigma}(\Omega)\|}{\|u|L_p(\Omega)\|}\le\frac{1}{\varepsilon}\right\} \tag{1.8}$$

where the maximum is taken over all finite-dimensional linear subspaces S of $E^m_{p,\sigma}(\Omega)$. Let $\Omega=\Big(\bigcup\limits_{j=1}^{J}\overline{\Omega}_j\Big)^\circ$ be a finite decomposition with disjoint domains Ω_j, $j=1,...,J$. The essential tool is the bracketing property from Proposition 4.6 which reads as

$$\sum_{j=1}^{J}\mu_0(\varepsilon,\Omega_j)\le\nu_0(\varepsilon,\Omega)\le\sum_{j=1}^{J}\nu_0(\varepsilon,\Omega_j). \tag{1.9}$$

This can be seen as an L_p-version of the Dirichlet-Neumann technique from spectral L_2-theory which was introduced and used in Section 3.3. In deed, Proposition 4.7 shows that $\nu_0(\varepsilon,\Omega)$ and $\mu_0(\varepsilon,\Omega)$ coincide with this technique if $p=2$. One can use (1.9) to cut off the singularity of the weight $|x|^m\psi(|x|)$ at the origin and consider the corresponding domains separately. In Section 4.1 we apply this approach to the setting from [Tr12b] where

$$\psi(t)=(1+|\log t|)^\sigma,\ \sigma>0, \tag{1.10}$$

and get in Proposition 4.10 sharp results for the approximation numbers

$$a_k(\mathrm{id}:E^m_{p,\psi}(B)\hookrightarrow L_p(B))\sim\begin{cases}k^{-\frac{m}{n}}, & \text{if } \sigma>\frac{m}{n},\\ k^{-\frac{m}{n}}(\log k)^{\frac{m}{n}}, & \text{if } \sigma=\frac{m}{n},\\ k^{-\sigma}, & \text{if } 0<\sigma<\frac{m}{n},\end{cases} \tag{1.11}$$

now valid for all $1\le p<\infty$. Generalisations in the sense of a slowly varying function are done in Section 4.2. The asymptotic behaviour of the approximation numbers is then given by

$$a_k(\mathrm{id})\sim\begin{cases}k^{-\frac{m}{n}}, & \text{if } \big([\psi(2^{-j})]^{-1}\big)_{j\in\mathbb{N}}\in\ell_{\frac{n}{m}},\\ k^{-\frac{m}{n}}H(k)^{-\frac{1}{2}}, & \text{otherwise},\end{cases} \tag{1.12}$$

with some function H depending on ψ and n, m, see Theorem 4.13.

Section 5 is devoted to the entropy numbers of the embedding id. Fortunately endowed with (1.11) and (1.12), we can apply Carl's inequality to get immediately upper bounds for the corresponding entropy numbers. On the other hand, similar constructions of basis functions as used for the lower bounds of (1.11) or rather (1.12) lead to lower estimates for the entropy numbers. Finally (1.11) and (1.12) also hold true for the entropy numbers $e_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B))$, see Theorems 5.1 and 5.2. In the logarithmic case this outcome confirms the Conjecture 3.8 made in [Tr12b].

In Section 6 we apply our results to slowly varying functions of the form

$$\begin{aligned}
&\textbf{i)}\ \psi(t) = (1 + |\log t|)^{\sigma}(1 + \log(1 + |\log t|))^{\gamma} \\
&\qquad\quad \text{with } \sigma > 0, \gamma \in \mathbb{R} \text{ or } \sigma = 0, \gamma > 0 \\
&\textbf{ii)}\ \psi(t) = \exp(|\log t|^{c}) \\
&\qquad\quad \text{with } 0 < c < 1 \\
&\textbf{iii)}\ \psi(t) = \exp\big((\log(1 + |\log t|))^{a}\big) \\
&\qquad\quad \text{with } a > 0.
\end{aligned}$$

Thereby special attention is paid to the influence of the growth rate of the weight, since it turns out that one can make applications sometimes easier if one considers the setting from that point of view. As (1.12) shows, the function $\psi(t)$ may influence the quality of the compactness if $\psi(t)$ tends to infinity too weakly. In particular, the determination of the function H requires primitives of $\psi(2^{-\cdot})^{-\frac{n}{m}}$ and inverting operations. In Corollary 6.7 we formulate simpler assertions (only using derivatives of ψ) to achieve

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \psi(2^{-k})^{-1}. \tag{1.13}$$

Measuring the growth rate of $\psi(t)$ in the number

$$c := \lim_{t \to 0^+} \frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})} \tag{1.14}$$

we get (roughly speaking) the following two cases in Corollary 6.8

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{if } 1 < c \leq \infty, \\ \psi(2^{-k})^{-1}, & \text{if } 0 \leq c < 1, \end{cases} \tag{1.15}$$

skipping the limiting case $c = 1$.

The main part of the results from this thesis is published in [Mi15a, Mi15b, Mi15c]. In [Mi15a] we give some first generalisation of [Tr12b] considering admissible functions instead of the logarithm. Continuing this line of investigation we deal with slowly varying functions in [Mi15b] including the discussion on the growth rate as described in Section 6. But in both last named papers sharp results are only obtained if $p = 2$. In [Mi15c] we close this gap for $1 \leq p < \infty$ with help of the bracketing method from Section 4.1 if ψ is a positive power of the logarithmic function. The corresponding results for slowly varying functions from Section 4.2 and partly from Section 5 are not published yet.

2 Slowly varying functions

First, we fix some notation. By $\mathbb{N}$ we mean the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components, by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}^n$, $n \in \mathbb{N}$, the Euclidean $n-$space. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ we fix its length $|\alpha| = \sum_{j=1}^n |\alpha_j|$ and put $\alpha! = \alpha_1! \cdots \alpha_n!$. The derivatives $\mathrm{D}^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})$ have the usual meaning. Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.
$C^\infty(\mathbb{R}^n)$ collects all infinitely continuously differentiable complex valued functions on $\mathbb{R}^n$ and $C_0^m(\mathbb{R}^n)$ all complex valued compactly supported functions on $\mathbb{R}^n$ having classical derivatives up to order $m \in \mathbb{N}_0$. Furthermore, $u \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ means that the complex-valued Lebesgue measurable function u on $\mathbb{R}_+ = (0, \infty)$ is Lebesgue integrable on each interval $(0, a)$ with $a > 0$.
All unimportant positive constants will be denoted by c, occasionally with subscripts. For two positive real sequences $\{\alpha_k\}_{k\in\mathbb{N}}$, $\{\beta_k\}_{k\in\mathbb{N}}$ or two positive functions $\phi(x)$, $\psi(x)$ we mean by

$$\alpha_k \sim \beta_k \quad \text{or} \quad \phi(x) \sim \psi(x)$$

that there exist constants $c_1, c_2 > 0$ such that

$$c_1\ \alpha_k \le \beta_k \le c_2\ \alpha_k \quad \text{or} \quad c_1\ \phi(x) \le \psi(x) \le c_2\ \phi(x).$$

A real function on the interval $(0, \infty]$ is called monotone if it is either decreasing or increasing, where decreasing (increasing) means non-increasing (non-decreasing). A real function on $(0, 1]$ is called *locally decreasing (or increasing) at zero*, if there is a $t_0 \in (0, 1]$ such that it is decreasing (or increasing) on $(0, t_0]$. Finally, log is always taken with respect to base 2 and $\exp(t) := \exp_2(t) = 2^t, t \in \mathbb{R}$.

2.1 Admissible functions

Since the seventies there was a growing interest among function spaces of generalised smoothness of Besov and Triebel-Lizorkin type from many

different starting points and in different contexts. These spaces cover the classical function spaces but the main smoothness parameter is finely tuned with some function ψ. In this field Edmunds and Triebel introduced (in connection with fractal sets) the class of admissible functions in [ET98, ET99]. Thereby the so called admissible functions were defined in terms of their qualitative behaviour, see Definition 2.1 below. We refer to [Mo01, Br01, Br02, CM04a, CM04b] where the authors followed this approach associated with an intense study of the class of admissible weights. A short overview of references and remarks about the history of function spaces of generalised smoothness may be found in [Tr06, Sect. 1.9.5, p.52-55].
Roughly speaking admissible functions are functions that have at most a logarithmic decay or growth near zero. This concept turns out to be very useful in various asymptotic investigations. In this work admissible functions in the sense of [ET98, ET99] will serve as fine adjustments for the compensations of power weights. This section serves as a brief introduction to that class of functions and as an overview about some useful properties.

Definition 2.1. *A positive monotone function $\psi : (0,1] \to \mathbb{R}$ is called* admissible *if for all $j \in \mathbb{N}_0$*

$$\psi(2^{-j}) \sim \psi(2^{-2j}). \tag{2.1}$$

Remark 2.2. Note that from (2.1) it follows for all $j \in \mathbb{N}_0$

$$\psi(2^{-(j+1)}) \sim \psi(2^{-j}). \tag{2.2}$$

Furthermore, we distinguish between the cases of monotonicity:

(i) If ψ is decreasing then there exists $\theta \geq 0$ such that

$$\psi(2^{-j}) \geq 2^{-\theta k}\psi(2^{-2^k j}), \quad j,k \in \mathbb{N}_0. \tag{2.3}$$

(ii) If ψ is increasing then there exists $\theta' \geq 0$ such that

$$\psi(2^{-j}) \leq 2^{\theta' k}\psi(2^{-2^k j}), \quad j,k \in \mathbb{N}_0. \tag{2.4}$$

We briefly derive (2.4). If ψ is increasing then there exists a constant $\theta > 0$ such that

$$\psi(2^{-2j}) \leq \psi(2^{-j}) \leq 2^{\theta}\psi(2^{-2j})$$

and (2.4) follows by iteration

$$\psi(2^{-j}) \leq 2^{\theta}\psi(2^{-2j}) \leq 2^{\theta}2^{\theta}\psi(2^{-2\cdot 2j}) \leq \ldots \leq 2^{\theta k}\psi(2^{-2^k j}).$$

The verification of (2.3) is similar.

Examples 2.3. Trivial examples of admissible functions are given by positive constants or positive functions with $\lim_{t\to 0}\psi(t) = c \in (0,\infty)$. Non-trivial examples are given by the logarithm function, the iterated logarithm and powers of it. That means that for any $m \in \mathbb{N}$, $\alpha = (\alpha_1, ...\alpha_m) \in \mathbb{R}^m$, the function

$$\psi(t) = \prod_{i=1}^{m} \ell_i(t)^{\alpha_i}, \qquad t \in (0,1], \tag{2.5}$$

is admissible, where $\ell_1, ..., \ell_m$ are defined by $\ell_1(t) = 1 + |\log t|$ and $\ell_i(t) = 1 + \log(\ell_{i-1}(t))$, $i = 2, ..., m$.

We collect a few properties of admissible functions. In particular, the next proposition shows that one can think of an admissible function as a function of at most logarithmic decay or growth near zero.

Proposition 2.4. *Let ψ be an admissible function on $(0,1]$.*

(i) *Let $a > 0$. Then it holds*

$$\lim_{t\to 0^+} t^a\, \psi(t) = 0.$$

(ii) *There exist constants $\theta \geq 0$ and $c_1, c_2 > 0$ such that for all $t \in (0,1]$ it holds*

$$c_1(1+|\log t|)^{-\theta} \leq \inf_{0<s\leq 1} \frac{\psi(ts)}{\psi(s)} \leq \psi(t) \leq \sup_{0<s\leq 1} \frac{\psi(ts)}{\psi(s)} \leq c_2\, (1+|\log t|)^{\theta}.$$

(iii) *For any $a > 0$ and $d > 0$ there is a constant $0 < \delta < 1$ such that for all $t \in (0,\delta)$ it holds*

$$\psi(at^d) \sim \psi(t).$$

(iv) *Let $0 < \delta < 1$. Then there exists a constant $c = c\,(\psi,\delta) > 0$ such that for all $t \in (0,1]$ it holds*

$$\psi(\delta\, t) \ \leq \ c\, \psi(t).$$

(v) *Let $\gamma > 1$. Then there exists a constant $c = c(\psi,\gamma) > 0$ such that for all $t \in (0, \frac{1}{\gamma}]$ it holds*

$$\psi(\gamma\, t) \ \leq \ c\, \psi(t).$$

Proof. *Step 1.* (i) follows from (ii). We prove (ii), see also [CM04a, Lemma 2.3 (i)] and [Mo01, Proposition 1.1.4 (vi)]. If ψ is an admissible increasing function then ψ^{-1} is still admissible but decreasing. Hence we may assume without loss of generality that ψ is monotone decreasing. Clearly

$$1 \leq \inf_{0<s\leq 1} \frac{\psi(ts)}{\psi(s)} \qquad \forall\ t \in (0,1]. \tag{2.6}$$

Let $\theta \geq 0$ be the constant from (2.3) such that for all $j, k \in \mathbb{N}_0$

$$\psi(2^{-2j}) \geq \psi(2^{-j}) \geq 2^{-\theta k}\psi(2^{-2^k j}). \tag{2.7}$$

Fix some $t \in (0,1]$. We show the existence of a constant $c_2 > 0$, independent of t, such that

$$\frac{\psi(ts)}{\psi(s)} \leq c_2\,(1 + |\log t|)^\theta \qquad \forall s \in (0,1]. \tag{2.8}$$

Therefore, let $s \in (0,1]$ and choose $k, j \in \mathbb{N}_0$ where $j \geq k$ such that

$$2^{-(k+1)} < s \leq 2^{-k} \qquad \text{and} \qquad 2^{-(j+1)} < ts \leq 2^{-j}.$$

We define the number $\nu := [\log(2+j-k)]+1$. Here $[x]$ denotes the largest integer smaller than x, i.e. $[x] = N \in \mathbb{N}_0$ if, and only if, $N < x \leq N+1$ and thus

$$\nu - 1 < \log(2+j-k) \leq \nu. \tag{2.9}$$

If $k \neq 0$ we claim

$$2^{-(j+1)} \geq 2^{-2^\nu k}. \tag{2.10}$$

The inequality (2.10) is clear for $j = k$, since

$$j + 1 \leq 2 \cdot j \leq 2^\nu j.$$

If $j > k$, recall the fact that

$$\frac{x}{y} \leq 1 + x - y \qquad \text{if } 1 \leq y \leq x$$

where we put $x = j+1$ and $y = k$. Then

$$\log\left(\frac{j+1}{k}\right) \leq \log(2+j-k) \leq \nu$$

what finishes the proof of (2.10). Now we use (2.10) and (2.7) to get

$$\psi(ts) \leq \psi(2^{-(j+1)}) \leq \psi(2^{-2^\nu k}) \leq 2^{\theta\nu}\psi(2^{-k}) \leq 2^{\theta\nu}\psi(s).$$

A similar estimate holds if $k = 0$. Namely, the assumption (2.9) with $k = 0$ also ensures $j + 1 \leq 2^\nu$ and we obtain

$$\psi(ts) \leq \psi(2^{-(j+1)}) \leq \psi(2^{-2^\nu}) \leq 2^{\theta\nu}\psi(2^{-1}) \leq \frac{\psi(2^{-1})}{\psi(1)}2^{\theta\nu}\psi(s).$$

Consequently there exists a constant $c > 0$, independent of s and t, that satisfies

$$\psi(ts) \leq c \cdot 2^{\theta\nu}\psi(s).$$

Then

$$\frac{\psi(ts)}{\psi(s)} \leq c \cdot 2^{\theta\nu} \leq c \cdot 2^{\theta}(2+j-k)^{\theta} \leq c \cdot 2^{\theta}(2+|\log t|+1)^{\theta}$$

completes the proof of (2.8).
Step 2. We prove (iii), see also [CM04a, Lemma 2.3]. One may assume again that ψ is decreasing. Due to [Mo01, Proposition 1.1.4 (iv)] there exists a number $j_0 \in \mathbb{N}_0$ such that for any $j_0 \in \mathbb{N}_0$ with $j \geq j_0$

$$\psi(a2^{-j}) \sim \psi(2^{-j}) \qquad \text{and} \qquad \psi(2^{-jd}) \sim \psi(2^{-j}).$$

For $t \in (0,1]$ let $k \in \mathbb{N}_0$ be such that

$$2^{-(k+1)} < t \leq 2^{-k}.$$

Then it follows

$$\psi(at) \leq \psi(a\,2^{-(k+1)}) \sim \psi(2^{-k}) \leq \psi(t) \qquad \text{if } k \geq j_0$$

and on the other hand

$$\psi(t) \leq \psi(2^{-(k+1)}) \sim \psi(a\,2^{-k}) \leq \psi(at) \qquad \text{if } k \geq j_0.$$

Hence there is a constant $\delta > 0$ such that

$$\psi(at) \sim \psi(t) \qquad \forall\, t \in (0,\delta).$$

Analogously one can prove that

$$\psi(t^d) \sim \psi(t) \qquad \forall\, t \in (0,\delta).$$

Step 3. We prove (iv). If ψ is increasing one may choose any $c \geq 1$ and (iv) is valid. We assume that ψ is decreasing. Fix a number $k \in \mathbb{N}$ such that $k \geq \ln(1-\ln\delta)$. This yields for all $j \in \mathbb{N}$

$$\begin{aligned} 2^k \geq 1 - \frac{\ln\delta}{j} &\iff \ln\delta \geq -j(2^k-1) \\ &\iff \delta \geq 2^j\, 2^{-j2^k}. \end{aligned}$$

For $t \in (0,1]$ let $j_0 \in \mathbb{N}$ be such that $2^{-j_0} \leq t \leq 2^{-(j_0-1)}$. Now we obtain due to (2.2) and (2.3) that

$$\psi(\delta t) \leq \psi(\delta 2^{-j_0}) \leq \psi(2^{-2^k j_0}) \leq 2^{\theta' k}\,\psi(2^{-j_0}) \leq c\,2^{\theta' k}\,\psi(t).$$

Step 4. We prove (v). One may assume that ψ is increasing. Let $j_0, k_0 \in \mathbb{N}_0$ be such that for $\gamma > 1$ and $t \in (0,\frac{1}{\gamma})$ it holds

$$2^{-(j_0+1)} \leq \gamma t \leq 2^{-j_0} \qquad \text{and} \qquad 2^{-(k_0+1)} \leq \gamma^{-1} \leq 2^{-k_0}.$$

Furthermore, let $k \in \mathbb{N}$ be with $k \geq \log(1 + k_0 + 2)$. This yields

$$k \geq \log(1 + \frac{k_0 + 2}{j_0}) \iff 2^k j_0 \geq j_0 + k_0 + 2.$$

With (2.4) we have

$$\begin{aligned}
\psi(\gamma t) \leq \psi(2^{-j_0}) &= \frac{\psi(2^{-j_0})}{\psi(2^{-(j_0+k_0+2)})} \psi(2^{-(k_0+1)}\, 2^{-(j_0+1)}) \\
&\leq \frac{\psi(2^{-j_0})}{\psi(2^{-(j_0+k_0+2)})}\, \psi(t) \\
&\leq 2^{\theta k} \frac{\psi(2^{-2^k j_0})}{\psi(2^{-(j_0+k_0+2)})} \psi(t) \\
&\leq 2^{\theta k}\, \psi(t).
\end{aligned}$$

Since k does only depend on γ and not on t, the proof of (v) is finished. This is an extension of [Mo01, Prop. 1.1.4 (v)] where $\gamma = 2$. $\square$

2.2 Slowly varying functions

This section is devoted to slowly varying functions which extend the class of admissible functions from Section 2.1. They were introduced by Karamata in the early 1930s. The theory of slowly varying functions is useful for various asymptotic investigations in analysis and has also revealed its use in probability theory. Furthermore, slowly varying functions emphasised their importance in the field of function spaces with generalised smoothness. Some remarks are already given at the beginning of Section 2.1. Investigations in that area are done by many authors. Besides the extensive Russian literature we want to mention Farkas/Leopold, Edmunds/Triebel, Moura, Bricchi and Cobos/Kühn. For more detailed references and historical remarks we refer to [KaL87, FaL06, Tr06]. Throughout this work slowly varying functions are of interest in that they enable to compensate local singularities of weighted function spaces. We start with the definition of slowly varying functions.

Definition 2.5. *A positive and measurable function $\psi : (0,1] \to (0,\infty)$ is called* slowly varying *if for all $s \in (0,1]$*

$$\lim_{t \to 0^+} \frac{\psi(st)}{\psi(t)} = 1. \tag{2.11}$$

We refer to Karamata [Ka30] where the author introduced slowly varying functions but required continuity instead of measurability. The monograph

[BGT87] offers a detailed treatment of slowly varying functions defined in a neighbourhood of infinity. In Definition 2.5 the function $\psi(t)$ is defined in a neighbourhood of the origin but one can adapt all statements from [BGT87] appropriately. To do so one should replace every occurrence of $\psi(\cdot)$ where ψ is slowly varying at the origin, with $\Psi(1/\cdot)$ where Ψ is slowly varying at infinity.

Slowly varying functions are often informally referred to *as logarithmic corrections.* In fact, the logarithmic function $\psi(t) = 1 + |\log t|$, $t \in (0, 1]$, is the most basic example. Keep in mind that all functions converging to a positive constant are trivially slowly varying. Hence the concept of *slow variation* must be understood according to the asymptotic behaviour of the function near zero - not to the function itself.

Examples 2.6. **(i)** Let $\psi(t) = (1 + |\log t|)^\sigma$, $\sigma \in \mathbb{R}$, defined for $t \in (0, 1]$. Then $\psi(t)$ is slowly varying since for all $s \in (0, 1]$

$$\lim_{t\to 0^+} \frac{\psi(st)}{\psi(t)} = \lim_{t\to 0^+} \Big(\frac{1 + |\log t| + |\log s|}{1 + |\log t|}\Big)^\sigma = \lim_{t\to 0^+} \Big(1 + \frac{|\log s|}{1 + |\log t|}\Big)^\sigma = 1.$$

(ii) Let $\psi(t) = (1 + \log(1 + |\log t|))^\gamma$, $\gamma \in \mathbb{R}$, defined for $t \in (0, 1]$. Then $\psi(t)$ is slowly varying since for all $s \in (0, 1]$

$$\begin{aligned}
\lim_{t\to 0^+} \frac{\psi(st)}{\psi(t)} &= \lim_{t\to 0^+} \Big(\frac{1 + \log(1 + |\log t| + |\log s|)}{1 + \log(1 + |\log t|)}\Big)^\gamma \\
&= \lim_{t\to 0^+} \Big(\frac{1 + \log(1 + |\log t|) + \log(1 + \frac{|\log s|}{1+|\log t|})}{1 + \log(1 + |\log t|)}\Big)^\gamma \\
&= \lim_{t\to 0^+} \Big(1 + \frac{\log(1 + \frac{|\log s|}{1+|\log t|})}{1 + \log(1 + |\log t|)}\Big)^\gamma \\
&= 1.
\end{aligned}$$

(iii) Let $\psi(t) = \exp(|\log t|^c)$, $0 < c < 1$, defined for $t \in (0, 1]$. To show that $\psi(t)$ is slowly varying we define for $t \in (0, 1)$ the function

$$h_t(s) := (|\log t| + |\log s|)^c, \qquad s \in (0, 1].$$

We express

$$\frac{\psi(st)}{\psi(t)} = \frac{\exp((|\log t| + |\log s|)^c)}{\exp(|\log t|^c)} = \exp(h_t(s) - h_t(1)).$$

Due to the mean value theorem there exists some $s_0 \in (s,1)$ such that

$$\begin{aligned} 0 < h_t(s) - h_t(1) = (s-1)(h_t)'(s_0) &= \frac{c\,(1-s)}{s_0}\,(|\log t| + |\log s_0|)^{c-1} \\ &< \frac{c}{s}\,(|\log t| + |\log s_0|)^{c-1} \\ &\le c_s \frac{1}{|\log t|^{1-c}}. \end{aligned}$$

The right-hand side tends to zero as $t \to 0$ for all values $s \in (0,1]$. This finishes the proof of

$$\lim_{t\to 0^+} \frac{\psi(st)}{\psi(t)} = \lim_{t\to 0^+} \exp(h_t(s) - h_t(1)) = 1.$$

Note that the function $\exp(|\log t|^c), 0 < c < 1$, has the interesting property of growing more rapidly to infinity than any positive power of the logarithm, but decays faster than any negative power function, i.e. for any $\varepsilon > 0$

$$\lim_{t\to 0^+} \frac{\exp(|\log t|^c)}{|\log t|^{\varepsilon}} = \infty \qquad \text{and} \qquad \lim_{t\to 0^+} \frac{\exp(|\log t|^c)}{t^{-\varepsilon}} = 0.$$

(iv) Let $\psi(t) = \exp([\log(1+|\log t|)]^a)$, $a \in \mathbb{R}$, defined for $t \in (0,1]$. Then $\psi(t)$ is slowly varying. Therefore, one should proceed analogously to (iii) now with a function $h_t(s) := [\log(1+|\log t| + |\log s|)]^a$, $s \in (0,1]$.

The following proposition is known as the *Representation Theorem* stated in [BGT87, Theorem 1.3.1].

Proposition 2.7. *A positive, measurable function $\psi : (0,1] \to (0,\infty)$ is slowly varying if, and only if, it can be written as*

$$\psi(t) = b(t)\,\exp\Big(-\int_t^1 \varepsilon(u)\frac{\mathrm{d}u}{u}\Big) \tag{2.12}$$

where b and ε are measurable functions on $(0,1]$ with

$$\lim_{t\to 0^+} b(t) =: b \in (0,\infty) \qquad \textit{and} \qquad \lim_{t\to 0^+} \varepsilon(t) = 0.$$

Examples 2.8. The representation (2.12) is not unique. All functions considered in the Examples 2.6 admit to choose the function $b(t) \equiv 1$. In that case we can easily determine a suitable function $\varepsilon(t)$ that satisfies

$$\log \psi(t) = -\int_t^1 \varepsilon(u)\frac{\mathrm{d}u}{u}.$$

(i) Let $\psi(t) = (1 + |\log t|)^\sigma (1 + \log(1 + |\log t|))^\gamma$ where $\sigma, \gamma \in \mathbb{R}$. Then we can express (2.12) with $b(t) = 1$ and

$$\begin{aligned}
\varepsilon(t) &= t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[\log \psi(t)\Big] \\
&= t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[\sigma \log(1 + |\log t|) + \gamma \log(1 + \log(1 + |\log t|))\Big] \\
&= \frac{-\sigma}{\log(1 + |\log t|)} + \frac{-\gamma}{(1 + \log(1 + |\log t|))(1 + |\log t|)}.
\end{aligned}$$

(ii) Let $\psi(t) = \exp(|\log t|^c)$ where $0 < c < 1$. Then we can express (2.12) with $b(t) = 1$ and

$$\varepsilon(t) = t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[\log \psi(t)\Big] = t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[|\log t|^c\Big] = -c\,|\log t|^{c-1}.$$

(iii) Let $\psi(t) = \exp\big((\log(1 + |\log t|))^a\big)$ where $a \in \mathbb{R}$. Then we can express (2.12) with $b(t) = 1$ and

$$\begin{aligned}
\varepsilon(t) &= t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[\log \psi(t)\Big] \\
&= t\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[(\log(1 + |\log t|))^a\Big] \\
&= \frac{-a}{(1 + |\log t|)(\log(1 + |\log t|))^{1-a}}.
\end{aligned}$$

From the representation (2.12) one can derive many useful properties of slowly varying functions. Nevertheless, in some cases another characterisation that is connected to monotone equivalents is more beneficial. We state this interrelation in the next proposition.

Proposition 2.9. *Let ψ be a positive, measurable function on* $(0, 1]$.

(i) *The function ψ is slowly varying if, and only if, for any $\varepsilon > 0$ there exist a decreasing function ϕ and an increasing function φ, defined on $(0, 1]$, such that there exits $t_0 \in (0, 1]$ and*

$$t^{-\varepsilon}\psi(t) \sim \phi(t) \qquad \text{and} \qquad t^{\varepsilon}\psi(t) \sim \varphi(t) \qquad \forall\, t \le t_0. \tag{2.13}$$

(ii) *If ψ is slowly varying such that*

$$\forall\, t_0 \in (0, 1]\ \exists c_1, c_2 > 0\ \forall\, t \in [t_0, 1] : c_1 \le \psi(t) \le c_2 \tag{2.14}$$

then (2.13) holds true for all $t \in (0, 1]$.

