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Decision Uncertainty in Multiobjective Optimization

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Abstract

In many real-world optimization problems, a solution cannot be realized in practice exactly as computed, e.g., it may be impossible to produce a board of exactly 3.546 mm width. Whenever computed solutions are not realized exactly but in a perturbed way, we speak of *decision uncertainty.* We study decision uncertainty in multiobjective optimization problems and we propose the concept *decision robust efficiency* for evaluating the robustness of a solution in this case. Therefore, we address decision uncertainty within the framework of set-valued maps. First, we prove that convexity and continuity are preserved by the resulting set-valued mappings. Second, we obtain specific results for particular classes of objective functions that are relevant for solving the set-valued problem. We furthermore prove that decision robust efficient solutions can be found by solving a deterministic problem in case of linear objective functions. We also investigate the relationship of the proposed concept to other concepts in the literature.

1 Introduction

When applying mathematical optimization methods to real world problems, several difficulties have to be considered. We study a class of problems where two specific difficulties occur simultaneously. First, the problem can have several conflicting objectives, leading to a *multiobjective optimization problem.* Secondly, a calculated solution may only be realized within some accuracy instead of being put into practice exactly. Realizations of calculated solutions have to be considered uncertain in this case. In the following, we refer to this kind of uncertainty as *decision uncertainty* since the decision variables are the source of uncertainty inside the problem. We distinguish decision uncertainty from *parameter uncertainty* in optimization problems, see, e.g., [BTGN09], where the values of parameters are not known at the time a problem is solved.

In recent years, multiobjective optimization problems including parameter uncertainty have been studied in various ways. In [KL12] and [GJLVP14], the authors consider for each solution the vector consisting in each component of the worst case of the corresponding objective in

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place of the uncertain objective. [EIS14] and [AB08] compare solutions by the sets of their possible outcome vectors. Solutions that are effficient for all realizations of uncertainty are studied, e.g., in [GLP13]. For a survey of different concepts of robustness in the literature, we refer the reader to [IS16].

Single-objective optimization problems including decision uncertainty have been treated with minmax robustness for instance in [Das97], [LP09] and [BS07]. Decision uncertainty in multi-objective optimization has been addressed by evaluating the mean or integral of each objective over the set of possible values of a solution, see, e.g. [DG06], or by sensitivity analysis, see, e.g., [BA06]. However, until now, there exists no worst-case robustness approach for decision uncertainty in multiobjective optimization.

In this work, we investigate decision uncertainty in multiobjective optimization with a minmax robustness approach based on set-valued optimization, i.e., we define and analyze a robust counterpart for this kind of problem and we present approaches for solving it.
The conceptual idea of the presented robustness concept, which we call *decision robust efficiency*, is based on a minmax robustness approach for parameter uncertainty in multiobjective optimization introduced by [EIS14]. However, exploiting the specific structure arising in problems with decision uncertainty, we obtain results that are generally not valid for problems with parameter uncertainty.

In the following paragraphs we sketch practical applications of multiobjective problems that are affected by decision uncertainty. Definitions from set-valued optimization are recalled in Section 2. Following on that, the robustness concept *decision robust efficiency* is developed. In Section 3, the set-valued functions that correspond to decision robust efficiency are investigated with respect to continuity and convexity properties. Section 4 relates the presented robustness concept to approaches in the literature. Solution approaches for problem classes with specific objective functions are studied in Section 5.

Applications

We present two real-world applications for multiobjective decision uncertainty. First, we present an application in the Lorentz force velocimetry framework and secondly, we present the Growing Media Mixing Problem for plant nurseries. Both applications can be modeled as biobjective optimization problems with decision uncertainty.

Lorentz Force Velocimetry (LFV) Framework The LFV is an electromagnetic non-contact flow measurement technique for electrically conducting fluids. This is especially suited for corrosive or extremely hot fluids like glass melts or acidic mixtures which can damage other measurement setups [DTE14]. The magnetic field of the permanent magnets interacts with an electrically conducting fluid which moves through a channel, see Figure 1. Eddy

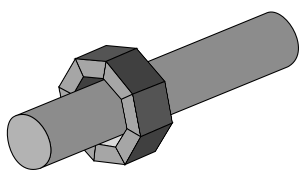

Figure 1: Illustration of a non-contact flow measurement system with 8 magnets.

currents develop and the resulting secondary magnetic field acts on the magnet system. The Lorentz force breaks the fluid and an equal but opposite force deflects the magnet system, which can be measured. Fluids with a small electrical conductivity produce only very small Lorentz forces. Thus, a sensitive balance system is required for a reliable measurement. This limits the weight of the external magnet system. Therefore, we obtain two conflicting goals: maximize the Lorentz force and minimize the magnetization. As variables, the direction of the magnetization of each magnet and the magnetization have to be optimized.
Figure 1 gives a simple arrangement of eight permanent magnets around a channel. In practice an (optimally) chosen magnetic direction cannot be realized in the desired accuracy, as magnets can only be produced with a guarantee of the magnetic directions within some tolerance interval. Therefore, decision uncertainty has to be taken into account here, in order to avoid solutions with practical realizations that are far from being efficient in the worst case.

Growing Media Mixing Problem for a Plant Nursery For each plant species, a plant nursery that produces potted plants has to decide the type of planter pot and the type of growing media to be used. First, possible types of planter pots are plastic pots made from fossil ressources and bio-degradable planter pots. The former are cheaper, while the latter are more eco-friendly. Secondly, the growing media can be chosen as a mixture of peat and compost, where the feasible mixing ratio lies between 100% peat and 33% peat plus 66% compost. The corresponding decision problem consists of determining one type of planter pot and one mixing ratio for each plant species such that two objectives, i.e., costs and global warming potential, are minimized.
However, a chosen mixing ratio can usually not be put into practice exactly. Workers in plant nurseries are used to work rather fast than exactly and the different types of soil are often mixed in a rough fashion only. Hence, the mixing ratio will vary in each planter pot. Because the mixing ratio influences the quality of the raised plant and therefore its selling price, decision uncertainty has to be taken into account in this biobjective optimization problem. A detailed study of this problem can be found in [CKGS16].

2 Decision Robust Efficiency

We consider the (deterministic) multiobjective optimization problem

$$(\mathcal{P}) \qquad \min_{x\in\Omega} f(x)$$

with feasible set $\Omega \subseteq \mathbb{R}^n$ and objective function $f\colon \Omega \to \mathbb{R}^k$. We want to take into account the fact that the decision variables in $(\mathcal{P})$ are uncertain, i.e., they can not be realized exactly as computed.
The practical realization of a solution $x \in \mathbb{R}^n$ is composed of x and a perturbation from a fixed perturbation set $Z \subseteq \mathbb{R}^n$, i.e., the realization of a solution x is an element of the set $\{x\} + Z = \{x + z \mid z \in Z\}$. Because a solution might as well be put into practice exactly as planned, we require $0 \in Z$. Decision uncertainty in $(\mathcal{P})$ is modeled by considering the family of optimization problems

$$\left(\mathcal{P}(z) \qquad \min_{x+z\in\Omega} f(x+z), \quad z \in Z\right).$$

The goal is to find a solution x without knowing the value of the perturbation z. We want to hedge against the worst case. Addressing decision uncertainty within the framework of robust optimization, we define:

Definition 1. *A point* $x \in \mathbb{R}^n$ *is called* decision robust feasible *for* $(\mathcal{P}(z), z \in Z)$ *if* $\{x + z \mid z \in Z\} \subseteq \Omega$, *i.e., if all realizations of* x *are feasible solutions. We call*

$$X := \{x \in \mathbb{R}^n \mid \ x + z \in \Omega \textit{ for all } z \in Z\}$$

the set of decision robust feasible solutions.

Definition 1 takes a conservative point of view since x is required to be feasible for every perturbation that may occur. This conceptual idea corresponds to strict robust feasibility by [BTGN09] for single-objective robust optimization with parameter uncertainty.

In the remainder of the paper, we make the following two assumptions:

- We assume $Z \subseteq \mathbb{R}^n$ is a compact set and $0 \in Z$. Hence, $X \subseteq \Omega$.
- We furthermore assume $X \neq \emptyset$.

For each decision robust feasible solution $x \in X$ we define the image set of all realizations of x as

$$f_Z(x) := \{f(x + z) \mid z \in Z\}.$$

In order to develop a solution concept for $(\mathcal{P}(z), z \in Z)$, we recall solution concepts from both, multiobjective optimization and set-valued optimization.

2.1 Approach using Multiobjective Optimization

In the following, we write $\mathrm{cl}(\cdot)$, $\mathrm{int}\,(\cdot)$ and $\mathrm{bd}(\cdot)$ for the closure, the interior and the boundary of a set, respectively. Furthermore, we denote the open and closed ball of radius $\varepsilon > 0$ for a given norm $\|\cdot\|$ on $\mathbb{R}^k$ around a point $y \in \mathbb{R}^k$ as $B(y, \varepsilon) := \{y \in \mathbb{R}^k \mid \|y\| < \varepsilon\}$ and $\overline{B}(y, \varepsilon) := \{y \in \mathbb{R}^k \mid \|y\| \leq \varepsilon\}$.
Recall that a nonempty set K is called a *cone* if $\lambda k \in K$ for all $\lambda \geq 0$ and all $k \in K$, *pointed* if $K \cap (-K) = \{0\}$, and *solid* if $\mathrm{int}\,(K) \neq \emptyset$. A cone K is convex if and only if $K + K = K$. It is well known that for a convex pointed cone K we have $K + \mathrm{int}\,(K) = \mathrm{int}\,(K)$ as well as $K\backslash\{0\} + K = K\backslash\{0\}$ and that the sets $\mathrm{int}\,(K)$ and $K\backslash\{0\}$ are also convex (see for instance [Jah11]). A closed convex pointed and solid cone $K \subseteq \mathbb{R}^k$ defines a partial order on $\mathbb{R}^k$. We consider the following relations:

$$x\ [</\leq/\leqq]\ y \quad \Leftrightarrow \quad y - x \in [\mathrm{int}(K)/K\backslash\{0\}/K]$$

A possible choice for $K \subseteq \mathbb{R}^k$ is $K = \mathbb{R}^k_+ := \left\{y \in \mathbb{R}^k \mid y_i \geq 0,\ \forall 1 \leq i \leq k\right\}$. Then the partial order introduced by K is also called the *component-wise* or *natural* order on $\mathbb{R}^k$.
Given a closed, convex, pointed and solid cone $K \subseteq \mathbb{R}^k$, the classical optimality concepts for the deterministic multiobjective problem $(\mathcal{P})$ are given in the following definition. We remark that efficiency is often refered to as (Edgeworth-) Pareto minimality.

