Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 14/10

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November 2014

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The invertibility of 2×2 operator matrices

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November 11, 2014

Abstract

In this paper the properties of right invertible row operators, i.e., of 1×2 surjective operator matrices are studied. This investigation is based on a specific space decomposition. Using this decomposition, we characterize the invertibility of a 2×2 operator matrix. As an application, the invertibility of Hamiltonian operator matrices is investigated.

Keywords: 2×2 operator matrix, Hamiltonian operator matrix, invertibility, row operator

MSC 2010: 47A05, 47A10.

1 Introduction

The invertibility of a linear operator is one of the most basic problems in operator theory, and, obviously, appears in the study of the linear equation $Tx = y$ with a linear operator T.

This problem becomes even more involved if one considers the invertibility of 2×2 operator matrices. For this let A, B, C and D be bounded linear operators on a Hilbert space. If, e.g., they are pairwise commutative, then the operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

is invertible if and only if $AD - BC$ is invertible (cf. [3, Problem 70]). If only C and D are commutative, and if, in addition, D is invertible, then the operator matrix M is invertible if and only if $AD-BC$ is invertible (cf. [3, Problem 71]). In fact, the commutativity is essential in the above characterization, see [3, Problem 71]. The situation is even more involved if A and D are not defined on the same space and, hence, the formal expression $AD - BC$ has no meaning.

In general, there is no complete description of the invertibility of operator matrices in the non-commutative case. But if at least one of the entries A or D of

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the operator matrix M is invertible, one can describe the invertibility of M in terms of the Schur complement. A similar statement holds also in the case of invertible entries B or C. Moreover, the Schur complement method can be effectively used also in the case where the entries of M are unbounded operators under additionally assumptions on the domain of the entries, such as the diagonally (or off-diagonally) dominant or upper (lower) dominant cases, see, e.g., the monograph [7]. We also refer to [5, 8] for sufficient conditions for nonnegative Hamiltonian operators to have bounded inverses.

However, it is easy to see that there are many invertible 2×2 operator matrices with non invertible entries A, B, C and D (see, e.g., Theorem 2.11 below). Obviously, in such cases, the Schur complement method is not applicable.

It is the aim of the present article to give a full characterization for the invertibility of bounded 2×2 operator matrices. We do this in the following manner: A necessary condition for the invertibility of a 2×2 operator matrix M in (1.1) is the fact that the row operator $(A\ B)$ is right invertible (that is, the range $\mathcal{R}((A\ B))$ of the operator $(A\ B)$ covers all of the spaces). A further necessary condition is $\mathcal{N}((A\ B)) \neq \{0\}$, where $\mathcal{N}((A\ B))$ denotes the kernel of $(A\ B)$ (see Corollary 3.3 below). This non-zero kernel $\mathcal{N}((A\ B))$ plays a crucial role. Its projection $P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ onto the first component is a subset of the kernel of $P_{\mathcal{R}(B)^\perp} A$, where $P_{\mathcal{R}(B)^\perp}$ denotes the orthogonal projection onto $\mathcal{R}(B)^\perp$. Similarly, the projection of $\mathcal{N}((A\ B))$ onto the second component is a subset of $\mathcal{N}(P_{\mathcal{R}(A)^\perp} B)$.

Therefore we investigate a right invertible row operator $(A\ B)$ and choose a decomposition of the space into six parts which is built out of the subspaces $\mathcal{N}(A), \mathcal{N}(B), \mathcal{N}(P_{\mathcal{R}(B)^\perp} A)$ and $\mathcal{N}(P_{\mathcal{R}(A)^\perp} B)$. As a result, we show that the operator $B_2^{-1} \widetilde{A}_2$ considered as an operator from $P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ to $\mathcal{N}(B)^\perp \ominus \mathcal{N}(P_{\mathcal{R}(A)^\perp} B)^\perp$ is correctly defined. Here $\widetilde{A}_2$ (B_2) denote the restriction of A (B, respectively) to $\mathcal{N}(P_{\mathcal{R}(B)^\perp} A)$ ($\mathcal{N}(B)^\perp \ominus \mathcal{N}(P_{\mathcal{R}(A)^\perp} B)^\perp$, respectively).

The main result of the present article is a full characterization of the invertibility of a 2×2 matrix operator M in terms of its entries A, B, C, D, or to be more precise, in terms of the restrictions $\widetilde{A}_2, B_2, C_2$ and D_2 which are, in some sense, all related to $\mathcal{N}((A\ B))$: A 2×2 operator matrix M is invertible if and only if the following two statements are satisfied

(i) The restriction $D|_{\mathcal{N}(B)}$ is left invertible and

(ii) the operator

$$C_2 - D_2 B_2^{-1} \widetilde{A}_2 : P_{\mathcal{X}}(\mathcal{N}((A\,B))) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \text{ is one-to-one and surjective.}$$

Here C_2 (D_2) is the restriction of C (D, respectively) to $\mathcal{N}(P_{\mathcal{R}(B)^\perp} A)$ ($\mathcal{N}(B)^\perp \ominus \mathcal{N}(P_{\mathcal{R}(A)^\perp} B)^\perp$, respectively) projected onto $(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$.

This characterization is especially helpful if the spaces $\mathcal{N}((A\ B)), \mathcal{N}(P_{\mathcal{R}(B)^\perp} A)$ or $\mathcal{N}(P_{\mathcal{R}(A)^\perp} B)$ are known explicitly, see, e.g., Theorem 2.11 in Section 2. Moreover, we use it to derive a characterization for isomorphic row operators in Section 3. Finally, in Section 4 we give an application to Hamiltonian operators.

2 Main result

We always assume that $\mathcal{X}$ and $\mathcal{Y}$ are complex separable Hilbert spaces. Let T be a bounded operator between $\mathcal{X}$ and $\mathcal{Y}$. We write $T \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ and, if $\mathcal{X} = \mathcal{Y}$, $T \in \mathcal{B}(\mathcal{X})$. The range of T is denoted by $\mathcal{R}(T)$, the kernel by $\mathcal{N}(T)$. The term *isomorphism* is reserved for linear bijections $T : \mathcal{X} \to \mathcal{Y}$ that are homeomorphisms, i.e., $T \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ and $T^{-1} \in \mathcal{B}(\mathcal{Y},\mathcal{X})$.

A subspace in $\mathcal{Y}$ is an operator range if it coincides with the range of some bounded operator $T \in \mathcal{B}(\mathcal{X},\mathcal{Y})$. The following lemma is from [2, Theorem 2.4].

Lemma 2.1 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be operator ranges in $\mathcal{Y}$ such that $\mathcal{R}_1 + \mathcal{R}_2$ is closed.*
(i) *If $\mathcal{R}_1 \cap \mathcal{R}_2$ is closed, then $\mathcal{R}_1$ and $\mathcal{R}_2$ are closed.*
(ii) *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are dense in $\mathcal{Y}$, then $\mathcal{R}_1 \cap \mathcal{R}_2$ is dense in $\mathcal{Y}$.*

From [1, Proposition 2.14, Theorem 2.16], we have the following basic facts, which are important in the proofs of our main results.

