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LINEAR RELATIONS AND THE KRONECKER CANONICAL FORM

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Abstract. We show that the Kronecker canonical form (which is a canonical decomposition for pairs of matrices) is the representation of a linear relation in a finite dimensional space. This provides a new geometric view upon the Kronecker canonical form. Each of the four entries of the Kronecker canonical form has a natural meaning for the linear relation which it represents. These four entries represent the Jordan chains at finite eigenvalues, the Jordan chains at infinity, the so-called singular chains and the multi-shift part. Or, to state it more concise: *For linear relations the Kronecker canonical form is the analogue of the Jordan canonical form for matrices.*

Key words. Linear relations, Jordan chains, Kronecker canonical form, Wong sequences

AMS subject classifications. Primary 15A21, 47A06; Secondary 15A22, 47A15.

1. Introduction. Solutions of linear ordinary differential equations of the form

$$\dot{x}(t) = Ax(t)$$

can be completely characterized by the eigenvalues and generalized eigenvectors of the matrix A, i.e., by the Jordan canonical form of A. If E is an invertible matrix, the same applies to the equation

$$E\dot{x}(t) = Ax(t), \tag{1.1}$$

and the solutions are encoded in the Jordan canonical form of

$$E^{-1}A. \tag{1.2}$$

The situation is more challenging when E is not invertible. Then (1.1) may contain purely algebraic equations; for instance if E has a zero row, then the corresponding equation does not contain any derivatives. Thus (1.1) is called a *differential-algebraic equation* (DAE), see e.g. [10, 18, 20]. A characterization of the solutions of (1.1) (see e.g. [6]) is done via the *Kronecker canonical form* (*KCF*). The KCF is a canonical form for a matrix pair (E, A) (often considered in the form of a matrix pencil $sE - A$) and, hence, a generalization of the Jordan canonical form. It has its origin in [17], see also [15].

But even in the case of a non-invertible matrix E, the expression (1.2) can be given a meaning. For this we use the theory of linear relations (or, what is the same, of multi-valued mappings), see [1, 26] for instance. Each matrix (or linear mapping) is considered via its graph as a subspace in $\mathbb{C}^n \times \mathbb{C}^n$. Addition and multiplication of two subspaces are defined in analogy to the addition and multiplication of two linear mappings, for details see Section 2. In the sense of linear relations, the inverse E^{-1} of a non-invertible matrix E is given as the subspace of all tuples $\left(\begin{smallmatrix} Ex \\ x \end{smallmatrix}\right)$ in $\mathbb{C}^n \times \mathbb{C}^n$. Then expression (1.2) has a natural meaning,

$$E^{-1}A = \{ \ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \ \mid Ax = Ey \ \},$$

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which was already studied in [4, 5]. An eigenvector at $\lambda \in \mathbb{C}$ of $E^{-1}A$ is a tuple of the form $(x, \lambda x) \in E^{-1}A$, $x \neq 0$, and thus satisfies $Ax = \lambda Ex$. It follows that the (point) spectrum of $E^{-1}A$ and the (point) spectrum of the matrix pencil $sE - A$ coincide. This shows a deep connection between the matrix pair (E, A) and the linear relation $E^{-1}A$.

It is the aim of the present paper to explore this connection. While the connection between the (point) spectra (and the Jordan chains) of $E^{-1}A$ and the matrix pencil $sE - A$ is quite obvious and in a certain sense due to the "right" definition of Jordan chains of $E^{-1}A$, another aspect is more stunning and the main objective of this paper: The connection of the KCF of a matrix pair and the linear relation $\mathcal{A}$ represented with the help of these matrices. For this we show in Section 3 that an arbitrary linear relation $\mathcal{A}$ in $\mathbb{C}^n \times \mathbb{C}^n$can be represented with matrices A, E, F, G in the following way

$$\mathcal{A} = GF^{-1} = E^{-1}A. \tag{1.3}$$

We restrict ourselves to the representation $\mathcal{A} = GF^{-1}$ and show that the KCF of the matrix pair (F, G) is indeed a canonical form for the linear relation $\mathcal{A}$. We show that each of the four entries of the KCF has a natural meaning for the linear relation $\mathcal{A}$. These are the Jordan chains at finite eigenvalues, the Jordan chains at infinity, the singular chains and the multishifts. This is the main result of the present paper:

The Kronecker canonical form of a matrix pair (F, G)
is the canonical form for the linear relation $\mathcal{A} = GF^{-1}$.

This provides new geometric insight for the entries of the KCF. Moreover, as a byproduct, we obtain a decomposition result for linear relations, which completes the considerations in [24].

The present paper can be viewed as the link between two different fields in linear algebra: linear relations and matrix pairs (or matrix pencils).

The paper is organized as follows: In Section 2 we give a short but self-contained introduction to the theory of linear relations which covers all relevant notions like Jordan chains, singular chains and multishifts. In particular, for readers not so familiar with linear relations, we illustrate these concepts with simple examples. In Section 3 we show that any linear relation has a representation of the form (1.3) which is called image and kernel representation, respectively. The KCF is recalled in Section 4 and its block entries are related to some properties of linear relations. Using the KCF of the image representation of a linear relation, we obtain a full characterization of a linear relation in terms of its Jordan chains at finite eigenvalues, its Jordan chains at infinity, its singular chains and its multi-shift part. As a byproduct, a canonical decomposition for linear relations is shown. In Section 5 we recall the notion of Wong sequences and exploit them to derive representations for the root and Jordan chain manifolds of a linear relation.

2. Preliminaries: Linear relations. Let $\mathfrak{H}$ and $\mathfrak{G}$ be linear spaces. A *linear relation* $\mathcal{A}$ in $\mathfrak{H} \times \mathfrak{G}$ is a (linear) subspace of $\mathfrak{H} \times \mathfrak{G}$. A linear relation $\mathcal{A}$ is usually viewed as a multivalued mapping. We restrict ourselves to finite dimensional spaces $\mathfrak{H} = \mathbb{C}^n$ and $\mathfrak{G} = \mathbb{C}^m$. Moreover, if $\mathfrak{H}$ and $\mathfrak{G}$ coincide, i.e., if $\mathfrak{H} = \mathfrak{G} = \mathbb{C}^n$, then we briefly say that $\mathcal{A}$ is a *linear relation in* $\mathbb{C}^n$ instead of $\mathbb{C}^n \times \mathbb{C}^n$. Most of the definitions below remain valid for infinite-dimensional spaces, see e.g. [13].

Linear mappings (e.g. given via a matrix) are always identified with linear relations via their graphs. For the general study of linear relations we refer to the monographs [12, 16], see also [1, 26].

It is usual to write the elements of $\mathcal{A}$ as (x, y) for $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$. In the older literature it is also usual to write the elements of $\mathcal{A}$ as column vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Here, we agree not to distinguish between these two notions.

By $\operatorname{dom} \mathcal{A}$ and $\operatorname{ran} \mathcal{A}$ we denote the *domain* and the *range* of a linear relation $\mathcal{A}$ in $\mathbb{C}^n \times \mathbb{C}^m$,

$$\begin{aligned} &\operatorname{dom} \mathcal{A} = \{\ x \in \mathbb{C}^n \ \mid \exists\, y \in \mathbb{C}^m :\ (x, y) \in \mathcal{A}\ \} \\ \text{and}\quad &\operatorname{ran} \mathcal{A} = \{\ y \in \mathbb{C}^m \ \mid \exists\, x \in \mathbb{C}^n :\ (x, y) \in \mathcal{A}\ \}. \end{aligned}$$

Furthermore, $\ker \mathcal{A}$ and $\operatorname{mul} \mathcal{A}$ denote the *kernel* and the *multivalued* part of $\mathcal{A}$,

$$\ker \mathcal{A} = \{\ x \in \mathbb{C}^n \ \mid (x, 0) \in \mathcal{A}\ \} \quad \text{and} \quad \operatorname{mul} \mathcal{A} = \{\ y \in \mathbb{C}^m \ \mid (0, y) \in \mathcal{A}\ \}.$$

A linear relation $\mathcal{A}$ is the graph of an operator if, and only if, $\operatorname{mul} \mathcal{A} = \{0\}$. The inverse $\mathcal{A}^{-1}$ is given by

$$\mathcal{A}^{-1} = \{\ (y, x) \in \mathbb{C}^m \times \mathbb{C}^n \ \mid (x, y) \in \mathcal{A}\ \}. \tag{2.1}$$

For relations $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{C}^n \times \mathbb{C}^m$ the *operator-like sum* $\mathcal{A} + \mathcal{B}$ is the relation defined by

$$\mathcal{A} + \mathcal{B} = \{\ (x, y + z) \in \mathbb{C}^n \times \mathbb{C}^m \ \mid (x, y) \in \mathcal{A}, (x, z) \in \mathcal{B}\ \}.$$

and for $\lambda \in \mathbb{C}$ the relation $\lambda\mathcal{A}$ is defined by

$$\lambda\mathcal{A} = \{\ (x, \lambda y) \in \mathbb{C}^n \times \mathbb{C}^m \ \mid (x, y) \in \mathcal{A}\ \},$$

We illustrate the above definitions by a simple example.

Example 2.1. *Let e_1, e_2 be the two linearly independent unit vectors in $\mathbb{C}^2$. Define*

$$\begin{aligned} &\mathcal{A} := \operatorname{span}\{(0, e_1), (e_1, e_2), (e_2, 0)\}, && \mathcal{B} := \operatorname{span}\{(e_2, e_1), (e_1, 0)\}, \\ &\mathcal{C} := \operatorname{span}\{(e_1, 2e_1)\} && \textit{and} \quad \mathcal{D} := \operatorname{span}\{(e_1, e_2)\}, \end{aligned}$$

which are subspaces in $\mathbb{C}^2 \times \mathbb{C}^2$, and hence linear relations in $\mathbb{C}^2$. We have

$$\begin{array}{llll} \operatorname{dom} \mathcal{A} = \mathbb{C}^2 & \operatorname{dom} \mathcal{B} = \mathbb{C}^2 & \operatorname{dom} \mathcal{C} = \operatorname{span}\{e_1\} & \operatorname{dom} \mathcal{D} = \operatorname{span}\{e_1\} \\ \operatorname{ran} \mathcal{A} = \mathbb{C}^2 & \operatorname{ran} \mathcal{B} = \operatorname{span}\{e_1\} & \operatorname{ran} \mathcal{C} = \operatorname{span}\{e_1\} & \operatorname{ran} \mathcal{D} = \operatorname{span}\{e_2\} \\ \ker \mathcal{A} = \operatorname{span}\{e_2\} & \ker \mathcal{B} = \operatorname{span}\{e_1\} & \ker \mathcal{C} = \{0\} & \ker \mathcal{D} = \{0\} \\ \operatorname{mul} \mathcal{A} = \operatorname{span}\{e_1\} & \operatorname{mul} \mathcal{B} = \{0\} & \operatorname{mul} \mathcal{C} = \{0\} & \operatorname{mul} \mathcal{D} = \{0\}. \end{array}$$

Moreover, $\mathcal{A}$ is not the graph of an operator whereas $\mathcal{B}$ is the graph of the linear mapping induced by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. The relations $\mathcal{C}$ and $\mathcal{D}$ are the graphs of operators which are defined only on the subset $\operatorname{span}\{e_1\} \subseteq \mathbb{C}^2$ and map e_1 to $2e_1$, resp. e_1 to e_2. The inverses are given by

$$\begin{aligned} &\mathcal{A}^{-1} := \operatorname{span}\{(0, e_2), (e_2, e_1), (e_1, 0)\}, && \mathcal{B}^{-1} := \operatorname{span}\{(0, e_1)(e_1, e_2)\}, \\ &\mathcal{C}^{-1} := \operatorname{span}\{(e_1, \tfrac{1}{2}e_1)\} && \textit{and} \quad \mathcal{D}^{-1} := \operatorname{span}\{(e_2, e_1)\}, \end{aligned}$$

and, the (operator-like) sum of $\mathcal{A}$ and $\mathcal{C}$, the sum of $\mathcal{C}$ and $5\mathcal{D}$ and the sum of $\mathcal{C}$ and $\mathcal{D}^{-1}$ are given by

$$\begin{aligned}\mathcal{A}+\mathcal{C} &= \operatorname{span}\{(0,e_1),(e_1,2e_1+e_2)\},\\ \mathcal{C}+5\mathcal{D} &= \operatorname{span}\{(e_1,2e_1+5e_2)\}, \quad \textit{and} \quad \mathcal{C}+\mathcal{D}^{-1}=\{0\}.\end{aligned}$$

