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Set approach for set optimization with variable ordering structures

Gabriele Eichfelder* and Maria Pilecka**

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Abstract

This paper aims at combining variable ordering structures with set relations in set optimization, which have been defined using the constant ordering cone before. Since the purpose is to connect these two important approaches in set optimization, we do not restrict our considerations to one certain relation. Conversely, we provide the reader with many new variable set relations generalizing the relations from [16, 25] and discuss their usefulness. After analyzing the properties of the introduced relations, we define new solution notions for set-valued optimization problems equipped with variable ordering structures and compare them with other concepts from the literature. In order to characterize the introduced solutions a nonlinear scalarization approach is used.

Key Words: set optimization; set relation; variable ordering structure; scalarization

Mathematics subject classifications (MSC 2000): 49J53, 90C29, 90C30, 54C60, 06A75

1 Introduction

In set optimization we deal with optimization problems with a set-valued objective function $F\colon S \subset X \to 2^Y$ (with real linear spaces X, Y, a nonempty set S and the power set 2^Y). Such problems arise for instance in case one considers a vector-valued objective map but allows (relative) errors [16], in bilevel optimization if neither optimistic nor pessimistic approach is regarded [6] or in some applications as the navigation of transportation robots, cf. [16]. For a detailed introduction to set optimization see [20].

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The topic of this paper are set optimization problems based on variable ordering structures. As discussed in [7, Remark 2.6] there are (at least) three approaches for defining solution concepts in set optimization: the vector approach [2, 27], the set approach [10, 19, 23] and the lattice approach [12, 33].

In the vector approach one directly generalizes the concepts known from vector optimization to set optimization: at first the union of all image sets $F(S) := \bigcup_{x \in S} F(x)$ of the set-valued objective map F over the feasible set S is determined. Then one tries to find a best element $\bar{y}$ in the set $F(S)$. This defines an optimal solution of the set-valued optimization problem by searching for a feasible element $\bar{x} \in S$ with $\bar{y} \in F(\bar{x})$. The drawback of this approach is that in most applications only one element $\bar{y}$ does not imply that the whole set $F(\bar{x})$ is in a certain sense optimal compared to the other sets $F(x)$, $x \in S$ in general. Therefore, in the set approach, one aims on defining binary relations $\preccurlyeq$ for comparing sets. Then one denotes $\bar{x} \in S$ an optimal solution if $F(x) \preccurlyeq F(\bar{x})$ for some $x \in S$ implies $F(\bar{x}) \preccurlyeq F(x)$, cf. [10, Definition 1.9].

For both approaches, the binary relations for comparing elements or sets in a linear space are in general based on a convex cone $K \subset Y$. Next to such fixed ordering structures, there is in the last years an increasing interest in vector optimization problems with a variable ordering structure (see for instance [3, 5, 8, 28, 32]). There, the partial ordering defined by a convex cone in the image space of the vector-valued objective function is replaced by a variable ordering structure. This variable ordering structure is defined by a set-valued map, called ordering map, which associates with each element of the space an individual set or cone of preferred or dominated directions.

We are interested in such variable ordering structures for set optimization problems. In contrast to [7] and [9], who follow the vector approach, we take the set approach and introduce binary relations to compare sets directly. This was also done in a recent paper by Köbis, see [21], who compared sets w.r.t. a variable ordering structure based on one specific set relation. We propose here a different approach for these relations and discuss this subject in more detail on page 8.

In Section 2 we introduce new set relations based on variable ordering structures, which generalize the relations defined in [16, 25] using constant ordering cones. Thereby we do not restrict our considerations to one chosen relation intentionally. Doubtlessly, some of these relations are more practically relevant than others, which we discuss on page 11. Section 3 presents some properties of the considered set relations including among others reflexivity, transitivity and antisymmetry of these relations. In Section 4 we define optimality notions using the introduced relations and provide characterizations of some of these solutions w.r.t. selected relations by means of nonlinear scalarization.

We end this introduction by discussing an application which illustrates the relevance of considering such set optimization problems. In intensity-modulated radiation therapy (IMRT) the importance of incorporating variable ordering structures to allow an improved modeling of the decision making problem is already discussed in [8, Chapter 10]. In IMRT one searches for an optimal treatment plan x for the irradiation of a tumor. Thereby one aims on reducing the radiation dose delivered to the neighbored healthy organs, while destroying the tumor. This is a multi-objective optimization problem with an objective f_i

($i \in \{1, \dots, k\}$) for each healthy neighbored organ measuring its dose stress.

For calculating the dose stress $f_i(x)$ in an organ i a model of the patients body is used. This model is based on a partition of the relevant area into voxels which are assigned to the different tissues and organs. For choosing an appropriate model of the patient a body scan can be used and the specific data can be compared with models from a data base. For safety purpose one might prefer to do the calculations based on several data sets. Additionally, for calculating the dose stress various approaches exist. Widely used is a concept based on the equivalent uniform dose in terms of p-norms. The organ dependent parameters p are determined based on statistical tests so it might also be of interest to study a range of parameters p. This leads to a dose level $f_i(x, p, m)$ for each choice of the model m and the parameter p, and for safety reasons it might be preferable to consider the whole sets

$$F(x) := \bigcup_{p,m} \{(f_1(x, p, m), \dots, f_k(x, p, m))^\top\}$$

for each irradiation plan x, i.e. a set optimization problem with a set-valued objective function $F(x) \subset \mathbb{R}^k$.

For comparing elements $(f_1(x, p, m), \dots, f_k(x, p, m))^\top$ in $\mathbb{R}^k$ it is already mentioned in [8, Chapter 10] that a variable ordering structure reflects the application problem better than a constant ordering cone. This is based on the fact that there is no linear (but an s-shaped) relation between the dose delivered to an organ and the effect on that organ. For small and high values of the dose a change in the dose level has hardly any effect on an organ, but there is a sensible dose range in which slight changes have a significant impact. As long as the response of the organ on dose variations is relatively small a rise of the dose delivered to that organ in favor of an improvement of the value for another organ would be preferred. Thus, dependent on the current values in the objective space, different preferences exist. According to [8, Chapter 10] this can be modeled by a variable ordering structure defined by a set-valued map $\mathcal{D}\colon \mathbb{R}^k \to 2^{\mathbb{R}^k}$ with $\mathbb{R}^k_+ \subset \mathcal{D}(y)$ for all $y \in \mathbb{R}^k$. In this paper we consider approaches to compare such sets $F(x)$ based on variable ordering structures associating to each element y of the image space $\mathbb{R}^k$ an individual set $\mathcal{D}(y)$.

2 Set relations

After introducing some basic notations, we present in this section new variable set relations generalizing the relations from [16, 25], and discuss their usefulness.

2.1 Preliminaries

Let Y be a real linear space. A nonempty set $C \subset Y$ is called a cone if $\lambda c \in C$ holds for all $\lambda \in \mathbb{R},\ \lambda \geq 0$ and $c \in C$. The set C is said to be pointed if $C \cap (-C) = \{0_Y\}$ where $-C := \{-c : c \in C\}$. We denote the Minkowski sum of two nonempty sets $A, B \subset Y$ by $A + B$ and the difference by $A - B := A + (-B)$. For the (algebraic) interior of a set $A \subset Y$ we write $\operatorname{cor}(A)$ and for the (algebraic) boundary ∂A. In case A is a subset of a real topological linear space Y, we denote by $\mathrm{int}(A)$ the (topological) interior and by $\mathrm{cl}(A)$ the (topological) closure of A.

Definition 2.1. *Let Y be an arbitrary nonempty set. A binary relation $\leq$ on Y is called a partial ordering on Y if for each $x, y, z \in Y$ the following properties are satisfied:*

(i) $y \leq y$ (reflexivity),

(ii) $y \leq x$ and $x \leq z$ imply $y \leq z$ (transitivity),

(iii) $y \leq x$ and $x \leq y$ imply $x = y$ (antisymmetry).

If a relation satisfies only the properties (i) and (ii), then it is a preorder. A convex cone $K \subset Y$ which introduces a preorder on a real linear space Y by

$$\forall y, z \in Y : \; y \leq^K z \; :\Leftrightarrow \; z - y \in K$$

is denoted an ordering cone.

Using the set-valued maps $\mathcal{D} : Y \to 2^Y$ and $\mathcal{P} : Y \to 2^Y$, two different binary relations are defined on Y by:

$$y \leq^{\mathcal{D}} \bar{y} \; :\Leftrightarrow \; \bar{y} \in \{y\} + \mathcal{D}(y) \tag{1}$$

and by

$$y \leq^{\mathcal{P}} \bar{y} \; :\Leftrightarrow \; y \in \{\bar{y}\} - \mathcal{P}(\bar{y}). \tag{2}$$

Be aware that we use the symbol $2^Y := \{A \subset Y \; : \; A \text{ is nonempty } \}$ to denote the power set of Y, while $\mathcal{P}$ is a set-valued map. The notation $\mathcal{P}(Y) := \bigcup_{y \in Y} \mathcal{P}(y)$ stands for the image of Y under $\mathcal{P}$.

As it will be discussed in the forthcoming Lemma 2.3, these binary relations define no preorder on Y in general. The binary relation (1) reflects the idea of domination: we have for $y, z \in Y$ that $y \leq^{\mathcal{D}} z$ if z is considered to be larger (or worse — in case we minimize) than y from the point of view of y. Whereas the binary relation (2) models the idea of preference. Then $y \leq^{\mathcal{P}} z$ can be interpreted by saying that y is smaller (or better) than z from the point of view of z. Of course, for any $y \in Y$ the set $\mathcal{D}(y)$, describing with $\{y\} + \mathcal{D}(y) \setminus \{0_Y\}$ the set of elements which are considered to be worse than y, does not have to be equal to $\mathcal{P}(y)$, depicting with $\{y\} - \mathcal{P}(y) \setminus \{0_Y\}$ the set of elements which are considered to be preferred to y.

Definition 2.2. *Let Y be a real linear space and $\mathcal{D}, \mathcal{P} : Y \to 2^Y$ set-valued maps where $\mathcal{D}(y), \mathcal{P}(y)$ are (algebraically) closed sets with $0_Y \in \partial \mathcal{D}(y)$ and $0_Y \in \partial \mathcal{P}(y)$ for all $y \in Y$. If elements in the space Y are compared using (1) or (2), then the map $\mathcal{D}$ or $\mathcal{P}$ is called ordering map, its images are ordering sets and relations (1), (2) are variable ordering relations.*

In the following we are going to make use of some properties of relations (1) and (2), which are summarized in the lemma below and can be found in [8, Lemma 1.10].

Lemma 2.3. *Let Y be a real linear space with ordering maps $\mathcal{D}$ and $\mathcal{P}$ where $\mathcal{D}(y)$ and $\mathcal{P}(y)$ are nonempty convex cones for all $y \in Y$.*

(i) The relations defined in (1) and (2) are reflexive.

(ii) The relation $\leq^{\mathcal{D}}$ is transitive if and only if

$$\forall y \in Y \ \forall d \in \mathcal{D}(y) : \quad \mathcal{D}(y+d) \subset \mathcal{D}(y). \tag{3}$$

(iii) The relation $\leq^{\mathcal{P}}$ is transitive if and only if

$$\forall y \in Y \ \forall d \in \mathcal{P}(y) : \quad \mathcal{P}(y-d) \subset \mathcal{P}(y). \tag{4}$$

(iv) The relation $\leq^{\mathcal{D}}$ ($\leq^{\mathcal{P}}$) is antisymmetric if $\mathcal{D}(Y) = \bigcup\limits_{y \in Y} \mathcal{D}(y)$ ($\mathcal{P}(Y) = \bigcup\limits_{y \in Y} \mathcal{P}(y)$) is pointed.

Let X, Y be real linear spaces, $S \subset X$ a nonempty set and $F : X \to 2^Y$ a set-valued map. We assume that the space Y is not equipped with a partial ordering but that we have some kind of variable ordering structure (which we discuss and introduce next) and study the set optimization problem

$$\min_{x \in S} \ F(x) \ . \tag{SOP}$$

It is possible to use variable ordering structures introduced by ordering maps as given in Definition 2.2 for set optimization problems. This can be done by application of results from vector optimization with variable ordering structures, cf. [3, 7, 9], based on the so called vector approach. Using this concept a pair $(\bar{x}, \bar{y})$ is an optimal solution of (SOP) if $\bar{y} \in F(\bar{x})$ and $\bar{y}$ is an optimal element of $F(S) = \bigcup_{x \in S} F(x)$ in a certain sense, see also Definition 4.9. If instead the set approach is regarded, we need to define set-valued maps depending on a whole set in order to introduce binary relations between elements of the power set of Y analogously to (1) and (2). Next we discuss the possibilities of how to define the ordering maps (cf. Definition 2.2) in order to clarify the choice of approach that we use in the sequel.

I. $\widetilde{\mathcal{D}} : 2^Y \to 2^Y$ and $\widetilde{\mathcal{P}} : 2^Y \to 2^Y$.
Let us consider an arbitrary set $A \subset Y$ and the corresponding image of the ordering map $\widetilde{\mathcal{D}}(A)$. Since $\widetilde{\mathcal{D}}$ is defined on 2^Y, the ordering set associated with some set $B \subset A$ could be completely different from $\widetilde{\mathcal{D}}(A)$. In the worst case we may obtain $\widetilde{\mathcal{D}}(A) \cap \widetilde{\mathcal{D}}(B) = \{0_Y\}$. The same problem arises while considering the map $\widetilde{\mathcal{P}}$.

II. $\widehat{\mathcal{D}} : X \to 2^Y$ and $\widehat{\mathcal{P}} : X \to 2^Y$.
By regarding the map $\widehat{\mathcal{D}}$ without any additional assumptions, we notice the following: it is possible that there exist $x_1, x_2 \in X$ with $F(x_1) = F(x_2)$. If we now take the map $\widehat{\mathcal{D}}$ into consideration, we note that, because it is defined on X, it can take different values for x_1 and x_2. Consequently, the set $F(x_1)$ would be characterized by two diverse ordering cones in the image space. Moreover, also in this case the same problem as in I. may occur.

III. $\mathcal{D} : Y \to 2^Y$ and $\mathcal{P} : Y \to 2^Y$.
We may unify all sets $\mathcal{D}(a)$, $a \in A$ in order to obtain $\mathcal{D}(A) = \bigcup\limits_{a \in A} \mathcal{D}(a)$. Additionally, by considering images of a given map F we have $\mathcal{D}(F(x_1)) = \mathcal{D}(F(x_2))$ whenever $F(x_1) = F(x_2)$ and $\mathcal{D}(B) \subset \mathcal{D}(A)$ if $B \subset A$.

We can observe the following relationships between the set-valued maps defined above:

(i) If F is bijective, then we may consider $\widehat{\mathcal{D}}(x) := \widetilde{\mathcal{D}}(F(x))$ and the maps $\widetilde{\mathcal{D}}$ and $\widehat{\mathcal{D}}$ may coincide.

(ii) From a practical point of view it is important to require one of the following properties of $\widetilde{\mathcal{D}}$ (and also $\widehat{\mathcal{D}}$): for all $B \subset A$ we have $\widetilde{\mathcal{D}}(B) \subset \widetilde{\mathcal{D}}(A)$ or $\widetilde{\mathcal{D}}(A) \subset \widetilde{\mathcal{D}}(B)$. By setting $\widetilde{\mathcal{D}}(A) := \bigcup_{a \in A} \mathcal{D}(a)$ we note that the first condition is obviously satisfied. This could be interpreted as considering the ordering set for the set A such that if $d \in \widetilde{\mathcal{D}}(A)$, then there exists at least one element $a \in A$ with $d \in \widetilde{\mathcal{D}}(a)$ — so $d \in \widetilde{\mathcal{D}}(A)$ as soon as for at least one value $a \in A$ adding d means a worsening of the current value. If we define $\widetilde{\mathcal{D}}(A) := \bigcap_{a \in A} \mathcal{D}(a)$, we obtain the second property (for all $B \subset A$ we have $\widetilde{\mathcal{D}}(A) \subset \widetilde{\mathcal{D}}(B)$). In this case $d \in \widetilde{\mathcal{D}}(A)$ means that $d \in \widetilde{\mathcal{D}}(a)$ for each $a \in A$ — so $d \in \widetilde{\mathcal{D}}(A)$ only if for all values $a \in A$ adding d means a worsening of the current value.