Lemma 2.2 *Let Ω_1 and Ω_2 be two closed subspaces in $\mathcal{X}$. Then*

$$\Omega_1 \cap \Omega_2 = (\Omega_1^\perp + \Omega_2^\perp)^\perp, \quad \Omega_1^\perp \cap \Omega_2^\perp = (\Omega_1 + \Omega_2)^\perp,$$

and we further have the following equivalent descriptions:
(i) *$\Omega_1 + \Omega_2$ is closed;*
(ii) *$\Omega_1^\perp + \Omega_2^\perp$ is closed;*
(iii) *$\Omega_1 + \Omega_2 = (\Omega_1^\perp \cap \Omega_2^\perp)^\perp$;*
(iv) *$(\Omega_1 \cap \Omega_2)^\perp = \Omega_1^\perp + \Omega_2^\perp$.*

As usual, the symbol $\oplus$ denotes the orthogonal sum of two closed subspaces in a Hilbert space whereas the symbol $\dot{+}$ denotes the direct sum of two (not necessarily closed) subspaces in a Hilbert space. If Ω, Ω_1 are closed subspaces, $\Omega_1 \subset \Omega$, we denote by $\Omega \ominus \Omega_1$ the uniquely determined closed subspace Ω_2 in Ω with $\Omega = \Omega_1 \oplus \Omega_2$.

The next lemma is well known, see, e.g., [7, Proposition 1.6.2] or [4, 6].

Lemma 2.3 *Let $A \in \mathcal{B}(\mathcal{X}), B \in \mathcal{B}(\mathcal{Y},\mathcal{X}), C \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, and $D \in \mathcal{B}(\mathcal{Y})$. Let A (B) be an isomorphism. Then the 2×2 operator matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y})$$

is an isomorphism if and only if $D - CA^{-1}B$ (resp. $C - DB^{-1}A$) is an isomorphism.

Recall that an operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called right invertible if there exists an operator $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ with $TS = I_{\mathcal{Y}}$, where $I_{\mathcal{Y}}$ stands for the identity mapping in $\mathcal{Y}$. Hence, if T is right invertible then it is surjective. Conversely, if $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ then the restriction $T|_{\mathcal{N}(T)^\perp}$ maps $\mathcal{N}(T)^\perp$ onto $\mathcal{R}(T)$ and, if $\mathcal{R}(T) = \mathcal{Y}$, then $T|_{\mathcal{N}(T)^\perp} : \mathcal{N}(T)^\perp \to \mathcal{Y}$ is an isomorphism. Then with

$$S := \begin{pmatrix} 0 \\ \left(T|_{\mathcal{N}(T)^\perp}\right)^{-1} \end{pmatrix} : \mathcal{Y} \to \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp \tag{2.1}$$

considered as an operator in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$ we see that T is right invertible. This shows the equivalence of (i)-(iii) in the following (well-known) lemma.

Lemma 2.4 *For $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ the following assertions are equivalent.*

(i) *The operator T is right invertible.*

(ii) $\mathcal{R}(T) = \mathcal{Y}$.

(iii) *The operator $T|_{\mathcal{N}(T)^\perp}$ considered as an operator from $\mathcal{N}(T)^\perp$ into $\mathcal{Y}$ is an isomorphism.*

(iv) *There exists an isomorphism $U \in \mathcal{B}(\mathcal{Y})$ such that UT is a right invertible operator.*

Proof. It remains to show the equivalence of (iv) with (i)-(iii). Choose $U = I_{\mathcal{Y}}$ and we see that (i) implies (iv). Conversely, let $U \in \mathcal{B}(\mathcal{Y})$ be an isomorphism. If UT is right invertible, then by (ii) $\mathcal{R}(UT) = \mathcal{Y}$. As $\mathcal{R}(T) = \mathcal{R}(UT)$, again (ii) shows that T is right invertible. □

Similarly, $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called left invertible if there exists an operator $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ with $ST = I_{\mathcal{X}}$. Hence, if T is left invertible then it is injective and for a sequence (y_n) in $\mathcal{R}(T)$ with $y_n \to y$ as $n \to \infty$ we find (x_n) with $Tx_n = y_n$ and

$$x_n = STx_n = Sy_n \to Sy \quad \text{and} \quad y_n = Tx_n \to TSy,$$

which shows the closedness of $\mathcal{R}(T)$.

Conversely, if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is closed, then T considered as an operator from $\mathcal{X}$ into $\mathcal{R}(T)$ is an isomorphism and its inverse T^{-1} acts from $\mathcal{R}(T)$ into $\mathcal{X}$. Then with

$$S := \begin{pmatrix} 0 & T^{-1} \end{pmatrix} : \mathcal{R}(T)^\perp \oplus \mathcal{R}(T) \to \mathcal{X}, \tag{2.2}$$

considered as an operator in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$, we see that T is left invertible. We collect these statements in the following lemma, where the equivalence of (i)-(iii) follows from the above considerations and the equivalence of (i)-(iii) with (iv) is obvious.

Lemma 2.5 *For $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ the following assertions are equivalent.*

(i) *The operator T is left invertible.*

(ii) $\mathcal{N}(T) = \{0\}$ *and* $\mathcal{R}(T)$ *is closed.*

(iii) *The operator T considered as an operator from $\mathcal{X}$ into $\mathcal{R}(T)$ is an isomorphism.*

(iv) *There exists an isomorphism $V \in \mathcal{B}(\mathcal{X})$ such that TV is a left invertible operator.*

Remark 2.6 *The following observation for $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ follows immediately from the Lemmas 2.4 and 2.5. If T is right invertible, then there exists a left invertible operator $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ (cf.* (2.1)*) with $TS = I_{\mathcal{Y}}$ and $\mathcal{R}(S) = \mathcal{N}(T)^{\perp}$. If T is left invertible, then there exists a right invertible operator $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ (cf.* (2.2)*) with $ST = I_{\mathcal{X}}$.*

For the orthogonal projection onto a closed subspace Ω in some Hilbert space we shortly write P_{Ω}.

Theorem 2.7 *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and assume that the row operator $(A\ B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{X})$ is right invertible. Then $\mathcal{X}$ admits the decomposition*

$$\mathcal{X} = (\mathcal{R}(A)^{\perp} \dot{+} \mathcal{R}(B)^{\perp}) \oplus \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \tag{2.3}$$

and the space $\mathcal{X} \oplus \mathcal{Y}$ admits the decomposition

$$\mathcal{X} \oplus \mathcal{Y} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{Y}_3 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_1, \tag{2.4}$$

where

$$\begin{array}{lll} \mathcal{X}_1 := \mathcal{N}(A), & \mathcal{X}_2 := \mathcal{N}(A)^{\perp} \ominus \mathcal{N}(P_{\mathcal{R}(B)^{\perp}} A)^{\perp}, & \mathcal{X}_3 := \mathcal{N}(P_{\mathcal{R}(B)^{\perp}} A)^{\perp}; \\ \mathcal{Y}_1 := \mathcal{N}(B), & \mathcal{Y}_2 := \mathcal{N}(B)^{\perp} \ominus \mathcal{N}(P_{\mathcal{R}(A)^{\perp}} B)^{\perp}, & \mathcal{Y}_3 := \mathcal{N}(P_{\mathcal{R}(A)^{\perp}} B)^{\perp}. \end{array} \tag{2.5}$$