Definition 2. *A solution* $x^* \in \Omega$ *is called* [weakly/$\cdot$/strictly] efficient *for* $(\mathcal{P})$ *if there is no* $x \in \Omega \setminus \{x^*\}$ *with the property* $f(x) \in \{f(x^*)\} - [\mathrm{int}(K)/K\backslash\{0\}/K]$.

In order define a concept for a solution of the decision uncertain problem $(\mathcal{P}(z), z \in Z)$, we replace the outcome vector $f(x)$ in Definition 2 by the set $f_Z(x)$.

Definition 3. *A solution $x^* \in X$ is called a* decision robust [weakly/·/strictly] efficient *solution of $(\mathcal{P}(z), z \in Z)$, if there is no $x \in X\backslash\{x^*\}$ with the property*

$$f_Z(x) \subseteq f_Z(x^*) - [\mathrm{int}(K)/K\backslash\{0\}/K]. \tag{1}$$

Therefore, by decision robust efficiency, we obtain a solution concept for the decision uncertain multiobjecive problem $(\mathcal{P}(z), z \in Z)$.

2.2 Approach using Set-Valued Optimization

Considering the image set of all realizations $f_Z(x)$ for all $x \in X$ defines a set-valued map $f_Z\colon X \rightrightarrows \mathbb{R}^k$. We define the *Robust Counterpart* (RC) of the decision uncertain multiobjective problem $(\mathcal{P}(z),\ z \in Z)$ as the set-valued optimization problem

$$\text{(RC)} \qquad \min_{x \in X} f_Z(x).$$

Next, we see that the decision robust efficient solutions of $(\mathcal{P}(z), z \in Z)$ are exactly the optimal solutions to the set-valued problem (RC). In set-valued optimization, i.e., in optimization with a set-valued objective function, the so called set approach [EJ12, HJ11, Kur98] uses order relations to compare the sets that are the images of the objective function. One widely used order relation is the u-type less order relation $\preccurlyeq_K$, see [HJ11, Def. 3.2], which is defined for arbitrary nonempty sets $A, B \subseteq \mathbb{R}^k$ by

$$A \preccurlyeq_K B \ :\Leftrightarrow\ A \subseteq B - K. \tag{2}$$

The order relation $\preccurlyeq_K$ is a reflexive and transitive binary relation. It can also be written in the following form:

$$A \preccurlyeq_K B \ \Leftrightarrow\ (\forall a \in A\ \exists b \in B : a \leqq b). \tag{3}$$

If we replace K by $K\backslash\{0\}$ or $\mathrm{int}\,(K)$ in (2), and thus $\leqq$ by $\leq$ or by $<$ in (3), we analogously obtain the set relations $\preccurlyeq_{K\backslash\{0\}}$ and $\preccurlyeq_{\mathrm{int}(K)}$. However, these are no longer reflexive.

In order to specify optimal solutions of the set-valued optimization problem (RC), we make use of the following definition, see, e.g., [RMS07].

Definition 4. *Let a nonempty set $X \subseteq \mathbb{R}^n$ and a set-valued map $H\colon X \rightrightarrows \mathbb{R}^k$ be given with $H(x) \neq \emptyset$ for all $x \in X$. The element $x^* \in X$ is called* strictly optimal solution *of the set-valued optimization problem*

$$\min_{x \in X} H(x)$$

w.r.t. $\preccurlyeq$, where $\preccurlyeq \in \{\preccurlyeq_K, \preccurlyeq_{\mathrm{int}(K)}, \preccurlyeq_{K\backslash\{0\}}\}$, if there exists no $x \in X\backslash\{x^*\}$ with $H(x) \preccurlyeq H(x^*)$.

Using the set relation (2) we can replace

$$f_Z(x) \subseteq f_Z(x^*) - [\mathrm{int}(K)/K\backslash\{0\}/K]$$

in (1) by

$$f_Z(x) \quad [\preccurlyeq_{\mathrm{int}(K)} / \preccurlyeq_{K\backslash\{0\}} / \preccurlyeq_K] \quad f_Z(x^*)$$

as done similarly in [IKK+14] for multi-objective parameter uncertainty. Consequently, we receive equivalence of the optimal solutions to (RC) using Definition 4 and regularization robust efficient solutions according to Definition 3.

Proposition 5. *The point $x^* \in X$ is a decision robust [weakly/·/strictly] efficient solution of $(\mathcal{P}(z), z \in Z)$ if and only if $x^* \in X$ is a* strictly optimal solution *of the set-valued optimization problem (RC) w.r.t.* $[\preccurlyeq_{\text{int}(K)} / \preccurlyeq_{K\setminus\{0\}} / \preccurlyeq_K]$.

The following example illustrates decision robust efficiency. It shows that decision robust efficient solutions are not necessarily efficient for the deterministic problem $(\mathcal{P})$.

Example 6. *Let*

$$(\mathcal{P}) \qquad \min_{x\in\Omega} \begin{pmatrix} x_1^2 \\ -x_2 \end{pmatrix}$$

and

$$(\mathcal{P}(z), z \in Z) = \left(\min_{x+z\in\Omega} \begin{pmatrix} (x_1+z_1)^2 \\ -x_2-z_2 \end{pmatrix}, z \in Z \right),$$

where

$$\begin{aligned} & \Omega = \operatorname{conv}\left\{ \begin{pmatrix} -0.6 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.6 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.7 \\ -0.3 \end{pmatrix}, \begin{pmatrix} -0.3 \\ -0.3 \end{pmatrix} \right\} \\ \text{and} \quad & Z = \operatorname{conv}\left(\left\{ \begin{pmatrix} -0.6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.3 \\ -0.3 \end{pmatrix} \right\} \right). \end{aligned}$$

Consequently, we obtain $X = [0,1]^2$ as the set of decision robust feasible solutions. The feasible set Ω and the decision robust feasible set X are visualized in Figure 2.

Figure 2: Illustration of the sets Ω and X in Example 6.

Furthermore, let $K = \mathbb{R}^2_+$. We obtain

$$\text{(RC)} \quad \min_{x\in X} f_Z(x) \qquad \text{where} \qquad f_Z(x) = \left\{ \begin{pmatrix} (x_1+z_1)^2 \\ -x_2-z_2 \end{pmatrix},\ z \in Z \right\}.$$

Consider $x^0 := \begin{pmatrix} 0.3 \\ 1 \end{pmatrix}$ and $x^1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In this example:

- *The solution x^1 is efficient for $(\mathcal{P})$, but not decision robust efficient for $(\mathcal{P}(z), z \in Z)$.*
- *The solution x^0 is decision robust efficient for $(\mathcal{P}(z), z \in Z)$, but not efficient for $(\mathcal{P})$.*

One can check that there exists no $x \in X$ with $f(x) \in \{f(x^1)\} - \mathbb{R}^2_+\setminus\{0\}$. Hence, x^1 is efficient for the deterministic problem. In particular,

$$f(x^1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \leq \begin{pmatrix} 0.09 \\ -1 \end{pmatrix} = f(x^0).$$

Therefore, x^0 is not efficient for the deterministic problem. Looking at the set $f_Z(x)$ for each $x \in X$, one can check that there is no $x \in X\setminus\{x^0\}$ with $f_Z(x) \subseteq f_Z(x^0) - \mathbb{R}^2_+\setminus\{0\}$ and x^0 is decision robust efficient according to Definition 3. In particular, it holds $f_Z(x^0) \subseteq f_Z(x^1) - \mathbb{R}^2_+\setminus\{0\}$ as illustrated in Figure 3. Hence, x^1 is not decision robust efficient even though being efficient for the deterministic problem.

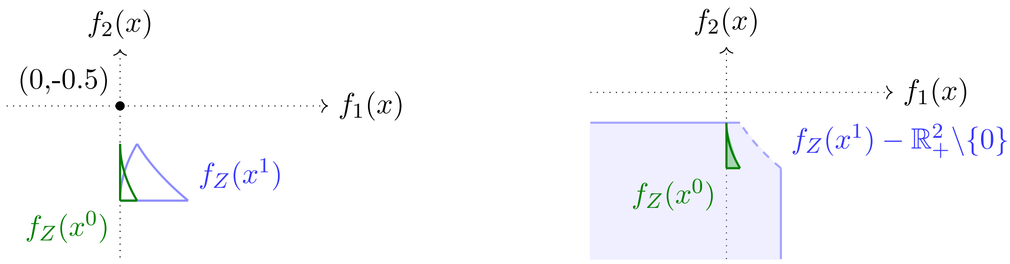

Figure 3: Relationship of the set $f_Z(x^0)$ and the set $f_Z(x^1)$ in Example 6.

2.3 Approach using Supremal Sets

In minmax robust optimization, solutions are determined that perform best in the objective function when evaluated together with worst-case realizations of uncertainty. Therefore, we investigate suprema of the sets $f_Z(x)$ for $x \in X$. We show that minimizing the supremal sets of the sets f_Z is equivalent to solving (RC).
We use a concept for the supremum of a set that is defined and studied thoroughly in [Nie80] and [Löh11].

Definition 7. *For a bounded nonempty subset $A \subsetneq \mathbb{R}^k$ the supremal set of A is defined as*

$$\mathrm{Sup}(A) = \{y \in \mathrm{cl}(A - K) \mid (\{y\} + \mathrm{int}\,(K)) \cap \mathrm{cl}\,(A - K) = \emptyset\}.$$

According to [Löh11, Cor. 1.48, Cor. 1.49], for a bounded nonempty subset $A \subsetneq \mathbb{R}^k$ it holds

$$\mathrm{Sup}(A) = \mathrm{bd}\,(\mathrm{Sup}(A) - K). \tag{4}$$

For an illustration of a supremal set, we refer to Figure 3. The boundary of the blue set, iclud-ing the broken and unbrocken blue lines, is exactly the supremal set of $f_Z(x^1)$ in Example 6. Following Definition 7, we formulate another set-valued optimization problem

$$\min_{x \in X} f_{\mathrm{Sup}}(x) \quad \text{with} \quad f_{\mathrm{Sup}} \colon X \rightrightarrows \mathbb{R}^k,\ x \mapsto \mathrm{Sup}\,(f_Z(x)). \tag{5}$$

In general set-valued optimization problems, the image set of the objective function, such as $f_{\mathrm{Sup}}(x)$ in (5), is assumed to be nonempty. Therefore, we additionally assume f to be continuous in the remainder of the section.