In [7] the ordering map was defined on X, i.e. $\widehat{\mathcal{D}} : X \to 2^Y$, and an application in the theory of consumer demand was also presented. Hereby, based on a map $g \colon X \to Y^*$ with $X = \mathbb{R}^n$, the ordering map $\widehat{\mathcal{D}}(x) := \{d \in Y \;:\; g(x)(d) \geq 0\}$ is defined. Here, Y^* denotes the dual space. Of course, for $Y = \mathbb{R}^m$ (which is in general the case in the application, see [8]) and any $y \in g(\mathbb{R}^n)$ we can instead define the map $\mathcal{D} : g(\mathbb{R}^n) \to 2^Y$ by $\mathcal{D}(y) := \{d \in \mathbb{R}^m \;:\; y^\top d \geq 0\}$.

Based on the properties of the discussed ordering maps we define binary relations on 2^Y. Before we introduce these binary relations, let us recall the definition of the l-type less order relation (l-less order relation), see e.g. [12, 15, 24, 29], between sets $A, B \in 2^Y$ with a constant ordering cone $K \subset Y$:

$$A \preccurlyeq_l B \;:\Leftrightarrow\; \big(\forall b \in B \; \exists a \in A : \; a \leq^K b\big) \;\Leftrightarrow\; B \subset A + K. \tag{5}$$

If we regard the set $A + K$ on the right hand side of the last inclusion and try to replace the cone K with the set $\mathcal{D}(A)$, we notice that by considering the map $\mathcal{D}$ we have another possibility, namely, the set $\bigcup\limits_{a \in A} (\{a\} + \mathcal{D}(a))$ with $\bigcup\limits_{a \in A} (\{a\} + \mathcal{D}(a)) \subset A + \mathcal{D}(A)$. It is also important to mention that there exists in general no set $C \subset Y$ such that the equation $\bigcup\limits_{a \in A} (\{a\} + \mathcal{D}(a)) = A + C$ holds.

Since the different binary relations for set-valued maps, which can be found in [16, 25], are defined pointwise, in our opinion the maps $\mathcal{D}$ and $\mathcal{P}$ seem to provide the best possibility to generalize these relations using variable ordering structures.

2.2 Definition of set relations

Following the publications [16] and [25] we can define 16 set relations (in Definitions 2.4 and 2.5), which are not equivalent in general. Note that for instance from the binary relation defined in (5) for a partially ordered space Y we obtain two different binary relations w.r.t. a variable ordering structure dependent on whether we replace $\leq^K$ by $\leq^{\mathcal{D}}$ (see the binary relation number 7) or by $\leq^{\mathcal{P}}$ (see the binary relation 8).

Definition 2.4. *Let Y be a real linear space and $A, B \in 2^Y$ be nonempty arbitrarily chosen sets and $\mathcal{D}, \mathcal{P} : Y \to 2^Y$ be ordering maps. Then we define binary relations on 2^Y as follows:*

1. *the certainly less order relation of type $\mathcal{D}$ ($\preccurlyeq_c^{\mathcal{D}}$) is defined by:*

$$A \preccurlyeq_c^{\mathcal{D}} B \;:\Leftrightarrow\; \big(\forall a \in A\ \forall b \in B:\ a \leq^{\mathcal{D}} b\big) \;\Leftrightarrow\; B \subset \bigcap_{a \in A} (\{a\} + \mathcal{D}(a)).$$

2. *the certainly less order relation of type $\mathcal{P}$ ($\preccurlyeq_c^{\mathcal{P}}$) is defined by:*

$$A \preccurlyeq_c^{\mathcal{P}} B \;:\Leftrightarrow\; \big(\forall a \in A\ \forall b \in B:\ a \leq^{\mathcal{P}} b\big) \;\Leftrightarrow\; A \subset \bigcap_{b \in B} (\{b\} - \mathcal{P}(b)).$$

3. *the l-existence less order relation (there exists an element which is "smaller") of type $\mathcal{D}$ ($\preccurlyeq_{e_l}^{\mathcal{D}}$) is defined by:*

$$A \preccurlyeq_{e_l}^{\mathcal{D}} B \;:\Leftrightarrow\; \big(\exists a \in A \ \text{ such that } \ \forall b \in B:\ a \leq^{\mathcal{D}} b\big) \;\Leftrightarrow\; \exists a \in A:\ B \subset \{a\} + \mathcal{D}(a).$$

4. *the l-existence less order relation of type $\mathcal{P}$ ($\preccurlyeq_{e_l}^{\mathcal{P}}$) is defined by:*

$$A \preccurlyeq_{e_l}^{\mathcal{P}} B \;:\Leftrightarrow\; \big(\exists a \in A \ \text{ such that } \ \forall b \in B:\ a \leq^{\mathcal{P}} b\big) \;\Leftrightarrow\; A \cap \left(\bigcap_{b \in B} (\{b\} - \mathcal{P}(b))\right) \neq \emptyset.$$

5. *the u-existence less order relation (there exists an element which is "larger") of type $\mathcal{D}$ ($\preccurlyeq_{e_u}^{\mathcal{D}}$) is defined by:*

$$A \preccurlyeq_{e_u}^{\mathcal{D}} B \;:\Leftrightarrow\; \big(\exists b \in B \ \text{ such that } \ \forall a \in A:\ a \leq^{\mathcal{D}} b\big) \;\Leftrightarrow\; B \cap \left(\bigcap_{a \in A} (\{a\} + \mathcal{D}(a))\right) \neq \emptyset.$$

6. *the u-existence less order relation of type $\mathcal{P}$ ($\preccurlyeq_{e_u}^{\mathcal{P}}$) is defined by:*

$$A \preccurlyeq_{e_u}^{\mathcal{P}} B \;:\Leftrightarrow\; \big(\exists b \in B \ \text{ such that } \ \forall a \in A:\ a \leq^{\mathcal{P}} b\big) \;\Leftrightarrow\; \exists b \in B:\ A \subset \{b\} - \mathcal{P}(b).$$

7. *the l-less order relation of type $\mathcal{D}$ ($\preccurlyeq_l^{\mathcal{D}}$) is defined by:*

$$A \preccurlyeq_l^{\mathcal{D}} B \;:\Leftrightarrow\; \big(\forall b \in B\ \exists a \in A:\ a \leq^{\mathcal{D}} b\big) \;\Leftrightarrow\; B \subset \bigcup_{a \in A} (\{a\} + \mathcal{D}(a)).$$

8. *the l-less order relation of type $\mathcal{P}$ ($\preccurlyeq_l^{\mathcal{P}}$) is defined by:*

$$A \preccurlyeq_l^{\mathcal{P}} B \;:\Leftrightarrow\; \big(\forall b \in B\ \exists a \in A:\ a \leq^{\mathcal{P}} b\big) \;\Leftrightarrow\; \forall b \in B:\ A \cap (\{b\} - \mathcal{P}(b)) \neq \emptyset.$$

9. *the u-less order relation of type $\mathcal{D}$ ($\preccurlyeq_u^{\mathcal{D}}$) is defined by:*

$$A \preccurlyeq_u^{\mathcal{D}} B \;:\Leftrightarrow\; \big(\forall a \in A\ \exists b \in B:\ a \leq^{\mathcal{D}} b\big) \;\Leftrightarrow\; \forall a \in A:\ B \cap (\{a\} + \mathcal{D}(a)) \neq \emptyset.$$

10. *the u-less order relation of type $\mathcal{P}$ ($\preccurlyeq_u^{\mathcal{P}}$) is defined by:*

$$A \preccurlyeq_u^{\mathcal{P}} B \;:\Leftrightarrow\; \big(\forall a \in A\ \exists b \in B:\ a \leq^{\mathcal{P}} b\big) \;\Leftrightarrow\; A \subset \bigcup_{b \in B} (\{b\} - \mathcal{P}(b))\,.$$

11. *the possibly less order relation of type $\mathcal{D}$ ($\preccurlyeq_p^{\mathcal{D}}$) is defined by:*

$$A \preccurlyeq_p^{\mathcal{D}} B \;:\Leftrightarrow\; \big(\exists a \in A\ \exists b \in B:\ a \leq^{\mathcal{D}} b\big) \;\Leftrightarrow\; B \cap \left(\bigcup_{a \in A} (\{a\} + \mathcal{D}(a))\right) \neq \emptyset.$$

12. *the possibly less order relation of type $\mathcal{P}$ ($\preccurlyeq_p^{\mathcal{P}}$) is defined by:*

$$A \preccurlyeq_p^{\mathcal{P}} B \;:\Leftrightarrow\; \big(\exists a \in A\ \exists b \in B:\ a \leq^{\mathcal{P}} b\big) \;\Leftrightarrow\; A \cap \left(\bigcup_{b \in B} (\{b\} - \mathcal{P}(b))\right) \neq \emptyset.$$

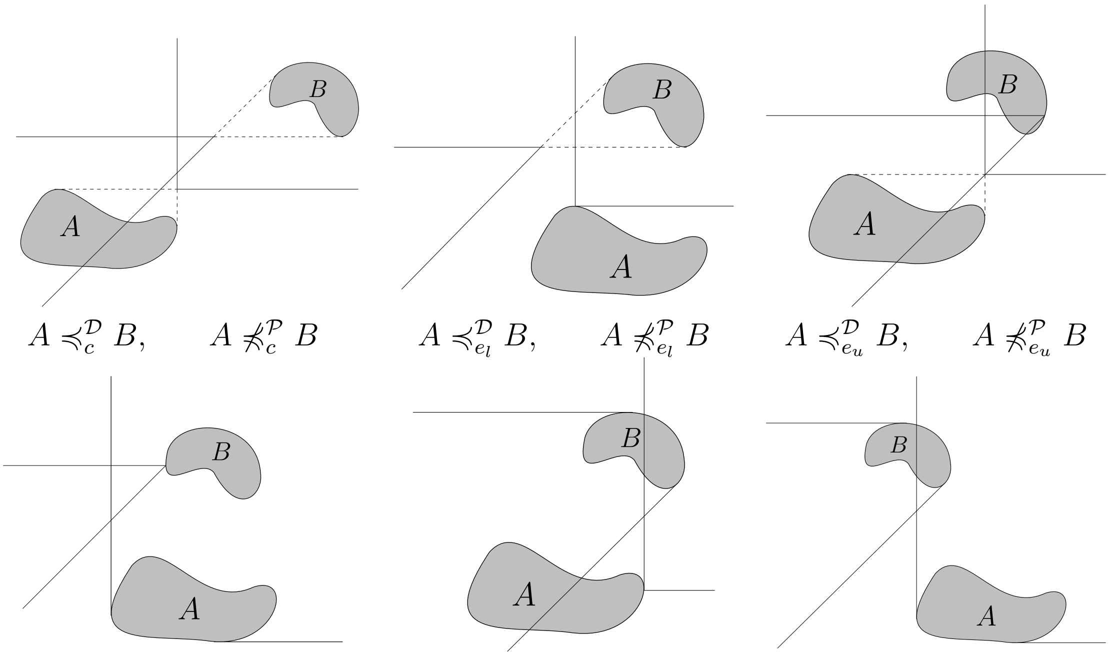
Provided the constant maps $\mathcal{D}(y) = \mathcal{P}(y) = K$ for all $y \in Y$ with a convex cone K in Y are considered, the above relations $\preccurlyeq_c^{\mathcal{D}}$ and $\preccurlyeq_c^{\mathcal{P}}$, $\preccurlyeq_{e_l}^{\mathcal{D}}$ and $\preccurlyeq_{e_l}^{\mathcal{P}}$, $\preccurlyeq_{e_u}^{\mathcal{D}}$ and $\preccurlyeq_{e_u}^{\mathcal{P}}$, $\preccurlyeq_l^{\mathcal{D}}$ and $\preccurlyeq_l^{\mathcal{P}}$, $\preccurlyeq_u^{\mathcal{D}}$ and $\preccurlyeq_u^{\mathcal{P}}$, and $\preccurlyeq_p^{\mathcal{D}}$ and $\preccurlyeq_p^{\mathcal{P}}$ coincide pairwise with each other. Moreover, their definitions correspond exactly to those introduced for a constant ordering cone in [25]. Therefore, in that case we omit the superindex.

Note that in [19] the certainly less order relation — assuming the space Y to be partially ordered by a constant pointed ordering cone $K \subset Y$ — is defined as follows:

$$A \preccurlyeq_{c,r} B \;:\Leftrightarrow\; \big(A \neq B,\ \forall a \in A\ \forall b \in B:\ a \leq^K b\big) \text{ or } (A = B)\,.$$

Based on this definition the obtained relation is reflexive, which is otherwise not the case in general. We instead generalize here the certainly less order relation as originally introduced in [4] and discuss the properties of the above defined binary relations in more detail in Section 3.

For an illustration of the concepts for two sets A and B in case of constant (but different) maps $\mathcal{D}$ and $\mathcal{P}$ on A and B respectively, see Figures 1 and 2. Figure 3 depicts a survey on the implications between the different binary relations, which follow directly from the definitions.



$A \preccurlyeq_c^{\mathcal{D}} B, \quad A \not\preccurlyeq_c^{\mathcal{P}} B \qquad A \preccurlyeq_{e_l}^{\mathcal{D}} B, \quad A \not\preccurlyeq_{e_l}^{\mathcal{P}} B \qquad A \preccurlyeq_{e_u}^{\mathcal{D}} B, \quad A \not\preccurlyeq_{e_u}^{\mathcal{P}} B$

$A \preccurlyeq_l^{\mathcal{D}} B, \quad A \not\preccurlyeq_l^{\mathcal{P}} B \qquad A \preccurlyeq_u^{\mathcal{D}} B, \quad A \not\preccurlyeq_u^{\mathcal{P}} B \qquad A \preccurlyeq_p^{\mathcal{D}} B, \quad A \not\preccurlyeq_p^{\mathcal{P}} B$

Figure 1: Illustration of the concepts of Definition 2.4.

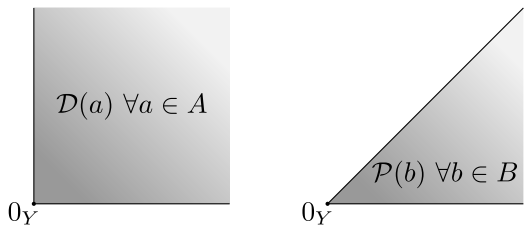


Figure 2: Illustration of maps $\mathcal{D}$ and $\mathcal{P}$ used in Figure 1.