The row operator $(A\ B)$ from $\mathcal{X} \oplus \mathcal{Y}$ into $\mathcal{X}$ admits the following representation with respect to the decompositions (2.3) *and* (2.4)

$$\begin{pmatrix} 0 & 0 & 0 & B_3 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & A_2 & A_0 & B_0 & B_2 & 0 \end{pmatrix}, \tag{2.6}$$

where

$$A_0 \in \mathcal{B}\left(\mathcal{X}_3, \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}\right),\ A_2 \in \mathcal{B}\left(\mathcal{X}_2, \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}\right),\ A_3 \in \mathcal{B}\left(\mathcal{X}_3, \mathcal{R}(B)^{\perp}\right);$$
$$B_0 \in \mathcal{B}\left(\mathcal{Y}_3, \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}\right),\ B_2 \in \mathcal{B}\left(\mathcal{Y}_2, \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}\right),\ B_3 \in \mathcal{B}\left(\mathcal{Y}_3, \mathcal{R}(A)^{\perp}\right).$$

Then the operators A_3 and B_3 are isomorphisms and the row operator $(A_2\ B_2)$: $\mathcal{X}_2 \oplus \mathcal{Y}_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ is right invertible and

$$\overline{\mathcal{R}(A_2)} = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(B_2)}. \tag{2.7}$$

Proof. *Step 1. We prove* (2.3)–(2.6).

The row operator $(A\ B) : \mathcal{X} \oplus \mathcal{Y} \to \mathcal{X}$ is right invertible and we have with Lemma 2.4

$$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{X}. \tag{2.8}$$

We claim

$$P_{\mathcal{R}(A)^\perp}(\mathcal{R}(B)) = \mathcal{R}(A)^\perp. \tag{2.9}$$

To see this, it suffices to show the inclusion $P_{\mathcal{R}(A)^\perp}(\mathcal{R}(B)) \supset \mathcal{R}(A)^\perp$. Let $x \in \mathcal{R}(A)^\perp$. Then there exist $x_1 \in \mathcal{R}(A)$ and $x_2 \in \mathcal{R}(B)$ such that $x = x_1 + x_2$, so $x = P_{\mathcal{R}(A)^\perp} x_2 \in P_{\mathcal{R}(A)^\perp}(\mathcal{R}(B))$. This proves the claim. Similarly, we obtain

$$P_{\mathcal{R}(B)^\perp}(\mathcal{R}(A)) = \mathcal{R}(B)^\perp. \tag{2.10}$$

Moreover, by (2.8), we have

$$\{0\} = \mathcal{X}^\perp = (\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)})^\perp = \overline{\mathcal{R}(A)}^\perp \cap \overline{\mathcal{R}(B)}^\perp$$

and also the sum $\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}$ is closed. By Lemma 2.2 (iv) it follows that

$$\left(\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}\right)^\perp = \overline{\mathcal{R}(A)}^\perp + \overline{\mathcal{R}(B)}^\perp.$$

To sum up, we have the space decomposition (2.3). As $\mathcal{N}(A) \subset \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$, we have $\mathcal{N}(P_{\mathcal{R}(B)^\perp}A)^\perp \subset \mathcal{N}(A)^\perp$. Analogously we see $\mathcal{N}(P_{\mathcal{R}(A)^\perp}B)^\perp \subset \mathcal{N}(B)^\perp$ and, hence, decomposition (2.4) follows.

For $x \in \mathcal{X}_3^\perp = \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$ we have

$$Ax = \left(I - P_{\mathcal{R}(B)^\perp}\right) Ax = P_{\overline{\mathcal{R}(B)}} Ax.$$

Hence, $x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$ if and only if

$$Ax \in \overline{\mathcal{R}(B)}. \tag{2.11}$$

Similarly, $y \in \mathcal{N}(P_{\mathcal{R}(A)^\perp}B)$ if and only if $By \in \overline{\mathcal{R}(A)}$. Therefore, if $x_2 \in \mathcal{X}_2$ ($y_2 \in \mathcal{Y}_2$), then it follows that $x_2 \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$ (resp. $y_2 \in \mathcal{N}(P_{\mathcal{R}(A)^\perp}B)$) and, by (2.11)

$$Ax_2 \in \overline{\mathcal{R}(B)} \quad \text{(resp. } By_2 \in \overline{\mathcal{R}(A)}\text{)}. \tag{2.12}$$

Then the zero entries in (2.6) follow from the fact that $Ax = 0$ for $x \in \mathcal{N}(A)$, $By = 0$ for $y \in \mathcal{N}(B)$, $Ax \in \mathcal{R}(A)$, $By \in \mathcal{R}(B)$, and (2.12).

Step 2. We show that $(A_2\ B_2)$ *is right invertible.*

We have $\mathcal{N}(A) \subset \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$, $\mathcal{N}(B) \subset \mathcal{N}(P_{\mathcal{R}(A)^\perp}B)$ and by (2.8) and (2.3) we see that A_3 and B_3 are isomorphisms. Thus, there exists an isomorphism $U \in \mathcal{B}((\mathcal{R}(A)^\perp \dot{+} \mathcal{R}(B)^\perp) \oplus \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)})$

$$U := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -B_0 B_3^{-1} & -A_0 A_3^{-1} & 1 \end{pmatrix}$$

such that

$$U\begin{pmatrix} 0 & 0 & 0 & B_3 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & A_2 & A_0 & B_0 & B_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & B_3 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & B_2 & 0 \end{pmatrix}.$$

As $(A\ B)$ is right invertible, Lemma 2.4 shows that $(A_2\ B_2) : \mathcal{X}_2 \oplus \mathcal{Y}_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ is right invertible.

Step 3. We show (2.7).

By definition, we have $\mathcal{R}(A_2) \subset \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ and $\mathcal{R}(B_2) \subset \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$. We will only show $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \subset \overline{\mathcal{R}(B_2)}$. The proof for $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \subset \overline{\mathcal{R}(A_2)}$ is the same and, hence, we omit this proof.

Let $z \in \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$. Then there exists a sequence (z_n) in $\mathcal{R}(B)$ which converges to z. By the block representation (2.6) for B we find $z_{1,n}$ in $\mathcal{R}(A)^\perp$ and $z_{3,n} \in \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ with

$$z_n = z_{1,n} + z_{3,n}, \quad n \in \mathbb{N}, \tag{2.13}$$

where we have

$$z_{1,n} = B_3 y_{3,n} \quad \text{and} \quad z_{3,n} = B_0 y_{3,n} + B_2 y_{2,n} \quad \text{for } n \in \mathbb{N} \tag{2.14}$$

for some $y_{2,n} \in \mathcal{Y}_2$ and $y_{3,n} \in \mathcal{Y}_3$. The convergence of (z_n) implies the convergence of $(z_{1,n})$ to some $z_1 \in \mathcal{R}(A)^\perp$ and of $(z_{3,n})$ to some $z_3 \in \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$,

$$z = z_1 + z_3.$$

The vectors z and z_3 belong to $\overline{\mathcal{R}(A)}$, thus $z_1 \in \overline{\mathcal{R}(A)}$ and $z_1 = 0$ follows. Therefore $(B_3 y_{3,n})$ in (2.14) converges to zero. The fact that B_3 is an isomorphism implies $y_{3,n} \to 0$ as $n \to \infty$. We conclude

$$z = z_3 = \lim_{n\to\infty} z_{3,n} = \lim_{n\to\infty} B_2 y_{2,n}$$

and $z \in \overline{\mathcal{R}(B_2)}$ follows. Relation (2.7) is proved. □

The following proposition will be used in the proof of the main result.

