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Random Approximations in Multiobjective Optimization

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Abstract

Often decision makers have to cope with a programming problem with unknown quantitities. Then they will estimate these quantities and solve the problem as it then appears - the 'approximate problem'. Thus there is a need to establish conditions which will ensure that the solutions to the approximate problem will come close to the solutions to the true problem in a suitable manner. Confidence sets, i.e. sets that cover the true sets with a given prescribed probability, provide useful quantitative information. In this paper we consider multiobjective problems and derive confidence sets for the sets of efficient points, weakly efficient points, and the corresponding solution sets.

Besides the crucial convergence conditions for the objective and/or constraint functions, one approach for the derivation of confidence sets requires some knowledge about the true problem, which may be not available. Therefore also another method, called relaxation, is suggested. This approach works without any knowledge about the true problem.

The results are applied to the Markowitz model of portfolio optimization.

Keywords: multiobjective programming, stability, confidence sets, estimated functions, relaxation, Markowitz model

1 Introduction

Often decision makers face a problem where not all quantities are completely known. Then they usually estimate the unknown parameters or probability distributions and solve the problem as it then arises. They hope that the decisions obtained in that way come close to the true optimal decisions. Hence there is a need for assertions that can justify this hope, so-called stability assertions. Regarding the estimates as random variables, the decision problem which is really solved, is a realization of a random problem. As reasonable estimates approximate the true value in some random sense if the sample size n tends to infinity, one can ask in what sense and under what conditions the random decision problems approximate the true ones. Qualitative stability statements provide conditions that ensure convergence (almost surely, in probability, in distribution) of optimal values and solution sets of the random surrogate problems.

In parametric statistics, in addition to the convergence (consistency) of the estimators, confidence sets play an important rule. Confidence sets are random sets that cover the true parameter with a prescribed high probability. They are derived from samples of size n and they should shrink to the true parameter if n increases. Thus they can provide important quantitative information.

In the present paper we will derive confidence sets in the framework of multiobjective decision problems. Decision makers usually have to take into account more than one goals. Then the sets of efficient points in the image set are investigated instead of the optimal values. We will use a minimization framework, hence the sets of minimal points with respect to the usual partial ordering in R^k and the corresponding decisions - the solutions - will be investigated. And again, estimates will have to be used instead of unknown quantities and one arrives at a sequence of random multiobjective problems.

Estimates for unknown quantities are not the only framework where random approximations occur. Random surrogate problems come also into play if completely known decision problems are solved with an algorithm that uses random steps. Sample size approximation [15] is an important example. Furthermore, bootstrap procedures (in a wider sense) often sample from an approximate model, which is obtained via an estimation procedure, hence confidence sets for the parameters of the model are of interest.

Qualitative stability results for deterministic multiobjective parametric programming problems can be found in [7], [8], [9], [13], [19], and [20]. The authors consider, among others, 'semicontinuous' behaviour of the sets of efficient points and the corresponding decisions. Multiobjective stochastic optimization problems are investigated in [3] and the probability measure is regarded as parameter. In [28] stability results for sequences of deterministic problems are formulated in a unifying framework.

The results from deterministic multiobjective programming, particularly from multiobjective parametric programming, can be employed to derive statements about convergence almost surely, see [23]. Note that the stochastic approach usually requires considerably weaker conditions than the deterministic approach, while in real-life situations an assertion that holds 'almost surely' is usually not worse than an assertion that holds in the deterministic sense. Weaker convergence modes, such as convergence in probability or convergence in distribution, require even weaker assumptions and hence apply to a larger class of problems.

In [23] a method was suggested which opens the possibility to derive results about the convergence in probability from assertions for convergence almost surely. In his PhD thesis Gersch [1] investigated convergence in distribution for single-objective and multiobjective problems and considered also ε-efficient points.

Quantitative assertions for single-objective problems which provide bounds for the distance between solution sets in terms of probability metrics are given e.g. in [12]. Confidence sets for solution sets of single-objective optimization problems were derived in [11] and [25]. Like in statistics, in our framework a confidence set is understood as a set that covers the true set with a prescribed high probability. Recall that many statistical estimates are obtained as solutions of random optimization problems and hence fit into this setting, cf. [24]. In statistics, however, confidence sets are usually derived for single-valued solutions of an estimation-optimization procedure. Often so-called identifiability conditions are imposed to enforce single-valuedness. In the multiobjective setting we can not confine to sets that are single-valued. The approach, which will be used here, relies on a quantified version of convergence in probability. Such 'quantified' convergence assertions can not be obtained in the way described in [23]. We therefore extend a method suggested in [21]. While in [21] only a rate for the convergence in probability was taken into account, the derivation of confidence sets also requires further information.

In order to derive a confidence set for a set $\Psi_0 \subset R^p$, we can proceed as follows: Assume that a sequence $(\Psi_n)_{n\in N}$ of random sets with the following property is available:

$$\forall \kappa > 0: \sup_{n \in N} P\{\omega: \ \Psi_0 \setminus U_{\beta_{n,\kappa}} \Psi_n(\omega) \neq \emptyset\} \leq \mathcal{H}(\kappa). \quad (1)$$

Here $(\beta_{n,\kappa})_{n \in N}$ is a sequence of nonnegative numbers tending to zero and $\mathcal{H}$ satisfies $\lim_{\kappa \to \infty} \mathcal{H}(\kappa) = 0$.

Given a prescribed confidence level $1 - \eta$, one determines κ_0 such that $\mathcal{H}(\kappa_0) \leq \eta$. Then for each sample size n the set $U_{\beta_{n,\kappa_0}} \Psi_n$ covers the true set Ψ_0 at least with probability $1 - \eta$. Note that no knowledge about the exact distribution or the asymptotic distribution is needed. Because confidence sets for each sample size n can be derived in this way, Pflug [11], who first derived confidence sets in the framework of stochastic programming, introduced the denotion 'universal confidence' sets.

We call sequences of random sets which satisfy (1) *outer approximations in probability with convergence rate* $\beta_{n,\kappa}$ *and tail behavior function* $\mathcal{H}$. The denotations 'convergence rate' and 'tail behavior function' were suggested in [11]. Note that the convergence rate ist different from the rate for convergence in probability considered in [21].

Once an outer approximation is found, each sequence of supersets also forms an outer approximation. Since one is interested in small confidence sets, one could therefore ask for sequences which tend to be contained in the true set. So-called *inner approximations* with convergence rate and tail behavior function, defined by

$$\forall \kappa > 0: \sup_{n \in N} P\{\omega: \ \Psi_n(\omega) \setminus U_{\beta_{n,\kappa}} \Psi_0 \neq \emptyset\} \leq \mathcal{H}(\kappa), \quad (2)$$

have this property.

Note that a sequence which is an inner and an outer approximation with the same convergence rate and tail behavior function is Kuratowski-Painlevè-convergent with this convergent rate and tail behavior function, see [25] for the relation to Kuratowski-Painlevè-convergence in probability.

However, inner approximations need not be contained in the true set for fixed n. Approximations which are contained in the true set for each n with a high probability will be called *subset-approximations*, see [27]. These kind of approximations will be needed in the relaxation approach.

In multiobjective decision problems the sets of efficient points and the corresponding solution sets are of main interest. However, it is well-known from parametric multiobjective programming that the sets of efficient points of the approximate problems usually do not approximate the set of efficient points of the true problem, they tend to be contained in a superset, the set so-called weakly efficient points. At the end of Section 2 an example with a deterministic sequence of approximating problems is provided which shows a typical situation. Hence, additionally to the sets of efficient points, also the sets of weakly efficient points and the corresponding decisions (weak solutions) have to be taken into account. We will provide outer and inner approximations for the image sets, the sets of efficient points, the sets of weakly efficient points, and the corresponding solution sets. In this paper we will not consider approximately efficient solutions, as for instance dealt with in [18] and [28]. This topic will be considered elsewhere.

The assertions will be illustrated by the Markowitz model of portfolio optimization. As some of the conditions usually imposed in stability theory in multiobjective deterministic programming do not apply to the Markowitz model, we will also prove results which are particularly useful for linear and quadratic objective functions with estimated parameters.

The results assume certain convergence properties of the objective and/or the constraint functions. Sufficient conditions for these assumptions are considered in [26] for functions which are expectations. Regression functions are dealt with in [16] and [17]. The case of estimated parameters for a Lipschitz function will be added in this paper.

Besides the convergence of the objective and perhaps the constraint functions some knowledge about the true problem is needed. In many cases bounds for the continuity functions or growth functions, employed in the following, are available. If one can or will not rely on information about the true problem, one can use an approach called relaxation. It was investigated in [27] for the single-objective case. In this paper it will be elaborated in the multiobjective framework.

The paper is organized as follows. The mathematical model is provided in section 2. The Markowitz model is introduced in section 3. Section 4 investigates the image sets. Moreover, further sufficient conditions, which particularly apply to the Markowitz model, are derived. In section 5 results about outer and inner approximations of the sets of efficient points and the sets of weakly efficient points are proved. Section 6 deals with the solution sets and section 7 explains the relaxation approach.

2 Mathematical Model

Suppose that we are given the deterministic multiobjective programming problem

$$(P_0) \quad \min_{x\in\Gamma_0} f_0(x)$$

where $\Gamma_0 \subset R^p$ is a nonempty closed set and $f_0|R^p \to R^k$. Minimization is understood with respect to the usual partial ordering "$\leq$" in R^k, which is generated by the cone R^k_+. By $(a_1 \ldots a_k)^T < (b_1 \ldots b_k)^T; \quad a_i, b_i \in R$, we mean $a_i < b_i \; \forall i \in \{1, \ldots, k\}$.