Köbis introduces in [21] two set relations called upper set less order relations with variable ordering based on a set-valued map $C\colon Y \to 2^Y$:

$$\begin{aligned} A \preceq^u_{C,1} B \quad &:\Leftrightarrow \quad \forall a \in A\ \exists b \in B\colon \quad b \in a + C(a) \\ A \preceq^u_{C,2} B \quad &:\Leftrightarrow \quad \forall a \in A\ \exists b \in B\colon \quad a \in b - C(b). \end{aligned}$$

By using the set-valued map $\mathcal{D}$ ($\mathcal{P}$) instead of C for the definition of $\preceq^u_{C,1}$ ($\preceq^u_{C,2}$) we obtain exactly the u-less order relation of type $\mathcal{D}$ ($\preccurlyeq^{\mathcal{D}}_u$) and the u-less order relation of type $\mathcal{P}$ ($\preccurlyeq^{\mathcal{P}}_u$) as defined above. Also a stronger relation was introduced in [21]:

$$\begin{aligned} A \preceq^u_{C,1,2} B \quad &:\Leftrightarrow \quad \forall a \in A\ \exists b \in B\colon \quad b \in a + C(a) \text{ and } a \in b - C(b) \\ &\Leftrightarrow \quad \forall a \in A\ \exists b \in B\colon \quad b - a \in C(a) \cap C(b). \end{aligned}$$

Based on a new map $\mathcal{C}\colon Y \times Y \to 2^Y$ the so called variable upper set less order relation is then defined by

$$A \preceq^u_{\mathcal{C}} B :\Leftrightarrow \forall a \in A\ \exists b \in B\colon b - a \in \mathcal{C}(a, b).$$

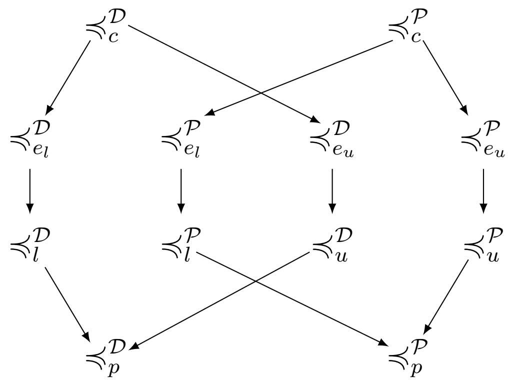


Figure 3: Implications between the certainly less order relations, the l-existence and u-existence less order relations, the l-less and u-less order relations and the possibly less order relations.

This approach generalizes the relations $\preceq_{C,1}$ and $\preceq_{C,2}$ in the following sense: the binary relation $\preceq^u_{C,1}$ is the special case of $\preceq^u_{\mathcal{C}}$ with $\mathcal{C}(a,b) = \mathcal{C}(a,c) =: C(a)$ for arbitrary $a, b, c \in Y$, i.e. the case when $\mathcal{C}$ depends on the first variable only. The binary relation $\preceq^u_{C,2}$ is the special case of $\preceq^u_{\mathcal{C}}$ when $\mathcal{C}$ depends on the second variable only (cf. [21, Remark 2]).

We do not follow this approach by [21] based on the following considerations. Let us assume that $\mathcal{C}(a,b)$ depends on both variables a and b for all $a, b \in Y$ (and not on the first or the second variable only), i.e. for $(a,b) \neq (a,c)$ we have $\mathcal{C}(a,b) \neq \mathcal{C}(a,c)$ and for $(a,c) \neq (b,c)$ we have $\mathcal{C}(a,c) \neq \mathcal{C}(b,c)$ in general. Then this means the following: for a pair $(a,b) \in Y \times Y$ all but one element of the set $\mathcal{C}(a,b)$ are not of interest and do not play any role for the defined set relations. It only matters whether $b - a \in \mathcal{C}(a,b)$ or $b - a \notin \mathcal{C}(a,b)$. The remaining elements of the set $\mathcal{C}(a,b)$ are not of interest for the definition of the binary relation $\{a\} \preceq^u_{\mathcal{C}} \{b\}$. In other words, all other elements $z \in \mathcal{C}(a,b)$ with $z \neq a - b$ are superfluous since they give no information about domination or preference for a given pair $(a,b) \in Y \times Y$. Since we cannot find a suitable interpretation for all elements of the set $\mathcal{C}(a,b)$ we do not use such a more general set-valued map $\mathcal{C}$ in this manuscript.

The forthcoming Definition 2.5 contains four other relations which are defined with the aid of relations 7-10 from Definition 2.4. These relations seem to be particularly practically relevant (cf. [19]), since they consider the set as a whole. Definition 2.5 is based on the definition of the set less relation with a constant ordering cone K (see e.g. [16]):

$$A \preccurlyeq_s B \;:\Leftrightarrow\; B \subset A + K \text{ and } A \subset B - K. \tag{6}$$

This definition was motivated by a set relation for intervals in [4] denoted there set less or equal relation.

Definition 2.5. *Let Y be a linear space and $A, B \in 2^Y$ be nonempty arbitrarily chosen sets. Then we define binary relations on 2^Y as follows:*

13. the set less order relation of type $\mathcal{D}$ ($\preccurlyeq^{\mathcal{D}}_s$) is defined by:

$$A \preccurlyeq^{\mathcal{D}}_s B \;:\Leftrightarrow A \preccurlyeq^{\mathcal{D}}_l B \;\; \textit{and} \;\; A \preccurlyeq^{\mathcal{D}}_u B.$$

$$
\begin{tikzcd}
(\preccurlyeq_{e_l}^{\mathcal{D}} \wedge \preccurlyeq_{e_u}^{\mathcal{D}}) \arrow[d] & (\preccurlyeq_{e_l}^{\mathcal{D}} \wedge \preccurlyeq_{e_u}^{\mathcal{P}}) \arrow[d] & (\preccurlyeq_{e_l}^{\mathcal{P}} \wedge \preccurlyeq_{e_u}^{\mathcal{D}}) \arrow[d] & (\preccurlyeq_{e_l}^{\mathcal{P}} \wedge \preccurlyeq_{e_u}^{\mathcal{P}}) \arrow[d] \\
\preccurlyeq_s^{\mathcal{D}} \arrow[d] \arrow[drr] & \preccurlyeq_s^{\mathcal{DP}} \arrow[dl] \arrow[drr] & \preccurlyeq_s^{\mathcal{PD}} \arrow[dl] \arrow[d] & \preccurlyeq_s^{\mathcal{P}} \arrow[dll] \arrow[d] \\
\preccurlyeq_l^{\mathcal{D}} & \preccurlyeq_l^{\mathcal{P}} & \preccurlyeq_u^{\mathcal{D}} & \preccurlyeq_u^{\mathcal{P}}
\end{tikzcd}
$$

Figure 4: Implications between the l-existence and u-existence order relations, the l-less and u-less order relations and the set less order relations.

14. the set less order relation of type $\mathcal{DP}$ ($\preccurlyeq_s^{\mathcal{DP}}$) is defined by:

$$A \preccurlyeq_s^{\mathcal{DP}} B \;:\Leftrightarrow A \preccurlyeq_l^{\mathcal{D}} B \;\; \textit{and} \;\; A \preccurlyeq_u^{\mathcal{P}} B.$$

15. the set less order relation of type $\mathcal{PD}$ ($\preccurlyeq_s^{\mathcal{PD}}$) is defined by:

$$A \preccurlyeq_s^{\mathcal{PD}} B \;:\Leftrightarrow A \preccurlyeq_l^{\mathcal{P}} B \;\; \textit{and} \;\; A \preccurlyeq_u^{\mathcal{D}} B.$$

16. the set less order relation of type $\mathcal{P}$ ($\preccurlyeq_s^{\mathcal{P}}$) is defined by:

$$A \preccurlyeq_s^{\mathcal{P}} B \;:\Leftrightarrow A \preccurlyeq_l^{\mathcal{P}} B \;\; \textit{and} \;\; A \preccurlyeq_u^{\mathcal{P}} B.$$

If the constant maps $\mathcal{D} = \mathcal{P} = K$ for all $y \in Y$ with a convex cone $K \subset Y$ are considered, the relations 13-16 are equivalent. For a survey on the implications between the binary relations of Definitions 2.4 and 2.5 see Figure 4. The shown implications follow again directly from the definitions.

2.3 Discussion on the different set relations

We study in this paper a wide range of set relations and generalize them to variable ordering structures. There are even more concepts for set relations which are based, for instance, on comparing the sets of minimal or maximal elements of the considered sets as defined in [19, Section 3.2]. In this section we discuss the usefulness of the different set relations to motivate introducing all the relations in Definitions 2.4 and 2.5.

We start with a result which allows a simple intuitive interpretation of the set less relation: $A \preccurlyeq_s B$ if A is "less or equal" B and B is "greater or equal" A at the same time. We give this characterization in a partially ordered space but it still illustrates the meaning of the set less order relations also in the case of a variable ordering. For this result we need the notion of the infimal set and supremal set for a cone-bounded set which were defined in a more general setting by Löhne [26]. Thereby, a subset A of a real topological linear space Y is K-bounded w.r.t. some ordering cone $K \subset Y$ if there exist $a \in Y$ and $b \in Y$ with

$$A \subset \{a\} + K \quad \text{and} \quad A \subset \{b\} - K.$$

If $\emptyset \neq \operatorname{int}(K) \neq Y$ is satisfied, then it holds for a K-bounded set A: $\operatorname{cl}(A + K) \neq Y$.

Definition 2.6. *Let Y be a real topological linear space with a (constant) ordering cone K satisfying $\emptyset \neq int(K) \neq Y$. Assume that $A \subset Y$ is nonempty and bounded, then the infimal set and supremal set of A with respect to K are defined by*

$$InfA := \{y \in cl(A + K) : (\{y\} - int(K)) \cap cl(A + K) = \emptyset\}$$

$$SupA := \{y \in cl(A - K) : (\{y\} + int(K)) \cap cl(A - K) = \emptyset\}$$

respectively.

In the next lemma we observe that by considering the $\preccurlyeq_s$ relation both infimal and supremal sets are compared with each other:

Lemma 2.7. *Let Y be a partially ordered real topological linear space with a pointed ordering cone K satisfying $\emptyset \neq int(K) \neq Y$. Moreover, assume that $A, B \subset Y$ are nonempty and bounded sets. Then*

$$A \preccurlyeq_s B \quad \Rightarrow \quad InfA \preccurlyeq_l InfB \;\; \text{and} \;\; SupA \preccurlyeq_l SupB.$$

Proof. Let $A \preccurlyeq_s B$ hold. Then it follows $B+K \subset A+K$ and consequently, using Corollary 1.48 (ix) from [26] we obtain

$$\mathrm{Inf}B \subset \mathrm{cl}(B+K) \subset \mathrm{cl}(A+K) = \mathrm{Inf}A \cup (\mathrm{Inf}A + \mathrm{int}(K)) \subset \mathrm{Inf}A \cup (\mathrm{Inf}A + K) = \mathrm{Inf}A + K.$$

Due to the fact that $A \subset B - K$ if and only if $-A \subset -B + K$ and for a given set C it follows $\mathrm{Sup}C = -\mathrm{Inf}(-C)$ (see [26, Chapter 1]), we obtain analogously

$$\begin{aligned} \mathrm{Sup}A &= -\mathrm{Inf}(-A) \subset -\mathrm{cl}(-A + K) \subset -\mathrm{cl}(-B + K + K) \\ &= -\mathrm{cl}(-B + K) = -\left(\mathrm{Inf}(-B) \cup (\mathrm{Inf}(-B) + \mathrm{int}(K)\right) \subset \mathrm{Sup}B - K. \end{aligned}$$

Now we only need to show that the last inclusion implies $\mathrm{Sup}B \subset \mathrm{Sup}A + K$. Suppose on the contrary that there exists $\bar{b} \in \mathrm{Sup}B$ with $\bar{b} \notin \mathrm{Sup}A + K$. Using Corollary 1.48 (xi) from [26] we have $\bar{b} \in Y = \mathrm{Sup}A \cup (\mathrm{Sup}A - \mathrm{int}(K)) \cup (\mathrm{Sup}A + \mathrm{int}(K))$. Therefore, it follows $\bar{b} \in \mathrm{Sup}A - \mathrm{int}(K)$ and there exist $\bar{a} \in \mathrm{Sup}A$ and $\bar{k} \in \mathrm{int}(K)$ with $\bar{b} = \bar{a} - \bar{k}$. Since $\mathrm{Sup}A \subset \mathrm{Sup}B - K$ holds, if we take $\bar{a} \in \mathrm{Sup}A$, then there exist $b \in \mathrm{Sup}B$ and $k \in K$ with $\bar{a} = b - k$. Now we have $b - k = \bar{a} \in (\{\bar{b}\} + \mathrm{int}(K)) \cap \mathrm{cl}(B - K)$, which is a contradiction to $\bar{b} \in \mathrm{Sup}B$. □

Based on the above result the set less order relations (Definition 2.5) seem to be most realistic and practically relevant, cf. [18] (where a constant ordering cone is considered). Especially, the set less order relation of type $\mathcal{DP}$ provides the following simple interpretation. Let us consider two nonempty sets $A, B \subset Y$. Then A is less than B if each element in B is dominated by some element in A and also each element in $a \in A$ is preferred to some element in B.

While the practical relevance of the set less order relations may be true in many cases, there might also be some applications where other concepts are of interest as well. We illustrate this with a simple example (Example 2.8) assuming for simplicity again a partially ordered space. Moreover, in Example 2.9 we discuss a special situation where there is no need to consider the set less order relation.

Furthermore, even when the set less order relation is the appropriate concept for an application, other (stronger or weaker) concepts might be used to derive necessary and sufficient conditions, which could be easier to evaluate, see Example 2.10. For instance Theorem 4.18 presents such simple sufficient criteria using the stronger concept of the certainly less order relation.

Example 2.8. *Let us assume that there is a head of a school with 20 school classes. For a competition the head can send classes which will be examined in math and geography. The head is only interested in whether a class of his school wins the competition or not. In the competition, each student of the class has to write a test in both subjects and gets marks (1 is best, 6 is worst). A final mark is calculated as a weighted sum* $w_1x + w_2y$ *(with* $w_1, w_2 \geq 0$*,* $w_1 + w_2 = 1$*) of the two marks* x *and* y*. The weights are not known before the head of the school has to decide which classes should be sent to take part in the competition. Finally, the result of a class is identified with the best result achieved by one of its students. Moreover, it is expected that the students will have the same marks as they are known to have in school. Assume class* A *has 10 students with marks*

$$(math = 2, geography = 1), (1, 2), (3, 1), (1, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 5), (5, 6)$$

and class B *has 10 students with marks*

$$(2, 2), (1, 3), (3, 1), (2, 3), (3, 2), (3, 3), (4, 1), (1, 4), (2, 4), (3, 4), (4, 4).$$

Then no matter which weights w_1, w_2 *will be chosen, class* A *will always reach a better or equal result than class* B*. Thus, class* B *is not going to be sent to the competition.*

The classes A *and* B *can be identified with discrete subsets from* $\mathbb{R}^2$*. Then* $A \preccurlyeq_l B$ *w.r.t. the natural ordering but* $A \not\preccurlyeq_s B$*, so the l-less order relation (and not the set less order relation) seems to be appropriate for modeling this problem.*

The following example shows that for some special situations (unbounded sets) there is no need to consider the set less order relation.

Example 2.9. *Let* Y *be a real linear space and let* K *be a cone with* $cor(K) \neq \emptyset$ *and* $K \subset \mathcal{P}(y)$ *for all* $y \in Y$*. Then* K *is reproducing, i.e.* $K - K = Y$*. Let* $\mathcal{M} \subset 2^Y$ *be a family of nonempty sets with the property*

$$A + K = A \quad \textit{for all} \quad A \in \mathcal{M}.$$

Then for arbitrary sets $A, B \in \mathcal{M}$ *it holds*

$$A \subset Y = B + Y = B + K - K = B - K \subset \bigcup_{b \in B} (\{b\} - \mathcal{P}(b)),$$

i.e. $A \preccurlyeq_u^{\mathcal{P}} B$. *Therefore, for arbitrary* $A, B \in \mathcal{M}$

$$A \preccurlyeq_s^{\mathcal{D}\mathcal{P}} B \;\Leftrightarrow A \preccurlyeq_l^{\mathcal{P}} B \quad \text{and} \quad A \preccurlyeq_u^{\mathcal{P}} B \;\Leftrightarrow A \preccurlyeq_l^{\mathcal{P}} B$$

and it suffices to study the l-less order relation of type $\mathcal{D}$.

Analogous argumentation can be also used to motivate the usefulness of the u-less order relations.

It is known that the certainly less order relation is a very strong concept and that in practical applications it might happen that no sets can be compared by this set relation. Therefore, defining a solution using this relation seems not to be reasonable. However, if there is a high number of sets for instance in a low dimensional space as $\mathbb{R}^2$, there might be many sets which can be compared. In this case the certainly less order relation (or characterization results for this relation) can be used to pre-select some sets. This is illustrated in the following example. For simplicity we assume the space $\mathbb{R}^2$ to be partially ordered by the natural ordering.