Proposition 2.8 *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and let the row operator $(A\ B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{X})$ be right invertible. The following assertions are equivalent.*

(i) *$\mathcal{R}(B)$ is closed.*

(ii) *$P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ is a closed subspace in $\mathcal{X}$.*

(iii) *$\mathcal{R}(B_2)$ is closed.*

Proof. Let $\mathcal{R}(B)$ be closed. We have

$$P_{\mathcal{X}}(\mathcal{N}((A\ B))) = \{x \in \mathcal{X} : Ax \in \mathcal{R}(A) \cap \mathcal{R}(B)\} = \{x \in \mathcal{X} : Ax \in \mathcal{R}(B)\}$$

and $P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ is the pre-image of $\mathcal{R}(B)$ under A, and, hence, it is a closed subspace and (ii) holds.

If $P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ is closed, then also

$$\Omega := P_{\mathcal{X}}(\mathcal{N}((A\ B))) \cap \mathcal{N}(A)^{\perp} = \{x \in \mathcal{X} : x \in \mathcal{N}(A)^{\perp},\ Ax \in \mathcal{R}(A) \cap \mathcal{R}(B)\}$$

is closed. Decompose $x \in \Omega$ with respect to the decomposition, cf. Theorem 2.7, $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ as $x = x_1 + x_2 + x_3$ with $x_j \in \mathcal{X}_j$ for $j = 1, 2, 3$. Then $x_1 = 0$ and for some $y \in \mathcal{Y}$ we have $Ax = By$. Decompose y with respect to $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_3$ (cf. Theorem 2.7) as $y = y_1 + y_2 + y_3$ with $y_j \in \mathcal{Y}_j$ for $j = 1, 2, 3$. Relation (2.6) shows

$$Ax = A \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ A_3 x_3 \\ A_2 x_2 + A_0 x_3 \end{pmatrix} = \begin{pmatrix} B_3 y_3 \\ 0 \\ B_0 y_3 + B_2 y_2 \end{pmatrix} = B \begin{pmatrix} y_3 \\ y_2 \\ y_1 \end{pmatrix} = By$$

and, as A_3 is an isomorphism, we obtain $x_3 = 0$. Therefore $\Omega \subset \mathcal{X}_2$ and we write

$$\mathcal{X}_2 = \Omega \oplus (\mathcal{X}_2 \ominus \Omega).$$

By Theorem 2.7 $(A_2\ B_2)$ is right invertible and we obtain with Lemma 2.4

$$A_2(\mathcal{X}_2 \ominus \Omega) + B_2(\mathcal{Y}_2) = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}, \quad A_2(\mathcal{X}_2 \ominus \Omega) \cap B_2(\mathcal{Y}_2) = \{0\}.$$

Thus, using Lemma 2.1, we deduce that $A_2(\mathcal{X}_2 \ominus \Omega)$ and $\mathcal{R}(B_2)$ are closed.

Assume that (iii) holds. Then, by (2.7), the operator B_2 is an isomorphism. Let $z \in \overline{\mathcal{R}(B)}$. Then there exists a sequence (z_n) in $\mathcal{R}(B)$ which converges to z. By the block representation (2.6) for B we find $z_{1,n}$ in $\mathcal{R}(A)^{\perp}$ and $z_{3,n} \in \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ such that (2.13) and (2.14) hold for some $y_{2,n} \in \mathcal{Y}_2$ and $y_{3,n} \in \mathcal{Y}_3$. The convergence of (z_n) implies the convergence of $(z_{1,n})$ to some $z_1 \in \mathcal{R}(A)^{\perp}$ and of $(z_{3,n})$ to some $z_3 \in \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$, $z = z_1 + z_3$. As the operators B_2 and B_3 (cf. Theorem 2.7) are isomorphisms, we have

$$y_{3,n} \to B_3^{-1} z_1 \quad y_{2,n} \to -B_2^{-1} B_0 B_3^{-1} z_1 + B_2^{-1} z_3 \quad \text{as } n \to \infty.$$

Thus, with (2.6),

$$B \begin{pmatrix} B_3^{-1} z_1 \\ -B_2^{-1} B_0 B_3^{-1} z_1 + B_2^{-1} z_3 \\ 0 \end{pmatrix} = \begin{pmatrix} z_1 \\ 0 \\ z_3 \end{pmatrix} = z,$$

and $z \in \mathcal{R}(B)$. □

Lemma 2.9 *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and assume that the row operator $(A\ B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{X})$ is right invertible. Let A_2 and B_2 be as in Theorem 2.7. Then B_2 considered as an operator from $\mathcal{Y}_2$ to $\mathcal{R}(B_2)$ is one-to-one and has an inverse $B_2^{-1} : \mathcal{R}(B_2) \to \mathcal{Y}_2$. Define*

$$\widetilde{A}_2 := (0\ A_2) : \mathcal{X}_1 \oplus \mathcal{X}_2 \to \overline{\mathcal{R}(A)} \cap \overline{R(B)}.$$

Then $\widetilde{A}_2|_{P_{\mathcal{X}}(\mathcal{N}((A\ B)))}$ maps to $\mathcal{R}(B_2)$ and the operator

$$B_2^{-1}\widetilde{A}_2|_{P_{\mathcal{X}}(\mathcal{N}((A\ B)))} : P_{\mathcal{X}}(\mathcal{N}((A\ B))) \to \mathcal{Y}_2$$

is correctly defined.

If $\mathcal{R}(B)$ is closed, then B_2 is an isomorphism and we have

$$\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = P_{\mathcal{X}}(\mathcal{N}((A\ B)))$$

and the operator

$$B_2^{-1}\widetilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \to \mathcal{Y}_2 \tag{2.15}$$

is correctly defined.