For relations $\mathcal{A}$ in $\mathbb{C}^n\times\mathbb{C}^m$ and $\mathcal{B}$ in $\mathbb{C}^p\times\mathbb{C}^n$ the *product* $\mathcal{AB}$ is defined as the relation

$$\mathcal{AB}=\{\ (x,y)\in\mathbb{C}^p\times\mathbb{C}^m \ \mid\ (x,z)\in\mathcal{B},(z,y)\in\mathcal{A} \text{ for some } z\in\mathbb{C}^n\ \}. \tag{2.2}$$

We recall the notion of eigenvalues, root manifolds and point spectrum of linear relations. Therefore, let $\mathcal{A}$ be a linear relation in $\mathbb{C}^n$, i.e., $m=n$. Then, with the notion of operator-like sum from above, the expression $\mathcal{A}-\lambda$ stands for $\mathcal{A}-\lambda I$, where I is the identity operator on $\mathbb{C}^n$,

$$\mathcal{A}-\lambda=\{\ (x,y-\lambda x)\in\mathbb{C}^n\times\mathbb{C}^n\ \mid (x,y)\in\mathcal{A}\ \}.$$

A point $\lambda\in\mathbb{C}$ is called an *eigenvalue* of $\mathcal{A}$ if $\ker(\mathcal{A}-\lambda)\neq\{0\}$ and ∞ is called an *eigenvalue* of $\mathcal{A}$ if $\operatorname{mul}\mathcal{A}\neq\{0\}$. The *point spectrum* $\sigma_p(\mathcal{A})$ is the set of all eigenvalues $\lambda\in\mathbb{C}\cup\{\infty\}$ of $\mathcal{A}$. The *root manifolds* $\mathcal{R}_\lambda(\mathcal{A})$ and $\mathcal{R}_\infty(\mathcal{A})$ are defined by

$$\mathcal{R}_\lambda(\mathcal{A}):=\bigcup_{i\in\mathbb{N}}\ker(\mathcal{A}-\lambda)^i, \qquad \mathcal{R}_\infty(\mathcal{A}):=\bigcup_{i\in\mathbb{N}}\operatorname{mul}\mathcal{A}^i.$$

It is clear that $\ker(\mathcal{A}-\lambda)^k\subseteq\ker(\mathcal{A}-\lambda)^{k+1}$ and $\operatorname{mul}\mathcal{A}^k\subseteq\operatorname{mul}\mathcal{A}^{k+1}$ for any $k\in\mathbb{N}$. By [25, Lemma 3.4] and the fact that $\mathcal{A}$ is a linear relation in a finite dimensional space $\mathbb{C}^n$, there exists a natural number $n_0\leq n$ such that $\ker(\mathcal{A}-\lambda)^k=\ker(\mathcal{A}-\lambda)^{k+1}$ for all $k\geq n_0$. A similar statement holds for $\operatorname{mul}\mathcal{A}^k$. For $x\in\ker(\mathcal{A}-\lambda)^l\setminus\ker(\mathcal{A}-\lambda)^{l-1}$, $l\geq1$, we find (cf. (2.2)) $x_1,\ldots,x_{l-1}\in\mathbb{C}^n$ such that

$$(x,x_{l-1}),(x_{l-1},x_{l-2}),\ldots,(x_2,x_1),(x_1,0)\in\mathcal{A}-\lambda$$

or, equivalently,

$$(x,x_{l-1}+\lambda x),(x_{l-1},x_{l-2}+\lambda x_{l-1}),\ldots,(x_2,x_1+\lambda x_2),(x_1,\lambda x_1)\in\mathcal{A}. \tag{2.3}$$

The vectors $x_1,\ldots,x_{l-1},x\in\mathbb{C}^n$ are linearly independent and (2.3) is called a *Jordan chain at* λ, see [24, Lemma 2.1]. Moreover, we say that it is a *Jordan chain of length* l. Similarly, for $y\in\operatorname{mul}\mathcal{A}^m\setminus\operatorname{mul}\mathcal{A}^{m-1}$, $m\geq1$, there are $y_1,\ldots,y_{m-1}\in\mathbb{C}^n$ such that

$$(0,y_1),(y_1,y_2),\ldots,(y_{m-2},y_{m-1}),(y_{m-1},y)\in\mathcal{A}. \tag{2.4}$$

The vectors $y_1,\ldots,y_{n-1},y\in\mathbb{C}^n$ are linearly independent and (2.4) is called a *Jordan chain at* ∞ (cf. [24, Lemma 2.1]). Moreover, we say that it is a *Jordan chain of length* m. Obviously, a chain of the form (2.4) is a Jordan chain at ∞ (of length m) if, and only if,

$$(y,y_{m-1}),(y_{m-1},y_{m-2}),\ldots,(y_2,y_1),(y_1,0)$$

is a Jordan chain of $\mathcal{A}^{-1}$ at 0 (of length m).

EXAMPLE 2.2. *For the linear relations $\mathcal{B}$ and $\mathcal{C}$ from Example 2.1 we have*

$$\sigma_p(\mathcal{B}) = \{0\}, \quad \text{and} \quad \sigma_p(\mathcal{C}) = \{2\}.$$

In addition, for the linear relation $\mathcal{B}$ the chain $(e_2, e_1), (e_1, 0)$ is a Jordan chain of length two at 0 and for $\mathcal{C}$ the chain $(e_1, 2e_1)$ is a Jordan chain of length one at 2 with

$$\mathcal{R}_0(\mathcal{B}) = \mathbb{C}^2, \quad \text{and} \quad \mathcal{R}_2(\mathcal{C}) = \operatorname{span}\{e_1\}.$$

We define the *Jordan chain manifold* $\mathcal{R}_J(\mathcal{A})$ as the linear span of all root manifolds,

$$\mathcal{R}_J(\mathcal{A}) := \operatorname{span}\ \{\ \mathcal{R}_\lambda(\mathcal{A})\ \mid \lambda \in \sigma_p(\mathcal{A})\ \},$$

and the *finite Jordan chain manifold* $\mathcal{R}_f(\mathcal{A})$ as the linear span of all root manifolds $\mathcal{R}_\lambda(\mathcal{A})$ with $\lambda \neq \infty$,

$$\mathcal{R}_f(\mathcal{A}) := \operatorname{span}\ \{\ \mathcal{R}_\lambda(\mathcal{A})\ \mid \lambda \in \mathbb{C}\ \}.$$

Obviously, if $\mathcal{A}$ is the graph of an operator in the finite dimensional space $\mathbb{C}^n$, then $\mathcal{R}_f(\mathcal{A}) = \mathcal{R}_J(\mathcal{A}) = \mathbb{C}^n$. The converse is not true in general, which is illustrated by the following example.

EXAMPLE 2.3. *The linear relation $\mathcal{A}$ from Example 2.1 is not the graph of an operator. Its chain $(e_1, e_2), (e_2, 0)$ is a Jordan chain at 0 and $(0, e_1), (e_1, e_2)$ is a Jordan chain at ∞. Therefore $0, \infty \in \sigma_p(\mathcal{A})$. We have $\mathcal{R}_0(\mathcal{A}) = \mathbb{C}^2$ and $\mathcal{R}_\infty(\mathcal{A}) = \mathbb{C}^2$, thus*

$$\mathcal{R}_f(\mathcal{A}) = \mathcal{R}_J(\mathcal{A}) = \mathbb{C}^2.$$

Moreover, we obtain

$$\mathcal{R}_0(\mathcal{A}) \cap \mathcal{R}_\infty(\mathcal{A}) \neq \{0\}. \tag{2.5}$$

In the following example we compute the point spectrum of the linear relation $\mathcal{A}$ from Example 2.1.

EXAMPLE 2.4. *From Example 2.3 we conclude $0, \infty \in \sigma_p(\mathcal{A})$. For $\lambda \in \mathbb{C} \setminus \{0\}$ we have*

$$(e_1 + \lambda^{-1} e_2, \lambda e_1 + e_2) = \lambda(0, e_1) + (e_1, e_2) + \lambda^{-1}(e_2, 0) \in \mathcal{A},$$

and, hence, $e_1 + \lambda^{-1} e_2 \in \ker(\mathcal{A} - \lambda)$. This implies $\lambda \in \sigma_p(\mathcal{A})$ and we conclude

$$\sigma_p(\mathcal{A}) = \mathbb{C} \cup \{\infty\}.$$

It is well-known [24, Proposition 3.2 and Theorem 4.4] that it is the property in (2.5) which is equivalent to $\sigma_p(\mathcal{A}) = \mathbb{C} \cup \{\infty\}$. We recall this important fact from [24] in the following lemma.

LEMMA 2.5. *Let $\mathcal{A}$ be a linear relation in $\mathbb{C}^n$. Then $\sigma_p(\mathcal{A}) = \mathbb{C} \cup \{\infty\}$ if, and only if, $\mathcal{R}_0(\mathcal{A}) \cap \mathcal{R}_\infty(\mathcal{A}) \neq \{0\}$.*

In the sequel the set $\mathcal{R}_0(\mathcal{A}) \cap \mathcal{R}_\infty(\mathcal{A})$ is useful in the study of linear relations and it is called the *singular chain manifold*,

$$\mathcal{R}_c(\mathcal{A}) := \mathcal{R}_0(\mathcal{A}) \cap \mathcal{R}_\infty(\mathcal{A}).$$

Moreover, $\mathcal{A}$ is called *completely singular*, if

$$\mathcal{A} = \mathcal{A} \cap (\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A})). \tag{2.6}$$

For any $x \in \mathcal{R}_c(\mathcal{A}) \setminus \{0\}$ there exist linearly independent $x_1, \ldots, x_k \in \mathbb{C}^n$ (see [24, Lemma 3.1]) such that $x = x_j$ for some $j \in \{1, \ldots, k\}$ and

$$(0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k), (x_k, 0) \in \mathcal{A}. \tag{2.7}$$

A chain of this form is called a *singular chain*. Moreover, we say that it is a *singular chain of length k*. A linear relation $\mathcal{A}$ is completely singular if, and only if, it is the span of singular chains of the form (2.7), see [24, Section 7].

If $\mathcal{R}_c(\mathcal{A}) = \{0\}$ (and hence $\sigma_p(\mathcal{A}) \neq \mathbb{C} \cup \{\infty\}$ by Lemma 2.5), then (see e.g. [24, Theorem 4.6]) the number of eigenvalues in $\sigma_p(\mathcal{A})$ is bounded by the dimension of the linear subspace $\mathcal{A}$. If, in addition, the linear relation $\mathcal{A}$ consists only of Jordan chains at the (finitely many) eigenvalues, then we call $\mathcal{A}$ a *Jordan relation*.

Apart from linear relations in $\mathbb{C}^n$ with finite point spectrum and with point spectrum equal to $\mathbb{C} \cup \{\infty\}$ there exist also linear relations with no point spectrum.

EXAMPLE 2.6. *For the linear relation $\mathcal{D}$ from Example 2.1 we have* $\operatorname{mul} \mathcal{D} = \{0\}$ *and hence $\infty \notin \sigma_p(\mathcal{D})$. Moreover, for $\lambda \in \mathbb{C}$,*

$$\mathcal{D} - \lambda = \operatorname{span}\{(e_1, e_2 - \lambda e_1)\}.$$

As e_1, e_2 are linearly independent unit vectors, we have $e_2 - \lambda e_1 \neq 0$ for all $\lambda \in \mathbb{C}$, thus $\lambda \notin \sigma_p(\mathcal{A})$ which implies

$$\sigma_p(\mathcal{A}) = \emptyset. \tag{2.8}$$

We use property (2.8) for the definition of a subclass of all linear relations: A linear relation $\mathcal{A}$ in $\mathbb{C}^n$ with $\sigma_p(\mathcal{A}) = \emptyset$ is called a *multishift* (see, e.g., [24, Section 8]).