We consider random surrogate problems

$$(P_n(\omega)) \quad \min_{x\in\Gamma_n(\omega)} f_n(x, \omega)$$

where $\Gamma_n, n \in N$, are multifunctions defined on a given complete probability space $[\Omega, \mathcal{A}, P]$ with values in the $\sigma-$field of Borel sets Σ^p. $f_n|R^p \times \Omega \to R^k$ is taken as $(\Sigma^p \otimes \mathcal{A}, \Sigma^k)-$ measurable. Sufficient conditions for this property are given by Vogel [21]. To avoid restricting the model to closed-valued multifunctions we, additionally, assume that the graphs Graph $\Gamma_n, n \in N$, belong to $\mathcal{A} \otimes \Sigma^p$. In our setting multifunctions with measurable graphs are measurable, i.e. $\Gamma_n^{-1}(M) := \{\omega \in \Omega : \Gamma_n(\omega) \cap M \neq \emptyset\} \in \mathcal{A}$ for every closed set $M \in \Sigma^p$.

Γ_0 and $\Gamma_n(\omega)$ may be specified by inequality constraints:

$$\Gamma_0 := \{x \in R^p : g_0^j(x) \leq 0, j \in J\},$$

$$\Gamma_n(\omega) := \{x \in R^p : g_n^j(x, \omega) \leq 0, j \in J\},$$

where $g_0^j|R^p \to R^1; \quad g_n^j|R^p \times \Omega \to R^1$ is $(\Sigma^p \otimes \mathcal{A}, \Sigma^1)$-measurable and J is a finite index set. Multifunctions Γ_n of the above form have measurable graphs.

For sake of simplicity we assume that there is a compact set $K \subset R^p$ such that $\Gamma_0 \subset K$ and $\Gamma_n(\omega) \subset K \; \forall n \in N \; \forall \omega \in \Omega$.

When a single component of f_0 or f_n or other vector-valued functions is dealt with, the same letter is used with a superscript: f_n^j denotes the j-th component of f_n. For elements of R^p, R^m, or R^k, however, we use subscripts: x_j denotes the j-th component of x.

Firstly, we have to deal with the image sets. We will denote them by F_0 and F_n:

$$F_0 := \{f_0(x) : x \in \Gamma_0\} = f_0(\Gamma_0),$$

$$F_n(\omega) := \{f_n(x,\omega) : x \in \Gamma_n(\omega)\} = f_n(\Gamma_n(\omega),\omega).$$

The sets of efficient points (or efficiency sets) for the original problem (P_0) and the approximate problems $(P_n(\omega))$ are explained by

$$E_0 := \{y \in F_0 : \nexists\, \bar{y} \in F_0 \text{ with } (\bar{y} \le y \wedge \bar{y} \neq y)\} = \{y \in F_0 : (y - R^k_+) \cap F_0 = \{y\}\},$$

$$\begin{aligned} E_n(\omega) &:= \{y \in F_n(\omega) : \nexists\, \bar{y} \in F_n(\omega) \text{ with } (\bar{y} \le y \wedge \bar{y} \neq y)\} \\ &= \{y \in F_n(\omega) : (y - R^k_+) \cap F_n(\omega) = \{y\}\}. \end{aligned}$$

By S_0 and S_n we denote the corresponding solution sets:

$$S_0 := \{x \in \Gamma_0 : \nexists\, \bar{x} \in \Gamma_0 \text{ with } (f_0(\bar{x}) \le f_0(x) \wedge f_0(\bar{x}) \neq f_0(x))\},$$

$$S_n(\omega) := \{x \in \Gamma_n(\omega) : \nexists\, \bar{x} \in \Gamma_n(\omega) \text{ with } (f_n(\bar{x},\omega) \le f_n(x,\omega) \wedge f_n(\bar{x},\omega) \neq f_n(x,\omega))\}.$$

Moreover, we introduce the sets of weakly efficient points

$$W_0 := \{y \in F_0 : \nexists\, \bar{y} \in F_0 \text{ with } \bar{y} < y\} = \{y \in F_0 : (y - \text{int}R^k_+) \cap F_0 = \emptyset\}$$

and

$$W_n(\omega) := \{y \in F_n(\omega) : \nexists \bar{y} \in F_n(\omega) \text{ with } \bar{y} < y\} = \{y \in F_n(\omega) : (y - \text{int}R^k_+) \cap F_n(\omega) = \emptyset\}$$

and the corresponding 'weak' solution sets

$$S_0^W := \{x \in \Gamma_0 : \nexists\, \bar{x} \in \Gamma_0 \text{ with } f_0(\bar{x}) < f_0(\omega)\},$$

$$S_n^W(\omega) := \{x \in \Gamma_n(\omega) : \nexists\, \bar{x} \in \Gamma_n(\omega) \text{ with } f_n(\bar{x},\omega) < f_n(x,\omega)\}$$

. By definition, the sets of efficient points are contained in the sets of weakly efficient points and a corresponding relation holds for the solution sets.

The following deterministic example shows that, in general, one can only expect that the sets of efficient points of the approximating problems tend to a subset of the set of weakly efficient points of the true problem.

Example. Suppose that $f_0^1(x_1,x_2) = x_1$, $f_0^2(x_1,x_2) = x_2$ and $0 \le x_i \le 1$, $i = 1,2$. Then the set of images is the set

$$F_0 := \{(f_0^1(x_1,x_2), f_0^2(x_1,x_2)) : 0 \le x_i \le 1,\ i = 1,2\} = [0,1] \times [0,1]$$

and the only efficient point is the point $(0,0)$. The approximating functions are assumed to have the form $f_n^1(x_1,x_2) = x_1$ and

$$f_n^2(x_1,x_2) = \begin{cases} 1-(n-1)x_1+x_2, & \text{if } 0 \le x_1 \le \frac{1}{n},\ 0 \le x_2 \le 1, \\ \frac{1-x_1}{n-1}+x_2, & \text{if } \frac{1}{n} < x_1 \le 1,\ 0 \le x_2 \le 1. \end{cases}$$

Then the set of efficient points of

$$F_n := \{(f_n^1(x_1,x_2), f_n^2(x_1,x_2)) : 0 \le x_i \le 1,\ i=1,2\}$$

is the set

$$E_n = \{(x_1,x_2) : 0 \le x_1 \le \frac{1}{n}, x_2 = 1-(n-1)x_1\} \cup \{(x_1,x_2) : \frac{1}{n} < x_1 \le 1,\ x_2 = \frac{1-x_1}{n-1}\},$$

which, for $n \to \infty$ approximates the set $W_0 = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ in the Hausdorff metric and in the Kuratowski-Painlevé sense.

In [23] we showed that measurability of the sets under consideration is ensured under the assumptions of this paper.

3 The Markowitz Model

As an example we will consider the well-known Markowitz model of portfolio optimization. An investor has an amount of money of value 1 and can invest his money into a set of p assets. Markowitz suggested to maximize the expected return and minimize the variance of the return as an indicator of risk. Thus, denoting the proportion invested into the i-th asset with x_i and the random return of the i-th asset with ρ_i, we have the decision-vector $x = (x_1,\dots,x_p)^T$ and the random vector of returns $\rho = (\rho_1,\dots,\rho_p)^T$. Let $\mu = (\mu_1,\dots,\mu_p)^T$ be the expectation of ρ and B its covariance matrix. Then, in our minimization setting, we obtain the objective functions

$$f_0^1(x) = -\mathrm{E}(x^T\rho) = -x^T\mu$$

and

$$f_0^2(x) = \mathrm{var}(x^T\rho) = x^TBx.$$

We assume that estimates

$$\hat{\mu}_n = (\hat{\mu}_{n,1},\dots,\hat{\mu}_{n,p})^T = \frac{1}{n}\sum_{l=1}^{n}\rho^{(l)}$$

for the expectation and

$$\hat{B}_n = \frac{1}{n-1}\sum_{l=1}^{n}(\rho^{(l)} - \hat{\mu}_n)(\rho^{(l)} - \hat{\mu}_n)^T$$

for the covariance matrix, based on i.i.d. random samples $\rho^{(l)} = (\rho_1^{(l)}, \ldots, \rho_p^{(l)})^T$ $l = 1, \ldots, n$, are available. Thus the approximating objective functions have the following form:

$$f_n^1(x, \omega) = -x^T \hat{\mu}_n(\omega),$$

$$f_n^2(x, \omega)) = x^T \hat{B}_n(\omega)x.$$

The constraint set is given by

$$\Gamma_0 := \{x \in R^p | x_i \geq 0, i = 1, \ldots, p; \sum_{i=1}^{n} x_i \leq 1\}$$

and in this form there is no need for approximations. We would like to note, however, that further restrictions, e.g. in form of shortfall constraints, could be incorporated. Then also approximations of the constraint set would come into play.

Note that the measurability assumptions imposed in section 2 are satisfied.

4 Approximation of the Image Set

The sets of efficient points and the sets of weakly efficient points are subsets of the image sets. Therefore we start with the investigation of the behavior of the image sets F_n. We present an assertion for the inner approximations and an assertion for the outer approximations.