Example 2.10. *We consider the family of sets* $\mathcal{M}$ *which contains* n *nonempty closed convex subsets of* $\mathbb{R}^2$. *We assume the space* $\mathbb{R}^2$ *to be partially ordered by* $\mathbb{R}^2_+$. *These subsets are randomly constructed disks with random radius in the open interval* $(0, 2)$ *and random center points* $(x_i, y_i) \in (0, 10)$. *(The implementation uses the Matlab function rand which constructs pseudorandom values drawn from the standard uniform distribution on the open interval* $(0, 1)$*). Figure 5 shows such a family of sets* $\mathcal{M}$ *with* $n = 100$ *disks. We determined those disks which are larger w.r.t. the certainly less order relation than any other set and marked them in the figure in gray — these have been* $k = 72$ *disks. So only the remaining* 28 *disks have to be considered for other binary relations as the l-less or the set less order relation in case one wants to find the "minimal" sets (see Definition 4.1).*

In a simulation run we considered the families $\mathcal{M}$ *of random disks with* $n = 50, 100, 500,$ 1000 *and* 5000 *disks, respectively. As the runs are based on random data, we repeated each of them* 100 *times and took the average results. Table 1 gives the average number (in total* k *and the ratio* k/n*) of those disks which are larger than any other of the* n *sets w.r.t. the certainly less order relation. As it can be seen, for larger* n *the number of the sets which cannot be optimal w.r.t. the l-less or the set less order relation increases. For* $n = 5000$ *disks it turns out that an average of* 79 *percent of random sets is larger than another set w.r.t. the certainly less order relation and thus, these sets can be deleted in a pre-selection.*

The same purpose — to sort out the sets which cannot be optimal — may be attacked using the existence relations.

The possibly less relation is very week and therefore, it seems indeed not to be practically relevant. On the other hand, the negation of this relation might be useful to see in which case one set is definitely not less or equal another using any of the relations considered before.

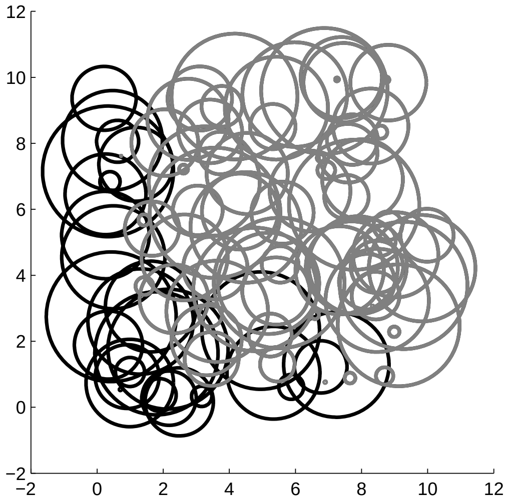


Figure 5: Sets of Example 2.10. The gray circles are the boundaries of those sets which are larger than some other set w.r.t. the certainly less order relation.

n	k	$\frac{k}{n}$
50	30	0.61
100	66	0.66
500	371	0.74
1000	755	0.76
5000	3941	0.79

Table 1: Average simulation results (100 runs each) for families of sets $\mathcal{M}$ with $n \in \{50, 100, 500, 1000, 5000\}$ subsets (disks) of $\mathbb{R}^2$.

3 Properties of the set relations

This section is devoted to the discussion of the basic properties of the introduced set relations as reflexivity or transitivity. Next to that we provide some basic calculation rules. We make use of the following assumption:

Assumption 1. *Let Y be a real linear space and $\mathcal{D}, \mathcal{P} : Y \to 2^Y$ be ordering maps (see Definition 2.2) where $\mathcal{D}(y), \mathcal{P}(y)$ are convex cones for all $y \in Y$.*

Lemma 3.1. *Let Assumption 1 hold.*

(i) The binary relations $\preccurlyeq_l^{\mathcal{P}}$, $\preccurlyeq_l^{\mathcal{P}}$, $\preccurlyeq_u^{\mathcal{D}}$, $\preccurlyeq_u^{\mathcal{P}}$, $\preccurlyeq_p^{\mathcal{D}}$,$\preccurlyeq_p^{\mathcal{P}}$ (Relations 7-12 in Definition 2.4) are reflexive.

(ii) Any of the binary relations $\preccurlyeq_c^{\mathcal{D}}$, $\preccurlyeq_{e_l}^{\mathcal{D}}$, $\preccurlyeq_{e_u}^{\mathcal{D}}$, $\preccurlyeq_l^{\mathcal{P}}$, $\preccurlyeq_u^{\mathcal{D}}$ (Relations 1,3,5,7,9 in Definition

2.4) is transitive if and only if $\mathcal{D}$ satisfies (3), i.e. if and only if the binary relation $\leq^{\mathcal{D}}$ is transitive.

(iii) Any of the binary relations $\preccurlyeq_c^{\mathcal{P}}$, $\preccurlyeq_{e_l}^{\mathcal{P}}$, $\preccurlyeq_{e_u}^{\mathcal{P}}$, $\preccurlyeq_l^{\mathcal{P}}$, $\preccurlyeq_u^{\mathcal{P}}$ (Relations 2,4,6,8,10 in Definition 2.4) is transitive if and only if $\mathcal{P}$ satisfies (4), i.e. if and only if the binary relation $\leq^{\mathcal{P}}$ is transitive.

(iv) If the cone $\mathcal{D}(Y)$ is pointed, then the binary relation $\preccurlyeq_c^{\mathcal{D}}$ (Relation 1 in Definition 2.4) is antisymmetric. If the cone $\mathcal{P}(Y)$ is pointed, then the binary relation $\preccurlyeq_c^{\mathcal{P}}$ (Relation 2 in Definition 2.4) is antisymmetric.

Proof. (i) The reflexivity follows from the fact that $0_Y \in \mathcal{D}(y)$ and $0_Y \in \mathcal{P}(y)$ for all $y \in Y$.

(ii) We only show the equivalence for the certainly less order relation of type $\mathcal{D}$. The proofs of the other equivalences are analogous.

First assume that $\mathcal{D}$ possesses the property given by (3) and we have $A, B, C \subset Y$ with $A \preccurlyeq_c^{\mathcal{D}} B$ and $B \preccurlyeq_c^{\mathcal{D}} C$. Then from the definition it follows for all $a \in A$ and for all $b \in B$ that $b - a \in \mathcal{D}(a)$. Furthermore, we obtain also for all $b \in B$ and for all $c \in C$ that $c \in \{b\} + \mathcal{D}(b)$. Due to the property (3) of $\mathcal{D}$ and since $\mathcal{D}(a)$ is a convex cone, we have for all $a \in A$, for all $b \in B$, and for all $c \in C$ that

$$c \in \{b\} + \mathcal{D}(a + b - a) \subset \{b\} + \mathcal{D}(a) \subset \{a\} + \mathcal{D}(a) + \mathcal{D}(a) = \{a\} + \mathcal{D}(a).$$

Consequently, $A \preccurlyeq_c^{\mathcal{D}} C$ is satisfied and since the sets were chosen arbitrarily, the relation $\preccurlyeq_c^{\mathcal{D}}$ is transitive. For the opposite direction we only need to state that if $\preccurlyeq_c^{\mathcal{D}}$ is transitive for all elements of the power set of Y, then it is also transitive for singletons. Hence, we obtain immediately the transitivity of the binary relation $\leq^{\mathcal{D}}$.

(iii) Analogous to the proof of (ii).

(iv) We only show the implication for the certainly less order relation of type $\mathcal{D}$. The proof for the other implication is analogous.

Let us consider $A, B \subset Y$ satisfying the conditions $A \preccurlyeq_c^{\mathcal{D}} B$ and $B \preccurlyeq_c^{\mathcal{D}} A$ but $A \neq B$. W.l.o.g. there exists $a \in A$ such that $a \notin B$. Take an arbitrary $b \in B$ and observe that due to relations between A and B, we obtain the inclusions $b - a \in \mathcal{D}(a) \subset \mathcal{D}(Y)$ and $a - b \in \mathcal{D}(b) \subset \mathcal{D}(Y)$. Since $a \neq b$ holds, we have a contradiction to the pointedness of $\mathcal{D}(Y)$.

□

The certainly less order relation of type $\mathcal{D}$ is obviously not reflexiv in general. If $\mathcal{D}$ is a cone valued map and $\mathcal{D}(Y)$ is pointed, the reflexivity of this relation implies that the compared sets have to be singletons, which can be shown by contradiction. The same property holds for the certainly less order relation of type $\mathcal{P}$. The l-existence less order relation of type $\mathcal{D}$ and of type $\mathcal{P}$ as well as the u-existence less order relation of type $\mathcal{D}$

and of type $\mathcal{P}$ are also not reflexiv in general. In order to analyze the reflexivity of these binary relations, we need the notions of infimum and supremum of a considered set. These definitions using constant ordering cone can be found in [26].

Definition 3.2. *Let $A \subset Y$ and consider a variable ordering relation $\leq$. An element $l \in Y$ is called a lower (upper) bound of A if $l \leq a$ ($a \leq l$) for all $a \in A$. Furthermore, an element $k \in Y$ is called infimum (supremum) of A if k is a lower (upper) bound of A and for every other lower (upper) bound l of A we have $l \leq k$ ($k \leq l$). We use the notation* $\inf A$ ($\sup A$) *for the set of infima (suprema) of A.*

By considering Example 3.3 we note that the set $\inf A$ of a given set A is no singleton in general.

Example 3.3. *Let the set A be defined by: $A := \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 \leq 1\}$. We consider the following ordering map*

$$\mathcal{D}(z) := \begin{cases} \mathbb{R}^2_+ & \forall z \in \mathbb{R}^2 \setminus \{(2,0)\} \\ \mathit{cone}\{(-1,0),(0,1)\} & \text{if } z = (2,0)\ . \end{cases}$$

Using Definition 3.2 we obtain that the set A has two infima w.r.t. the variable ordering relation $\leq^{\mathcal{D}}$ at points $(0,0)$ and $(2,0)$ since for each lower bound l of the set A we have $(0,0) \in \{l\} + \mathcal{D}(l)$ as well as $(2,0) \in \{l\} + \mathcal{D}(l)$.

If we assume that the maps $\mathcal{D}$ and $\mathcal{P}$ satisfy the properties given by (3) and (4) respectively and that $\mathcal{D}(Y)$ and $\mathcal{P}(Y)$ are pointed, then we obtain that for a given set $A \subset Y$ $\inf A$ and $\sup A$ are singletons if they exist. This follows directly from Proposition 1.4 in [26] since in this case $\leq^{\mathcal{D}}$ or $\leq^{\mathcal{P}}$ are transitive and antisymmetric and therefore, they are partial orderings. Note that if in the definition of the infimum (Definition 3.2) the condition "for every other lower bound l of A we have $l \leq k$" is replaced by "for every other lower bound l of A we have $k \nleq l$", then there could exist more than one infimum even if the ordering cone is constant (for an example see [26, Example 1.9]).

For the reflexivity of the l-existence less order relation of type $\mathcal{D}$ we obtain the following characterization:

Lemma 3.4. *Let Y be a real linear space, $A \subset Y$ be a nonempty set and $\mathcal{D} : Y \to 2^Y$ be an ordering map. Then $A \preccurlyeq^{\mathcal{D}}_{e_l} A$ if and only if* $(\inf A) \cap A \neq \emptyset$*, where* $\inf A$ *is considered w.r.t. the variable ordering relation $\leq^{\mathcal{D}}$.*

Proof. First we assume that the set A contains an infimum with respect to the relation $\leq^{\mathcal{D}}$. Since any infimum of A is also a lower bound of A, for all $a \in A$ we obtain $\inf A \leq^{\mathcal{D}} a$ and hence, $A \preccurlyeq^{\mathcal{D}}_{e_l} A$ holds. On the other hand, by considering $A \subset Y$ and assuming that $A \preccurlyeq^{\mathcal{D}}_{e_l} A$ is satisfied, from the definition of the relation $\preccurlyeq^{\mathcal{D}}_{e_l}$ we obtain immediately that there is a lower bound $\bar{a}$ of A contained in this set. If we now assume that $\bar{a}$ is not an infimum of A, then there exists another lower bound k of A with $k \nleq^{\mathcal{D}} \bar{a}$. However, due to the fact that k is a lower bound of A and $\bar{a} \in A$ holds, we obtain $k \leq^{\mathcal{D}} \bar{a}$, which is a contradiction. Therefore, it follows $(\inf A) \cap A \neq \emptyset$. $\square$

The reflexivity of the l-existence less order relation of type $\mathcal{P}$ can be obtained analogously. For studying the reflexivity of the u-existence less order relations of type $\mathcal{D}$ or type $\mathcal{P}$ we obtain analogously the condition $(\sup A) \cap A \neq \emptyset$, where $\sup A$ is considered w.r.t. the variable ordering relation $\leq^{\mathcal{D}}$ or $\leq^{\mathcal{P}}$, respectively.

The possibly less order relations of type $\mathcal{D}$ and $\mathcal{P}$ are both not transitive because they are also not transitive if a constant ordering cone is considered ($\mathcal{D}(\cdot) = \mathcal{P}(\cdot) = K \subset Y$), cf. [19]. Note that all introduced relations except the certainly less order relations of type $\mathcal{D}$ and $\mathcal{P}$ (under assumption $\mathcal{D}(Y)$ and $\mathcal{P}(Y)$ are pointed) are not antisymmetric in general. This follows from the fact that they are also not antisymmetric if a constant ordering cone is considered (cf. [19] for u-less, l-less and possibly less order relations).

For the l-existence order relation of type $\mathcal{D}$ we obtain the following result:

Lemma 3.5. *Let Y be a real linear space and $\mathcal{D} : Y \to 2^Y$ be an ordering map. Assume additionally that $\mathcal{D}(Y)$ is pointed and $A, B \subset Y$ are arbitrary nonempty sets. Then $A \preccurlyeq_{e_l}^{\mathcal{D}} B$ and $B \preccurlyeq_{e_l}^{\mathcal{D}} A$ if and only if there exists $\bar{y} \in (\inf A) \cap (\inf B)$, where $\inf A$ and $\inf B$ are considered w.r.t. the variable ordering relation $\leq^{\mathcal{D}}$.*

Proof. First assume $A \preccurlyeq_{e_l}^{\mathcal{D}} B$ and $B \preccurlyeq_{e_l}^{\mathcal{D}} A$. The definition of the relation $\preccurlyeq_{e_l}^{\mathcal{D}}$ implies the existence of $\bar{a} \in A$ and $\bar{b} \in B$ satisfying $B \subset \{\bar{a}\} + \mathcal{D}(\bar{a})$ and $A \subset \{\bar{b}\} + \mathcal{D}(\bar{b})$. Therefore, we have $\bar{a} - \bar{b} \in \mathcal{D}(\bar{b})$ as well as $\bar{b} - \bar{a} \in \mathcal{D}(\bar{a})$. Since $D(Y)$ is pointed, it follows $\bar{a} = \bar{b}$ and hence, the obtained conditions $B \subset \{\bar{b}\} + \mathcal{D}(\bar{b})$ and $A \subset \{\bar{a}\} + \mathcal{D}(\bar{a})$ imply $\bar{a} \in \inf A$ and $\bar{b} \in \inf B$ with respect to the relation $\leq^{\mathcal{D}}$.

On the other hand, if there exist $\bar{a} \in \inf A$ and $\bar{b} \in \inf B$ with respect to the relation $\leq^{\mathcal{D}}$ satisfying $\bar{a} = \bar{b}$, then the definitions of $\inf A$ and the relation $\preccurlyeq_{e_l}^{\mathcal{D}}$ imply $A \preccurlyeq_{e_l}^{\mathcal{D}} B$ and $B \preccurlyeq_{e_l}^{\mathcal{D}} A$. $\square$

Similar properties characterize also the l-existence order relation of type $\mathcal{P}$ and the u-existence order relations of type $\mathcal{D}$ and $\mathcal{P}$. Therefore, these relations are not antisymmetric in general as well.