3. Image and kernel representations of linear relations. In this section we derive two representations for a linear relation $\mathcal{A}$ in $\mathbb{C}^n$. It is well-known that if there exists a complex number μ with $\operatorname{ran}(\mathcal{A} - \mu) = \mathbb{C}^n$ and $\ker(\mathcal{A} - \mu) = \{0\}$, then the inverse of $\mathcal{A} - \mu$ is a linear mapping defined on $\mathbb{C}^n$, that is $(\mathcal{A} - \mu)^{-1}$ can be identified with a matrix in $\mathbb{C}^{n \times n}$ and $\mathcal{A}$ admits a representation of the form (see, e.g., [14, Proposition 2.2])

$$\mathcal{A} = \left\{ \ ((\mathcal{A} - \mu)^{-1} x, (I + \mu(\mathcal{A} - \mu)^{-1})x) \in \mathbb{C}^n \times \mathbb{C}^n \ \middle| \ x \in \mathbb{C}^n \ \right\}. \tag{3.1}$$

Definition 3.1. *A representation of a linear relation $\mathcal{A}$ in $\mathbb{C}^n$ of the form*

$$\mathcal{A} = \left\{ (Fz, Gz) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; z \in \mathbb{C}^d \right\} = \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix}$$

with $d \in \mathbb{N}$ and matrices $F, G \in \mathbb{C}^{n \times d}$ is called an image representation *of $\mathcal{A}$. A representation of $\mathcal{A}$ of the form*

$$\mathcal{A} = \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid Ax = Ey \} = \ker [A, -E]$$

with matrices $A, E \in \mathbb{C}^{r \times n}$ and $r \in \mathbb{N}$ is called a kernel representation *of $\mathcal{A}$.*

Observe that (3.1) is a special case of an image representation where $d = n$.

Let us consider a linear relation $\mathcal{A}$ in $\mathbb{C}^n$ with image and kernel representation as in Definition 3.1. As usual, we identify the matrices F and G with the corresponding relations via their graphs,

$$F = \left\{ (x, Fx) \in \mathbb{C}^d \times \mathbb{C}^n \;\middle|\; x \in \mathbb{C}^d \right\} \quad \text{and} \quad G = \left\{ (y, Gy) \in \mathbb{C}^d \times \mathbb{C}^n \;\middle|\; y \in \mathbb{C}^d \right\}.$$

We have, see (2.1), $F^{-1} = \left\{ (Fy, y) \in \mathbb{C}^n \times \mathbb{C}^d \;\middle|\; y \in \mathbb{C}^d \right\}$ and with (2.2)

$$\begin{aligned} GF^{-1} &= \left\{ (x, y) \;\middle|\; (x, z) \in F^{-1}, (z, y) \in G \text{ for some } z \in \mathbb{C}^d \right\} \\ &= \left\{ (x, Gz) \;\middle|\; (z, x) \in F, (z, Gz) \in G \text{ for some } z \in \mathbb{C}^d \right\} \\ &= \left\{ (Fz, Gz) \;\middle|\; z \in \mathbb{C}^d \right\} = \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} = \mathcal{A}. \end{aligned} \tag{3.2}$$

Similarly, if we identify A and E with the corresponding relations $A = \{ (x, Ax) \in \mathbb{C}^n \times \mathbb{C}^r \mid x \in \mathbb{C}^n \}$ and $E = \{ (y, Ey) \in \mathbb{C}^n \times \mathbb{C}^r \mid y \in \mathbb{C}^n \}$, then, with $E^{-1} = \{ (Ey, y) \in \mathbb{C}^r \times \mathbb{C}^n \mid y \in \mathbb{C}^n \}$ and with (2.2), we obtain

$$\begin{aligned} E^{-1}A &= \left\{ (x, y) \;\middle|\; (x, z) \in A, (z, y) \in E^{-1} \text{ for some } z \in \mathbb{C}^r \right\} \\ &= \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid Ax = Ey \} = \ker [A, -E] = \mathcal{A}. \end{aligned} \tag{3.3}$$

We have thus proved the following result.

Lemma 3.2. *Let $\mathcal{A}$ be a linear relation in $\mathbb{C}^n$ with image and kernel representation as in Definition 3.1. Then we have*

$$\mathcal{A} = GF^{-1} = E^{-1}A.$$

In the following we show that for every relation in $\mathbb{C}^n$ an image and a kernel representation exist.

Theorem 3.3. *Let $\mathcal{A}$ be a linear relation in $\mathbb{C}^n$ with $\dim \mathcal{A} = d$. Then there exist matrices $F, G \in \mathbb{C}^{n \times d}$ with*

$$\operatorname{rk} \begin{bmatrix} F \\ G \end{bmatrix} = d \tag{3.4}$$

such that

$$\mathcal{A} = \left\{ (Fz, Gz) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; z \in \mathbb{C}^d \right\} = \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} = GF^{-1}. \tag{3.5}$$

Moreover, for $r = 2n - d$ *there exist matrices* $A, E \in \mathbb{C}^{r\times n}$ *with*

$$\operatorname{rk}[A, E] = r$$

such that

$$\mathcal{A} = \{\ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \ \mid Ax = Ey\ \} = \ker[A, -E] = E^{-1}A. \tag{3.6}$$

Proof. The third equality in (3.5) and in (3.6) follows from Lemma 3.2. We have $\mathcal{A} \oplus \mathcal{A}^{\perp} = \mathbb{C}^n \times \mathbb{C}^n$ and $\dim \mathcal{A}^{\perp} = r$. Therefore, we find vector space isomorphisms

$$V : \mathbb{C}^d \to \mathcal{A} \quad \text{and} \quad W : \mathbb{C}^r \to \mathcal{A}^{\perp}. \tag{3.7}$$

Denote by P_1 and P_2 the orthogonal projection in $\mathbb{C}^n \times \mathbb{C}^n$ onto the first and second component, respectively. Then we obtain

$$\mathcal{A} = \operatorname{ran} \begin{bmatrix} P_1V \\ P_2V \end{bmatrix} \quad \text{and} \quad \mathcal{A}^{\perp} = \operatorname{ran} \begin{bmatrix} P_1W \\ P_2W \end{bmatrix} \tag{3.8}$$

and (3.5) is shown. In order to show (3.6) we continue with

$$\begin{aligned} \mathcal{A} = \left(\operatorname{ran} \begin{bmatrix} P_1W \\ P_2W \end{bmatrix}\right)^{\perp} &= \{\ (y, z) \in \mathbb{C}^n \times \mathbb{C}^n \ \mid \forall x \in \mathbb{C}^r : \ y^*P_1Wx + z^*P_2Wx = 0\ \} \\ &= \ker[(P_1W)^*, (P_2W)^*] = \ker[W^*P_1, W^*P_2]. \end{aligned}$$

□

The kernel representation (3.6) of a linear relation $\mathcal{A}$ was already considered in [4, 5] using the notation

$$E\backslash A := \{\ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \ \mid Ax = Ey\ \}.$$

However, Lemma 3.2 and Theorem 3.3 show

$$E\backslash A = E^{-1}A.$$

Remark 3.4. *It seems natural to single out the cases when* $\mathcal{A}$ *allows an image representation (a kernel representation) with square matrices* F *and* G *(*A *and* E*, respectively). Obviously, if* $\dim \mathcal{A} \le n$*, then we can choose in* (3.7) *a mapping* $V : \mathbb{C}^n \to \mathcal{A}$ *such that* V *is surjective but not necessarily injective. Note that in this case* V *is an isomorphism if, and only if,* $\dim \mathcal{A} = n$*. Then we obtain as in* (3.8) *an image representation with square matrices. Conversely, if* $\mathcal{A}$ *has an image representaion with square matrices* $F, G \in \mathbb{C}^{n\times n}$*, then*

$$\dim \mathcal{A} = \dim \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} \le n.$$

Therefore there exists an image representation of $\mathcal{A}$ *with square matrices if, and only if,* $\dim \mathcal{A} \le n$*. By similar arguments, there exists a kernel representation of* $\mathcal{A}$ *with square matrices if, and only if,* $\dim \mathcal{A} \ge n$*.*

4. Kronecker canonical form. In this section we recall the KCF and show how it is related to the decomposition of an associated linear relation. We introduce the following notation: Let α be a multi-index, $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$. As usual, the absolute value of α is $|\alpha| = \sum_{i=1}^{l} \alpha_i$. For $k \in \mathbb{N}$ we define the matrices

$$N_k = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad N_\alpha = \operatorname{diag}(N_{\alpha_1}, \ldots, N_{\alpha_l}) \in \mathbb{C}^{|\alpha| \times |\alpha|}$$

For $k \in \mathbb{N}$, $k > 1$, we define the rectangular matrices

$$K_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \; L_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{C}^{(k-1) \times k}.$$

For some multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$, with $\alpha_j > 1$ for all $j = 1, \ldots, l$, we define

$$K_\alpha = \operatorname{diag}(K_{\alpha_1}, \ldots, K_{\alpha_l}), \; L_\alpha = \operatorname{diag}(L_{\alpha_1}, \ldots, L_{\alpha_l}) \in \mathbb{C}^{(|\alpha|-l) \times |\alpha|}. \tag{4.1}$$

We extend the above notion to multi-indices $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$ where some entries are equal to 1 but $\alpha \neq (1, 1, \ldots, 1)$. In this case, we define K_α and L_α in the following way: Collect in a multi-index α_0 all entries of α which are larger than 1, $\alpha_0 = (\alpha_{j_1}, \ldots, \alpha_{j_k})$ with $1 \leq k \leq l-1$. We have $|\alpha_0| - k = |\alpha| - l$ and K_{α_0} (L_{α_0}) is defined via (4.1) and is of size $(|\alpha_0| - k) \times |\alpha_0|$. Then K_α (L_α, respectively) is defined by augmenting K_{α_0} (L_{α_0}, respectively) by $|\alpha| - |\alpha_0|$ zero columns without changing the number of rows in such a way that to each entry $\alpha_j = 1$ in α there corresponds a zero column located at the j-th position. Then K_α and L_α are of size $(|\alpha| - l) \times |\alpha|$. As an example consider

$$K_{(1,2)} := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad K_{(1,2,1,3)} := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Moreover, we extend the above notion to the case $\alpha = (1, 1, \ldots, 1) \in \mathbb{N}^l$, if K_α and L_α are diagonal blocks entries of a larger matrix with other proper defined matrix elements. For instance, let $M \in \mathbb{C}^{m \times n}$, $\alpha = (1, 1, \ldots, 1) \in \mathbb{N}^l$. Then we define

$$\begin{aligned} \operatorname{diag}(M, K_\alpha) &:= [M, 0_{m \times l}] \in \mathbb{C}^{m \times (n+l)}, \quad \operatorname{diag}(K_\alpha, M) := [0_{m \times l}, M] \in \mathbb{C}^{m \times (n+l)}, \\ \operatorname{diag}(M, L_\alpha) &:= [M, 0_{m \times l}] \in \mathbb{C}^{m \times (n+l)}, \quad \operatorname{diag}(L_\alpha, M) := [0_{m \times l}, M] \in \mathbb{C}^{m \times (n+l)}. \end{aligned}$$

Similarly,

$$\operatorname{diag}(M, K_\alpha^\top) := \begin{bmatrix} M \\ 0_{l \times n} \end{bmatrix} \in \mathbb{C}^{(m+l) \times n}, \quad \operatorname{diag}(M, L_\alpha^\top) := \begin{bmatrix} M \\ 0_{l \times n} \end{bmatrix} \in \mathbb{C}^{(m+l) \times n}.$$

Finally, we define for $\beta = (1, 1, \ldots, 1) \in \mathbb{N}^l$ and $\gamma = (1, 1, \ldots, 1) \in \mathbb{N}^m$

$$\operatorname{diag}(K_\beta, K_\gamma^\top) = 0_{m \times l} \in \mathbb{C}^{m \times l}.$$

Some of the properties of the matrices K_α, L_α and N_α are collected in the following lemma.

LEMMA 4.1. *For $\alpha \in \mathbb{N}^l$ we have*

$$\operatorname{rk} K_\alpha = \operatorname{rk} L_\alpha = \operatorname{rk} N_\alpha = |\alpha| - l.$$

Furthermore, for all $\lambda \in \mathbb{C}$,

$$\operatorname{rk}(\lambda K_\alpha - L_\alpha) = |\alpha| - l,$$

and in particular

$$\ker K_\alpha^\top = \ker(\lambda K_\alpha^\top - L_\alpha^\top) = \{0\}.$$

Kronecker proved in [17] that any pair of matrices F, G can be transformed into a canonical form, see also [8, 9, 15]. Here we refer to the version in [15].