Proof. When dealing with slow variation at infinity a related proof of (i) can be found in [BGT87, Theorems 1.5.3, 1.5.4].
Step 1. Let ψ be slowly varying. We show (2.13). The function

$$f_\rho(x) := x^\rho \psi(\tfrac{1}{x}), \qquad x \in [1,\infty),$$

is regular varying with index $\rho \in \mathbb{R}$ in the sense of [BGT87, Sect. 1.4, p.18]. Hence the *Uniform Convergence Theorem* [BGT87, Theorem 1.5.2, p.22] tells us that

$$\sup_{1\le\lambda<\infty}\Big(\frac{f_\rho(\lambda x)}{f_\rho(x)} - \lambda^\rho\Big) \overset{x\to\infty}{\longrightarrow} 0 \qquad \text{if } \rho < 0. \tag{2.15}$$

For $\varepsilon > 0$ we define an increasing function φ by

$$\varphi(t) := \sup_{0<s\le t} s^\varepsilon \psi(s), \qquad t \in (0,1].$$

Then we get

$$\begin{aligned}
0 \le \frac{\varphi(t)}{t^\varepsilon\psi(t)} \;-\; 1 &= \frac{\sup_{0<s\le t} s^\varepsilon\psi(s)}{t^\varepsilon\psi(t)} \;-\; 1\\
&= \frac{\sup_{0<\mu\le 1}(\mu t)^\varepsilon\psi(\mu t)}{t^\varepsilon\psi(t)} \;-\; 1\\
&= \frac{\sup_{0<\mu\le 1} f_{-\varepsilon}(\frac{1}{\mu t})}{f_{-\varepsilon}(\frac{1}{t})} \;-\; 1\\
&\le \sup_{1\le\lambda<\infty}\Big(\frac{f_{-\varepsilon}(\lambda\frac{1}{t})}{f_{-\varepsilon}(\frac{1}{t})} - \lambda^{-\varepsilon}\Big) + \sup_{1\le\lambda<\infty} \lambda^{-\varepsilon} - 1\\
&\le \sup_{1\le\lambda<\infty}\Big(\frac{f_{-\varepsilon}(\lambda\frac{1}{t})}{f_{-\varepsilon}(\frac{1}{t})} - \lambda^{-\varepsilon}\Big).
\end{aligned}$$

Due to (2.15) this shows that

$$\lim_{t\to0^+}\frac{\varphi(t)}{t^\varepsilon\psi(t)} = 1.$$

Similarly one can define a decreasing function ϕ by

$$\phi(t) := \sup_{t\le s\le 1} s^{-\varepsilon}\psi(s), \qquad t \in (0,1],$$

such that it holds

$$\lim_{t\to0^+}\frac{\phi(t)}{t^{-\varepsilon}\psi(t)} = 1.$$

Step 2. Assume that (2.13) is true. We show

$$\lim_{t\to0^+}\frac{\psi(st)}{\psi(t)} = 1 \qquad \forall\; s \in (0,1]. \tag{2.16}$$

For every $\varepsilon > 0$ there exist functions $b_1(t)$ and $b_2(t)$ such that $\lim\limits_{t\to 0^+} b_1(t) = \lim\limits_{t\to 0^+} b_2(t) = 1$ and

$$\psi(t) = b_1(t)\, t^{-\varepsilon}\varphi(t) = b_2(t)\, t^{\varepsilon}\phi(t).$$

For $s \in (0,1]$ we get

$$\frac{b_2(st)}{b_2(t)} s^{-\varepsilon} = \frac{\psi(st)}{\psi(t)}\ \frac{\phi(t)}{\phi(st)} \le \frac{\psi(st)}{\psi(t)} \le \frac{\psi(st)}{\psi(t)}\ \frac{\varphi(t)}{\varphi(st)} = \frac{b_1(st)}{b_1(t)} s^{\varepsilon}$$

and hence

$$s^{-\varepsilon} \le \liminf_{t\to 0^+} \frac{\psi(st)}{\psi(t)} \le \limsup_{t\to 0^+} \frac{\psi(st)}{\psi(t)} \le s^{\varepsilon}.$$

Now letting ε tend to zero the proof of (2.16) is complete.
Step 3. We assume in addition (2.14). Let φ be an increasing function according to (2.13) for some fixed $\varepsilon > 0$. Then there exists a constant $t_0 \in (0,1]$ depending on $\varepsilon > 0$ and ψ such that

$$\forall t \in (0, t_0] : (1-\varepsilon)\varphi(t) < t^{\varepsilon}\psi(t) < (1+\varepsilon)\varphi(t).$$

For the remaining values $t \in [t_0, 1]$ we apply (2.14)

$$\forall t \in [t_0, 1] : c_1\, t_0^{\varepsilon}\, \frac{\varphi(t)}{\varphi(1)} \le c_1\, t_0^{\varepsilon} \le \psi(t) t^{\varepsilon} \le c_2 \le \frac{c_2}{\varphi(t_0)}\, \varphi(t).$$

Hence it holds for $t \in (0,1]$

$$\min\Big\{\frac{c_1 t_0^{\varepsilon}}{\varphi(1)}, 1-\varepsilon\Big\}\varphi(t) \le t^{\varepsilon}\psi(t) \le \max\Big\{\frac{c_2}{\varphi(t_0)}, 1+\varepsilon\Big\}\varphi(t).$$

A similar argumentation with a decreasing function ϕ lead to the validity of (2.13) for all values $t \in (0,1]$. □

Note that if we want to assume (2.13) on the whole interval $(0,1]$, a condition like (2.14) is necessary. As already mentioned the definition of slowly varying functions only refers to an asymptotic property near zero and in particular does not see what happens near 1. Hence we can easily construct examples of slowly varying functions such that (2.13) does not hold in a neighbourhood of 1. Simply consider the function

$$\psi(t) := \begin{cases} 1 & \text{if } 0 < t \le \frac{1}{2}, \\ 2(1-t) & \text{if } \frac{1}{2} < t < 1, \\ 1 & \text{if } t = 1. \end{cases}$$

If we assume an increasing function φ such that $c_1 t^{\varepsilon}\psi(t) \le \varphi(t) \le c_2 t^{\varepsilon}\psi(t)$ holds for all $t \in (0,1]$ then we get for the sequence $t_k = 1 - \frac{1}{k}, k \ge 3$, that

$\lim_{k\to\infty} \varphi(t_k) = \lim_{k\to\infty} 2(1-t_k) = \lim_{k\to\infty} \frac{2}{k} = 0$. But then it follows $\varphi(t_k) = 0$ for all $k \geq 3$ since φ is increasing. This is a contradiction.

Within the scope of this work we will consider slowly varying functions that are continuous and hence comply in particular with condition (2.14). That means that a continuous function $\psi : (0,1] \to (0,\infty)$ is slowly varying if, and only if, for any $\varepsilon > 0$ there exist a decreasing function ϕ and an increasing function φ such that

$$t^{-\varepsilon}\psi(t) \sim \phi(t) \qquad \text{and} \qquad t^{\varepsilon}\psi(t) \sim \varphi(t) \qquad \forall t \in (0,1]. \tag{2.17}$$

In the previous Section 2.1 we considered the class of admissible functions. Those functions are, up to equivalence, also slowly varying. It is known that for any admissible function ψ there exists an admissible function $\tilde{\psi} \sim \psi$ which is slowly varying, see [Br02, Prop.4.16] and [Br01]. Hence an admissible function need not to be slowly varying, but there is always an equivalent representative which is slowly varying. Conversely, there are examples for slowly varying functions which are not equivalent to any admissible function. The functions from (ii) and (iii) in the Examples 2.8 are of this type whereas the functions from (i) are also admissible.
The following proposition provides that the class of slowly varying functions is closed under the operations addition, multiplication and exponentiation.

Proposition 2.10. *Let ψ, ψ_1, ψ_2 be slowly varying functions on* $(0,1]$.

- **(i)** *The function ψ^a is slowly varying for every $a \in \mathbb{R}$.*
- **(ii)** *The product $\psi_1 \cdot \psi_2$ is slowly varying.*
- **(iii)** *The sum $\psi_1 + \psi_2$ is slowly varying.*

Proof. The assertions (i) and (ii) are obvious. (iii) emerges from straightforward argumentation observing that

$$\frac{\psi_1(st)+\psi_2(st)}{\psi_1(t)+\psi_2(t)} = \frac{\psi_1(st)}{\psi_1(t)} + \left[\frac{\psi_2(st)}{\psi_2(t)} - \frac{\psi_1(st)}{\psi_1(t)}\right]\frac{\psi_2(t)}{\psi_1(t)+\psi_2(t)}.$$

□

We list some elementary, but significant properties of slowly varying functions that will be used in the sequel.

Proposition 2.11. *Let ψ be a slowly varying function on* $(0,1]$.

- **(i)** *Let $a > 0$. Then it holds*

$$\lim_{t\to 0^+} t^a\,\psi(t) = 0. \tag{2.18}$$

(ii) *Let $\varepsilon > 0$. Then there exist constants $c > 1$ and $t_0 \in (0,1]$ such that for all $s \in (0,1]$*

$$\frac{1}{c}\, s^{\varepsilon} \le \frac{\psi(st)}{\psi(t)} \le c\, s^{-\varepsilon} \qquad \forall t \in (0, t_0]. \tag{2.19}$$

Furthermore, if ψ is bounded away from zero and satisfies (2.14) then (2.19) holds for all $t \in (0,1]$.

(iii) *Let $0 < \delta < 1$. Then there exist constants $c_1, c_2 > 0$, only depending on ψ, and $t_0 \in (0,1]$ such that*

$$c_1\, \delta\, \psi(t) \;\le\; \psi(\delta t) \;\le\; c_2\, \delta^{-1} \psi(t) \qquad \forall t \in (0, t_0]. \tag{2.20}$$

Furthermore, if ψ satisfies (2.14) then (2.20) holds for all $t \in (0,1]$.

(iv) *Let $\gamma > 1$. Then there exist constants $c_1, c_2 > 0$, only depending on ψ, and $t_0 \in (0, \frac{1}{\gamma}]$ such that*

$$c_1\, \gamma^{-1}\, \psi(t) \;\le\; \psi(\gamma t) \;\le\; c_2\, \gamma\, \psi(t) \qquad \forall t \in (0, t_0]. \tag{2.21}$$

Furthermore, if ψ satisfies (2.14) then (2.21) holds for all $t \in (0, \frac{1}{\gamma}]$.

Proof. (i) is stated in [BGT87, Proposition 1.3.6]. It is a simple consequence of the representation (2.12).
Step 1. We show (ii) what is known as *Potter's bound*, see [BGT87, Theorem 1.5.6, (i)]. Using the notation from (2.12), let $t_0 \in (0,1]$ be such that for all $0 < t \le t_0$

$$|\varepsilon(t)| \le \varepsilon \qquad \text{and} \qquad 1 - \varepsilon \le \frac{b(t)}{b} \le 1 + \varepsilon.$$

Then

$$\frac{\psi(st)}{\psi(t)} = \frac{b(st)}{b(t)} \exp\Big(- \int_{st}^{t} \varepsilon(u) \frac{\mathrm{d}u}{u} \Big) \le \frac{b(st)}{b(t)} \exp\Big(\varepsilon \int_{st}^{t} \frac{\mathrm{d}u}{u} \Big) \le \frac{1+\varepsilon}{1-\varepsilon}\, s^{-\varepsilon}$$

and similarly

$$\frac{\psi(st)}{\psi(t)} = \frac{b(st)}{b(t)} \exp\Big(- \int_{st}^{t} \varepsilon(u) \frac{\mathrm{d}u}{u} \Big) \ge \frac{b(st)}{b(t)} \exp\Big(- \varepsilon \int_{st}^{t} \frac{\mathrm{d}u}{u} \Big) \ge \frac{1-\varepsilon}{1+\varepsilon}\, s^{\varepsilon}.$$

This shows (2.19) for $0 < t \le t_0$. Next we consider the values $t > t_0$ under the additional assumption that ψ is bounded on the interval $[t_0, 1]$ and bounded away from zero. We denote

$$C_{max} := \max_{t_0 \le u \le 1} \psi(u) \qquad \text{and} \qquad C_{min} := \min_{0 < u \le 1} \psi(u) > 0.$$

If $t_0 \le st$ then

$$\frac{C_{min}}{\max\limits_{t_0\le u\le 1}\psi(u)} \le \frac{\psi(st)}{\psi(t)} \le \frac{\max\limits_{t_0\le u\le 1}\psi(u)}{C_{min}}.$$

In case of $t_0 > st$ we apply (2.19) with $\tilde{s} = \frac{st}{t_0} \le 1$ and $\tilde{t} = t_0$ to get

$$\frac{\psi(st)}{\psi(t)} = \frac{\psi(\tilde{s}\,t_0)}{\psi(t_0)}\,\frac{\psi(t_0)}{\psi(t)} \le c\left(\frac{st}{t_0}\right)^{-\varepsilon}\frac{\psi(t_0)}{\psi(t)} \le \frac{c\,C_{max}}{C_{min}}\left(\frac{st}{t_0}\right)^{-\varepsilon} \le \frac{c\,C_{max}}{C_{min}}\;s^{-\varepsilon}$$

Likewise we get

$$\frac{\psi(st)}{\psi(t)} \ge \frac{C_{min}}{c\,C_{max}}\;s^{\varepsilon}$$

and the proof of (ii) is complete.
Step 2. Using (2.13) (and assuming in addition (2.14)) the proof of (iii) is very simple. For $0 < \delta < 1$ we have

$$\psi(\delta t) \sim (\delta t)^{-1}\varphi(\delta t) \le (\delta t)^{-1}\varphi(t) \sim \delta^{-1}\psi(t)$$

and conversely

$$\psi(\delta t) \sim \delta t)\,\phi(\delta t) \ge \delta\, t\phi(t) \sim \delta\psi(t).$$

The proof of (iv) is carried out analogously. These are special cases of [GOT05, Proposition 2.2]. □

The last proposition shows that slowly varying functions are dominated by power functions. In particular,

$$\forall\varepsilon>0\,\exists c>0, t_0\in(0,1]\,\forall t\in(0,t_0]:\;\; c^{-1}t^{\varepsilon} \le \psi(t) \le c\,t^{-\varepsilon}. \tag{2.22}$$

For the sake of completeness we recall *Karamata's Theorem (direct half)* and refer to [BGT87, Proposition 1.5.8, Proposition 1.5.10]. Roughly speaking, one can treat slowly varying functions as constants when considering integrals of type

$$\int_t^1 s^{-\alpha}\psi(s)\mathrm{d}s \sim \psi(t)\int_t^1 s^{-\alpha}\mathrm{d}s \qquad \text{if } \alpha > 1$$

and t tends to zero. For $\alpha < 1$ the same holds true but integrating from 0 to t. The following proposition makes this precise.

Proposition 2.12. *Let $\psi : (0,1] \to (0,\infty)$ be a continuous, slowly varying function.*

(i) *For $\alpha > 1$ it holds*

$$\psi(t) \sim (\alpha-1)t^{\alpha-1}\int_t^1 \psi(s)s^{-\alpha}\mathrm{d}s \qquad \textit{as } t \searrow 0.$$

(ii) *For $\alpha < 1$ it holds*

$$\psi(t) \sim (1-\alpha)t^{\alpha-1}\int_0^t \psi(s)s^{-\alpha}\mathrm{d}s \qquad \text{as } t \searrow 0.$$

Proof. Let $\alpha > 1$. Then for $t \in (0,1]$

$$\begin{aligned}\frac{\int_t^1 \psi(s)s^{-\alpha}\mathrm{d}s}{\psi(t)t^{1-\alpha}} &= \int_t^1 \frac{\psi(s)}{\psi(t)}\Big(\frac{t}{s}\Big)^{\alpha}\frac{\mathrm{d}s}{t} = \int_1^{1/t}\frac{\psi(ut)}{\psi(t)}u^{-\alpha}\mathrm{d}u \\ &= \int_1^{\infty}\frac{\psi(ut)}{\psi(t)}\chi_{[1,1/t]}(u)u^{-\alpha}\mathrm{d}u.\end{aligned}$$

The integrand $f_t(u) := \frac{\psi(ut)}{\psi(t)}\chi_{[1,1/t]}(u)u^{-\alpha}$ converges pointwise to $u^{-\alpha}$ as $t \to 0$. Potter's bound (2.19) delivers an integrable majorant

$$|f_t(u)| \le c_\varepsilon u^{-(\alpha+\varepsilon)}, \qquad \varepsilon > 0.$$

Thus by dominated convergence we get for $\alpha > 1$

$$\lim_{t\to 0}\frac{\int_t^1 \psi(s)s^{-\alpha}\mathrm{d}s}{\psi(t)t^{1-\alpha}} = \int_1^{\infty} u^{-\alpha}\mathrm{d}u = \frac{1}{\alpha-1}.$$

This proves (i). Similarly one can show (ii) by deriving for $\alpha < 1$ that

$$\lim_{t\to 0}\frac{\int_0^t \psi(s)s^{-\alpha}\mathrm{d}s}{\psi(t)t^{1-\alpha}} = \int_0^{1} u^{-\alpha}\mathrm{d}u = \frac{1}{1-\alpha}.$$

□

2.3 Hardy inequalities

We turn towards some Hardy inequalities. The origin of this matter traces back to the early 20th century when Hardy proved in [Ha20, Ha25] the famous *classical Hardy inequality*

$$\int_0^{\infty}\Big(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\Big)^p\mathrm{d}x \le c_p\int_0^{\infty}|f(x)|^p\mathrm{d}x, \tag{2.23}$$

$1 < p < \infty$, for non-negative measurable functions f. From this point many authors improved and modified this pioneering result. The reader may consult [OK90, KMP07] and references given there. One can interpret (2.23) in the following sense: Let

$$(Hf)(x) := \frac{1}{x}\int_0^x f(t)\mathrm{d}t.$$

Then (2.23) expresses that the operator $H : L_p(0,\infty) \to L_p(0,\infty)$ is bounded with norm $\|H\| \le (c_p)^{1/p}$. The so-called *general Hardy Inequality* reads as

$$\Big(\int_a^b \big(\int_a^x f(t)\mathrm{d}t\big)^q u(x)\mathrm{d}x\Big)\Big)^{1/q} \le c_{p,q}\Big(\int_a^b f(x)^p v(x)\mathrm{d}x\Big)^{1/p} \tag{2.24}$$

where $-\infty \le a < b \le \infty$, $0 < q \le \infty$, $1 \le p < \infty$ and u, v are positive measurable functions. Note that the classical case (2.23) is covered with $p = q$, $a = 0$, $b = \infty$, $v(x) = 1$ and $u(x) = x^{-q}$. Putting $g(x) := \int_a^x f(t)\mathrm{d}t$, the inequality (2.24) admits an equivalent *differential version*

$$\Big(\int_a^b |g(x)|^q u(x)\mathrm{d}x\Big)^{1/q} \le c_{p,q}\Big(\int_a^b |g'(x)|^p v(x)\mathrm{d}x\Big)^{1/p} \tag{2.25}$$

where g is a differentiable function such that $g(a) = 0$. That is why Hardy inequalities are inequalities that estimate weighted L_p-norms of a function by weighted L_p-norms of derivatives of the function. The higher-dimensional analogue of (2.25) with $p = q$ is

$$\int_\Omega |g(x)|^p u(x)\mathrm{d}x \le c_p \int_\Omega |\nabla g(x)|^p v(x)\mathrm{d}x, \qquad 1 \le p < \infty, \tag{2.26}$$

where $\Omega \ne \mathbb{R}^n$ is a bounded domain. For the purpose of this work we establish some weighted Hardy inequalities of higher order $m \in \mathbb{N}$ such as

$$\int_{B_\delta} |g(x)|^p u(x)\mathrm{d}x \le c \int_{B_\delta} \sum_{|\alpha|=m} |\mathrm{D}^\alpha g(x)|^p v(x)\mathrm{d}x \tag{2.27}$$

where B_δ is the scaled unit ball in $\mathbb{R}^n$ of radius $\delta > 0$

$$B_\delta = \{x \in \mathbb{R}^n : |x| < \delta\}. \tag{2.28}$$

The weight function $u(x)$ is of type $|\log|x||^\sigma w(|x|)$ whereas $v(x)$ is of type $|x|^{mp}|\log|x||^\sigma w(|x|)$ with a suitable function w. This is stated below in Proposition 2.13. In Corollary 2.15 it turns out that one can choose the function w such that

$$u(x) = \psi(|x|) \qquad \text{and} \qquad v(x) = |x|^{mp}\psi(|x|)$$

where $\psi : (0,1] \to (0,\infty)$ is slowly varying in the sense of Definition 2.5. We denote by $C_0^m(B_\delta)$ the collection of all complex-valued functions f having classical derivatives up to order $m \in \mathbb{N}$ with compact support $\operatorname{supp} f \subset B_\delta$. The following logarithmic Hardy inequality is proved in [Tr12b, Theorem 2.4].

Proposition 2.13. *Let $n, m \in \mathbb{N}$, $1 \le p < \infty$ and $\sigma \in \mathbb{R}$. Let w be a Lebesgue measurable function on $\mathbb{R}_+$ with $0 < w(t) < \infty$, $t > 0$, such that*

$$\max(1, -\log t)^{\sigma p}\ t^{n-1} w(t) \in L_1^{\mathrm{loc}}\big([0,\infty)\big) \tag{2.29}$$

and

$$\int_1^\infty t^{-\frac{n}{p}-1}\Big(\sup_{\tau>0}\frac{w(\tau)}{w(\tau t)}\Big)^{\frac{1}{p}}\mathrm{d}t \ < \ \infty. \tag{2.30}$$

Then there are numbers $0 < \delta < 1$ and $c > 0$ such that for all $f \in C_0^m(B_\delta)$

$$\Big(\int_{\mathbb{R}^n} |\log|x||^{\sigma p}|f(x)|^p w(|x|)\mathrm{d}x\Big)^{1/p}$$
$$\le c\,\Big(\int_{\mathbb{R}^n} |x|^{mp}|\log|x||^{\sigma p}\sum_{|\alpha|=m}|\mathrm{D}^\alpha f(x)|^p\, w(|x|)\mathrm{d}x\Big)^{1/p}. \tag{2.31}$$

We want to generalise these inequalities and involve slowly varying weights. First we need to extend a slowly varying function defined on $(0,1]$ conveniently to $(0,\infty)$ as done in the next proposition.

Proposition 2.14. *Let ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1) = 1$. The function $\Psi : (0,\infty) \to (0,\infty)$ is defined by*

$$\Psi(t) := \begin{cases}\psi(t), & 0 < t \le 1,\\ \psi(t^{-1})^{-1}, & t > 1.\end{cases} \tag{2.32}$$

Then for every $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that for all $t \in (0,\infty)$

$$c_1 \min(t^{-\varepsilon}, t^{\varepsilon}) \le \sup_{\tau>0}\frac{\Psi(\tau)}{\Psi(t\tau)} \le c_2 \max(t^{-\varepsilon}, t^{\varepsilon}). \tag{2.33}$$

If ψ is in addition admissible, then there exist constants $b \ge 0$ and $c_1, c_2 > 0$ such that for all $t \in (0,\infty)$

$$c_1(1 + |\log t|)^{-b} \le \sup_{\tau>0}\frac{\Psi(\tau)}{\Psi(t\tau)} \le c_2(1 + |\log t|)^{b}. \tag{2.34}$$

Proof. We illustrate the proof of (2.33) for $t > 1$. We use ideas from [CM04a, Lemma 2.5] where the inequalities (2.34) have been shown. For $\varepsilon > 0$ let ϕ be decreasing and φ increasing according to (2.17). Then

$$\sup_{0<\tau\le\frac{1}{t}}\frac{\Psi(\tau)}{\Psi(t\tau)} = \sup_{0<\tau\le\frac{1}{t}}\frac{\psi(\tau)}{\psi(t\tau)} \le c\,t^{\varepsilon}\sup_{0<\tau\le\frac{1}{t}}\frac{\varphi(\tau)}{\varphi(t\tau)} \le c\,t^{\varepsilon}$$

as well as

$$\begin{aligned}\sup_{\frac{1}{t}<\tau\le 1}\frac{\Psi(\tau)}{\Psi(t\tau)} &= \sup_{\frac{1}{t}<\tau\le 1}\psi(\tau)\,\psi((t\tau)^{-1})\\ &\le c\,t^{\varepsilon}\sup_{\frac{1}{t}<\tau\le 1}\varphi(\tau)\sup_{\frac{1}{t}<\tau\le 1}\varphi((t\tau)^{-1})\\ &\le c\,t^{\varepsilon}\Big[\sup_{\frac{1}{t}<\sigma\le 1}\varphi(\sigma)\Big]^2\\ &\le c\,t^{\varepsilon}\end{aligned}$$

and

$$\begin{aligned}\sup_{\tau>1}\frac{\Psi(\tau)}{\Psi(t\tau)} &= \sup_{\tau>1}\frac{\psi((t\tau)^{-1})}{\psi(\tau^{-1})}\\ &\le c\,t^{\varepsilon}\sup_{\tau>1}\frac{\phi((t\tau)^{-1})}{\phi(\tau^{-1})}\\ &\le c\,t^{\varepsilon}.\end{aligned}$$

Similarly one gets the lower estimates of (2.33). □

Note that a slight modification of the last result can be found in [GOT05, Prop. 2.2,(iii)]. Now we can give Hardy inequalities involving slowly varying weights.

Corollary 2.15. *Let $m_1, m_2 \in \mathbb{N}_0$ with $m_1 \le m_2$ and $1 \le p < \infty$. Let ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1) = 1$. Then there are numbers $0 < \delta < 1$ and $c > 0$ such that for all $f \in C_0^{m_2}(B_\delta)$ it holds*

$$\int_{\mathbb{R}^n}|x|^{m_1 p}\psi(|x|)^p\sum_{|\alpha|=m_1}|\mathrm{D}^\alpha f(x)|^p\mathrm{d}x \le c\int_{\mathbb{R}^n}|x|^{m_2 p}\psi(|x|)^p\sum_{|\alpha|=m_2}|\mathrm{D}^\alpha f(x)|^p\mathrm{d}x.$$

In particular, for $m \in \mathbb{N}$ and $f \in C_0^m(B_\delta)$ it holds

$$\int_{\mathbb{R}^n}\psi(|x|)^p|f(x)|^p\mathrm{d}x \;\le\; c\int_{\mathbb{R}^n}|x|^{mp}\psi(|x|)^p\sum_{|\alpha|=m}|\mathrm{D}^\alpha f(x)|^p\mathrm{d}x. \qquad (2.35)$$

Proof. We apply Proposition 2.13 with $n = 1$, $\sigma = 0$ and $w(t) = t^{m_1 p}\ \Psi^p(t)$, $t > 0$, according to (2.32). Then

$$\max(1, -\log t)^{\sigma p}\ t^{n-1}w(t) = w(t) = t^{m_1 p}\ \Psi^p(t)$$

and the local integrability (2.29) follows by the fact that $t^{m_1 p}\ \Psi^p(t)$ is bounded on every interval $[a,b] \subset [0,\infty)$. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{p} + m_1$. Recall (2.33) and choose $c_\varepsilon > 0$ such that

$$\sup_{\tau>0}\frac{\Psi(\tau)}{\Psi(t\tau)} \le c_\varepsilon\, t^{\varepsilon}, \qquad t > 1.$$

Now we derive the condition (2.30) from

$$\int_1^\infty t^{-\frac{1}{p}-1}\Big(\sup_{\tau>0}\frac{w(\tau)}{w(\tau t)}\Big)^{\frac{1}{p}}\mathrm{d}t = \int_1^\infty t^{-(1+\frac{1}{p}+m_1)}\Big(\sup_{\tau>0}\frac{\Psi(\tau)}{\Psi(\tau t)}\Big)^{\frac{1}{p}}\mathrm{d}t$$
$$\leq c_\varepsilon{}^{\frac{1}{p}}\int_1^\infty t^{-(1+\frac{1}{p}+m_1-\varepsilon)} < \infty$$

and all required assumptions of Proposition 2.13 are fulfilled. So if $f \in C_0^{m_2}(B_\delta)$ (what means $\mathrm{D}^\alpha f \in C_0^{m_2-|\alpha|}(B_\delta)$) then the inequality (2.31) with $\sigma = 0$ and $w(t) = t^{m_1 p}\ \Psi^p(t)$ results in

$$\sum_{|\alpha|=m_1}\int_{\mathbb{R}^n}|\mathrm{D}^\alpha f(x)|^p\,|x|^{m_1p}\psi(|x|)^p\mathrm{d}x = \sum_{|\alpha|=m_1}\int_{B_\delta}|\mathrm{D}^\alpha f(x)|^p\,w(|x|)\mathrm{d}x$$
$$\leq c\sum_{|\alpha|=m_1}\int_{B_\delta}|x|^{(m_2-m_1)p}\sum_{|\gamma|=m_2-m_1}|\mathrm{D}^\gamma\mathrm{D}^\alpha f(x)|^p\,w(|x|)\mathrm{d}x$$
$$\leq c\sum_{|\alpha|=m_1}\int_{\mathbb{R}^n}|x|^{m_2p}\sum_{|\gamma|=m_2-m_1}|\mathrm{D}^\gamma\mathrm{D}^\alpha f(x)|^p\,\psi(|x|)^p\mathrm{d}x$$
$$\leq c\int_{\mathbb{R}^n}|x|^{m_2p}\,\psi(|x|)^p\sum_{|\beta|=m_2}|\mathrm{D}^\beta f(x)|^p\mathrm{d}x.$$

□

3 Compact embeddings of weighted Sobolev spaces

The aim of this work is to study compact embeddings of weighted Sobolev spaces, defined on the unit ball, into Lebesgue spaces. The weight is of polynomial growth perturbed by a slowly varying function and has a singularity at the origin. Let the unit ball in $\mathbb{R}^n$ be denoted by

$$B = \{x \in \mathbb{R}^n : |x| < 1\}.$$

We start collecting some classical settings for retrieval. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let $L_p(\Omega)$ with $1 \leq p < \infty$ be the Lebesgue space of all complex-valued Lebesgue measurable functions on Ω such that

$$\|f|L_p(\Omega)\| := \Big(\int_\Omega |f(x)|^p \mathrm{d}x \Big)^{1/p} < \infty.$$

Furthermore, $D(\Omega) = C_0^\infty(\Omega)$ and the set of all complex distributions $D'(\Omega)$ have their usual meaning. $L_p(\Omega)$ and all other spaces introduced below are considered in the standard setting of $D'(\Omega)$.
For $m \in \mathbb{N}$ the classical Sobolev space $W_p^m(\Omega) = \big\{ f \in L_p(\Omega) : \mathrm{D}^\alpha f \in L_p(\Omega) \ , |\alpha| \leq m \big\}$, $1 \leq p < \infty$, is normed by

$$\big\| f \mid W_p^m(\Omega) \big\| := \Big(\sum_{|\alpha| \leq m} \|\mathrm{D}^\alpha f | L_p(\Omega)\|^p \ \Big)^{1/p}. \tag{3.1}$$

Here we consider the Sobolev space $W_p^m(\Omega)$ defined via some intrinsic way. Another usual approach is the definition by restriction from $\mathbb{R}^n$ to corresponding domains. But as long as one deals with domains with a 'smooth' boundary these spaces coincide. Let $\overset{\circ}{W}{}_p^m(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in $W_p^m(\Omega)$. Thus $f \in L_p(\Omega)$ belongs to $\overset{\circ}{W}{}_p^m(\Omega)$ if, and only if, there exists a sequence of functions $(f_k)_{k\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ such that $f_k \overset{k\to\infty}{\longrightarrow} f$ in $W_p^m(\Omega)$. In view of Friedrichs' inequality the space $\overset{\circ}{W}{}_p^m(\Omega)$ can be equivalently normed by

$$\big\| f \mid \overset{\circ}{W}{}_p^m(\Omega) \big\| := \Big(\sum_{|\alpha| = m} \|\mathrm{D}^\alpha f | L_p(\Omega)\|^p \ \Big)^{1/p}. \tag{3.2}$$

In the next proposition we recall the non-trivial, but well-known fact that $\overset{\circ}{W}{}^m_p(\Omega)$ is compactly embedded in $L_p(\Omega)$. This compactness is fundamental for many applications of linear and non-linear functional analysis to partial differential equations.

Proposition 3.1. *Let $1 \leq p < \infty$ and $m \in \mathbb{N}$. Then the embedding*

$$\overset{\circ}{W}{}^m_p(\Omega) \hookrightarrow L_p(\Omega) \tag{3.3}$$

is continuous and compact.