Proof. As $\mathcal{Y}_2 \subset \mathcal{N}(B)^\perp$ the operator B_2 is one-to-one, hence its inverse $B_2^{-1} : \mathcal{R}(B_2) \to \mathcal{Y}_2$ exists. From

$$P_{\mathcal{X}}(\mathcal{N}((A\ B))) = \{x \in \mathcal{X} : Ax \in \mathcal{R}(A) \cap \mathcal{R}(B)\} \subset \{x \in \mathcal{X} : Ax \in \overline{\mathcal{R}(B)}\} \tag{2.16}$$

we conclude

$$P_{\mathcal{X}}(\mathcal{N}((A\ B))) \subset \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = \mathcal{X}_1 \oplus \mathcal{X}_2.$$

Moreover, we decompose $x \in P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ with respect to the decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ (cf. Theorem 2.7) as $x = x_1 + x_2 + x_3$ with $x_j \in \mathcal{X}_j$ for $j = 1, 2, 3$. Then $x_3 = 0$ and for some $y \in \mathcal{Y}$ we have $Ax = By$. Decompose y with respect to $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_3$ (cf. Theorem 2.7) as $y = y_1 + y_2 + y_3$ with $y_j \in \mathcal{Y}_j$ for $j = 1, 2, 3$. Relation (2.6) shows

$$Ax = A\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A_2 x_2 \end{pmatrix} = \begin{pmatrix} B_3 y_3 \\ 0 \\ B_0 y_3 + B_2 y_2 \end{pmatrix} = B\begin{pmatrix} y_3 \\ y_2 \\ y_1 \end{pmatrix} = By$$

and, as B_3 is an isomorphism, we obtain $y_3 = 0$ and $A_2 x_2 = B_2 y_2$. Thus $\widetilde{A}_2 x \in \mathcal{R}(B_2)$ for $x \in P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ and $B_2^{-1}\widetilde{A}_2|_{P_{\mathcal{X}}(\mathcal{N}((A\ B)))}$ is correctly defined. If $\mathcal{R}(B)$ is closed, then by Proposition 2.8 also $\mathcal{R}(B_2)$ is closed and by (2.7) we see that B_2 is an isomorphism. Moreover, from (2.16) we see in this case $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ and (2.15) follows. □

The following theorem is the main result. It provides a full characterization of isomorphic 2×2 operator matrices in terms of their entries.

Theorem 2.10 *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Assume that the row operator $(A\ B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{X})$ is right invertible and, hence, adopt the notions A_2, B_2, and $\mathcal{X}_j$, $\mathcal{Y}_j$, $j = 1, 2, 3$, as in Theorem 2.7 and $\widetilde{A}_2$ as in Lemma 2.9. Let $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{B}(\mathcal{Y})$. Define the operator matrix M by*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Define the operator $B_2^{-1} \widetilde{A}_2|_{P_\mathcal{X}(\mathcal{N}((A\ B)))}$ as in Lemma 2.9 and define

$$C_2 := P_{(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp} C|_{\mathcal{X}_1 \oplus \mathcal{X}_2} : X_1 \oplus \mathcal{X}_2 \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

and

$$D_2 := P_{(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp} D|_{\mathcal{Y}_2} : \mathcal{Y}_2 \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp.$$

Then M is an isomorphism if and only if the following two statements are satisfied:

(i) *The restriction $D|_{\mathcal{N}(B)} : \mathcal{N}(B) \to \mathcal{Y}$ is left invertible.*

(ii) *The operator*

$$\left(C_2 - D_2 B_2^{-1} \widetilde{A}_2\right)\Big|_{P_\mathcal{X}(\mathcal{N}((A\ B)))} : P_\mathcal{X}(\mathcal{N}((A\ B))) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

is one-to-one and surjective.

Proof. Let M be an isomorphism. Then the row operator $(A\ B) : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is right invertible, see Lemma 2.4, and the column operator $\left(\begin{smallmatrix} B \\ D \end{smallmatrix}\right) : \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ is injective. Moreover, if the range of $\left(\begin{smallmatrix} B \\ D \end{smallmatrix}\right)$ is not closed then there exists a sequence (y_n) in $\mathcal{Y}$ with $\|y_n\| = 1$, $n \in \mathbb{N}$, and $\left(\begin{smallmatrix} B \\ D \end{smallmatrix}\right) y_n \to 0$ as $n \to \infty$. But this implies $M \left(\begin{smallmatrix} 0 \\ y_n \end{smallmatrix}\right) \to 0$, a contradiction as M is assumed to be an isomorphism. Therefore the column operator $\left(\begin{smallmatrix} B \\ D \end{smallmatrix}\right)$ is left invertible, cf. Lemma 2.5.

Now let $z \in \overline{\mathcal{R}(D|_{\mathcal{N}(B)})}$. Then, there exists $z_n \in \mathcal{N}(B)$ such that $Dz_n \to z$ as $n \to \infty$, and we further have

$$\begin{pmatrix} B \\ D \end{pmatrix} z_n = \begin{pmatrix} 0 \\ Dz_n \end{pmatrix} \to \begin{pmatrix} 0 \\ z \end{pmatrix},$$

which together with Lemma 2.5 implies

$$\begin{pmatrix} B \\ D \end{pmatrix} x = \begin{pmatrix} 0 \\ z \end{pmatrix}$$

for some $x \in \mathcal{N}(B)$, and hence $D|_{\mathcal{N}(B)} x = z$. This proves that $\mathcal{R}(D|_{\mathcal{N}(B)})$ is closed, hence, $D|_{\mathcal{N}(B)}$ is left invertible by Lemma 2.5 and (i) is proved.

As $\mathcal{R}(D|_{\mathcal{N}(B)})$ is a closed subspace in $\mathcal{Y}$, we decompose $\mathcal{Y}$,

$$\mathcal{Y} = (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \oplus \mathcal{R}(D|_{\mathcal{N}(B)}). \tag{2.17}$$

Similar to the proof of Theorem 2.7, M as an operator from $\mathcal{N}(P_{\mathcal{R}(B)^\perp}A)\oplus \mathcal{X}_3\oplus\mathcal{Y}_3\oplus\mathcal{Y}_2\oplus\mathcal{Y}_1$ into

$$(\mathcal{R}(A)^\perp\dot{+}\mathcal{R}(B)^\perp)\oplus\overline{\mathcal{R}(A)}\cap\overline{R(B)}\oplus(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp\oplus\mathcal{R}(D|_{\mathcal{N}(B)})$$

has the following block representation

$$M=\begin{pmatrix} 0 & 0 & B_3 & 0 & 0\\ 0 & A_3 & 0 & 0 & 0\\ \widetilde{A}_2 & A_0 & B_0 & B_2 & 0\\ C_2 & C_3 & D_1 & D_2 & 0\\ C_4 & C_5 & D_3 & D_4 & D_5 \end{pmatrix}. \tag{2.18}$$

By Theorem 2.7, A_3 and B_3 are isomorphisms. Additionally, as M is an isomorphism, D_5 is also an isomorphism. Then there exist isomorphisms

$$U\in\mathcal{B}\left((\mathcal{R}(A)^\perp\dot{+}\mathcal{R}(B)^\perp)\oplus\overline{\mathcal{R}(A)}\cap\overline{R(B)}\oplus(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp\oplus\mathcal{R}(D|_{\mathcal{N}(B)})\right),$$
$$V\in\mathcal{B}\left(\mathcal{N}(P_{\mathcal{R}(B)^\perp}A)\oplus\mathcal{X}_3\oplus\mathcal{Y}_3\oplus\mathcal{Y}_2\oplus\mathcal{Y}_1\right)$$

with

$$U:=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ -B_0B_3^{-1} & -A_0A_3^{-1} & 1 & 0 & 0\\ -D_1B_3^{-1} & -C_3A_3^{-1} & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V:=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ -D_5^{-1}C_4 & -D_5^{-1}C_5 & -D_5^{-1}D_3 & -D_5^{-1}D_4 & 1 \end{pmatrix}$$

such that

$$UMV=\begin{pmatrix} 0 & 0 & B_3 & 0 & 0\\ 0 & A_3 & 0 & 0 & 0\\ \widetilde{A}_2 & 0 & 0 & B_2 & 0\\ C_2 & 0 & 0 & D_2 & 0\\ 0 & 0 & 0 & 0 & D_5 \end{pmatrix}. \tag{2.19}$$

Thus, M is an isomorphism if and only if

$$\Delta:=\begin{pmatrix} \widetilde{A}_2 & B_2\\ C_2 & D_2 \end{pmatrix}:\mathcal{N}(P_{\mathcal{R}(B)^\perp}A)\oplus\mathcal{Y}_2\to(\overline{\mathcal{R}(A)}\cap\overline{\mathcal{R}(B)})\oplus(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \tag{2.20}$$

is an isomorphism.