In the following theorems sets H, B and Λ occur. B is the set of sequences of positive numbers that converge monotonously to zero. H denotes the set of functions $\mathcal{H}|R^+ \to R^+$ with the property $\lim\limits_{\kappa \to \infty} \mathcal{H}(\kappa) = 0$.
Λ is defined by

$$\Lambda := \{\lambda | R^+ \to R^+ \colon \lambda \text{ is increasing, not constant, right-continuous, and satisfies } \lambda(0) = 0\}.$$

The functions in Λ allow for a generalized inverse in the following form: $\lambda^{-1}(y) := \inf\{z \in R^1 : \ \lambda(z) > y\}$. $||\cdot||$ denotes the Euclidean norm and d the metric induced by this norm. A neighborhood $U_\varepsilon A$ of a set $A \subset R^l$ is defined by $U_\varepsilon A = \{x \in R^l : d(x, A) < \varepsilon\}$. $U\Gamma_0$ denotes a suitable neighborhood of Γ_0.

Theorem 1-(i). Suppose that the following conditions are satisfied:

(CΓ-i) There exist a function $\mathcal{H}_1 \in H$ and for all $\kappa > 0$ a sequence $(\beta^{(1)}_{n,\kappa})_{n\in N} \in B$ such that
$\forall\kappa > 0 \ \sup_{n\in N} P\{\omega : \ \Gamma_n(\omega) \setminus U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \neq \emptyset\} \leq \mathcal{H}_1(\kappa)$.

(Cf) There exist a function $\mathcal{H}_2 \in H$ and for all $\kappa > 0$ a sequence $(\beta^{(2)}_{n,\kappa})_{n\in N} \in B$ such that
$\forall\kappa > 0 \ \sup_{n\in N} P\{\omega : \ \sup_{x\in U\Gamma_0} ||f_n(x, \omega) - f_0(x)|| \geq \beta^{(2)}_{n,\kappa}\} \leq \mathcal{H}_2(\kappa)$.

(Sf) There exists a function $\lambda \in \Lambda$ such that for all $\varepsilon > 0$
$\forall \ x, y \in U\Gamma_0 : ||f_0(x) - f_0(y)|| \geq \varepsilon \Rightarrow ||x - y|| \geq \lambda(\varepsilon)$.

Then for $\beta^{(3)}_{n,\kappa} := \max\{2\beta^{(2)}_{n,\kappa}, 2\lambda^{-1}(\beta^{(1)}_{n,\kappa})\}$ the following relation holds:
$\forall\kappa > 0$
$\sup_{n\in N} P\{\omega : U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0 \text{ and } F_n(\omega) \setminus U_{\beta^{(3)}_{n,\kappa}}F_0 \neq \emptyset\} \leq \mathcal{H}_1(\kappa) + \mathcal{H}_2(\kappa)$.

Proof. Assume that for given $0 < \kappa, \ n \in N$, and $\omega \in \Omega$ the relations $U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0$ and $F_n(\omega) \setminus U_{\beta^{(3)}_{n,\kappa}}F_0 \neq \emptyset$ are satisfied. Then there exists a $y_n(\omega) \in F_n(\omega)$ which does not belong to $U_{\beta^{(3)}_{n,\kappa}}F_0$. To $y_n(\omega)$ we find $x_n(\omega) \in \Gamma_n(\omega)$ with $f_n(x_n(\omega), \omega) = y_n(\omega)$. Hence $\forall x_0 \in \Gamma_0 : \ ||f_n(x_n(\omega) - f_0(x_0)|| \geq \beta^{(3)}_{n,\kappa}$.

Firstly, assume that $d(x_n(\omega), \Gamma_0) \geq \beta^{(1)}_{n,\kappa}$. Then $\Gamma_n(\omega) \setminus U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \neq \emptyset$ and we can employ condition (CΓ-i).

Secondly, assume that here exists $x_0(\omega) \in \Gamma_0$ with $x_0(\omega) \in U_{\beta^{(1)}_{n,\kappa}}(x_n(\omega))$. Then, by (Sf) and $||x_n(\omega) - x_0(\omega)|| < \beta^{(1)}_{n,\kappa}$, the inequality $||f_0(x_n(\omega)) - f_0(x_0(\omega))|| < \frac{\beta^{(3)}_{n,\kappa}}{2}$ follows. From $||f_n(x_n(\omega), \omega) - f_0(x_0(\omega))|| > \beta^{(3)}_{n,\kappa}$ we obtain $||f_n(x_n(\omega), \omega) - f_0(x_n(\omega))|| > \frac{\beta^{(3)}_{n,\kappa}}{2}$. As $x_n(\omega) \in U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0$ we have $\sup_{x\in U\Gamma_0} ||f_n(x, \omega) - f_0(x)|| \geq \beta^{(2)}_{n,\kappa}$ and the assertion follows by (Cf). $\square$

In the conclusion we use the condition $U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0$. This is no restriction, we only have to make sure that $U\Gamma_0$ is large enough to cover all sets $U_{\beta^{(1)}_{n,\kappa}}\Gamma_0$. If $U\Gamma_0$ does not have this property, we have to replace $\sup\limits_{n\in N} P\{\ldots\}$ by $\sup\limits_{n\geq n_0} P\{\ldots\}$.

Theorem 1-(ii). Suppose that (Cf), (Sf), and the following condition are satisfied:

(CΓ-o) There exist a function $\mathcal{H}_1 \in H$ and for all $\kappa > 0$ a sequence $(\beta^{(1)}_{n,\kappa})_{n\in N} \in B$ such that
$\forall \kappa > 0 \ \sup\limits_{n\in N} P\{\omega : \ \Gamma_0 \setminus U_{\beta^{(1)}_{n,\kappa}}\Gamma_n(\omega) \neq \emptyset\} \leq \mathcal{H}_1(\kappa).$

Then for $\beta^{(3)}_{n,\kappa} := \max\{2\beta^{(2)}_{n,\kappa}, 2\lambda^{-1}(\beta^{(1)}_{n,\kappa})\}$ the following relation holds:
$\forall \kappa > 0$

$$\sup_{n\in N} P\{\omega : U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0 \text{ and } F_0 \setminus U_{\beta^{(3)}_{n,\kappa}}F_n(\omega) \neq \emptyset\} \leq \mathcal{H}_1(\kappa) + \mathcal{H}_2(\kappa).$$

Proof. Assume that for given $0 < \kappa$, $n \in N$, and $\omega \in \Omega$ the relations $U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0$ and $F_0 \setminus U_{\beta^{(3)}_{n,\kappa}}F_n(\omega) \neq \emptyset$ are satisfied. Then there exists a $y_0(\omega) \in F_0$ which does not belong to $U_{\beta^{(3)}_{n,\kappa}}F_n(\omega)$. To $y_0(\omega)$ we find $x_0(\omega) \in \Gamma_0$ with $f_n(x_0(\omega), \omega) = y_0(\omega)$.

Firstly, assume that $d(x_0(\omega), \Gamma_n(\omega)) \geq \beta^{(1)}_{n,\kappa}$. Then $\Gamma_0 \setminus U_{\beta^{(1)}_{n,\kappa}}\Gamma_n(\omega) \neq \emptyset$. Otherwise we choose $x_n(\omega) \in \Gamma_n(\omega)$ with $||x_0(\omega) - x_n(\omega)|| < \beta^{(1)}_{n,\kappa}$ and proceed as in the proof of Theorem 1-(i). □

Sufficient conditions for (CΓ-i) and (CΓ-o) are available for the case that the constraint set is given by inequality constraints. Then assumptions similar to (Cf) are imposed for the constraint functions, see [25]. The condition (Sf) is a 'continuity' condition. In many cases it should be possible to give at least a rough bound for the function λ. Note that, once a function λ satisfying (Sf) is found, each function in Λ with smaller positive values also satisfies (Sf). Smaller values for λ, however, result in larger confidence sets.

Sufficient conditions for the condition (Cf) can be derived from sufficient conditions for single-objective functions, see the introduction for references. The case of estimated parameters has not been considered so far and will be added in the following. We confine the investigation to functions with values in R^1.

Suppose that there exist a function $f|R^p \times R^m \to R^1$ such that $f_0(x) := f(x, y_0)$ for some $y_0 \in R^m$. If y_0 is estimated by a sequence $(Y_n)_{n\in N}$ of random variables, we obtain $f_n(x,\omega) = f(x, Y_n(\omega))$.

Lemma 4.1. Suppose that Γ_0 is compact and that f is continuous on $R^p \times \{y_0\}$. Furthermore, assume that there exist a neighborhood $U\Gamma_0$ and for all $\kappa > 0$ a sequence $(\beta_{n,\kappa})_{n\in N} \in B$ such that the following conditions are satisfied:

(Lf) To each $\kappa > 0$ there exists a constant $L(\kappa)$ such that for all $x \in U\Gamma_0$ and all $y \in U_{\beta_{1,\kappa}}\{y\}$ the relation $|f(x,y) - f(x,y_0)| < L(\kappa)||y - y_0||$ holds.

(CY) There exist a function $\mathcal{H} \in H$ such that $\forall \kappa > 0 \sup\limits_{n\in N} P\{\omega : \ ||Y_n(\omega) - y_0|| \geq \beta_{n,\kappa}\} \leq \mathcal{H}(\kappa)$.

Then (Cf) is satisfied with $\beta^{(2)}_{n,\kappa} = \frac{1}{L(\kappa)}\beta_{n,\kappa}$ and $\mathcal{H}_2 = \mathcal{H}$.