The following corollary is a simple consequence of Lemma 3.1.

Corollary 3.6. *Let Assumption 1 hold.*

(i) The set less order relation of type $\mathcal{D}$ is reflexive and it is transitive if and only if the map $\mathcal{D}$ possesses the property given by (3).

(ii) The set less order relations of types $\mathcal{DP}$ and $\mathcal{PD}$ are reflexive and they are transitive if and only if the maps $\mathcal{D}$ and $\mathcal{P}$ possess the properties given by (3) and (4) respectively.

(iii) The set less order relation of type $\mathcal{P}$ is reflexive and it is transitive if and only if the map $\mathcal{P}$ possesses the property given by (4).

In the following we collect some basic calculation rules and implications for the discussed set relations. We omit the simple proofs as they easily follow from the definitions.
For the first result let the sets $A, B \subset Y$ be comparable with respect to one of the relations

introduced in Definitions 2.4 and 2.5. If additionally the binary relations $\leq^{\mathcal{D}}$ and $\leq^{\mathcal{P}}$ are transitive, we can note the following implications between the corresponding images of the maps $\mathcal{D}$ and $\mathcal{P}$ of these sets.

Lemma 3.7. *Suppose that Assumption 1 holds. Let $A, B \subset Y$ be arbitrary nonempty sets, the intersections of the corresponding cones be denoted by $\overline{\mathcal{D}}(A) := \bigcap_{a\in A} \mathcal{D}(a)$ and $\overline{\mathcal{P}}(A) := \bigcap_{a\in A} \mathcal{P}(a)$ and the maps $\mathcal{D}$ and $\mathcal{P}$ satisfy the conditions (3) and (4) respectively. Moreover, assume that for each $y \in Y$ it holds $\mathcal{D}(y) \neq \{0_Y\}$ and $\mathcal{P}(y) \neq \{0_Y\}$. Then*

(i) if $A \preccurlyeq_c^{\mathcal{D}} B$, then $\mathcal{D}(B) \subset \overline{\mathcal{D}}(A)$,

(ii) if $A \preccurlyeq_c^{\mathcal{P}} B$, then $\mathcal{P}(A) \subset \overline{\mathcal{P}}(B)$,

(iii) if $A \preccurlyeq_{e_l}^{\mathcal{D}} B$, then $\exists a \in A: \mathcal{D}(B) \subset \mathcal{D}(a)$,

(iv) if $A \preccurlyeq_{e_l}^{\mathcal{P}} B$, then $\exists a \in A: \mathcal{P}(a) \subset \overline{\mathcal{P}}(B)$,

(v) if $A \preccurlyeq_{e_u}^{\mathcal{D}} B$, then $\exists b \in B: \mathcal{D}(b) \subset \overline{\mathcal{D}}(A)$,

(vi) if $A \preccurlyeq_{e_u}^{\mathcal{P}} B$, then $\exists b \in B: \mathcal{P}(A) \subset \mathcal{P}(b)$,

(vii) if $A \preccurlyeq_l^{\mathcal{D}} B$, then $\mathcal{D}(B) \subset \mathcal{D}(A)$,

(viii) if $A \preccurlyeq_l^{\mathcal{P}} B$, then $\forall b \in B: \mathcal{P}(A) \cap \mathcal{P}(b) \neq \{0_Y\}$,

(ix) if $A \preccurlyeq_u^{\mathcal{D}} B$, then $\forall a \in A: \mathcal{D}(B) \cap \mathcal{D}(a) \neq \{0_Y\}$,

(x) if $A \preccurlyeq_u^{\mathcal{P}} B$, then $\mathcal{P}(A) \subset \mathcal{P}(B)$,

(xi) if $A \preccurlyeq_p^{\mathcal{D}} B$, then $\mathcal{D}(B) \cap \mathcal{D}(A) \neq \{0_Y\}$,

(xii) if $A \preccurlyeq_p^{\mathcal{P}} B$, then $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{0_Y\}$,

(xiii) if $A \preccurlyeq_s^{\mathcal{D}} B$, then $\mathcal{D}(B) \subset \mathcal{D}(A)$ and $\forall a \in A: \mathcal{D}(B) \cap \mathcal{D}(a) \neq \{0_Y\}$,

(xiv) if $A \preccurlyeq_s^{\mathcal{D}\mathcal{P}} B$, then $\mathcal{D}(B) \subset \mathcal{D}(A)$ and $\mathcal{P}(A) \subset \mathcal{P}(B)$,

(xv) if $A \preccurlyeq_s^{\mathcal{P}\mathcal{D}} B$, then $\forall b \in B: \mathcal{P}(A) \cap \mathcal{P}(b) \neq \{0_Y\}$ and $\forall a \in A: \mathcal{D}(B) \cap \mathcal{D}(a) \neq \{0_Y\}$,

(xvi) if $A \preccurlyeq_s^{\mathcal{P}} B$, then $\forall b \in B: \mathcal{P}(A) \cap \mathcal{P}(b) \neq \{0_Y\}$ and $\mathcal{P}(A) \subset \mathcal{P}(B)$.

The following lemma states which relations hold if we regard subsets or supersets of considered sets:

Lemma 3.8. *Let Y be a real linear space and $\mathcal{D}, \mathcal{P} : Y \to 2^Y$ be ordering maps. Assume that $A, B, C \subset Y$ are nonempty sets. Then*

(i) if $A \preccurlyeq_c^{\mathcal{D}} B$ and $C \subset A$, then $C \preccurlyeq_c^{\mathcal{D}} B$,
if $A \preccurlyeq_c^{\mathcal{D}} B$ and $C \subset B$, then $A \preccurlyeq_c^{\mathcal{D}} C$,

(ii) if $A \preccurlyeq_{e_l}^{\mathcal{D}} B$ and $A \subset C$, then $C \preccurlyeq_{e_l}^{\mathcal{D}} B$,
if $A \preccurlyeq_{e_l}^{\mathcal{D}} B$ and $C \subset B$, then $A \preccurlyeq_{e_l}^{\mathcal{D}} C$,

(iii) if $A \preccurlyeq_{e_u}^{\mathcal{D}} B$ and $C \subset A$, then $C \preccurlyeq_{e_u}^{\mathcal{D}} B$,
if $A \preccurlyeq_{e_u}^{\mathcal{D}} B$ and $B \subset C$, then $A \preccurlyeq_{e_u}^{\mathcal{D}} C$,

(iv) if $A \preccurlyeq_l^{\mathcal{D}} B$ and $A \subset C$, then $C \preccurlyeq_l^{\mathcal{D}} B$,
if $A \preccurlyeq_l^{\mathcal{D}} B$ and $C \subset B$, then $A \preccurlyeq_l^{\mathcal{D}} C$,

(v) if $A \preccurlyeq_u^{\mathcal{D}} B$ and $C \subset A$, then $C \preccurlyeq_u^{\mathcal{D}} B$,
if $A \preccurlyeq_u^{\mathcal{D}} B$ and $B \subset C$, then $A \preccurlyeq_u^{\mathcal{D}} C$,

(vi) if $A \preccurlyeq_p^{\mathcal{D}} B$ and $A \subset C$, then $C \preccurlyeq_p^{\mathcal{D}} B$,
if $A \preccurlyeq_p^{\mathcal{D}} B$ and $B \subset C$, then $A \preccurlyeq_p^{\mathcal{D}} C$,

(vii) if in assertions (i)-(vi) the relations of type $\mathcal{D}$ are replaced by the corresponding relations of type $\mathcal{P}$, the same implications hold.

Note that for the set less order relations of all types (see Definition 2.5) it is not possible to obtain similar results as in Lemma 3.8. In the next lemma we consider the intersections and unions of the images of the ordering maps $\mathcal{D}, \mathcal{P} : Y \to 2^Y$. The relations follow directly from the fact that for arbitrary nonempty sets $A, B \subset Y$ we have: for all $a \in A$ ($b \in B$) the condition $\overline{\mathcal{D}}(A) \subset \mathcal{D}(a) \subset \mathcal{D}(A)$ ($\overline{\mathcal{P}}(B) \subset \mathcal{P}(b) \subset \mathcal{P}(B)$) is satisfied.

Lemma 3.9. *Let Y be a real linear space and $\mathcal{D}, \mathcal{P} : Y \to 2^Y$ be ordering maps. Assume that $A, B \subset Y$ are arbitrary nonempty sets. The following implications hold:*

$$A \preccurlyeq^{\overline{\mathcal{D}}(A)} B \;\Rightarrow\; A \preccurlyeq^{\mathcal{D}} B \;\Rightarrow\; A \preccurlyeq^{\mathcal{D}(A)} B \tag{7}$$

where $\preccurlyeq^{\cdot}$ denotes one of relations 1, 3, 5, 7, 9, 11, 13 (from Definitions 2.4 and 2.5) with one of the constant cones $\overline{\mathcal{D}}(A)$, $\mathcal{D}(A)$ or map $\mathcal{D}$,

$$A \preccurlyeq^{\overline{\mathcal{P}}(B)} B \;\Rightarrow\; A \preccurlyeq^{\mathcal{P}} B \;\Rightarrow\; A \preccurlyeq^{\mathcal{P}(B)} B \tag{8}$$

where $\preccurlyeq^{\cdot}$ denotes one of relations 2, 4, 6, 8, 10, 12, 16 with one of the constant cones $\overline{\mathcal{P}}(A)$, $\mathcal{P}(A)$ or map $\mathcal{P}$. Additionally, for relations 14 and 15 we have:

$$A \preccurlyeq_s^{\overline{\mathcal{D}}(A)\overline{\mathcal{P}}(B)} B \;\Rightarrow\; A \preccurlyeq^{\mathcal{D}\mathcal{P}} B \;\Rightarrow\; A \preccurlyeq^{\mathcal{D}(A)\mathcal{P}(B)} B, \tag{9}$$

$$A \preccurlyeq^{\overline{\mathcal{P}}(B)\overline{\mathcal{D}}(A)} B \;\Rightarrow\; A \preccurlyeq^{\mathcal{P}\mathcal{D}} B \;\Rightarrow\; A \preccurlyeq^{\mathcal{P}(B)\mathcal{D}(A)} B. \tag{10}$$

4 Optimal solutions of a set optimization problem

Based on the introduced set relations we define in this section optimal solutions of a set optimization problem. As discussed in the introduction we use thereby the set approach in set optimization. We also compare the introduced optimality notions with those obtained by the vector approach.

A well known tool in vector optimization is to use scalarization approaches to determine optimal solutions. Jahn proposes in [18] a vectorization approach for set optimization based on the set less order relation with a constant ordering cone. This approach uses linear scalarizations for each of the two set relations (the l- and the u-less order relation) on which the definition of the set less order relation is based. Köbis gives in [21] linear scalarizations for the variable upper set less oder relation based on a variable ordering structure. Hernández and Rodríguez-Marín [14] and Gutiérrez et al. [11] discuss also nonlinear scalarization functionals for the l-less order relation in partially ordered spaces based among others on the Tammer-Weidner functional. We study in the following shortly linear scalarization approaches for the introduced set relations before we discuss a nonlinear scalarization based on a representation of the images of the ordering maps $\mathcal{D}$ and $\mathcal{P}$ as Bishop-Phelps cones.

In this section, next to Assumption 1, we make use of the following assumption:

Assumption 2. *Let in addition to Assumption 1 X be a real linear space, $S \subset X$ be a nonempty set and $F\colon X \to 2^Y$ be a set-valued map with $F(x) \neq \emptyset$ for all $x \in S$.*

Recall that by Assumption 1, Y is a real linear space and $\mathcal{D}, \mathcal{P} : Y \to 2^Y$ are ordering maps where $\mathcal{D}(y), \mathcal{P}(y)$ are convex cones for all $y \in Y$.

4.1 Definition of optimal solutions

In this section we use the notation $\preccurlyeq$ in order to indicate an arbitrary relation from Definitions 2.4 and 2.5. Provided we use only one or a subset of these relations, we remark it explicitly. Based on the set relations from Definitions 2.4 and 2.5 we can define optimal elements of a family of sets:

Definition 4.1. *Let Assumption 1 hold and let $\mathcal{F} \subset 2^Y$ be a family of nonempty sets.*

(i) The set $\bar{A} \in \mathcal{F}$ is an optimal element of $\mathcal{F}$ w.r.t. $\preccurlyeq$ if $A \preccurlyeq \bar{A}$ for some set $A \in \mathcal{F}$ implies $\bar{A} \preccurlyeq A$.

(ii) The set $\bar{A} \in \mathcal{F}$ is a strongly optimal element of $\mathcal{F}$ w.r.t. $\preccurlyeq$ if $\bar{A} \preccurlyeq A$ for all sets $A \in \mathcal{F} \setminus \{\bar{A}\}$.

We denote the set of optimal elements and of strongly optimal elements of $\mathcal{F}$ by $\mathrm{Min}(\mathcal{F}, \preccurlyeq)$ and $\mathrm{SMin}(\mathcal{F}, \preccurlyeq)$, respectively. Note that if a considered set relation $\preccurlyeq$ is reflexive, then we have: a set $\bar{A} \in \mathrm{SMin}(\mathcal{F}, \preccurlyeq)$ if and only if $\bar{A} \preccurlyeq A$ for all sets $A \in \mathcal{F}$. Based on optimal elements of a family of sets we can define optimal solutions of the set optimization problem (SOP) as preimages of optimal elements of the family of sets $\{F(x) : x \in S\}$, see part (i) in the definition below. We also give the definition of a strict optimal solution:

Definition 4.2. *Let Assumption 2 hold and let $\bar{x}$ be an element of S.*

(i) The element $\bar{x}$ is a (strongly) optimal solution of (SOP) w.r.t. one of the set relations $\preccurlyeq$ if $F(\bar{x})$ is a (strongly) optimal element of $\mathcal{F} := \{F(x) : x \in S\}$ w.r.t. this set relation $\preccurlyeq$, i.e. if $F(\bar{x}) \in Min(\mathcal{F}, \preccurlyeq)$ ($F(\bar{x}) \in SMin(\mathcal{F}, \preccurlyeq)$).

(ii) The element $\bar{x}$ is a strict optimal solution of (SOP) w.r.t. one of the set relations $\preccurlyeq$ if there exists no $x \in S \setminus \{\bar{x}\}$ with $F(x) = F(\bar{x})$ or with $F(x) \preccurlyeq F(\bar{x})$ (i.e. we have $F(x) \not\preccurlyeq F(\bar{x})$ and $F(x) \neq F(\bar{x})$ for all $x \in S \setminus \{\bar{x}\}$).

The definition above is based on the set approach. See [19, 31] for the analogous definitions regarding binary relations using a constant ordering cone. Note that the reason why we modified the definition of the strict optimal solutions slightly is the lack of reflexivity of some introduced relations, cf. Lemma 3.1. In the scalar-valued case the optimal and strongly optimal solutions both coincide with the common minimal solutions and the strict optimal solutions correspond to the unique minimal solutions.

4.2 Relation to the vector approach

For deriving some relationships of the solution concepts from Definition 4.2 and solutions using the vector approach with variable ordering structures in [7, 9], we define a new property for a given family of sets. We introduce the weak domination property, which is based on the concept of the domination property as used for instance in [19, Definition 4.9] for studying set relations with constant ordering cones.