Proof. *Step 1.* In order to show the continuity of the embedding (3.3) we claim

$$\|f\,|L_p(\Omega)\| \leq c\,\|f\,|\overset{\circ}{W}{}^m_p(\Omega)\|, \qquad f \in C_0^\infty(\Omega) \tag{3.4}$$

Referring to [EE87, Theorem 3.22, p. 242], we first assume without loss of generality

$$\Omega \subset Q := \{x \in \mathbb{R}^n : 0 < x_j < d,\, j = 1, ..., n\}, \qquad d > 0,$$

and extend $f \in C_0^\infty(\Omega)$ by zero in $Q\backslash\Omega$. Then obviously for $x = (x', x_n) \in Q$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$

$$f(x) = \int_0^{x_n} \frac{\partial f}{\partial x_n}(x', y)\mathrm{d}y.$$

If $1 < p < \infty$ we get by Hölder's inequality

$$|f(x)|^p \leq d^{\frac{p}{p'}} \int_0^d \left|\frac{\partial f}{\partial x_n}(x', y)\right|^p \mathrm{d}y$$

where $1 < p' < \infty$ is the Hölder conjugate of p. Integration over Q yields

$$\int_\Omega |f(x)|^p \mathrm{d}x \leq d^{\frac{p}{p'}} \int_Q \int_0^d \left|\frac{\partial f}{\partial x_n}(x', y)\right|^p \mathrm{d}y\mathrm{d}x \leq \quad d^{\frac{p}{p'}} \cdot d \cdot \int_\Omega \left|\frac{\partial f}{\partial x_n}(x)\right|^p \mathrm{d}x.$$

Likewise, corresponding estimates hold for $p = 1$. One can apply the last steps to $x_j, j = 1, ..., n$ instead of x_n. This shows

$$\int_\Omega |f(x)|^p \mathrm{d}x \leq d^p \int_\Omega \left|\frac{\partial f}{\partial x_j}(x)\right|^p \mathrm{d}x, \qquad j = 1, ..., n,$$

what proves (3.4) if $m = 1$. If $m > 1$ we iterate the last inequality

$$\int_\Omega |f(x)|^p \mathrm{d}x \leq d^{mp} \int_\Omega \left|\frac{\partial^m f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}(x)\right|^p \mathrm{d}x$$

for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = m$ and the proof of (3.4) is complete. By standard density arguments the inequality (3.4) holds for all

$f \in \overset{\circ}{W}{}_p^m(\Omega)$.
Step 2. For a proof of the compactness of the embedding (3.3) we refer to [EE87, Theorem 4.18, p.269]. Furthermore, we give some remarks depending upon whether

$$\begin{array}{rl} \text{(I)} & 1 \le p < \frac{n}{m}, \\ \text{(II)} & p = \frac{n}{m}, \\ \text{(III)} & \frac{n}{m} < p < \infty. \end{array}$$

One of the most classical results in context with Sobolev embedding theorems goes back to Kondrachov in 1945 and earlier to Rellich in case of Hilbert spaces. As stated in [EE87, Theorem 3.7, p. 230], if $1 \le p < \infty$ with $mp < n$ then the embedding

$$\overset{\circ}{W}{}_p^m(\Omega) \hookrightarrow L_q(\Omega), \qquad p \le q < \frac{np}{n-mp}, \tag{3.5}$$

is compact. Hence we can choose $q = p$ in case of (I). Note that any bounded sequence $(f_j)_{j\in\mathbb{N}}$ in $\overset{\circ}{W}{}_p^m(\Omega)$ is also bounded in $\overset{\circ}{W}{}_{p-\varepsilon}^m(\Omega)$ if $\varepsilon > 0$. If $p = \frac{n}{m}$ we choose $\varepsilon > 0$ such that

$$p < \frac{n(p-\varepsilon)}{n-m(p-\varepsilon)} \iff \varepsilon < \frac{n}{2m}.$$

Now by (I) applied to $p - \varepsilon < \frac{n}{m}$, we can use the compactness of the embedding $\overset{\circ}{W}{}_{p-\varepsilon}^m(\Omega) \hookrightarrow L_p(\Omega)$ and (II) follows. For the proof of (III) we refer to [EE87, Theorem 3.20, p. 241] with $l = 0$. There it is proved that for $\gamma \in (0,1]$ with $(m-\gamma)p > n$ the embedding

$$\overset{\circ}{W}{}_p^m(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$$

is compact where the Hölder space $C^{0,\gamma}(\overline{\Omega})$, $\gamma \in (0,1]$, consists of those functions f that are Hölder continuous with exponent γ, i.e. $|f(x)-f(y)| \le C|x-y|^\gamma$, $x, y \in \Omega$. Now we can choose in case of (III) the constant $\gamma \in (0,1]$ such that

$$(m-\gamma)p > n \iff p > \frac{n}{m-\gamma}.$$

Therefore, the embedding $\overset{\circ}{W}{}_p^m(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega}) \hookrightarrow L_p(\Omega)$ is compact if (III).
□

3.1 Weighted Sobolev spaces

We introduce a weighted version of $\overset{\circ}{W}{}_p^m(\Omega)$. Let ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1)=1$. We define for $f \in C_0^m(B)$

$$\|f \;|E^m_{p,\psi}(B)\| := \left(\int_B |x|^{mp}\; \psi(|x|)^p \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \right)^{1/p} \tag{3.6}$$

where $1 \le p < \infty$ and $m \in \mathbb{N}$. Then it follows for all $0 \le m_1 \le m_2$ and all functions $f \in C_0^{m_2}(B)$

$$\|f|E^{m_1}_{p,\psi}(B)\| \le c\; \|f|E^{m_2}_{p,\psi}(B)\|. \tag{3.7}$$

This can be seen by applying Corollary 2.15 to $f(\cdot/\delta) \in C_0^{m_2}(B_\delta)$

$$\begin{aligned}
\|f|E^{m_1}_{p,\psi}(B)\|^p &= \int_B |x|^{m_1 p}\; \psi(|x|)^p \sum_{|\alpha|=m_1} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \\
&= \delta^{-n} \int_{B_\delta} \delta^{-m_1 p} |y|^{m_1 p}\; \psi(\delta^{-1}|y|)^p \sum_{|\alpha|=m_1} |\mathrm{D}^\alpha (f(\tfrac{y}{\delta}))|^p \mathrm{d}y \\
&\le c\; \delta^{-(n+m_1 p)} \int_{B_\delta} |y|^{m_2 p}\; \psi(|y|)^p \sum_{|\alpha|=m_2} |\mathrm{D}^\alpha (f(\tfrac{y}{\delta}))|^p \mathrm{d}y \\
&= c\; \delta^{-m_1 p} \int_B |x|^{m_2 p} \psi(\delta|x|)^p \sum_{|\alpha|=m_2} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \\
&\le c\; \delta^{-m_1 p}\, \|f|E^{m_2}_{p,\psi}(B)\|^p
\end{aligned}$$

where we also used (iii) and (iv) from Proposition 2.11. In particular, if ψ is bounded from below by a positive constant, i.e.

$$\exists\, \gamma > 0\; \forall\, t \in (0,1] : \psi(t) \ge \gamma, \tag{3.8}$$

then we get from (3.7) with $m_1 = 0$ and $m_2 = m$ that

$$\|f|L_p(B)\| \le c\|f|E^m_{p,\psi}(B)\|. \tag{3.9}$$

Hence $\|\cdot|E^m_{p,\psi}(B)\|$ defines a norm on the space $C_0^m(B)$.

Definition 3.2. *Let $1 \le p < \infty$, $m \in \mathbb{N}_0$ and ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1) = 1$ and bounded from below by a positive constant, i.e.* (3.8)*. Then the* weighted Sobolev space $E^m_{p,\psi}(B)$ *is the closure of $C_0^m(B)$ in $L_p(B)$ with respect to the norm*

$$\|f \;|E^m_{p,\psi}(B)\| := \left(\int_B |x|^{mp}\; \psi(|x|)^p \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \right)^{1/p}. \tag{3.10}$$

Remark 3.3. Hereby we mean that $E^m_{p,\psi}(B)$ is the collection of all functions in $L_p(B)$ that are limit elements of convergent sequences in $C_0^m(B)$ in the norm (3.10). This definition goes back to [Tr12b] in which $\psi(t) = (1+|\log t|)^\sigma, \sigma \geq 0$, was considered. Nevertheless, there is an alternative way to introduce the space $E^m_{p,\psi}(B)$. We denote by $W_p^{m,loc}(\Omega)$ the set of all distributions in $L_p(\Omega)$ that belong to $W_p^m(K)$ for every compact subset $K \subset \Omega$. Then the space $\overset{*}{E}{}^m_{p,\psi}(B)$ is given by

$$\overset{*}{E}{}^m_{p,\psi}(B) := \Big\{ f \in W_p^{m,loc}(\mathbb{R}^n \setminus \{0\}) : \operatorname{supp} f \subset \overline{B}, \tag{3.11}$$
$$\|f|\overset{*}{E}{}^m_{p,\psi}(B)\| := \Big(\int_B \psi(|x|)^p \sum_{|\gamma|\leq m} |x|^{|\gamma|p} \, |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}} < \infty \Big\}.$$

We claim

(1) $\overset{*}{E}{}^m_{p,\psi}(B)$ is a Banach space,

(2) $C_0^\infty(\dot{B})$ is dense in $\overset{*}{E}{}^m_{p,\psi}(B)$ where $\dot{B} = \{x \in \mathbb{R}^n : 0 < |x| < 1\}$,

(3) (3.10) is an equivalent norm on $\overset{*}{E}{}^m_{p,\psi}(B)$.

Part (1) follows by standard arguments. We prove (2). Let $\varphi \in C^\infty(\mathbb{R}^n)$ be such that $\varphi(x) \equiv 1$ on $|x| \leq 1$ and $\varphi(x) \equiv 0$ on $|x| \geq 2$. For $j \in \mathbb{N}_0$ we put

$$\varphi_j(x) := \varphi(2^j x), \qquad x \in \mathbb{R}^n.$$

Then $\varphi_j(x) \equiv 1$ on $|x| \leq 2^{-j}$, $\varphi_j(x) \equiv 0$ on $|x| \geq 2^{-j+1}$ and $|\mathrm{D}^\beta \varphi_j| \leq c 2^{j|\beta|}$, $\beta \in \mathbb{N}_0^n$. In particular,

$$\operatorname{supp} \varphi_j \subseteq \{x \in \mathbb{R}^n : |x| < 2^{-j+1}\} =: B^j$$

and for $|\beta| \geq 1$

$$\operatorname{supp} \mathrm{D}^\beta \varphi_j \subseteq \{x \in \mathbb{R}^n : 2^{-j} \leq |x| < 2^{-j+1}\}.$$

One has for $f \in \overset{*}{E}{}^m_{p,\psi}(B)$

$$\|\varphi_j f|\overset{*}{E}{}^m_{p,\psi}(B)\|^p = \int_{B^j} \psi(|x|)^p \sum_{|\gamma|\leq m} |x|^{|\gamma|p} \, |\mathrm{D}^\gamma (f\varphi_j(x))|^p \mathrm{d}x$$
$$\leq \int_{B^j} \psi(|x|)^p \sum_{|\gamma|\leq m} |x|^{|\gamma|p} \sum_{|\gamma_1|+|\gamma_2|=|\gamma|} c_{\gamma_1,\gamma_2} |\mathrm{D}^{\gamma_1} \varphi_j(x)|^p |\mathrm{D}^{\gamma_2} f(x)|^p \mathrm{d}x.$$

If $|\gamma_1| \geq 1$ in the second sum then the integration is only over the set $\{x \in \mathbb{R}^n : 2^{-j} \leq |x| < 2^{-j+1}\}$ and hence $|\mathrm{D}^{\gamma_1} \varphi_j(x)|^p \leq c\, 2^{j|\gamma_1|p} \sim |x|^{-|\gamma_1|p}$.

We get

$$
\begin{aligned}
&\|\varphi_j f|\overset{*}{E}{}^m_{p,\psi}(B)\|^p \\
&\quad\le c\int_{B^j}\psi(|x|)^p\sum_{|\gamma|\le m}|x|^{|\gamma|p}\Big[|\mathrm{D}^\gamma f(x)|^p+\sum_{0\le|\gamma_2|<|\gamma|}|x|^{-(|\gamma|-|\gamma_2|)p}|\mathrm{D}^{\gamma_2}f(x)|^p\Big]\mathrm{d}x \\
&\quad\le c\int_{B^j}\psi(|x|)^p\sum_{|\gamma|\le m}\Big[|x|^{|\gamma|p}|\mathrm{D}^\gamma f(x)|^p+\sum_{0\le|\gamma_2|<|\gamma|}|x|^{|\gamma_2|p}|\mathrm{D}^{\gamma_2}f(x)|^p\Big]\mathrm{d}x \\
&\quad\le c\int_{B^j}\psi(|x|)^p\sum_{|\gamma|\le m}|x|^{|\gamma|p}\,|\mathrm{D}^\gamma f(x)|^p\mathrm{d}x.
\end{aligned}
$$

Since $|B^j|\overset{j\to\infty}{\longrightarrow}0$ this shows

$$\|\varphi_j f|\overset{*}{E}{}^m_{p,\psi}(B)\|=\|f-(1-\varphi_j)f|\overset{*}{E}{}^m_{p,\psi}(B)\|\overset{j\to\infty}{\longrightarrow}0.$$

Therefore the set $\left\{g\in\overset{*}{E}{}^m_{p,\psi}(B):\operatorname{supp}g\subset\dot{B}\text{ compact}\right\}$ is dense in $\overset{*}{E}{}^m_{p,\psi}(B)$. Since $\psi(t)$ is continuous on $(0,1]$ we get for any such $g\in\overset{*}{E}{}^m_{p,\psi}(B)$ with compact support in $\dot{B}$

$$\|g|\overset{*}{E}{}^m_{p,\psi}(B)\|\sim\|g|\overset{\circ}{W}{}^m_p(\dot{B})\|$$

and thus g can be approximated by C_0^∞-functions. Hence $C_0^\infty(\dot{B})$ is dense in $\overset{*}{E}{}^m_{p,\psi}(B)$. Finally part (3) follows from (3.7), first for $f\in C_0^\infty(\dot{B})$ and then by completion arguments.

Due to the last remark there is no need to distinguish between the spaces $\overset{*}{E}{}^m_{p,\psi}(B)$ and $E^m_{p,\psi}(B)$. We will now look at the continuity and compactness of embeddings of the just defined weighted Sobolev spaces $E^m_{p,\psi}(B)$ into Lebesgue spaces $L_p(B)$.

Theorem 3.4. *Let $1\le p<\infty$, $m\in\mathbb{N}$ and ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1)=1$ and bounded from below by a positive constant. Then it holds for $f\in E^m_{p,\psi}(B)$*

$$\|f|L_p(B)\|\le c\,\|f|E^m_{p,\psi}(B)\|. \tag{3.12}$$

Furthermore, the embedding

$$\mathrm{id}:E^m_{p,\psi}(B)\hookrightarrow L_p(B)$$

is compact if, and only if, ψ is unbounded on $(0,1]$, i.e. $\lim\limits_{t\to0^+}\psi(t)=\infty$.

Proof. Since ψ is bounded from below we get as an immediate consequence of (3.7) with $m_1 = 0$ and $m_2 = m$

$$\begin{aligned}\int_B |f(x)|^p \mathrm{d}x &\leq \sup_{t\in(0,1]} \frac{1}{\psi(t)^p} \int_B |f(x)|^p \psi(|x|)^p \mathrm{d}x \\ &\leq c\, \|f|E^m_{p,\psi}(B)\|^p, \qquad\qquad f \in C^m_0(B).\end{aligned}$$

To prove the compactness we follow the idea of [Tr12a], where $m = n = p = 2$ and $\psi(t) = (1 + |\log t|)^\lambda$, $\lambda \geq 0$, and complete it with constructions found in the proof of [Tr12b, Theorem 3.3].

Step 1. We show that $\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$ is not compact if ψ is bounded (from above) on $(0, 1]$. We put

$$\varphi_j(x) := 2^{j\frac{n}{p}} \varphi(2^j x), \qquad j \in \mathbb{N},$$

where we choose a function $\varphi \in C^1(B)$ such that

- $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k = \emptyset, \quad j \neq k$
- $\|\varphi_j|L_p(B)\| = 1, \quad j \in \mathbb{N}$
- $\displaystyle\int_B |x|^{mp} \sum_{|\alpha|=m} |\mathrm{D}^\alpha \varphi_j(x)|^p \mathrm{d}x \ \sim\ 1, \quad j \in \mathbb{N}$

The sequence $(\varphi_j)_{j\in\mathbb{N}}$ is not precompact in $L_p(B)$ since $\|\varphi_j - \varphi_k|L_p(B)\| = 2$ for $j \neq k$. But the sequence $(\varphi_j)_{j\in\mathbb{N}}$ is bounded in $E^m_{p,\psi}(B)$

$$\|\varphi_j|E^m_{p,\psi}(B)\|^p \leq c \int_B |x|^{mp} \sum_{|\alpha|=m} |\mathrm{D}^\alpha \varphi_j(x)|^p \mathrm{d}x \ \sim\ 1.$$

Therefore, id is not compact.

Step 2. We show that id is compact if $\lim_{t\to 0^+} \psi(t) = \infty$. The idea is to cut off the singularity at the origin and achieve compactness in that way. Let $B_J := \{x \in \mathbb{R}^n : |x| \leq 2^{-(J+1)}\}$, $J \in \mathbb{N}$. We shall construct a function $\varphi \in C^m_0(B)$ with $\varphi \equiv 1$ on B_{J+1} and $\operatorname{supp}\varphi \subseteq B_J$. Then we shall decompose

$$\mathrm{id} \ = \underbrace{(1-\varphi)\ \mathrm{id}}_{=:\ \mathrm{Id}^J} + \underbrace{\varphi\,\mathrm{id}}_{=:\ \mathrm{Id}_J}. \tag{3.13}$$

and show that Id^J is compact while Id_J is bounded with (arbitrarily) small norm. Since the set of compact operators is closed, the compactness of id then follows. This topic was treated in [Mi15a, Proposition 2.5] in the context of admissible functions. In comparison, the difference is to handle the loss of monotonicity in case of slowly varying functions. It comes out that property (2.17) takes this upcoming role, but losing some qualitative assertions about

$$\lim_{J\to\infty} \| \operatorname{Id}_J \| = 0. \tag{3.14}$$

Let $B^j := \{x \in \mathbb{R}^n : 2^{-(j+2)} \leq |x| \leq 2^{-j}\}, j \in \mathbb{N}_0$, and choose $\varphi_j \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi_j \subseteq B^j$, $|\mathrm{D}^\beta \varphi_j(x)| \leq c\, 2^{j|\beta|}$, $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1, x \in \dot{B}$. We consider the decomposition (3.13) where the function $\varphi(x)$ is defined by

$$\varphi(x) = 1 - \sum_{j=0}^{J} \varphi_j(x),\ x \in B.$$

Step 2a. We show the compactness of the mapping

$$\operatorname{Id}^J : E^m_{p,\psi}(B) \hookrightarrow L_p(B \setminus B_{J+1}),\ f \mapsto \sum_{j=0}^{J} \varphi_j f. \tag{3.15}$$

We start with the following expressions

$$\begin{aligned}
\|\varphi_j f|\overset{\circ}{W}{}^m_p(B^j)\|^p &= \sum_{|\alpha|=m} \int_{B^j} |\mathrm{D}^\alpha(\varphi_j(x) f(x))|^p \mathrm{d}x \\
&\leq c\, \Big[\int_{B^j} \sum_{|\gamma|=m} |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x + \sum_{0\leq|\gamma|<m} 2^{j(m-|\gamma|)p} \int_{B^j} |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \Big].
\end{aligned}$$

The first term of the latter expression can be estimated with help of (2.17) and (2.20) by

$$\begin{aligned}
\int_{B^j} \sum_{|\gamma|=m} |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \ &\sim\ \int_{B^j} \varphi(|x|)^{-p} |x|^{mp} \psi(|x|)^p \sum_{|\gamma|=m} |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \\
&\leq c\, \varphi(2^{-(j+2)})^{-p} \int_{B^j} |x|^{mp} \psi(|x|)^p \sum_{|\gamma|=m} |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \\
&\sim\ 2^{jmp} \psi(2^{-(j+2)})^{-p} \int_{B^j} |x|^{mp} \psi(|x|)^p \sum_{|\gamma|=m} |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \\
&\leq c\ 2^{jmp} \psi(2^{-j})^{-p}\, \|f|\, E^m_{p,\psi}(B^j)\|^p.
\end{aligned}$$

Now let $|\gamma| < m$. Fix the constant $0 < \delta < 1$ from Corollary 2.15. First assume $f \in C^m_0(B_\delta)$ and without loss of generality $B^j \subset B_\delta$. Again using (2.17) we get similarly as above

$$\begin{aligned}
\int_{B^j} |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x &\leq c\, 2^{j|\gamma|p} \psi(2^{-j})^{-p} \int_{B^j} |x|^{|\gamma|p} \psi(|x|)^p |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \\
&\leq c\, 2^{j|\gamma|p} \psi(2^{-j})^{-p}\, \|f|\, E^m_{p,\psi}(B^j)\|^p
\end{aligned}$$

where we used (3.7) with $m_1 = |\gamma|$ and $m_2 = m$ in the last line. Dilation arguments verify the latter inequalities for $f \in C_0^m(B)$. If $(f_n)_n \subset C_0^m(B)$ is an approximating sequence for $f \in E^m_{p,\psi}(B)$, we obtain

$$\begin{aligned}\int_{B^j} &|\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \\ &\le c\Big[\|\mathrm{D}^\gamma(f-f_n)|L_p(B^j)\|^p + 2^{j|\gamma|p}\psi(2^{-j})^{-p}\,\|f_n|E^m_{p,\psi}(B^j)\|^p\Big] \\ &\le c\Big[\|\mathrm{D}^\gamma(f-f_n)|E^{m-|\gamma|}_{p,\psi}(B^j)\|^p + 2^{j|\gamma|p}\psi(2^{-j})^{-p}\,\|f_n|E^m_{p,\psi}(B^j)\|^p\Big].\end{aligned}$$

Here the constant c does not depend on j. Note that for $g \in E^m_{p,\psi}(B^j)$ it holds

$$\|\mathrm{D}^\gamma g|E^{m-|\gamma|}_{p,\psi}(B^j)\| \sim 2^{j|\gamma|}\,\|g|E^m_{p,\psi}(B^j)\|. \tag{3.16}$$

Endowed with the last line, we proceed with the above inequalities

$$\begin{aligned}\int_{B^j} &|\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \\ &\le c\Big[2^{j|\gamma|p}\|f-f_n|E^m_{p,\psi}(B)\|^p + 2^{j|\gamma|p}\psi(2^{-j})^{-p}\|f_n|E^m_{p,\psi}(B^j)\|^p\Big].\end{aligned}$$

Consequently for all $|\gamma| < m$ it holds

$$\int_{B^j} |\mathrm{D}^\gamma f(x)|^p \mathrm{d}x \le c\,2^{j|\gamma|p}\psi(2^{-j})^{-p}\|f|E^m_{p,\psi}(B^j)\|^p, \qquad f \in E^m_{p,\psi}(B).$$

Finally this shows

$$\|\varphi_j f|\mathring{W}^m_p(B^j)\| \le c\ 2^{jm}\psi(2^{-j})^{-1}\|f|E^m_{p,\psi}(B)\| \tag{3.17}$$

and in conclusion

$$\begin{aligned}\|\sum_{j=0}^{J}\varphi_j f|\mathring{W}^m_p(B\setminus B_{J+1})\| &\le c\sum_{j=0}^{J}\|\varphi_j f|\mathring{W}^m_p(B^j)\| \\ &\le c\sum_{j=0}^{J} c_j\,\|f|E^m_{p,\psi}(B)\|.\end{aligned}$$

Hence the map $f \mapsto \sum_{j=0}^{J}\varphi_j f$ considered from $E^m_{p,\psi}(B)$ to $\mathring{W}^m_p(B\setminus B_{J+1})$ is bounded. Now the compactness of Id^J follows by composition with the compact identity $\mathring{W}^m_p(B\setminus B_{J+1}) \hookrightarrow L_p(B\setminus B_{J+1})$.

Step 2b. To argue that (3.14) holds true, we start with the observation that

$$\|\,\mathrm{Id}_J\,f|L_p(B_J)\|^p \le c\Big[\|f|L_p(B_J)\|^p + \|\sum_{j=0}^{J}\varphi_j f\,|L_p(B_J)\|^p\Big]. \tag{3.18}$$

Fix $0 < \delta < 1$ according to (2.35) and let $\varepsilon > 0$. Due to the unboundedness of ψ the first term can be estimated by

$$\|f|L_p(B_J)\|^p \leq \varepsilon^p \int_{B_J} |f(x)|^p \psi(|x|)^p \mathrm{d}x \leq c\,\varepsilon^p \,\|f|E^m_{p,\psi}(B_J)\|^p, \quad f \in C^m_0(B)$$

if $J = J(\varepsilon)$ is sufficiently large. Furthermore, we observe that $\varphi_j f \equiv 0$ on B_J for all $j = 0, ..., J-1$ and $|\varphi_J(x)| \leq 1$. Thus with a decreasing function ϕ according to (2.17) we have

$$\begin{aligned}
\|\sum_{j=0}^{J} \varphi_j f\,|L_p(B_J)\|^p &\leq \int_{2^{-(J+2)}<|x|<2^{-(J+1)}} |f(x)|^p \mathrm{d}x \\
&\leq c \int_{2^{-(J+2)}<|x|<2^{-(J+1)}} \big[\,|x|\,\phi(|x|)\,\big]^{-p} \psi(|x|)^p |f(x)|^p \mathrm{d}x \\
&\leq c\,2^{(J+2)p} \phi(2^{-(J+1)})^{-p} \int_{2^{-(J+2)}<|x|<2^{-(J+1)}} \psi(|x|)^p |f(x)|^p \mathrm{d}x \\
&\leq c\,\psi(2^{-(J+1)})^{-p} \int_{B_J} |f(x)|^p \psi(|x|)^p \mathrm{d}x \\
&\leq c\,\varepsilon^p \,\|f|E^m_{p,\psi}(B_J)\|^p.
\end{aligned}$$

By completion one can extend this to all $f \in E^m_{p,\psi}(B)$. This means that if $J = J(\varepsilon)$ is sufficiently large then $\|\,\mathrm{Id}_J\,\| \leq \varepsilon$. □

3.2 Entropy and approximation numbers

We want to measure the compactness of the embedding from Theorem 3.4

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

in terms of so called *entropy and approximation numbers*. The underlying concept has a long history. The interest of this subject is due to the arising opportunity of estimating eigenvalues in terms of entropy numbers and approximation numbers.

Recall that a linear and bounded operator $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is compact, if every bounded set in $\mathbb{X}$ is mapped to a precompact set in $\mathbb{Y}$. This is equivalent to the circumstance that for every $\varepsilon > 0$ the image of the unit ball $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} \leq 1\}$ is covered by a finite $\varepsilon-$net, i.e.

$$\forall\, \varepsilon > 0\; \exists\, y_1, ..., y_N \in \mathbb{Y} : T(B_{\mathbb{X}}) \subseteq \bigcup_{i=1}^{N} B_{\mathbb{Y}}(y_i, \varepsilon).$$

Here $B_{\mathbb{Y}}(y, \varepsilon) = \{z \in \mathbb{Y} : \|y - z\|_{\mathbb{Y}} < \varepsilon\}$ denotes the ball in $\mathbb{Y}$ with radius ε centered at $y \in \mathbb{Y}$. When quantifying the compactness a natural question arises: Given a fixed number N what is the least radius ε such that N balls cover $T(B_{\mathbb{X}})$? This leads to the following definition.

Definition 3.5. *Let $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ and $k \in \mathbb{N}$. The k-th* entropy number *of the operator T is defined by*

$$e_k(T) := \inf \Big\{ \varepsilon > 0 \,\Big|\, \exists\, y_1, ..., y_{2^{k-1}} \in \mathbb{Y} : \, T(B_{\mathbb{X}}) \subseteq \bigcup_{i=1}^{2^{k-1}} B_{\mathbb{Y}}(y_i, \varepsilon) \Big\}.$$

Accordingly we have the following characterisation for compactness of the operator T

$$T \quad \textit{is compact} \iff \lim_{k\to\infty} e_k(T) = 0. \tag{3.19}$$

We can quantify compactness in the sense that the rate of decay of the monotonically decreasing sequence $(e_k(T))_{k\in\mathbb{N}}$ can be interpreted as a degree of compactness. Hence the faster the sequence $(e_k(T))_{k\in\mathbb{N}}$ tends to zero the better is the compactness of T.
A second approach in that area relies on that fact the any limit of finite rank operators is compact. In that case one can naturally ask: Given a fixed number N how good can T be approximated by operators of finite rank N? This motivates the following definition.

Definition 3.6. *Let $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ and $k \in \mathbb{N}$. The k-th* approximation number *of the operator T is defined by*

$$a_k(T) := \inf \big\{ \|T - S\| : \; S \in \mathcal{L}(\mathbb{X}, \mathbb{Y}), \; \mathrm{rank} S < k \big\}$$

where $\mathrm{rank}\, S = \dim S(\mathbb{X})$.

The following fact is called the *rank property of approximation numbers*

$$a_m(T) = 0 \qquad \text{if, and only if,} \qquad \mathrm{rank}\, T < m. \tag{3.20}$$

Furthermore, it follows a sufficient condition for compactness

$$\lim_{k\to\infty} a_k(T) = 0 \implies T \quad \textit{is compact.} \tag{3.21}$$

Whether the converse of (3.21) is true was an outstanding problem of Banach space theory for a long time. Finally in 1973, P. Enflo gave the negative answer with a counterexample in [Enf73].
Standard references of the underlying abstract theory of entropy and approximation numbers in Banach spaces (including proofs of the listed properties below) are [Pi78, Koe86, EE87, Pi87, CS90]. We briefly recall some basic properties.

1. *Monotonicity:* For all operators $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ it holds

$$\|T\| = e_1(T) \geq e_2(T) \geq ... \geq e_k(T) \geq e_{k+1}(T), \quad k \in \mathbb{N}. \tag{3.22}$$

2. *Additivity:* For all operators $T, S \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ it holds

$$e_{k+l-1}(T+S) \leq e_k(T) + e_l(S), \quad k, l \in \mathbb{N}. \tag{3.23}$$

3. *Multiplicativity:* For all operators $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y}), R \in \mathcal{L}(\mathbb{Y}, \mathbb{Z})$ it holds

$$e_{k+l-1}(R \circ T) \leq e_k(R)\, e_l(T), \quad k, l \in \mathbb{N}. \tag{3.24}$$

All properties (3.22)-(3.24) also hold true for $a_k(T)$ instead of $e_k(T)$. Some generalisations to quasi-Banach spaces may be found in [ET96]. The statement (3.24) implies

$$e_k(R \circ T) \leq \|R\|\, e_k(T), \quad k \in \mathbb{N}. \tag{3.25}$$

This motivates the investigation of many compact embeddings between function spaces since interesting maps (such as those coming from integral operators) can often be factorised into the composition of compact embedding maps and continuous maps.
Let $T \in \mathcal{L}(\mathbb{X}, \mathbb{X})$ be a compact operator. Then the spectrum of T consists only of eigenvalues of finite algebraic multiplicity. Let $(\lambda_k(T))_{k \in \mathbb{N}}$ denote the sequence of eigenvalues, monotonically ordered according to their geometric multiplicities, such that

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq ... \geq |\lambda_k(T)| \overset{k \to \infty}{\longrightarrow} 0.$$

Hereby we put $\lambda_k(T) = 0$ for all $k > N$ if T has only finitely many non-vanishing eigenvalues and the sum of their multiplicities is N. We recall Carl's inequality, often also called Carl-Triebel inequality,

$$|\lambda_k(T)| \leq \sqrt{2}\, e_k(T), \quad k \in \mathbb{N}. \tag{3.26}$$

This remarkable relation (3.26) between spectral properties of T and its geometrical characteristics was originally proved in [CT80, Ca81b]. A proof can also be found in [CS90, Theorem 4.2.1, p. 143]. In many applications one can decompose the operator T into $T_1 \circ \mathrm{id} \circ T_2$ with some bounded operators T_1, T_2 and a suitable embedding id such that

$$|\lambda_k(T)| \leq c\, \|T_1\| \cdot \|T_2\| \cdot e_k(\mathrm{id}).$$

Hence knowledge of compact embeddings by means of entropy numbers can be interpreted as knowledge about related eigenvalues.

In the Hilbert space setting one can achieve further reaching results for the relation between approximation numbers and spectral properties, especially for self-adjoint operators. Let $T \in \mathcal{L}(\mathbb{H})$ be a compact and self-adjoint operator acting on a Hilbert space $\mathbb{H}$. Then it turns out that

$$a_k(T) = |\lambda_k(T)|, \qquad k \in \mathbb{N}, \tag{3.27}$$

see for example [CS90, Proposition 4.2.1, p.152]. This spectral property goes back to E. Schmidt. An early proof of (3.27) can be found in [GK65] with reference to [All57].