Proof. Assume that for given $0 < \kappa$, $n \in N$, and $\omega \in \Omega$ the relation $\sup\limits_{x\in U\Gamma_0} |f_n(x,\omega) - f_0(x)| \geq \beta^{(2)}_{n,\kappa}$ is satisfied. Hence there is an $x \in U\Gamma_0$ such that $|f(x, Y_n(\omega)) - f(x, y_0)| \geq \frac{1}{L(\kappa)}\beta_{n,\kappa}$. If $Y_n(\omega) \notin U_{\beta_{n,\kappa}}\{y_0\}$ condition (CY) can be employed. Otherwise, because of (Lf), $|f_n(x,\omega) - f_0(x)| < L(\kappa)||Y_n(\omega) - y_0|| < \beta^{(2)}_{n,\kappa}$ in contradiction to assumption. □

Proposition 2.1 could be applied to the Markowitz model and used to derive uniform concentration of measure results for the expected return and the variance of the return under a uniform boundedness condition of the random returns. Because of the simple structure of the constraint set in the Markowitz model, we will instead provide a direct proof for a bound. Note that we do not need to approximate the constraint set, hence instead of taking the supremum over a neighborhood of Γ_0 it suffices to take the supremum over the set Γ_0.

Lemma 4.2. Assume that there exists a constant $C \in R$ such that $|\rho_i| \leq C$ a.s. Then we have
$P\{\omega : \sup\limits_{x\in\Gamma_0} |\hat{\mu}^T_n(\omega) - \mu^T|x \geq \frac{\kappa}{\sqrt{n}}\} \leq 2pe^{-\frac{\kappa^2}{2p^2C^2}}$ and

$P\{\omega : \sup_{x\in\Gamma_0} |x^T(\hat{B}_n(\omega) - B)x| \geq \frac{\kappa}{\sqrt{n}}\} \leq 2p^2 e^{-\frac{\kappa^2}{8C^4}}$.

Proof. Firstly, we use Hoeffding's inequality [2] and obtain

$P\{\omega : \sup_{x\in\Gamma_0} | \sum_{j=1}^{p} [\frac{1}{n} \sum_{l=1}^{n} \rho_j^{(l)} - \mathrm{E}\rho_j^{(l)}] x_j| \geq \eta\}$

$\leq P\{\omega : \ \exists j \in \{1, \dots, p\} \text{ with } \sup_{x\in\Gamma_0} |\frac{1}{n} \sum_{l=1}^{n} \rho_j^{(l)} - \mathrm{E}\rho_j^{(l)}| x_j \geq \frac{\eta}{p}\}$

$\leq P\{\omega : \ \exists j \in \{1, \dots, p\} \text{ with } |\frac{1}{n} \sum_{l=1}^{n} \rho_j^{(l)} - \mathrm{E}\rho_j^{(l)}| \geq \frac{\eta}{p}\} \leq 2pe^{-\frac{n\eta^2}{2p^2C^2}}$.

With $\eta = \frac{\kappa}{\sqrt{n}}$ the first conclusion follows.

For the second assertion we consider $A(\omega) := \hat{B}_n(\omega) - B$ and make use of the following inequalities:

$$|x^T A(\omega) x| \leq \max_{i,j\in\{1,\dots,p\}} |a_{i,j}(\omega)| x^T x \leq \max_{i,j\in\{1,\dots,p\}} |a_{i,j}(\omega)|.$$

Hence $|x^T A(\omega)x| \geq \eta$ implies $\max_{i,j\in\{1,\dots,p\}} |a_{i,j}(\omega)| \geq \eta$.

Now we use a concentration-of-measure result for

$$a_{i,j}(\omega) = \frac{1}{n-1} \sum_{l=1}^{n} (\rho_i^{(l)}(\omega) - \hat{\mu}_i(\omega))(\rho_j^{(l)}(\omega) - \hat{\mu}_j(\omega)) - \mathrm{cov}(\rho_i, \rho_j),$$

where $\hat{\mu}_i(\omega)$ stands for $\hat{\mu}_{n,i}(\omega)$ and $\mathrm{cov}(\rho_i, \rho_j) = \mathrm{E}[(\rho_i - \mathrm{E}\rho_i)(\rho_j - \mathrm{E}\rho_j)]$. Since the summands in the above sum are not independent, we can not make use of Hoeffding's inequality. Instead we use McDiarmid's inequality [6]. $a_{i,j}$ can be regarded as a function of the i.i.d. vectors $(\rho_i^{(1)}, \rho_j^{(1)}), \dots, (\rho_i^{(n)}, \rho_j^{(n)})$:

$$\begin{aligned} a_{i,j} &= g((\rho_i^{(1)}, \rho_j^{(1)}), \dots, (\rho_i^{(n)}, \rho_j^{(n)})) \\ &= \frac{1}{n-1} \sum_{l=1}^{n} (\rho_i^{(l)} - \frac{1}{n} \sum_{k=1}^{n} \rho_i^{(k)})(\rho_j^{(l)} - \frac{1}{n} \sum_{r=1}^{n} \rho_j^{(r)}) - \mathrm{cov}(\rho_i, \rho_j). \end{aligned}$$

Note that $\mathrm{E}(a_{i,j}) = 0$. $\sum_{l=1}^{n} (\rho_i^{(l)}(\omega) - \hat{\mu}_i(\omega))(\rho_j^{(l)}(\omega) - \hat{\mu}_j(\omega))$ can be rewritten as

$\sum_{l=1}^{n} (\rho_i^{(l)}(\omega) - \hat{\mu}_i(\omega))(\rho_j^{(l)}(\omega) - \hat{\mu}_j(\omega)) = \bar{\rho}_i(\omega)^T M \bar{\rho}_j(\omega)$

where $\bar{\rho}_i = (\rho_i^{(1)} - \mu_i, \dots, \rho_i^{(n)} - \mu_i)^T$, and $M = (m_{i,j})_{i,j=1,\dots,n}$ with $m_{i,i} = 1 - \frac{1}{n}$ and $m_{i,j} = \frac{1}{n}$ if $i \neq j$.

Consequently we have for all (y_j, z_j) with $|y_j| \leq C$ and $|z_j| \leq C$ the inequality

$|g((y_1,z_1),\ldots,(y_r,z_r),\ldots(y_n,z_n)) - g((y_1,z_1),\ldots,(y'_r,z'_r),\ldots(y_n,z_n))|$
$\le \frac{1}{n-1}(1-\frac{1}{n})4C^2 = \frac{1}{n}4C^2.$

Hence, by McDiarmid's inequality, for each pair (i,j),
$P\{\omega : |a_{i,j}(\omega)| \ge \eta) \le 2e^{-\frac{2n\eta^2}{16C^4}}$, and finally
$P\{\omega : \max_{i,j\in\{1,\ldots,p\}} |a_{i,j}(\omega)| \ge \eta\} \le p^2 P\{\omega : |a_{i,j}(\omega)| \ge \eta\} \le 2p^2 e^{-\frac{n\eta^2}{8C^4}}$.
With $\eta = \frac{\kappa}{\sqrt{n}}$ the conclusion follows. □

5 Approximation of the Sets of Efficient Points and the Sets of Weakly Efficient Points

As mentioned in the introduction and section 2 the sets of efficient points of the approximate problems tend to be contained in the set of weakly efficient points of the true problem. Therefore problems for which the set of efficient points and the set of weakly efficient points coincide are of special interest. Fortunately, there are some important cases where this property is fulfilled, see e.g. [20] or [21]. Particularly for the Markowitz model the following condition (VE) is useful:

(VE) For all y_λ with $y_\lambda = \lambda y_1 + (1-\lambda)y_2,\ y_1 \in F_0,\ y_2 \in F_0,$ $y_1 \ne y_2,\ \lambda \in (0,1)$, the set $(y_\lambda - \text{int}R^k_+) \cap F_0$ is nonempty.

Lemma 5.1. If (VE) is satisfied the equality $E_0 = W_0$ holds.

Proof. Suppose that there is a $y_0 \in W_0$ which does not belong to E_0. Hence there exists $y \in F_0$ such that $y \le y_0$ and $y_j < y_{0,j}$ for some $j \in \{1,\ldots,k\}$. Consider $y_\lambda = \lambda y_0 + (1-\lambda)y$. Then to y_λ there is $\tilde{y}_\lambda \in F_0$ with $\tilde{y}_\lambda < y_\lambda$ and consequently, because of $y_\lambda \le y_0$, also $\tilde{y}_\lambda < y_0$ in contradiction to $y_0 \in W_0$. □

Lemma 5.2. In the Markowitz model condition (VE) is satisfied if the returns have pairwise different positive expectations and the covariance matrix is positive definite.

Proof: Consider $y_1,\ y_2 \in F_0,\ y_1 \ne y_2, y_\lambda = \lambda y_1 + (1-\lambda)y_2$ with $\lambda \in (0,1)$, and the preimages $x_1 \in \Gamma_0$ and $x_2 \in \Gamma_0$ with $f_0(x_i) = y_i,\ i = 1,2$. We construct $y \in (y_\lambda - \text{int}R^k_+) \cap F_0$. Because of the strict convexity of f_0^2 we have for $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ the relation $f_0^2(x_\lambda) < y_{\lambda,2}$. Note that there is a neighborhood $U\{x_\lambda\}$ of x_λ with $f_0^2(x) < y_{\lambda,2}$ for all $x \in U\{x_\lambda\}$.

If x_λ is an inner point of Γ_0, we find $x \in U\{x_\lambda\} \cap \Gamma_0$ with $f_0^1(x) < f_0^1(x_\lambda)$ and can choose $y = f_0(x)$.