Definition 4.3. *Let a family $\mathcal{F} \subset 2^Y$ of nonempty sets and one of the reflexive set relations $\preccurlyeq$ be given.*

(i) We say that $\mathcal{F}$ satisfies the weak domination property w.r.t. $\preccurlyeq$ if for each set $A \in \mathcal{F}$ there exists a family of sets $\overline{\mathcal{F}}^A \subset \mathcal{F}$ such that $\overline{\mathcal{F}}^A \subset Min(\mathcal{F}, \preccurlyeq)$ and we have

$$\bigcup\{B : B \in \overline{\mathcal{F}}^A\} \preccurlyeq A.$$

(ii) We say that $\overline{\mathcal{F}} \subset \mathcal{F}$ satisfies the domination property for $\mathcal{F}$ w.r.t. $\preccurlyeq$ if $Min(\overline{\mathcal{F}}, \preccurlyeq) \neq \emptyset$ and for each set $A \in \mathcal{F}$ there exists a set $B \in Min(\overline{\mathcal{F}}, \preccurlyeq)$ such that $B \preccurlyeq A$.

The following lemma clarifies, why we speak of a weaker notion.

Lemma 4.4. *Let a family $\mathcal{F} \subset 2^Y$ of nonempty sets and one of the reflexive set relations $\preccurlyeq$ be given.*

(i) Let the binary relation $\preccurlyeq$ be transitive. If there exists $\overline{\mathcal{F}} \subset \mathcal{F}$ such that $\overline{\mathcal{F}}$ satisfies the domination property for $\mathcal{F}$ w.r.t. $\preccurlyeq$, then $\mathcal{F}$ satisfies the weak domination property w.r.t. $\preccurlyeq$.

(ii) If Min$(\mathcal{F}, \preccurlyeq)$ satisfies the domination property for $\mathcal{F}$ w.r.t. $\preccurlyeq$, then $\mathcal{F}$ satisfies the weak domination property w.r.t. $\preccurlyeq$.

Proof. (i) We first show that $\mathrm{Min}(\overline{\mathcal{F}}, \preccurlyeq) \subset \mathrm{Min}(\mathcal{F}, \preccurlyeq)$. Assume to the contrary that $B \in \mathrm{Min}(\overline{\mathcal{F}}, \preccurlyeq)$ but $B \notin \mathrm{Min}(\mathcal{F}, \preccurlyeq)$. Then there exists $A \in \mathcal{F} \setminus \overline{\mathcal{F}}$ with $A \preccurlyeq B$ and $B \not\preccurlyeq A$. Since we have $A \in \mathcal{F}$, there is a set $C \in \mathrm{Min}(\overline{\mathcal{F}}, \preccurlyeq)$ such that $C \preccurlyeq A$ holds. As $\preccurlyeq$ is transitive, it follows $C \preccurlyeq B$. Due to the inclusions $B \in \mathrm{Min}(\overline{\mathcal{F}}, \preccurlyeq)$ and $C \in \overline{\mathcal{F}}$, we obtain also $B \preccurlyeq C$. Again, since $\preccurlyeq$ is transitive, we get $B \preccurlyeq A$ which is a contradiction.

Next, let $A \in \mathcal{F}$ be arbitrarily chosen. As $\overline{\mathcal{F}} \subset \mathcal{F}$ satisfies the domination property for $\mathcal{F}$ w.r.t. $\preccurlyeq$, there exists $B \in \mathrm{Min}(\overline{\mathcal{F}}, \preccurlyeq) \subset \mathrm{Min}(\mathcal{F}, \preccurlyeq) \subset \mathcal{F}$ such that $B \preccurlyeq A$ is satisfied. Let us define $\overline{\mathcal{F}}^A := \{B\}$. Then $\overline{\mathcal{F}}^A \subset \mathrm{Min}(\mathcal{F}, \preccurlyeq)$ holds and we have $\bigcup\{C : C \in \overline{\mathcal{F}}^A\} = B \preccurlyeq A$. Hence, the proof is completed.

(ii) Let $A \in \mathcal{F}$ be arbitrarily chosen. As $\mathrm{Min}(\mathcal{F}, \preccurlyeq)$ satisfies the domination property for $\mathcal{F}$ w.r.t. $\preccurlyeq$, there exists $B \in \mathrm{Min}(\mathcal{F}, \preccurlyeq) \subset \mathcal{F}$ such that $B \preccurlyeq A$. We set $\overline{\mathcal{F}}^A := \{B\}$. Then we have $\overline{\mathcal{F}}^A \subset \mathrm{Min}(\mathcal{F}, \preccurlyeq)$ as well as $\bigcup\{C : C \in \overline{\mathcal{F}}^A\} = B \preccurlyeq A$ and we are done.

□

The following example gives a family of sets $\mathcal{F}$ which possesses the weak domination property but for which there exists no $\overline{\mathcal{F}} \subset \mathcal{F}$ such that $\overline{\mathcal{F}}$ satisfies the domination property for $\mathcal{F}$. We use the l-less order relation of type $\mathcal{D}$ in this example.

Example 4.5. *Consider the constant ordering map $D(y) = \mathbb{R}^2_+$ for each $y \in \mathbb{R}^2_+$ and the l-less order relation of type $\mathcal{D}$. Let the set-valued map $F : [0,1] \to 2^{\mathbb{R}^2}$ be defined by:*

$$F(x) = \begin{cases} \{(\frac{1}{2}, 0)\} & \textit{if } \ x = 1 \\ \{(0, \frac{1}{2})\} & \textit{if } \ x = 0 \\ \textit{conv}\{(0,1), (1,x)\} & \textit{otherwise.} \end{cases} \tag{11}$$

Then the points $F(0)$ and $F(1)$ are the only optimal elements of the family of sets $\mathcal{F} := \{F(x) \ : \ x \in [0,1]\}$. On the other hand, for each set $F(x)$ with $x \in (0,1)$ we have $F(0) \not\preccurlyeq^{\mathcal{D}}_l F(x)$ as well as $F(1) \not\preccurlyeq^{\mathcal{D}}_l F(x)$ and there is no $\overline{\mathcal{F}} \subset \mathcal{F}$ such that $\overline{\mathcal{F}}$ satisfies the domination property for $\mathcal{F}$. However, by taking $\overline{\mathcal{F}}^{F(x)} := \{F(0), F(1)\}$ for each $x \in [0,1]$, $\mathcal{F}$ satisfies the weak domination property.

The fact that we need the transitivity of the binary relation $\preccurlyeq$ for the result (i) of Lemma 4.4 is illustrated in the next example using again the l-less order relation of type $\mathcal{D}$.

Example 4.6. *Consider the l-less order relation of type $\mathcal{D}$ and let a family of nonempty sets $\mathcal{F} \subset 2^{\mathbb{R}^2}$ be given by $\mathcal{F} := \{(0,1),(1,0),(1,1)\}$. The set-valued map $\mathcal{D}$ possesses the following images: $\mathcal{D}(y) = cone\{(0,1),(1,1)\}$ for all $y \in \mathbb{R}^2 \setminus \{(1,1)\}$ and $\mathcal{D}(1,1) = cone\{(0,1),(-1,0)\}$. It holds*

$$\{(1,1)\} \preccurlyeq_l^{\mathcal{D}} \{(0,1)\} \quad \textit{and} \quad \{(1,0)\} \preccurlyeq_l^{\mathcal{D}} \{(1,1)\}$$

but

$$\{(1,0)\} \not\preccurlyeq_l^{\mathcal{D}} \{(0,1)\}.$$

Thus $\preccurlyeq_l^{\mathcal{D}}$ is not transitive. The set $\overline{\mathcal{F}} = \{(0,1),(1,0)\} = Min(\overline{\mathcal{F}}, \preccurlyeq_l^{\mathcal{D}})$ satisfies the domination property for $\mathcal{F}$ w.r.t. $\preccurlyeq_l^{\mathcal{D}}$. On the other hand, $\mathcal{F}$ does not possess the weak domination property since we have $Min(\mathcal{F}, \preccurlyeq_l^{\mathcal{D}}) = \{(1,0)\}$. Hence, if we set $A = \{(0,1)\}$, there is no family of sets $\overline{\mathcal{F}}^A \subset \mathcal{F}$ with $\overline{\mathcal{F}}^A \subset Min(\mathcal{F}, \preccurlyeq_l^{\mathcal{D}})$ such that the condition $\bigcup\{B : B \in \overline{\mathcal{F}}^A\} \preccurlyeq A$ is satisfied. This example illustrates also that the pointedness of $\mathcal{D}(Y)$ is not a sufficient condition for the first relation between the domination properties from Lemma 4.4.

The next property, the lower quasi domination property for families of sets, generalizes Definition 3.7 in [19] and is equivalent to the weak domination property for certain set relations, cf. Lemma 4.8.

Definition 4.7. *Let a family $\mathcal{F} \subset 2^Y$ of nonempty sets and one of the reflexive set relations $\preccurlyeq$ be given. We say that $\mathcal{F}$ satisfies the lower quasi domination property w.r.t. $\preccurlyeq$ if for each $A \in \mathcal{F}$ it follows $\bigcup\{B : B \in Min(\mathcal{F}, \preccurlyeq)\} \preccurlyeq A$.*

Lemma 4.8. *Let a family $\mathcal{F} \subset 2^Y$ of nonempty sets and one of the reflexive set relations $\preccurlyeq_l^{\mathcal{D}}, \preccurlyeq_l^{\mathcal{P}}, \preccurlyeq_p^{\mathcal{D}}, \preccurlyeq_p^{\mathcal{P}}$ be given. Then $\mathcal{F} \subset 2^Y$ has the lower quasi domination property w.r.t. $\preccurlyeq$ iff it satisfies the weak domination property w.r.t. $\preccurlyeq$.*

Proof. Let $\mathcal{F}$ possess the lower quasi domination property w.r.t. $\preccurlyeq$. Then by defining $\overline{\mathcal{F}}^A := \text{Min}(\mathcal{F}, \preccurlyeq)$ for each $A \in \mathcal{F}$ we obtain immediately that for all $A \in \mathcal{F}$ the relation $\bigcup\{B : B \in \text{Min}(\mathcal{F}, \preccurlyeq)\} \preccurlyeq A$ holds. Hence, $\mathcal{F}$ satisfies the weak domination property w.r.t. $\preccurlyeq$.
On the other hand, if $\mathcal{F}$ satisfies the weak domination property w.r.t. $\preccurlyeq$, then for each set $A \in \mathcal{F}$ there exists a family of sets $\overline{\mathcal{F}}^A \subset \mathcal{F}$ such that $\overline{\mathcal{F}}^A \subset \text{Min}(\mathcal{F}, \preccurlyeq)$ and we have $\bigcup\{B : B \in \overline{\mathcal{F}}^A\} \preccurlyeq A$. Now, using Lemma 3.8 it follows $\bigcup\{B : B \in \text{Min}(\mathcal{F}, \preccurlyeq)\} \preccurlyeq A$ and the proof is completed. $\square$

Obviously, if the considered relation $\preccurlyeq$ is not reflexive, none of the introduced domination properties can be satisfied.

Definitions of solutions of (SOP) which generalize the vector approach for a variable ordering given by a set-valued map $\mathcal{D}\colon Y \to 2^Y$ can be found in [9]:

Definition 4.9. *Let Assumption 2 hold. Let $\bar{x}$ be an element of S and $\bar{y} \in F(\bar{x})$.*

(i) The pair $(\bar{x}, \bar{y})$ is called a nondominated solution of the problem (SOP) with respect to the ordering map $\mathcal{D}$ if $\bar{y}$ is a nondominated element of the set $F(S)$, i.e. there is no $y \in F(S) \setminus \{\bar{y}\}$ such that $\bar{y} \in \{y\} + \mathcal{D}(y)$.

(ii) The pair $(\bar{x}, \bar{y})$ is called a strongly minimal solution of the problem (SOP) with respect to the ordering map $\mathcal{D}$ if $\bar{y}$ is a strongly minimal element of the set $F(S)$, i.e. if $F(S) \subset \{\bar{y}\} + \mathcal{D}(\bar{y})$.

Analogously one can define (weakly/strongly/properly) minimal and nondominated solutions [8]. We restrict ourselves here to the two notions given in (i) and (ii) in Definition 4.9 and give relations to the optimality notions from Definition 4.2.

Theorem 4.10. *Let Assumption 2 hold and consider problem (SOP). Then the following assertions hold:*

(i) Assume that $(\bar{x}, \bar{y})$ is a nondominated solution (from Definition 4.9) of problem (SOP) with respect to the ordering map $\mathcal{D}$ and $\mathcal{F} := \{F(x), x \in S\}$ has the weak domination property w.r.t. $\preccurlyeq_l^{\mathcal{D}}$. Then there exists an optimal solution $x' \in S$ of (SOP) w.r.t. the relation $\preccurlyeq_l^{\mathcal{D}}$ with $\bar{y} \in F(x')$.

(ii) Assume additionally that $\mathcal{D}$ satisfies condition (3) and that $\mathcal{D}(y)$ is pointed for all $y \in S$. Let $\bar{x}$ be a strongly optimal solution of (SOP) w.r.t. the relation $\preccurlyeq_l^{\mathcal{D}}$ and the set $F(\bar{x})$ possesses at least one nondominated element $\bar{y}$ with respect to the ordering map $\mathcal{D}$, i.e. it holds $\bar{y} \notin \{y\} + \mathcal{D}(y)$ for all $y \in F(\bar{x}) \setminus \{\bar{y}\}$. Then $(\bar{x}, \bar{y})$ is a nondominated solution (from Definition 4.9) of problem (SOP) with respect to the ordering map $\mathcal{D}$.

(iii) If $(\bar{x}, \bar{y})$ is a strongly minimal solution (from Definition 4.9) of problem (SOP) with respect to the ordering map $\mathcal{D}$, then $\bar{x}$ is a strongly optimal solution of (SOP) w.r.t. the relation $\preccurlyeq_{e_l}^{\mathcal{D}}$.

(iv) Let $\bar{x}$ be a strongly optimal solution of (SOP) w.r.t. the relation $\preccurlyeq_c^{\mathcal{D}}$ and a strongly minimal element $\bar{y}$ of $F(\bar{x})$ exist, i.e. $F(\bar{x}) \subset \{\bar{y}\} + \mathcal{D}(\bar{y})$. Then $(\bar{x}, \bar{y})$ is a strongly minimal solution (from Definition 4.9) of problem (SOP) with respect to the ordering map $\mathcal{D}$.

Proof. (i) It holds $\bar{y} \notin \{y\} + \mathcal{D}(y)$ for all $y \in F(S) \setminus \{\bar{y}\}$. Furthermore, due to the weak domination property, there exists a family of sets $\overline{\mathcal{F}}^{F(\bar{x})} \subset \mathcal{F}$ such that each x with $F(x) \in \overline{\mathcal{F}}^{F(\bar{x})}$ is an optimal solution of (SOP) w.r.t. $\preccurlyeq_l^{\mathcal{D}}$. Thus, there exists $\bar{S} \subset S$ with $\overline{\mathcal{F}}^{F(\bar{x})} = \{F(x) \mid x \in \bar{S}\}$ and each $x \in \bar{S}$ is an optimal solution of (SOP) w.r.t. $\preccurlyeq_l^{\mathcal{D}}$. Moreover, we have

$$F(\bar{x}) \subset \bigcup_{x \in \bar{S}} \bigcup_{y \in F(x)} (\{y\} + \mathcal{D}(y)) \,.$$

Hence, as $\bar{y} \in F(\bar{x})$ holds, we obtain that there is $y \in F(\bar{S}) \subset F(S)$ with $\bar{y} \in \{y\}+\mathcal{D}(y)$. However, by assumption we have $\bar{y} \notin \{y\}+\mathcal{D}(y)$ for all $y \in F(S)\setminus\{\bar{y}\}$ and therefore, $y = \bar{y}$ follows. Consequently, there exists $x' \in \bar{S}$ with $\bar{y} \in F(x')$.