One may ask about the relation between entropy and approximation numbers. Note that (universal) estimates of the form

$$e_k(T) \le C\ a_k(T) \qquad \text{or} \qquad a_k(T) \le c\ e_k(T) \tag{3.28}$$

cannot exist with constants $C, c > 0$ independent of $k \in \mathbb{N}$ and $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$. To disprove the first estimate of (3.28) consider an operator $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, acting between real Banach spaces $\mathbb{X}, \mathbb{Y}$ with finite rank $m \in \mathbb{N}$. Then $a_{m+1}(T) = 0$. But on the other hand, the larger the rank of an operator the smaller the rate of decrease of its entropy numbers, namely

$$\operatorname{rank} T = m \iff \exists\, c > 0\ \forall\, k \in \mathbb{N} : c\, 2^{-\frac{k}{m}} \le e_{k+1}(T) \le 4\|T\|\, 2^{-\frac{k}{m}}$$

as shown in [CS90, Proposition 1.3.1, p.15]. Hence there cannot exist a constant $C > 0$ as in the first estimate of (3.28). Furthermore, the existence of compact operators T such that $\lim_{k\to\infty} e_k(T) = 0$ but $\lim_{k\to\infty} a_k(T) > 0$, as stated in connection with (3.19) and (3.21), disproves the second inequality of (3.28). Despite universal estimates of the form (3.28) cannot be true, there exit rather general inequalities in that direction. One very useful observation is that for every $L \in \mathbb{N}$ and $0 < \nu < \infty$

$$\sup_{k=1,\dots,L} k^\nu e_k(T) \le c \sup_{k=1,\dots,L} k^\nu a_k(T) \tag{3.29}$$

where the constant $c > 0$ may depend on ν but not on L. This goes back to [Ca81b], see [CS90, Theorem 3.1.1, p.96]. For the purpose of this work we also want to mention [Tr94], see also [ET96, Section 1.3.3, p.15], where it is proved that if for a compact $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$

$$a_{2^{j-1}}(T) \le c\, a_{2^j}(T)\ \forall j \in \mathbb{N}$$

then there is a constant $C > 0$ such that for all $k \in \mathbb{N}$

$$e_k(T) \le C\, a_k(T). \tag{3.30}$$

We turn to the classical setting of embeddings of Sobolev spaces $W_p^m(\Omega)$ and have a look on well known results for their entropy and approximation numbers. We refer in particular to [ET96, Section 3.3.5, p.126] where one can find some remarks on the history of the following substantial results.

Proposition 3.7. *Let $W_p^m(\Omega)$ be the classical Sobolev space according to* (3.1)*. Then it follows for $1 < p_1, p_2 < \infty$ and $m > n(\frac{1}{p_1} - \frac{1}{p_2})$ that the embedding $W_{p_1}^m(\Omega) \hookrightarrow L_{p_2}(\Omega)$ is compact and*

$$e_k(W_{p_1}^m(\Omega) \hookrightarrow L_{p_2}(\Omega)) \sim k^{-\frac{m}{n}}, \quad k \in \mathbb{N}. \tag{3.31}$$

Furthermore, if $1 \leq p < \infty$ then the embedding $W_p^m(\Omega) \hookrightarrow L_p(\Omega)$ is compact and

$$e_k(W_p^m(\Omega) \hookrightarrow L_p(\Omega)) \sim a_k(W_p^m(\Omega) \hookrightarrow L_p(\Omega)) \sim k^{-\frac{m}{n}}, \quad k \in \mathbb{N}. \tag{3.32}$$

The result (3.31) is due to Birman/Solomjak [BS67, BS72] using piecewise-polynomial approximations. Initially in [Tr78] extensions in the scale of Besov and Triebel-Lizorkin spaces $A_{p,q}^s(\Omega)$ have been established on the basis of Fourier-analytical techniques. We refer to Theorem 3.3.3/2, p. 118, in [ET96] and references given there. Concerning the approximation numbers first results were known by Kolmogorov in [Ko36]. We refer to [Tr78, Theorem 4.10.2, p. 348], [EE87, Sect.V.6, p. 292] and references given in subsequent remarks there. Note that Proposition 3.7 also holds true for $\overset{\circ}{W}{}_p^m(\Omega)$ instead of $W_p^m(\Omega)$.

To deal with the setting of weighted Sobolev spaces $E_{p,\psi}^m(B)$ we refine the arguments in the proof of Theorem 3.4 and get some first informations about the asymptotic behaviour of entropy and approximation numbers of the embedding $\mathrm{id} : E_{p,\psi}^m(B) \hookrightarrow L_p(B)$. We will see in the next proposition that if the growth rate of $\psi(t) \overset{t\to 0}{\longrightarrow} \infty$ is above a certain critical bound then the weight has no influence on the rate of compactness. In detail, if the sequence $(\psi(2^{-j}))_{j\in\mathbb{N}}$ pointwise multiplied by a (tending to zero) ℓ_1-sequence tends strongly enough to infinity, then the entropy and approximation numbers of $\mathrm{id} : E_{p,\psi}^m(B) \hookrightarrow L_p(B)$ behave as in case of the unweighted setting (3.32).

Proposition 3.8. *Let $1 \leq p < \infty$, $m \in \mathbb{N}$ and ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1) = 1$, bounded from below by a positive constant and locally decreasing at zero with $\lim_{t\to 0} \psi(t) = \infty$. Then the embedding*

$$\mathrm{id} : E_{p,\psi}^m(B) \hookrightarrow L_p(B)$$

is compact. If there exists a sequence $(c_j)_{j\in\mathbb{N}} \in \ell_1$ *such that*

$$\Big([\psi(2^{-j})\, c_j]^{-1}\Big)_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}} \tag{3.33}$$

then it holds for all $k \in \mathbb{N}$

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim k^{-\frac{m}{n}}. \tag{3.34}$$

Proof. The assumption (3.33) implies $\lim\limits_{j\to\infty} \psi(2^{-j}) = \infty$. The compactness then follows by Theorem 3.4. Choose $J \in \mathbb{N}$ such that $k^{\frac{m}{n}} \sim \psi(2^{-J})$. As in (3.13) we decompose

$$\mathrm{id} = \mathrm{Id}_J + \mathrm{Id}^J = (1 - \sum_{j=0}^{J} \varphi_j)\,\mathrm{id} + \sum_{j=0}^{J} \varphi_j\,\mathrm{id}\,.$$

As before Id_J is bounded with $\|\,\mathrm{Id}_J\,\| \le c\,[\psi(2^{-J})]^{-1}$ and Id^J is compact. We have shown in (3.17) that $\mathrm{id}_W^j : E^m_{p,\psi}(B) \hookrightarrow \overset{\circ}{W}{}^m_p(B^j), f \mapsto \varphi_j f$ is bounded with

$$\|\,\mathrm{id}_W^j\,\| \le c\; 2^{jm}\psi(2^{-j})^{-1}. \tag{3.35}$$

We fix $k_j \in \mathbb{N}$ such that $k_j \sim \psi(2^{-J})^{\frac{n}{m}}\psi(2^{-j})^{-\frac{n}{m}}c_j{}^{-\frac{n}{m}}, j = 1, ..., J$ and $k_0 \sim \psi(2^{-J})^{\frac{n}{m}}$. Because of (3.33) we get

$$\sum_{j=0}^{J} k_j \sim \psi(2^{-J})^{\frac{n}{m}} \sim k.$$

We denote $\mathrm{id}^j : E^m_{p,\psi}(B) \hookrightarrow L_p(B^j), f \mapsto \varphi_j f$ what means $\mathrm{Id}^J = \sum\limits_{j=0}^{J} \mathrm{id}^j$. Then it follows

$$\begin{aligned}
e_k(\mathrm{id}) &\le \|\,\mathrm{Id}_J\,\| + e_k(\mathrm{Id}^J) \le \|\,\mathrm{Id}_J\,\| + \sum_{j=0}^{J} e_{k_j}(\mathrm{id}^j)\\
&\le \|\,\mathrm{Id}_J\,\| + \sum_{j=0}^{J} \|\,\mathrm{id}_W^j\,\| \;\cdot e_{k_j}(\overset{\circ}{W}{}^m_p(B^j) \hookrightarrow L_p(B^j))\\
&\le c\Big[\psi(2^{-J})^{-1} + \sum_{j=1}^{J} 2^{jm}\psi(2^{-j})^{-1}\cdot 2^{-jm}k_j{}^{-\frac{m}{n}}\Big]\\
&\le c\Big[\psi(2^{-J})^{-1} + \psi(2^{-J})^{-1}\sum_{j=1}^{J} c_j\Big] \le c\,k^{-\frac{m}{n}}.
\end{aligned}$$

On the other hand

$$k^{-\frac{m}{n}} \sim e_k(\overset{\circ}{W}{}^m_p(B) \hookrightarrow L_p(B)) \le \| \overset{\circ}{W}{}^m_p(B) \hookrightarrow E^m_{p,\psi}(B)\| \; e_k(\mathrm{id}) \le c \; e_k(\mathrm{id}).$$

One can transfer the above arguments to approximation numbers $a_k(\mathrm{id})$ instead of $e_k(\mathrm{id})$. $\square$

Examples 3.9. **i)** Let $\psi(t) = (1+|\log t|)^\sigma$, $\sigma > 0$. We put $c_j := j^{-\kappa} \in \ell_1$ for $\kappa > 1$. Then

$$\left([\psi(2^{-j})\, c_j]^{-1}\right)_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}} \iff \sum_{j=1}^{\infty} j^{-(\sigma-\kappa)\frac{n}{m}} < \infty$$

$$\iff \sigma > \tfrac{m}{n} + \kappa.$$

If $\sigma > \frac{m+n}{n}$ we choose $\kappa > 1$ such that $\sigma > \frac{m}{n} + \kappa$ and the last proposition then provides

$$e_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)) \sim a_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)) \sim k^{-\frac{m}{n}}.$$

This illustrates that Proposition 3.8 generalises [Tr12b, Theorem 3.3].

ii) Let $\psi(t) = (1+|\log t|)^\sigma(1+\log(1+|\log t|))^\gamma$, $\sigma > 0, \gamma \neq 0$. If $\sigma > \frac{m+n}{n}$ one can again choose $c_j := j^{-\kappa} \in \ell_1$ for $\kappa > 1$ with $\sigma > \frac{m}{n} + \kappa$. Then

$$\left([\psi(2^{-j})\, c_j]^{-1}\right)_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}} \iff \sum_{j=2}^{\infty} j^{-(\sigma-\kappa)\frac{n}{m}} (\log j)^{-\gamma\frac{n}{m}} < \infty$$

$$\iff \sigma > \tfrac{m}{n} + \kappa.$$

If $\sigma = \frac{m+n}{n}$ we put $c_j := j^{-1}(\log j)^{-\kappa} \in \ell_1$ with $\kappa > 1$. Then

$$\left([\psi(2^{-j})\, c_j]^{-1}\right)_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}} \iff \sum_{j=2}^{\infty} j^{-1} (\log j)^{-(\gamma-\kappa)\frac{n}{m}} < \infty$$

$$\iff \gamma > \tfrac{m}{n} + \kappa.$$

Thus if $\sigma > \frac{m+n}{n}, \gamma \in \mathbb{R}$, or $\sigma = \frac{m+n}{n}, \gamma > \frac{m+n}{n}$, we have

$$e_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)) \sim a_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)) \sim k^{-\frac{m}{n}}.$$

iii) Let $\psi(t) = (1+\log(1+|\log t|))^\gamma$, $\gamma > 0$. We cannot apply Proposition 3.8. This can be seen as follows. Assume a sequence $\{c_j\}_j \in \ell_1$. This implies ${c_j}^{-\frac{n}{m}} \ge 1, \; j \ge j_0$. Now

$$\left([\psi(2^{-j})\, c_j]^{-1}\right)_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}} \iff \sum_{j=j_0}^{\infty} (\log j)^{-\gamma\frac{n}{m}} {c_j}^{-\frac{n}{m}} < \infty$$

is never true by comparison with $\sum\limits_{j=j_0}^{\infty} (\log j)^{-\gamma\frac{n}{m}} = \infty$.

iv) Let $\psi(t) = \exp(|\log t|^c), 0 < c < 1$. For $\beta > \frac{m+n}{n}$ one can always find a constant $\kappa > 1$ such that $\frac{m+n}{n} < \frac{m}{n} + \kappa < \beta$. Put $c_j := j^{-\kappa} \in \ell_1$. Then

$$\exists j_0 \in \mathbb{N}\ \forall j \geq j_0 : \ \psi(2^{-j}) = \exp(j^c) \geq j^\beta,$$

and (3.33) follows by comparison with a summable series

$$\forall j \geq j_0 : \left[\psi(2^{-j})c_j\right]^{-\frac{n}{m}} = \exp(j^c)^{-\frac{n}{m}} j^{\kappa\frac{n}{m}} \leq j^{-\frac{n}{m}(\beta-\kappa)}.$$

Thus for all $0 < c < 1$ it holds

$$e_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)) \sim a_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)) \sim k^{-\frac{m}{n}}.$$

v) Let $\psi(t) = \exp((\log(1 + |\log t|))^a), a > 0$. Assuming $a > 1$, the fact that for every $\beta > \frac{m+n}{n}$

$$\exists j_0 \in \mathbb{N}\ \forall j \geq j_0 : \ \psi(2^{-j}) = \exp((\log j)^a) \geq j^\beta$$

leads to (3.33) similarly as in case of iv). Thus for all $a > 1$ it holds

$$e_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)) \sim a_k(\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)) \sim k^{-\frac{m}{n}}.$$

As already indicated with the latter examples, the assumption (3.33) is closely connected to the growth rate of ψ. We formulate a direct consequence of Proposition 3.8.

Corollary 3.10. *Let $1 \leq p < \infty$, $m \in \mathbb{N}$ and ψ be a continuous, slowly varying function on $(0, 1]$ with $\psi(1) = 1$, bounded from below by a positive constant and locally decreasing at zero. Let there exist a constant $\beta > \frac{m+n}{n}$ such that*

$$\exists\, c > 0, t_0 \in (0, 1]\ \forall\, t \leq t_0 : \ \psi(t) \geq c\,|\log t|^\beta. \tag{3.36}$$

Then the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact and it holds for all $k \in \mathbb{N}$

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim k^{-\frac{m}{n}}. \tag{3.37}$$

Proof. It follows for all $j \geq j_0$ that $\psi(2^{-j}) \geq c\, j^\beta$ if $j_0 \in \mathbb{N}$ is sufficiently large. Choose $\kappa > 1$ such that $\beta > \frac{m}{n} + \kappa$ and put $c_j := j^{-\kappa} \in \ell_1$. Since

$$\left([\psi(2^{-j})\, c_j]^{-1}\right)_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}} \iff \sum_{j=j_0}^{\infty} \psi(2^{-j})^{-\frac{n}{m}} j^{\kappa\frac{n}{m}} < \infty,$$

we can apply Proposition 3.8 due to comparison of the series. □

Remark 3.11. Closest to the results from this section among the recent papers mentioned in the introduction seems Theorem 4.7. in [HS11b]. Apart from the obviously different setting - dealing with Muckenhoupt weights in spaces on $\mathbb{R}^n$ - we could recover Example 3.9 i), but only for $\sigma > \frac{2m}{n}$. Otherwise the global decay of the corresponding weights in [HS11b] always dominates the influence of the local part of the weight. Moreover, the results in [HS08, HS11a, HS11b] as well as in [KLSS06a, KLSS06b, KLSS07] are essentially related to Besov spaces (due to their easier structure and the proof techniques). This causes some less sharp results for Sobolev-type spaces, especially in limiting cases.

3.3 Quadratic forms and eigenvalue distribution of degenerate elliptic operators

The methods utilised in the previous section do not provide exact estimates for entropy and approximation numbers for the whole range. They are too rough to handle the singular behaviour of the weight functions. To seal this gap Triebel proposed in [Tr12b] an approach via quadratic forms at least in case of Hilbert spaces. In this section we introduce these specific Hilbert space arguments and give an application to the general case of slowly varying functions. Mainly Courant's Max-min-principle for positive definite self-adjoint operators comes into play. The main results are then related to the distribution of eigenvalues of some degenerate elliptic operators.

We start with some preliminaries. Let $\mathbb{H}$ be a separable infinite-dimensional complex Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\| = (\cdot,\cdot)^{\frac{1}{2}}$.

Definition 3.12. *Let $D \subset \mathbb{H}$ be a dense linear subset. A bilinear symmetric map $E : D \times D \to \mathbb{C}$ is called* quadratic form, *i.e. for $f, g, h \in D$ and $\lambda, \mu \in \mathbb{C}$*

$$E(\lambda g + \mu h, f) = \lambda\, E(g,f) + \mu\, E(h,f), \qquad E(f,g) = \overline{E(g,f)}.$$

E is called positive definite *if*

$$\exists\, c > 0\ \forall f \in D : E(f,f) \geq c\,\|f\|^2.$$

E is called closed *if D is complete with respect to the norm*

$$\|f\|_E := \big(\, E(f,f) + \|f\|^2 \,\big)^{\frac{1}{2}}. \tag{3.38}$$

E is called closable *if E has a closed extension.*

For more informations about closable and closed (positive definite) quadratic forms we refer to [MR92, Section I.3].

Remark 3.13. A positive definite quadratic form $E : D \times D \to \mathbb{C}$ is closable if, and only if, for all sequences $(f_n)_{n\in\mathbb{N}} \subset D$ with $\|f_n\| \overset{n\to\infty}{\longrightarrow} 0$ and $E(f_n - f_m, f_n - f_m) \overset{n,m\to\infty}{\longrightarrow} 0$ it holds $E(f_n, f_n) \overset{n\to\infty}{\longrightarrow} 0$. If $f, g \in D$ then

$$(f, g)_E := E(f, g) + (f, g)$$

defines an inner product on D. Recall that there exists a complete linear space $\overline{D}$ (equipped with an inner product that extends $(\cdot,\cdot)_E$ by continuity) such that D is isomorph to a dense subset of $\overline{D}$. Furthermore, E extends by continuity to a continuous quadratic form on $\overline{D}$. We call $\overline{D}$ the *abstract completion* of D. Formally, $\overline{D}$ is the set of all $\|\cdot\|_E-$Cauchy sequences in D factorised by null sequences. We refer to [Tr92, Theorem 2/1.1.6]. In general the abstract completion $\overline{D}$ cannot be identified with a subset in $\mathbb{H}$, but if E is closable this is the case. Then the domain of the *closure* $\hat{E}$ is the collection of all $v \in \mathbb{H}$ such that there is a sequence $(v_n)_{n\in\mathbb{N}} \subset D$ with

$$E(v_n - v_m, v_n - v_m) \overset{n,m\to\infty}{\longrightarrow} 0, \qquad \|v_n - v\| \overset{n\to\infty}{\longrightarrow} 0$$

and

$$\hat{E}(u, v) := \lim_{n\to\infty} E(u_n, v_n). \tag{3.39}$$

Every positive definite closed quadratic form Q is uniquely generated by a positive definite self-adjoint operator A, i.e.

$$\operatorname{dom} Q = \operatorname{dom} A^{1/2} \tag{3.40}$$

$$Q(f, g) = (Af, g). \tag{3.41}$$

If E arises from a positive definite symmetric operator A in the sense of (3.40) and (3.41), then its closure is associated in the same way with a self-adjoint extension A_F of A. A_F is called *Friedrichs extension.* The *energy space* $\operatorname{dom} A_F^{1/2}$ coincides with the domain of the closure of E. Details and proofs may be found in [Da95, Section 4.4], [EE87, Section IV.2] and [Tr92, Sections 4.1.9, 4.4.3].

If the spectrum of a positive definite self-adjoint operator $A : \operatorname{dom} A \to \mathbb{H}$, $\operatorname{dom} A \subseteq \mathbb{H}$, consists solely of eigenvalues of finite (geometric) multiplicity, we call A an *operator with pure point spectrum.* We refer to [Tr92, Section IV]. We denote by $(\lambda_k(A))_{k\in\mathbb{N}}$ its eigenvalues ordered (including geometric multiplicities) according to

$$0 < \lambda_1(A) \le \lambda_2(A) \le ... \le \lambda_k(A) \overset{k\to\infty}{\longrightarrow} \infty. \tag{3.42}$$

Due to Rellich's criterion A has pure point spectrum if, and only if, the embedding of the energy space $H_A = \operatorname{dom} A^{1/2}$ into $\mathbb{H}$ is compact. In that case, we have

$$a_k(\mathrm{id} : H_A \hookrightarrow \mathbb{H}) \sim \lambda_k(A)^{-1/2}. \tag{3.43}$$

One can order densely defined, positive definite self-adjoint operators A, B with pure point spectrum in the sense of quadratic forms, i.e.

$$0 \leq (Af, f) \leq (Bf, f), \qquad f \in \operatorname{dom} B \subseteq \operatorname{dom} A. \tag{3.44}$$

Then the eigenvalues obey the same order

$$\lambda_k(A) \leq \lambda_k(B), \qquad k \in \mathbb{N}. \tag{3.45}$$

This follows from the Max-min-principle for positive definite self-adjoint operators, that the kth eigenvalue of A can be written as

$$\lambda_k(A) = \sup_{M_{k-1}} \inf_{\substack{f \in \operatorname{dom}(A) \cap M_{k-1}^{\perp} \\ \|f\|=1}} \big(Af, f\big)$$

where the supremum is taken over all linear subspaces M_{k-1} of dimension at most $k-1$, respectively for B. A proof can be found in [EE87, Section XI, p.489].

We turn towards our setting assuming $p = 2$. We put $E^m_\psi(B) := E^m_{2,\psi}(B)$ according to Definition 3.2 and ask for the quality of compactness of the embedding

$$\mathrm{id} : E^m_\psi(B) \hookrightarrow L_2(B) \tag{3.46}$$

in terms of entropy and approximation numbers. Consider the positive definite, closed quadratic form $E^m_\psi : E^m_\psi(B) \times E^m_\psi(B) \to \mathbb{C}$ in $\mathbb{H} = L_2(B)$ given by

$$E^m_\psi(f, g) := \int_B b_{m,\psi}(x) \sum_{|\alpha|=m} \mathrm{D}^\alpha f(x) \overline{\mathrm{D}^\alpha g(x)} \mathrm{d}x, \qquad f, g \in E^m_\psi(B) \tag{3.47}$$

where $b_{m,\psi}(x) := |x|^{2m}\, \psi(|x|)^2$, $x \in B$. Then $E^m_\psi(\cdot,\cdot)$ is generated by a positive definite self-adjoint operator A^m_ψ, i.e.

$$\big(A^m_\psi f, g\big)_{L_2(B)} = E^m_\psi(f, g), \qquad f, g \in E^m_\psi(B). \tag{3.48}$$

The operator A^m_ψ is given by

$$\operatorname{dom}((A^m_\psi)^{1/2}) = E^m_\psi(B) \tag{3.49}$$

$$A^m_\psi f = (-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha(b_{m,\psi} \mathrm{D}^\alpha f), \qquad f \in C_0^\infty(\dot{B}). \tag{3.50}$$

If $\lim\limits_{t\to 0^+} \psi(t) = \infty$ then the embedding

$$\operatorname{dom}(A^m_\psi)^{-1/2} \hookrightarrow L_2(B)$$

is compact due to Theorem 3.4. As described in Remark 3.13 the operator A^m_ψ has pure point spectrum and it holds due to (3.43) that

$$a_k(\mathrm{id} : E^m_\psi(B) \hookrightarrow L_2(B)) \sim \lambda_k(A^m_\psi)^{-\frac{1}{2}}. \tag{3.51}$$

This interrelation enables us to concentrate on the distribution of eigenvalues of the operator A^m_ψ in order to analyse the asymptotic behaviour of the approximation numbers. This is the background of the next proposition.

Proposition 3.14. *Let $m \in \mathbb{N}$ and ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1) = 1$, bounded from below by a positive constant and locally decreasing at zero with $\lim\limits_{t\to 0^+} \psi(t) = \infty$.*
Let A^m_ψ be the positive definite, self-adjoint operator generated by (3.47) with pure point spectrum $(\lambda_k(A^m_\psi))_{k\in\mathbb{N}}$ ordered according to

$$0 < \lambda_1(A^m_\psi) \le \lambda_2(A^m_\psi) \le ... \le \lambda_k(A^m_\psi) \overset{k\to\infty}{\longrightarrow} \infty. \tag{3.52}$$

For $\lambda > 1$ let $J = J_\lambda \in \mathbb{N}$ be such that $\psi(2^{-J_\lambda}) \sim \lambda^{\frac{1}{2}}$. Let Φ be a function equivalent to a primitive of $[\psi(2^{-\cdot})]^{-\frac{n}{m}}$. Then it holds

$$N(\lambda, A^m_\psi) \sim \lambda^{\frac{n}{2m}}\, \Phi(J_\lambda) \tag{3.53}$$

where $N(\lambda, A^m_\psi)$ denotes the number of eigenvalues of A^m_ψ smaller than λ. In particular, if $([\psi(2^{-j})]^{-1})_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}}$ then

$$N(\lambda, A^m_\psi) \sim \lambda^{\frac{n}{2m}}.$$

Before moving on to the proof of the last proposition, we start with some preparatory remarks. To prove Proposition 3.14 we will use the Courant-Weyl method of *Dirichlet-Neumann bracketing*. We refer the reader to [EE87, Chap.XI] and [EE04, Chap.3]. The essence of this method are the inequalities

$$\sum_{j=1}^{J} N(\lambda, -\Delta_{D,\Omega^j}) \le N(\lambda, -\Delta_{D,\Omega}) \le N(\lambda, -\Delta_{N,\Omega}) \le \sum_{j=1}^{J} N(\lambda, -\Delta_{N,\Omega^j})$$

where $(\Omega^j)_{j=1}^J$ is a finite non-overlapping tesselation of Ω and Δ_{D,Ω^j} or Δ_{N,Ω^j} denotes appropriately the scaled Dirichlet- or Neumann- Laplace operator. Thus the idea is to incorporate the corresponding operator in between Dirichlet- and Neumann-Laplacians in the sense of (3.44). The known eigenvalues of Δ_{D,Ω^j} and Δ_{N,Ω^j} then provide lower and upper bounds.

We deliver a density assertion that is needed in the sequel.

Proposition 3.15. *Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$ and let $\partial\Omega$ denote its boundary. Let ν be the C^∞-vector field of outer normals. Then the set*

$$C^{m,\nu}(\Omega) = \{f \in C^\infty(\overline{\Omega}) : \frac{\partial^j f}{\partial \nu^j}(x) = 0,\ x \in \partial\Omega,\ j = m, ..., 2m-1\}$$

is dense in $W_p^m(\Omega), 1 \le p < \infty, m \in \mathbb{N}$.

Proof. We recall [HT08, Prop. 5.19] where Sobolev spaces of type $W_2^1(\mathbb{R}^n_+)$ are considered. It suffices to approximate functions in $\mathcal{D}(\Omega)$ with functions belonging to $C^{m,\nu}(\Omega)$. As standard, we first deal with the case of the half space $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}$. Let $f \in \mathcal{D}(\mathbb{R}^n_+)$ and $\varepsilon > 0$. We choose a function $f^\varepsilon \in C^\infty(\overline{\mathbb{R}^n_+})$ such that

$$\operatorname{supp} f^\varepsilon \subseteq \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, 0 \le x_n \le 2\varepsilon\}$$

and

$$f^\varepsilon(x', x_n) = \sum_{j=m}^{2m-1} \frac{1}{j!} \frac{\partial^j f}{\partial x_n^j}(x', 0)\ x_n^j, \qquad 0 \le x_n \le \varepsilon,\ x' \in \mathbb{R}^{n-1}.$$

For a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| \le m$ it holds

$$\begin{aligned}
&\Big|\mathrm{D}^\alpha \Big[\sum_{j=m}^{2m-1} \frac{1}{j!} \frac{\partial^j f}{\partial x_n^j}(x', 0)\ x_n^j \Big] \Big|^p \\
&\qquad \sim \ \Big| \sum_{j=m}^{2m-1} \frac{1}{j!} \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}} \Big[\frac{\partial^j f}{\partial x_n^j}(x', 0) \cdot x_n^j \Big] \Big|^p \\
&\qquad \sim \ \Big| \sum_{j=m}^{2m-1} \frac{1}{j!} \frac{\partial^{|\alpha|-\alpha_n+j} f}{\partial {x_1}^{\alpha_1} \cdots \partial {x_{n-1}}^{\alpha_{n-1}} \partial {x_n}^j}(x', 0)\ \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} \big[x_n^j \big] \Big|^p \\
&\qquad \le \ c \sum_{j=m}^{2m-1} x_n^{(j-\alpha_n)p}.
\end{aligned}$$

Integration over $\{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, 0 \le x_n \le \varepsilon\}$ leads to an upper estimate equivalent to ε and hence one can choose f^ε such that

$$\|f^\varepsilon | W_p^m(\mathbb{R}^n_+)\| \le c\,\varepsilon.$$

Furthermore, we observe that for $k = m, ..., 2m-1$

$$\begin{aligned}
\frac{\partial^k}{\partial x_n^k} \Big[\sum_{j=m}^{2m-1} \frac{1}{j!} \frac{\partial^j f}{\partial x_n^j}(x', 0) \cdot x_n^j \Big](y', 0) &= \sum_{j=m}^{2m-1} \frac{1}{j!} \frac{\partial^j f}{\partial x_n^j}(x', 0) \cdot \Big[\frac{\partial^k}{\partial x_n^k} x_n^j \Big](y', 0) \\
&= \frac{\partial^k f}{\partial x_n^k}(y', 0).
\end{aligned}$$

Finally this shows that $f_\varepsilon := f - f^\varepsilon$ belongs to $C^{m,\nu}(\mathbb{R}^n_+)$. Standard modifications verify the general case of a bounded C^∞-domain $\Omega \subset \mathbb{R}^n$. $\square$

Proof. [Proposition 3.14] Let $B_J := \{x \in \mathbb{R}^n : |x| < 2^{-J}\}$, $J \in \mathbb{N}$, and $B^j := \{x \in \mathbb{R}^n : 2^{-(j+1)} < |x| < 2^{-j}\}$, $j = 0, ..., J-1$. The aim is to derive a decomposition

$$\left(A^m_\psi f, f\right)_{L_2(B)} \sim \sum_{j=0}^{J-1} \left(A^{m,j}_\psi f, f\right)_{L_2(B^j)} + \left(A^m_{\psi,J} f, f\right)_{L_2(B_J)}, \tag{3.54}$$

for all $f \in E^m_\psi(B)$ where the eigenvalues of the elliptic operators $A^{m,j}_\psi$, acting on the annuli B^j, are known and those of the operator $A^m_{\psi,J}$, acting on the small ball B_J, do not influence the asymptotic behaviour of $(\lambda_k(A^m_\psi))_{k\in\mathbb{N}}$. In fact, we need two decompositions (3.54). One related to Dirichlet boundary conditions to get lower bounds of $N(\lambda, A^m_\psi)$ and another with Neumann boundary conditions for upper bounds.