Case 1: $\mathcal{R}(B)$ is closed. In this case, from Lemma 2.9, $B_2 : \mathcal{Y}_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ is an isomorphism and $B_2^{-1}\widetilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \to \mathcal{Y}_2$ is correctly defined, see Lemma 2.9. According to Lemma 2.3, Δ is an isomorphism if and only if

$$C_2 - D_2B_2^{-1}\widetilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

is an isomorphism. By Lemma 2.9 $\mathcal{N}(P_{\mathcal{R}(B)^\perp}A) = P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ and (ii) is satisfied.

Case 2: $\mathcal{R}(B)$ is not closed. By Proposition 2.8 also $\mathcal{R}(B_2)$ is not closed which implies $\dim \mathcal{R}(B_2) = \infty$ and $\dim \mathcal{Y}_2 = \infty$. The dimension does not change when we close a subspace, therefore we conclude from (2.7)

$$\dim \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \dim \overline{\mathcal{R}(B_2)} = \dim \mathcal{R}(B_2) = \infty. \tag{2.21}$$

By Theorem 2.7 $(A_2\ B_2)$ is right invertible, (2.7) and Lemma 2.1 imply

$$\overline{\mathcal{R}(A_2) \cap \mathcal{R}(B_2)} = \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}.$$

Obviously, $\mathcal{R}(A_2) \cap \mathcal{R}(B_2) \subset \mathcal{R}(A) \cap \mathcal{R}(B)$ and we obtain $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \subset \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$. Thus

$$\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}.$$

From this and from $\mathcal{R}(A) \cap \mathcal{R}(B) \subset \mathcal{R}(A) \cap \overline{\mathcal{R}(B)} \subset \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ we conclude with (2.21)

$$\infty = \dim \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} = \dim \mathcal{R}(A) \cap \mathcal{R}(B) = \dim \mathcal{R}(A) \cap \overline{\mathcal{R}(B)}. \tag{2.22}$$

We will use (2.22) to show

$$\dim \mathcal{N}((\widetilde{A}_2\ B_2)) = \dim \mathcal{N}(P_{\mathcal{R}(B)^\perp}A). \tag{2.23}$$

For this we consider

$$\mathcal{N}((A\ B)) = \{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) : x \in \mathcal{N}(A)\} \oplus \left\{\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) : y \in \mathcal{N}(A)^\perp, Ay = -Bz\right\} \tag{2.24}$$

and

$$\mathcal{N}(P_{\mathcal{R}(B)^\perp}A) = \mathcal{N}(A) \oplus \left\{x : x \in \mathcal{N}(A)^\perp, Ax \in \overline{\mathcal{R}(B)}\right\}.$$

As A restricted to $\mathcal{N}(A)^\perp$ is injective, we obtain with (2.22)

$$\begin{aligned}\dim \left\{\left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) : y \in \mathcal{N}(A)^\perp, Ay = -Bz\right\} &= \dim \mathcal{R}(A) \cap \mathcal{R}(B) = \dim \mathcal{R}(A) \cap \overline{\mathcal{R}(B)} \\ &= \dim \left\{x : x \in \mathcal{N}(A)^\perp, Ax \in \overline{\mathcal{R}(B)}\right\}.\end{aligned}$$

Therefore

$$\dim \mathcal{N}((A\ B)) = \dim \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$$

and with (2.19) we obtain $\dim \mathcal{N}((\widetilde{A}_2\ B_2)) = \dim \mathcal{N}(P_{\mathcal{R}(B)^\perp} A)$, hence (2.23) is proved. Two separable Hilbert spaces of the same dimension are unitarily equivalent, therefore there exists a left invertible operator

$$\begin{pmatrix} G \\ H \end{pmatrix} : \mathcal{Y}_2 \to \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \oplus \mathcal{Y}_2 \text{ with range } \mathcal{N}((\widetilde{A}_2\ B_2)). \tag{2.25}$$

Since $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{N}(P_{\mathcal{R}(B)^\perp} A)$ and by Theorem 2.7 and Lemma 2.9 $(\widetilde{A}_2\ B_2) : \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \oplus \mathcal{Y}_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ is a right invertible operator. Then, see Remark 2.6, there exists a left invertible operator

$$\begin{pmatrix} E \\ F \end{pmatrix} : \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \to \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \oplus \mathcal{Y}_2 \tag{2.26}$$

such that

$$\widetilde{A}_2 E + B_2 F = I_{\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}} \quad \text{with } \mathcal{R}\left(\begin{pmatrix} E \\ F \end{pmatrix}\right) = (\mathcal{N}((\widetilde{A}_2\ B_2)))^\perp \tag{2.27}$$

Define

$$W = \begin{pmatrix} E & G \\ F & H \end{pmatrix} : \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} \oplus \mathcal{Y}_2 \to \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \oplus \mathcal{Y}_2. \tag{2.28}$$

As $\left(\begin{smallmatrix} G \\ H \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} E \\ F \end{smallmatrix}\right)$ are left invertible and from (2.25) and (2.27) we obtain easily that W is an isomorphism. We have

$$\Delta W = \begin{pmatrix} I_{\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}} & 0 \\ C_2 E + D_2 F & C_2 G + D_2 H \end{pmatrix}. \tag{2.29}$$

As M is an isomorphism, Δ is an isomorphism (see (2.20)) and the operator $C_2 G + D_2 H : \mathcal{Y}_2 \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$ is an isomorphism. Moreover, the operator B_2 considered as an operator from $\mathcal{Y}_2$ to $\mathcal{R}(B_2)$ is one-to-one and has an inverse, see Lemma 2.9. From $\widetilde{A}_2 G + B_2 H = 0$ we conclude $-B_2^{-1} \widetilde{A}_2 G = H$ and

$$C_2 G + D_2 H = (C_2 - D_2 B_2^{-1} \widetilde{A}_2) G. \tag{2.30}$$

Therefore, $C_2 - D_2 B_2^{-1} \widetilde{A}_2 : \mathcal{R}(G) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$ is one-to-one with range equal to $(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$. From

$$\begin{aligned} \mathcal{R}(\left(\begin{smallmatrix} G \\ H \end{smallmatrix}\right)) &= \mathcal{N}((\widetilde{A}_2\ B_2)) \\ &= \begin{pmatrix} \mathcal{N}(A) \\ 0 \end{pmatrix} \oplus \left\{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : x \in \mathcal{N}(A)^\perp, y \in \mathcal{N}(B)^\perp, Ax = -By \right\} \\ &= \mathcal{N}((A\ B)), \end{aligned} \tag{2.31}$$

see (2.24), it follows that $\mathcal{R}(G) = P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ and (ii) is shown.