Remark 8. *Due to our assumptions, the perturbation set Z is nonempty and compact and $f\colon \Omega \to \mathbb{R}^k$ is continuous. Hence, $f_Z(x)$ is compact and nonempty for all $x \in X \subseteq \Omega$ and*

$$f_Z(x) - K = \mathrm{cl}\,(f_Z(x) - K) \notin \{\mathbb{R}^k, \emptyset\}.$$

According to Corollary 1.44 in [Löh11], this directly implies

$$f_{\mathrm{Sup}}(x) = \mathrm{Sup}(f_Z(x)) \neq \emptyset.$$

The following lemma is fundamental for our main result which relates the solutions of (5) to the solutions of (RC) and the decision robust [weakly/·/strictly] efficient solutions of $(\mathcal{P}(z), z \in Z)$.

Lemma 9. *Let $A, B \subsetneq \mathbb{R}^k$ be compact sets. Then*

$$A - [\mathrm{int}(K)/K\backslash\{0\}/K] = \mathrm{Sup}\,(A) - [\mathrm{int}(K)/K\backslash\{0\}/K]$$

and thus for any $\preccurlyeq \in \{\preccurlyeq_K, \preccurlyeq_{\mathrm{int}(K)}, \preccurlyeq_{K\backslash\{0\}}\}$ it holds

$$A \preccurlyeq B \qquad \Leftrightarrow \qquad \mathrm{Sup}\,(A) \preccurlyeq \mathrm{Sup}\,(B).$$

Proof. In [Löh11, Chapter 1] it was shown that $\mathrm{Sup}\,(A) - \mathrm{int}\,(K) = \mathrm{cl}(A-K) - \mathrm{int}\,(K)$. As A is compact and K is closed we have $\mathrm{cl}(A-K) = A-K$ and we get $\mathrm{Sup}\,(A) - \mathrm{int}\,(K) = A - \mathrm{int}\,(K)$. Next we show

$$A - K = \mathrm{Sup}\,(A) - K. \tag{6}$$

By [Löh11] we have $\mathrm{cl}\,(A-K) = \mathrm{Sup}\,(A) \cup (\mathrm{Sup}\,(A) - \mathrm{int}\,(K))$. And thus

$$A - K = \mathrm{cl}\,(A-K) = \mathrm{Sup}\,(A) \cup (\mathrm{Sup}\,(A) - \mathrm{int}\,(K)) \subseteq \mathrm{Sup}\,(A) - K.$$

By the definition of the supremal set and the compactness of A we also have $\mathrm{Sup}\,(A) - K \subseteq \mathrm{cl}\,(A-K) - K = A - K$ and we have shown (6). Finally we get from (6)

$$A - K\backslash\{0\} = A - K - K\backslash\{0\} = \mathrm{Sup}\,(A) - K - K\backslash\{0\} = \mathrm{Sup}\,(A) - K\backslash\{0\}.$$

Due to Relation (2) we obtain the last assertion:

$$\begin{aligned} A \subseteq B - [\mathrm{int}(K)/K\backslash\{0\}/K] \;&\Leftrightarrow\; A - K \subseteq B - [\mathrm{int}(K)/K\backslash\{0\}/K] \\ &\Leftrightarrow\; \mathrm{Sup}\,(A) - K \subseteq \mathrm{Sup}\,(B) - [\mathrm{int}(K)/K\backslash\{0\}/K] \\ &\Leftrightarrow\; \mathrm{Sup}\,(A) \subseteq \mathrm{Sup}\,(B) - [\mathrm{int}(K)/K\backslash\{0\}/K]. \end{aligned}$$

□

Due to Z being compact, the set $f_Z(x)$ is compact for a continuous objective function f and for each $x \in X$.

Corollary 10. *Let f be continuous. Then it holds for all $x, x' \in X$ and $\preccurlyeq \in \{\preccurlyeq_K, \preccurlyeq_{\mathrm{int}(K)}, \preccurlyeq_{K\backslash\{0\}}\}$*

$$f_Z(x) \preccurlyeq f_Z(x') \quad \text{if and only if} \quad \mathrm{Sup}\,(f_Z(x)) \preccurlyeq \mathrm{Sup}\,\big(f_Z(x')\big),$$

i.e., the set-valued optimization problems (RC) *and (5) are equivalent in the sense that the sets of strictly optimal solutions w.r.t. $[\preccurlyeq_{\mathrm{int}(K)} / \preccurlyeq_{K\backslash\{0\}} / \preccurlyeq_K]$ are the same.*

Using Proposition 5 and Corollary 10 we directly obtain our main result of this section.

Theorem 11. *Let f be continuous. The point $x^* \in X$ is a decision robust [weakly/$\cdot$/strictly] efficient solution of $(\mathcal{P}(z), z \in Z)$ if and only if $x^* \in X$ is a* strictly optimal solution *for the set-valued optimization problem (5) w.r.t. $[\preccurlyeq_{\mathrm{int}(K)} / \preccurlyeq_{K\backslash\{0\}} / \preccurlyeq_K]$.*

3 Properties of the Set-Valued Functions $f_Z(\cdot)$ and Sup$(f_Z(\cdot))$

We prove that convexity of the vector-valued function f implies convexity of the set-valued mapping f_Z. Furthermore, we prove that continuity of the vector-valued function f is preserved when considering the set-valued mapping f_Z or the respective supremal sets of its images instead of f.

3.1 Convexity

We start by recalling the definition of a K-convex single-valued and a K-convex set-valued map for a convex cone K. See here, e.g., [Kur96, Def. 2.1],[AF90, Lemma 2.1.2] as well as [BP03] and the references therein.

Definition 12. *Let $Y \subseteq \mathbb{R}^n$ be a convex set. A map $f\colon Y \to \mathbb{R}^k$ is called K-convex, if for all $x^1, x^2 \in Y$ and $\lambda \in [0,1]$ it holds that $\lambda f(x^1) + (1-\lambda) f(x^2) \in \{f(\lambda x^1 + (1-\lambda)x^2)\} + K$.*

Definition 13. *Let $Y \subseteq \mathbb{R}^n$ be a convex set. A set-valued map $H\colon Y \rightrightarrows \mathbb{R}^k$ is called K-convex, if for all $x^1, x^2 \in Y$ and $\lambda \in [0,1]$ it holds that $\lambda H(x^1) + (1-\lambda) H(x^2) \subseteq H(\lambda x^1 + (1-\lambda)x^2) + K$.*

Proposition 14. *Let Ω and Z be convex. Let $f\colon \Omega \to \mathbb{R}^k$ be a K-convex single-valued map. Then, X is a convex set and $f_Z\colon X \rightrightarrows \mathbb{R}^k$ is a K-convex set-valued map.*

Proof. For fixed $z \in Z$, the set $\{x \in \mathbb{R}^n \mid x + z \in \Omega\}$ is convex. Hence, $X = \bigcap_{z\in Z}\{x \in \mathbb{R}^n \mid x + z \in \Omega\}$ is convex as an intersection of convex sets.
Let $a \in f_Z(x^1)$ and $b \in f_Z(x^2)$ be arbitrarily chosen. Then there exist $z^1, z^2 \in Z$ with $a = f(x^1 + z^1)$ and $b = f(x^2 + z^2)$. With $x := \lambda\, x^1 + (1-\lambda)\, x^2$ and $z := \lambda\, z^1 + (1-\lambda)\, z^2 \in Z$ we obtain

$$\begin{array}{rcl} \lambda\, a + (1-\lambda)\, b & = & \lambda\, f(x^1 + z^1) + (1-\lambda)\, f(x^2 + z^2) \\ & \in & \{f(\lambda\,(x^1 + z^1) + (1-\lambda)\,(x^2 + z^2))\} + K \\ & = & \{f(x + z)\} + K \\ & \subseteq & f_Z(x) + K. \end{array}$$

□

Note that $\mathrm{Sup}(f_Z(\cdot))$ is not a K-convex map and that, in contrast to K-convexity, linearity of a function $f\colon \Omega \to \mathbb{R}^k$ does not necessarily lead to linearity in the corresponding set-valued function $f_Z\colon X \rightrightarrows \mathbb{R}^k$.

3.2 Semicontinuity

We prove that, if f is continuous everywhere, the set-valued functions f_Z and $\mathrm{Sup}(f_Z(\cdot))$ inherit continuity properties. We show that continuity of f leads to semicontinuity of f_Z. Moreover, $\mathrm{Sup}(f_Z(\cdot))$ is lower but not necessarily upper semicontinuous if f is continuous.

Throughout this section, we consider the feasible set $\Omega \subseteq \mathbb{R}^n$ as a metric space. Ω is equipped with the metric derived from an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$. Hence, we can use the following definition of [AF90, Section 1.4] to define semicontinuity on subsets of $\mathbb{R}^n$, such as Ω.

Definition 15. *Let S be a metric space and let $H\colon S \rightrightarrows \mathbb{R}^k$ be a set-valued map.*

(a) H is called lower semicontinuous at $x^0 \in S$, if for all open sets $V \subseteq \mathbb{R}^k$ with $H(x^0) \cap V \neq \emptyset$ there is a neighborhood U of x^0 such that $H(x) \cap V \neq \emptyset$ for all $x \in U$. H is lower semicontinuous if it is lower semicontinuous at any $x^0 \in S$.

(b) H is called upper semicontinuous at $x^0 \in S$ if for all open sets $V \subseteq \mathbb{R}^k$ with $H(x^0) \subseteq V$ there is a neighborhood U of x^0 such that $H(x) \subseteq V$ for all $x \in U$. H is upper semicontinuous if it is upper semicontinuous at any $x^0 \in S$.

An equivalent definition can be found in [KTZ15] under the name of lower and upper continuity. For a survey of the different and related notions for upper and lower semicontinuity used in the literature, we refer to [DD79].
If the function f is not continuous everywhere, the set-valued map f_Z might be neither lower nor upper semicontinuous. The following example shows: if f is continuous everywhere except at one point, the set-valued map f_Z does not need to be upper semicontinuous. By a similar example, it is also easy to see that the set-valued map f_Z does not need to be upper semicontinuous if f is continuous everywhere except at one point.

Example 16. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 0 & \textit{if } x \geq -1, \\ 1 & \textit{else} \end{cases}$$

and $Z := [-1, 1]$. Then

$$f_Z(x) = \begin{cases} \{0\} & \textit{if } x \geq 0, \\ \{0, 1\} & \textit{if } x < 0. \end{cases}$$

Let $x^0 = 0$ and $V = (-0.5, 0.5)$. Then $f_Z(x^0) \subseteq V$ but there is no neighborhood U of x^0 such that $f_Z(x) \subset V$ for all $x \in U$. However, f_Z is lower semicontinuous.

Next, we show that f_Z is both upper and lower semicontinuous if f is continuous.