Step 1. We start with Dirichlet bracketing. On the annuli B^j the weight is equivalent to a constant depending on j. That is

$$|x|^m \psi(|x|) \sim 2^{-jm} \psi(2^{-j}), \quad x \in B^j. \tag{3.55}$$

On the one hand, this follows by

$$\psi(|x|) \sim |x|^{-1} \varphi(|x|) \le 2^{(j+1)} \varphi(2^{-j}) \sim \psi(2^{-j})$$

where the increasing function φ is as in (2.17). On the other hand, using (2.20), it is

$$\psi(|x|) \sim |x|^{-1} \varphi(|x|) \ge 2^j \varphi(2^{-(j+1)}) \sim \psi(2^{-(j+1)}) \ge c\, \psi(2^{-j}).$$

Let $J_\lambda \in \mathbb{N}$ be such that $\psi(2^{-J_\lambda}) \sim \lambda^{\frac{1}{2}}$. A decomposition in the spirit of (3.54) is given by

$$\begin{aligned}\left(A^m_\psi f, f\right)_{L_2(B)} \sim\ & \sum_{j=0}^{J_\lambda - 1} \underbrace{2^{-2jm} \psi(2^{-j})^2 \int_{B^j} \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^2 \mathrm{d}x}_{=:\ \left(A^{m,j}_\psi f, f\right)_{L_2(B^j)}} \\ & + \underbrace{\int_{B_{J_\lambda}} |x|^{2m} \psi(|x|)^2 \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^2 \mathrm{d}x}_{=:\ \left(A^m_{\psi,J_\lambda} f, f\right)_{L_2(B_J)}}\end{aligned} \tag{3.56}$$

with corresponding domains

$$\begin{aligned}\mathrm{dom}(A^{m,j}_\psi)^{\frac{1}{2}} &= \mathring{W}^m_2(B^j),\ j = 0, ..., J_\lambda - 1 \\ \mathrm{dom}(A^m_{\psi,J_\lambda})^{\frac{1}{2}} &= E^m_\psi(B_{J_\lambda}).\end{aligned}$$

One can consider A^m_{ψ,J_λ} as the positive definite, self-adjoint operator generated by (3.47) for $f, g \in E^m_\psi(B_{J_\lambda})$. By the following, the operator A^m_{ψ,J_λ} has no eigenvalues smaller than λ

$$\|f|L_2(B_{J_\lambda})\|^2 \le \psi(2^{-J_\lambda})^{-2} \int_{B_{J_\lambda}} |f(x)|^2 \psi(|x|)^2 \,\mathrm{d}x \le \lambda^{-1} E^m_\psi(f,f) \quad (3.57)$$

where $\psi(2^{-J_\lambda})^2 \sim \lambda$ and λ is sufficiently large. We used the monotonicity of ψ near zero and Corollary 2.15. The domains of the operators in (3.56) satisfy

$$E^m_\psi(B) \supset \bigoplus_{j=0}^{J_\lambda - 1} \mathring{W}^m_2(B^j) \oplus E^m_\psi(B_{J_\lambda}),$$

and it follows as described in (3.45) that

$$N(\lambda, A^m_\psi) \ge c \sum_{j=0}^{J_\lambda-1} N(\lambda, A^{m,j}_\psi) + N(\lambda, A^m_{\psi,J_\lambda}) = c \sum_{j=0}^{J_\lambda-1} N(\lambda, A^{m,j}_\psi). \quad (3.58)$$

We need to find lower bounds for $N(\lambda, A^{m,j}_\psi)$. Therefore, let

$$\begin{aligned} \mathrm{dom}\left((-\Delta)^m_{D,j}\right)^{1/2} &= \mathring{W}^m_2(B^j) \\ A^{m,j}_{\psi,D} &= 2^{-2jm}\psi(2^{-j})^2\,(-\Delta)^m_{D,j}, \end{aligned}$$

be the scaled Dirichlet operator $(-\Delta)^m_D$ on B^j. The corresponding eigenvalues $\lambda_k(A^{m,j}_{\psi,D})$ of $A^{m,j}_{\psi,D}$ satisfy

$$\lambda_k(A^{m,j}_{\psi,D})^{\frac{1}{2}} \;\sim\; \psi(2^{-j})\,k^{\frac{m}{n}}. \quad (3.59)$$

This follows by a corresponding result for $(-\Delta)^m_{D,0}$ in the annulus B^0, see [HT08, Theorem 7.15], and a subsequent reduction of $(-\Delta)^m_{D,j}$ to $(-\Delta)^m_{D,0}$ by dilation. Applying again Max-Min-Principle arguments, we get

$$N(\lambda, A^{m,j}_\psi) \ge N(\lambda, A^{m,j}_{\psi,D}) \sim \lambda^{\frac{n}{2m}} \psi(2^{-j})^{-\frac{n}{m}}.$$

Together with (3.58) it holds

$$N(\lambda, A^m_\psi) \ge c\,\lambda^{\frac{n}{2m}} \sum_{j=0}^{J_\lambda-1} \psi(2^{-j})^{-\frac{n}{m}} \sim \lambda^{\frac{n}{2m}}\,\Phi(J_\lambda).$$

Step 2. We continue with Neumann bracketing. The decomposition analogous to (3.56) is

$$
\begin{aligned}
\big(A_\psi^m f, f\big)_{L_2(B)} &\sim \int_B \psi(|x|)^2\Big[|f(x)|^2 + |x|^{2m}\sum_{|\alpha|=m}|\mathrm{D}^\alpha f(x)|^2\Big]\mathrm{d}x\\
&\sim \sum_{j=0}^{J_\lambda-1}\psi(2^{-j})^2\int_{B^j}\Big[|f(x)|^2 + 2^{-2jm}\sum_{|\alpha|=m}|\mathrm{D}^\alpha f(x)|^2\Big]\mathrm{d}x\\
&\qquad + \int_{B_{J_\lambda}}\psi(|x|)^2\Big[|f(x)|^2 + |x|^{2m}\sum_{|\alpha|=m}|\mathrm{D}^\alpha f(x)|^2\Big]\mathrm{d}x
\end{aligned}
$$

with domains $W_2^m(B^j)$, $j = 0, ..., J_\lambda - 1$, and $\overline{C^{m,\nu}(B_{J_\lambda})}^{\|\cdot|E_\psi^m(B_{J_\lambda})\|}$ of the corresponding operators with Neumann conditions. Here $C^{m,\nu}(B_{J_\lambda})$ consists of all $f \in C^m(B_{J_\lambda})$ such that $\dfrac{\partial^j f}{\partial\nu^j}(x) = 0$ for $|x| = 2^{-J_\lambda}$, $j = m, ..., 2m-1$. Note that $\overline{C^{m,\nu}(B_{J_\lambda})}^{\|\cdot|E_\psi^m(B_{J_\lambda})\|} = \overline{C^m(B_{J_\lambda})}^{\|\cdot|E_\psi^m(B_{J_\lambda})\|}$. This is a consequence of Proposition 3.15. Similarly to (3.57), we argue by

$$
\begin{aligned}
&\|f|L_2(B_{J_\lambda})\|^2\\
&\quad\le \lambda^{-1}\Big[\int_{B_{J_\lambda}}|f(x)|^2\psi(|x|)^2\mathrm{d}x + \int_{B_{J_\lambda}}\psi(|x|)^2|x|^{2m}\sum_{|\alpha|=m}|\mathrm{D}^\alpha f(x)|^2\mathrm{d}x\Big]
\end{aligned}
$$

$f \in C^m(B_{J_\lambda})$, that the corresponding operator acting on B_{J_λ} has no eigenvalues smaller than λ. Note that we did not use Corollary 2.15 for this last claim, since we cannot assume boundary values here. We define the scaled Neumann operator by

$$
\begin{aligned}
&\operatorname{dom}\big((-\Delta)_{N,j}^m\big)^{1/2} = W_2^m(B^j)\\
&A_{\psi,N}^{m,j} = \psi(2^{-j})^2\ \big[\ \mathrm{id} + 2^{-2jm}(-\Delta)_{N,j}^m\big].
\end{aligned}
$$

Because of

$$
E_\psi^m(B) \subset \bigoplus_{j=0}^{J_\lambda-1} W_2^m(B^j) \oplus \overline{C^{m,\nu}(B_{J_\lambda})}^{\|\cdot|E_\psi^m(B_{J_\lambda})\|}
$$

we obey

$$
N(\lambda, A_\psi^m) \le c\sum_{j=0}^{J_\lambda-1} N(\lambda, A_{\psi,N}^{m,j}) \le c\,\lambda^{\frac{n}{2m}}\sum_{j=0}^{J_\lambda-1}\psi(2^{-j})^{-\frac{n}{m}} \sim \lambda^{\frac{n}{2m}}\,\Phi(J_\lambda).
$$

□

With the help of Proposition 3.14, we can refine our results from Proposition 3.8 at least if $p = 2$. We give sharp growth conditions on ψ and a precise characterisation of their influence on the compactness in that case.

Theorem 3.16. *Let $m \in \mathbb{N}$ and ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1) = 1$, bounded from below by a positive constant and locally decreasing at zero with $\lim_{t\to 0} \psi(t) = \infty$. Then the embedding*

$$\mathrm{id} : E^m_\psi(B) \hookrightarrow L_2(B)$$

is compact. Furthermore, let ψ^{-1} be a positive function such that, with some $t_0 \in (0,1]$, it holds for all $t \leq t_0$

$$\psi(t) \sim s \iff \psi^{-1}(s) \sim t.$$

Let Φ be a continuous function equivalent to a primitive of $[\psi(2^{-\cdot})]^{-\frac{n}{m}}$ and let the function h be defined by

$$h(\lambda) := \lambda^{\frac{n}{2m}} \Phi\big(-\log(\psi^{-1}(\lambda^{\frac{1}{2}}))\big), \quad \lambda > 0.$$

Let H be a function such that for all $k \in \mathbb{N}, \lambda > 0$

$$h(\lambda) \sim k \iff k^{\frac{2m}{n}} H(k) \sim \lambda.$$

All preceding equivalence constants are independent of the variables. Then it holds for $k \geq 2$

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim k^{-\frac{m}{n}}\, H(k)^{-\frac{1}{2}}. \tag{3.60}$$

In particular, if $([\psi(2^{-j})]^{-1})_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}}$ then

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim k^{-\frac{m}{n}}. \tag{3.61}$$

Proof. Let A^m_ψ be the positive definite self-adjoint operator from Proposition 3.14. We have

$$\psi(2^{-J_\lambda}) \sim \lambda^{\frac{1}{2}} \iff J_\lambda \sim -\log(\psi^{-1}(\lambda^{\frac{1}{2}})) \tag{3.62}$$

if $J_\lambda \in \mathbb{N}$. It follows for sufficiently large λ that

$$\Phi(J_\lambda) \sim \Phi\big(-\log(\psi^{-1}(\lambda^{\frac{1}{2}}))\big) \tag{3.63}$$

where the equivalence constants of (3.63) depend on those of (3.62). This can be seen as follows

$$\begin{aligned}
\Phi\big(J_\lambda\big) &\sim \int_{j_0}^{J_\lambda} \psi(2^{-t})^{-\frac{n}{m}} \mathrm{d}t \\
&\leq \int_{j_0}^{c[-\log(\psi^{-1}(\lambda^{1/2}))]} \psi(2^{-t})^{-\frac{n}{m}} \mathrm{d}t = \; c \int_{j_0/c}^{-\log(\psi^{-1}(\lambda^{1/2}))} \psi(2^{-cs})^{-\frac{n}{m}} \mathrm{d}s \\
&\leq c \int_{j_0/c}^{-\log(\psi^{-1}(\lambda^{1/2}))} \psi(2^{-s})^{-\frac{n}{m}} \mathrm{d}s \\
&\sim \; \Phi(-\log(\psi^{-1}(\lambda^{\frac{1}{2}})))
\end{aligned}$$

if j_0 is sufficiently large. The last inequality follows from the monotonicity of ψ (assuming $c > 1$). The converse estimate is similar. We rewrite Proposition 3.14 as

$$N(\lambda, A_\psi^m) \sim h(\lambda).$$

Because of (3.51) we can conclude

$$a_k(\mathrm{id}) \sim k^{-\frac{m}{n}} H(k)^{-\frac{1}{2}}. \tag{3.64}$$

It remains to show the equivalence for the entropy numbers. For a constant $c > 0$ it is $H(k) \sim H(ck)$. Furthermore, (3.64) holds. Then one has in particular

$$a_{2^{j-1}}(\mathrm{id}) \sim a_{2^j}(\mathrm{id}),\ j \in \mathbb{N},$$

and it follows from (3.30)

$$e_k(\mathrm{id}) \leq c\, a_k(\mathrm{id}).$$

On the other hand

$$a_k(\mathrm{id}) \sim \lambda_k\big((A_\psi^m)^{-1/2}\big) \leq c\, e_k((A_\psi^m)^{-1/2}) \sim e_k(\mathrm{id}).$$

$\square$

Remark 3.17. If one is concerned with estimating eigenvalues of an operator a possible strategy is to deal with corresponding estimates of entropy numbers. Many operators (such as integral, differential or pseudodifferential operators) induce maps that can be decomposed into an embedding map between spaces of Sobolev type and various other continuous maps. Then due to the (sub-)multiplicativity knowledge about the entropy numbers of such Sobolev type embeddings together with knowledge about the norms of the remaining decomposition maps lead to the required knowledge about eigenvalues. In this view the strategy of this section pursues the opposite direction: the knowledge of the distribution of the eigenvalues of some degenerate elliptic operators results in knowledge about entropy and approximation numbers of Sobolev embeddings.

4 Approximation numbers via bracketing

We continue the discussion of the compact embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B), \qquad 1 \le p < \infty, m \in \mathbb{N}, \tag{4.1}$$

introduced in Section 3.1. Up to now we obtained in Theorem 3.16 sharp results for the corresponding entropy and approximation numbers - but only in case of Hilbert spaces. The strategy to gain this outcome for $p = 2$ hinges on the Courant-Weyl method of Dirichlet-Neumann bracketing. This technique is not available for $p \neq 2$. Nevertheless, we present a way to extend this approach to the general case of Banach spaces. By doing so, we improve the investigations from Section 3.2 and confirm a conjecture made in [Tr12b, Conjecture 3.8].

The examination of (4.1) has its roots in [Tr12b] where Triebel dealt with polynomial weights perturbed by a logarithmic term $\psi(t) = (1 + |\log t|)^\sigma$, $\sigma \ge 0$. This is why we first concentrate in Section 4.1 on the logarithmic setting and afterwards do some generalisation to slowly varying functions in Section 4.2.

4.1 The logarithmic case

Let $E^m_{p,\sigma}(B)$ be the closure of $C^m_0(B)$ in $L_p(B)$ with respect to the norm

$$\|f|E^m_{p,\sigma}(B)\| := \left(\int_B |x|^{mp}(1 + |\log|x||)^{\sigma p} \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x \right)^{1/p} \tag{4.2}$$

for $1 \le p < \infty, m \in \mathbb{N}, \sigma \ge 0$. This weighted Sobolev space was introduced in [Tr12b] concurrently with related logarithmic Hardy inequalities. Furthermore, the author investigated the setting in the sense of Section 3. Namely it is shown that the embedding

$$\mathrm{id} : E^m_{p,\sigma}(B) \hookrightarrow L_p(B) \tag{4.3}$$

is compact if, and only if, $\sigma > 0$. Note that this is also an immediate consequence of Theorem 3.4. We recall [Tr12b, Theorem 3.3, Theorem 3.4] dealing with the asymptotic behaviour of the entropy and approximation numbers. The results have some gaps since methods as described in Section 3.2 were used.

Proposition 4.1. *Let* $1 \leq p < \infty, m \in \mathbb{N}, \sigma > 0$ *and* $E^m_{p,\sigma}(B)$ *be the closure of* $C^m_0(B)$ *according to* (4.2). *Then the embedding*

$$\mathrm{id} : E^m_{p,\sigma}(B) \hookrightarrow L_p(B)$$

is compact.

(i) *If* $\sigma > \frac{n+m}{n}$ *then it holds for* $k \in \mathbb{N}$

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim k^{-\frac{m}{n}}.$$

(ii) *If* $0 < \sigma \leq \frac{n+m}{n}$ *then there are a number* $c > 0$ *and for any* $\varepsilon > 0$ *a number* $c_\varepsilon > 0$ *such that for* $k \in \mathbb{N}$

$$c\, k^{-\frac{m}{n}} \leq e_k(\mathrm{id}) \leq c_\varepsilon\, k^{-\sigma\frac{m}{n+m}+\varepsilon}.$$

(iii) *If* $0 < \sigma \leq \frac{n+m}{n}$ *then there are a number* $c > 0$ *and for any* $\varepsilon > 0$ *a number* $c_\varepsilon > 0$ *such that for* $k \in \mathbb{N}$

$$c\, k^{-\min(\sigma,\frac{m}{n})} \leq a_k(\mathrm{id}) \leq c_\varepsilon\, k^{-\sigma\frac{m}{n+m}+\varepsilon}.$$

As described in Section 3.3 one can refine these results if $p = 2$ and seal this gap using inclusions of related quadratic forms in Hilbert spaces. We recall [Tr12b, Theorem 3.6] which is also a consequence of Theorem 3.16.

Proposition 4.2. *Let* $m \in \mathbb{N}, \sigma > 0$. *Then the embedding*

$$\mathrm{id} : E^m_{2,\sigma}(B) \hookrightarrow L_2(B) \tag{4.4}$$

is compact and it holds for $k \in \mathbb{N}, k \geq 2$,

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}} & \text{if } \sigma > \frac{m}{n} \\ k^{-\frac{m}{n}}\,(\log k)^{\frac{m}{n}} & \text{if } \sigma = \frac{m}{n} \\ k^{-\sigma} & \text{if } 0 < \sigma < \frac{m}{n}. \end{cases} \tag{4.5}$$

As described in Section 3.3 the proof of the last proposition is based on the Courant-Weyl method of Dirichlet-Neumann bracketing in Hilbert spaces which is so effective in determining the asymptotic limit of eigenvalue counting functions if $p = 2$. We will extend this concept and confirm Conjecture 3.8 from [Tr12b] asserting that (4.5) holds for all $1 \leq p < \infty$. We follow the idea used in [EH93] and [EE04, Chapter 6.3] by Edmunds, Evans and Harris. Those authors developed a partial analogue of the Dirichlet-Neumann technique for estimating the asymptotic behaviour of the approximation numbers of (unweighted) Sobolev embeddings of type $W_p^1(\Omega)$ on a wide class of domains, i.e. rooms and passages domains or generalised ridged domains. We want to transfer this idea to control the singularity of the weight in the setting of (4.2).

We denote the restriction of (4.3) to subsets $\Omega \subseteq B$ by

$$\mathrm{id}_\Omega : E^m_{p,\sigma}(\Omega) \hookrightarrow L_p(\Omega). \tag{4.6}$$

We introduce the following quantities $\nu_0(\varepsilon, \Omega)$ and $\mu_0(\varepsilon, \Omega)$ overpassing the notation from [EH93].

Definition 4.3. *Let $1 \le p < \infty, m \in \mathbb{N}, \sigma > 0$ and $\varepsilon > 0$. We define*

$$\nu_0(\varepsilon, \Omega) := \max\{k \in \mathbb{N} : a_k(\mathrm{id}_\Omega) \ge \varepsilon\} \tag{4.7}$$

and put $\nu_0(\varepsilon, \Omega) = 0$ if $a_k(\mathrm{id}_\Omega) < \varepsilon$ for all $k \in \mathbb{N}$. Furthermore, let

$$\mu_0(\varepsilon, \Omega) := \max\left\{ \dim S : \alpha(S) := \sup_{u \in S \setminus \{0\}} \frac{\|u|E^m_{p,\sigma}(\Omega)\|}{\|u|L_p(\Omega)\|} \le \frac{1}{\varepsilon} \right\} \tag{4.8}$$

where the maximum is taken over all finite-dimensional linear subspaces S of $E^m_{p,\sigma}(\Omega)$.

Remark 4.4. In the framework of $L_p(\Omega)$ the approximation numbers $a_k(\mathrm{id}_\Omega)$ tend to zero due to the compactness of (4.6). Hence the maximum in (4.7) is attained for some natural number $N = N(\varepsilon)$. We will see in Proposition 4.6 that this implies $\mu_0(\varepsilon, \Omega) < \infty$ for every $\varepsilon > 0$ and the maximum in (4.8) is also attained.
The embedding $\mathrm{id}_\Omega : E^m_{p,\sigma}(\Omega) \hookrightarrow L_p(\Omega)$ is injective. Thus for every finite-dimensional linear subspace $S \subset E^m_{p,\sigma}(\Omega)$ the restriction id^S of id_Ω to S

$$\mathrm{id}^S : S \to \mathrm{id}_\Omega(S)$$

is bijective and bounded. We have

$$\|(\mathrm{id}^S)^{-1}\| = \alpha(S) < \infty.$$

Clearly

$$\nu_0(\varepsilon, \Omega) \to \infty \qquad \text{as } \varepsilon \to 0$$

describes the asymptotic behaviour of $a_k(\mathrm{id}_\Omega) \to 0$ as $k \to \infty$. So the main concern of this section is to obtain upper and lower bounds for $\nu_0(\varepsilon, B)$.

Remark 4.5. Let $\lim\limits_{k\to\infty} a_k(\mathrm{id}) = 0$ and $\nu_0(\varepsilon,\Omega) \sim f(\varepsilon)$ with some continuous function f. Then it follows

$$k \sim \nu_0(a_k(\mathrm{id}), B). \tag{4.9}$$

Clearly $k \leq \nu_0(a_k(\mathrm{id}), B)$. To verify the opposite estimate let for $k \in \mathbb{N}$ the numbers $N, M \in \mathbb{N}, N \leq k \leq M$ be such that

$$a_{M+1}(\mathrm{id}) < a_M(\mathrm{id}) = ... = a_k(\mathrm{id}) = ... = a_N(\mathrm{id}) < a_{N-1}(\mathrm{id}).$$

If $\varepsilon > 0$ is sufficiently small we obtain

$$f(a_N(\mathrm{id}) + \varepsilon) \sim \nu_0(a_N(\mathrm{id}) + \varepsilon, B) = N - 1 < k \leq M = \nu_0(a_k(\mathrm{id}), B).$$

Now letting $\varepsilon \to 0$ finishes the proof of (4.9).

The essential observation is that $\nu_0(\varepsilon,\Omega)$ and $\mu_0(\varepsilon,\Omega)$ provide a bracketing property which allows to decompose the domain appropriately. The next proposition can be seen as an L_p-version of the Dirichlet-Neumann bracketing method from spectral L_2-theory.

Proposition 4.6. *Let* $\Omega = \big(\bigcup\limits_{j=1}^{J} \overline{\Omega}_j\big)^{\circ}$ *with disjoint domains* $\Omega_j, j = 1, ..., J$. *Then for* $\varepsilon > 0$ *it holds*

$$\sum_{j=1}^{J} \mu_0(\varepsilon,\Omega_j) \leq \mu_0(\varepsilon,\Omega) \leq \nu_0(\varepsilon,\Omega) \leq \sum_{j=1}^{J} \nu_0(\varepsilon,\Omega_j). \tag{4.10}$$

Proof. *Step 1.* We prove the first inequality. Let $j \in \{1, ..., J\}$. We assume $\mu_0(\varepsilon,\Omega_j) < \infty$. Otherwise the assertion is trivial. Then there exists a subspace $S_j \subset E^m_{p,\sigma}(\Omega_j)$ with $\dim S_j = \mu_0(\varepsilon,\Omega_j)$ such that

$$\|u|E^m_{p,\sigma}(\Omega_j)\| \leq \frac{1}{\varepsilon}\, \|u|L_p(\Omega_j)\|, \qquad u \in S_j.$$

We consider the following subspace S of $E^m_{p,\sigma}(\Omega)$

$$S := \bigoplus_{j=1}^{J} S_j \subseteq E^m_{p,\sigma}(\Omega).$$

For $v \in S$, say $v = \sum\limits_{j=1}^{J} u_j, u_j \in S_j$, we get according to the disjointness of the domains Ω_j

$$\begin{aligned}\|v \mid E^m_{p,\sigma}(\Omega)\|^p &= \sum_{j=1}^{J} \|u_j \mid E^m_{p,\sigma}(\Omega_j)\|^p \\ &\leq \sum_{j=1}^{J} \frac{1}{\varepsilon^p}\|u_j \mid L_p(\Omega_j)\|^p = \frac{1}{\varepsilon^p}\|v \mid L_p(\Omega)\|^p.\end{aligned}$$

Thus S is an admitted subspace in (4.8) and we conclude

$$\mu_0(\varepsilon, \Omega) \geq \dim S = \sum_{j=1}^{J} \mu_0(\varepsilon, \Omega_j).$$

Step 2. We prove the second inequality. Let $S \subset E^m_{p,\sigma}(\Omega)$ be a finite-dimensional subspace and let $P : E^m_{p,\sigma}(\Omega) \to L_p(\Omega)$ be a finite rank operator with

$$\operatorname{rank} P < \dim S =: d.$$

Then there is an element $0 \neq f^* \in S$ with $P(f^*) = 0$. Denote $f^* = \sum_{i=1}^{d} \lambda_i e_i$ where $\sum_{i=1}^{d} |\lambda_i| \neq 0$ and $S = \operatorname{span}\{e_1, ..., e_d\}$. Then

$$\|(\mathrm{id} - P) f^* \mid L_p(\Omega)\| = \|f^* \mid L_p(\Omega)\| \geq \alpha(S)^{-1} \|f^* \mid E^m_{p,\sigma}(\Omega)\|$$

and hence

$$\|\,\mathrm{id} - P\| \geq \alpha(S)^{-1}.$$

At this point we have seen that

$$a_d(\mathrm{id}) \geq \alpha(S)^{-1}$$

for all finite dimensional subspaces $S \subset E^m_{p,\sigma}(\Omega)$ with $\dim S = d$. This means

$$a_d(\mathrm{id}) \geq \varepsilon$$

for all finite dimensional subspaces $S \subset E^m_{p,\sigma}(\Omega)$ with $\dim S = d$ and in addition $\alpha(S) \leq \frac{1}{\varepsilon}$. Hence,

$$\nu_0(\varepsilon, \Omega) \geq \dim S$$

for all finite dimensional subspaces $S \subset E^m_{p,\sigma}(\Omega)$ with $\alpha(S) \leq \frac{1}{\varepsilon}$. This finishes the verification of $\nu_0(\varepsilon, \Omega) \geq \mu_0(\varepsilon, \Omega)$.

Step 3. We prove the last inequality. For $k := \nu_0(\varepsilon, \Omega_j) + 1$ we have

$$a_k(\mathrm{id}^j : E^m_{p,\sigma}(\Omega_j) \hookrightarrow L_p(\Omega_j)) < \varepsilon.$$

In other words, for every $j = 1, ..., J$ there exist linear and bounded operators $P_j : E^m_{p,\sigma}(\Omega_j) \to L_p(\Omega_j)$, $\operatorname{rank} P_j \le \nu_0(\varepsilon, \Omega_j)$ such that

$$\| \operatorname{id}^j - P_j | E^m_{p,\sigma}(\Omega_j) \to L_p(\Omega_j)\| < \varepsilon.$$

Let P be the operator defined by

$$(Pf)(x) := \sum_{j=1}^{J} \chi_{\Omega_j}(x)\ (P_j f_{\Omega_j})(x), \quad f \in E^m_{p,\sigma}(\Omega)$$

where f_{Ω_j} denotes the restriction of f to Ω_j. Then it holds for all $f \in E^m_{p,\sigma}(\Omega)$

$$\begin{aligned}
\|f - Pf|L_p(\Omega)\|^p &= \int_\Omega |f(x) - \sum_{j=1}^{J} \chi_{\Omega_j}(x)\ (P_j f_{\Omega_j})(x)|^p \mathrm{d}x \\
&= \sum_{j=1}^{J} \int_{\Omega_j} |f(x) - P_j f(x)|^p \mathrm{d}x \\
&\le \sum_{j=1}^{J} \| \operatorname{id}^j - P_j\|^p\ \|f|E^m_{p,\sigma}(\Omega_j)\|^p \\
&< \varepsilon^p\ \|f|E^m_{p,\sigma}(\Omega)\|^p.
\end{aligned}$$

Thus for $L := 1 + \sum_{j=1}^{J} \nu_0(\varepsilon, \Omega_j)$, we have $a_L(\mathrm{id}) < \varepsilon$. Hence

$$\nu_0(\varepsilon, \Omega) = \max\{l : a_l(\mathrm{id}) \ge \varepsilon\} \ \le\ L - 1 = \sum_{j=1}^{J} \nu_0(\varepsilon, \Omega_j).$$

□

Next we discuss some facts that show the accordance with the Dirichlet-Neumann bracketing technique in case of $p = 2$. Recall that the main results for $p = 2$ in [Tr12b], or rather in the more general setting of Section 3.3, are related to the eigenvalue distribution of the degenerate elliptic operator A^m_σ, defined by

$$A^m_\sigma f = (-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha (b_{m,\sigma} \mathrm{D}^\alpha f) \tag{4.11}$$

$$\operatorname{dom}(A^m_\sigma)^{\frac{1}{2}} = E^m_{2,\sigma}(B) \tag{4.12}$$

where $b_{m,\sigma}(x) = |x|^{2m}(1 + |\log|x||)^{2\sigma}, x \in B$. The operator A^m_σ is positive definite, self-ajdoint and has pure point spectrum $(\lambda_k(A^m_\sigma))_{k\in\mathbb{N}}$ (monotonically ordered). In the next proposition we will see that the quantities

$\nu_0(\varepsilon, B)$ and $\mu_0(\varepsilon, B)$ coincide if $p = 2$ and recover the method of Dirichlet-Neumann bracketing used in Section 3.3 or rather [Tr12b].

Proposition 4.7. *Let $m \in \mathbb{N}, \sigma > 0$. Then the embedding*

$$\mathrm{id} : E^m_{2,\sigma}(B) \hookrightarrow L_2(B)$$

is compact. Furthermore, for every $\varepsilon > 0$ it holds for $p = 2$

$$\mu_0(\varepsilon, B) = \nu_0(\varepsilon, B) = N(\varepsilon^{-2}, A^m_\sigma) \tag{4.13}$$

where $N(\lambda, A^m_\sigma)$ denotes the number of eigenvalues $\lambda_k(A^m_\sigma)$ smaller than or equal to $\lambda > 0$.