Finally we assume that x_λ belongs to the boundary of Γ_0. We construct an $x \in \Gamma_0$ such that $y = f_0(x)$ has the desired property. We distinguish two cases:

(i) Let $x_{\lambda,i} = 0$ for all i in an index set $I \subset \{1,\dots,p\}$ and $\sum\limits_{i=1}^{p} x_{\lambda,i} < 1$.

Then, taking into account that $f_0^1(x) = -x^T\vec{\mu}$ we find $\gamma > 0$ and x such that $x := x_\lambda + \gamma \sum\limits_{i \notin I} e_i \in \Gamma_0$ and $f_0^1(x) < y_{1,1}$. Here e_i denotes the unit vector with 1 in the i-th position.

(ii) Let $x_{\lambda,i} = 0$ for all i in an index set $I \subset \{1,\dots,p\}$ and $\sum\limits_{i=1}^{p} x_{\lambda,i} = 1$. Then the set $\bar{I} = \{1,\dots,p\} \setminus I$ contains at least two elements. Let $i_0 \in \bar{I}$ be such that $\text{E}(\rho_{i_0}) > \text{E}(\rho_i)$ for all $i \in \bar{I} \setminus \{\rho_{i_0}\}$. Then, with $i_1 \in \bar{I} \setminus \{\rho_{i_0}\}$, we choose $x := x_\lambda + \gamma(e_{i_0} - e_{i_1}) \in \Gamma_0$. $\square$

Theorem 2. Suppose that there exist a function $\mathcal{H} \in H$ and for all $\kappa > 0$ a sequence $(\beta_{n,\kappa})_{n\in N} \in B$ such that

(CF) $\forall \varepsilon > 0 \ \sup\limits_{n \in N} P\{\omega : \ (F_n(\omega) \setminus U_{\beta_{n,\kappa}} F_0) \cup (F_0 \setminus U_{\beta_{n,\kappa}} F_n(\omega)) \neq \emptyset\} \leq \mathcal{H}(\kappa)$.

(i) If, additionally, there exists a function $\delta_1 \in \Lambda$ such that

(C1) $\forall \varepsilon > 0 \ \forall y \in (U_{\delta_1(\varepsilon)} F_0 \setminus U_\varepsilon W_0) \ \exists y_0 \in W_0 : \ U_{\delta_1(\varepsilon)}\{y_0\} \subset y - \text{int}R^k_+$

is satisfied, then for $\beta^{(i)}_{n,\kappa} = \delta_1^{-1}(\beta_{n,\kappa})$ the following relation holds:
$\forall \kappa > 0 \ \sup\limits_{n\in N} P\{\omega : \ W_n(\omega) \setminus U_{\beta^{(i)}_{n,\kappa}} W_0) \neq \emptyset\} \leq \mathcal{H}(\kappa)$.

(ii) If, additionally, there exists a function $\delta_2 \in \Lambda$ such that

(C2) $\forall \varepsilon > 0 \ \forall y_0 \in E_0 \ \exists y_1 \in E_0 : U_{\delta_2(\varepsilon)}(y_1) \subset U_\varepsilon(y_0)$ and
$[(U_{\delta_2(\varepsilon)}(y_1) - R^k_+) \setminus U_\varepsilon(y_0)] \cap U_{\delta_2(\varepsilon)} F_0 = \emptyset$

is satisfied, then for $\beta^{(o)}_{n,\kappa} = \delta_2^{-1}(\beta_{n,\kappa})$ the following relation holds:
$\forall \kappa > 0 \ \sup\limits_{n\in N} P\{\omega : E_n(\omega) \neq \emptyset \text{ and } E_0 \setminus U_{\beta^{(o)}_{n,\kappa}} E_n(\omega) \neq \emptyset\} \leq \mathcal{H}(\kappa)$.

Proof. (i) Assume that for given $\kappa > 0$, $n \in N$, and $\omega \in \Omega$ the relation $W_n(\omega) \setminus U_{\beta^{(i)}_{n,\kappa}} W_0 \neq \emptyset$ is satisfied. Then there exists a $y_n(\omega) \in W_n(\omega) \subset F_n(\omega)$ which does not belong to $U_{\beta^{(i)}_{n,\kappa}} W_0$. Firstly, assume that $y_n(\omega) \notin U_{\beta_{n,\kappa}} F_0$. Then, because of $y_n(\omega) \in F_n(\omega)$ we immediately obtain $F_n(\omega) \setminus U_{\beta_{n,\kappa}} F_0 \neq \emptyset$. Secondly, suppose that $y_n(\omega) \in U_{\beta_{n,\kappa}} F_0$. Hence $y_n(\omega) \in U_{\beta_{n,\kappa}} F_0 \setminus U_{\beta^{(i)}_{n,\kappa}} W_0$. By condition (C1) with $\varepsilon = \beta^{(i)}_{n,\kappa}$ there is $y_0 \in W_0$ such that $U_{\beta_{n,\kappa}}(y_0) \subset y_n(\omega) - R^k_+$. By definition of W_n, $y_n(\omega) - \text{int}R^k_+$ can not contain elements of $F_n(\omega)$. Consequently we obtain $U_{\beta_{n,\kappa}}(y_0) \cap F_n(\omega) = \emptyset$ and, taking into account that $y_0 \in F_0$, also $F_0 \setminus U_{\beta_{n,\kappa}} F_n(\omega) \neq \emptyset$.

(ii) Assume that for given $\kappa > 0$, $n \in N$, and $\omega \in \Omega$ the relations $E_n(\omega) \neq \emptyset$ and $E_0 \setminus U_{\beta^{(o)}_{n,\kappa}} E_n(\omega) \neq \emptyset$ is satisfied. Then there is $y_0(\omega) \in E_0$ which does not belong to $U_{\beta^{(o)}_{n,\kappa}} E_n(\omega)$. By condition (C2) to $y_0(\omega)$ we find $y_1(\omega) \in E_0$ such that $U_{\beta_{n,\kappa}}\{y_1(\omega)\} \subset U_{\beta^{(o)}_{n,\kappa}}\{y_0(\omega)\}$ and $[(U_{\beta_{n,\kappa}}\{y_1(\omega)\} - R^k_+) \setminus U_{\beta^{(o)}_{n,\kappa}}(y_0(\omega)] \cap U_{\beta_{n,\kappa}} F_0 = \emptyset$.

Firstly, assume that $U_{\beta_{n,\kappa}}\{y_1(\omega)\} \cap F_n(\omega) = \emptyset$. As $y_1(\omega) \in E_0 \subset F_0$ we obtain $F_0 \setminus U_{\beta_{n,\kappa}} F_n(\omega) \neq \emptyset$.

Secondly, suppose that there exists $y_n(\omega) \in U_{\beta_{n,\kappa}}\{y_1(\omega)\} \cap F_n(\omega)$. Because of $y_0(\omega) \notin U_{\beta^{(o)}_{n,\kappa}} E_n(\omega)$ we have $U_{\beta^{(o)}_{n,\kappa}}\{y_0(\omega)\} \cap E_n(\omega) = \emptyset$. Consequently, since $U_{\beta_{n,\kappa}}\{y_1(\omega)\} \subset U_{\beta^{(o)}_{n,\kappa}}\{y_0(\omega)\}$, $y_n(\omega)$ does not belong to $E_n(\omega)$.

Hence we find an element $\tilde{y}_n(\omega) \in E_n(\omega)$ which 'dominates' $y_n(\omega)$ with respect to the order relation, i.e. $\tilde{y}_n(\omega) \in (y_n(\omega) - R^k_+) \cap E_n(\omega)$ and further $\tilde{y}_n(\omega) \in (U_{\beta_{n,\kappa}}\{y_1(\omega)\} - R^k_+) \cap E_n(\omega)$.

Because of $y_0(\omega) \notin U_{\beta^{(o)}_{n,\kappa}} E_n(\omega)$, $\tilde{y}_n(\omega)$, which belongs to $E_n(\omega)$, can not be an element of $U_{\beta^{(o)}_{n,\kappa}}\{y_0(\omega)\}$. Consequently $\tilde{y}_n(\omega) \in (U_{\beta_{n,\kappa}}\{y_1(\omega)\} - R^k_+) \setminus U_{\beta^{(o)}_{n,\kappa}}\{y_0(\omega)\})$. Because of (C2), we obtain $\tilde{y}_n(\omega) \notin U_{\beta_{n,\kappa}} F_0$, hence $F_n(\omega) \setminus U_{\beta_{n,\kappa}} F_0 \neq \emptyset$. □

Since the sets of efficient points are contained in the sets of weakly efficient points, under the assumptions of Theorem 2-(i) we immediately obtain: $\forall \kappa > 0 \ \sup_{n \in N} P\{\omega : E_n(\omega) \setminus U_{\beta^{(3)}_{n,\kappa}} W_0 \neq \emptyset\} \leq \mathcal{H}(\kappa)$.

The existence of functions δ_2 was already considered in [21]. However in the setting of [21] the additional property $\delta_2 \in \Lambda$ is not needed. In [14] the existence of functions $\delta_1 \in \Lambda$ and $\delta_2 \in \Lambda$ is shown under compactness conditions for F_0 and E_0. Particularly, the function δ_2 may be hard to determine in the general case. The problem becomes considerably easier under convexity conditions, e.g. (VE). Special cases will be considered elsewhere. If it is not possibly to determine δ_2 or δ_2, the relaxation approach, presented in section 7, can be employed.