THEOREM 4.2 (Kronecker canonical form). *For any pair of matrices* $F, G \in \mathbb{C}^{n\times d}$ *there exist invertible matrices* $W \in \mathbb{C}^{n\times n}$ *and* $T \in \mathbb{C}^{d\times d}$ *such that*

$$WFT = \begin{bmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix} \quad \text{and} \quad WGT = \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix} \tag{4.2}$$

for some $A_0 \in \mathbb{C}^{n_0\times n_0}$ *in Jordan canonical form and multi-indices* $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\beta}$, $\gamma \in \mathbb{N}^{n_\gamma}$. *The multi-indices* α, β, γ *are unique up to a permutation of their respective entries. Further, the matrix* A_0 *is unique up to a permutation of its Jordan blocks.*

The entries of the multi-indices α, β, γ are called minimal indices and elementary divisors and play an important role in the analysis of matrix pairs (F, G), see e.g. [8, 9, 21, 22, 23], where the entries of α are the orders of the infinite elementary divisors, the entries of β are the column minimal indices and the entries of γ are the row minimal indices.

In what follows, we investigate the relationship between a linear relation $\mathcal{A}$ and the KCF of the matrices F and G from its image representation (cf. Definition 3.1). Then with the notation from Theorem 4.2 we find

$$\mathcal{A} = \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} = \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix} \\ W^{-1} \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix} \end{bmatrix}. \tag{4.3}$$

In the following proposition we collect some properties of the multi-index β from Theorem 4.2 and obtain a characterization of the dimension of $\mathcal{A}$ in terms of the indices which appear in the KCF (4.2).

PROPOSITION 4.3. *Let* $\mathcal{A}$ *be a linear relation in* $\mathbb{C}^n$ *with* $\dim \mathcal{A} = d \geq 1$. *Let* $F, G \in \mathbb{C}^{n\times d}$ *with* $\operatorname{rk}\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] = d$ *be such that* $\mathcal{A} = \operatorname{ran}\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ *and let* $W \in \mathbb{C}^{n\times n}$, $T \in \mathbb{C}^{d\times d}$ *be invertible matrices such that* WFT *and* WGT *are in KCF* (4.2). *Then the following statements hold.*

(i) *Either* $n_\beta = 0$ *or* $\beta_i \geq 2$ *for all* $i = 1, \ldots, n_\beta$.

(ii) *The dimension of* $\mathcal{A}$ *satisfies* $\dim \mathcal{A} = n_0 + |\alpha| + |\beta| + |\gamma| - n_\gamma$.
(iii) *We have* $\dim \mathcal{A} \geq n$ *if, and only if,* $n_\beta \geq n_\gamma$.
(iv) *We have* $\dim \mathcal{A} \leq n$ *if, and only if,* $n_\beta \leq n_\gamma$.

Proof. Since the entries of the multi-index β which equal 1 correspond to zero columns in $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$, statement (i) follows from $\operatorname{rk}\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] = d$. In order to show (ii), observe that for $k > 1$

$$\begin{bmatrix} K_k \\ L_k \end{bmatrix} \in \mathbb{C}^{2(k-1)\times k} \quad \text{and} \quad \begin{bmatrix} K_k^\top \\ L_k^\top \end{bmatrix} \in \mathbb{C}^{2k\times(k-1)}$$

with

$$\operatorname{rk}\begin{bmatrix} K_k \\ L_k \end{bmatrix} = k \quad \text{and} \quad \operatorname{rk}\begin{bmatrix} K_k^\top \\ L_k^\top \end{bmatrix} = k-1.$$

Therefore, for $\beta \in \mathbb{N}^{n_\beta}$ and $\gamma \in \mathbb{N}^{n_\gamma}$ and with (i) we see

$$\operatorname{rk}\begin{bmatrix} K_\beta \\ L_\beta \end{bmatrix} = |\beta| \quad \text{and} \quad \operatorname{rk}\begin{bmatrix} K_\gamma^\top \\ L_\gamma^\top \end{bmatrix} = |\gamma| - n_\gamma.$$

Then (ii) follows from (4.3). As n is the number of rows in the KCF (4.2),

$$n = n_0 + |\alpha| + |\beta| - n_\beta + |\gamma|.$$

and a comparison with (ii) yields (iii) and (iv) ∎

The properties of the linear relation $\mathcal{A}$ are encoded in the different blocks of the KCF. We start with the following simple example.

Example 4.4. *Let* $\mathcal{A}$ *be a linear relation in* $\mathbb{C}^n$ *in the form* (4.3). *Assume, for simplicity, that* W *and* T *in* (4.2) *are equal to the identity map.*

(i) *Assume that in* (4.2) *only the first row appears, i.e. the matrices* F *and* G *are of the form*

$$F = I_{n_0} \quad \textit{and} \quad G = A_0,$$

for some $A_0 \in \mathbb{C}^{n_0\times n_0}$ *in Jordan canonical form. Then* $\mathcal{A}$ *is given by*

$$\mathcal{A} = \operatorname{ran}\begin{bmatrix} F \\ G \end{bmatrix} = \operatorname{ran}\begin{bmatrix} I_{n_0} \\ A_0 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ A_0 x \end{bmatrix} \;\middle|\; x \in \mathbb{C}^{n_0} \right\}.$$

For $\lambda \in \mathbb{C}$ *and* $x \in \mathbb{C}^{n_0} \setminus \{0\}$ *we have* $A_0 x = \lambda x$ *if, and only if,* $(x, \lambda x) \in \mathcal{A}$. *This is equivalent to* $(x, 0) \in \mathcal{A} - \lambda$, *hence*

$$\sigma_p(\mathcal{A}) = \sigma(A_0),$$

where $\sigma(A_0)$ *denotes the spectrum of the matrix* A_0. *In particular, the point spectrum of* $\mathcal{A}$ *consists of finitely many points.*

(ii) *Assume that in* (4.2) *only the second row appears, i.e., the matrices* F *and* G *are of the form*

$$F = N_\alpha \quad \textit{and} \quad G = I_{|\alpha|}$$

and we have

$$\mathcal{A} = \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} = \operatorname{ran} \begin{bmatrix} N_\alpha \\ I_{|\alpha|} \end{bmatrix} = \left\{ \begin{bmatrix} N_\alpha y \\ y \end{bmatrix} \;\middle|\; y \in \mathbb{C}^{|\alpha|} \right\}.$$

The matrix N_α has only the eigenvalue zero. Hence

$$\mathcal{A}^{-1} = \operatorname{ran} \begin{bmatrix} I_{|\alpha|} \\ N_\alpha \end{bmatrix} = \left\{ \begin{bmatrix} y \\ N_\alpha y \end{bmatrix} \;\middle|\; y \in \mathbb{C}^{|\alpha|} \right\}$$

has only the eigenvalue zero and $\mathcal{A}$ is a Jordan relation with only eigenvalue ∞.

(iii) *Assume that in* (4.2) *only the third row appears and that the multi-index β consists of one entry only, $\beta = k$ for some $k \in \mathbb{N}$, $k > 1$,*

$$F = K_k \quad \text{and} \quad G = L_k.$$

Then

$$\mathcal{A} = \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} = \operatorname{ran} \begin{bmatrix} K_k \\ L_k \end{bmatrix} = \left\{ \begin{bmatrix} K_k x \\ L_k x \end{bmatrix} \;\middle|\; x \in \mathbb{C}^k \right\}.$$

For standard unit vectors $e_i \in \mathbb{C}^k$ we calculate $K_k e_k = 0 = L_k e_1$ and for $i = 1, \ldots, k-1$ we denote the unit vectors in $\mathbb{C}^{k-1}$ by $\widehat{e}_i$ and obtain

$$K_k e_i = \widehat{e}_i, \quad L_k e_{i+1} = \widehat{e}_i.$$

Therefore,

$$\begin{aligned} (0, \widehat{e}_{k-1}) &= (K_k e_k, L_k e_k) \in \mathcal{A}, \\ (\widehat{e}_{k-1}, \widehat{e}_{k-2}) &= (K_k e_{k-1}, L_k e_{k-1}) \in \mathcal{A}, \\ &\vdots \\ (\widehat{e}_2, \widehat{e}_1) &= (K_k e_2, L_k e_2) \in \mathcal{A}, \\ (\widehat{e}_1, 0) &= (K_k e_1, L_k e_1) \in \mathcal{A}, \end{aligned}$$

which is a singular chain in $\mathcal{A}$.

(iv) *Finally, assume that in* (4.2) *only the fourth row appears and that the multi-index γ consists of one entry only, $\gamma = k$ for some $k \in \mathbb{N}$, $k > 1$,*

$$F = K_k^\top \quad \text{and} \quad G = L_k^\top.$$

Then

$$\mathcal{A} = \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} = \operatorname{ran} \begin{bmatrix} K_k^\top \\ L_k^\top \end{bmatrix} = \left\{ \begin{bmatrix} K_k^\top y \\ L_k^\top y \end{bmatrix} \;\middle|\; y \in \mathbb{C}^{k-1} \right\}.$$

For $\lambda \in \mathbb{C}$ we have $(x, \lambda x) \in \mathcal{A}$ with $x \neq 0$ if, and only if, there exists $y \in \mathbb{C}^{k-1} \setminus \{0\}$ with $x = K_k^\top y$ and $\lambda x = L_k^\top y$. This is equivalent to $\ker(\lambda K_k^\top - L_k^\top) \neq \{0\}$. But Lemma 4.1 implies $\ker(\lambda K_k^\top - L_k^\top) = \{0\}$ for all $\lambda \in \mathbb{C}$, thus

$$\ker(\mathcal{A} - \lambda) = \{0\}.$$

Similarly, $x \in \operatorname{mul} \mathcal{A}$ if, and only if, there exists $y \in \mathbb{C}^{k-1}$ with $0 = K_k^\top y$ and $x = L_k^\top y$. But Lemma 4.1 implies $\ker K_k^\top = \{0\}$ and hence $\operatorname{mul} \mathcal{A} = \{0\}$. Therefore, $\sigma_p(\mathcal{A}) = \emptyset$ and $\mathcal{A}$ is a multishift.

Example 4.4 indicates that in the first block of the KCF the (finite) eigenvalues of $\mathcal{A} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ are encoded. The second block represents the eigenvalue ∞, the third block the singular chains and the fourth the multishifts. This relationship is exploited in all details (i.e., with emphasis on the number and length of the chains of different types) in the next theorem which is the main result of this paper. In what follows two chains are called linearly independent if their entries are linearly independent.

THEOREM 4.5. *Let $\mathcal{A}$ be a linear relation in $\mathbb{C}^n$ with $\dim \mathcal{A} = d \geq 1$. Let $F, G \in \mathbb{C}^{n\times d}$ with $\operatorname{rk} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] = d$ be such that $\mathcal{A} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ and let $W \in \mathbb{C}^{n\times n}$, $T \in \mathbb{C}^{d\times d}$ be invertible matrices such that WFT and WGT are in KCF (4.2) with $\alpha = (\alpha_1, \ldots, \alpha_{n_\alpha})$, $\beta = (\beta_1, \ldots, \beta_{n_\beta})$, and $\gamma = (\gamma_1, \ldots, \gamma_{n_\gamma})$.*

(i) *For the singular chain manifold we have*

$$\mathcal{R}_c(\mathcal{A}) = W^{-1}\left(\{0\}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right).$$

Moreover, $\mathcal{A} \cap \big(\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A})\big)$ is spanned by n_β linearly independent singular chains of lengths $\beta_1 - 1, \beta_2 - 1, \ldots, \beta_{n_\beta} - 1$.

(ii) *For the root manifold at ∞ we have*

$$\mathcal{R}_\infty(\mathcal{A}) = W^{-1}\left(\{0\}^{n_0} \times \mathbb{C}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right).$$

Moreover, $\mathcal{A} \cap \big(\mathcal{R}_\infty(\mathcal{A}) \times \mathcal{R}_\infty(\mathcal{A})\big)$ is spanned by $n_\alpha + n_\beta$ linearly independent chains with n_β singular chains of lengths $\beta_1 - 1, \beta_2 - 1, \ldots, \beta_{n_\beta} - 1$ and n_α Jordan chains at ∞ of lengths $\alpha_1, \alpha_2, \ldots, \alpha_{n_\alpha}$.

(iii) *For the finite Jordan chain manifold we have*

$$\mathcal{R}_f(\mathcal{A}) = W^{-1}\left(\mathbb{C}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right).$$

Moreover, $\mathcal{A} \cap \big(\mathcal{R}_f(\mathcal{A}) \times \mathcal{R}_f(\mathcal{A})\big)$ is spanned by a set of linearly independent chains consisting of n_β singular chains of lengths $\beta_1 - 1, \beta_2 - 1, \ldots, \beta_{n_\beta} - 1$ and the Jordan chains constituted by the Jordan chain vectors of the matrix A_0. In particular, we have $\sigma(A_0) \subseteq \sigma_p(\mathcal{A})$.