Proof. Let $\mathrm{id}^* : L_2(B) \hookrightarrow E^m_{2,\sigma}(B)$ be the dual map of id defined by

$$\big(\mathrm{id}\, f, g\big)_{L_2(B)} = \big(f, \mathrm{id}^*\, g\big)_{E^m_{2,\sigma}(B)} \qquad \forall f \in E^m_{2,\sigma}(B), g \in L_2(B). \tag{4.14}$$

Here $(\cdot,\cdot)_{L_2(B)}$ denotes the inner product in $L_2(B)$. Respectively the inner product $(\cdot,\cdot)_{E^m_{2,\sigma}(B)}$ in $E^m_{2,\sigma}(B)$ is given by

$$\int_B b_{m,\sigma}(x) \sum_{|\alpha|=m} \overline{\mathrm{D}^\alpha f(x)} \mathrm{D}^\alpha g(x) \mathrm{d}x, \qquad f, g \in E^m_{2,\sigma}(B). \tag{4.15}$$

Then (4.15) generates the positive definite, self-adjoint operator A^m_σ given by (4.11) and (4.12). That means

$$\big(A^m_\sigma f, \mathrm{id}\, g\big)_{L_2(B)} = \big(f, g\big)_{E^m_{2,\sigma}(B)}, \qquad f \in \mathrm{dom}\, A^m_\sigma, g \in E^m_{2,\sigma}(B).$$

The embedding $\mathrm{id} : E^m_{2,\sigma}(B) \hookrightarrow L_2(B)$ is compact. Hence the approximation numbers coincide with its singular values, see for instance [EE87, Theorem II.5.10, p.91]. That is

$$a_k(\mathrm{id}) = \lambda_k(|\,\mathrm{id}\,|) = \lambda_k([\,\mathrm{id}^* \circ \mathrm{id}\,]^{1/2}) \tag{4.16}$$

where $\lambda_k(\cdot)$ denotes the kth eigenvalue of the corresponding operator. Furthermore,

$$\mathrm{id}^* \circ \mathrm{id} : E^m_{2,\sigma}(B) \to E^m_{2,\sigma}(B)$$

is a non-negative, compact and selfadjoint operator. Respectively we apply [EE87, Theorem II.5.6, p.84] to $T = \mathrm{id}^* \circ \mathrm{id}$. We get that

$$\#\{k : \lambda_k(T) \geq \varepsilon^2\} = \max \dim S$$

where the maximum is taken over all closed linear subspaces S of $E^m_{2,\sigma}(B)$ such that for all $f \in S$

$$\big(Tf, f\big)_{E^m_{2,\sigma}(B)} \geq \varepsilon^2 \|f | E^m_{2,\sigma}(B)\|^2.$$

Due to (4.14) the last line is equivalent to

$$\alpha(S) = \sup_{f\in S, f\neq 0} \frac{\|f|E^m_{2,\sigma}(B)\|}{\|f|L_2(B)\|} \leq \frac{1}{\varepsilon}.$$

We have shown

$$\nu_0(\varepsilon, B) = \#\{k : a_k(\mathrm{id}) \geq \varepsilon\} = \#\{k : \lambda_k(T) \geq \varepsilon^2\} \leq \mu_0(\varepsilon, B).$$

The converse inequality was already shown in (4.10). This proves $\nu_0(\varepsilon,\Omega) = \mu_0(\varepsilon,\Omega)$. Recall that id and id* have the same singular values, see [EE87, Theorem II.5.7, p. 85]. Next we prove that

$$\mathrm{id}\circ\mathrm{id}^* = (A^m_\sigma)^{-1} \tag{4.17}$$

where we consider the operators from $L_2(B)$ to $L_2(B)$. Therefore, let $f \in \mathrm{dom}\, A^m_\sigma$ and $g \in E^m_{2,\sigma}(B)$. Then

$$\big(\mathrm{id}^*(A^m_\sigma f), g\big)_{E^m_{2,\sigma}(B)} = \big(A^m_\sigma f, \mathrm{id}\, g\big)_{L_2(B)} = \big(f,g\big)_{E^m_{2,\sigma}(B)}.$$

Hence for all $f \in \mathrm{dom}\, A^m_\sigma$

$$\mathrm{id}^*(A^m_\sigma f) = f.$$

The inverse operator $(A^m_\sigma)^{-1}$ acting in $L_2(B)$ is given by

$$\begin{aligned} \mathrm{dom}(A^m_\sigma)^{-1} &:= \{g \in L_2(B) : \exists\, f \in \mathrm{dom}\, A^m_\sigma, A^m_\sigma f = g\} \\ (A^m_\sigma)^{-1} g &:= \mathrm{id}\, f. \end{aligned}$$

Now

$$(A^m_\sigma)^{-1} g = \mathrm{id}\, f = \mathrm{id}(\mathrm{id}^*\, A^m_\sigma f) = \mathrm{id}(\mathrm{id}^*\, g).$$

what proves (4.17). Finally in view of (4.16) we get

$$\begin{aligned} \nu_0(\varepsilon, B) &= \#\{k : a_k(\mathrm{id}) \geq \varepsilon\} \\ &= \#\{k : \lambda_k(\mathrm{id}^*\circ\mathrm{id}) \geq \varepsilon^2\} \\ &= \#\{k : \lambda_k((A^m_\sigma)^{-1}) \geq \varepsilon^2\} \\ &= \#\{k : \lambda_k(A^m_\sigma) \leq \varepsilon^{-2}\} \\ &= N(\varepsilon^{-2}, A^m_\sigma). \end{aligned}$$

□

We turn to the situation of Banach spaces, $1 \leq p < \infty$, and concentrate on the asymptotic behaviour of $\nu_0(\varepsilon, B)$ as $\varepsilon \to 0$. First we will show that one can cut off the singularity of the weight $|x|^m(1+\log|x||)^\sigma$ at $x = 0$ without affecting the asymptotic behaviour of $\nu_0(\varepsilon, B)$ and $\mu_0(\varepsilon, B)$. This is the crucial point to control the singularity.

Proposition 4.8. *Let $\varepsilon > 0$ and put $B_J := \{x \in \mathbb{R}^n : |x| < 2^{-J}\}$ with $J = J(\varepsilon) \in \mathbb{N}$ such that $J \sim \varepsilon^{-\frac{1}{\sigma}}$. Then it holds*

$$\nu_0(\varepsilon, B_J) = \mu_0(\varepsilon, B_J) = 0. \tag{4.18}$$

Proof. Due to (4.10) it suffices to prove that $\nu_0(\varepsilon, B_J) = 0$. Let $0 < \delta < 1$ be the constant from (2.35). Without loss of generality we assume $2^{-J} \leq \delta$. Then for $f \in E^m_{p,\sigma}(B_J)$

$$\begin{aligned} \|f|L_p(B_J)\|^p &\leq c\, J^{-\sigma p} \int_{B_J} (1 + |\log|x||)^{\sigma p} |f(x)|^p \mathrm{d}x \\ &\leq c\, J^{-\sigma p} \|f|E^m_{p,\sigma}(B_J)\|^p. \end{aligned}$$

If $J \sim \varepsilon^{-\frac{1}{\sigma}}$ that means

$$\|\operatorname{id}_J |E^m_{p,\sigma}(B_J) \hookrightarrow L_p(B_J)\| < \varepsilon.$$

This proves the assertion since the approximation numbers $a_k(\operatorname{id}_J)$ are bounded by the norm $\|\operatorname{id}_J\|$. □

We are now able to mimic the Dirichlet-Neumann bracketing in $L_2(B)$ used in [Tr12b] to prove (4.5) for general values $1 \leq p < \infty$. We decompose the domain B into finitely many annuli leaving a small ball around the origin and consider restricted operators separately. In that way we get rid of the singularity due to the last proposition. Then $\nu_0(\varepsilon, B^j)$ and $\mu_0(\varepsilon, B^j)$ restricted to annuli B^j deliver lower and upper bounds for $\nu_0(\varepsilon, B)$. The result reads as follows.

Proposition 4.9. *Let $m \in \mathbb{N}$ and $\varepsilon > 0$ small. Then it holds*

$$\nu_0(\varepsilon, B) \sim \begin{cases} \varepsilon^{-\frac{n}{m}} & \text{if } \sigma > \frac{m}{n} \\ \varepsilon^{-\frac{n}{m}} |\log \varepsilon| & \text{if } \sigma = \frac{m}{n} \\ \varepsilon^{-\frac{1}{\sigma}} & \text{if } 0 < \sigma < \frac{m}{n}. \end{cases} \tag{4.19}$$

Proof. Let $J \in \mathbb{N}$ with $J \sim \varepsilon^{-\frac{1}{\sigma}}$ and denote finitely many annuli $B^j := \{x \in B :\ 2^{-j} \leq |x| < 2^{-j+1}\}, j = 1, ..., J$ and $B_J := \{x \in B :\ |x| < 2^{-J}\}$. Consider the disjoint partition of the unit ball

$$B = B_J \ \cup\ (B \setminus B_J)$$

where $B \setminus B_J := \bigcup_{j=1}^{J} B^j$. By (4.10) and (4.18) one has

$$\begin{aligned} \mu_0(\varepsilon, B \setminus B_J) &= \mu_0(\varepsilon, B \setminus B_J) + \mu_0(\varepsilon, B_J) \\ &\leq \nu_0(\varepsilon, B) \\ &\leq \nu_0(\varepsilon, B \setminus B_J) + \nu_0(\varepsilon, B_J) = \nu_0(\varepsilon, B \setminus B_J) \end{aligned} \tag{4.20}$$

Step 1. We prove the upper bounds of (4.19). On the annuli B^j we can replace the weight by proportional constants

$$|x|^m(1+|\log|x||)^\sigma \sim 2^{-jm}j^\sigma, \qquad x \in B^j.$$

Hence we have

$$\|\operatorname{id}^j|E^m_{p,\sigma}(B^j) \hookrightarrow \overset{\circ}{W}{}^m_p(B^j)\| \le c\, 2^{jm}j^{-\sigma}.$$

From the well-known classical result (3.32) for smooth bounded domains $\Omega \subset \mathbb{R}^n$

$$a_k(\overset{\circ}{W}{}^m_p(\Omega) \hookrightarrow L_p(\Omega)) \sim k^{-\frac{m}{n}}$$

it follows with dilation arguments that

$$a_k(\overset{\circ}{W}{}^m_p(B^j) \hookrightarrow L_p(B^j)) \sim 2^{-jm}k^{-\frac{m}{n}}.$$

By decomposition of $\operatorname{id}^j : E^m_{p,\sigma}(B^j) \hookrightarrow L_p(B^j)$ we get

$$\begin{aligned} a_k(\operatorname{id}^j) &\le c\,\|\operatorname{id}^j|E^m_{p,\sigma}(B^j) \hookrightarrow \overset{\circ}{W}{}^m_p(B^j)\|\, a_k(\overset{\circ}{W}{}^m_p(B^j) \hookrightarrow L_p(B^j)) \\ &\le c\, j^{-\sigma}k^{-\frac{m}{n}} \end{aligned}$$

and thus $\nu_0(\varepsilon, B^j) \le c\, j^{-\frac{n}{m}\sigma}\varepsilon^{-\frac{n}{m}}$. The constants are independent of j. Thus it holds

$$\begin{aligned} \nu_0(\varepsilon, B\setminus B_J) &\le \sum_{j=1}^{J} \nu_0(\varepsilon, B^j) \\ &\le c\,\varepsilon^{-\frac{n}{m}} \sum_{j=1}^{J} j^{-\frac{n}{m}\sigma} \sim \varepsilon^{-\frac{n}{m}} \begin{cases} 1 & \text{if } \sigma > \frac{m}{n} \\ \log J & \text{if } \sigma = \frac{m}{n} \\ J^{1-\frac{n}{m}\sigma} & \text{if } 0<\sigma<\frac{m}{n}. \end{cases} \end{aligned}$$

Using $J \sim \varepsilon^{-\frac{1}{\sigma}}$ and (4.20) the estimates from above (4.19) are proved.
Step 2. We prove the estimates from below in (4.19). Because of the inequalities (4.20) we shall therefore construct suitable finite-dimensional subspaces of $E^m_{p,\sigma}(B\setminus B_J)$ and estimate $\mu_0(\varepsilon, B\setminus B_J)$ from below. We use basis functions similar to those in [HT94], see also [ET96, Section 4.3.2, p.170-173, Step 1-2]. Let $f \in S'(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq [-1,1]^n$. Put

$$\operatorname{span}^l_j := \operatorname{span}\{f(2^l\cdot -k) : k \in \mathbb{Z}^n, 2^{-l}k \in B^j\} \qquad j,l \in \mathbb{N}, l \ge j. \quad (4.21)$$

The number of admitted lattice points $k \in \mathbb{Z}^n$ such that $2^{-l}k \in B^j$ is $2^{n(l-j)}$ (neglecting constants). Furthermore, we may assume that the functions $f(2^l\cdot -k)$ have disjoint supports and so

$$\dim \operatorname{span}^l_j \sim 2^{n(l-j)}.$$

For every $g \in \mathrm{span}_j^l$, say

$$g(x) = \sum\nolimits^{l,j} d_k f(2^l x - k), \qquad d_k \in \mathbb{C},\, j,l \in \mathbb{N}, l \geq j,$$

where the sum $\sum^{l,j}$ is taken over all lattice points $k \in \mathbb{Z}^n$ such that $2^{-l}k \in B^j$, it holds

$$\begin{aligned} \|g\,|L_p(B)\|^p &\sim \sum\nolimits^{l,j} |d_k|^p \, \|f(2^l \cdot -k)|L_p(B)\|^p \\ &\sim 2^{-ln} \sum\nolimits^{l,j} |d_k|^p. \end{aligned} \tag{4.22}$$

Then $\operatorname{supp} f(2^l \cdot -k) \subset B^j$ leads to

$$\begin{aligned} \|g\,|E^m_{p,\sigma}(B)\|^p &\sim \sum\nolimits^{l,j} |d_k|^p \, \|f(2^l \cdot -k)|E^m_{p,\sigma}(B)\|^p \\ &\sim j^{\sigma p}\, 2^{m(l-j)p}\, 2^{-ln} \sum\nolimits^{l,j} |d_k|^p. \end{aligned} \tag{4.23}$$

In particular, for all $g \in \mathrm{span}_j^l$

$$\|g\,|E^m_{p,\sigma}(B)\| \;\sim\; j^{\sigma} 2^{m(l-j)} \;\|g\,|L_p(B)\|. \tag{4.24}$$

Note that one can replace B by B^j in (4.24) due to the construction of span_j^l. We will deal with three different subspaces to obtain the three estimates in (4.19). First let $L \in \mathbb{N}$ such that $L \sim -\frac{1}{m}\log\varepsilon$ and put

$$S_1 := \mathrm{span}_1^L .$$

Then for every $g \in S_1$ we have due to (4.24)

$$\|g\,|E^m_{p,\sigma}(B)\| \;\sim\; 2^{mL} \;\|g\,|L_p(B)\|.$$

Hence $\alpha(S_1) \leq 2^{mL} \sim \frac{1}{\varepsilon}$. This ensures

$$\mu_0(\varepsilon, B \setminus B_J) \geq \dim S_1 \sim 2^{nL} \sim \varepsilon^{-\frac{n}{m}}.$$

The second subspace is defined by

$$S_2 := \bigoplus_{j=1}^{J} \mathrm{span}_j^j .$$

Then for every $g \in S_2$, say $g = \sum\limits_{j=1}^{J} g_j$ with $g_j \in \mathrm{span}_j^j$, we get with (4.24)

$$\|g\,|E^m_{p,\sigma}(B)\|^p \;\sim\; \sum_{j=1}^{J} \|g_j\,|E^m_{p,\sigma}(B^j)\|^p \sim \; \sum_{j=1}^{J} j^{\sigma p} \|g_j\,|L_p(B^j)\|^p.$$

Large-scale estimating of the term j^σ by J^σ gives $\alpha(S_2) \le J^\sigma \sim \frac{1}{\varepsilon}$. Consequently we obtain

$$\mu_0(\varepsilon, B \setminus B_J) \ge \dim S_2 \sim J \sim \varepsilon^{-\frac{1}{\sigma}}.$$

Up to now we have shown

$$\mu_0(\varepsilon, B \setminus B_J) \ge c\,\varepsilon^{-\max\{\frac{1}{\sigma},\frac{n}{m}\}}.$$

In the limiting case $\sigma = \frac{m}{n}$ we can refine the decomposition of $B \setminus B_J$ to obtain the log-factor in (4.19). Define the subspace

$$S_3 := \bigoplus_{j=1}^{J} \operatorname{span}_j^{l_j}$$

where $l_j \sim j + \frac{1}{n}(\log J - \log j)$. Then for every $g \in S_3$, say $g = \sum\limits_{j=1}^{J} g_j$ with $g_j \in \operatorname{span}_j^{l_j}$, it holds because of (4.24)

$$\begin{aligned} \|g|E^m_{p,\sigma}(B)\|^p \sim\ & \sum_{j=1}^{J} \|g_j|E^m_{p,\sigma}(B^j)\|^p \\ \sim\ & \sum_{j=1}^{J} j^{\sigma p}\, 2^{m(l_j-j)p}\, \|g_j|L_p(B^j)\|^p \\ \sim\ & J^{\frac{m}{n}p} \sum_{j=1}^{J} j^{(\sigma-\frac{m}{n})p}\, \|g_j|L_p(B^j)\|^p. \end{aligned}$$

If $\sigma = \frac{m}{n}$ then $j^{(\sigma-\frac{m}{n})p} = 1$ and so $\alpha(S_3) \le J^{\frac{m}{n}} \sim \frac{1}{\varepsilon}$. Thus we conclude for $\sigma = \frac{m}{n}$ that

$$\mu_0(\varepsilon, B \setminus B_J) \ge \dim S_3 \sim \sum_{j=1}^{J} 2^{n(l_j-j)} \sim J \sum_{j=1}^{J} \frac{1}{j} \sim J \log J \sim \varepsilon^{-\frac{n}{m}} |\log \varepsilon|.$$

Now the estimates from below in (4.19) follow from (4.20). □

We transfer the just gained asymptotic behaviour of $\nu_0(\varepsilon, B)$ as $\varepsilon \to 0$ from the last proposition to the corresponding approximation numbers.

Theorem 4.10. *Let $m \in \mathbb{N}, 1 \le p < \infty$ and $\sigma > 0$. Then the embedding*

$$\mathrm{id} : E^m_{p,\sigma}(B) \hookrightarrow L_p(B)$$

is compact. Furthermore, it holds for $k \in \mathbb{N}, k \geq 2$,

$$a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}} & \text{if } \sigma > \frac{m}{n} \\ k^{-\frac{m}{n}} (\log k)^{\frac{m}{n}} & \text{if } \sigma = \frac{m}{n} \\ k^{-\sigma} & \text{if } 0 < \sigma < \frac{m}{n}. \end{cases} \tag{4.25}$$

Proof. From $\nu_0(\varepsilon, B) \sim \varepsilon^{-\kappa} |\log \varepsilon|^{\rho}$, $\kappa > 0, \rho \in \mathbb{R}$, it follows $\nu_0(a_k(\mathrm{id}), B) \sim k$, see Remark 4.5. Hence

$$k \sim a_k(\mathrm{id})^{-\kappa} |\log a_k(\mathrm{id})|^{\rho}. \tag{4.26}$$

Then one has in particular

$$\log k \sim |\log a_k(\mathrm{id})| \left[\kappa + \rho \frac{\log|\log a_k(\mathrm{id})|}{|\log a_k(\mathrm{id})|}\right] \sim |\log a_k(\mathrm{id})|.$$

Inserting this in (4.26) one obtains for $k \geq 2$

$$a_k(\mathrm{id}) \sim k^{-\frac{1}{\kappa}} (\log k)^{\frac{\rho}{\kappa}}.$$

Now (4.25) follows from (4.19) with the suitable choice of $\varkappa$ and ρ. □

4.2 Generalisations

We extend the content of the last section to a wider class of weights. We replace the logarithmic term in (4.2) by a slowly varying function and establish corresponding bracketing results. Recall Definition 2.5 of a slowly varying function $\psi : (0,1] \to (0,\infty)$ and Definition 3.2 of the space $E^m_{p,\psi}(B), m \in \mathbb{N}, 1 \leq p < \infty$. Then, as shown in Theorem 3.4, the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact if, and only if, $\lim_{t \to 0^+} \psi(t) = \infty$. Let

$$\mathrm{id}_\Omega : E^m_{p,\psi}(\Omega) \hookrightarrow L_p(\Omega)$$

be the restriction to a subset $\Omega \subseteq B$. One can define values $\nu_0(\varepsilon, \Omega)$ and $\mu_0(\varepsilon, \Omega)$ in the same way as in Definition 4.3. Then an equivalent argumentation to that in Section 4.1 can be applied to the space $E^m_{p,\psi}(\Omega)$ instead of $E^m_{p,\sigma}(\Omega)$. We summarise the arising bracketing technique in the following proposition.

Proposition 4.11. *Let* $1 \leq p < \infty$, $m \in \mathbb{N}$ *and* ψ *be a continuous, slowly varying function on* $(0,1]$ *with* $\psi(1) = 1$, *bounded from below by a positive constant and* $\lim_{t \to 0^+} \psi(t) = \infty$. *Then the embedding*

$$\mathrm{id}_\Omega : E^m_{p,\psi}(\Omega) \hookrightarrow L_p(\Omega),$$

$\Omega \subseteq B$, is compact. For $\varepsilon > 0$ let

$$\nu_0(\varepsilon, \Omega) := \max\{k \in \mathbb{N} : a_k(\mathrm{id}_\Omega) \geq \varepsilon\}$$
$$\mu_0(\varepsilon, \Omega) := \max\Big\{ \dim S : \alpha(S) = \sup_{u \in S\setminus\{0\}} \frac{\|u|E^m_{p,\psi}(\Omega)\|}{\|u|L_p(\Omega)\|} \leq \frac{1}{\varepsilon} \Big\}$$

where the last maximum is taken over all finite-dimensional linear subspaces S of $E^m_{p,\psi}(\Omega)$. Put $\nu_0(\varepsilon, \Omega) = 0$ if $a_k(\mathrm{id}_\Omega) < \varepsilon$ for all $k \in \mathbb{N}$.

(i) *If $\Omega = \big(\bigcup_{j=1}^{J} \overline{\Omega}_j\big)^\circ$ with disjoint domains Ω_j then it holds*

$$\sum_{j=1}^{J} \mu_0(\varepsilon, \Omega_j) \leq \mu_0(\varepsilon, \Omega) \leq \nu_0(\varepsilon, \Omega) \leq \sum_{j=1}^{J} \nu_0(\varepsilon, \Omega_j). \tag{4.27}$$

(ii) *Let A^m_ψ be the positive definite self-adjoint operator in $L_2(B)$ generated by the closed quadratic form*

$$\int_B b_{m,\psi}(x) \sum_{|\alpha|=m} \mathrm{D}^\alpha f(x)\overline{\mathrm{D}^\alpha g(x)}\mathrm{d}x, \qquad f, g \in E^m_{2,\psi}(B) \tag{4.28}$$

where $b_{m,\psi}(x) := |x|^{2m}\,\psi(|x|)^2, x \in B$. Then A^m_ψ has pure point spectrum and

$$\mathrm{dom}((A^m_\psi)^{1/2}) = E^m_{2,\psi}(B)$$
$$A^m_\psi f = (-1)^m \sum_{|\alpha|=m} \mathrm{D}^\alpha(b_{m,\psi}\mathrm{D}^\alpha f), \qquad f \in C^\infty_0(\dot{B}).$$

Furthermore, it holds for $p = 2$

$$\nu_0(\varepsilon, B) = \mu_0(\varepsilon, B) = N(\varepsilon^{-2}, A^m_\psi) \tag{4.29}$$

where $N(\lambda, A^m_\psi)$ denotes the number of eigenvalues of A^m_ψ smaller than or equal to $\lambda > 0$.

(iii) *Let $j \in \mathbb{N}$. Then it holds*

$$\nu_0(\varepsilon, B^j) \sim \psi(2^{-j})^{-\frac{n}{m}}\,\varepsilon^{-\frac{n}{m}} \tag{4.30}$$

where $B^j = \{x \in B : 2^{-j} \leq |x| < 2^{-j+1}\}$.

(iv) *Let $\varepsilon > 0$ and choose $J \in \mathbb{N}$ such that $\varepsilon \sim \psi(2^{-J})^{-1}$. If ψ is locally decreasing at zero, then for sufficiently small $\varepsilon > 0$ it holds*

$$\nu_0(\varepsilon, B_J) = \mu_0(\varepsilon, B_J) = 0 \tag{4.31}$$

where $B_J = \{x \in \mathbb{R}^n : |x| < 2^{-J}\}$.

Proof. One can transfer the line of arguments from Proposition 4.6 to the space $E^m_{p,\psi}(\Omega)$ and obtains (4.27). We have seen in Section 3.3 that the operator A^m_ψ is positive definite, self-adjoint and has pure point spectrum $(\lambda_k(A^m_\psi))_{k\in\mathbb{N}}$ monotonically ordered (including geometric multiplicities) according to

$$\lambda_1(A^m_\psi) \le \lambda_2(A^m_\psi) \le \ldots \le \lambda_k(A^m_\psi) \stackrel{k\to\infty}{\longrightarrow} \infty.$$

Then it holds

$$a_k(\mathrm{id} : E^m_{2,\psi}(B) \hookrightarrow L_2(B)) \sim \lambda_k(A^m_\psi)^{-\frac{1}{2}}.$$

By analogy with Proposition 4.7, one gets (4.29). We prove (4.30). Recall that

$$a_k(\overset{\circ}{W}{}^m_p(B^j) \hookrightarrow L_p(B^j)) \sim 2^{-jm} k^{-\frac{m}{n}}.$$

Hence for $\mathrm{id}_{B^j} : E^m_{p,\psi}(B^j) \hookrightarrow L_p(B^j)$ we find with (3.55) that

$$\begin{aligned} a_k(\mathrm{id}_{B^j}) &\le \|E^m_{p,\psi}(B^j) \hookrightarrow \overset{\circ}{W}{}^m_p(B^j)\| \; 2^{-jm}\, k^{-\frac{m}{n}} \\ &\le c\; 2^{jm}\psi(2^{-j})^{-1} \cdot 2^{-jm}\, k^{-\frac{m}{n}} \\ &\le c\; \psi(2^{-j})^{-1} k^{-\frac{m}{n}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} 2^{-jm}\, k^{-\frac{m}{n}} &\sim a_k(\overset{\circ}{W}{}^m_p(B^j) \hookrightarrow L_p(B^j)) \\ &\le \|\overset{\circ}{W}{}^m_p(B^j) \hookrightarrow E^m_{p,\psi}(B^j)\| \; a_k(\mathrm{id}_{B^j}) \\ &\le c\; 2^{-jm}\psi(2^{-j})\, a_k(\mathrm{id}_{B^j}). \end{aligned}$$

Now (4.30) follows from $a_k(\mathrm{id}_{B^j}) \sim \psi(2^{-j})^{-1} k^{-\frac{m}{n}}$. It remains to show (4.31). Let $0 < \delta < 1$ be the constant from (2.35). Assume $2^{-J} \le \delta$. Due to the monotonicity of ψ near zero one has for $f \in C^m_0(B_J)$

$$\begin{aligned} \|f|L_p(B_J)\|^p &\le \psi(2^{-J})^{-p} \int_{B_J} \psi(|x|)^p |f(x)|^p \mathrm{d}x \\ &\le c\; \psi(2^{-J})^{-p}\, \|f|E^m_{p,\psi}(B_J)\|^p. \end{aligned}$$

Hence we may assume

$$\|\,\mathrm{id}_J : E^m_{p,\psi}(B_J) \hookrightarrow L_p(B_J)\| < \varepsilon$$

and consequently $a_k(\mathrm{id}) < \varepsilon$ for all $k \in \mathbb{N}$. □

We state the resulting asymptotic behaviour of $\nu_0(\varepsilon, B)$ in the next proposition.

Proposition 4.12. *Let $1 \le p < \infty, m \in \mathbb{N}$ and ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1) = 1$, bounded from below by a positive constant and locally decreasing at zero with $\lim_{t\to 0^+} \psi(t) = \infty$. Let Φ be a function equivalent to a primitive of $\psi(2^{-\cdot})^{-\frac{m}{n}}$. Then for $\varepsilon > 0$ it holds*

$$\nu_0(\varepsilon, B) \sim \varepsilon^{-\frac{n}{m}} \Phi(J) \tag{4.32}$$

where $J = J(\varepsilon) \in \mathbb{N}$ is such that $\varepsilon \sim \psi(2^{-J})^{-1}$.

Proof. We generalise the proof of Proposition 4.9 for slowly varying functions. Let $B^j := \{x \in B : \; 2^{-j} \le |x| < 2^{-j+1}\}$, $j = 1, ..., J$ and $B_J := \{x \in B : \; |x| < 2^{-J}\}$. Consider the disjoint partition of the unit ball

$$B = B_J \;\cup\; (B \setminus B_J).$$

Then with (4.27) and (4.31)

$$\mu_0(\varepsilon, B \setminus B_J) \le \nu_0(\varepsilon, B) \le \nu_0(\varepsilon, B \setminus B_J). \tag{4.33}$$

We estimate the right-hand side of (4.33) from above with the help of (4.27) and (4.30)

$$\nu_0(\varepsilon, B \setminus B_J) \le \sum_{j=1}^{J} \nu_0(\varepsilon, B^j) \le \; c\, \varepsilon^{-\frac{n}{m}} \sum_{j=1}^{J} \psi(2^{-j})^{-\frac{n}{m}} \;\sim\; \varepsilon^{-\frac{n}{m}} \Phi(J).$$

As before, let $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq [-1,1]^n$ and put

$$\operatorname{span}_j^l := \operatorname{span}\{f(2^l \cdot -k) : k \in \mathbb{Z}^n, 2^{-l}k \in B^j\} \qquad j, l \in \mathbb{N}, l \ge j. \tag{4.34}$$

We may assume disjoint supports of the functions $f(2^l \cdot -k)$ and thus

$$\dim \operatorname{span}_j^l \sim 2^{n(l-j)} =: N_{l-j}.$$

For every $g \in \operatorname{span}_j^l$, say $g(x) = \sum^{l,j} d_k\, f(2^l x - k)$, $d_k \in \mathbb{C}$, $j, l \in \mathbb{N}$, $l \ge j$ where the sum $\sum^{l,j}$ is taken over all $k \in \mathbb{N}$ such that $2^{-l}k \in B^j$, it holds

$$\begin{aligned}
\|g|E^m_{p,\psi}(B)\|^p &\sim \sum\nolimits^{l,j} |d_k|^p \; \|f(2^l \cdot -k)|E^m_{p,\psi}(B^j)\|^p \\
&\sim \sum\nolimits^{l,j} |d_k|^p \; 2^{-jmp}\psi(2^{-j})^p \; \|f(2^l \cdot -k)|\mathring{W}^m_p(B^j)\|^p \\
&\sim \sum\nolimits^{l,j} |d_k|^p \; 2^{-jmp}\psi(2^{-j})^p \; 2^{lmp}2^{-ln} \\
&\sim 2^{m(l-j)p}\psi(2^{-j})^p \sum\nolimits^{l,j} 2^{-ln}|d_k|^p
\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}
\|g|L_p(B)\|^p &\sim \sum\nolimits^{l,j} |d_k|^p \; \|f(2^l \cdot -k)|L_p(B^j)\|^p \\
&\sim \sum\nolimits^{l,j} 2^{-ln}|d_k|^p.
\end{aligned} \tag{4.36}$$

Hence for all $g \in \mathrm{span}_j^l$ we have

$$\|g|E^m_{p,\psi}(B)\| \sim 2^{m(l-j)}\psi(2^{-j})\ \|g|L_p(B)\|. \tag{4.37}$$

Note that one can replace B by B^j in (4.37). Consider the subspace

$$S := \bigoplus_{j=1}^{J} \mathrm{span}_j^{l_j} \tag{4.38}$$

where we choose the number $l_j \in \mathbb{N}$ such that

$$l_j \sim j + \frac{1}{m}\log\Big(\frac{\psi(2^{-J})}{\psi(2^{-j})}\Big). \tag{4.39}$$

This assumption is natural since ψ is decreasing near zero, i.e. we can assume $\psi(2^{-J}) \geq \psi(2^{-j}), j = 1, ..., J$. Note that (4.39) is equivalent to

$$n(l_j - j) \sim \frac{n}{m}\big(\log\psi(2^{-J}) - \log\psi(2^{-j})\big)$$

and

$$2^{n(l_j-j)} \sim \psi(2^{-J})^{\frac{n}{m}}\,\psi(2^{-j})^{-\frac{n}{m}}. \tag{4.40}$$

Then one has

$$\begin{aligned} \dim S &\sim \sum_{j=1}^{J} \dim \mathrm{span}_j^{l_j} \\ &\sim \sum_{j=1}^{J} 2^{n(l_j-j)} \\ &\sim \psi(2^{-J})^{\frac{n}{m}} \sum_{j=1}^{J} \psi(2^{-j})^{-\frac{n}{m}} \\ &\sim \varepsilon^{-\frac{n}{m}}\Phi(J). \end{aligned} \tag{4.41}$$

Furthermore, for every $h \in S$, say $h = \sum_{j=1}^{J} g_j$ with $g_j \in \mathrm{span}_j^{l_j}$, one obtains with (4.37)

$$\begin{aligned}\|h|E^m_{p,\psi}(B\setminus B_J)\|^p &\sim \sum_{j=1}^{J}\|g_j|E^m_{p,\psi}(B^j)\|^p\\ &\sim \sum_{j=1}^{J}2^{m(l_j-j)p}\psi(2^{-j})^p\ \|g_j|L_p(B^j)\|^p\\ &\sim \sum_{j=1}^{J}\psi(2^{-J})^p\,\psi(2^{-j})^{-p}\ \psi(2^{-j})^p\ \|g_j|L_p(B^j)\|^p\\ &\sim \psi(2^{-J})^p\,\|h|L_p(B\setminus B_J)\|^p.\end{aligned}$$

This shows $\alpha(S)\sim\psi(2^{-J})\sim\frac{1}{\varepsilon}$. Now we can estimate the left-hand side of (4.33) from below by

$$\mu_0(\varepsilon,B\setminus B_J)\geq \dim S\sim\varepsilon^{-\frac{n}{m}}\Phi(J).$$

□

We transfer the asymptotic behaviour from (4.32) to the corresponding approximation numbers in the next theorem.