Proposition 17. *Let $x^0 \in X$ be given and let $f\colon \Omega \to \mathbb{R}^k$ be continuous in the set $\{x^0\} + Z$. Then $f_Z\colon X \rightrightarrows \mathbb{R}^k$ is lower semicontinuous at x^0.*

Proof. Let $V \subseteq \mathbb{R}^k$ be an arbitrary open set such that $f_Z(x^0) \cap V \neq \emptyset$. Then there exists $z \in Z$ such that $f(x^0 + z) \in V$. As V is an open set, there exists $\varepsilon > 0$ such that $B(f(x^0 + z), \varepsilon) \subseteq V$. Because f is continuous in $x^0 + z$, there exists $\delta > 0$ such that $\|f(x^0 + z) - f(w)\| < \varepsilon$ for all $w \in \Omega$ with $\|x^0 + z - w\| < \delta$. We define the neighborhood $U(x^0) = \{x \in X \mid \|x - x^0\| < \delta\}$ of x^0 in X. Then we have for all $x \in U(x^0)$ an element $w := x + z \in \{x\} + Z$ and

$$\|x^0 + z - w\| = \|x - x^0\| < \delta.$$

Hence, $f(x + z) = f(w) \in B(f(x^0 + z), \varepsilon) \subseteq V$ and $f_Z(x) \cap V \neq \emptyset$. □

Proposition 18. *Let $f\colon \Omega \to \mathbb{R}^k$ be continuous. Then $f_Z\colon X \rightrightarrows \mathbb{R}^k$ is upper semicontinuous.*

Proof. Let $x^0 \in X$ be given and let $V \subseteq \mathbb{R}^k$ be open such that $f_Z(x^0) \subseteq V$. Because f is continuous, the inverse image $W := \{y \in \Omega \mid f(y) \in V\}$ is open in Ω. If $W = \Omega$, we immediately obtain $\{x\} + Z \subseteq W$ and $f_Z(x) \subseteq V$ for all $x \in X$ and our claim is true.
Otherwise, $A := \Omega \backslash W$ is nonempty and closed in the metric space Ω and disjoint from the compact set $\{x^0\} + Z \subseteq W$. Consequently, there exists $\delta > 0$ such that for all $y \in \{x^0\} + Z$ and $a \in A$

$$\|y - a\| > \delta.$$

Thus,

$$\bigcup_{z \in Z} B(x^0 + z, \delta) \subseteq W.$$

We define the neighborhood $U := B(x^0, \delta) \subseteq X$. For all $x \in U$ and $z \in Z$ we have $\|x + z - (x^0 + z)\| = \|x - x^0\| < \delta$ and

$$\{x\} + Z \subseteq \bigcup_{z \in Z} B(x^0 + z, \delta) \subseteq W,$$

where $B(x^0 + z, \delta)$ denotes an open ball in the metric space Ω for all $z \in Z$. Therefore, for all $x \in U$

$$f_Z(x) \subseteq \{f(w) \mid w \in W\} \subseteq V.$$

□

As we have seen in Theorem 11, determining whether a decision robust feasible solution is decision robust efficient is equivalent to a comparison of supremal sets. Therefore, we continue by examining semicontinuity properties of the set-valued function

$$f_{\mathrm{Sup}} \colon X \rightrightarrows \mathbb{R}^k, \quad x \mapsto \operatorname{Sup}(f_Z(x))$$

that was already introduced as (5) in Section 2.3.
The following lemma serves as a preparation for proving the lower semicontinuity of the function f_{Sup} in Proposition 20.

Lemma 19. *Let $\varepsilon > 0$, and let $a \in \operatorname{int}(K)$ with $\|a\| < \varepsilon$. Then*

$$T = B(0, \varepsilon) \cap (\{a\} - \operatorname{int}(K)) \cap (\{-a\} + \operatorname{int}(K))$$

(a) is convex and open,

(b) $B(0, \tilde{\varepsilon}) \subseteq T \subseteq B(0, \varepsilon)$ for some $0 < \tilde{\varepsilon} < \varepsilon$,

(c) T is symmetric, i.e. $T = -T$.

(d) For all $s, u \in \mathbb{R}^k$ it holds

$$\{s\} + T \subseteq \{u\} - \operatorname{int}(K) \quad \Leftrightarrow \quad \{u\} + T \subseteq \{s\} + \operatorname{int}(K).$$

(e) $a \in \operatorname{cl}(T)$.

Proof. The set T is convex and open as an intersection of three convex sets, hence (a) holds. Furthermore, $T \subseteq B(0, \varepsilon)$. To see the second part of (b), note that $0 \in T$ because $0 \in B(0, \varepsilon)$, $0 = a - a \in \{a\} - \operatorname{int}(K)$ and $0 = -a + a \in \{-a\} + \operatorname{int}(K)$. Because T is open, there exists $\tilde{\varepsilon} > 0$ such that $B(0, \tilde{\varepsilon}) \subseteq T$.
Next, (c) results from $B(0, \varepsilon) = -B(0, \varepsilon)$ and from $\{a\} - \operatorname{int}(K) = -(\{-a\} + \operatorname{int}(K))$, and (d) is a direct consequence of (c). Finally, for (e), note that $a \in \operatorname{int}(B(0, \varepsilon))$ and that $a \in \operatorname{int}(\{-a\} + \operatorname{int}(K))$ since $a = -a + 2 \cdot a$ and $a \in \operatorname{int}(K)$. Because a is in the interior of both sets, every sequence that is convergent to a has a subsequence that converges to a in $B(0, \varepsilon) \cap (\{-a\} + \operatorname{int}(K))$. Furthermore, $a \in \operatorname{cl}(\{a\} - \operatorname{int}(K))$ and the result follows. □

Proposition 20. *Let $f \colon \Omega \to \mathbb{R}^k$ be continuous. Then $f_{\mathrm{Sup}} \colon X \rightrightarrows \mathbb{R}^k$ with $x \mapsto \operatorname{Sup}(f_Z(x))$ is lower semicontinuous.*

Proof. Let $x^0 \in X$ and let $V \subseteq \mathbb{R}^k$ be an open set such that $\operatorname{Sup}(f_Z(x^0)) \cap V \neq \emptyset$. Choose $s \in \operatorname{Sup}(f_Z(x^0)) \cap V \neq \emptyset$. Due to Lemma 9 we have for all $x \in X$

$$f_Z(x) - K = \operatorname{Sup}(f_Z(x)) - K. \tag{7}$$

Therefore, there exist $z \in Z$ and $k \in K$ such that $s := f(x^0 + z) - k$. Furthermore, there exists $\varepsilon > 0$ such that $\{s\} + B(0, \varepsilon) \subseteq V$.
Additionally, there exists a convex neighborhood T of $0 \in \mathbb{R}^k$ that satisfies the properties of Lemma 19. In accordance with Property (a) of Lemma 19, let $a \in \mathbb{R}^k$ such that

$$T = B(0, \varepsilon) \cap (\{a\} - \operatorname{int}(K)) \cap (\{-a\} + \operatorname{int}(K)). \tag{8}$$

According to Property (b) of Lemma 19, $\{s\} + T$ is a convex neighborhood of s satisfying $\{s\} + B(0, \tilde{\varepsilon}) \subseteq \{s\} + T \subseteq \{s\} + B(0, \varepsilon) \subseteq V$. Because f is continuous, there exists for $\tilde{\varepsilon} > 0$ some $\delta_0 > 0$ such that

$$f(y) \in \{f(x^0 + z)\} + T \text{ for all } \; y \in \Omega \; \text{ with } \; \|y - (x^0 + z)\| < \delta_0. \tag{9}$$

Furthermore, f_Z is upper semicontinuous in x^0 due to Proposition 18. Consequently, there exists $\delta_1 > 0$ such that $f_Z(x) \subseteq f_Z(x^0) + T$ for all $x \in X$ with $\|x - x^0\| < \delta_1$. We define $\delta := \min\{\delta_0, \delta_1\}$. The set $U := \{x \in X \mid \|x - x^0\| < \delta\}$ is a neighborhood of x^0 in X with respect to the subset topology inherited from $\mathbb{R}^n$.
Assume that there exists $x \in U$ such that

$$\operatorname{Sup}(f_Z(x)) \cap (\{s\} + T) = \emptyset. \tag{10}$$

Next, we show that this assumption leads to

$$\{s\} + T \subseteq \operatorname{Sup}(f_Z(x)) - K. \tag{11}$$

We have $\|x + z - (x^0 + z)\| < \delta \leq \delta_0$, hence, using (9) and the definition of s, we get

$$f(x + z) - k \in \{s\} + T.$$

Together with (7), we obtain

$$f(x + z) - k \in (\{s\} + T) \cap (f_Z(x) - K) = (\{s\} + T) \cap (\operatorname{Sup}(f_Z(x)) - K). \tag{12}$$

By our assumption (10), $f(x+z) - k \notin \operatorname{Sup}(f_Z(x))$. Next, we assume that (11) does not hold. Then, there exists $t \in T$ such that $s + t \notin \operatorname{Sup}(f_Z(x)) - K$. Due to (12) and because of $\{s\} + T$ being convex due to Lemma 19, there exists $w \in \operatorname{conv}\{f(x + z) - k, s + t\}$ such that, using (4),

$$w \in \operatorname{bd}(\operatorname{Sup}(f_Z(x)) - K) \cap (\{s\} + T) = \operatorname{Sup}(f_Z(x)) \cap (\{s\} + T)$$

in contradiction to assumption (10). Hence, (11) holds and by using (7) and Remark 8 we obtain

$$\operatorname{cl}(\{s\} + T) \subseteq \operatorname{cl}(\operatorname{Sup}(f_Z(x)) - K) = \operatorname{cl}(f_Z(x) - K) = f_Z(x) - K.$$

By Property (e) in Lemma 19 we get

$$s + a \in \operatorname{cl}(\{s\} + T) \subseteq f_Z(x) - K.$$

Therefore, there exists $\tilde{z} \in Z$ such that $s + a \in \{f(x + \tilde{z})\} - K$. Consequently, due to (8),

$$\{s\} + T \subseteq \{s + a\} - \operatorname{int}(K) \subseteq \{f(x + \tilde{z})\} - \operatorname{int}(K).$$

According to Property (d) in Lemma 19 this is equivalent to

$$\{f(x + \tilde{z})\} + T \subseteq \{s\} + \operatorname{int}(K).$$

Because of $\|x - x^0\| < \delta \leq \delta_1$, there exists $z^0 \in Z$ such that $f(x^0 + z^0) \in \{f(x + \tilde{z})\} + T$. As a consequence, we finally obtain

$$f(x^0 + z^0) \in \{f(x + \tilde{z})\} + T \subseteq \{s\} + \operatorname{int}(K)$$

in contradiction to $s \in \operatorname{Sup}(f_Z(x^0))$. Thus, (10) does not hold and we have for all $x \in U$

$$\emptyset \neq \operatorname{Sup}(f_Z(x)) \cap (\{s\} + T) \subseteq \operatorname{Sup}(f_Z(x)) \cap (\{s\} + B(0, \varepsilon)) \subseteq \operatorname{Sup}(f_Z(x)) \cap V.$$

$\square$

While the set-valued function $f_{\operatorname{Sup}}(\cdot) = \operatorname{Sup}(f_Z(\cdot))$ is lower semicontinuous, it is not upper semicontinuous as the following counterexample shows.