(iv) *For the Jordan chain manifold we have*

$$\mathcal{R}_J(\mathcal{A}) = W^{-1}\left(\mathbb{C}^{n_0} \times \mathbb{C}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right).$$

Moreover, $\mathcal{A} \cap \big(\mathcal{R}_f(\mathcal{A}) \times \mathcal{R}_f(\mathcal{A})\big)$ is spanned by a set of linearly independent chains consisting of n_β singular chains of lengths $\beta_1 - 1, \beta_2 - 1, \ldots, \beta_{n_\beta} - 1$ and n_α Jordan chains at ∞ of lengths $\alpha_1, \alpha_2, \ldots, \alpha_{n_\alpha}$ and the Jordan chains constituted by the Jordan chain vectors of the matrix A_0.

Proof.

Step 1. We show (i). Let $x \in \mathcal{R}_c(\mathcal{A}) \setminus \{0\}$. Then there exists a singular chain of the form (2.7) with linearly independent $x_1, \ldots, x_k \in \mathbb{C}^n$ and $x = x_j$ for some $j \in \{1, \ldots, k\}$. By $\mathcal{A} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ there exist $z_1, \ldots, z_{k+1} \in \mathbb{C}^d$ such that

$$\begin{array}{rcl} (0, x_1) & = & (Fz_1, Gz_1), \\ (x_1, x_2) & = & (Fz_2, Gz_2), \\ & \vdots & \\ (x_{k-1}, x_k) & = & (Fz_k, Gz_k), \\ (x_k, 0) & = & (Fz_{k+1}, Gz_{k+1}). \end{array} \tag{4.4}$$

For $i \in \{1,\dots,k+1\}$ define $y_i = T^{-1}z_i$. Partitioning $y_i = (y_{i,1}^\top,\dots,y_{i,4}^\top)^\top$ with $y_{i,1}\in\mathbb{C}^{n_0}$, $y_{i,2}\in\mathbb{C}^{|\alpha|}$, $y_{i,3}\in\mathbb{C}^{|\beta|}$, $y_{i,4}\in\mathbb{C}^{|\gamma|-n_\gamma}$ according to the decomposition (4.2), we obtain from the first equation in (4.4) that

$$0 = Fz_1 = W^{-1}(WFT)T^{-1}z_1 = W^{-1}\begin{bmatrix} I_{n_0} & 0 & 0 & 0\\ 0 & N_\alpha & 0 & 0\\ 0 & 0 & K_\beta & 0\\ 0 & 0 & 0 & K_\gamma^\top\end{bmatrix}\begin{pmatrix} y_{1,1}\\ y_{1,2}\\ y_{1,3}\\ y_{1,4}\end{pmatrix}$$

and hence $y_{1,1}=0$ and $K_\gamma^\top y_{1,4}=0$, thus, by Lemma 4.1, $y_{1,4}=0$. Furthermore,

$$\begin{aligned} x_1 &= Gz_1 = W^{-1}(WGT)T^{-1}z_1\\ &= W^{-1}\begin{bmatrix} A_0 & 0 & 0 & 0\\ 0 & I_{|\alpha|} & 0 & 0\\ 0 & 0 & L_\beta & 0\\ 0 & 0 & 0 & L_\gamma^\top\end{bmatrix}\begin{pmatrix} y_{1,1}\\ y_{1,2}\\ y_{1,3}\\ y_{1,4}\end{pmatrix} = W^{-1}\begin{pmatrix} 0\\ y_{1,2}\\ L_\beta y_{1,3}\\ 0\end{pmatrix}. \end{aligned} \tag{4.5}$$

The second equation in (4.4) gives

$$x_1 = Fz_2 = W^{-1}(WFT)T^{-1}z_2 = W^{-1}\begin{bmatrix} I_{n_0} & 0 & 0 & 0\\ 0 & N_\alpha & 0 & 0\\ 0 & 0 & K_\beta & 0\\ 0 & 0 & 0 & K_\gamma^\top\end{bmatrix}\begin{pmatrix} y_{2,1}\\ y_{2,2}\\ y_{2,3}\\ y_{2,4}\end{pmatrix}$$

and a comparison with (4.5) yields $y_{2,1}=0$ and $K_\gamma^\top y_{2,4}=0$, thus, by Lemma 4.1, $y_{2,4}=0$. Furthermore,

$$\begin{aligned} x_2 &= Gz_2 = W^{-1}(WGT)T^{-1}z_2\\ &= W^{-1}\begin{bmatrix} A_0 & 0 & 0 & 0\\ 0 & I_{|\alpha|} & 0 & 0\\ 0 & 0 & L_\beta & 0\\ 0 & 0 & 0 & L_\gamma^\top\end{bmatrix}\begin{pmatrix} y_{2,1}\\ y_{2,2}\\ y_{2,3}\\ y_{2,4}\end{pmatrix} = W^{-1}\begin{pmatrix} 0\\ y_{2,2}\\ L_\beta y_{2,3}\\ 0\end{pmatrix}. \end{aligned}$$

Proceeding in this way, we see

$$x_i = W^{-1}\begin{pmatrix} 0\\ y_{i,2}\\ L_\beta y_{i,3}\\ 0\end{pmatrix},\quad i=1,\dots,k. \tag{4.6}$$

From the last equation in (4.4) we conclude that

$$0 = Gz_{k+1} = W^{-1}(WGT)T^{-1}z_{k+1} = W^{-1}\begin{bmatrix} A_0 & 0 & 0 & 0\\ 0 & I_{|\alpha|} & 0 & 0\\ 0 & 0 & L_\beta & 0\\ 0 & 0 & 0 & L_\gamma^\top\end{bmatrix}\begin{pmatrix} y_{k+1,1}\\ y_{k+1,2}\\ y_{k+1,3}\\ y_{k+1,4}\end{pmatrix}$$

and hence $y_{k+1,2}=0$ and $L_\gamma^\top y_{k+1,4}=0$, thus, by Lemma 4.1, $y_{k+1,4}=0$. Furthermore,

$$\begin{aligned} x_k &= Fz_{k+1} = W^{-1}(WFT)T^{-1}z_{k+1}\\ &= W^{-1}\begin{bmatrix} I_{n_0} & 0 & 0 & 0\\ 0 & N_\alpha & 0 & 0\\ 0 & 0 & K_\beta & 0\\ 0 & 0 & 0 & K_\gamma^\top\end{bmatrix}\begin{pmatrix} y_{k+1,1}\\ y_{k+1,2}\\ y_{k+1,3}\\ y_{k+1,4}\end{pmatrix} = W^{-1}\begin{pmatrix} y_{k+1,1}\\ 0\\ K_\beta y_{k+1,3}\\ 0\end{pmatrix}. \end{aligned}$$

Proceeding in this way gives

$$x_i = W^{-1}\begin{pmatrix} y_{i+1,1} \\ 0 \\ K_\beta y_{i+1,3} \\ 0 \end{pmatrix}, \quad i = 1,\dots,k,$$

and together with (4.6) we find

$$W^{-1}\begin{pmatrix} 0 \\ y_{i,2} \\ L_\beta y_{i,3} \\ 0 \end{pmatrix} = x_i = W^{-1}\begin{pmatrix} y_{i+1,1} \\ 0 \\ K_\beta y_{i+1,3} \\ 0 \end{pmatrix}$$

for all $i = 1,\dots,k$. This implies that

$$x = x_j = W^{-1}\begin{pmatrix} 0 \\ 0 \\ L_\beta y_{j,3} \\ 0 \end{pmatrix} \in W^{-1}\left(\{0\}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right).$$

Conversely, let us start with the case that K_β and L_β consist of one block only. That is, $n_\beta = 1$ and $\beta = k$ with some $k \in \mathbb{N}$, $k \geq 2$, cf. Proposition 4.3. Define

$$y_1 := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},\ y_2 := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix},\ \dots,\ y_k := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^k. \tag{4.7}$$

Then $0 = K_k y_1$, $L_k y_k = 0$, and

$$L_k y_1 = K_k y_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^{k-1}, \dots,\ L_k y_{k-1} = K_k y_k = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{k-1}. \tag{4.8}$$

Set

$$x_0 := 0,\ x_i := W^{-1}(0,0,(L_k y_i)^\top,0)^\top,\ \text{for } i = 1,\dots,k-1, \quad \text{and } x_k := 0.$$

As WFT, WGT are in KCF (4.2), we find that, invoking (4.8),

$$\begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix} = \begin{bmatrix} W^{-1}\begin{bmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_k & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix} \\ \\ W^{-1}\begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_k & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ y_i \\ 0 \end{pmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} T \begin{pmatrix} 0 \\ 0 \\ y_i \\ 0 \end{pmatrix} \in \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} = \mathcal{A}$$

for $i = 1, \dots, k$, and hence

$$(0, x_1), (x_1, x_2), \dots, (x_{k-2}, x_{k-1}), (x_{k-1}, 0) \in \mathcal{A}. \tag{4.9}$$

We see from (4.8) that $L_k y_1, \dots, L_k y_{k-1}$ are the unit vectors in $\mathbb{C}^{k-1}$. Hence, $x_1, \dots, x_{k-1}$ are linearly independent and (4.9) constitutes a singular chain as in (2.7). Therefore, if the multi-index β has only one entry and if this entry equals k, then (4.9) is a singular chain of length $k-1$ and in particular

$$\mathcal{R}_c(\mathcal{A}) = W^{-1}\left(\{0\}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{k-1} \times \{0\}^{|\gamma|}\right).$$

This, (4.3), and the fact that $[y_k, \dots, y_1] = I_k \in \mathbb{C}^{k \times k}$ yield

$$\begin{aligned}
&\mathcal{A} \cap \big(\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A})\big) \\
&= \operatorname{ran} \begin{bmatrix} W^{-1}\begin{bmatrix} 0 \\ 0 \\ K_k \\ 0 \end{bmatrix} \\[2em] W^{-1}\begin{bmatrix} 0 \\ 0 \\ L_k \\ 0 \end{bmatrix} \end{bmatrix} [y_k, \dots, y_1] = \operatorname{ran} \begin{bmatrix} W^{-1}\begin{bmatrix} 0 \\ 0 \\ [L_k y_{k-1}, \dots, L_k y_1, 0] \\ 0 \end{bmatrix} \\[2em] W^{-1}\begin{bmatrix} 0 \\ 0 \\ [0, L_k y_{k-1}, \dots, L_k y_1] \\ 0 \end{bmatrix} \end{bmatrix} \\
&= \operatorname{span}\{(0, x_1), (x_1, x_2), \dots, (x_{k-2}, x_{k-1}), (x_{k-1}, 0)\}.
\end{aligned}$$

In the general case $\beta \in \mathbb{N}^{n_\beta}$ with $n_\beta > 1$ we have n_β decoupled blocks in K_β and L_β and for each block the above construction leads to a singular chain of $\mathcal{A}$. In this manner we obtain n_β linearly independent singular chains of lengths $\beta_1 - 1, \beta_2 - 1, \dots, \beta_{n_\beta} - 1$, resp., which span $\mathcal{A} \cap \big(\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A})\big)$.

Step 2. We show (ii). Let $y \in \mathcal{R}_\infty(\mathcal{A})$. Then there exists a Jordan chain at ∞ of the form (2.4) with linearly independent $y_1, \dots, y_{m-1}, y \in \mathbb{C}^n$. Set $y_m := y$. As in the proof of (4.6) it follows that for some $z_{1,2}, \dots, z_{m,2} \in \mathbb{C}^{|\alpha|}$, $z_{1,3}, \dots, z_{m,3} \in \mathbb{C}^{|\beta|}$ we have

$$y_i = W^{-1}\begin{pmatrix} 0 \\ z_{i,2} \\ L_\beta z_{i,3} \\ 0 \end{pmatrix}, \quad i = 1, \dots, m.$$

This proves $y \in W^{-1}\left(\{0\}^{n_0} \times \mathbb{C}^{|\alpha|} \times \mathbb{C}^{|\beta| - n_\beta} \times \{0\}^{|\gamma|}\right)$.