Theorem 4.13. *Let $1\leq p<\infty, m\in\mathbb{N}$ and ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1)=1$, bounded from below by a positive constant and locally decreasing at zero with $\lim\limits_{t\to0^+}\psi(t)=\infty$. Then the embedding*

$$id: E^m_{p,\psi}(B)\hookrightarrow L_p(B)$$

is compact. Furthermore, let ψ^{-1} be a positive function such that, with some $t_0\in(0,1]$, for all $t\leq t_0$

$$\psi(t)\sim s\iff\psi^{-1}(s)\sim t.$$

Let Φ be a continuous function equivalent to a primitive of $[\psi(2^{-\cdot})]^{-\frac{n}{m}}$ and let the function h be defined by

$$h(\varepsilon):=\varepsilon^{-\frac{n}{m}}\Phi\big(-\log(\psi^{-1}(\varepsilon^{-1}))\big),\quad \varepsilon>0.$$

Let H be a function such that for all $k\in\mathbb{N},\varepsilon>0$

$$h(\varepsilon)\sim k\iff\varepsilon\sim k^{-\frac{m}{n}}H(k).$$

All preceding equivalence constants are independent of the variables.
Then it holds for $k\in\mathbb{N}$

$$a_k(\mathrm{id})\sim k^{-\frac{m}{n}}\,H(k).\tag{4.42}$$

In particular, if $([\psi(2^{-j})]^{-1})_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}}$ *then*

$$a_k(\mathrm{id}) \sim k^{-\frac{m}{n}}. \tag{4.43}$$

Proof. Let $J \in \mathbb{N}$ be such that

$$\psi(2^{-J}) \sim \tfrac{1}{\varepsilon} \iff J \sim -\log(\psi^{-1}(\varepsilon^{-1})). \tag{4.44}$$

It follows

$$\Phi(J) \sim \Phi(-\log(\psi^{-1}(\varepsilon^{-1}))) \tag{4.45}$$

where the equivalence constants of (4.45) depend on those of (4.44). This can be seen as follows

$$\begin{aligned}
\Phi(J) &\sim \int_{j_0}^{J} \psi(2^{-t})^{-\frac{n}{m}} \mathrm{d}t \\
&\le \int_{j_0}^{c[-\log(\psi^{-1}(\varepsilon^{-1}))]} \psi(2^{-t})^{-\frac{n}{m}} \mathrm{d}t = \; c \int_{j_0/c}^{-\log(\psi^{-1}(\varepsilon^{-1}))} \psi(2^{-cs})^{-\frac{n}{m}} \mathrm{d}s \\
&\le c \int_{j_0/c}^{-\log(\psi^{-1}(\varepsilon^{-1}))} \psi(2^{-s})^{-\frac{n}{m}} \mathrm{d}s \\
&\sim \; \Phi(-\log(\psi^{-1}(\varepsilon^{-1})))
\end{aligned}$$

where the last inequality follows from the monotonicity of ψ (assuming $c > 1$) and j_0 is sufficiently large. The converse estimate is similar. We insert (4.45) in (4.32)

$$\nu_0(\varepsilon, B) \sim \varepsilon^{-\frac{n}{m}} \Phi(-\log(\psi^{-1}(\varepsilon^{-1}))) = h(\varepsilon).$$

Now it follows from $\nu_0(a_k(\mathrm{id}), B) \sim k$, see (4.9), that

$$k \sim h(a_k(\mathrm{id})) \iff a_k(\mathrm{id}) \sim k^{-\frac{m}{n}} H(k).$$

□

5 Entropy numbers

So far we gained in Section 4 the exact behaviour of the approximation numbers of the compact embedding $\mathrm{id} : E^m_{p,\sigma}(B) \hookrightarrow L_p(B)$ in Theorem 4.10 or rather $\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$ in Theorem 4.13. We turn to the corresponding entropy numbers. We will find out that upper bounds for entropy numbers follow from Carl's inequality [Ca81b] or immediately from [ET96, Section 1.3.3]. On the other hand, similar constructions to those used in Section 4 lead to estimates from below for $e_k(\mathrm{id})$.
We proceed as before. First we concentrate on the logarithmic case in Theorem 5.1 and afterwards we have a look at the general setting of slowly varying functions in Theorem 5.2.

Theorem 5.1. *Let $1 \le p < \infty$, $m \in \mathbb{N}$ and $\sigma > 0$. Then the embedding*

$$\mathrm{id} : E^m_{p,\sigma}(B) \hookrightarrow L_p(B)$$

is compact and

$$e_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}} & \text{if } \sigma > \frac{m}{n} \\ k^{-\frac{m}{n}} (\log k)^{\frac{m}{n}} & \text{if } \sigma = \frac{m}{n} \\ k^{-\sigma} & \text{if } 0 < \sigma < \frac{m}{n}. \end{cases} \tag{5.1}$$

Proof. *Step 1.* We prove the upper bounds of (5.1). Recall (3.29), that for every $0 < \nu < \infty$ and $L \in \mathbb{N}$

$$\sup_{k=1,\dots,L} k^\nu e_k(\mathrm{id}) \le c \sup_{k=1,\dots,L} k^\nu a_k(\mathrm{id}).$$

If $0 < \sigma < \frac{m}{n}$ we put $\nu := \sigma$. Using (4.25)

$$\sup_{k=1,\dots,L} k^\sigma e_k(\mathrm{id}) \le c$$

and hence for all $k \in \mathbb{N}$

$$e_k(\mathrm{id}) \le c\, k^{-\sigma}.$$

If $\sigma > \frac{m}{n}$ we put $\nu := \frac{m}{n}$ and proceed similarly. If $\sigma = \frac{m}{n} = \nu$ we get for $L \in \mathbb{N}$

$$\begin{aligned} L^{\frac{m}{n}} e_L(\mathrm{id}) &\leq \sup_{k=1,\dots,L} k^{\frac{m}{n}} e_k(\mathrm{id}) \\ &\leq c \sup_{k=1,\dots,L} k^{\frac{m}{n}} a_k(\mathrm{id}) \\ &\leq c \sup_{k=1,\dots,L} (\log k)^{\frac{m}{n}} \\ &\leq c\, (\log L)^{\frac{m}{n}}. \end{aligned}$$

Step 2. By decomposition of id and (3.32), we clearly get due to the multiplicativity of entropy numbers the classical lower estimate

$$k^{-\frac{m}{n}} \sim e_k(\overset{\circ}{W}{}^m_p(B) \hookrightarrow L_p(B)) \leq c\, e_k(\mathrm{id}).$$

We claim

$$e_k(\mathrm{id}) \geq c\ k^{-\sigma}. \tag{5.2}$$

We adapt arguments from [HT94] or [ET96, Theorem 4.3.2, Step 1]. We use the same basis functions as in (4.21). Let

$$f_j^l(x) := \sum\nolimits^{l,j} d_k f(2^l x - k), \qquad d_k \in \mathbb{C},\ j, l \in \mathbb{N}, l \geq j, \tag{5.3}$$

where $f \in \mathcal{S}'(\mathbb{R}^n)$ is such that $\operatorname{supp} f \subset [-1,1]^n$. The sum $\sum^{l,j}$ is taken over all lattice points $k \in \mathbb{Z}^n$ such that $2^{-l}k \in B^j$. The number of summands is $N_{l-j} := 2^{n(l-j)}$ (neglecting constants). As before we assume that the functions $f(2^l \cdot - k)$ have disjoint supports. In view of (4.22) and (4.23) we have

$$\|f_j^l\,|L_p(B)\| \sim 2^{-l\frac{n}{p}} \big(\sum\nolimits^{l,j} |d_k|^p\big)^{\frac{1}{p}} \tag{5.4}$$

and

$$\|f_j^l\,|E^m_{p,\sigma}(B)\| \sim j^{\sigma} 2^{m(l-j)} 2^{-l\frac{n}{p}} \big(\sum\nolimits^{l,j} |d_k|^p\big)^{\frac{1}{p}}. \tag{5.5}$$

Due to the definition of entropy numbers, there exist $2^{N_{l-j}}$ balls K^i, $i = 1, ..., 2^{N_{l-j}}$ in $L_p(B)$ with radius $\tilde{\varepsilon} = 2e_{N_{l-j}}(\mathrm{id})$ which cover the unit ball U of $E^m_{p,\sigma}(B)$. For any one of these balls $K = K^i$ it holds

$$\mathrm{vol}(K \cap \mathrm{span}_j^l) \leq c\big[2e_{N_{l-j}}(\mathrm{id}) 2^{l\frac{n}{p}}\big]^{N_{l-j}} \mathrm{vol}(U_p^{N_{l-j}})$$

where U_p^N is the unit ball in ℓ_p^N. We obtain

$$\begin{aligned} \mathrm{vol}(U \cap \mathrm{span}_j^l) &\leq \sum_{i=1}^{2^{N_{l-j}}} \mathrm{vol}(K^i \cap \mathrm{span}_j^l) \\ &\leq c\, 2^{N_{l-j}} \big[2\, e_{N_{l-j}}(\mathrm{id})\, 2^{l\frac{n}{p}}\big]^{N_{l-j}} \mathrm{vol}(U_p^{N_{l-j}}). \end{aligned} \tag{5.6}$$

The left-hand side is equivalent to $[j^{-\sigma}2^{-m(l-j)}2^{l\frac{n}{p}}]^{N_{l-j}}\,\mathrm{vol}(U_p^{N_{l-j}})$ and hence

$$j^{-\sigma}2^{-m(l-j)} \le c\, e_{N_{l-j}}(\mathrm{id}). \tag{5.7}$$

If $\sigma = \frac{m}{n}$ we choose $j=1$. Then

$$2^{-ml} \le c\, e_{N_l}(\mathrm{id}).$$

Otherwise if $\sigma \neq \frac{m}{n}$, let l and j be such that

$$\begin{aligned} l \sim j + \tfrac{\sigma}{n\sigma - m}\log j \iff & (n - \tfrac{m}{\sigma})(l-j) \sim \log j \\ \iff & 2^{n(l-j)}2^{-\frac{m}{\sigma}(l-j)} \sim j \\ \iff & 2^{n(l-j)} \sim j\, 2^{\frac{m}{\sigma}(l-j)} \\ \iff & [N_{l-j}]^{-\sigma} \sim j^{-\sigma}\, 2^{-m(l-j)}. \end{aligned}$$

Then (5.7) leads to

$$[N_{l-j}]^{-\sigma} \sim j^{-\sigma}\, 2^{-m(l-j)} \le c\, e_{N_{l-j}}(\mathrm{id}).$$

and (5.2) is verified.

Step 3. We prove the limiting case $\sigma = \frac{m}{n}$ of (5.1). We fix $J \in \mathbb{N}$ and construct in each annulus $B^j, j = 1, ..., J$ functions of type (5.3) such that the size of the lattice depends on j. Namely, consider

$$f^J(x) := \sum_{j=1}^{J} f_j^{l_j}(x), \qquad f_j^{l_j} \in \mathrm{span}_j^{l_j},$$

where $f_j^{l_j}(x) = \sum^{l_j,j} d_k^j\, f(2^{l_j}x - k), d_k^j \in \mathbb{C}$. Choose $l_j \in \mathbb{N}$ such that

$$l_j \sim j + \tfrac{1}{n}(\log J - \log j).$$

Denote

$$\mathrm{span}^J := \mathrm{span}\,\big\{f(2^{-l_j}x -\, k) : k \in \mathbb{Z}^n, 2^{-l_j}k \in B^j,\, j = 1, ..., J\big\}.$$

Then

$$\dim \mathrm{span}^J \sim \sum_{j=1}^{J} 2^{n(l_j - j)} \sim\, J \sum_{j=1}^{J} \frac{1}{j} \sim\, J\log J.$$

We have the following counterparts of (5.4) and (5.5) with modified coefficients $b_k^j = 2^{-l_j\frac{n}{p}}d_k^j$

$$\begin{aligned} \|f^J|L_p(B)\| \sim \sum_{j=1}^{J} \|f_j^{l_j}|L_p(B)\| &\sim \sum_{j=1}^{J} 2^{-l_j\frac{n}{p}}\big(\sum\nolimits^{l_j,j} |d_k^j|^p\big)^{\frac{1}{p}} \\ &\sim \big(\sum\nolimits^{*} |b_k^j|^p\big)^{\frac{1}{p}}. \end{aligned}$$

The sum $\sum^* = \sum\limits_{j=1}^{J} \sum^{l_j,j}$ is taken over $N^J \sim J \log J$ summands. If $\sigma = \frac{m}{n}$ it holds $j^\sigma 2^{m(l_j-j)} \sim J^{\frac{m}{n}}$. Then

$$\begin{aligned} \|f^J|E^m_{p,\sigma}(B)\| &\sim \sum_{j=1}^{J} \|f_j^{l_j}|E^m_{p,\sigma}(B)\| \sim \sum_{j=1}^{J} j^\sigma 2^{m(l_j-j)} 2^{-l_j \frac{n}{p}} \Big(\sum\nolimits^{l_j,j} |d_k^j|^p \Big)^{\frac{1}{p}} \\ &\sim J^{\frac{m}{n}} \Big(\sum\nolimits^{*} |b_k^j|^p \Big)^{\frac{1}{p}}. \end{aligned}$$

Now we are in the same situation as in Step 2. For 2^{N^J} balls K^i, $i = 1, ..., 2^{N^J}$, with radius $\tilde{\varepsilon} = 2e_{N^J}(\mathrm{id})$, which cover the unit ball U it holds as a counterpart of (5.6)

$$\mathrm{vol}(U \cap \mathrm{span}^J) \le \sum_{i=1}^{2^{N^J}} \mathrm{vol}(K^i \cap \mathrm{span}^J) \le c\, 2^{N^J} \big[2e_{N^J}(\mathrm{id})\big]^{N^J} \mathrm{vol}(U_p^{N^J}).$$

Similarly as in Step 2, the left-hand side is equivalent to $[J^{-\frac{m}{n}}]^{N^J} \mathrm{vol}(U_p^{N^J})$. Hence we showed that

$$J^{-\frac{m}{n}} \le c\, e_{N^J}(\mathrm{id}).$$

Finally we complete the proof with

$$N^J \sim J \log J \iff J^{-\frac{m}{n}} \sim (N^J)^{-\frac{m}{n}} (\log N^J)^{\frac{m}{n}}.$$

what shows

$$k^{-\frac{m}{n}} (\log k)^{\frac{m}{n}} \le c\, e_k(\mathrm{id}), \qquad k \in \mathbb{N}, k \ge 2.$$

□

We extend the last theorem to slowly varying functions.

Theorem 5.2. *Let $1 \le p < \infty, m \in \mathbb{N}$ and ψ be a continuous, slowly varying function on $(0,1]$ with $\psi(1) = 1$, bounded from below by a positive constant and locally decreasing at zero with $\lim\limits_{t \to 0^+} \psi(t) = \infty$. Then the embedding*

$$id : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact. Furthermore, let ψ^{-1} be a positive function such that, with some $t_0 \in (0,1]$, for all $t \le t_0$

$$\psi(t) \sim s \iff \psi^{-1}(s) \sim t.$$

Let Φ be a continuous function equivalent to a primitive of $[\psi(2^{-\cdot})]^{-\frac{n}{m}}$ and let the function h be defined by

$$h(\varepsilon) := \varepsilon^{-\frac{n}{m}} \Phi\Big(-\log(\psi^{-1}(\varepsilon^{-1}))\Big), \quad \varepsilon > 0.$$

Let H be a function such that for all $k \in \mathbb{N}, \varepsilon > 0$

$$h(\varepsilon) \sim k \iff k \sim \varepsilon^{-\frac{m}{n}} H(k).$$

All preceding equivalence constants are independent of the variables. Then it holds for $k \in \mathbb{N}$

$$e_k(\mathrm{id}) \sim k^{-\frac{m}{n}} H(k). \tag{5.8}$$

In particular, if $([\psi(2^{-j})]^{-1})_{j \in \mathbb{N}} \in \ell_{\frac{n}{m}}$ then

$$e_k(\mathrm{id}) \sim k^{-\frac{m}{n}}. \tag{5.9}$$

Proof. Since $H(ck) \sim H(k)$ for fixed $c > 0$ one has $a_{2^{j-1}}(\mathrm{id}) \sim a_{2^j}(\mathrm{id})$. Then it follows from [ET96, Section 1.3.3] and (4.42)

$$e_k(\mathrm{id}) \leq \ c\, a_k(\mathrm{id}) \sim k^{-\frac{m}{n}} H(k).$$

Note that $\lim\limits_{\varepsilon \to 0} h(\varepsilon) = \infty$, i.e. for every $k \in \mathbb{N}$ there exists a number $\varepsilon > 0$ such that $k \sim h(\varepsilon)$. Hence to prove the estimate from below in (5.8), it suffices to show

$$e_k(\mathrm{id}) \geq \varepsilon \qquad \text{if} \ \ k \sim h(\varepsilon). \tag{5.10}$$

Let $J \in \mathbb{N}$ with $\varepsilon \sim \psi(2^{-J})^{-1}$. Consider the subspace from (4.38)

$$S^J = \bigoplus_{j=1}^{J} \mathrm{span}_j^{l_j} \tag{5.11}$$

where span_j^l is defined by (4.34) and

$$l_j \sim j + \frac{1}{m} \log \Big(\frac{\psi(2^{-J})}{\psi(2^{-j})} \Big).$$

Let $g \in S^J$, say $g = \sum\limits_{j=1}^{J} g_j$ and $g_j(x) = \sum^{l_j,j} d_k^j f(2^l x - k)$, $d_j^k \in \mathbb{C}$. The sum $\sum^{l_j,j}$ is taken over all $k \in \mathbb{N}$ such that $2^{-l_j} k \in B^j$ and has $N_{l_j - j} = 2^{n(l_j - j)} \sim \psi(2^{-J})^{\frac{n}{m}} \psi(2^{-j})^{-\frac{n}{m}}$ summands, see (4.40). Then with (4.35)

$$\begin{aligned}
\|g|E^m_{p,\psi}(B)\|^p &\sim \sum_{j=1}^{J} \|g_j|E^m_{p,\psi}(B^j)\|^p \\
&\sim \sum_{j=1}^{J} 2^{m(l_j - j)} \psi(2^{-j})^p \sum\nolimits^{l_j,j} 2^{-l_j n} |d_k^j|^p \\
&\sim \psi(2^{-J})^p \sum\nolimits^{*} |b_k^j|^p
\end{aligned}$$

and with (4.36)

$$\begin{aligned}\|g|L_p(B)\|^p &\sim \sum_{j=1}^{J} \|g_j|L_p(B^j)\|^p \\ &\sim \sum_{j=1}^{J} \sum\nolimits^{l_j,j} 2^{-l_j n}|d_k^j|^p \\ &\sim \sum\nolimits^{*} |b_k^j|^p\end{aligned}$$

where the coefficients $b_k^j = 2^{-l_j \frac{n}{p}} d_k^j$ and the sum $\sum^* = \sum\limits_{j=1}^{J} \sum^{l_j,j}$ is taken over $N^J \sim \sum\limits_{j=1}^{J} 2^{n(l_j - j)} \sim \varepsilon^{-\frac{n}{m}} \Phi(J)$ summands, see (4.41). There exist 2^{N^J} balls K^i, $i = 1, ..., 2^{N^J}$ in $L_p(B)$ with radius $\tilde{\varepsilon} = 2e_{N^J}(\mathrm{id})$ which cover the unit ball U of $E^m_{p,\psi}(B)$. For any one of these balls it holds

$$\mathrm{vol}(K^i \cap S^J) \leq c\big[2e_{N^J}(\mathrm{id})\big]^{N^J} \mathrm{vol}(U_p^{N^J}).$$

U_p^N is the unit ball in ℓ_p^N. Now we can estimate

$$\begin{aligned}\mathrm{vol}(U \cap S^J) &\leq \sum_{i=1}^{2^{N^J}} \mathrm{vol}(K^i \cap S^J) \\ &\leq \sum_{i=1}^{2^{N^J}} \big[2e_{N^J}(\mathrm{id})\big]^{N^J} \mathrm{vol}(U_p^{N^J}) \\ &\leq \ c\,\big[2e_{N^J}(\mathrm{id})\big]^{N^J} \mathrm{vol}(U_p^{N^J})\, 2^{N^J}.\end{aligned}$$

The left-hand side is equivalent to $\big[\psi(2^{-J})^{-1}\big]^{N^J} \mathrm{vol}(U_p^{N^J})$. Hence

$$\varepsilon \sim \psi(2^{-J})^{-1} \leq \ c\, e_{N^J}(\mathrm{id})$$

where $N^J \sim \varepsilon^{-\frac{n}{m}} \Phi(J) \sim h(\varepsilon)$. This proves (5.10). □

Remark 5.3. We want to point out that Theorem 4.13 and Theorem 5.2 extend Proposition 3.8. To verify we check that if ψ satisfies (3.33) for some sequence $(c_j)_{j\in\mathbb{N}} \in \ell_1$ then the sequence $(\psi(2^{-j})^{-1})_j$ belongs to $\ell_{\frac{n}{m}}$. This can be proved by contradiction. Suppose the opposite $(\psi(2^{-j})^{-1})_{j\in\mathbb{N}} \notin \ell_{\frac{n}{m}}$ and let $(c_j)_{j\in\mathbb{N}} \in \ell_1$ be such that

$$\sum_{j\in\mathbb{N}} (\psi(2^{-j})c_j)^{-\frac{n}{m}} < \infty.$$

Since ${c_j}^{-\frac{n}{m}} \geq 1$ for $j \geq j_0$ we have

$$(\psi(2^{-j})c_j)^{-\frac{n}{m}} \geq \psi(2^{-j})^{-\frac{n}{m}}, \qquad j \geq j_0,$$

what leads to a contradiction by comparison of the series.

6 Concrete examples and growth rates

We have seen in Theorem 4.13 and Theorem 5.2 that the function $\psi(t)$ may influence the quality of the compactness of the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B), \qquad 1 \le p < \infty, m \in \mathbb{N}.$$

The outcome depends on the growth rate of $\psi(t)$. Indeed there is a function H depending on ψ and n, m such that

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim k^{-\frac{m}{n}} H(k). \tag{6.1}$$

If the function $\psi(t)$ tends to infinity near zero in the sense of the condition $([\psi(2^{-j})]^{-1})_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}}$ then the function $H(k)$ is equivalent to a constant. That means that if the growth rate of $\psi(t)$ is strong enough then the degree of compactness does not depend on $\psi(t)$ and one gains the same behaviour of $e_k(\mathrm{id})$ and $a_k(\mathrm{id})$ as in the classical setting (3.32). In the case where the growth rate of $\psi(t)$ is below a certain critical bound, the function $H(k)$ is built upon primitives of $\psi(2^{-\cdot})^{-\frac{n}{m}}$ and inverting operations. In the Examples 6.2, 6.4, 6.5 and 6.6 we illustrate this method in some concrete cases. Nevertheless, the application is sometimes very complex, although the result (6.1) is sharp. That is why we formulate some simpler assertions only using derivatives of ψ if the sequence $([\psi(2^{-j})]^{-1})_{j\in\mathbb{N}}$ does not belong to $\ell_{\frac{n}{m}}$. This is done in Corollary 6.7 where we achieve

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \psi(2^{-k})^{-1}.$$

Here the entropy and approximation numbers depend only on the function ψ and no longer on the parameters of the spaces. Finally Corollary 6.8 summarises these results in view of the growth rate of $\psi(t)$ compared to $|\log t|^{\frac{m}{n}}$. Roughly speaking we measure the growth rate of $\psi(t)$ in the number

$$c := \lim_{t\to 0} \frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})}.$$

Skipping the limiting case $c = 1$ we get

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{if } 1 < c \le \infty, \\ \psi(2^{-k})^{-1}, & \text{if } 0 \le c < 1. \end{cases}$$

In the first part of this Section, we apply the outcome described in (6.1) to iterated logarithm weights of the form

$$\psi(t) = (1 + |\log t|)^\sigma (1 + \log(1 + |\log t|))^\gamma, \qquad \sigma \geq 0, \gamma \in \mathbb{R}. \tag{6.2}$$

We start with analysing the asymptotic growth of some needed integrals.

Lemma 6.1. *Let $a, b \in \mathbb{R}$. Then for $T \geq T_0 > 1$*

$$\int_{T_0}^{T} \frac{\mathrm{d}t}{t^a (\log t)^b} \sim \begin{cases} T^{1-a}(\log T)^{-b}, & \text{if } a < 1, b \in \mathbb{R}, \\ \log(\log T), & \text{if } a = 1, b = 1, \\ (\log T)^{1-b}, & \text{if } a = 1, b < 1 \end{cases}$$

where the equivalence constants are independent of T.

Proof. The last two cases are obvious. Let $a < 1$ and $b > 0$. Then for any $\varepsilon > 0$ with $a + \varepsilon < 1$

$$\begin{aligned} \int_{T_0}^{T} t^{-a} (\log t)^{-b} \mathrm{d}t &\leq \sup_{T_0 \leq t \leq T} t^{\varepsilon} (\log t)^{-b} \int_{T_0}^{T} t^{-(a+\varepsilon)} \mathrm{d}t \\ &\leq c\, (\log T)^{-b}\, T^{1-a}. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_{T_0}^{T} t^{-a} (\log t)^{-b} \mathrm{d}t &\geq (\log T)^{-b} \int_{T_0}^{T} t^{-a} \mathrm{d}t \\ &\geq c\, (\log T)^{-b}\, T^{1-a}. \end{aligned}$$

Similar arguments apply for $b < 0$. □

Example 6.2. Let $\psi(t) = (1 + |\log t|)^\sigma$, $\sigma > 0$, $t \in (0, 1]$. This setting has been studied in Theorem 4.10 as well as in Theorem 5.1. It is also the object of investigation in [Tr12b]. Nevertheless, we incorporate it into our results in context with slowly varying functions due to Theorem 4.13 and Theorem 5.2. The sequence

$$\left(\psi(2^{-j})^{-1}\right)_{j \in \mathbb{N}} = \left((1 + j)^{-\sigma}\right)_{j \in \mathbb{N}}$$

belongs to $\ell_{\frac{n}{m}}$ if, and only if, $\sigma > \frac{m}{n}$. In that case (4.43) and (5.9) can be applied. For $T \geq T_0 > 1$ let the function Φ be such that

$$\Phi(T) \sim \int_{T_0}^{T} \psi(2^{-t})^{-\frac{n}{m}} \mathrm{d}t \sim \int_{T_0}^{T} (1 + t)^{-\sigma \frac{n}{m}} \mathrm{d}t \sim \begin{cases} \log T, & \text{if } \sigma = \frac{m}{n}, \\ T^{1 - \sigma \frac{n}{m}}, & \text{otherwise}. \end{cases}$$

We define

$$\psi^{-1}(s) := \exp(-s^{1/\sigma})$$

and put
$$h(\varepsilon) := \varepsilon^{-\frac{n}{m}}\Phi\big(-\log(\psi^{-1}(\varepsilon^{-1}))\big) = \varepsilon^{-\frac{n}{m}}\Phi\big(\varepsilon^{-\frac{1}{\sigma}}\big).$$
Hence if $\sigma = \frac{m}{n}$ we get

$$\begin{aligned} h(\varepsilon) \sim k &\iff k \sim \varepsilon^{-\frac{n}{m}} \log(\varepsilon^{-\frac{1}{\sigma}}) \\ &\iff k \sim \varepsilon^{-\frac{n}{m}} |\log \varepsilon|. \end{aligned}$$

In that case we have

$$\begin{aligned} \log k &\sim \log(\varepsilon^{-\frac{n}{m}}) + \log|\log \varepsilon| \\ &\sim |\log \varepsilon| \Big(1 + \frac{\log|\log \varepsilon|}{|\log \varepsilon|}\Big) \\ &\sim |\log \varepsilon| \end{aligned}$$

as $\varepsilon \to 0$. Therefore,

$$h(\varepsilon) \sim k \iff \varepsilon \sim k^{-\frac{m}{n}} (\log k)^{\frac{m}{n}}.$$

If $0 < \sigma < \frac{m}{n}$ we get

$$\begin{aligned} h(\varepsilon) \sim k &\iff k \sim \varepsilon^{-\frac{n}{m}} \varepsilon^{-\frac{1}{\sigma} + \frac{n}{m}} \\ &\iff \varepsilon \sim k^{-\sigma}. \end{aligned}$$

Now we define

$$H(k) := \begin{cases} k^{\frac{m}{n} - \sigma}, & \text{if } 0 < \sigma < \frac{m}{n}, \\ (\log k)^{\frac{m}{n}}, & \text{if } \sigma = \frac{m}{n}, \\ 1, & \text{if } \sigma > \frac{m}{n}. \end{cases}$$

Using (4.42) and (5.8) verifies that the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact and it holds for $k \geq 2$

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{if } \sigma > \frac{m}{n}, \\ k^{-\frac{m}{n}} (\log k)^{\frac{m}{n}}, & \text{if } \sigma = \frac{m}{n}, \\ k^{-\sigma}, & \text{if } 0 < \sigma < \frac{m}{n}. \end{cases}$$

There is a breaking point at $\sigma = \frac{m}{n}$. For all values σ greater than $\frac{m}{n}$ the compactness of the embedding id is the same as in (3.32) where the unweighted setting $\overset{\circ}{W}{}^m_p(B) \hookrightarrow L_p(B)$ is considered. Otherwise if σ is below the critical value $\frac{m}{n}$ the decay of the entropy and approximation numbers is worse. Then $e_k(\mathrm{id})$ and $a_k(\mathrm{id})$ tend to zero only with a rate of $k^{-\sigma}$.