Example 21. *Let* $\Omega = [-1, 1]^2$, $Z = \{0\}$, $K = \mathbb{R}^2_+$ *and* $f = \operatorname{id}\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad x \mapsto x$. *Then* $X = \Omega$ *and we have for all* $x \in X$

$$\operatorname{Sup}(f_Z(x)) = \{x\} - \operatorname{bd}(\mathbb{R}^2_+).$$

The set

$$V := \left\{z \in \mathbb{R}^2 \mid z_1 \leq 0\right\} \cup \left\{z \in \mathbb{R}^2 \mid z_1 > 0,\ z_2 > -\frac{1}{z_1}\right\}$$

is a neighborhood of $\operatorname{Sup}(f_Z(x^0))$ *with* $x^0 = (0, 0)^\top$. *We show that for each* $\delta > 0$ *there exists* $x \in X$ *with* $\|x^0 - x\|_\infty < \delta$ *and with* $\operatorname{Sup}(f_Z(x)) \not\subseteq V$. *Choose an arbitrary* $\delta > 0$ *with* $\delta < 2$. *Then* $x' := \frac{\delta}{2}(1, 0)^\top \in X$ *satisfies* $\|x^0 - x'\|_\infty < \delta$. *We have* $w := (\delta/2, -4/\delta)^\top \in \operatorname{Sup}(f_Z(x'))$ *but due to*

$$w_2 = -\frac{4}{\delta} \leq -\frac{2}{\delta} = -\frac{1}{w_1}$$

it holds $w \notin V$. *Because* $\delta > 0$ *can be chosen arbitrarily small, we see that* $\operatorname{Sup}(f_Z(\cdot))$ *is not upper semicontinuous.*

The results on the semicontinuity of the functions considered in this section are summarized in the following table. For continuous objective functions f we have:

	$f_Z(\cdot)$	$\operatorname{Sup}(f_Z(\cdot))$
lower semicontinuous	✓	✓
upper semicontinuous	✓	✗

Even though $\operatorname{Sup}(f_Z(\cdot))$ is not upper semicontinuous, we finally show that $\operatorname{Sup}(f_Z(x^0))$ and $\operatorname{Sup}(f_Z(x))$ are arbitrary close for $x^0, x \in X$ with $\|x^0 - x\| < \delta$ if we chose $\delta > 0$ small enough.

Theorem 22. *Let $f\colon \Omega \to \mathbb{R}^k$ be continuous. Then for each $x^0 \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\mathrm{Sup}(f_Z(x)) \subseteq \mathrm{Sup}(f_Z(x^0)) + B(0,\varepsilon)$$

for all $x \in X$ with $\|x^0 - x\| < \delta$.

Proof. Let $\varepsilon > 0$ and $x^0 \in X$.
Due to Lemma 19, there exists a convex neighborhood T of $0 \in \mathbb{R}^k$ that satisfies the properties of Lemma 19. In accordance with of Lemma 19, let $a \in \mathbb{R}^k$ such that

$$T = B(0,\varepsilon) \cap (\{a\} - \mathrm{int}\,(K)) \cap (\{-a\} + \mathrm{int}\,(K)). \tag{13}$$

According to Property (b) of Lemma 19, for every $v \in \mathbb{R}^k$ the set $\{v\} + T$ is a convex neighborhood of v satisfying $\{v\} + T \subseteq B(v,\varepsilon)$.
Because $f_Z(x^0)$ is compact in $\mathbb{R}^k$ and $\bigcup_{z\in Z}\{f(x^0+z)\} + \frac{1}{2}T$ is an open cover of $f_Z(x^0)$, there exists a finite subcover

$$f_Z(x^0) \subseteq \bigcup_{\tilde{z}\in\tilde{Z}} \{f(x^0+\tilde{z})\} + \frac{1}{2}T,$$

which will be indexed by $\tilde{Z} \subseteq Z$ with $|\tilde{Z}| < \infty$. Furthermore, there exists $\delta_1 > 0$ such that

$$f_Z(x) \subseteq \bigcup_{\tilde{z}\in\tilde{Z}} \{f(x^0+\tilde{z})\} + \frac{1}{2}T \tag{14}$$

for all $x \in X$ with $\|x - x^0\| < \delta_1$ because f_Z is upper semicontinuous due to Proposition 18.
Because f is continuous on Ω, there exists $\delta(\tilde{z}) > 0$ for each $\tilde{z} \in \tilde{Z}$ such that

$$f(y) \in \{f(x^0+\tilde{z})\} + \frac{1}{2}T \quad \text{for all} \quad y \in \Omega \quad \text{with} \quad \|y - (x^0+\tilde{z})\| < \delta(\tilde{z}). \tag{15}$$

Let $\delta_2 = \min\{\delta(\tilde{z}) \mid \tilde{z} \in \tilde{Z}\}$ and define $\delta := \min\{\delta_1, \delta_2\}$.
We hence have that $f(x+\tilde{z}) \in \{f(x^0+\tilde{z})\} + \frac{1}{2}T$ for all $x \in X$ with $\|x^0 - x\| < \delta \leq \delta_2$ and all $\tilde{z} \in \tilde{Z}$ and, because of the symmetry of T due to Property (c) in Lemma 19,

$$f(x^0+\tilde{z}) \in \{f(x+\tilde{z})\} + \frac{1}{2}T. \tag{16}$$

Additionally, for each $z \in Z$ there exists $\tilde{z} \in \tilde{Z}$ such that

$$f(x^0+z) \in \{f(x^0+\tilde{z})\} + \frac{1}{2}T$$

because our finite subcover is indexed by $\tilde{Z}$. Together with Relation (16) we see that for each $x \in X$ with $\|x^0 - x\| < \delta$ and $z \in Z$ there exists $\tilde{z} \in \tilde{Z}$ such that

$$f(x^0+z) \in \{f(x^0+\tilde{z})\} + \frac{1}{2}T \subseteq (\{f(x+\tilde{z})\} + \frac{1}{2}T) + \frac{1}{2}T = \{f(x+\tilde{z})\} + T$$

because T is convex. Hence, using again the symmetry of T, we obtain for each $z \in Z$ the existence of some $\tilde{z} \in \tilde{Z}$ such that

$$f(x+\tilde{z}) \in \{f(x^0+z)\} + T. \tag{17}$$

In order to prove Theorem 22, we assume on the contrary that there exists $x \in X$ with $\|x^0 - x\| < \delta$ such that

$$\mathrm{Sup}(f_Z(x)) \not\subseteq \mathrm{Sup}(f_Z(x^0)) + T. \tag{18}$$

Then there exists $y \in \mathrm{Sup}(f_Z(x))$ such that $y \notin \mathrm{Sup}(f_Z(x^0)) + T$ and, due to the symmetry of T,

$$(\{y\} + T) \cap \mathrm{Sup}(f_Z(x^0)) = \emptyset. \tag{19}$$

Recall from Lemma 9 that for all $x' \in X$

$$f_Z(x') - K = \mathrm{Sup}(f_Z(x')) - K. \tag{20}$$

From (20) we know that there exist $z \in Z$ and $k \in K$ such that

$$y = f(x + z) - k. \tag{21}$$

Next, we show that Assumption (19) leads to

$$\{y\} + T \subseteq \mathrm{Sup}(f_Z(x^0)) - K. \tag{22}$$

We have $\|x - x^0\| < \delta \leq \delta_1$. Due to (14) together with the symmetry of T, there exists $\tilde{z} \in \tilde{Z}$ such that $f(x^0 + \tilde{z}) \in \{f(x + z)\} + \frac{1}{2}T \subseteq \{f(x + z)\} + T$, because T is convex and $0 \in T$. Hence, we get from (21)

$$f(x^0 + \tilde{z}) - k \in (\{y\} + T) \cap \left(\mathrm{Sup}(f_Z(x^0)) - K\right). \tag{23}$$

By assumption (19), $f(x^0 + \tilde{z}) - k \notin \mathrm{Sup}(f_Z(x^0))$.
We assume that (22) does not hold. Then, there exists $t \in T$ such that $y + t \notin \mathrm{Sup}(f_Z(x^0)) - K$. Because $\{y\} + T$ is convex and due to (23), there exists

$$w \in \mathrm{conv}\{f(x^0 + \tilde{z}) - k, y + t\} \subseteq \{y\} + T$$

such that, using (4),

$$w \in \mathrm{bd}\left(\mathrm{Sup}(f_Z(x^0)) - K\right) \cap (\{y\} + T) = \mathrm{Sup}(f_Z(x^0)) \cap (\{y\} + T)$$

in contradiction to (19). Hence, (22) holds and by using (20) we obtain

$$\mathrm{cl}\,(\{y\} + T) \subseteq \mathrm{cl}\left(\mathrm{Sup}(f_Z(x^0)) - K\right) = \mathrm{cl}\left(f_Z(x^0) - K\right).$$

By Property (e) in Lemma 19 and Remark 8 we get

$$y + a \in \mathrm{cl}\,(\{y\} + T) \subseteq \mathrm{cl}\left(f_Z(x^0) - K\right) = f_Z(x^0) - K.$$

Therefore, there exists $z' \in Z$ such that $y + a \in \{f(x^0 + z')\} - K$. Then, due to (13), it holds

$$\{y\} + T \subseteq \{y + a\} - \mathrm{int}\,(K) \subseteq \{f(x^0 + z')\} - \mathrm{int}\,(K).$$

According to Property (d) in Lemma 19 this is equivalent to

$$\{f(x^0 + z')\} + T \subseteq \{y\} + \mathrm{int}\,(K).$$

Due to (17), there exists $\tilde{z} \in Z$ such that $f(x + \tilde{z}) \in \{f(x^0 + z')\} + T$. As a consequence, we finally obtain

$$f(x + \tilde{z}) \in \{f(x^0 + z')\} + T \subseteq \{y\} + \mathrm{int}\,(K)$$

in contradiction to $y \in \mathrm{Sup}(f_Z(x))$. Thus, (18) does not hold and we have

$$\mathrm{Sup}(f_Z(x)) \subseteq \mathrm{Sup}(f_Z(x^0)) + T \subseteq \mathrm{Sup}(f_Z(x^0)) + B(0, \varepsilon).$$

□

4 Relationship to Robustness Concepts in the Literature

This section compares decision robust efficiency to two types of robustness concepts in the literature. First, a minmax robust counterpart for decision uncertainty in single objective optimization is investigated. We show that decision robust efficiency can be considered as a general approach to decision uncertainty because it reduces to a scalar-valued decision robustness concept when applied to single objective problems. Secondly, we investigate a robustness concept for parameter uncertainty in multiobjective optimization. We show that decision robust efficiency can be considered a special case of the presented concept if the corresponding multiobjective problem with decision uncertainty is reformulated accordingly.