(ii) As $\bar{x}$ is a strongly optimal solution of (SOP) w.r.t. the relation $\preccurlyeq_l^{\mathcal{P}}$, for each $x \in S$ we have $F(\bar{x}) \preccurlyeq_l^{\mathcal{P}} F(x)$. That means $F(x) \subset \bigcup_{y\in F(\bar{x})} (\{y\} + \mathcal{D}(y))$ for all $x \in S$. Now we take an arbitrary nondominated element $\bar{y}$ of $F(\bar{x})$ and assume on the contrary that $\bar{y}$ is not a nondominated element of $F(S)$. Then there are $x \in S$ and $y \in F(x) \setminus \{\bar{y}\}$ with

$$\bar{y} \in \{y\} + \mathcal{D}(y).$$

On the other hand, there exists $\hat{y} \in F(\bar{x})$ such that $y \in \{\hat{y}\} + \mathcal{D}(\hat{y})$ (since $\bar{x}$ is a strongly optimal solution of (SOP)). Using condition (3) we obtain $\mathcal{D}(y) \subset \mathcal{D}(\hat{y})$ and consequently, (due to convexity of images of $\mathcal{D}$) we have

$$\{y\} + \mathcal{D}(y) \subset \{\hat{y}\} + \mathcal{D}(\hat{y}).$$

Now it follows $\bar{y} \in \{\hat{y}\}+\mathcal{D}(\hat{y})$ with $\hat{y} \in F(\bar{x})$. Provided $\bar{y} \neq \hat{y}$, we have a contradiction since $\bar{y}$ is a nondominated element of $F(\bar{x})$. If $\hat{y} = \bar{y}$ holds, then due to property (3) of $\mathcal{D}$ we have $\mathcal{D}(\bar{y}) \subset \mathcal{D}(y)$ and we obtain

$$\{\bar{y}\} + \mathcal{D}(\bar{y}) \subset \{y\} + \mathcal{D}(y) \subset \{\bar{y}\} + \mathcal{D}(\bar{y}).$$

Moreover, it follows $\mathcal{D}(\bar{y}) \subset \mathcal{D}(y) \subset \mathcal{D}(\hat{y}) = \mathcal{D}(\bar{y})$ and hence, $\mathcal{D}(y) = \mathcal{D}(\bar{y})$ is satisfied. Thus, $\{y\} + \mathcal{D}(\bar{y}) = \{\bar{y}\} + \mathcal{D}(\bar{y})$ holds and as $\mathcal{D}(\bar{y})$ is pointed, we obtain $y = \bar{y}$. This is a contradiction to $y \in F(x) \setminus \{\bar{y}\}$. Therefore, $\bar{y}$ is a nondominated element of $F(S)$ and $(\bar{x}, \bar{y})$ is a nondominated solution of problem (SOP) with respect to the ordering map $\mathcal{D}$.

(iii) Since $F(S) \subset \{\bar{y}\}+\mathcal{D}(\bar{y})$ holds, it follows $F(x) \subset \{\bar{y}\}+\mathcal{D}(\bar{y})$ for all $x \in S$. Therefore, we have $F(\bar{x}) \preccurlyeq_{e_l}^{\mathcal{D}} F(x)$ for all $x \in S$ and $\bar{x}$ is a strongly optimal solution of (SOP) w.r.t. the relation $\preccurlyeq_{e_l}^{\mathcal{D}}$.

(iv) From $F(\bar{x}) \preccurlyeq_c^{\mathcal{D}} F(x)$ for all $x \in S$ with $F(x) \neq F(\bar{x})$ it follows

$$F(x) \subset \bigcap_{y\in F(\bar{x})} (\{y\} + \mathcal{D}(y)) \quad \text{for all } \; x \in S \text{ with } \; F(x) \neq F(\bar{x}).$$

Choose an arbitrary strongly minimal element $\bar{y}$ of $F(\bar{x})$. Then we obtain $F(x) \subset \{\bar{y}\} + \mathcal{D}(\bar{y})$ for all $x \in S$ and consequently, $F(S) \subset \{\bar{y}\} + \mathcal{D}(\bar{y})$ holds. Thus, $(\bar{x}, \bar{y})$ is a strongly minimal solution of (SOP) with respect to the ordering map $\mathcal{D}$.

□

In [7, Definition 2.1] the following vector approach based on an ordering map $\widehat{\mathcal{D}}\colon X \to 2^Y$ is used (in fact, a local version of the following definition):

Definition 4.11. *Let real linear spaces X and Y, a nonempty set $S \subset X$, a set-valued map $F : X \to 2^Y$, an ordering map $\widehat{\mathcal{D}}\colon X \to 2^Y$ and the pair $(\bar{x}, \bar{y}) \in S \times Y$ with $\bar{y} \in F(\bar{x})$ be given. The pair $(\bar{x}, \bar{y})$ is called a nondominated solution of the problem (SOP) with respect to the ordering map $\widehat{\mathcal{D}}$ if*

$$\bar{y} \notin \bigcup_{x \in S} \left(F(x) + \widehat{\mathcal{D}}(x) \setminus \{0_Y\} \right) .$$

Lemma 4.12. *Let Assumption 2 hold. Additionally, assume that $\bar{x}$ is a strict optimal solution of (SOP) w.r.t. the relation $\preccurlyeq_p^{\mathcal{D}}$ and define $\widehat{\mathcal{D}}(x) := \bigcap_{y \in F(x)} \mathcal{D}(y)$ for all $x \in X$. If there exists $\bar{y} \in F(\bar{x})$ with $\bar{y} \notin F(\bar{x}) + \widehat{\mathcal{D}}(\bar{x}) \setminus \{0_Y\}$, then the pair $(\bar{x}, \bar{y})$ is a nondominated solution of (SOP) w.r.t. $\widehat{\mathcal{D}}$ (from Definition 4.11).*

Proof. As $\bar{x}$ is a strict optimal solution of (SOP) w.r.t. the set relation $\preccurlyeq_p^{\mathcal{D}}$, we have

$$F(x) \not\preccurlyeq_p^{\mathcal{D}} F(\bar{x}) \ \text{ for all } x \in S \setminus \{\bar{x}\}.$$

Therefore, for all $x \in S \setminus \{\bar{x}\}$, for all $y \in F(x)$ and for all $\hat{y} \in F(\bar{x})$ it holds $\hat{y} \notin \{y\} + \mathcal{D}(y)$. Consequently, since for each $x \in S \setminus \{\bar{x}\}$ we have $\hat{y} \notin \bigcup_{y \in F(x)} (\{y\} + \mathcal{D}(y))$, it follows for all $x \in S \setminus \{\bar{x}\} : \hat{y} \notin \bigcup_{y \in F(x)} \left(\{y\} + \bigcap_{\tilde{y} \in F(x)} \mathcal{D}(\tilde{y}) \right)$. Now we obtain

$$\hat{y} \notin \bigcup_{x \in S \setminus \{\bar{x}\}} \left(F(x) + \widehat{\mathcal{D}}(x) \right) \quad \text{for all } \ \hat{y} \in F(\bar{x}).$$

Let us now take $\bar{y} \in F(\bar{x})$ satisfying $\bar{y} \notin F(\bar{x}) + \widehat{\mathcal{D}}(\bar{x}) \setminus \{0_Y\}$. Then it follows

$$\bar{y} \notin \bigcup_{x \in S} \left(F(x) + \widehat{\mathcal{D}}(x) \setminus \{0_Y\} \right)$$

and the pair $(\bar{x}, \bar{y})$ is a nondominated solution of (SOP) w.r.t. $\widehat{\mathcal{D}}$ (from Definition 4.11). $\square$

Lemma 4.12 allows to use the necessary optimality conditions developed in [7] for characterizing strict optimal solutions w.r.t. the relation $\preccurlyeq_p^{\mathcal{D}}$.

4.3 Characterization by scalarization

Next we study characterizations of optimal solutions by scalarization. We start with a short examination of linear scalarizations. These were used by Jahn in [18] for the set less order relation, cf. (6), with a constant ordering cone resulting in a so-called vectorization approach. Vectorization with variable ordering structures can also be found in the paper by Köbis [21] for the special binary relation mentioned above.

The vectorization approach by Jahn in [18] uses the assumption that for a given subset A of a real locally convex linear space Y and the ordering cone $K \subset Y$ the Minkowski sum of these sets $A + K$ is closed and convex. He proved the relation

$$A \subset B + K \;\Leftrightarrow\; \sup_{\ell \in K^* \setminus \{0_{Y^*}\}} (\inf_{a \in A} \ell(a) - \inf_{b \in B} \ell(b)) \leq 0 \;, \tag{12}$$

where $K^* := \{\ell \in Y^* : \ell(y) \geq 0 \text{ for all } y \in K\}$ denotes the dual cone of K and Y^* denotes the (topological) dual space of Y.
In order to derive an analogous result for the l-less order relation of type $\mathcal{D}$, we need to define the cone $\overline{\mathcal{D}} := \bigcap_{y \in Y} \mathcal{D}(y)$. Applying the result by Jahn, cf. [18, Lemma 2.1], we obtain immediately

$$\sup_{\ell \in (\overline{\mathcal{D}})^* \setminus \{0_{Y^*}\}} (\inf_{a \in A} \ell(a) - \inf_{b \in B} \ell(b)) \leq 0 \Leftrightarrow A \preccurlyeq_l^{\overline{\mathcal{D}}} B.$$

Because $\overline{\mathcal{D}} \subset \mathcal{D}(a)$ holds for each $a \in A$, we can use the analogous argumentation as in Lemma 3.9 and this implies the following corollary.

Corollary 4.13. *Let Assumption 1 hold and let Y be additionally locally convex. Let A and B be arbitrary nonempty subsets of Y. If the set $A + \overline{\mathcal{D}}$ is closed and convex, then*

$$\sup_{\ell \in (\overline{\mathcal{D}})^* \setminus \{0_{Y^*}\}} (\inf_{a \in A} \ell(a) - \inf_{b \in B} \ell(b)) \leq 0 \;\Rightarrow\; A \preccurlyeq_l^{\mathcal{D}} B. \tag{13}$$

The result in Corollary 4.13 resembles the one derived by Köbis [21] for a different binary relation. Note that for comparing fixed nonempty sets $A \subset Y$ and $B \subset Y$ as in Corollary 4.13 it is enough to take the intersection of the images of the ordering maps over A or B, i.e. $\overline{\mathcal{D}}(A) = \bigcap_{a \in A} \mathcal{D}(a)$ or $\overline{\mathcal{D}}(B) = \bigcap_{b \in B} \mathcal{D}(b)$.

By using the cone $\mathcal{D}(Y) = \bigcup_{y \in Y} \mathcal{D}(y)$ we can derive similarly a necessary condition. If we assume that $\mathcal{D}(Y)$ is convex and $A + \mathcal{D}(Y)$ is closed and convex, we have:

$$A \preccurlyeq_l^{\mathcal{D}} B \Rightarrow \sup_{\ell \in (\mathcal{D}(Y))^* \setminus \{0_{Y^*}\}} (\inf_{a \in A} \ell(a) - \inf_{b \in B} \ell(b)) \leq 0. \tag{14}$$

Doubtlessly, it would be interesting to obtain similar assertions without considering constant unions or intersections of images of the ordering map. Unfortunately, the results that we derive in this case are more difficult to evaluate than the above ones.

Before we illustrate this issue using the l-less order relation of type $\mathcal{D}$, we need to recall some well known results. For any closed convex cone $K \neq \{0_Y\}$ of a locally convex linear space Y we have

$$K = \{y \in Y : \ell(y) \geq 0 \text{ for all } \ell \in K^* \setminus \{0_{Y^*}\}\}$$

[16, Lemma 3.21]. Therefore, for any $c \in Y$ we obtain

$$\inf_{\ell \in K^* \setminus \{0_{Y^*}\}} \ell(c) = \begin{cases} 0 & \text{if } c \in K \\ -\infty & \text{if } c \notin K. \end{cases} \tag{15}$$

We use this in order to prove the following characterization of the l-less order relation of type $\mathcal{D}$:

Lemma 4.14. *Let Assumption 1 hold, $\mathcal{D}(y) \neq \{0_Y\}$ for all $y \in Y$ and Y be locally convex. Assume that A and B are arbitrary nonempty subsets of Y. Then*

$$A \preccurlyeq_l^{\mathcal{D}} B \;\Leftrightarrow\; \inf_{b\in B} \sup_{a\in A} \inf_{\ell\in(\mathcal{D}(a))^*\setminus\{0_{Y^*}\}} \ell(b-a) = 0. \tag{16}$$

Proof. Using (15) we obtain the following chain of relationships:

$$\begin{aligned} & A \preccurlyeq_l^{\mathcal{D}} B \\ \Leftrightarrow\quad & \forall b \in B\ \exists a \in A:\ b - a \in \mathcal{D}(a) \\ \Leftrightarrow\quad & \forall b \in B\ \exists a \in A:\ \inf_{\ell\in(\mathcal{D}(a))^*\setminus\{0_{Y^*}\}} \ell(b-a) = 0 \\ \Leftrightarrow\quad & \forall b \in B: \sup_{a\in A} \inf_{\ell\in(\mathcal{D}(a))^*\setminus\{0_{Y^*}\}} \ell(b-a) = 0 \\ \Leftrightarrow\quad & \inf_{b\in B} \sup_{a\in A} \inf_{\ell\in(\mathcal{D}(a))^*\setminus\{0_{Y^*}\}} \ell(b-a) = 0. \end{aligned}$$

□

For the certainly less order relation of type $\mathcal{D}$ we have the following with analogous argumentation:

$$A \preccurlyeq_c^{\mathcal{D}} B \;\Leftrightarrow\; \inf_{b\in B} \inf_{a\in A} \inf_{l\in(\mathcal{D}(a))^*\setminus\{0_{Y^*}\}} l(b-a) = 0. \tag{17}$$

To give another example, for the set less order relation of type $\mathcal{DP}$ we have

$$\begin{aligned} A \preccurlyeq_s^{\mathcal{DP}} B \quad \Leftrightarrow \quad & \inf\nolimits_{b\in B}\ \sup\nolimits_{a\in A} \inf\nolimits_{\ell\in(\mathcal{D}(a))^*\setminus\{0_{Y^*}\}} \ell(b-a) = 0 \\ & \wedge\ \inf\nolimits_{a\in A}\ \sup\nolimits_{b\in B} \inf\nolimits_{\ell\in(\mathcal{P}(b))^*\setminus\{0_{Y^*}\}} \ell(b-a) = 0. \end{aligned} \tag{18}$$

From such necessary and sufficient conditions one may obtain characterizations of optimal solutions of set optimization problems by scalar-valued problems. Since the nonlinear scalarization constitutes our main focus, we deal with this issue in more detail using the nonlinear scalarization functionals, which we discuss next.

We use functionals introduced in [9] and therefore, we need to assume that the images of the ordering map $\mathcal{D}$ are representable as Bishop-Phelps cones.
A cone K in a real normed space $(Y, \|\cdot\|_Y)$ is said to be representable as a Bishop-Phelps (BP) cone if there exists a norm $\|\cdot\|$, equivalent to the norm of the space, and a continuous linear functional ℓ such that we can write $K = \{z \in Y \ :\ \|l\| \leq \ell(z)\}$. For that reason we need the following assumption:

Assumption 3. *Let $(Y, \|\cdot\|)$ be a real normed space and let $\mathcal{D}, \mathcal{P} : Y \to 2^Y$ be ordering maps where $\mathcal{D}(y), \mathcal{P}(y)$ are representable as BP cones for all $y \in Y$. That means, to any $y \in Y$ there exists a norm $\|\cdot\|_y$ equivalent to but eventually different from the norm of the space and for a given map $\ell\colon Y \to Y^*$ we can write*

$$\mathcal{D}(y) = \{z \in Y \ :\ \ell(y)(z) \geq \|z\|_y\}. \tag{19}$$

Moreover, let $\mathcal{D}(y) = \mathcal{P}(y)$ for all $y \in Y$.

Note that a wide range of different cones can be represented as BP cones (cf. [13] for examples and [9] for a discussion on this topic). For instance every nontrivial convex cone in $\mathbb{R}^n$ is representable as a BP cone if and only if it is closed and pointed. Moreover, in general real normed spaces every nontrivial convex cone with a closed and bounded base is representable as a BP cone (cf. [17, 30]). Examples for cone-valued maps satisfying condition (19) are given in [9].
We assume here that $\mathcal{D}(y) = \mathcal{P}(y)$ holds for all $y \in Y$ in order to avoid introducing to many notations. Obviously, all the results may be obtained analogously for maps which do not satisfy this assumption.