Remark 6.3. Inter alia in this work we consider the embedding

$$\mathrm{id}_{w^\sigma} : E^m_{p,w^\sigma}(B) \hookrightarrow L_p(B), \quad m \in \mathbb{N}, 1 \le p < \infty,$$

where $w^\sigma(t) := (1 + |\log t|)^\sigma$, $\sigma > 0$, is involved in the source space. One could also transfer the weight to the target space and consider

$$\mathrm{id}^{w_\varkappa} : E^m_p(B) \hookrightarrow L_p(B, w_\varkappa), \quad m \in \mathbb{N}, 1 \le p < \infty,$$

where $w_\varkappa(t) := (1 + |\log t|)^{-\varkappa}$, $\varkappa > 0$. Here $L_p(B, w)$ is the weighted Lebesgue space normed by

$$\|f|L_p(B,w)\| = \Big(\int_B |f(x)|^p w(x)^p \mathrm{d}x\Big)^{\frac{1}{p}}$$

and $E^m_p(B)$ denotes the closure of $C^m_0(B)$ with respect to the norm

$$\|f|E^m_p(B)\| := \Big(\int_B |x|^{mp} \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x\Big)^{\frac{1}{p}}.$$

We have for all $f \in E^m_p(B)$

$$\int_B |f(x)|^p w_\varkappa(|x|)^p \mathrm{d}x \le c \int_B |x|^{mp} \sum_{|\alpha|=m} |\mathrm{D}^\alpha f(x)|^p \mathrm{d}x.$$

All presented methods can be applied to that case and $\varkappa$ takes the role of σ. That means for $k \in \mathbb{N}$, $k \ge 2$,

$$e_k(\mathrm{id}^{w_\varkappa}) \sim a_k(\mathrm{id}^{w_\varkappa}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{if } \varkappa > \frac{m}{n}, \\ k^{-\frac{m}{n}}(\log k)^{\frac{m}{n}}, & \text{if } \varkappa = \frac{m}{n}, \\ k^{-\varkappa}, & \text{if } 0 < \varkappa < \frac{m}{n}. \end{cases}$$

So it does not matter if one compensates the singularity of the non-compact embedding $E^m_p(B) \hookrightarrow L_p(B)$ in the source or in the target space. Of course also generalised slowly varying weights can be transferred in that way.

Example 6.4. Let $\psi(t) = (1+|\log t|)^\sigma(1+\log(1+|\log t|))^\gamma$, $\sigma > 0, \gamma \neq 0$, $t \in (0,1]$. The sequence

$$\big(\psi(2^{-j})^{-1}\big)_{j\in\mathbb{N}} = \big((1+j)^{-\sigma}(1+\log(1+j))^{-\gamma}\big)_{j\in\mathbb{N}}$$

belongs to $\ell_{\frac{n}{m}}$ if, and only if, $\sigma > \frac{m}{n}$ or $\sigma = \frac{m}{n}$ and $\gamma > \frac{m}{n}$. In that case

(4.43) and (5.9) hold true. For $T \geq T_0 > 1$ let the function Φ be such that

$$\begin{aligned}\Phi(T) &\sim \int_{T_0}^{T} \psi(2^{-t})^{-\frac{n}{m}} \mathrm{d}t \\ &\sim \int_{T_0}^{T} (1+t)^{-\sigma\frac{n}{m}}(1+\log(1+t))^{-\gamma\frac{n}{m}} \mathrm{d}t \\ &\sim \begin{cases} T^{1-\sigma\frac{n}{m}}(\log T)^{-\gamma\frac{n}{m}}, & \text{if } 0<\sigma<\frac{m}{n}, \\ \log(\log T), & \text{if } \sigma=\frac{m}{n}, \gamma=\frac{m}{n}, \\ (\log T)^{1-\gamma\frac{n}{m}}, & \text{if } \sigma=\frac{m}{n}, \gamma<\frac{m}{n}. \end{cases}\end{aligned}$$

First we determine the function ψ^{-1}

$$\begin{aligned}\psi(t)\sim s &\iff s \sim (1+|\log t|)^{\sigma}(1+\log(1+|\log t|))^{\gamma} \\ &\iff s^{\frac{1}{\sigma}} \sim |\log t|\,(\log(1+|\log t|))^{\frac{\gamma}{\sigma}}.\end{aligned}$$

In that case we have

$$\begin{aligned}\log s &\sim \sigma\log(1+|\log t|) + \gamma\log(1+\log(1+|\log t|)) \\ &\sim \log(1+|\log t|)\Big[\sigma + \gamma\,\frac{\log(1+\log(1+|\log t|))}{\log(1+|\log t|)}\Big] \\ &\sim \log(1+|\log t|)\end{aligned}$$

as $t \to 0$. Hence we continue with

$$\begin{aligned}\psi(t)\sim s &\iff |\log t| \sim s^{\frac{1}{\sigma}}(\log s)^{-\frac{\gamma}{\sigma}} \\ &\iff \exp\big(-s^{\frac{1}{\sigma}}(\log s)^{-\frac{\gamma}{\sigma}}\big) \sim t\end{aligned}$$

and define

$$\psi^{-1}(s) := \exp\big(-s^{\frac{1}{\sigma}}(\log s)^{-\frac{\gamma}{\sigma}}\big).$$

The function $h(\varepsilon)$ is given by

$$h(\varepsilon) := \varepsilon^{-\frac{n}{m}}\,\Phi\big(-\log(\psi^{-1}(\varepsilon^{-1}))\big) = \varepsilon^{-\frac{n}{m}}\,\Phi(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}}).$$

If $0<\sigma<\frac{m}{n}$ we have

$$\begin{aligned}h(\varepsilon)\sim k &\iff k \sim \varepsilon^{-\frac{n}{m}}\,\Phi(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}}) \\ &\iff k \sim \varepsilon^{-\frac{n}{m}}\,(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}})^{1-\sigma\frac{n}{m}}\Big(\log(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}})\Big)^{-\gamma\frac{n}{m}} \\ &\iff k \sim \varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{\gamma(\frac{n}{m}-\frac{1}{\sigma})}\big(\tfrac{1}{\sigma}|\log\varepsilon| - \tfrac{\gamma}{\sigma}\log|\log\varepsilon|\big)^{-\gamma\frac{n}{m}} \\ &\iff k \sim \varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{\gamma(\frac{n}{m}-\frac{1}{\sigma})}|\log\varepsilon|^{-\gamma\frac{n}{m}}\Big(\frac{1}{\sigma} - \frac{\gamma}{\sigma}\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\Big)^{-\gamma\frac{n}{m}} \\ &\iff k \sim \varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}}\end{aligned}$$

as $\varepsilon \searrow 0$. In that case it holds

$$\begin{aligned}\log k &\sim \log\left(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}}\right)\\ &\sim \tfrac{1}{\sigma}|\log\varepsilon| - \tfrac{\gamma}{\sigma}\log|\log\varepsilon|\\ &\sim |\log\varepsilon|\Big(\tfrac{1}{\sigma} - \tfrac{\gamma}{\sigma}\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\Big)\\ &\sim |\log\varepsilon|\end{aligned}$$

as $\varepsilon \searrow 0$ and we obtain

$$\begin{aligned}h(\varepsilon) \sim k \iff& \varepsilon^{-\frac{1}{\sigma}} \sim k\,(\log k)^{\frac{\gamma}{\sigma}}\\ \iff& \varepsilon \sim k^{-\sigma}(\log k)^{-\gamma}.\end{aligned}$$

In case of $\sigma = \gamma = \frac{m}{n}$ we derive

$$\begin{aligned}h(\varepsilon) \sim k \iff& k \sim \varepsilon^{-\frac{n}{m}}\,\Phi(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}})\\ \iff& k \sim \varepsilon^{-\frac{n}{m}}\,\log\left(\log(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}})\right)\\ \iff& k \sim \varepsilon^{-\frac{n}{m}}\,\log\left(\tfrac{1}{\sigma}|\log\varepsilon| - \tfrac{\gamma}{\sigma}\log|\log\varepsilon|\right)\\ \iff& k \sim \varepsilon^{-\frac{n}{m}}\,\log|\log\varepsilon|\\ \iff& \varepsilon \sim k^{-\frac{m}{n}}(\log\log k)^{\frac{m}{n}}\end{aligned}$$

as $\varepsilon \searrow 0$. Similarly as before we used $\log k \sim |\log\varepsilon|$ what implies $\log\log k \sim \log|\log\varepsilon|$ in the last line. For the remaining values $\sigma = \frac{m}{n}$ and $\gamma < \frac{m}{n}$ we use

$$\begin{aligned}h(\varepsilon) \sim k \iff& k \sim \varepsilon^{-\frac{n}{m}}\,\Phi(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}})\\ \iff& k \sim \varepsilon^{-\frac{n}{m}}\left(\log(\varepsilon^{-\frac{1}{\sigma}}|\log\varepsilon|^{-\frac{\gamma}{\sigma}})\right)^{1-\gamma\frac{n}{m}}\\ \iff& k \sim \varepsilon^{-\frac{n}{m}}\left(\tfrac{1}{\sigma}|\log\varepsilon| - \tfrac{\gamma}{\sigma}\log|\log\varepsilon|)\right)^{1-\gamma\frac{n}{m}}\\ \iff& k \sim \varepsilon^{-\frac{n}{m}}\,|\log\varepsilon|^{1-\gamma\frac{n}{m}}\\ \iff& \varepsilon \sim k^{-\frac{m}{n}}(\log k)^{\frac{m}{n}-\gamma}\end{aligned}$$

where again $|\log\varepsilon| \sim \log k$ is applied. Hence we set

$$H(k) := \begin{cases} 1, & \text{if } \sigma > \frac{m}{n} \text{ or } \sigma = \frac{m}{n}, \gamma > \frac{m}{n},\\ (\log\log k)^{\frac{m}{n}}, & \text{if } \sigma = \frac{m}{n}, \gamma = \frac{m}{n},\\ (\log k)^{\frac{m}{n}-\gamma}, & \text{if } \sigma = \frac{m}{n}, \gamma < \frac{m}{n},\\ k^{\frac{m}{n}-\sigma}(\log k)^{-\gamma}, & \text{if } 0 < \sigma < \frac{m}{n}.\end{cases}$$

Now by Theorem 4.13 and Theorem 5.2 the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact and it holds for $k \in \mathbb{N}, k \geq 2$,

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{if } \sigma > \frac{m}{n} \text{ or } \sigma = \frac{m}{n}, \gamma > \frac{m}{n}, \\ k^{-\frac{m}{n}}(\log\log k)^{\frac{m}{n}}, & \text{if } \sigma = \frac{m}{n}, \gamma = \frac{m}{n}, \\ k^{-\frac{m}{n}}(\log k)^{\frac{m}{n}-\gamma}, & \text{if } \sigma = \frac{m}{n}, \gamma < \frac{m}{n}, \\ k^{-\sigma}(\log k)^{-\gamma}, & \text{if } 0 < \sigma < \frac{m}{n}. \end{cases}$$

The breaking point for σ and γ is $\frac{m}{n}$. The σ-log term plays the key role and influences the asymptotic behaviour with a rate of $k^{-\sigma}$ whereas the γ-log log term influences the setting similarly with a rate of $(\log k)^{-\gamma}$. Roughly speaking we can observe the same effect as before.

Example 6.5. Let $\psi(t) = (1 + \log(1 + |\log t|))^\gamma$, $\gamma > 0$, $t \in (0, 1]$. The sequence

$$\big(\psi(2^{-j})^{-1}\big)_{j\in\mathbb{N}} = \big((1 + \log(1 + j))^{-\gamma}\big)_{j\in\mathbb{N}}$$

never belongs to the space $\ell_{\frac{n}{m}}$. For $T \geq T_0 > 1$ let the function Φ be such that

$$\begin{aligned} \Phi(T) &\sim \int_{T_0}^{T} \psi(2^{-t})^{-\frac{n}{m}} \mathrm{d}t \\ &\sim \int_{T_0}^{T} (1 + \log(1 + t))^{-\gamma\frac{n}{m}} \mathrm{d}t \\ &\sim T\,(\log T)^{-\gamma\frac{n}{m}}. \end{aligned}$$

We define

$$\psi^{-1}(s) := \exp\big(-\exp(s^{1/\gamma})\big)$$

and put

$$h(\varepsilon) := \varepsilon^{-\frac{n}{m}}\Phi\big(-\log(\psi^{-1}(\varepsilon^{-1}))\big) = \varepsilon^{-\frac{n}{m}}\Phi\big(\exp(\varepsilon^{-\frac{1}{\gamma}})\big).$$

We observe

$$\begin{aligned} k \sim h(\varepsilon) \quad &\Longleftrightarrow \quad k \sim \varepsilon^{-\frac{n}{m}} \exp(\varepsilon^{-\frac{1}{\gamma}})\big(\log(\exp(\varepsilon^{-\frac{1}{\gamma}}))\big)^{-\gamma\frac{n}{m}} \\ &\Longleftrightarrow \quad k \sim \exp(\varepsilon^{-\frac{1}{\gamma}}) \\ &\Longleftrightarrow \quad \varepsilon \sim (\log k)^{-\gamma}. \end{aligned}$$

Hence we identify $H(k) := k^{\frac{m}{n}}(\log k)^{-\gamma}$. Now by Theorem 4.13 and Theorem 5.2 the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact and it holds for $k \in \mathbb{N}, k \geq 2$,

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim (\log k)^{-\gamma}.$$

This fits into the scheme of the preceding Example 6.4 letting σ tend to zero. We have seen that the unbounded growth near zero of an (arbitrarily large) power of an iterated logarithm like

$$(1+\log(1+|\log t|))^{\gamma},\ (1+\log(1+\log(1+|\log t|)))^{\gamma},\ ...,\ \gamma>0$$

is too weak to attain a degree of compactness like $k^{-\frac{m}{n}}$ as in the unweighted classical setting (3.32). Therefore, one needs at least sufficiently strong simple logarithmic growth as shown in Examples 6.2 or 6.4.

Example 6.6. Let $\psi(t) = \exp(|\log t|^c),\ 0 < c < 1,\ t \in (0,1]$. Then the sequence

$$\left(\psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}} = (\exp(-j^c))_{j\in\mathbb{N}}$$

belongs to the space $\ell_{\frac{n}{m}}$ for all values $0 < c < 1$. Hence the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact and it holds for $k \in \mathbb{N}$

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim k^{-\frac{m}{n}}.$$

To overcome the partially complex determination of the function $H(k)$ in (6.1) we focus our attention on the growth rate of the function $\psi(t)$. We are able to formulate some simpler assertions to describe the asymptotic behaviour of $e_k(\mathrm{id})$ and $a_k(\mathrm{id})$ except some limiting cases.

Corollary 6.7. *Let $1 \le p < \infty, m \in \mathbb{N}$ and ψ be a differentiable, slowly varying function on $(0,1]$ with $\psi(1) = 1$, bounded from below by a positive constant and locally decreasing at zero with $\lim_{t\to 0^+} \psi(t) = \infty$. Then the embedding*

$$id : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact. If the following limits exist

$$\lim_{t\to 0^+} \psi(t)|\log t|^{-\frac{m}{n}} = 0 \tag{6.3}$$

$$\lim_{t\to 0^+} -t|\log t|\frac{\psi'(t)}{\psi(t)} \in [0, \tfrac{m}{n}) \tag{6.4}$$

then it holds for $k \in \mathbb{N}$

$$e_k(\mathrm{id}) \sim a_k(\mathrm{id}) \sim \psi(2^{-k})^{-1}. \tag{6.5}$$

Proof. We use the notation from Theorem 4.13 and Theorem 5.2. That means

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim k^{-\frac{m}{n}} H(k).$$

So (6.5) holds true if, and only if,

$$\begin{aligned} k^{-\frac{m}{n}}H(k) \sim \psi(2^{-k})^{-1} \iff & \; k \sim h\big(\psi(2^{-k})^{-1}\big) \\ \iff & \; k \sim \psi(2^{-k})^{\frac{n}{m}}\Phi(k). \end{aligned}$$

But this is the case if

$$|\log t| \sim \psi(t)^{\frac{n}{m}} \int_0^{|\log t|} \psi(2^{-s})^{-\frac{n}{m}} \mathrm{d}s \sim \psi(t)^{\frac{n}{m}} \int_t^1 \psi(s)^{-\frac{n}{m}} \frac{\mathrm{d}s}{s}$$

as $t \to 0$. Hence it suffices to prove that the limit

$$c := \lim_{t\to 0^+} \frac{\psi(t)^{\frac{n}{m}}}{|\log t|} \int_t^1 \psi(s)^{-\frac{n}{m}} \frac{\mathrm{d}s}{s} \tag{6.6}$$

exists and is positive, i.e. that $c \in (0, \infty)$. To do so we firstly remark that since $\lim\limits_{t\to 0} \psi(t)^{-\frac{n}{m}} |\log t| = \infty$ it follows

$$\lim_{t\to 0^+} \int_t^1 \psi(s)^{-\frac{n}{m}} \frac{\mathrm{d}s}{s} = \infty.$$

We can analyse the limit c with l'Hospital's rule

$$\begin{aligned} \frac{\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_t^1 \psi(s)^{-\frac{n}{m}} \frac{\mathrm{d}s}{s}\Big]}{\frac{\mathrm{d}}{\mathrm{d}t}\Big[|\log t|\psi(t)^{-\frac{n}{m}}\Big]} &= \frac{-t^{-1}\psi(t)^{-\frac{n}{m}}}{\psi(t)^{-2\frac{n}{m}}\Big[-t^{-1}\psi(t)^{\frac{n}{m}} - \frac{n}{m}|\log t|\psi(t)^{\frac{n}{m}-1}\psi'(t)\Big]} \\ &= \Big[1 + \tfrac{n}{m} t|\log t| \frac{\psi'(t)}{\psi(t)}\Big]^{-1}. \end{aligned}$$

Because we require (6.4) this proves $1 \le c < \infty$. □

In the following corollary we describe the setting in view of the growth rate of $\psi(t)$ compared to $|\log t|^{\frac{m}{n}}$ as $t \to 0$. It turns out that a growth rate of $|\log t|^{\frac{m}{n}}$ for $\psi(t)$ is a limiting situation. We distinguish between two situations: if the quotient

$$\frac{\psi(t)}{|\log t|^{\frac{m}{n}}} \tag{6.7}$$

tends to infinity at least with a rate of $|\log t|^{\varepsilon}$ or if it tends to zero at most with a rate of $|\log t|^{-\varepsilon}$ for some $\varepsilon > 0$. This is expressed in (6.10) and (6.12). We prove that in the first case there is no influence of the weight and the rate of compactness is $k^{-\frac{m}{n}}$. In the second case the growth rate is directly reflected in a rate of compactness of $\psi(2^{-k})^{-1}$.

Corollary 6.8. *Let $1 \le p < \infty, m \in \mathbb{N}$ and ψ be a differentiable, slowly varying function on $(0, 1]$ with $\psi(1) = 1$, bounded from below by a positive constant and locally decreasing at zero with $\lim\limits_{t\to 0^+} \psi(t) = \infty$. Then the* *embedding*

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B)$$

is compact. Furthermore, assume

$$\psi(t) = \exp\Big(- \int_t^1 \varepsilon(u) \frac{\mathrm{d}u}{u} \Big) \tag{6.8}$$

for some function $\varepsilon(u)$ which is non-positive locally at zero, i.e.

$$\exists\, u_0 \in (0,1] \forall\, u \leq u_0 : \varepsilon(u) \leq 0. \tag{6.9}$$

(i) *Let the limit*

$$\lim_{t \to 0} \frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})} \in (1, \infty] \tag{6.10}$$

exist (possibly also in the sense of an improper limit).
Then it holds for $k \in \mathbb{N}$

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim k^{-\frac{m}{n}}. \tag{6.11}$$

(ii) *Let the limit*

$$\lim_{t \to 0^+} \frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})} \in [0, 1) \tag{6.12}$$

exist. Furthermore, assume that the limit $\lim\limits_{t \to 0^+} \varepsilon(t) |\log t|$ exists (possibly also in the sense of an improper limit).
Then it holds for $k \in \mathbb{N}$

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim \psi(2^{-k})^{-1}. \tag{6.13}$$

Proof. *Step 1.* We prove (i). Due to (4.43) and (5.9) it suffices to show that the sequence $\big(\psi(2^{-j})^{-1}\big)_{j \in \mathbb{N}}$ belongs to $\ell_{\frac{n}{m}}$. There exist $\delta > 0$ and $t_0 \in (0,1]$ such that for all $t \leq t_0$

$$\frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})} \geq 1 + \delta$$

and thus

$$\psi(t) \geq |\log t|^{\frac{m}{n}(1+\delta)}.$$

Hence there exist $\varkappa > \frac{m}{n}$ and $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$

$$\psi(2^{-j}) \geq j^{\varkappa}. \tag{6.14}$$

This shows $\big(\psi(2^{-j})^{-1}\big)_{j \in \mathbb{N}} \in \ell_{\frac{n}{m}}$ by comparison.
Step 2. We prove (ii). Note that (6.3) follows from (6.12). Hence we need to prove that

$$c := \lim_{t \to 0^+} f(t) \in [0, 1) \tag{6.15}$$

where $f(t) = -\frac{n}{m} t |\log t| \frac{\psi'(t)}{\psi(t)} = -\frac{n}{m} |\log t| \varepsilon(t)$. It is preconditioned that the limit c exists in $[0, \infty]$ and it remains to show that $c < 1$. Assume $c \geq 1$. Then we get with l'Hospital's rule a contradiction by

$$\begin{aligned}
1 \leq \lim_{t \to 0^+} f(t) &= \lim_{t \to 0^+} \frac{\psi'(t)}{\psi(t)} \Big[-\tfrac{m}{n} \frac{1}{t |\log t|} \Big]^{-1} \\
&= \lim_{t \to 0^+} \frac{\mathrm{d}}{\mathrm{d}t} \big(\log \psi(t) \big) \Big[\frac{\mathrm{d}}{\mathrm{d}t} \big(\log(|\log t|^{\frac{m}{n}}) \big) \Big]^{-1} \\
&= \lim_{t \to 0^+} \log \psi(t) \Big[\log(|\log t|^{\frac{m}{n}}) \Big]^{-1} \\
&< 1.
\end{aligned} \tag{6.16}$$

□

We discuss whether the assumptions made in Corollary 6.8 are natural.

1. **Smoothness of $\psi(t)$:** The representation (6.8) also implies that $\psi(t)$ is differentiable and
$$\psi'(t) = \psi(t) \frac{\varepsilon(t)}{t} \quad a.e.$$
For every slowly varying function there exist an equivalent C^∞-function that is slowly varying. We refer to [BGT87, Theorem 1.3.3]. In that sense one can assume $\psi(t)$ (and likewise the function $\varepsilon(t)$) to be arbitrarily smooth.
2. **$b(t)$ is constant:** Slowly varying functions that have a representation (2.12) with a constant function $b(t) \equiv b \in (0, \infty)$ are called *normalised slowly varying functions.* Due to [BGT87, Theorem 1.5.5] those functions coincide with functions from the *Zygmund class.* This class consists of all functions $f(t)$ such that for all $\varepsilon > 0$ the function $t^{-\varepsilon} f(t)$ is locally non-increasing at zero and the function $t^{\varepsilon} f(t)$ is locally non-decreasing at zero.
3. **$\varepsilon(t)$ is non-positive:** Non-positivity of the function $\varepsilon(t)$ is a characterisation of the monotonicity of the function $\psi(t)$. The function ψ is locally decreasing at zero, i.e.
$$\exists\, t_0 \in (0,1]\ \forall\, t_1 \leq t_2 \leq t_0 :\ \psi(t_1) \geq \psi(t_2),$$
if, and only if, it holds (6.9). To prove this latter claim we observe that
$$\begin{aligned}
&\exists\, t_0 \in (0,1]\ \forall\, t_1 \leq t_2 \leq t_0 :\ \psi(t_1) \geq \psi(t_2) \\
&\quad \iff \exists\, t_0 \in (0,1]\ \forall\, t_1 \leq t_2 \leq t_0 :\ \log \psi(t_1) - \log \psi(t_2) \geq 0 \\
&\quad \iff \exists\, t_0 \in (0,1]\ \forall\, t_1 \leq t_2 \leq t_0 :\ \int_{t_1}^{t_2} \varepsilon(u) \frac{\mathrm{d}u}{u} \leq 0.
\end{aligned}$$

Remark 6.9. We can replace the condition (6.10) in the last corollary by

$$\exists \varkappa > \tfrac{m}{n}, c > 0, t_0 \in (0,1]\ \forall\, t \le t_0 : \psi(t) \ge c\,|\log t|^{\varkappa}.$$

This is a reformulation of (6.14) and it follows $\big(\psi(2^{-j})^{-1}\big)_{j\in\mathbb{N}} \in \ell_{\frac{n}{m}}$ again by comparison. On the other hand the condition (6.12) may be substituted by

$$\exists 0 < \gamma < \tfrac{m}{n}, c > 0, t_0 \in (0,1]\ \forall\, t \le t_0 : \psi(t) \le c\,|\log t|^{\gamma}.$$

Then it follows that the quotient (6.7) tends to zero at most with a rate of $|\log t|^{-\varepsilon}$, i.e. there exists $0 < \varepsilon < \frac{m}{n}$ such that

$$\frac{\psi(t)}{|\log t|^{\frac{m}{n}}} \le c\,|\log t|^{-\varepsilon}, \qquad t \le t_0.$$

Hence

$$\log \psi(t) \le (1 - \varepsilon \tfrac{n}{m}) \log(|\log t|^{\frac{m}{n}}) + \log c, \qquad t \le t_0$$

and we get

$$\exists 0 < \tilde{\varepsilon} < 1, \tilde{t_0} \in (0,1]\ \forall\, t \le \tilde{t_0} : \frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})} \le 1 - \tilde{\varepsilon}.$$

Now we can derive a similar contradiction as in (6.16) as long as we assume the existence of the limit (6.15).

Remark 6.10. In Theorem 3.4 we have seen that the embedding id : $E^m_{p,\psi}(B) \hookrightarrow L_p(B)$ is not compact if $\psi(t)$ tends to a (finite) constant as $t \to 0$. In other words one needs the slowly varying perturbation of the weight

$$w(x) := |x|^m \psi(|x|), \qquad m \in \mathbb{N}, x \in B,$$

to obtain compactness. In this regard the unboundedness of $\psi(t)$ decompensates the singular behaviour of the polynomial part. Hence it is not surprising that the quality of compactness depends on the growth rate of the function $\psi(t)$ in the sense that the stronger ψ tends to infinity the better the compactness (measured by means of entropy and approximation numbers). Especially in the situation of (6.12) one can make this observation clear. Here $a_k(\mathrm{id})$ and $e_k(\mathrm{id})$ tend with the same rate to zero as with which the function $\psi(t)$ tends to infinity.

We want to illustrate our results from Corollary 6.8 with some examples. Thereby we will see that this application is often easier than the investigation in the sense of Theorem 4.13 and Theorem 5.2 as done in the Examples 6.2, 6.4 - 6.6 .

Example 6.11. Let $\psi(t) = (1 + |\log t|)^\sigma (1 + \log(1 + |\log t|))^\gamma, t \in (0, 1]$, where $\sigma > 0$, $\gamma \in \mathbb{R}$ or $\sigma = 0$, $\gamma > 0$. Then the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B), \qquad 1 \le p < \infty, m \in \mathbb{N},$$

is compact. The function $\psi(t)$ can be represented by (2.12) with $b(t) \equiv 1$ and

$$\varepsilon(u) = \frac{-\sigma}{1 + |\log u|} + \frac{-\gamma}{(1 + |\log u|)(1 + \log(1 + |\log u|))}.$$

Hence the limit $\lim_{t \to 0^+} \varepsilon(t) |\log t| = -\sigma$ exists. Note that

$$\frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})} = \sigma \frac{n}{m} \frac{\log(1 + |\log t|)}{\log |\log t|} + \gamma \frac{n}{m} \frac{\log(1 + \log(1 + |\log t|))}{\log |\log t|}$$

tends to $\sigma \frac{n}{m}$ as $t \to 0$. If $\sigma > \frac{m}{n}$ then we obtain (6.10) while (6.12) holds for $0 \le \sigma < \frac{m}{n}$. In the limiting situation $\sigma = \frac{m}{n}$ we proceed as in Example 6.2 and Example 6.4 where

$$H(k) \sim \begin{cases} 1, & \text{if } \sigma = \frac{m}{n}, \gamma > \frac{m}{n}, \\ (\log \log k)^{\frac{m}{n}}, & \text{if } \sigma = \frac{m}{n}, \gamma = \frac{m}{n}, \\ (\log k)^{\frac{m}{n} - \gamma}, & \text{if } \sigma = \frac{m}{n}, 0 < \gamma < \frac{m}{n}. \end{cases}$$

Due to (4.42), (5.8), (6.11) and (6.13) we summarize the results

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{if } \sigma > \frac{m}{n} \text{ or } \sigma = \frac{m}{n}, \gamma > \frac{m}{n} \\ k^{-\frac{m}{n}} (\log \log k)^{\frac{m}{n}}, & \text{if } \sigma = \frac{m}{n}, \gamma = \frac{m}{n}, \\ k^{-\frac{m}{n}} (\log k)^{\frac{m}{n} - \gamma}, & \text{if } \sigma = \frac{m}{n}, \gamma < \frac{m}{n}, \\ k^{-\sigma} (\log k)^{-\gamma}, & \text{if } 0 < \sigma < \frac{m}{n} \text{ or } \sigma = 0, \gamma > 0. \end{cases}$$

Example 6.12. Let $\psi(t) = \exp(|\log t|^c), t \in (0, 1]$, where $0 < c < 1$. Then the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B), \qquad 1 \le p < \infty, m \in \mathbb{N},$$

is compact. The limit according to (6.10) is

$$\lim_{t \to 0^+} \frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})} = \lim_{t \to 0^+} \frac{|\log t|^c}{\log(|\log t|^{\frac{m}{n}})} = \infty.$$

Using (6.11) leads for every $k \in \mathbb{N}$ to

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim k^{-\frac{m}{n}}.$$

Example 6.13. Let $\psi(t) = \exp([\log(1 + |\log t|)]^a), t \in (0, 1]$, where $a > 0$. Then the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B), \qquad 1 \le p < \infty, m \in \mathbb{N},$$

is compact. If $a = 1$ the setting coincides with Example 6.11 where $\sigma = a = 1$ and $\gamma = 0$. Consider the limit according to (6.10)

$$\lim_{t\to 0^+} \frac{\log \psi(t)}{\log(|\log t|^{\frac{m}{n}})} = \lim_{t\to 0^+} \frac{n}{m} \frac{(\log(1+|\log t|))^a}{\log|\log t|} = \begin{cases} 0, & \text{if } 0 < a < 1, \\ \infty, & \text{if } a > 1. \end{cases}$$

The function $\psi(t)$ can be represented by (2.12) with $b(t) \equiv 1$ and

$$\varepsilon(u) = -\frac{a}{(1+|\log t|)(1+\log(1+|\log t|))^{1-a}}.$$

Hence the limit $\lim\limits_{t\to 0^+} \varepsilon(t)|\log t|$ exists. We apply Corollary 6.8 to get for $k \in \mathbb{N}$

$$a_k(\mathrm{id}) \sim e_k(\mathrm{id}) \sim \begin{cases} k^{-\frac{m}{n}}, & \text{if } a > 1 \text{ or } a = 1 > \frac{m}{n}, \\ k^{-1}\log k, & \text{if } a = 1 = \frac{m}{n}, \\ k^{-1}, & \text{if } a = 1 < \frac{m}{n}, \\ \exp(-(\log k)^a), & \text{if } 0 < a < 1. \end{cases}$$

Remark 6.14. So far we considered within the scope of this thesis the asymptotic behaviour of entropy and approximation numbers of the embedding

$$\mathrm{id} : E^m_{p,\psi}(B) \hookrightarrow L_p(B), \qquad 1 \le p < \infty. \tag{6.17}$$

It seems reasonable to check whether the presented methods can be applied to a similar setting involving different parameters of integrability such as

$$\mathrm{id} : E^m_{p_1,\psi}(B) \hookrightarrow L_{p_2}(B), \qquad 1 \le p_2 \le p_1 < \infty. \tag{6.18}$$

If one wishes to extend the setting to a more general scale of functions spaces, i.e. (weighted) Besov spaces, the approach might be modified substantially expecting that wavelet techniques will be more efficient in this context.

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