4.1 Relationship to Minmax Robustness for Decision Uncertainty in Single-Objective Optimization

Single-objective optimization with decision uncertainty has been studied for years in the areas robust optimization and robust optimal control, see [BS07] for a survey.
In single-objective robust optimization, the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is replaced by the mapping

$$x \mapsto \bar{f}(x) := \sup_{z\in Z} f(x+z),$$

leading to the single-objective minmax robust counterpart

$$(\mathrm{RC}^{so}) \qquad \min_{x\in X} \bar{f}(x).$$

The mapping $\bar{f}$ is studied in more detail under the name *robust regularization* in [Lew02] and [LP09]. The definition of the robust regularization $\bar{f}$ is given under the assumption that for all $x \in X$ the supremum satisfies $\sup_{z\in Z} f(x+z) < \infty$ and hence $\bar{f}\colon X \to \mathbb{R}$.
In the literature on single-objective optimization with decision uncertainty, Z is widely defined as a convex compact neighborhood of 0. However, we only require $0 \in Z$ and Z compact. Proposition 23 shows that f_{Sup}, which is studied in Section 2.3, reduces to $\bar{f}$ in single-objective problems. Hence, the set of decision robust efficient solutions corresponds exactly to the minimizers of (RC^{so}) if $k = 1$ and $K = \mathbb{R}_+$. Therefore, our definition of decision robust efficiency can be considered as a generalization of the single-objective framework.

Proposition 23. *Let $f\colon \Omega \to \mathbb{R}$ be continuous and let $K = \mathbb{R}_+$. Then we have for all $x^* \in X$*

(i) $f_{\mathrm{Sup}}(x^*) = \{\bar{f}(x^*)\} = \{\sup_{z\in Z} f(x^*+z)\}$,

(ii) *x^* is decision robust [weakly/·/strictly] efficient in the sense of Definition 3 if and only if x^* is [an/an/the unique] optimal solution of* (RC^{so}).

Proof. We consider dimension $k = 1$, hence $\mathrm{int}\,(\mathbb{R}_+) = \mathbb{R}_+\backslash\{0\}$ and decision robust efficiency coincides with decision robust weak efficiency. The maximum $\max_{z\in Z} f(x^*+z) = \bar{f}(x^*)$ exists because f is continuous and Z is compact. Consequently, for all $x^* \in X$ it holds according to Definition 7

$$\begin{aligned} f_{\mathrm{Sup}}(x) &= \{y \in \mathrm{cl}\,(f_Z(x) - \mathbb{R}_+) \mid \forall y' > y\colon y' \notin \mathrm{cl}\,(f_Z(x) - \mathbb{R}_+)\} \\ &= \{\sup_{z\in Z} f(x+z)\} = \{\max_{z\in Z} f(x+z)\}. \end{aligned}$$

Applying Theorem 11, we therefore obtain that x^* is decision robust [weakly/·/strictly] efficient if and only if $\max_{z\in Z} f(x+z)[\geq / \geq / >] \max_{z\in Z} f(x^*+z)$ for all $x \in X\backslash\{x^*\}$, i.e., if and only if x^* is [an/an/the unique] optimal solution of (RC^{so}). $\square$

Remark 24. *Result (i) in Proposition 23 also holds if f is not continuous, but (ii) is less strong in that case. If f is not continuous, it is straightforward to prove for all $x^* \in X$:*

- *If x^* is decision robust [weakly/·] efficient, then x^* is an optimal solution of* (RC^{so}).
- *If x^* is the unique optimal solution of* (RC^{so})*, then x^* is decision robust strictly efficient.*

4.2 Relationship to Parameter Uncertainty in Multiobjective Optimization

Decision robust efficiency is a methodology to handle decision uncertainty in multiobjective optimization. Parameter uncertainty in multiobjective optimization has recently been addressed with minmax robustness, see, e.g., [EIS14]. From a mathematical point of view, both concepts are related, as Proposition 26 shows. We first give the definition of *robust efficiency* for parameter uncertainty in multiobjective optimization problems.

Definition 25 ([EIS14],[IKK$^+$14])**.** *Let $(\mathcal{P}(\xi), \xi \in \mathcal{U})$ be a family of optimization problems of the type*

$$(\mathcal{P}(\xi)) \qquad \min_{x \in \mathbb{R}^n} \{g(x,\xi) \mid G_j(x,\xi) \le 0, \forall 1 \le j \le m\},$$

where $\mathcal{U} \subseteq \mathbb{R}^p$, $g\colon \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^k$ and $G\colon \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^m$. A point $x \in \mathbb{R}^n$ is called robust feasible *for $(\mathcal{P}(\xi), \xi \in \mathcal{U})$ if x is feasible for $(\mathcal{P}(\xi))$ for all $\xi \in \mathcal{U}$.*
A robust feasible solution $x \in \mathbb{R}^n$ is called robust [weakly/·/strictly] efficient *for $(\mathcal{P}(\xi), \xi \in \mathcal{U})$ if there is no other robust feasible $y \neq x$ such that*

$$g_{\mathcal{U}}(y) \subseteq g_{\mathcal{U}}(x) - [\operatorname{int}(K)/K\backslash\{0\}/K],$$

where $g_{\mathcal{U}}(y) = \{g(y,\xi) \mid \xi \in \mathcal{U}\}$ for $y \in \mathbb{R}^n$. In case of $K = \mathbb{R}^k_+$, each robust [weakly/·/strictly] efficient solution is called a minmax robust [weakly/·/strictly] efficient *solution.*

In the following we show that we can rewrite the multiobjective problem with decision uncertainty $(\mathcal{P}(z), z \in Z)$ to an equivalent problem with parameter uncertainty and that the decision robust [weakly/·/strictly] efficient solutions for the former coincide with the robust [weakly/·/strictly] efficient solutions for the latter.

Proposition 26. **(i)** *Every family $(\mathcal{P}(z), z \in Z)$, where*

$$\mathcal{P}(z) = \min_{x \in \mathbb{R}^n} \{f(x+z) \mid F_j(x+z) \le 0 \ \forall 1 \le j \le m\}$$

can be written as a family $(\mathcal{P}(\xi), \xi \in \mathcal{U})$ of uncertain multiobjective optimization problems in the sense of Definition 25, where $\mathcal{U} = Z$, $g(x,\xi) = f(x+\xi)$ and $G_j(x,\xi) = F_j(x+\xi)$ for $x \in \Omega$, $\xi \in \mathcal{U}$.

(ii) *A solution $x \in \Omega$ is decision robust [weakly/·/strictly] efficient for* (RC) *if and only if it is robust [weakly/·/strictly] efficient w.r.t. the family of uncertain multiobjective optimization problems according to* (i)*.*

Proof. **(i)** Plugging the definitions of $\mathcal{U}, g\colon \Omega \times Z \to \mathbb{R}^k$, and $G\colon \Omega \times \mathcal{U} \to \mathbb{R}^m$ into $(\mathcal{P}(\xi))$ results exactly in $(\mathcal{P}(z))$ for all $z \in Z$.

(ii) a solution x is robust feasible in the sense of Definition 25 if and only if $G_j(x,\xi) \leq 0$ for all $1 \leq j \leq m$ and $\xi \in \mathcal{U}$. Then

$$\begin{aligned} & x \text{ is decision robust feasible} \\ \Leftrightarrow \quad & \forall z \in \mathcal{U} = Z \; \forall 1 \leq j \leq m \colon 0 \geq G_j(x+z) = F_j(x,z) \\ \Leftrightarrow \quad & x \text{ is robust feasible in the sense of Definition 25.} \end{aligned}$$

From the definitions of g and $\mathcal{U}$ in (i) we have $f_Z(x) = g_{\mathcal{U}}(x)$ for all $x \in X = \{y \in \mathbb{R}^n \mid F_j(y+z) \leq 0 \; \forall z \in Z \; \forall 1 \leq j \leq m\} = \{y \in \mathbb{R}^n \mid G_j(y,z) \leq 0 \; \forall z \in \mathcal{U} \; \forall 1 \leq j \leq m\}$. Hence,

$$\begin{aligned} & x \in X \text{ is decision robust [weakly/}\cdot\text{/strictly] efficient for (RC)} \\ \Leftrightarrow \quad & \forall y \in X \backslash \{x\} \colon f_Z(y) \nsubseteq f_Z(x) - [\mathrm{int}(K)/K\backslash\{0\}/K] \\ \Leftrightarrow \quad & \forall y \in X \backslash \{x\} \colon g_{\mathcal{U}}(y) \nsubseteq g_{\mathcal{U}}(x) - [\mathrm{int}(K)/K\backslash\{0\}/K] \\ \Leftrightarrow \quad & x \in X \text{ is robust [weakly/}\cdot\text{/strictly] efficient w.r.t. Definition 25.} \end{aligned}$$

□

Hence, from the mathematical point of view, decision uncertainty can be considered as a special case of parameter uncertainty and the results of [EIS14] apply. However, the setting of decision uncertainty in multiobjective optimization adds a significant amount of structure to the robust counterpart. In Section 2.3, the specific structure of decision uncertainty leads to the presented continuity properties of the set-valued functions f_Z and f_{Sup}. Moreover, in the following section, this specific structure benefits the study of (RC) with respect to specific classes of objective functions. In the following, we present solution approaches for our setting that do not hold for multiobjective problems with parameter uncertainty.

5 Results for Special Classes of Objective Functions

In the following, we investigate properties of the decision robust counterpart (RC) for three distinct types of objective functions. The types are linear, Lipschitz continuous and monotonic objective functions. For linear objective functions, we show that the decision robust [weakly/·/strictly] efficient solutions can be determined by methods from deterministic linear multiobjective optimization. A necessary condition for decision robust efficient solutions is given in case of Lipschitz continuous objective functions. We also present a sufficient condition for decision robust efficient solutions in case of monotonic objective functions.

5.1 Linear Objective Functions

In this section we look at the case where the objective function f is linear. We prove that each decision robust feasible solution is decision robust [weakly/·/strictly] efficient for $(\mathcal{P}(z), z \in Z)$ if and only if it is [weakly/·/strictly] efficient for the deterministic multiobjective problem

$$(\mathcal{P}|_X) \qquad \min_{x \in X} f(x).$$

As a preparation for the main result of this section, we state the following lemma.