Conversely, let $z_2 \in \mathbb{C}^{|\alpha|}$, $z_3 \in \mathbb{C}^{|\beta| - n_\beta}$ and set $x := W^{-1}(0, z_2^\top, z_3^\top, 0)^\top$. It follows from (i) that

$$W^{-1}(0, 0, z_3^\top, 0)^\top \in \mathcal{R}_c(\mathcal{A}) \subseteq \mathcal{R}_\infty(\mathcal{A}).$$

Assume that N_α consists of one block only. That is, $n_\alpha = 1$ and $\alpha = k$ for some $k \geq 1$. Choose $y_1, \dots, y_k$ as in (4.7). Then

$$0 = N_k y_1, \ y_1 = N_k y_2, \ \dots, \ y_{k-1} = N_k y_k. \tag{4.10}$$

Set $x_0 := 0$, $x_i := W^{-1}(0, y_i^\top, 0, 0)^\top$ for $i = 1, \dots, k$, we obtain with (4.2) and (4.10),

$$\begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix} = \begin{bmatrix} W^{-1} \begin{bmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_k & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix} \end{bmatrix} \begin{pmatrix} 0 \\ y_i \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} T \begin{pmatrix} 0 \\ y_i \\ 0 \\ 0 \end{pmatrix} \in \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix} = \mathcal{A}$$

for $i = 1, \dots, k$ and therefore

$$(0, x_1), (x_1, x_2), \dots, (x_{k-2}, x_{k-1}), (x_{k-1}, x_k) \in \mathcal{A}. \tag{4.11}$$

The vectors $y_1, \dots, y_k$ in (4.7) are linearly independent and then the same holds for $x_1, \dots, x_k$. Thus, (4.11) constitutes a Jordan chain of $\mathcal{A}$ at ∞ of length k. In particular, since $z_2 \in \operatorname{span}\{y_1, \dots, y_k\}$ it follows that $x \in \mathcal{R}_c(\mathcal{A}) + \operatorname{span}\{x_1, \dots, x_k\}$ and thus we have shown that

$$\mathcal{R}_\infty(\mathcal{A}) = W^{-1}\left(\{0\}^{n_0} \times \mathbb{C}^k \times \mathbb{C}^{|\beta| - n_\beta} \times \{0\}^{|\gamma|}\right).$$

This and the first step of this proof yield

$$\begin{aligned} \mathcal{A} \cap \big(\mathcal{R}_\infty(\mathcal{A}) \times \mathcal{R}_\infty(\mathcal{A})\big) &= \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} 0 & 0 \\ N_k & 0 \\ 0 & K_\beta \\ 0 & 0 \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} 0 & 0 \\ I_k & 0 \\ 0 & L_\beta \\ 0 & 0 \end{bmatrix} \end{bmatrix} \\ &= \mathcal{A} \cap \big(\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A})\big) \dotplus \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} 0 \\ N_k \\ 0 \\ 0 \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} 0 \\ I_k \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} [y_k, \dots, y_1], \end{aligned}$$

where we used that $[y_k, \dots, y_1] = I_k \in \mathbb{C}^{k \times k}$ and $\mathcal{A} \dotplus \mathcal{B}$ denotes the direct sum of two subspaces $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \cap \mathcal{B} = \{0\}$. Therefore

$$\begin{aligned}
&\mathcal{A} \cap \big(\mathcal{R}_\infty(\mathcal{A}) \times \mathcal{R}_\infty(\mathcal{A})\big) \\
&= \mathcal{A} \cap \big(\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A})\big) \dotplus \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} 0 \\ [y_{k-1}, \ldots, y_1, 0] \\ 0 \\ 0 \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} 0 \\ [y_k, \ldots, y_2, y_1] \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \\
&= \mathcal{A} \cap \big(\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A})\big) \dotplus \operatorname{span}\{(0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k)\},
\end{aligned}$$

Hence, (ii) is proved in the case $n_\alpha = 1$ and $\alpha = k$. In general, if $\alpha \in \mathbb{N}^{n_\alpha}$, then there are n_α decoupled blocks in N_α. For each block the above construction leads to a Jordan chain of $\mathcal{A}$ at ∞ and we obtain n_α linearly independent Jordan chains at ∞ of lengths $\alpha_1, \ldots, \alpha_{n_\alpha}$, respectively, which lead, together with the singular chains, to the span of $\mathcal{A} \cap \big(\mathcal{R}_\infty(\mathcal{A}) \times \mathcal{R}_\infty(\mathcal{A})\big)$.

Step 3. We show (iii). Let $x \in \mathcal{R}_f(\mathcal{A})$. Since $\mathcal{R}_f(\mathcal{A})$ has a finite basis, there exist $k \in \mathbb{N}$ and pairwise distinct $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ such that

$$x \in \sum_{i=1}^{k} \mathcal{R}_{\lambda_i}(\mathcal{A}).$$

Therefore, we find

$$v_i \in \ker(\mathcal{A} - \lambda_i)^n, \quad i = 1, \ldots, k,$$

such that $x = v_1 + \ldots + v_k$. We show that $v_i \in W^{-1}\left(\mathbb{C}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta| - n_\beta} \times \{0\}^{|\gamma|}\right)$ for all $i = 1, \ldots, k$. For simplicity, let $v := v_i$ and $\lambda := \lambda_i$ for some $i \in \{1, \ldots, k\}$. By (2.3) there exist $v^1, \ldots, v^{n-1} \in \mathbb{C}^n$ such that

$$(v, v^{n-1} + \lambda v),\ (v^{n-1}, v^{n-2} + \lambda v^{n-1}),\ \ldots,\ (v^2, v^1 + \lambda v^2),\ (v^1, \lambda v^1) \in \mathcal{A},$$

where, for simplicity, we do not assume that the vectors $v^1, \ldots, v^{n-1}, v$ are linearly independent. Set $v^n := v$. We obtain from $\mathcal{A} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ the existence of $z_1, \ldots, z_n \in \mathbb{C}^d$ such that

$$\begin{aligned}
(v^n, v^{n-1} + \lambda v^n) &= (F z_n, G z_n), \\
(v^{n-1}, v^{n-2} + \lambda v^{n-1}) &= (F z_{n-1}, G z_{n-1}), \\
&\ \vdots \\
(v^2, v^1 + \lambda v^2) &= (F z_2, G z_2), && (4.12) \\
(v^1, \lambda v^1) &= (F z_1, G z_1). && (4.13)
\end{aligned}$$

Define $y_i := T^{-1} z_i$ for $i = 1, \ldots, n$. Partitioning $y_i = (y_{i,1}^\top, \ldots, y_{i,4}^\top)^\top$ with $y_{i,1} \in \mathbb{C}^{n_0}$, $y_{i,2} \in \mathbb{C}^{|\alpha|}$, $y_{i,3} \in \mathbb{C}^{|\beta|}$, $y_{i,4} \in \mathbb{C}^{|\gamma| - n_\gamma}$ according to the decomposition (4.2), we obtain from (4.13)

$$v^1 = Fz_1 = W^{-1}(WFT)T^{-1}z_1 = W^{-1}\begin{bmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix}\begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ y_{1,4} \end{pmatrix}$$

and

$$\lambda v^1 = Gz_1 = W^{-1}(WGT)T^{-1}z_1 = W^{-1}\begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix}\begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ y_{1,4} \end{pmatrix}.$$

Therefore,

$$(\lambda N_\alpha - I_{|\alpha|})y_{1,2} = 0 \quad \text{and} \quad (\lambda K_\gamma^\top - L_\gamma^\top)y_{1,4} = 0,$$

thus, by invertibility of $\lambda N_\alpha - I_{|\alpha|}$ and Lemma 4.1, $y_{1,2} = 0$ and $y_{1,4} = 0$. Similarly (4.12) gives

$$(\lambda N_\alpha - I_{|\alpha|})y_{2,2} = -N_\alpha y_{1,2} = 0 \quad \text{and} \quad (\lambda K_\gamma^\top - L_\gamma^\top)y_{2,4} = -K_\gamma^\top y_{1,4} = 0,$$

and hence $y_{2,2} = 0$ and $y_{2,4} = 0$. Solving the remaining equations successively, we obtain finally $y_{n,2} = 0$ and $y_{n,4} = 0$, which implies

$$v = v^n = Fz_n = W^{-1}\begin{pmatrix} y_{n,1} \\ 0 \\ K_\beta y_{n,3} \\ 0 \end{pmatrix} \in W^{-1}\left(\mathbb{C}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right)$$

and $\mathcal{R}_f(\mathcal{A}) \subseteq W^{-1}\left(\mathbb{C}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right)$ follows.

We prove the converse inclusion. From (i) it follows

$$W^{-1}\left(\{0\}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right) = \mathcal{R}_c(\mathcal{A}) \subseteq \mathcal{R}_0(\mathcal{A}) \subseteq \mathcal{R}_f(\mathcal{A}).$$

We show $W^{-1}\left(\mathbb{C}^{n_0} \times \{0\}^{|\alpha|} \times \{0\}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right) \subseteq \mathcal{R}_f(\mathcal{A})$. The space $\mathbb{C}^{n_0}$ has a basis consisting of Jordan chains of the matrix A_0. Let $v_1, \dots, v_k$ be a Jordan chain of A_0 at $\lambda \in \mathbb{C}$ of length k. Then $v_1, \dots, v_k$ are linearly independent and satisfy

$$(A_0-\lambda)v_k = v_{k-1},\ (A_0-\lambda)v_{k-1} = v_{k-2},\ \dots,\ (A_0-\lambda)v_2 = v_1,\ (A_0-\lambda)v_1 = 0. \tag{4.14}$$

Set

$$x_j := W^{-1}((v_j)^\top, 0, 0, 0)^\top \quad \text{for } j = 1, \dots, k.$$

Then we obtain

$$x_j = W^{-1}\begin{bmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix}\begin{pmatrix} v_j \\ 0 \\ 0 \\ 0 \end{pmatrix} = FT\begin{pmatrix} v_j \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and we see with (4.14) that for $j = 2, \dots, k$,

$$\lambda x_j + x_{j-1} = W^{-1}\begin{pmatrix} \lambda v_j + v_{j-1} \\ 0 \\ 0 \\ 0 \end{pmatrix} = W^{-1}\begin{pmatrix} A_0 v_j \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$= W^{-1}\begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix}\begin{pmatrix} v_j \\ 0 \\ 0 \\ 0 \end{pmatrix} = GT\begin{pmatrix} v_j \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and for $j = 1$

$$\lambda x_1 = W^{-1}\begin{pmatrix} \lambda v_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = W^{-1}\begin{pmatrix} A_0 v_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$= W^{-1}\begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix}\begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = GT\begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, for $j = 2, \dots, k$,

$$\begin{pmatrix} x_j \\ \lambda x_j + x_{j-1} \end{pmatrix} \in \operatorname{ran}\begin{bmatrix} F \\ G \end{bmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ \lambda x_1 \end{pmatrix} \in \operatorname{ran}\begin{bmatrix} F \\ G \end{bmatrix},$$

hence

$$(x_k, \lambda x_k + x_{k-1}), (x_{k-1}, \lambda x_{k-1} + x_{k-2}), \dots, (x_2, \lambda x_2 + x_1), (x_1, \lambda x_1) \in \mathcal{A}. \tag{4.15}$$

The vectors $x_1, \dots, x_k$ are linear independent because W is invertible and the vectors $v_1, \dots, v_k$ are linear independent. Therefore, (4.15) constitutes a Jordan chain of $\mathcal{A}$ at λ of length k. Since the Jordan chain $v_1, \dots, v_k$ was arbitrary we have shown that

$$\mathcal{R}_f(\mathcal{A}) = W^{-1}\left(\mathbb{C}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta| - n_\beta} \times \{0\}^{|\gamma|}\right).$$

The remaining statements of (iii) follow from (i), the construction of (4.15) above and the observation that

$$\mathcal{A} \cap \big(\mathcal{R}_f(\mathcal{A}) \times \mathcal{R}_f(\mathcal{A})\big)$$
$$= \operatorname{ran}\begin{bmatrix} W^{-1}\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \\ 0 & K_\beta \\ 0 & 0 \end{bmatrix} \\ \\ W^{-1}\begin{bmatrix} A_0 & 0 \\ 0 & 0 \\ 0 & L_\beta \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \mathcal{A} \cap \big(\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A})\big) \dotplus \operatorname{ran}\begin{bmatrix} W^{-1}\begin{bmatrix} I_{n_0} \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \\ W^{-1}\begin{bmatrix} A_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}.$$

In particular, we see that $\sigma(A_0) \subseteq \sigma_p(\mathcal{A})$.