For the condition $E_n(\omega) \neq \emptyset$ in the conclusion of Theorem 2-(ii) there are several sufficient conditions, which can be found in textbooks on multiobjective optimization.

6 Approximation of the Solution Sets

In the deterministic parametric framework, stability results for the solution sets are usually derived for one-to-one objective functions. In our setting we need a quantification with a growth function μ for this property, see the following condition :

(Gf) There exist a function $\mu \in \Lambda$ and a neighborhood $U\Gamma_0$ such that
$\forall \varepsilon > 0$
$\forall\, x, y \in U\Gamma_0 : ||x - y|| \geq \varepsilon \Rightarrow ||f_0(x) - f_0(y)|| \geq \mu(\varepsilon)$.

Theorem 3. Suppose that (CΓ-i), (Cf), and (Gf) are satisfied. Furthermore assume that the following condition holds:

(CW) There exist a function $\mathcal{H}_3 \in H$ and for all $\kappa > 0$ a sequence $(\beta^{(3)}_{n,\kappa})_{n\in N} \in B$ such that
$\sup_{n\in N} P\{\omega : W_n(\omega) \setminus U_{\beta^{(3)}_{n,\kappa}} W_0 \neq \emptyset\} \leq \mathcal{H}_3(\kappa)$.

Then for $\beta^{(4)}_{n,\kappa} := \max\{\mu^{-1}(\beta^{(2)}_{n,\kappa} + \beta^{(3)}_{n,\kappa}), \beta^{(1)}_{n,\kappa}, \beta^{(2)}_{n,\kappa}\}$ the following relation holds:

$\forall \kappa > 0$
$\sup_{n\in N} P\{\omega : U_{\beta^{(1)}_{n,\kappa}} \Gamma_0 \subset U\Gamma_0 \text{ and } S^W_n(\omega) \setminus U_{\beta^{(4)}_{n,\kappa}} S^W_0 \neq \emptyset\} \leq \mathcal{H}_1(\kappa) + \mathcal{H}_2(\kappa) + \mathcal{H}_3(\kappa.)$

Proof. Assume that for given $\kappa > 0$, $n \in N$, and $\omega \in \Omega$ the relations $U_{\beta^{(1)}_{n,\kappa}} \Gamma_0 \subset U\Gamma_0$ and $S^W_n(\omega) \setminus U_{\beta^{(4)}_{n,\kappa}} S^W_0 \neq \emptyset$ are satisfied. Then there exists an $x_n(\omega) \in S^W_n(\omega)$ which does not belong to $U_{\beta^{(4)}_{n,\kappa}} S^W_0$. Firstly, assume that $x_n(\omega) \notin U_{\beta^{(1)}_{n,\kappa}} \Gamma_0$. Then we can employ (CΓ-i).

Secondly, assume that $x_n(\omega) \in U_{\beta^{(1)}_{n,\kappa}}\Gamma_0$. If $||f_n(x_n(\omega),\omega) - f_0(x_n(\omega))|| \geq \beta^{(2)}_{n,\kappa}$ we make use of (Cf). Otherwise we have $||f_n(x_n(\omega),\omega) - f_0(x_n(\omega))|| < \beta^{(2)}_{n,\kappa}$ and can proceed as follows: For $x_0 \in S^W_0$ we obtain $||f_n(x_n(\omega),\omega) - f_0(x_0)|| \geq ||f_0(x_0) - f_0(x_n(\omega)|| - ||f_n(x_n(\omega),\omega) - f_0(x_n(\omega))|| \geq \mu(\beta^{(4)}_{n,\kappa}) - \beta^{(2)}_{n,\kappa} \geq \beta^{(3)}_{n,\kappa}$. Since the last inequality holds for all $x_0 \in S^W_0$ we have $W_n(\omega) \setminus U_{\beta^{(3)}_{n,\kappa}} W_0 \neq \emptyset$, and the conclusion follows. □

Theorem 4. Suppose that (CΓ-i), (Cf), and (Gf) are satisfied. Furthermore assume that the following condition is fulfilled:

(CE) There exist a function $\mathcal{H}_3 \in H$ and for all $\kappa > 0$ a sequence $(\beta^{(3)}_{n,\kappa})_{n\in N} \in B$ such that
$\sup_{n\in N} P\{\omega : E_0 \setminus U_{\beta^{(3)}_{n,\kappa}} E_n(\omega) \neq \emptyset\} \leq \mathcal{H}_3(\kappa)$.

Then for $\beta^{(4)}_{n,\kappa} := \max\{\mu^{-1}(\beta^{(2)}_{n,\kappa} + \beta^{(3)}_{n,\kappa}), \beta^{(1)}_{n,\kappa}, \beta^{(2)}_{n,\kappa}\}$ the following relation holds:
$\forall \kappa > 0$
$\sup_{n\in N} P\{\omega : U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0$ and $S^E_0 \setminus U_{\beta^{(4)}_{n,\kappa}} S^E_n(\omega) \neq \emptyset\} \leq \mathcal{H}_1(\kappa) + \mathcal{H}_2(\kappa) + \mathcal{H}_3(\kappa)$.

Proof. Assume that for given $\kappa > 0$, $n \in N$, and $\omega \in \Omega$ the relations $U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0$ and $S^E_0 \setminus U_{\beta^{(4)}_{n,\kappa}} S^E_n(\omega) \neq \emptyset$ are satisfied. Then there exists an $x_0(\omega) \in S^E_0$ which does not belong to $U_{\beta^{(4)}_{n,\kappa}} S^E_n(\omega)$. Consider an $x_n(\omega) \in S^E_n(\omega)$. Firstly, assume that $x_n(\omega) \notin U_{\beta^{(1)}_{n,\kappa}}\Gamma_0$. Then we can employ (CΓ-i).
Secondly, assume that $x_n(\omega) \in U_{\beta^{(1)}_{n,\kappa}}\Gamma_0$. If $||f_n(x_n(\omega),\omega) - f_0(x_n(\omega))|| \geq \beta^{(2)}_{n,\kappa}$ we make use of (Cf). Otherwise we have that $||f_n(x_n(\omega),\omega) - f_0(x_n(\omega))|| < \beta^{(2)}_{n,\kappa}$ and can proceed as follows: For all $x_n(\omega) \in S^E_n(\omega)$ we obtain $||f_n(x_n(\omega),\omega) - f_0(x_0(\omega))|| \geq ||f_0(x_0(\omega)) - f_0(x_n(\omega))|| - ||f_n(x_n(\omega),\omega) - f_0(x_n(\omega))|| \geq \mu(\beta^{(4)}_{n,\kappa}) - \beta^{(2)}_{n,\kappa} \geq \beta^{(3)}_{n,\kappa}$. Since the last inequality holds for all $x_n(\omega) \in S^E_n(\omega)$ we have $E_0 \setminus U_{\beta^{(3)}_{n,\kappa}} E_n(\omega) \neq \emptyset$ and the conclusion follows. □

The existence of a growth function μ is considered in the following lemma.

Lemma 6.1. Let f_0 be continuous. Furthermore assume that for a compact neighborhood $U\Gamma_0$ the following condition is satisfied:

$\forall x \in U\Gamma_0 \; \forall y \in (U\Gamma_0 \setminus \{x\}): \; ||f_0(x) - f_0(y)|| > 0.$
Then (Gf) holds.

Proof. Firstly we show that condition (C) is fulfilled:

$$\text{(C)} \; \forall \varepsilon > 0 \; \exists \tilde{\mu}(\varepsilon) > 0 \; \forall x, y \in U\Gamma_0\text{: } ||x - y|| \geq \varepsilon \Rightarrow ||f_0(x) - f_0(y)|| \geq \tilde{\mu}(\varepsilon).$$

Assume, to the contrary,
$\exists \varepsilon > 0 \; \forall n \in N \; \exists x_n, \; y_n \in U\Gamma_0 : ||x_n - y_n|| \geq \varepsilon \wedge ||f_0(x_n) - f_0(y_n)|| < \frac{1}{n}.$

$U\Gamma_0$ being compact, there are subsequences $(x_{n_l})_{l\in N}$ and $(y_{n_l})_{l\in N}$ of $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$, respectively, which converge to $x_0 \in U\Gamma_0$ and $y_0 \in U\Gamma_0$. Because of the continuity of f_0 we have $f_0(x_0) = f_0(y_0)$ but $||x_0 - y_0|| \geq \varepsilon$ in contradiction to the assumption of the lemma.

Once a function $\varepsilon \rightarrow \tilde{\mu}(\varepsilon)$ with property (C) has been found, each function $\tilde{\mu}_1$ with smaller positive values satisfies condition (C). Hence by $\hat{\mu}$ we denote the function which assigns to each ε the supremum over all possible values. Furthermore, we define $\mu(\alpha) := \inf\limits_{\varepsilon > \alpha} \hat{\mu}(\varepsilon)$. The function μ is by definition increasing and right continuous. $\mu(0) = 0$ follows by (C) because of the continuity of f_0.