Lemma 27. *Let K be a closed convex pointed solid cone in $\mathbb{R}^k$. Let $v \in \mathbb{R}^k$ such that $v \notin [\mathrm{int}(K)/K\backslash\{0\}/K]$. Then there exists $c \in \mathbb{R}^k$ such that $\langle c, v\rangle < \langle c, k\rangle$ for all $k \in [\mathrm{int}(K)/K\backslash\{0\}/K]$.*

Proof. In case of $v \notin K$ or $v \notin \mathrm{int}\,(K)$ the claim corresponds to the Theorem of Hahn-Banach, see, e.g., [Rud91].
For the case $v \notin K\backslash\{0\}$ we distinguish two cases: If $v \neq 0$ we can separate v from K, hence also from $K\backslash\{0\}$. Now, let $v = 0$. Because K is closed, convex and pointed, we have

$$K^{\#} = \{d \in \mathbb{R}^k \mid \langle d, k\rangle > 0,\ \forall k \in K\backslash\{0\}\} \neq \emptyset$$

due to Theorem 3.38 in [Jah11]. Hence, for each $c \in K^{\#}$ and all $k \in K\backslash\{0\}$ it holds $\langle c, k\rangle > 0 = \langle c, v\rangle$. □

Now, we can prove our main result about decision robust [weakly/·/strictly] efficient solutions to uncertain problems with linear objective functions.

Theorem 28. *Let $f\colon \Omega \to \mathbb{R}^k$ be linear. Then for each decision robust feasible solution $x \in X$*

$$\begin{aligned} & x \text{ is [weakly/}\cdot\text{/strictly] efficient for } (\mathcal{P}|_X) \\ \Leftrightarrow \quad & x \text{ is decision robust [weakly/}\cdot\text{/strictly] efficient for } (\mathcal{P}(z), z \in Z). \end{aligned}$$

According to this theorem, solving the deterministic linear multiobjective problem $(\mathcal{P}|_X)$, is equivalent to determining decision robust efficient solutions. Hence, linear multiobjective problems including decision uncertainty can be solved by any solution method for deterministic linear multiobjective optimization.

Proof. We begin by showing that every [weakly/·/strictly] efficient solution for $(\mathcal{P}|_X)$ is also decision robust [weakly/·/strictly] efficient for $(\mathcal{P}(z), z \in Z)$. Let $x \in X$ be [weakly/·/strictly] efficient for $(\mathcal{P}|_X)$ and assume that x is not decision robust [weakly/·/strictly] efficient for $(\mathcal{P}(z), z \in Z)$. Consequently, there exists $y \in X\backslash\{x\}$ such that $f_Z(y) \subseteq f_Z(x) - [\mathrm{int}(K)/K\backslash\{0\}/K]$. Since x is [weakly/·/strictly] efficient

$$v := f(x) - f(y) = f(x-y) \notin [\mathrm{int}(K)/K\backslash\{0\}/K].$$

According to Lemma 27 there exists $c \in \mathbb{R}^k$ such that

$$\langle c, v\rangle < \langle c, k\rangle \quad \forall k \in [\mathrm{int}(K)/K\backslash\{0\}/K]. \tag{24}$$

Choose $\tilde{z} \in \mathrm{argmax}_{z \in Z}\langle c, f(z)\rangle$, which is nonempty, because f and the inner product are continuous and Z is compact. Hence,

$$\langle c, f(z - \tilde{z})\rangle \leq 0 \quad \forall z \in Z. \tag{25}$$

Because $f_Z(y) \subseteq f_Z(x) - [\mathrm{int}(K)/K\backslash\{0\}/K]$, there exists $z \in Z$ and $k \in [\mathrm{int}(K)/K\backslash\{0\}/K]$ such that $f(y + \tilde{z}) = f(x + z) - k$. Then, by using (25),

$$\begin{aligned} & k = f(x-y) + f(z - \tilde{z}) = v + f(z - \tilde{z}) \\ \Rightarrow \quad & \langle c, k\rangle = \langle c, v\rangle + \langle c, f(z - \tilde{z})\rangle \leq \langle c, v\rangle \\ \Rightarrow \quad & \langle c, k\rangle \leq \langle c, v\rangle \end{aligned}$$

in contradiction to Equation (24). Therefore, x is decision robust [weakly/$\cdot$/strictly] efficient.

Now, the opposite implication is proven. Assume that x is not [weakly/$\cdot$/strictly] efficient for $(\mathcal{P}|_X)$. We show that x can not be decision robust [weakly/$\cdot$/strictly] efficient in this case.

$$
\begin{array}{ll}
& x \text{ is not [weakly/}\cdot\text{/strictly] efficient} \\
\Rightarrow & \exists y \in X\backslash\{x\}\colon f(y) \in \{f(x)\} - [\mathrm{int}(K)/K\backslash\{0\}/K] \\
\Rightarrow & \exists y \in X\backslash\{x\}\colon\ \forall z \in Z \quad f(y+z) \in \{f(x+z)\} - [\mathrm{int}(K)/K\backslash\{0\}/K] \\
\Rightarrow & \exists y \in X\backslash\{x\}\colon\ \forall z \in Z \exists z' = z \in Z\colon f(y+z) \in \{f(x+z')\} - [\mathrm{int}(K)/K\backslash\{0\}/K] \\
\Rightarrow & x \text{ is not decision robust [weakly/}\cdot\text{/strictly] efficient.}
\end{array}
$$

$\square$

5.2 Lipschitz Continuous Objective Functions

In general, decision robust efficient solutions for $(\mathcal{P}(z), z \in Z)$ are not efficient for $(\mathcal{P}|_X)$. As Example 6 shows, this is not even true for Lipschitz objective functions, since the objective function is Lipschitz on the compact feasible set Ω. Nevertheless, in case of Lipschitz objective functions, every decision robust efficient solution is at least approximately efficient. Therefore, we prove a necessary condition for a decision robust efficient solution.
Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}^k$ is called *Lipschitz continuous* or *Lipschitz* with *Lipschitz constant* $L > 0$ if for all $x, y \in \mathbb{R}^n$

$$\|f(x) - f(y)\| \leq L \cdot \|x - y\|.$$

Several concepts of approximate solutions for multiobjective optimization problems have been introduced in the literature see for instance [Dur07] and the references therein. We use the following concept which goes back to Kutateladze [Kut79].

Definition 29. *Let $k^0 \in K\backslash\{0\}$ and $\varepsilon > 0$ be given. The point $x^* \in X$ is a [weakly/$\cdot$/strictly]* (ε, k^0)-minimal solution *for $(\mathcal{P}|_X)$, i.e., $\min_{x\in X} f(x)$, if there exists no $x \in X$ such that*

$$f(x) \in \{f(x^*) - \varepsilon\, k^0\} - [\mathrm{int}(K)/K\backslash\{0\}/K].$$

In the following we assume w.l.o.g. $\|k^0\| = 1$ in the definition above.

Proposition 30. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be Lipschitz continuous with Lipschitz constant $L > 0$ and $Z \neq \{0\}$. Furthermore, set*

$$\tilde{L} := L \cdot \max_{z\in Z} \|z\| \quad \text{and} \quad S := \bigcap_{h \in \overline{B}(0,\tilde{L})} \{h\} + K.$$

If $x^* \in X$ is decision robust [weakly/$\cdot$/strictly] efficient for $(\mathcal{P}(z), z \in Z)$, then there exists no $\tilde{x} \in X$ and no $s \in S$ such that $f(\tilde{x}) \in \{f(x^*) - s\} - [\mathrm{int}(K)/K\backslash\{0\}/K]$. Hence, x^* is an [weakly/$\cdot$/strictly] (ε, k^0)-minimal solution for $(\mathcal{P}|_X)$, i.e., $\min_{x\in X} f(x)$, where

$$k^0 = \frac{s}{\|s\|} \quad \text{and} \quad \varepsilon = \|s\|$$

for all $s \in S$.

Proof. First we note that $S \subseteq K\backslash\{0\}$: as K is pointed and solid there exists $\tilde{h} \in \operatorname{int}(K) \cap \overline{B}(0, \tilde{L})$ and thus we have $S \subseteq \{\tilde{h}\} + K \subseteq \operatorname{int}(K)$ and $0 \notin S$.
Because f is Lipschitz continuous, we have for all $x \in X$ and the closed Ball of radius $\tilde{L}$, $\overline{B}(0, \tilde{L})$,

$$f_Z(x) \subseteq \{f(x)\} + \overline{B}(0, \tilde{L}).$$

Consequently, for each $x \in X$ and $z \in Z$ there exists $h \in \overline{B}(0, \tilde{L})$ such that $f(x+z) = f(x)+h$. Furthermore, it holds $s \in S$ if and only if $h \in \{s\} - K$ for all $h \in \overline{B}(0, \tilde{L})$.
Let $x^* \in X$ be decision robust [weakly/·/strictly] efficient and suppose there exists $\tilde{x} \in X$ and $s \in S$ such that

$$f(\tilde{x}) \in \{f(x^*) - s\} - [\operatorname{int}(K)/K\backslash\{0\}/K].$$

Then, for each $z \in Z$ there exists $h \in \overline{B}(0, \tilde{L})$ such that

$$\begin{aligned} f(\tilde{x} + z) &= f(\tilde{x}) + h \\ &\in \{f(\tilde{x})\} + (\{s\} - K) \\ &\subseteq \{f(x^*) - s\} + \{s\} - [\operatorname{int}(K)/K\backslash\{0\}/K] \\ &= \{f(x^*)\} - [\operatorname{int}(K)/K\backslash\{0\}/K] \\ &\subseteq f_Z(x^*) - [\operatorname{int}(K)/K\backslash\{0\}/K], \end{aligned}$$

in contradiction to x^* being decision robust [weakly/·/strictly] efficient. □

5.3 Monotonic Objective Functions

In the following, we look at the special case of monotonically decreasing and increasing objective functions. We show that, in case of at least one increasing objective, one decreasing objective and a partially ordered feasible set, all solutions are decision robust efficient.
Throughout this section, we assume that $\mathbb{R}^k$ is partially ordered by the closed convex pointed solid cone $\mathbb{R}^k_+$ and that $\mathbb{R}^n$ is ordered correspondingly by the cone $\mathbb{R}^n_+$. First, we recall the definition of a monotonically increasing function on $\mathbb{R}^n$, see, e.g., [Jah11, Mie12].

Definition 31. *Let S be a nonempty subset of $Y \subseteq \mathbb{R}^n$. Then the function $h\colon Y \to \mathbb{R}$ is called*

(a) monotonically increasing *on S if for all $x, y \in S$ it holds*
$x \leqq y \quad \Rightarrow \quad h(x) \leq h(y).$

(b) strictly monotonically increasing *on S if for all $x, y \in S$ it holds*
$x < y \quad \Rightarrow \quad h(x) < h(y).$

(c) strongly monotonically increasing *on S if for all $x, y \in S$ it holds*
$x \leq y \quad \Rightarrow \quad h(x) < h(y).$

Similarly, f is called decreasing if $(-f)$ is increasing.