Step 4. We show (iv). Since $\mathcal{R}_J(\mathcal{A}) = \mathcal{R}_\infty(\mathcal{A}) + \mathcal{R}_f(\mathcal{A})$ it follows from (i) and (iii) that

$$\mathcal{R}_J(\mathcal{A}) = W^{-1}\left(\mathbb{C}^{n_0} \times \mathbb{C}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right).$$

The remaining statements of (iv) follow from

$$\begin{aligned}
&\mathcal{A} \cap \left(\mathcal{R}_J(\mathcal{A}) \times \mathcal{R}_J(\mathcal{A})\right) \\
&= \operatorname{ran} \begin{bmatrix} W^{-1}\begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & N_\alpha & 0 \\ 0 & 0 & K_\beta \\ 0 & 0 & 0 \end{bmatrix} \\ \\ W^{-1}\begin{bmatrix} A_0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 \\ 0 & 0 & L_\beta \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} = \mathcal{A} \cap \left(\mathcal{R}_f(\mathcal{A}) \times \mathcal{R}_f(\mathcal{A})\right) \dotplus \operatorname{ran} \begin{bmatrix} W^{-1}\begin{bmatrix} I_{n_0} \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \\ W^{-1}\begin{bmatrix} A_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}
\end{aligned}$$

and the same arguments for the Jordan chains of A_0 as in Step 3. ∎

Corollary 4.6. *With the notation from Theorem 4.5 we have*

$$\begin{aligned}
n_0 &= \dim \mathcal{R}_f(\mathcal{A}) - \dim \mathcal{R}_c(\mathcal{A}), \\
|\alpha| &= \dim \mathcal{R}_\infty(\mathcal{A}) - \dim \mathcal{R}_c(\mathcal{A}), \\
|\beta| - n_\beta &= \dim \mathcal{R}_c(\mathcal{A}), \\
|\gamma| &= n - \dim \mathcal{R}_J(\mathcal{A}).
\end{aligned}$$

Corollary 4.7. *With the notation from Theorem 4.5 we have*

$$\sigma_p(\mathcal{A}) = \begin{cases} \mathbb{C} \cup \{\infty\}, & \text{if } n_\beta \neq 0, \\ \sigma(A_0) \cup \{\infty\}, & \text{if } n_\beta = 0 \text{ and } n_\alpha \neq 0 \\ \sigma(A_0), & \text{if } n_\beta = n_\alpha = 0 \\ \emptyset, & \text{if } n_\beta = n_\alpha = n_0 = 0. \end{cases} \tag{4.16}$$

Proof. First observe that by Lemma 2.5, $\sigma_p(\mathcal{A}) = \mathbb{C} \cup \{\infty\}$ if, and only if, $\mathcal{R}_c(\mathcal{A}) \neq \{0\}$. By Theorem 4.5 and Proposition 4.3, this is equivalent to $n_\beta \neq 0$ and the first line in (4.16) is shown.

For the rest of the proof we assume $n_\beta = 0$. A complex number $\lambda \in \mathbb{C}$ belongs to $\sigma_p(\mathcal{A})$ if, and only if, there exists $x \in \mathbb{C}^n \setminus \{0\}$ with $(x, \lambda x) \in \mathcal{A}$. Equivalently, by Theorem 4.5, there exist $y_1 \in \mathbb{C}^{n_0}$, $y_2 \in \mathbb{C}^{|\alpha|}$ and $y_4 \in \mathbb{C}^{|\gamma|-n_\gamma}$ such that at least one of them is non-zero, with

$$x = W^{-1}\begin{pmatrix} y_1 \\ N_\alpha y_2 \\ K_\gamma^\top y_4 \end{pmatrix} \quad \text{and} \quad \lambda x = W^{-1}\begin{pmatrix} A_0 y_1 \\ y_2 \\ L_\gamma^\top y_4 \end{pmatrix}.$$

With Lemma 4.1 we see that this is equivalent to $y_2 = y_4 = 0$, $y_1 \neq 0$ and $A_0 y_1 = \lambda y_1$, or, what is the same, $\lambda \in \sigma(A_0)$. Hence, in the case $n_\beta = 0$, we have

$$\sigma_p(\mathcal{A}) \setminus \{\infty\} = \sigma(A_0). \tag{4.17}$$

It remains to consider the point ∞. By definition, $\infty \in \sigma_p(\mathcal{A})$ if, and only if, $\mathcal{R}_\infty(\mathcal{A}) \neq \{0\}$ which is, by Theorem 4.5, equivalent to $n_\alpha \neq 0$. This and (4.17) show the second and third line in (4.16), whereas the last line in (4.16) is now obvious. ∎

Using Theorem 4.5 we may derive a characterization for $\mathcal{A}$ being completely singular, a Jordan relation or a multishift.

Proposition 4.8. *With the notation from Theorem 4.5 we have that the linear relation $\mathcal{A}$ is*

(i) *completely singular (see (2.6)) if, and only if, $n_0 = n_\alpha = 0$ and $\gamma = (1, \ldots, 1)$;*
(ii) *a Jordan relation if, and only if, $n_\beta = 0$ and $\gamma = (1, \ldots, 1)$;*
(iii) *a multishift if, and only if, $n_0 = n_\alpha = n_\beta = 0$.*

Proof. We show (i). By (2.6) $\mathcal{A}$ is completely singular if, and only if, $\mathcal{A} = \mathcal{A} \cap (\mathcal{R}_c(\mathcal{A}) \times \mathcal{R}_c(\mathcal{A}))$. Invoking Theorem 4.5 and (4.3) this is equivalent to

$$\operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix} \end{bmatrix} = \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} 0 \\ 0 \\ K_\beta \\ 0 \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} 0 \\ 0 \\ L_\beta \\ 0 \end{bmatrix} \end{bmatrix}.$$

This is true if, and only if,

$$n_0 = n_\alpha = 0 \quad \text{and} \quad \gamma = (1, \ldots, 1).$$

We show (ii). By definition (cf. Section 2), $\mathcal{A}$ is a Jordan relation if, and only if, $\mathcal{R}_c(\mathcal{A}) = \{0\}$ and $\mathcal{A} = \mathcal{A} \cap (\mathcal{R}_J(\mathcal{A}) \times \mathcal{R}_J(\mathcal{A}))$. Invoking Theorem 4.5 this is equivalent to $n_\beta = 0$ and

$$\operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & N_\alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} A_0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_\gamma^\top \end{bmatrix} \end{bmatrix} = \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} I_{n_0} & 0 \\ 0 & N_\alpha \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} A_0 & 0 \\ 0 & I_{|\alpha|} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}.$$

Similar to (i), this is true if, and only if, $n_\beta = 0$ and $\gamma = (1, \ldots, 1)$, thus (ii) follows.

We show (iii). By definition, $\mathcal{A}$ is a multishift if, and only if, $\sigma_p(\mathcal{A}) = \emptyset$. Then (iii) follows from Corollary 4.7. ∎

The KCF leads to a decomposition of a linear relation $\mathcal{A}$ in a natural way. For this we introduce the following notion. Here "$\dot{+}$" stands for the direct sum of subspaces.

DEFINITION 4.9. *Let $\mathcal{A}$ be a linear relation in $\mathbb{C}^n$. A decomposition $\mathcal{A} = \mathcal{A}_1 \dotplus \ldots \dotplus \mathcal{A}_k$, $k \in \mathbb{N}$, of a linear relation $\mathcal{A}$ is called* completely reduced, *if there exist subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_k$ of $\mathbb{C}^n$ such that*

(i) $\mathcal{V}_1 \dotplus \ldots \dotplus \mathcal{V}_k = \mathbb{C}^n$ *and*

(ii) $\mathcal{A}_j = \mathcal{A} \cap (\mathcal{V}_j \times \mathcal{V}_j)$ *for all $j = 1, \ldots, k$.*

In [24] it is shown that any linear relation can be decomposed into a direct sum of a Jordan relation, a completely singular relation and a multishift. However, this decomposition is in general not completely reduced. The KCF resolves this problem.

PROPOSITION 4.10. *With the notation from Theorem 4.5 define*

$$\mathcal{A}_S := \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} 0 \\ 0 \\ K_\beta \\ 0 \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} 0 \\ 0 \\ L_\beta \\ 0 \end{bmatrix} \end{bmatrix}, \quad \mathcal{A}_J := \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} I_{n_0} & 0 \\ 0 & N_\alpha \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} A_0 & 0 \\ 0 & I_{|\alpha|} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix},$$

$$\mathcal{A}_M := \operatorname{ran} \begin{bmatrix} W^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ K_\gamma^\top \end{bmatrix} \\ \\ W^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ L_\gamma^\top \end{bmatrix} \end{bmatrix}.$$

Then $\mathcal{A}_S$ is completely singular, $\mathcal{A}_J$ is a Jordan relation, $\mathcal{A}_M$ is a multishift and

$$\mathcal{A} = \mathcal{A}_S \dotplus \mathcal{A}_J \dotplus \mathcal{A}_M$$

is a completely reduced decomposition.

Proof. We show that the decomposition is completely reduced. Set

$$\begin{aligned} \mathcal{V}_1 &:= \mathcal{R}_c(\mathcal{A}) = W^{-1}\left(\{0\}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta| - n_\beta} \times \{0\}^{|\gamma|}\right), \\ \mathcal{V}_2 &:= W^{-1}\left(\mathbb{C}^{n_0} \times \mathbb{C}^{|\alpha|} \times \{0\}^{|\beta| - n_\beta} \times \{0\}^{|\gamma|}\right), \\ \mathcal{V}_3 &:= W^{-1}\left(\{0\}^{n_0} \times \{0\}^{|\alpha|} \times \{0\}^{|\beta| - n_\beta} \times \mathbb{C}^{|\gamma|}\right). \end{aligned}$$

Then it is clear that $\mathcal{V}_1 \dotplus \mathcal{V}_2 \dotplus \mathcal{V}_3 = \mathbb{C}^n$ and

$$\mathcal{A}_S = \mathcal{A} \cap (\mathcal{V}_1 \times \mathcal{V}_1), \quad \mathcal{A}_J = \mathcal{A} \cap (\mathcal{V}_2 \times \mathcal{V}_2), \quad \mathcal{A}_M = \mathcal{A} \cap (\mathcal{V}_3 \times \mathcal{V}_3).$$

It is clear that $\mathcal{R}_c(\mathcal{A}_S) = \mathcal{R}_c(\mathcal{A}) = \mathcal{V}_1$ and hence $\mathcal{A}_S$ is completely singular. Furthermore, $\mathcal{R}_c(\mathcal{A}_J) = \{0\}$ and $\mathcal{R}_J(\mathcal{A}_J) = \mathcal{V}_2$, thus $\mathcal{A}_J$ is a Jordan relation. Finally, it follows from Corollary 4.7 that $\sigma_p(\mathcal{A}_M) = \emptyset$ and hence $\mathcal{A}_M$ is a multishift. ∎

5. Wong sequences. Recently, see [7, 8, 9], Wong sequences are used to prove the KCF and compared to the proof by Gantmacher [15] they provide some geometrical insight. The Wong sequences have their origin in Wong [27]. Wong sequences are also useful to describe the different parts of a linear relation. In this section we derive representations for the root and Jordan chain manifolds of a linear relation in terms of the Wong sequences.