In order to show that $\mu \in \Lambda$ it remains to confirm that μ is not constant. Suppose that $\mu(\alpha) = 0$ for an $\alpha > 0$. Hence there are an x_0 and a sequence $(x_n)_{n\in N} \subset U\Gamma_0$ with $||x_n - x_0|| \geq \varepsilon_n$, $\liminf\limits_{n\to\infty} \varepsilon_n \geq \alpha$ and $\lim\limits_{n\to\infty} ||f_0(x_n) - f_0(x_0)|| = 0$. Because of the compactness assumption there are an $\hat{x}_0 \neq x_0$ and a subsequence $(x_{n_l})_{l\in N}$ with $\lim\limits_{l\to\infty} ||x_{n_l} - \hat{x}_0|| = 0$ and $\lim\limits_{l\to\infty} ||f_0(x_{n_l}) - f_0(\hat{x}_0)|| = 0$. Consequently we obtain $||f_0(x_0) - f_0(\hat{x}_0)|| = 0$ in contradiction to the assumption. □

In some applications, among them the Markowitz model, condition (Gf) is not satisfied. Therefore we we prove another result with the weaker condition (w-Gf), which applies to the Markowitz model if (VE) is fulfilled.

$$\text{(w-Gf)} \; \exists \varepsilon_0 > 0 \; \exists \delta \in \Lambda \; \forall 0 < \varepsilon \leq \varepsilon_0 \; \forall x_0 \in S_0^E$$
$$\exists x \in \Gamma_0 \text{ with } d(x, x_0) = \varepsilon \; \exists j \in J: \; f_0^j(x) - f_0^j(x_0) \geq \delta(\varepsilon).$$

Lemma 6.2. Suppose that S_0^E ist compact, connected, and not single-valued and that f_0 is continuous. Furthermore assume that (VE) is satisfied. Then (w-Gf) holds.

Proof. Firstly we show that
$\exists \varepsilon_0 > 0 \; \forall 0 < \varepsilon \leq \varepsilon_0 \; \exists \tilde{\delta} > 0 \; \forall x_0 \in S_0^E \; \exists x \in \Gamma_0$ with $d(x, x_0) = \varepsilon$
$\exists j \in J : \; f_0^j(x) - f_0^j(x_0) \geq \tilde{\delta}$.
Assume, to the contrary,
$\forall n \in N \; \exists \varepsilon_n$ with $0 < \varepsilon_n \leq \frac{1}{n} \; \forall r \in N \; \exists x_{n,r}^0 \in S_0^E \; \forall x_{n,r} \in \Gamma_0$ with $d(x_{n,r}^0, x_{n,r}) = \varepsilon_n$
$\forall j \in J : f_0^j(x_{n,r}) - f_0^j(x_{n,r}^0) < \frac{1}{r}$.

For each $n \in N$, let r tend to infinity. Then, because of the compactness of S_0^E, to $(x_{n,r}^0)_{r \in N}$ there is a subsequence $(x_{n,r_l}^0)_{l \in N}$ which converges to an $x_n^0 \in S_0^E$. Furthermore, to $(x_{n,r_l}^0)_{l \in N}$ we consider a sequence $(x_{n,r_l})_{l \in N}$ with $x_{n,r_l} \in S_0^E$. Such a sequence, which belongs to $S_0^E \subset \Gamma_0$, exists, because S_0^E is connected and not single-valued. $(x_{n,r_l})_{l \in N} \in S_0^E$ in turn contains a converging subsequence which tends to $x_n \in S_0^E$. We obtain $d(x_n^0, x_n) = \frac{1}{n}$, but $\forall j \in J \; \forall r \in N : \; f_0^j(x_n) - f_0^j(x_n^0) < \frac{1}{r}$. Hence to each $x_n^0 \in S_0^E$ there exists $x_n \in S_0^E$ with $d(x_n, x_n^0) = \frac{1}{n}$ for sufficiently large $n \in N$ and $\forall j \in J$: $f_0^j(x_n) - f_0^j(x_n^0) = 0$ which contradicts (VE).

In order to show that to the function $\varepsilon \to \tilde{\delta}(\varepsilon)$ there is a function $\delta \in \Lambda$ which satisfies (w-Gf) we can proceed as in the proof of Lemma 6.1. □

With this condition we obtain the following modification of Theorem 4.

Theorem 4-M. Suppose that the conditions (CΓ-i), (Cf), (w-Gf), and (CE) are satisfied.

Then for $\beta_{n,\kappa}^{(4)} := \max\{\delta^{-1}(2\beta_{n,\kappa}^{(2)} + \beta_{n,\kappa}^{(3)}), \beta_{n,\kappa}^{(1)}, \beta_{n,\kappa}^{(2)}\}$ the following relation holds:

$\forall \kappa > 0 \; \sup\limits_{n \in N} P\{\omega : U_{\beta_{n,\kappa}^{(1)}} \Gamma_0 \subset U\Gamma_0$ and $S_0^E \setminus U_{\beta_{n,\kappa}^{(4)}} S_n^E(\omega) \neq \emptyset\}$
$\leq \mathcal{H}_1(\kappa) + \mathcal{H}_2(\kappa) + \mathcal{H}_3(\kappa)$.

Proof. Assume that for given $\kappa > 0$, $n \in N$, and $\omega \in \Omega$ the relations $U_{\beta_{n,\kappa}^{(1)}} \Gamma_0 \subset U\Gamma_0$ and $S_0^E \setminus U_{\beta_{n,\kappa}^{(4)}} S_n^E(\omega) \neq \emptyset$ are satisfied. Then there exists an $x_0(\omega) \in S_0^E$ which does not belong to $U_{\beta_{n,\kappa}^{(4)}} S_n^E(\omega)$. Because of (w-Gf) there exists $\tilde{x}_0(\omega) \in \Gamma_0 \setminus U_{\frac{\beta_{n,\kappa}^{(4)}}{2}} S_n^E(\omega)$ such that for a $j \in J$ the relation $f_0^j(\tilde{x}_0(\omega)) - f_0(x_0(\omega)) \geq \delta(\frac{\beta_{n,\kappa}^{(4)}}{2})$ holds.

Then we can proceed as in the proof of Theorem 4, replacing $x_0(\omega)$ with $\tilde{x}_0(\omega)$. □

7 Relaxation

The assertions of the foregoing sections impose assumptions about the true problem, namely the knowledge of the continuity function λ, the growth function μ, and the functions δ_1, δ_2. If this information is not available, one could try to estimate these functions from the data. How this approach works in the case of one objective function is considered in [27]. In [27] also another approach is proposed, the so-called relaxation. It relies on the following consideration: If a convergence rate has to be taken into account it can be used to 'relax' the constraints and the objective functions with the error, which is 'usually' not exceeded, namely the convergence rate.

We will derive sequences of random sets $\tilde{S}^W_{R,n}$ and $S^W_{R,n}$ which cover S^W_0 with a probability which is bounded by a tail behavior function. These sets can be regarded as 'superset-approximations'. They depend on the argument κ of the tail behavior function, although this is not indicated in the denotation. Hence in order to derive a confidence set, one can proceed as explained in the introduction for outer approximations with convergence rate and tail behavior function. Because of $S^E_0 \subset S^W_0$, also confidence sets for S^E_0 are obtained in that way.

Firstly, we assume that the 'continuity' function λ is known. It will become clear from the proof of the next theorem that also a bound for λ will do, and such a bound is is often available, at least locally. The general case without any knowledge about the true problem will be considered in Theorem 6.

We introduce a 'shifted' order cone, which will be used to cope with the approximation of the objective function. For sake of simplicity we use the same convergence rate for all components of the objective function, one could, however, also deal with individual convergence rates for each component. Let, for a given 'continuity function' $\lambda \in \Lambda$, $\beta^\lambda_{n,\kappa} := 2\beta^{(2)}_{n,\kappa} + \lambda(\beta^{(1)}_{n,\kappa})$,
$R^k_{\beta^\lambda_{n,\kappa}} := R^k_+ + \beta^\lambda_{n,\kappa} 1^k$, and

$$\tilde{S}^W_{R,n}(\omega) := \{x \in U_{\beta^{(1)}_{n,\kappa}} \Gamma_n(\omega) : \ (f_n(x,\omega) - \text{int} R^k_{\beta^\lambda_{n,\kappa}}) \cap F_n(\omega) = \emptyset\}$$

. Here 1^k denotes the k-dimensional vector $(1, 1, \ldots, 1)^T$.

Then we have the following assertion.

Theorem 5. Suppose that the conditions (CΓ-i), (CΓ-o), (Cf), and (Sf) are satisfied. Then the following realtion holds:

$\forall \kappa > 0$:
$\sup_{n \in N} P\{\omega : U_{\beta^{(1)}_{n,\kappa}} \Gamma_0 \subset U\Gamma_0 \text{ and } S^W_0 \setminus \tilde{S}^W_{R,n}(\omega) \neq \emptyset\} \leq \mathcal{H}_1(\kappa) + \mathcal{H}_2(\kappa)$.

Proof. Assume that for given $\kappa > 0$, $n \in N$, and $\omega \in \Omega$ the relations $U_{\beta^{(1)}_{n,\kappa}} \Gamma_0 \subset U\Gamma_0$ and $S^W_0 \setminus \tilde{S}^W_{R,n}(\omega) \neq \emptyset$ are satisfied. Then there exists an $x_0(\omega) \in S^W_0$ which does not belong to $\tilde{S}^W_{R,n}(\omega)$. If $x_0(\omega) \notin U_{\beta^{(1)}_{n,\kappa}} \Gamma_n(\omega)$ we have $\Gamma_0 \setminus U_{\beta^{(1)}_{n,\kappa}} \Gamma_n(\omega) \neq \emptyset$ and can employ (CΓ-o). Hence, in the following we assume that $x_0(\omega) \in U_{\beta^{(1)}_{n,\kappa}} \Gamma_n(\omega)$.