Now let us assume that (i) and (ii) hold. Then $\mathcal{R}(D|_{\mathcal{N}(B)})$ is a closed subspace and $\mathcal{Y}$ admits a decomposition as in (2.17) and we obtain the representation of M as in (2.18), where A_3, B_3 and D_5 are isomorphisms. Then, taking the same U and V as above, we obtain the relation (2.19). Moreover, if Δ in (2.20) is an isomorphism, then M is an isomorphism.

If $\mathcal{R}(B)$ is closed, then from Lemma 2.9, $B_2 : \mathcal{Y}_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ is an isomorphism and $B_2^{-1}\widetilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \to \mathcal{Y}_2$ is correctly defined. Moreover, Lemma 2.9, $\mathcal{N}(P_{\mathcal{R}(B)^\perp}A) = P_{\mathcal{X}}(\mathcal{N}((A\ B)))$. Then, by (ii),

$$C_2 - D_2B_2^{-1}\widetilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

is an isomorphism and according to Lemma 2.3, Δ is an isomorphism and, hence, M is an isomorphism.

If $\mathcal{R}(B)$ is not closed, then as above, we define the operators G, H, E, F, and W as in (2.25), (2.26), (2.27), and (2.28). Moreover, the operator W in (2.28) is an isomorphism and also (2.30) and (2.31) hold. By (2.31) $\mathcal{R}(G) = P_{\mathcal{X}}(\mathcal{N}((A\ B)))$ and as B_2 is one-to-one, we see that the operator G in (2.25) is one-to-one. Hence, together with (ii), the operator $(C_2 - D_2B_2^{-1}\widetilde{A}_2)G : \mathcal{Y}_2 \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$ is one-to-one with range equal to $(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$. Therefore, by (2.30), $C_2G + D_2H$ is an isomorphism and, by (2.29) and as W is an isomorphism, also Δ is an isomorphism. Therefore, see (2.20), M is an isomorphism. □

Finally, we consider the following special case.

Theorem 2.11 *Let $A, B, C, D \in \mathcal{B}(\mathcal{X})$ and let $\mathcal{X}', \mathcal{X}''$ be closed subspaces of $\mathcal{X}$ with*

$$\mathcal{X} = \mathcal{X}' \oplus \mathcal{X}''$$

such that

$$\mathcal{R}(A) = \mathcal{X}', \quad \mathcal{N}(A) = \mathcal{X}'', \quad \mathcal{R}(B) = \mathcal{X}'', \quad \text{and} \quad \mathcal{N}(B) = \mathcal{X}'.$$

Moreover assume that the restriction $D|_{\mathcal{X}'} : \mathcal{X}' \to \mathcal{X}$ is left invertible. Then the 2×2 operator matrix M,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

is an isomorphism if and only if

$$C_2 := P_{(\mathcal{R}(D|_{\mathcal{X}'}))^\perp} C|_{\mathcal{X}''} : \mathcal{X}'' \to (\mathcal{R}(D|_{\mathcal{X}'}))^\perp$$

is an isomorphism.

In particular, if, in addition, $\mathcal{R}(B) \neq \{0\}$ and the operator $D|_{\mathcal{X}'} : \mathcal{X}' \to \mathcal{X}$ is an isomorphism, then for every operator $C \in \mathcal{B}(\mathcal{X})$ the 2×2 operator matrix M is not an isomorphism.

Proof. Denote by $P_{\mathcal{X}}$ the orthogonal projection in $\mathcal{X} \oplus \mathcal{X}$ onto the first component. Then

$$P_{\mathcal{X}}(\mathcal{N}((A\ B))) = \mathcal{N}(A) = \mathcal{X}''.$$

Moreover, we have $\mathcal{N}(P_{\mathcal{R}(B)^\perp}A)^\perp = \mathcal{N}(P_{\mathcal{X}'}A)^\perp = \mathcal{N}(A)^\perp$ and $\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \mathcal{X}' \cap \mathcal{X}'' = \{0\}$. Then the space $\mathcal{X}_2$ in Theorem 2.7 equals zero and the operators A_2 and $\widetilde{A}_2$ in Theorem 2.10 are zero. Then the statements of Theorem 2.11 follow from Theorem 2.10. □

3 A characterization of isomorphic row operators

In this section let A, B, C, D and M be as in Theorem 2.10. In the following we use Theorems 2.7 and 2.10 to characterize the case of an isomorphic row operator $(A\ B)$ and to derive a necessary condition for M to be an isomorphism.

Proposition 3.1 *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. The row operator $(A\ B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{X})$ is an isomorphism (i.e. $(A\ B)$ is left and right invertible) if and only if the following two statements are satisfied:*

(i) $\mathcal{N}(A) = \mathcal{N}(B) = \{0\}$.

(ii) $\mathcal{R}(A) = \mathcal{R}(B)^\perp$, $\mathcal{R}(B) = \mathcal{R}(A)^\perp$.

Proof. If (i) and (ii) hold, then $Ax + By = 0$ for some $x \in \mathcal{X}$, $y \in \mathcal{Y}$ implies $Ax = -By \in \mathcal{R}(B)$. By (ii), $Ax = 0$ and, hence, $By = 0$ follows. Then (i) implies $x = y = 0$ and $\mathcal{N}((A\ B)) = \{0\}$. Moreover, we have with (ii)

$$\mathcal{R}((A\ B)) \subset \mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A) + \mathcal{R}(A)^\perp = \mathcal{X}$$

and the row operator $(A\ B)$ is an isomorphism.

For the contrary let the row operator $(A\ B)$ be an isomorphism. If for some $x \in \mathcal{X}$ we have $Ax = 0$ then $(A\ B)\binom{x}{0} = 0$ and, as $\mathcal{N}(A\ B) = \{0\}$, $x = 0$ follows. That is, $\mathcal{N}(A) = \{0\}$ and, similarly, we see $\mathcal{N}(B) = \{0\}$. This shows (i). In order to show (ii) let $x \in \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$ and assume $x \neq 0$. Then there exists sequences (x_n) in $\mathcal{X}$ and (y_n) in $\mathcal{Y}$ such that (Ax_n) and (By_n) converge both to x with $\liminf_{n\to\infty} \|x_n\| > 0$ and $\liminf_{n\to\infty} \|y_n\| > 0$. But then $(A\ B)\binom{x_n}{-y_n} = Ax_n - By_n$ tends to zero and $\mathcal{R}((A\ B))$ is not closed, a contradiction. This shows

$$\overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)} = \{0\}. \tag{3.1}$$