For $E, A \in \mathbb{C}^{r\times n}$ the Wong sequences are defined as the sequences of subspaces $(\mathcal{V}_i)$ and $(\mathcal{W}_i)$,

$$\begin{aligned} \mathcal{V}_0 &= \mathbb{C}^n, & \mathcal{V}_{i+1} &= \{\, x\in\mathbb{C}^n \mid Ax\in E\mathcal{V}_i \,\}, \quad i\in\mathbb{N}_0,\\ \mathcal{W}_0 &= \{0\}, & \mathcal{W}_{i+1} &= \{\, x\in\mathbb{C}^n \mid Ex\in A\mathcal{W}_i \,\}, \quad i\in\mathbb{N}_0. \end{aligned}$$

The limits of the Wong sequences are denoted by

$$\mathcal{V}^* = \bigcap_{i\in\mathbb{N}_0} \mathcal{V}_i \quad \text{and} \quad \mathcal{W}^* = \bigcup_{i\in\mathbb{N}_0} \mathcal{W}_i.$$

For $F, G \in \mathbb{C}^{n\times d}$ alternative Wong sequences are defined by the sequences $(\widehat{\mathcal{V}}_i)$ and $(\widehat{\mathcal{W}}_i)$,

$$\begin{aligned} \widehat{\mathcal{V}}_0 &= \mathbb{C}^n, & \widehat{\mathcal{V}}_{i+1} &= \left\{\, Fx \;\middle|\; x\in\mathbb{C}^d,\ Gx\in\widehat{\mathcal{V}}_i \,\right\}, \quad i\in\mathbb{N}_0,\\ \widehat{\mathcal{W}}_0 &= \{0\}, & \widehat{\mathcal{W}}_{i+1} &= \left\{\, Gx \;\middle|\; x\in\mathbb{C}^d,\ Fx\in\widehat{\mathcal{W}}_i \,\right\}, \quad i\in\mathbb{N}_0, \end{aligned}$$

with corresponding limits

$$\widehat{\mathcal{V}}^* = \bigcap_{i\in\mathbb{N}_0} \widehat{\mathcal{V}}_i \quad \text{and} \quad \widehat{\mathcal{W}}^* = \bigcup_{i\in\mathbb{N}_0} \widehat{\mathcal{W}}_i.$$

THEOREM 5.1. *Let $\mathcal{A}$ be a linear relation in $\mathbb{C}^n$ with $\dim\mathcal{A} = d$ and let $A, E \in \mathbb{C}^{r\times n}$, $r = 2n-d$, with $\operatorname{rk}[A, E] = r$ and $F, G \in \mathbb{C}^{n\times d}$ with $\operatorname{rk}\left[\begin{smallmatrix}F\\G\end{smallmatrix}\right] = d$ be such that*

$$\mathcal{A} = \ker[A, -E] = \operatorname{ran}\left[\begin{smallmatrix}F\\G\end{smallmatrix}\right].$$

Then we have

$$\begin{aligned} \mathcal{V}^*\cap\mathcal{W}^* &= \widehat{\mathcal{V}}^*\cap\widehat{\mathcal{W}}^* &&= \mathcal{R}_c(\mathcal{A}),\\ \mathcal{V}^* &= \widehat{\mathcal{V}}^* &&= \mathcal{R}_f(\mathcal{A}) = \operatorname{dom}\mathcal{A}^n,\\ \mathcal{W}^* &= \widehat{\mathcal{W}}^* &&= \mathcal{R}_\infty(\mathcal{A}),\\ \mathcal{V}^*+\mathcal{W}^* &= \widehat{\mathcal{V}}^*+\widehat{\mathcal{W}}^* &&= \mathcal{R}_J(\mathcal{A}). \end{aligned}$$

Proof. We show that, for all $i\in\mathbb{N}_0$,

$$\operatorname{dom}\mathcal{A}^i = \mathcal{V}_i = \widehat{\mathcal{V}}_i, \tag{5.1}$$

$$\operatorname{mul}\mathcal{A}^i = \mathcal{W}_i = \widehat{\mathcal{W}}_i. \tag{5.2}$$

We prove the first equality in (5.1) by induction. For $i=0$ the statement is true, so assume that it holds for some $i\in\mathbb{N}_0$. Then

$$\begin{aligned} \operatorname{dom}\mathcal{A}^{i+1} &= \left\{\, x\in\mathbb{C}^n \;\middle|\; \exists\, y\in\mathbb{C}^n:\ (x,y)\in\mathcal{A} = \ker[A,-E] \ \text{and}\ y\in\operatorname{dom}\mathcal{A}^i \,\right\}\\ &= \{\, x\in\mathbb{C}^n \mid \exists\, y\in\mathbb{C}^n:\ Ax = Ey \ \text{and}\ y\in\mathcal{V}_i \,\} = \mathcal{V}_{i+1}. \end{aligned}$$

The second equality in (5.1) follows from

$$\begin{aligned}\operatorname{dom}\mathcal{A}^{i+1} &= \left\{\ x\in\mathbb{C}^n \ \middle|\ \exists\, y\in\mathbb{C}^n:\ (x,y)\in\mathcal{A}=\operatorname{ran}\left[\begin{smallmatrix}F\\G\end{smallmatrix}\right]\ \text{ and }\ y\in\operatorname{dom}\mathcal{A}^i=\widehat{\mathcal{V}}_i\ \right\}\\ &= \left\{\ x\in\mathbb{C}^n \ \middle|\ \exists\, y\in\mathbb{C}^n\ \exists\, z\in\mathbb{C}^d:\ x=Fz\ \text{ and }\ y=Gz\ \text{ and }\ y\in\widehat{\mathcal{V}}_i\ \right\}\\ &= \left\{\ x\in\mathbb{C}^n \ \middle|\ \exists\, z\in\mathbb{C}^d:\ x=Fz\ \text{ and }\ Gz\in\widehat{\mathcal{V}}_i\ \right\}=\widehat{\mathcal{V}}_{i+1}.\end{aligned}$$

The proof of (5.2) is analogous and omitted. From (5.1) and (5.2) and the fact that, by finite dimensionality, $\mathcal{V}^*=\mathcal{V}_n$, $\widehat{\mathcal{V}}^*=\widehat{\mathcal{V}}_n$, $\mathcal{W}^*=\mathcal{W}_n$, $\widehat{\mathcal{W}}^*=\widehat{\mathcal{W}}_n$ it now follows that

$$\operatorname{dom}\mathcal{A}^n=\mathcal{V}^*=\widehat{\mathcal{V}}^*\quad\text{ and }\quad\mathcal{R}_\infty(\mathcal{A})=\mathcal{W}^*=\widehat{\mathcal{W}}^*.$$

Next we show that $\widehat{\mathcal{V}}^*=\mathcal{R}_f(\mathcal{A})$. To this end, let $W\in\mathbb{C}^{n\times n}$, $T\in\mathbb{C}^{d\times d}$ be invertible matrices such that WFT and WGT are in KCF (4.2). We prove that

$$\forall\, i\in\mathbb{N}_0:\ \widehat{\mathcal{V}}_i=W^{-1}\left(\mathbb{C}^{n_0}\times\operatorname{ran}N_\alpha^i\times\mathbb{C}^{|\beta|-n_\beta}\times\operatorname{ran}(N_\gamma^\top)^i\right).$$

For $i=0$ the statement is true. Assume that the statement is true for some $i\in\mathbb{N}_0$. We obtain

$$\begin{aligned}\widehat{\mathcal{V}}_{i+1} &= \left\{\ Fx \ \middle|\ x\in\mathbb{C}^d,\ Gx\in\widehat{\mathcal{V}}_i\ \right\}\\ &= \left\{\ FTy \ \middle|\ W^{-1}\begin{bmatrix}A_0&0&0&0\\0&I_{|\alpha|}&0&0\\0&0&L_\beta&0\\0&0&0&L_\gamma^\top\end{bmatrix}y\in\widehat{\mathcal{V}}_i\ \right\}\end{aligned}$$

Now, by assumption, $\widehat{\mathcal{V}}_i=W^{-1}\left(\mathbb{C}^{n_0}\times\operatorname{ran}N_\alpha^i\times\mathbb{C}^{|\beta|-n_\beta}\times\operatorname{ran}(N_\gamma^\top)^i\right)$ and we have

$$\begin{aligned}\widehat{\mathcal{V}}_{i+1} &= W^{-1}\begin{bmatrix}I_{n_0}&0&0&0\\0&N_\alpha&0&0\\0&0&K_\beta&0\\0&0&0&K_\gamma^\top\end{bmatrix}\left\{\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}\ \middle|\ \begin{array}{l}y_1\in\mathbb{C}^{n_0},\ y_2\in\operatorname{ran}N_\alpha^i,\\ y_3\in\mathbb{C}^{|\beta|},\ L_\gamma^\top y_4\in\operatorname{ran}(N_\gamma^\top)^i\end{array}\right\}\\ &= W^{-1}\left(\mathbb{C}^{n_0}\times\operatorname{ran}N_\alpha^{i+1}\times\mathbb{C}^{|\beta|-n_\beta}\times\operatorname{ran}(N_\gamma^\top)^{i+1}\right),\end{aligned}$$

where for the last equality we need to show that

$$\left\{\ K_\gamma^\top y_4\ \middle|\ L_\gamma^\top y_4\in\operatorname{ran}(N_\gamma^\top)^i\ \right\}=\operatorname{ran}(N_\gamma^\top)^{i+1}.$$

Observe that $N_\gamma^\top L_\gamma^\top=K_\gamma^\top$ and $N_\gamma^\top K_\gamma^\top=K_\gamma^\top N_{\gamma-1}^\top$, where $\gamma-1=(\gamma_1-1,\ldots,\gamma_{n_\gamma}-1)$ and if $\gamma_j=1$ for some $j\in\{1,\ldots,n_\gamma\}$, then we define

$$N_{\gamma-1}=\operatorname{diag}\left(N_{\gamma_1},\ldots,N_{\gamma_j-1},N_{\gamma_j+1},\ldots,N_{\gamma_{n_\gamma}}\right).$$

Furthermore, we have that for any $j,k\in\mathbb{N}$, $k\geq 2$, and $v\in\mathbb{C}^{k-1}$, $v=(v_1,\ldots,v_{k-1})^\top$,

$$L_k^\top v\in\operatorname{ran}(N_k^\top)^j\quad\Longleftrightarrow\quad v_{k-j}=\ldots=v_{k-1}=0\quad\Longleftrightarrow\quad v\in\operatorname{ran}(N_{k-1}^\top)^j,$$

from which it follows that

$$\left\{\ v\in\mathbb{C}^{|\gamma|-n_\gamma}\ \middle|\ L_\gamma^\top v\in\operatorname{ran}(N_\gamma^\top)^j\ \right\}=\operatorname{ran}(N_{\gamma-1}^\top)^j.$$

Now we are in the position to infer that

$$\begin{aligned}\operatorname{ran}(N_\gamma^\top)^{i+1} = \operatorname{ran}(N_\gamma^\top)^{i+1}L_\gamma^\top &= \operatorname{ran}(N_\gamma^\top)^i K_\gamma^\top \\ &= \operatorname{ran} K_\gamma^\top (N_{\gamma-1}^\top)^i = \left\{ \ K_\gamma^\top v \ \middle| \ L_\gamma^\top v \in \operatorname{ran}(N_\gamma^\top)^i \ \right\}.\end{aligned}$$

Since $N_\alpha^n = 0$ and $(N_\gamma^\top)^n = 0$ it now follows with Theorem 4.5 that

$$\operatorname{dom} \mathcal{A}^n = \widehat{\mathcal{V}}^* = W^{-1}\left(\mathbb{C}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\beta|-n_\beta} \times \{0\}^{|\gamma|}\right) = \mathcal{R}_f(\mathcal{A}).$$

The remaining statements can now be concluded from Theorem 4.5. ∎

Equations (5.1) and (5.2) provide a direct connection between the Wong sequences and the corresponding linear relation $\mathcal{A}$. In [8] it is shown how the Wong sequences can be used to define a basis transformation which puts the matrix pair (F, G), where $\mathcal{A} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$, into KCF. Moreover, in [8, 9] it is shown that certain Wong sequences completely determine the KCF. In this sense, the KCF of F and G is completely determined by the linear relation $\mathcal{A} = \operatorname{ran} \left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$.

6. Conclusion. We have shown how the Kronecker canonical form provides a natural decomposition of a linear relation into a completely singular relation, a Jordan relation and a multishift. Furthermore, the KCF can be used for a complete description of the structure of a linear relation up to the structure of the multishift part. In this sense we have derived an analogue to the Jordan canonical form for matrices; this is new for linear relations. On the other hand, the KCF of a given matrix pair is completely described by the corresponding linear relation, which can be established using Wong sequences.

The KCF is widely used in the study of matrix pairs or, what is the same, in the investigation of DAEs. A famous unsolved problem in the theory of matrix pairs is the distance to the nearest singular pair, see [11]. It is possible to characterize regularity and singularity in terms of the induced linear relation. For linear relations, the effect of perturbations can be studied using the gap metric, see [3, 19], or by utilizing results for finite dimensional perturbations, see e.g. [2]. In future research one may use the deep connection between linear relations and matrix pencils presented in the present paper. In particular, existing perturbation theory for linear relations is now available for the study of matrix pencils and DAEs.

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