Because of $x_0(\omega) \notin \tilde{S}^W_{R,n}(\omega)$ and the definition of $S^W_{R,n}(\omega)$ there is an $x_n(\omega) \in \Gamma_n(\omega)$ with the property
$f_n(x_n(\omega), \omega) < f_n(x_0(\omega), \omega) - \beta^\lambda_{n,\kappa}$. (3)
If $x_n(\omega) \notin U_{\beta^{(1)}_{n,\kappa}} \Gamma_0$ we again have $\Gamma_n(\omega) \setminus U_{\beta^{(1)}_{n,\kappa}} \Gamma_0 \neq \emptyset$ and make use of (CΓ-i). Hence we can assume that $x_n(\omega) \in U_{\beta^{(1)}_{n,\kappa}} \Gamma_0$. To $x_n(\omega)$ we consider $\tilde{x}_0(\omega) \in \Gamma_0$ with minimal distance to $x_n(\omega)$. If $\tilde{x}_0(\omega) \notin U_{\beta^{(1)}_{n,\kappa}} \{x_n(\omega)\}$ we have $\Gamma_n(\omega) \setminus U_{\beta^{(1)}_{n,\kappa}} \Gamma_0 \neq \emptyset$ and can again employ (CΓ-i). Now we assume that $\tilde{x}_0(\omega) \in U_{\beta^{(1)}_{n,\kappa}} \{x_n(\omega)\}$. Then
$||f_0(\tilde{x}_0(\omega)) - f_0(x_n(\omega))|| \leq \lambda(\beta^{(1)}_{n,\kappa})$. (4)
Because of $x_n(\omega) \in U\Gamma_0$ we can either apply (Cf) or we have for $x_n(\omega)$ and $x_0(\omega)$ the relations $||f_n(x_n(\omega), \omega) - f_0(x_n(\omega))|| \leq \beta^{(2)}_{n,\kappa}$ and $||f_n(x_0(\omega), \omega) - f_0(x_0(\omega))|| \leq \beta^{(2)}_{n,\kappa}$. In the latter case we obtain from (3) and (4) $f_n(x_n(\omega), \omega) - f_0(x_n(\omega)) + f_0(\tilde{x}_0(\omega)) < f_n(x_0(\omega), \omega) - f_0(x_0(\omega)) + f_0(x_0(\omega)) - \beta^\lambda_{n,\kappa} + \lambda(\beta^{(1)}_{n,\kappa})$ and can by $||f_n(x_0(\omega), \omega) - f_0(x_0(\omega))|| \leq \beta^{(2)}_{n,\kappa}$ and $||f_n(x_n(\omega), \omega) - f_0(x_n(\omega))|| \leq \beta^{(2)}_{n,\kappa}$ conclude $f_0(\tilde{x}_0(\omega)) < f_0(x_0(\omega))$ which contradicts $x_0(\omega) \in S^W_0$. □

Now we will cope without any knowledge about the true problem. This, however, requires the knowledge of sets which are subsets of the constraint set with prescribed high probability. Inner approximations need not be subsets of the true set, hence we introduce so-called subset-approximations. Such approximations are also considered in [27].

Definition. A sequence $(\Gamma^{sub}_{n,\kappa})_{n \in N}$ which satisfies the condition
$\forall \kappa > 0 : \sup_{n \in N} P\{\omega : \Gamma^{sub}_{n,\kappa}(\omega) \setminus \Gamma_0 \neq \emptyset\} \leq \mathcal{H}(\kappa)$
is called *subset-approximation in probability to* Γ_0 *with tail behavior function* $\mathcal{H}$.

If Γ_0 is given by inequality constraints, it is easy to obtain a subset-appproximation, see [27] and a simple method at the end of this section.

In order to derive a superset approximation for the set of weakly efficient points, we consider the image set $F^{sub}_{n,\kappa}$ of a subset-approximation for Γ_n:

$$F^{sub}_{n,\kappa}(\omega) := \{f_n(x,\omega) : x \in \Gamma^{sub}_{n,\kappa}(\omega)\}.$$

Furthermore we introduce $R^k_{\beta_{n,\kappa}} := R^k_+ + 2\beta_{n,\kappa}1^k$ and

$$S^W_{R,n}(\omega) := \{x \in U_{\beta^{(1)}_{n,\kappa}}\Gamma_n(\omega) : (f_n(x,\omega) - \mathrm{int}R^k_{\beta_{n,\kappa}}) \cap F^{sub}_{n,\kappa}(\omega) = \emptyset\}.$$

Theorem 6. Suppose that $(\Gamma^{sub}_{n,\kappa})_{n\in N}$ is a subset approximation to Γ_0 with tail behavior function $\mathcal{H}_3$ and that the conditions (CΓ-o) and (Cf) are satisfied. Then the following relation holds:
$\forall\kappa > 0$:
$\sup\limits_{n\in N} P\{\omega : U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0 \text{ and } S^W_0 \setminus S^W_{R,n}(\omega) \neq \emptyset\} \leq \mathcal{H}_1(\kappa) + \mathcal{H}_2(\kappa).$

Proof. Assume that for given $\kappa > 0$, $n \in N$, and $\omega \in \Omega$ the relations $U_{\beta^{(1)}_{n,\kappa}}\Gamma_0 \subset U\Gamma_0$ and $S^W_0 \setminus S^W_{R,n}(\omega) \neq \emptyset$ are satisfied. Then there exists an $x_0(\omega) \in S^W_0$ which does not belong to $S^W_{R,n}(\omega)$. If $x_0(\omega) \notin U_{\beta^{(1)}_{n,\kappa}}\Gamma_n(\omega)$ we have $\Gamma_0 \setminus U_{\beta^{(1)}_{n,\kappa}}\Gamma_n(\omega) \neq \emptyset$ and can employ (CΓ-o). Hence, in the following we assume that $x_0(\omega) \in U_{\beta^{(1)}_{n,\kappa}}\Gamma_n(\omega)$.

Because of $x_0(\omega) \notin S^W_{R,n}(\omega)$ and the definition of $S^W_{R,n}(\omega)$ there is an $x_n(\omega) \in \Gamma^{sub}_{n,\kappa}(\omega)$ with the property $f_n(x_n(\omega),\omega) < f_n(x_0(\omega),\omega) - 2\beta_{n,\kappa}$.

If $x_n(\omega) \notin \Gamma_0$ we have $\Gamma^{sub}_{n,\kappa}(\omega) \setminus \Gamma_0 \neq \emptyset$. Otherwise, because of (Cf), the relation $f_0(x_n(\omega)) < f_0(x_0(\omega))$ follows, which contradicts the assumption $x_0(\omega) \in S^W_0$. □

If the constraint set is not approximated, like in the Markowitz model, only the approximation of the objective functions has to be taken into account. Hence, for $\Gamma_n(\omega) = \Gamma_0\ \forall n \in N\ \forall\omega \in \Omega$, we have $F_n(\omega) = \{f_n(x,\omega) : x \in \Gamma_0\}$ and define
$\hat{S}^W_{R,n}(\omega) := \{x \in \Gamma_0 : (f_n(x,\omega) - \mathrm{int}R^k_{2\beta_{n,\kappa}}) \cap F_n(\omega) = \emptyset\}.$

Lemma 7.1 Suppose that (Cf) is satisfied. Then
$\forall \kappa > 0$: $\sup\limits_{n \in N} P\{\omega : S_0^W \setminus \hat{S}_{R,n}^W(\omega) \neq \emptyset\} \leq \mathcal{H}_2(\kappa)$.

Lemma 7.1 can be proved similar to Theorem 6.

In order to derive a subset-approximation of $\Gamma_0 := \{x \in R^p : \ g_0(x) \leq 0\}$ we introduce the following sets for $\alpha < 0$:

$$\Gamma_n^\alpha(\omega) := \{x \in R^p : \ g_n(x, \omega) \leq \alpha\}.$$

Here g_0 and g_n can be regarded as the supremum of a set of constraint functions. Obviously, for $\alpha < 0$ we obtain a subset of Γ_n. Furthermore, we have the following assertion:

Lemma 7.2: If there exist a function $\mathcal{H}_2$ and to all $\kappa > 0$ a sequence $(\beta_{n,\kappa}^{(2)})_{n \in N}$ such that
$\sup\limits_{n \in N} P\{\omega : \inf\limits_{x \in U\Gamma_0} (g_n(x, \omega) - g_0(x)) \leq -\beta_{n,\kappa}^{(1)}\} \leq \mathcal{H}_2(\kappa)$
for a suitable neighborhood $U\Gamma_0$,
then $(\Gamma_n^{-\beta_{n,\kappa}^{(1)}})_{n \in N}$ is a subset-approximation for Γ_0.

Proof. (i) Let $\omega \in \Omega$, $n \in N$ and $\kappa > 0$ be such that $\Gamma_n^{-\beta_{n,\kappa}^{(1)}}(\omega) \setminus \Gamma_0 \neq \emptyset$. Then there is an $x_n(\omega) \in \Gamma_n^{-\beta_{n,\kappa}^{(1)}}(\omega)$ which does not belong to Γ_0. Hence $g_n(x_n(\omega), \omega) \leq -\beta_{n,\kappa}^{(1)}$ while $g_0(x_n(\omega)) > 0$ and we can employ the assumption. □

Another subset approximation could be obtained if a suitable neighborhood of the boundary of $\Gamma_n(\omega)$ is removed from $\Gamma_n(\omega)$.

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