Note that every strongly increasing function is also strictly increasing and increasing, while strictly increasing functions are not necessarily increasing if $n \geq 2$. For $n = 1$, strict and strong monotonicity coincide and every strictly increasing function is also increasing.

Theorem 32. *Let $X \subseteq \Omega \subseteq \mathbb{R}^n$ and let $x \in X$ be such that for all $y \in X$*

$$x \neq y \qquad \Rightarrow \qquad x \; [\leq / < / \lneq] \; y \quad \text{or} \quad y \; [\leq / < / \lneq] \; x$$

and let $Z \subseteq \mathbb{R}^n$ have the property

$$z^1 \neq z^2 \qquad \Rightarrow \qquad z^1 \; [\leq / < / \lneq] \; z^2 \quad \text{or} \quad z^2 \; [\leq / < / \lneq] \; z^1$$

for all $z^1, z^2 \in Z$.
Let $k \geq 2$ and let $f \colon \mathbb{R}^n \to \mathbb{R}^k$ have the property that there exist $j, l \in \{1, \ldots, k\}$, $j \neq l$, such that

(a) $f_j \colon \mathbb{R}^n \to \mathbb{R}$ is [./strictly/strongly] increasing on X

(b) $f_l \colon \mathbb{R}^n \to \mathbb{R}$ is [./strictly/strongly] decreasing on X.

Then x is decision robust [weakly/strictly/strictly]–efficient.

Proof. Choose $y \in X \backslash \{x\}$ arbitrarily. Because Z is compact and totally ordered with respect to $\leqq$, there exist a minimal element $z^{min} \in Z$ and a maximal element $z^{max} \in Z$ due to Zorn's Lemma.
We distinguish two cases. First, we study the case $x \; [\leq / < / \lneq] \; y$. This leads to

$$\begin{aligned}
& x + z \quad [\leq / < / \lneq] \quad y + z \quad \leqq \quad y + z^{max} \qquad \forall z \in Z \\
\overset{(a)}{\Rightarrow} \quad & f_j(x + z) \quad [\leq / < / <] \quad f_j(y + z^{max}) \qquad \forall z \in Z \\
\Rightarrow \quad & f(y + z^{max}) \notin f_Z(x) - [\operatorname{int}(\mathbb{R}^k_+)/\mathbb{R}^k_+/\mathbb{R}^k_+] \\
\Rightarrow \quad & f_Z(y) \nsubseteq f_Z(x) - [\operatorname{int}(\mathbb{R}^k_+)/\mathbb{R}^k_+/\mathbb{R}^k_+].
\end{aligned}$$

Secondly, we study the alternative case $y \; [\leq / < / \lneq] \; x$. This leads to

$$\begin{aligned}
& y + z^{min} \quad [\leq / < / \lneq] \quad x + z^{min} \quad \leqq \quad x + z \qquad \forall z \in Z \\
\overset{(b)}{\Rightarrow} \quad & f_l(y + z^{min}) \quad [\geq / > / >] \quad f_l(x + z) \qquad \forall z \in Z \\
\Rightarrow \quad & f(y + z^{min}) \notin f_Z(x) - [\operatorname{int}(\mathbb{R}^k_+)/\mathbb{R}^k_+/\mathbb{R}^k_+] \\
\Rightarrow \quad & f_Z(y) \nsubseteq f_Z(x) - [\operatorname{int}(\mathbb{R}^k_+)/\mathbb{R}^k_+/\mathbb{R}^k_+].
\end{aligned}$$

In summary, there is no $y \in X \backslash \{x\}$ such that $f_Z(y) \subseteq f_Z(x) - [\operatorname{int}(\mathbb{R}^k_+)/\mathbb{R}^k_+/\mathbb{R}^k_+]$. Therefore, x is decision robust [weakly/strictly/strictly]–efficient. □

For an illustarion of Theorem 32, we refer to Example 34 in Appendix A.
A direct consequence of Theorem 32 is the next Corollary.

Corollary 33. *Let $Z \subseteq \mathbb{R}$ and $X \subseteq \Omega \subseteq \mathbb{R}$ be compact intervals and $f \colon \mathbb{R} \to \mathbb{R}^k$ be such that $f_1 \colon \mathbb{R} \to \mathbb{R}$ is [./strictly] increasing and $f_2 \colon \mathbb{R} \to \mathbb{R}$ is [./strictly] decreasing.*
Then all $x \in X$ are decision robust [weakly/strictly]–efficient.

In particular, Corollary 33 can be applied whenever $f \colon I \to \mathbb{R}^k$ is a curve with at least one increasing and one decreasing objective, where $I \subseteq \mathbb{R}$ is an interval.

6 Conclusions

A robustness concept for decision uncertainty in multiobjective optimization is introduced and investigated. The corresponding decision robust counterpart (RC) is introduced as a set-valued optimization problem. Semicontinuity and convexity of the set-valued objective functions in (RC) and their supremal sets are proven under the assumption that the vector-valued objective function in ($\mathcal{P}$) is continuous.
Results are obtained that lead to the determination of decision robust efficient solutions by methods from deterministic multiobjective optimization for three types of objective functions. First, in case of linear objective functions, solving (RC) is equivalent to solving a deterministic linear multiobjective problem. Second, a necessary condition for decision robust efficient solutions in case of Lipschitz objective functions is presented. Third, a sufficient condition for decision robust efficient solutions in case of monotonic objective functions is obtained.
Decision robust efficiency and (RC) are consistent with the generalization of the supremum to supremal sets in set-valued optimization. Furthermore, decision robust efficiency is closely related to existing robustness concepts in the literature in two ways. On the one hand, it is a generalization of minmax robustness approaches for decision uncertainty in single-objective optimization and on the other hand, it can be regarded as a special case of minmax robust efficiency for parameter uncertainty in multiobjective optimization.

Following the presented theoretical results, there are different options for further research. The results for the different classes of objective functions indicate that extended study of problems with specific structure might lead to specific solution techniques for various applications. Solution techniques and numerical evaluations of the two applications presented in the introduction are currently under research, see [CKGS16] for a specific application.
Additionally, the problem description of decision uncertainty can be extended by allowing the perturbation set Z to depend on the solution. There are various applications for problems with variable perturbation sets, including perurbation sets whose size depend on a percentage of the solution aimed at. It will be interesting to see, which of our results can be extended to that case.

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Appendix A An Example for Monotonic Objective Functions

The following example illustrates Theorem 32 and shows that, after having excluded a specific part of the decision robust feasible set, all decision robust efficient solutions can be determined by Theorem 32.

Example 34. *Let the feasible set, the perturbation set and the objective function be*

$$\Omega = \operatorname{conv}\left\{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1.5\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1.5\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\}, \quad Z = [0, 0.5] \times \{0\}$$

and

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad x \mapsto \begin{pmatrix} -(x_1+x_2) \\ x_1^2+x_2^2 \end{pmatrix}.$$

Then f_1 is strongly decreasing and f_2 is strongly increasing on $\mathbb{R}^2_+$ and therefore also on the decision robust feasible subset X of Ω, which is illustrated in Figure 4.

Figure 4: The feasible set Ω and the decision robust feasible set X in Example 34.

We show next that the set of decision robust [·/strictly] efficient solutions is

$$Y = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

We first show that the elements of $X \backslash Y$ can not be decision robust efficient.
For all $s \in (0,2)$ we define the set $L_s = \{x \in X \mid x_1 + x_2 = s\}$ and $y^s = \frac{1}{2} \cdot \begin{pmatrix}s\\s\end{pmatrix} \in L_s$, which is illustrated in Figure 5.
Let $s \in (0,2)$ and let $x \in L_s \backslash \{y^s\}$. Then $x_1 > x_2$ and there exists $0 < t \leq \frac{s}{2}$ such that $x_1 = \frac{s}{2} + t$ and $x_2 = \frac{s}{2} - t$. For each $z \in Z$ it holds

$$f_1(x+z) = -(x_1 + x_2 + z_1) = -(s + z_1) = -(y_1^s + y_2^s + z_1) = f_1(y^s + z)$$

as well as

$$f_2(y^s + z) = \frac{s^2}{2} + s \cdot z_1 + z_1^2$$

and

$$f_2(x+z) = \underbrace{\frac{s^2}{2} + s \cdot z_1 + z_1^2}_{=f_2(y^s+z)} + \underbrace{2t^2}_{>0} + \underbrace{2tz_1}_{\geq 0} > f_2(y^s + z).$$

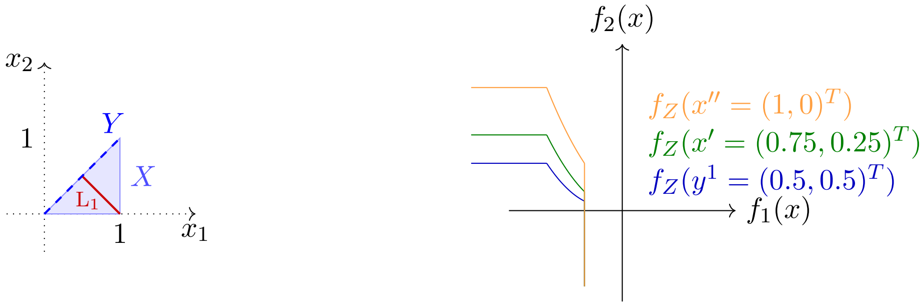

Figure 5: Illustration of Example 34. The sets Y and L_s for $s = 1$ are illustrated on the left hand side. On the right hand side, inclusions of the sets f_Z for different elements of L_1 are displayed.

Consequently, $f(y^s + z) \leq f(x+z)$ for all $z \in Z$. Since $x \in L_s$ was arbitrarily chosen, we have for all $x \in L_s \backslash \{y^s\}$

$$f_Z(y^s) \subseteq f_Z(x) - \mathbb{R}_+^2 \backslash \{0\},$$

which is visualized in Figure 5. Therefore, the set of decision robust [$\cdot$/strictly] efficient solutions is a subset of Y*. Furthermore, because of the transitivity of* $\leqq$*, every element in* Y *that is not decision robust [$\cdot$/strictly] efficient is dominated by another element of* Y*. Since* Y *and* Z *are totally ordered with respect to* $\leqq$*,* f_1 *is strongly de- and* f_2 *is strongly increasing, Theorem 32 can be applied, leading to the conclusion that all* $y \in Y$ *are decision robust [$\cdot$/strictly] efficient.*