As $x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$ if and only if $Ax \in \overline{\mathcal{R}(B)}$ (see also (2.11)), we conclude with $\mathcal{N}(A) = \{0\}$ and (3.1)

$$\mathcal{N}(P_{\mathcal{R}(B)^\perp}A) = \{0\}.$$

In the same way we obtain from (3.1) and $\mathcal{N}(B) = \{0\}$ that $\mathcal{N}(P_{\mathcal{R}(A)^\perp} B) = \{0\}$. Then for the spaces $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ from Theorem 2.7 we conclude

$$\mathcal{X}_1 = \{0\}, \quad \mathcal{X}_2 = \{0\}, \quad \mathcal{X}_3 = \mathcal{X}, \quad \mathcal{Y}_1 = \{0\}, \quad \mathcal{Y}_2 = \{0\}, \quad \text{and} \quad \mathcal{Y}_3 = \mathcal{Y}$$

and the row operator $(A\ B)$ admits a representation according to Theorem 2.7 with respect to the decompositions $\mathcal{X} \oplus \mathcal{Y}$ and $\mathcal{X} = \mathcal{R}(A)^\perp \dot{+} \mathcal{R}(B)^\perp$ of the form

$$\begin{pmatrix} 0 & B_3 \\ A_3 & 0 \end{pmatrix},$$

where $A_3 \in \mathcal{B}\left(\mathcal{X}, \mathcal{R}(B)^\perp\right)$ and $B_3 \in \mathcal{B}\left(\mathcal{Y}, \mathcal{R}(A)^\perp\right)$ are isomorphisms. This shows (ii). □

Example 3.2 *Let $\mathcal{X} = \mathcal{Y} = \ell^2(\mathbb{N})$ and consider the following operators A and B in X:*

$$A(x_n)_{n\in\mathbb{N}} := (x_1, 0, x_2, 0 \dots) \quad \textit{and} \quad B(x_n)_{n\in\mathbb{N}} := (0, x_1, 0, x_2 \dots).$$

Then the row operator $(A\ B)$ satisfies (i) *and* (ii) *of Proposition 3.1 and, hence, $(A\ B)$ is an isomorphism.*

As a consequence, we derive the following condition for M to be an isomorphism.

Corollary 3.3 *Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{B}(\mathcal{Y})$. If*

$$\mathcal{Y} \neq \{0\} \quad \textit{and} \quad \mathcal{N}((A\ B)) = \{0\}$$

then the operator matrix M

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is not a isomorphism.

Proof. If M is an isomorphism, then as noted in the proof of Theorem 2.10, the row operator $(A\ B)$ is right invertible. Assume $\mathcal{N}((A\ B)) = \{0\}$. Then $(A\ B)$ is an isomorphism, and, by Proposition 3.1, $\mathcal{N}(B) = \{0\}$. Hence, we obtain $(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp = \mathcal{Y}$ and (ii) in Theorem 2.10 cannot be true unless $\mathcal{Y} = \{0\}$. Therefore, either $\mathcal{Y} = \{0\}$ or $\mathcal{N}((A\ B)) \neq \{0\}$ holds. □

4 Application to Hamiltonian operators

In this section we consider the special case of Hamiltonian operators, i.e., in the situation of Theorem 2.10, $\mathcal{X} = \mathcal{Y}$, the operators B, C are self-adjoint and $D = -A^*$. Under these assumptions, Theorem 2.10 takes the following simple form.

Theorem 4.1 *Let $A, B, C \in \mathcal{B}(\mathcal{X})$. Assume that the row operator $(A\ B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{X}, \mathcal{X})$ is right invertible and that B and C are self-adjoint operators in $\mathcal{X}$, i.e. $B = B^*$ and $C = C^*$. Adopt the notions A_2, B_2, and $\mathcal{X}_j$, $\mathcal{Y}_j$, $j = 1, 2, 3$, as in Theorem 2.7 and $\widetilde{A}_2$ as in Lemma 2.9. Define the operator $B_2^{-1}\widetilde{A}_2|_{P_\mathcal{X}(\mathcal{N}((A\ B)))}$ as in Lemma 2.9 and define*

$$C_2 := P_{\mathcal{N}(P_{\mathcal{R}(B)^\perp} A)} C|_{\mathcal{X}_1 \oplus \mathcal{X}_2} : X_1 \oplus \mathcal{X}_2 \to \mathcal{N}(P_{\mathcal{R}(B)^\perp} A)$$

and

$$(-A^*)_2 := -P_{\mathcal{N}(P_{\mathcal{R}(B)^\perp} A)} A^*|_{\mathcal{Y}_2} : \mathcal{Y}_2 \to \mathcal{N}(P_{\mathcal{R}(B)^\perp} A).$$

Then the Hamiltonian operator

$$H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$$

is an isomorphism if and only if

(i) *the operator*

$$\left(C_2 - (-A^*)_2 B_2^{-1} \widetilde{A}_2\right)\Big|_{P_\mathcal{X}(\mathcal{N}((A\ B)))} : P_\mathcal{X}(\mathcal{N}((A\ B))) \to \mathcal{N}(P_{\mathcal{R}(B)^\perp} A)$$

is one-to-one and surjective.

If in this case we have, in addition, that $\mathcal{R}(B)$ is closed, then $C_2 - (-A^*)_2 B_2^{-1} \widetilde{A}_2 \in \mathcal{B}(\mathcal{N}(P_{\mathcal{R}(B)^\perp} A))$ is an isomorphism.

Proof. By assumption, the row operator $(A\ B)$ is right invertible, hence (see Lemma 2.4) its range is closed and $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{X}$. The same applies to $(B\ \ -A)$ and thus its adjoint,

$$(B\ \ -A)^* = \begin{pmatrix} B \\ -A^* \end{pmatrix},$$

has a closed range and is one-to-one. Let $z \in \overline{\mathcal{R}(-A^*|_{\mathcal{N}(B)})}$. Then, there exists $z_n \in \mathcal{N}(B)$ such that $-A^* z_n \to z$ as $n \to \infty$, and we further have

$$\begin{pmatrix} B \\ -A^* \end{pmatrix} z_n = \begin{pmatrix} 0 \\ -A^* z_n \end{pmatrix} \to \begin{pmatrix} 0 \\ z \end{pmatrix},$$

which together with the closedness of the range of $(B\ \ -A)^*$ implies

$$\begin{pmatrix} B \\ -A^* \end{pmatrix} x = \begin{pmatrix} 0 \\ z \end{pmatrix}$$

for some $x \in \mathcal{N}(B)$, and hence $-A^*|_{\mathcal{N}(B)} x = z$. This proves that $\mathcal{R}(-A^*|_{\mathcal{N}(B)})$ is closed and (i) in Theorem 2.10 is satisfied for $D = -A^*$.

Next, we verify

$$(\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp = \mathcal{N}(P_{\mathcal{R}(B)^\perp}A). \tag{4.1}$$

Indeed, if $x \in (\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp$, we have $(-Ax, y) = (x, -A^*y) = 0$ for every $y \in \mathcal{N}(B)$, hence $-Ax \in \mathcal{N}(B)^\perp$, which together with the self-adjointness of B deduces $Ax \in \overline{\mathcal{R}(B)}$, and hence $x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$; while if $x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A)$, then $Ax \in \overline{\mathcal{R}(B)}$, and hence we have for $y \in \mathcal{N}(B)$ that $(x, -A^*y) = (-Ax, y) = 0$, i.e., $x \in (\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp$.

Now the equivalence of (i) and the fact that H is an isomorphism follows from (4.1) and Theorem 2.10. The additional statement in the case of a closed range of B follows from Lemma 2.9. □

Acknowledgements

Junjie Huang gratefully acknowledges the support by the National Natural Science Foundation of China (No. 11461049), and the Natural Science Foundation of Inner Mongolia (No. 2013JQ01).

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