Following [9] we define the functionals $g : Y \times Y \to \mathbb{R}$ and $h : Y \times Y \to \mathbb{R}$ by

$$g(y,z) := \ell(z)(z-y) + \|z-y\|_z \qquad \text{for all } \; y \in Y,\; z \in Y \tag{20}$$
$$h(y,z) := \ell(y)(z-y) + \|z-y\|_y \qquad \text{for all } \; y \in Y,\; z \in Y. \tag{21}$$

Hereafter we need properties of these functionals depicted in the next lemma.

Lemma 4.15. *Let Assumption 3 hold and let elements $a, b \in Y$ be given. Then*

$$\begin{array}{rcl} a \leq^{\mathcal{D}} b & \Leftrightarrow & g(b,a) \leq 0 \\ a \leq^{\mathcal{P}} b & \Leftrightarrow & h(b,a) \leq 0. \end{array}$$

Proof. The first relationship holds due to

$$\begin{array}{rcl} a \leq^{\mathcal{D}} b & \Leftrightarrow & b - a \in \mathcal{D}(a) \\ & \Leftrightarrow & \ell(a)(b-a) \geq \|b-a\|_a \\ & \Leftrightarrow & \ell(a)(a-b) + \|a-b\|_a \leq 0 \\ & \Leftrightarrow & g(b,a) \leq 0\ . \end{array}$$

Analogously, we obtain the second relationship

$$\begin{array}{rcl} a \leq^{\mathcal{P}} b & \Leftrightarrow & b - a \in \mathcal{P}(b) \\ & \Leftrightarrow & \ell(b)(b-a) \geq \|b-a\|_b \\ & \Leftrightarrow & \ell(b)(a-b) + \|a-b\|_b \leq 0 \\ & \Leftrightarrow & h(b,a) \leq 0\ . \end{array}$$

□

As a consequence of Lemma 4.15 we observe that given a set $A \subset Y$ and an element $b \in Y$, we obtain

$$\forall\, a \in A : \quad a \leq^{\mathcal{D}} b \;\Leftrightarrow\; \sup_{a \in A} g(b,a) \leq 0.$$

Additionally, if the considered set A is weakly sequentially compact and $g(b,\cdot)$ is weakly lower semicontinuous on A, then we have also

$$\exists\, a \in A : \quad a \leq^{\mathcal{D}} b \;\Leftrightarrow\; \min_{a \in A} g(b,a) \leq 0.$$

Without assumption on the compactness of A we obviously obtain only the implication $\exists\, a \in A: \quad a \leq^{\mathcal{D}} b \Rightarrow \inf_{a\in A} g(b,a) \leq 0$.
Using these ideas we are able to characterize the set relations 1-16 introduced in Definitions 2.4 and 2.5 in terms of the functional (20) or (21). Due to the lack of space we will give only a few interesting examples. Let us consider the certainly less order relation of type $\mathcal{D}$ and assume that $A, B \subset Y$ are chosen arbitrarily. Then we obtain

$$\begin{aligned} A \preccurlyeq_c^{\mathcal{D}} B \Leftrightarrow &\quad \forall b \in B\ \forall a \in A\ :\ a \leq^{\mathcal{D}} b \\ \Leftrightarrow &\quad \forall b \in B\ :\ \sup_{a\in A} g(b,a) \leq 0 \\ \Leftrightarrow &\quad \sup_{b\in B} \sup_{a\in A} g(b,a) \leq 0 \\ \Leftrightarrow &\quad \sup_{a\in A,\, b\in B} g(b,a) \leq 0\ . \end{aligned} \tag{22}$$

The last equivalence relation follows from Lemma 7.3.2. in [22].

Similarly, we may consider the l-less order relation of type $\mathcal{D}$. If we choose two nonempty weakly sequentially compact sets $A, B \subset Y$ and assume that $g(b,\cdot)$ and $g(\cdot,a)$ are weakly lower semicontinuous on A or B respectively, we have

$$\begin{aligned} A \preccurlyeq_l^{\mathcal{D}} B \Leftrightarrow &\quad \forall b \in B\ \exists a \in A\ :\ a \leq^{\mathcal{D}} b \\ \Leftrightarrow &\quad \forall b \in B\ :\ \min_{a\in A} g(b,a) \leq 0 \\ \Leftrightarrow &\quad \sup_{b\in B} \min_{a\in A} g(b,a) \leq 0 \end{aligned} \tag{23}$$

and

$$A \preccurlyeq_u^{\mathcal{D}} B \ \Leftrightarrow\ \sup_{a\in A} \min_{b\in B} g(b,a) \leq 0.$$

In case we assume that the same norm $\|\cdot\|_y := \|\cdot\|$ can be chosen in Assumption 3 for all $y \in Y$, then it was shown in [9] that the function g inherits several properties from the properties of the map ℓ. To be more exact, according to [9, Proposition 3.13], if ℓ is continuous at a for each $a \in A$, then $g(b,\cdot)$ is continuous on A and the above assumptions on g are satisfied (as $g(\cdot,a)$ is obviously continuous on B without any additional assumptions).

For considering the possibly less order relation of type $\mathcal{P}$, we need to assume that the set $B \subset Y$ is compact, the set $A \subset Y$ is weakly sequentially compact and the functional $h(\cdot,\cdot)$ is lower semicontinuous on $A \times B$. Then using [1, Theorem 4.2.1] we obtain

$$\begin{aligned} A \preccurlyeq_p^{\mathcal{P}} B \Leftrightarrow &\quad \exists a \in A\ \exists b \in B\ :\ a \leq^{\mathcal{P}} b \\ \Leftrightarrow &\quad \exists a \in A\ :\ \min_{b\in B} h(b,a) \leq 0 \\ \Leftrightarrow &\quad \min_{a\in A,\, b\in B} h(b,a) \leq 0. \end{aligned} \tag{24}$$

Now we are ready to derive the characterization of the optimality notions introduced in Definition 4.2 using the functional (20) for the set less, the l-less and the certainly less order relations of type $\mathcal{D}$.

Theorem 4.16. *Consider problem (SOP). Let Assumptions 2 and 3 hold and let $\bar{x} \in S$. Additionally, suppose that for each $y \in F(S)$ the functionals $g(y,\cdot)$ and $g(\cdot,y)$ are weakly lower semicontinuous on $F(S)$ and that the images of F are weakly sequentially compact for all $x \in S$.*

(i) Then $\bar{x}$ is a strongly optimal solution of the problem (SOP) w.r.t. the set less order relation of type $\mathcal{D}$ iff

$$\sup_{y\in F(S)} \min_{\bar{y}\in F(\bar{x})} g(y,\bar{y}) \leq 0 \qquad \wedge \qquad \sup_{\bar{y}\in F(\bar{x})} \min_{y\in F(S)} g(y,\bar{y}) \leq 0.$$

(ii) Then $\bar{x}$ is a strict optimal solution of the problem (SOP) w.r.t. the set less order relation of type $\mathcal{D}$ iff

$$\forall x \in S\setminus\{\bar{x}\}: \quad \sup_{\bar{y}\in F(\bar{x})} \min_{y\in F(x)} g(\bar{y},y) > 0 \qquad \vee \qquad \sup_{y\in F(x)} \min_{\bar{y}\in F(\bar{x})} g(\bar{y},y) > 0$$

holds.

Proof. (i) This assertion follows immediately from the condition $F(\bar{x}) \preccurlyeq_s^{\mathcal{D}} F(x)$ for all $x \in S$ using (23).

(ii) As the set less order relation is reflexive, we have that $\bar{x} \in S$ is a strict optimal solution of the problem (SOP) if there exists no $x \in S\setminus\{\bar{x}\}$ with $F(x) \preccurlyeq_s^{\mathcal{D}} F(\bar{x})$. Using (23) this is equivalent to

$$\begin{aligned} \nexists x \in S\setminus\{\bar{x}\} \;:\; & \sup\nolimits_{\bar{y}\in F(\bar{x})} \min\nolimits_{y\in F(x)} g(\bar{y},y) \leq 0 \\ & \wedge \sup\nolimits_{y\in F(x)} \min\nolimits_{\bar{y}\in F(\bar{x})} g(\bar{y},y) \leq 0 \\ \Leftrightarrow \quad \forall x \in S\setminus\{\bar{x}\} \;:\; & \sup\nolimits_{\bar{y}\in F(\bar{x})} \min\nolimits_{y\in F(x)} g(\bar{y},y) > 0 \\ & \vee \sup\nolimits_{y\in F(x)} \min\nolimits_{\bar{y}\in F(\bar{x})} g(\bar{y},y) > 0 \;. \end{aligned}$$

Hence, the assertion holds. □

Similarly to Theorem 4.16 one may deduce analogous results for strong and strict optimal solutions w.r.t. the l-less or the u-less order relation of type $\mathcal{D}$. In the next theorem we give a result for optimal solutions w.r.t. the l-less order relation of type $\mathcal{D}$.

Theorem 4.17. *Consider problem (SOP). Let Assumptions 2 and 3 hold and let $\bar{x} \in S$. Additionally, suppose that for each $y \in F(S)$ the functionals $g(y,\cdot)$ and $g(\cdot,y)$ are weakly lower semicontinuous on $F(S)$ and that the images of F are weakly sequentially compact for all $x \in S$. Then $\bar{x}$ is an optimal solution of the problem (SOP) w.r.t. the l-less order relation of type $\mathcal{D}$ iff*

$$\forall x \in S: \; \sup_{y\in F(x)} \min_{\bar{y}\in F(\bar{x})} g(y,\bar{y}) \leq 0 \;\; \vee \;\; \sup_{\bar{y}\in F(\bar{x})} \min_{y\in F(x)} g(\bar{y},y) > 0. \tag{25}$$

Proof. Let $\bar{x}$ be an optimal solution of the problem (SOP) w.r.t. the l-less order relation of type $\mathcal{D}$. If we choose any $x \in S$, then we need to distinguish two cases:

- $F(x) \preccurlyeq_l^{\mathcal{D}} F(\bar{x})$ holds. Then it follows $F(\bar{x}) \preccurlyeq_l^{\mathcal{D}} F(x)$ and from (23) we obtain

$$\sup_{y \in F(x)} \min_{\bar{y} \in F(\bar{x})} g(y, \bar{y}) \leq 0.$$

- $F(x) \not\preccurlyeq_l^{\mathcal{D}} F(\bar{x})$ is satisfied or equivalently there exists an element $\bar{y} \in F(\bar{x})$ such that for all $y \in F(x)$ the condition $y \not\leq^{\mathcal{D}} \bar{y}$ holds. Using Lemma 4.15 it follows $g(\bar{y}, y) > 0$ for all $y \in F(x)$ and therefore, we have $\sup_{\bar{y} \in F(\bar{x})} \min_{y \in F(x)} g(\bar{y}, y) > 0$.

Let us now take $\bar{x} \in S$ such that the condition (25) is satisfied and assume that $\bar{x}$ is no optimal solution of (SOP) w.r.t. the l-less order relation of type $\mathcal{D}$. Then there exists $\hat{x} \in S$ such that $F(\hat{x}) \preccurlyeq_l^{\mathcal{D}} F(\bar{x})$ and $F(\bar{x}) \not\preccurlyeq_l^{\mathcal{D}} F(\hat{x})$ holds. Using (23) we obtain $\sup_{\bar{y} \in F(\bar{x})} \min_{y \in F(\hat{x})} g(\bar{y}, y) \leq 0$ and $\sup_{y \in F(\hat{x})} \min_{\bar{y} \in F(\bar{x})} g(y, \bar{y}) > 0$. This contradicts the condition (25). □

A result analogous to Theorem 4.17 can also be obtained for the set less order relation. However, due to the definition of this relation as a combination of the l-less and the u-less order relation, one needs to take many cases into consideration. Therefore, due to lack of space we do not regard it in this paper.

Theorem 4.18. *Consider problem (SOP). Let Assumptions 2 and 3 hold and let $\bar{x} \in S$.*

(i) $\bar{x}$ is a strongly optimal solution of the problem (SOP) w.r.t. the certainly less order relation of type $\mathcal{D}$ iff

$$\forall x \in S \text{ with } F(x) \neq F(\bar{x}) : \quad \sup_{y \in F(x), \bar{y} \in F(\bar{x})} g(y, \bar{y}) \leq 0.$$

(ii) Assume additionally that $F(x) \neq F(\bar{x})$ for all $x \in S \setminus \{\bar{x}\}$. Then $\bar{x}$ is a strict optimal solution of the problem (SOP) w.r.t. the certainly less order relation of type $\mathcal{D}$ iff

$$\forall x \in S \setminus \{\bar{x}\} : \quad \sup_{y \in F(x), \bar{y} \in F(\bar{x})} g(\bar{y}, y) > 0$$

holds.

(iii) $\bar{x}$ is an optimal solution of the problem (SOP) w.r.t. the certainly less order relation of type $\mathcal{D}$ iff

$$\forall x \in S : \quad \sup_{y \in F(x), \bar{y} \in F(\bar{x})} g(y, \bar{y}) \leq 0 \quad \vee \quad \sup_{y \in F(x), \bar{y} \in F(\bar{x})} g(\bar{y}, y) > 0. \tag{26}$$

We omit the simple proof, since it is based on the same ideas as the proofs of Theorems 4.16 and 4.17 using (22).

In case $\mathcal{D}(Y)$ is pointed and $|F(x)| \neq 1$ for each $x \in S$ we have that the set relation $\preccurlyeq_c^{\mathcal{D}}$ is antisymmetric and that $F(x) \not\preccurlyeq_c^{\mathcal{D}} F(x)$ for all $x \in S$. Under these additional assumptions obviously every optimal solution is also strict optimal provided $F(x) \neq F(\bar{x})$ for all $x \in S \setminus \{\bar{x}\}$. Hence, (ii) and (iii) coincide in this case.

One can derive analogous results for other set relations from Definitions 2.4 and 2.5 by adjusting the assumptions in order to guarantee that the supremum or infimum is attained (analogously as in (24)) depending on the considered relation.

Note that it is possible to apply the optimality conditions developed for the vector approach from [3] and [9] to set-valued optimization problems using the set criterion. This follows from the relations between the solutions of the set-valued optimization problems obtained in Theorem 4.10. Other nonlinear scalarization methods for problem (SOP) can also be derived if instead of the constant ordering cone in the scalarization functional in Gutiérrez et al. [11] a variable ordering map is considered. Let $A, B \subset Y$ be arbitrary sets and let $e \in -\left(\bigcap_{a\in A} \mathcal{D}(a)\right) \setminus \{0_Y\}$ hold. The Tammer-Weidner functional $G_e(A,B) : 2^Y \times 2^Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined by:

$$G_e(A,B) := \inf \left\{ t \in \mathbb{R} \ : B \subset t\, e + \bigcup_{a \in A} (\{a\} + \mathcal{D}(a)) \right\}$$

where $\inf\{\emptyset\} := +\infty$. The results in [8, Theorem 5.11] suggest that it may be possible to characterize the solutions of a set-valued optimization problem w.r.t. the l-less order relation of type $\mathcal{D}$ using this functional.

5 Conclusions

The introduced order relations establish an attempt to fill the gap between the set optimization with set relations and the problems equipped with variable ordering structure. This generalization of well-known set relations allows us to introduce optimality notions for set optimization problems with variable ordering structures, which seem to be more practice oriented than the notions based on the vector approach considered before. The obtained scalarization results for selected relations reveal the complexity of the regarded topic. Therefore, it would be interesting to consider set-valued maps with images characterized by a special structure as e.g. polyhedral-valued maps [34]. From the practical point of view some relations seem to be more appropriate for consideration than others. One of them may be the set less order relation of type $\mathcal{DP}$, which should be studied in the future in more detail.

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